Parametrically synthesized fast orthogonal transforms

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Abstract

A method of fast orthogonal linear transform algorithm synthesis by a single arbitrary vector is introduced. The method is based on factorization of a matrix and using the arbitrary input vector to define nonzero terms of the factors for building fast linear transform algorithms.

Introduction

A direct linear transform is a dot product of a transformation matrix by an input vector:

$$\vec{Z} = [H] \cdot \vec{X} \tag{1}$$

Here

$[H]$ - is an orthogonal normalized matrix,
$\vec{X}$ - is an input vector, and
$\vec{Z}$ - is an output vector.

Our goal is to build orthogonal matrix $[H]$ and its inverse providing desired properties of vector $\vec{Z}$ for a given arbitrary vector $\vec{X}$. Another goal is to build fast transform algorithms with low computational complexity.

1. Matrix factorization

Let's find such a matrix $[H]$ that for an arbitrary input vector $\vec{X}$ the output vector $\vec{Z}$ will have one and only one nonzero element situated at the first position and all other elements equal zero

$$\vec{Z} = \{E_x, 0, \dots, 0\} \tag{2}$$

and E_x is the energy of the input vector $\vec{X}$:

$$E_x = \vec{X} \cdot \vec{X}^{T*} \tag{3}$$

Superscript symbols 'T' and '*' depict transpose and complex conjugate operators correspondingly.

The inverse of (1) can be written as

$$\vec{X} = [H]^{-1} \cdot \vec{Z} \tag{4}$$

Matrix $[H]$ is orthogonal and normalized and therefore its inverse is equal to its conjugate transposition

$$[H]^{-1} = [H]^{T*} \tag{5}$$

and equation (4) becomes

$$\vec{X} = [H]^{T*} \cdot \vec{Z} \tag{6}$$

which is equivalent to the system of scalar equations

$$\begin{cases} x_0 = E_x \cdot {h_{0,0}}^* \\ x_1 = E_x \cdot {h_{1,0}}^* \\ \dots \\ x_i = E_x \cdot {h_{i,0}}^* \\ \dots \\ x_{N-1} = E_x \cdot {h_{N-1,0}}^* \end{cases} \tag{7}$$

Let's assume that matrix $[H]$ is defined as a product of sparsely populated matrices

$$[H] = \prod_{r=1}^{n} [G_r] \tag{8}$$

where $N = 2^n$, and

$$[G_r] = \begin{bmatrix} A_{r_0} & 0 & \cdots & 0 & 0 & \cdots & 0 & B_{r_0} & 0 & \dots & 0 & 0 \\ C_{r_0} & 0 & \cdots & 0 & 0 & \cdots & 0 & D_{r_0} & 0 & \cdots & 0 & 0 \\ 0 & A_{r_1} & \cdots & 0 & 0 & \cdots & 0 & 0 & B_{r_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & A_{r_{\frac{N}{2}-1}} & \cdots & 0 & 0 & 0 & \cdots & 0 & B_{r_{\frac{N}{2}-1}} \\ 0 & 0 & \cdots & 0 & C_{r_{\frac{N}{2}-1}} & \cdots & 0 & 0 & 0 & \cdots & 0 & D_{r_{\frac{N}{2}-1}} \end{bmatrix} \tag{9}$$

Nonzero elements of matrix $[G_r]$ form a set of generalized spectral cores represented by $2x2$ matrices:

$$\left\{ [M_r]_l = \begin{bmatrix} A_{r_l} & B_{r_l} \\ C_{r_l} & D_{r_l} \end{bmatrix} \middle| \, l \in \left[0, \frac{N}{2} - 1\right] \right\} \tag{10}$$

Let's represent matrix $[H]$ element indices in binary format as follows:

$$h_{x,u} = h_{x_{n-1,},\dots,x_0;u_{n-1,},\dots,u_0} \tag{11}$$

The elements of the matrix $[H]$ can be represented by products of the corresponding spectral core components:

$$h_{x_{n-1,},\dots,x_0;u_{n-1,},\dots,u_0} = \prod_{r=0}^{n-1} {A_r}^{\overline{x_r}\cdot\overline{u_r}} \cdot {B_r}^{\overline{x_r}\cdot u_r} \cdot {C_r}^{x_r\cdot\overline{u_r}} \cdot {D_r}^{x_r\cdot u_r} \tag{12}$$

where

$$x_r, u_r \in [0,1] \quad (13)$$

$$\overline{x_r} = 1 - x_r \quad (14)$$

Since matrix $[H]$ is orthogonal and normalized, the spectral core elements (10) are related to each other as follows [1,2]:

$$\begin{bmatrix} A_{r_l} & B_{r_l} \\ C_{r_l} & D_{r_l} \end{bmatrix} = \sqrt{2} \cdot \begin{bmatrix} \cos Q_{r_l} & \sin Q_{r_l} \\ \sin Q_{r_l} & -\cos Q_{r_l} \end{bmatrix} \quad (15)$$

Let's here and below define

$$C_{r_l} = \sqrt{2} \cdot \cos Q_{r_l} \quad (16)$$

and

$$S_{r_l} = \sqrt{2} \cdot \sin Q_{r_l} \quad (17)$$

Then (10) and (15) become

$$[M_r]_l = \begin{bmatrix} C_{r_l} & S_{r_l} \\ S_{r_l} & -C_{r_l} \end{bmatrix} \quad (18)$$

and nonzero terms of the matrices $[G_r]$ according to (9) will be:

$$\begin{cases} g_{r_{i,2\cdot i}} = C_{r_i} \\ g_{r_{i,2\cdot i+1}} = S_{r_i} \\ g_{r_{\frac{n}{2}+i,2\cdot i}} = S_{r_i} \\ g_{r_{\frac{n}{2}+i,2\cdot i+1}} = -C_{r_i} \end{cases} \quad (19)$$

where

$$r \in [0, n-1] \quad (20)$$

and

$$i \in [0, \tfrac{N}{2} - 1] \quad (21)$$

Now (7) becomes

$$
\begin{cases}
x_0 = E_x \cdot {h_{0,0}}^* = E_x \cdot {C_{n-1,_0}}^* \cdot {C_{n-2,_0}}^* \cdots \cdot {C_{0,_0}}^* \\
x_1 = E_x \cdot {h_{1,0}}^* = E_x \cdot {S_{n-1,_0}}^* \cdot {C_{n-2,_0}}^* \cdots \cdot {C_{0,_0}}^* \\
x_2 = E_x \cdot {h_{2,0}}^* = E_x \cdot {C_{n-1,_0}}^* \cdot {S_{n-2,_0}}^* \cdots \cdot {C_{0,_0}}^* \\
x_3 = E_x \cdot {h_{3,0}}^* = E_x \cdot {S_{n-1,_0}}^* \cdot {S_{n-2,_0}}^* \cdots \cdot {C_{0,_0}}^* \\
\vdots \\
x_{N-2} = E_x \cdot {h_{N-2,0}}^* = E_x \cdot {C_{n-1,_0}}^* \cdot {S_{n-2,_0}}^* \cdots \cdot {S_{0,_0}}^* \\
x_{N-1} = E_x \cdot {h_{N-1,0}}^* = E_x \cdot {S_{n-1,_0}}^* \cdot {S_{n-2,_0}}^* \cdots \cdot {S_{0,_0}}^*
\end{cases}
\tag{22}
$$

2. **Transform synthesis**

Using well known trigonometric identities [1,2], the solution of the system of equations (22) can be easily found:

$$
Q_{{n-r-1}_i} = \begin{cases}
\mathrm{atan}\left(\sqrt{\dfrac{\sum_{j=N\cdot 2^{-r-1}}^{N\cdot 2^{-r}-1} x^2_{N\cdot 2^{-r}\cdot i \bmod 2^r+j}}{\sum_{j=0}^{N\cdot 2^{-r-1}-1} x^2_{N\cdot 2^{-r}\cdot i \bmod 2^r+j}}}\right), \text{when } N > 2^{r+1} \\
\mathrm{atan}\left(\dfrac{x_{N\cdot 2^{-r}\cdot i \bmod 2^r+1}}{x_{N\cdot 2^{-r}\cdot i \bmod 2^r+1}}\right), \text{when } N = 2^{r+1}
\end{cases}
\tag{23}
$$

Similar derivation can be done for vector $\vec{Z}$ defined not as in (2) where its nonzero term was in the first position with index 0, but having its nonzero term at position $k \in [0, N-1]$:

$$
\vec{Z} = \{0, \ldots, 0, E_x, 0, \ldots, 0\} \tag{24}
$$

In this case (23) becomes

$$
Q_{{n-r-1}_{i,k}} = \begin{cases}
\mathrm{atan}\left((-1)^k \cdot \left(\dfrac{\sum_{j=N\cdot 2^{-r-1}}^{N\cdot 2^{-r}-1} x^2_{N\cdot 2^{-r}\cdot i \bmod 2^r+j}}{\sum_{j=0}^{N\cdot 2^{-r-1}-1} x^2_{N\cdot 2^{-r}\cdot i \bmod 2^r+j}}\right)^{\frac{(-1)^{\left[\frac{k}{N\cdot 2^{-r}}\right]}}{2}}\right), \text{when } N > 2^{r+1} \\
\mathrm{atan}\left((-1)^k \cdot \left(\dfrac{x_{N\cdot 2^{-r}\cdot i \bmod 2^r+1}}{x_{N\cdot 2^{-r}\cdot i \bmod 2^r+1}}\right)^{(-1)^{\left[\frac{k}{N\cdot 2^{-r}}\right]}}\right), \text{when } N = 2^{r+1}
\end{cases}
\tag{25}
$$

Only half of all spectral cores are defined in (23) or (25). The other half can be chosen arbitrarily. These spectral core parameter sets $\{Q\}$ define a unique pair of direct and inverse fast orthogonal transforms.

The direct transform is

$$
\{z_{2\cdot i}, z_{2\cdot i+1}\}_{r+1} = [M_r]_i \cdot \left\{z_i, z_{i+\frac{N}{2}}\right\}_r^T \tag{26}
$$

and the inverse transform is

$$\left\{z_i, z_{i+\frac{N}{2}}\right\}_r = [M_{r+1}]_i \cdot \{z_{2\cdot i}, z_{2\cdot i+1}\}_{r+1}{}^T \qquad (27)$$

where

r and i were defined in (20) and (21),
$\vec{Z}_r$ - intermitting vector ($r > 0$),
$\vec{Z}_0 = \vec{X}$ - input vector.

Conclusion

Transform pair (26) and (27) require each $2 \cdot N \cdot log_2 N$ two-argument multiplications and $N \cdot log_2 N$ two-argument additions and has computational complexity similar to fast Fourier transform algorithm.

References

[1] P. Dourbal, V. Shabalov. "Generator of Basis Functions" USSR Patent 1319013.

[2] V. Grigoriev, P. Dourbal, V. Shabalov. Generator of Basis Functions" USSR Patent 1413615.