PERMUTATIONS

16.1 THE FACTORIAL

In this section, we shall introduce the term and notation of factorial which will be often used in this chapter and the next three chapters.

FACTORIAL *The continued product of first n natural numbers is called the "n factorial" and is denoted by $n!$ or $\underline{\lfloor n}$.*

i.e. $\quad n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n.$

Thus, $\quad 3! = 1 \times 2 \times 3 = 6; \quad 4! = 1 \times 2 \times 3 \times 4 = 24, \quad 5! = 1 \times 2 \times 3 \times 4 \times 5 = 120$ etc.

Clearly, $n!$ is defined for positive integers only.

ZERO FACTORIAL *As we will require zero factorial in the later sections of this chapter and it does not make any sense to define it as the product of the integers from 1 to zero. So, we define 0! = 1.*

NOTE *Factorials of proper fractions or negative integers are not defined. Factorial n is defined only for whole numbers.*

DEDUCTION We have,

$$n! = 1 \times 2 \times 3 \times 4 \ldots \times (n-1) \times n = [1 \times 2 \times 3 \times 4 \ldots \times (n-1)]\, n = [(n-1)\,!]\, n = n \times (n-1)!$$

Thus, $\quad n! = n \times (n-1)!$

Similarly,

$$n! = n\,(n-1)\,(n-2)! = n\,(n-1)\,(n-2)\,(n-3)! = n\,(n-1)\,(n-2)\,(n-3)\,(n-4)! \text{ and so on.}$$

For example, $8! = 8\,(7!)$, $\quad 5! = 5\,(4!)\quad$ and $\quad 2! = 2\,(1!)$

Following examples will illustrate the use of this property of factorial n.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *Compute:* (i) $\dfrac{20!}{18!}$ $\qquad$ (ii) $\dfrac{10!}{6!\,4!}$

SOLUTION (i) We have,

$$\frac{20!}{18!} = \frac{20\,(19!)}{18!} = \frac{20 \times 19 \times 18!}{18!} \qquad [\because n! = n \times (n-1)\,!]$$

$$= 20 \times 19 = 380$$

(ii) $$\frac{10!}{6!\,4!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6! \times (4 \times 3 \times 2 \times 1)} = \frac{10 \times 9 \times 8 \times 7}{4 \times 3 \times 2 \times 1} = 210$$

EXAMPLE 2 *Convert the following products into factorials:*

(i) $6 \cdot 7 \cdot 8 \cdot 9 \cdot 10$ $\qquad$ (ii) $2 \cdot 4 \cdot 6 \cdot 8 \cdot 10$

SOLUTION (i) $$6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10}{1 \cdot 2 \cdot 3 \cdot 4 \; 5} = \frac{10!}{5!}$$

(ii) $$2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 = (2 \times 1)\,(2 \times 2)\,(2 \times 3)\,(2 \times 4)\,(2 \times 5) = 2^5 \times (1 \cdot 2 \cdot 3 \cdot 4 \cdot 5) = 2^5 \times 5!$$

EXAMPLE 3 *Find the LCM of 4!, 5! and 6!*

SOLUTION We have, $5! = 5 \times 4!$ and $6! = 6 \times 5 \times 4!$

$\therefore$ L.C.M. of $4!, 5!, 6! =$ L.C.M. $\{4!, 5 \times 4!, 6 \times 5 \times 4!\} = (4!) \times 5 \times 6 = 6! = 720$

EXAMPLE 4 *If* $\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$, *find* x.

SOLUTION We have,

$$\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$$

$$\Rightarrow \quad \frac{1}{9!} + \frac{1}{10 \times 9!} = \frac{x}{11 \times 10 \times 9!}$$

$$\Rightarrow \quad \frac{1}{9!}\left(1 + \frac{1}{10}\right) = \left(\frac{x}{11 \times 10}\right) \times \frac{1}{9!}$$

$$\Rightarrow \quad 1 + \frac{1}{10} = \frac{x}{11 \times 10} \Rightarrow \frac{11}{10} = \frac{x}{11 \times 10} \Rightarrow x = 11 \times 11 = 121.$$

<u>ALITER</u> We have,

$$\frac{1}{9!} + \frac{1}{10!} = \frac{x}{11!}$$

Multiplying both sides by the LCM of 9!, 10! and 11! i.e. by 11!, we obtain

$$\frac{11!}{9!} + \frac{11!}{10!} = \frac{x}{11!} \times 11!$$

$$\Rightarrow \quad \frac{11 \times 10 \times 9!}{9!} + \frac{11 \times 10!}{10!} = x$$

$$\Rightarrow \quad 11 \times 10 + 11 = x$$

$$\Rightarrow \quad x = 121.$$

EXAMPLE 5 *Find n, if:*

(i) $(n+2)! = 2550 \times n!$ (ii) $(n+1)! = 12 \times (n-1)!$

SOLUTION (i) We have,

$$(n+2)! = 2550 \times n!$$

$$\Rightarrow \quad (n+2)(n+1) \times n! = 2550 \times n!$$

$$\Rightarrow \quad (n+2) \times (n+1) = 2550$$

$$\Rightarrow \quad (n+2)(n+1) = 51 \times 50$$ [Expressing 2550 as the product of two consecutive natural numbers]

$$\Rightarrow \quad n+2 = 51 \text{ or } n+1 = 50$$ [By comparing]

$$\Rightarrow \quad n = 49$$

(ii) We have,

$$(n+1)! = 12 \times (n-1)!$$

$$\Rightarrow \quad (n+1) \times n \times (n-1)! = 12 \times (n-1)!$$

$$\Rightarrow \quad n(n+1) = 12$$

$$\Rightarrow \quad (n+1)n = 4 \times 3 \Rightarrow n = 3$$ [By comparing]

EXAMPLE 6 *If* $\frac{n!}{2!(n-2)!}$ *and* $\frac{n!}{4!(n-4)!}$ *are in the ratio 2 : 1, find the value of n.*

SOLUTION We have,

$$\frac{n!}{2!(n-2)!} : \frac{n!}{4!(n-4)!} = 2:1$$

$$\Rightarrow \quad \frac{n!}{2!(n-2)!} \times \frac{4!(n-4)!}{n!} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{4!(n-4)!}{2!(n-2)\times(n-3)\times(n-4)!} = \frac{2}{1}$$

$$\Rightarrow \quad \frac{4\times 3\times 2!}{2!(n-2)(n-3)} = \frac{2}{1}$$

$$\Rightarrow \quad (n-2)(n-3) = 6 \Rightarrow (n-2)(n-3) = 3\times 2 \Rightarrow n-2=3 \text{ and } n-3=2 \Rightarrow n=5$$

EXAMPLE 7 *Prove that:* $\frac{(2n)!}{n!} = \left\{1\cdot 3\cdot 5 \ldots (2n-1)\right\} 2^n.$

SOLUTION We have,

$$\frac{(2n)!}{n!} = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7\cdot 8\ldots(2n-2)(2n-1)(2n)}{n!}$$

$$= \frac{\{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)\}\cdot\{2\cdot 4\cdot 6\cdot 8\ldots(2n-2)(2n)\}}{n!}$$

$$= \frac{\{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)\}\, 2^n\, \{1\cdot 2\cdot 3\cdot 4\ldots(n-1)\, n\}}{n!}$$

$$= \frac{\{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)\}\cdot 2^n\cdot n!}{n!} = \{1\cdot 3\cdot 5\cdot 7\ldots(2n-1)\}\, 2^n$$

LEVEL-2

EXAMPLE 8 *Prove that (n! + 1) is not divisible by any natural number between 2 and n.*

SOLUTION Let m be divisible by k and r be any natural number between 1 and k. If $m+r$ is divided by k, then we obtain r as the remainder.

We have, $n! = 1\cdot 2\cdot 3\cdot 4\ldots(n-1)\cdot n.$

Therefore, $n!$ is divisible by every natural number between 2 and n. So, $(n!+1)$, when divided by any natural number between 2 and n, leaves 1 as the remainder.

Hence, $(n!+1)$ is not divisible by any natural number between 2 and n.

EXAMPLE 9 *Prove the inequalities* $(n!)^2 \le n^n (n!) < (2n)!$ *for all positive integers n.*

SOLUTION Clearly,

$$(n!)^2 = (n!)(n!) = (1\cdot 2\cdot 3\cdot 4\ldots(n-1)\, n)(n!)$$

Now,
$$\left.\begin{array}{r} 1\le n \\ 2\le n \\ 3\le n \\ \vdots\ \vdots\ \vdots \\ (n-1)\le n \\ n\le n \end{array}\right\} \Rightarrow 1\cdot 2\cdot (n-1)\, n \le \underbrace{n\cdot n\cdot n\ldots n}_{n\text{-times}}$$

$$\Rightarrow \quad n! \le n^n \Rightarrow (n!)(n!) \le n^n (n!) \Rightarrow (n!)^2 \le n^n (n!) \qquad \ldots(i)$$

We have, $(2n)! = 1\cdot 2\cdot 3\ldots(n-1)\, n\, (n+1)(n+2)\ldots(2n-1)(2n)$

$\Rightarrow \quad (2n)! = n!(n+1)(n+2)\ldots(2n-1)(2n)$

Now, $\left.\begin{array}{r} n+1>n \\ n+2>n \\ n+3>n \\ \vdots \quad \vdots \quad \vdots \\ n+(n-1)>n \\ n+n>n \end{array}\right\} \Rightarrow (n+1)(n+2)(n+3)\ldots(2n-1)(2n)>n^n$

$\Rightarrow \quad n!(n+1)(n+2)\ldots(2n-1)(2n)>n!\,n^n$ ...(ii)

$\Rightarrow \quad (2n)!>n!\,n^n \Rightarrow n!\,n^n<(2n)!$

From (i) and (ii), we get $(n!)^2 \leq n^n (n!) < (2n)!$

EXAMPLE 10 *Prove that 33! is divisible by 2^{15}. What is the largest integer n such that 33! is divisible by 2^n?*

SOLUTION Let $E_2(n)$ denote the index of 2 in n. Then,

$E_2(33!) = E_2(1.2.3.4.5.6.....32.33)$

$\Rightarrow \quad E_2(33!) = E_2(2.4.6.8.....30.32)$

$\Rightarrow \quad E_2(33!) = 16 + E_2(1.2.3......15.16)$

$\Rightarrow \quad E_2(33!) = 16 + E_2(2.4.6....14.16)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + E_2(1.2.3\ldots 8)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + E_2(2.4.6.8)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + 4 + E_2(1.2.3.4)$

$\Rightarrow \quad E_2(33!) = 16 + 8 + 4 + E_2(2.4) = 16 + 8 + 4 + 3 = 31.$

Thus, exponent of 2 in 33! is 31 i.e. $33! = 2^{31} \times$ an integer

This shows that 33! is divisible by 2^{15} and the largest integer n such that 33! is divisible by 2^n is 31.

EXERCISE 16.1

LEVEL-1

1. Compute:
   (i) $\dfrac{30!}{28!}$ (ii) $\dfrac{11!-10!}{9!}$ (iii) L.C.M. (6!, 7!, 8!)
2. Prove that $\dfrac{1}{9!}+\dfrac{1}{10!}+\dfrac{1}{11!}=\dfrac{122}{11!}$
3. Find x in each of the following:
   (i) $\dfrac{1}{4!}+\dfrac{1}{5!}=\dfrac{x}{6!}$ (ii) $\dfrac{x}{10!}=\dfrac{1}{8!}+\dfrac{1}{9!}$ [NCERT] (iii) $\dfrac{1}{6!}+\dfrac{1}{7!}=\dfrac{x}{8!}$ [NCERT]
4. Convert the following products into factorials:
   (i) $5\cdot6\cdot7\cdot8\cdot9\cdot10$ (ii) $3\cdot6\cdot9\cdot12\cdot15\cdot18$
   (iii) $(n+1)(n+2)(n+3)\ldots(2n)$ (iv) $1\cdot3\cdot5\cdot7\cdot9\ldots(2n-1)$
5. Which of the following are true:
   (i) $(2+3)!=2!+3!$ (ii) $(2\times3)!=2!\times3!$
6. Prove that : $n!(n+2)=n!+(n+1)!$
7. If $(n+2)!=60[(n-1)!]$, find n.
8. If $(n+1)!=90[(n-1)!]$, find n.

If $(n+3)! = 56\,[(n+1)!]$, find n.

If $\frac{(2n)!}{3!(2n-3)!}$ and $\frac{n!}{2!(n-2)!}$ are in the ratio 44 : 3, find n.

Prove that:

(i) $\frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-(r-1))$ (ii) $\frac{n!}{(n-r)!\,r!} + \frac{n!}{(n-r+1)!\,(r-1)!} = \frac{(n+1)!}{r!\,(n-r+1)!}$

Prove that: $\frac{(2n+1)!}{n!} = 2^n\left\{1\cdot 3\cdot 5\ldots(2n-1)(2n+1)\right\}$

ANSWERS

(i) 870 (ii) 100 (iii) 8! 3. (i) 36 (ii) 100 (iii) 64 4. (i) $\frac{10!}{4!}$ (ii) $3^6\,(6!)$

(iii) $\frac{(2n)!}{n!}$ (iv) $\frac{(2n)!}{2^n\,n!}$ 5.(i) False (ii) False 7. 3 8. 9 9. 5 10. 6

HINTS TO NCERT & SELECTED PROBLEMS

3. (i) We have,

$$\frac{1}{4!} + \frac{1}{5!} = \frac{x}{6!}$$

$$\Rightarrow \frac{6!}{4!} + \frac{6!}{5!} = x \qquad \text{[Multiplying both sides by 6!]}$$

$$\Rightarrow \frac{6\times 5\times 4!}{4!} + \frac{6\times 5!}{5!} = x \Rightarrow 6\times 5 + 6 = x \Rightarrow x = 36$$

(ii) We have,

$$\frac{x}{10!} = \frac{1}{8!} + \frac{1}{9!}$$

$$\Rightarrow x = \frac{10!}{8!} + \frac{10!}{9!} \qquad \text{[Multiplying both sides by 10!]}$$

$$\Rightarrow x = \frac{10\times 9\times 8!}{8!} + \frac{10\times 9!}{9!}$$

$$\Rightarrow x = 10\times 9 + 10 = 100$$

(iii) We have,

$$\frac{1}{6!} + \frac{1}{7!} = \frac{x}{8!}$$

$$\text{[Multiplying both sides by 8!]}$$

$$\Rightarrow \frac{8!}{6!} + \frac{8!}{7!} = x$$

$$\Rightarrow \frac{8\times 7\times 6!}{6!} + \frac{8\times 7!}{7!} = x$$

$$\Rightarrow 8\times 7 + 8 = x$$

$$\Rightarrow x = 64$$

7. $(n+2)! = 60(n-1)!$

$\Rightarrow (n+2)(n+1)(n)(n-1)! = 60\times(n-1)!$

$\Rightarrow (n+2)(n+1)(n) = 5\times 4\times 3$ [Expressing 60 as the product of three consecutive integers] [On comparing two sides]

$\Rightarrow n = 3$

8. $(n+1)! = 90\,(n-1)!$

$\Rightarrow n+1\,(n)\,(n-1)! = 90\,(n-1)!$ [Writing 90 as the product of consecutive integers]

$\Rightarrow (n+1)n = 10\times 9$

$\Rightarrow n = 9$

9. $(n+3)! = 56\,(n+1)!$

$\Rightarrow \quad (n+3)(n+2)(n+1)! = 56\,(n+1)!$

$\Rightarrow \quad (n+3)(n+2) = 8 \times 7$ [Writing 56 as the product of consecutive integers]

$\Rightarrow \quad n+2 = 7 \Rightarrow n = 5$

16.2 FUNDAMENTAL PRINCIPLES OF COUNTING

In this section, we shall discuss two fundamental principles viz. principle of addition and principle of multiplication. These two principles will enable us to understand permutations and combinations. In fact these two principles form the base of permutations and combinations.

FUNDAMENTAL PRINCIPLE OF MULTIPLICATION *If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.*

EXPLANATION If the first job is performed in any one of the m ways, we can associate with this any one of the n ways of performing the second job: and thus there are n ways of performing the two jobs without considering more than one way of performing the first; and so corresponding to each of the m ways of performing the first job, we have n ways of performing the second job. Hence, the number of ways in which the two jobs can be performed is $m \times n$.

ILLUSTRATION 1 *In a class there are 10 boys and 8 girls. The teacher wants to select a boy and a girl to represent the class in a function. In how many ways can the teacher make this selection?*

SOLUTION Here the teacher is to perform two jobs:

(i) selecting a boy among 10 boys, and (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore by the fundamental principle of multiplication, the required number of ways is $10 \times 8 = 80$.

REMARK *The above principle can be extended for any finite number of jobs as stated below:*
If there are n jobs $J_1, J_2, \ldots J_n$ such that job J_i can be performed independently in m_i ways; $i = 1, 2, \ldots, n$. Then the total number of ways in which all the jobs can be performed is $m_1 \times m_2 \times m_3 \times \ldots \times m_n$.

FUNDAMENTAL PRINCIPLE OF ADDITION *If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in $(m + n)$ ways.*

ILLUSTRATION 2 *In a class there are 10 boys and 8 girls. The teacher wants to select either a boy or a girl to represent the class in a function. In how many ways the teacher can make this selection?*

SOLUTION Here the teacher is to perform either of the following two jobs :

(i) selecting a boy among 10 boys. or, (ii) selecting a girl among 8 girls.

The first of these can be performed in 10 ways and the second in 8 ways. Therefore, by fundamental principle of addition either of the two jobs can be performed in $(10 + 8) = 18$ ways. Hence, the teacher can make the selection of either a boy or a girl in 18 ways.

DIFFERENCE BETWEEN THE TWO PRINCIPLES *As we have discussed in the principle of multiplication a job is divided or decomposed into a number of sub-jobs which are unconnected to each other and the job is said to be performed if each sub-job is performed. While in the principle of addition there are a number of independent jobs and we have to perform one of them. So, the total number of ways of completing any one of the sub-jobs is the sum of the number of ways of completing each sub-jobs.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *There are 3 candidates for a Classical, 5 for a Mathematical, and 4 for a Natural science scholarship.*

(i) In how many ways can these scholarships be awarded?

(ii) In how many ways one of these scholarships be awarded?

SOLUTION Clearly, Classical scholarship can be awarded to any one of the three candidates. So, there are 3 ways of awarding the Classical scholarship.

Similarly, Mathematical and Natural science scholarships can be awarded in 5 and 4 ways respectively. So, by Fundamental Principle of multiplication,

Number of ways of awarding three scholarships $= 3 \times 5 \times 4 = 60$

By Fundamental Principle of addition,

Number of way of awarding one of the three scholarships $= 3 + 5 + 4 = 12$

EXAMPLE 2 *A room has 6 doors. In how many ways can a man enter the room through one door and come out through a different door ?*

SOLUTION Clearly, a person can enter the room through any one of the six doors. So, there are six ways of entering into the room. After entering into the room, the man can come out through any one of the remaining five doors. So, he can come out through a different door in 5 ways.

Hence, the number of ways in which a man can enter a room through one door and come out through a different door $= 6 \times 5 = 30$.

EXAMPLE 3 *The flag of a newly formed forum is in the form of three blocks, each to be coloured differently. If there are six different colours on the whole to choose from, how many such designs are possible ?*

SOLUTION Since there are six colours to choose from, therefore, first block can be coloured in 6 ways. Now, the second block can be coloured by any one of the remaining colours in five ways. So, there are five ways to colour the second block.

After colouring first two blocks only four colours are left. The third block can now be coloured by any one of the remaining four colours. So, there are four ways to colour the third block.

Hence, by the fundamental principle of multiplication, the number of flag-designs is $6 \times 5 \times 4 = 120$.

EXAMPLE 4 *Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, when* **[NCERT]**

(i) the repetition of the letters is not allowed. *(ii) the repetition of the letters is allowed.*

SOLUTION (i) The total number of words is same as the number of ways of filling in 4 vacant places by the 4 letters. The first place can be filled in 4 different ways by any one of the 4 letters R, O, S, E. Since the repetition of letters is not allowed. Therefore, the second place can be filled in by any one of the remaining 3 letters in 3 different ways, following which the third place can be filled in by the remaining 2 letters in 2 different ways; following which the fourth place can be filled in by the remaining one letter in one way. Thus, by the fundamental principle of counting the required number of ways is $4 \times 3 \times 2 \times 1 = 24$.

Hence, required number of words $= 24$.

(ii) If the repetition of the letters is allowed, then each of the 4 vacant places can be filled in succession in 4 different ways.

Hence, required number of words $= 4 \times 4 \times 4 \times 4 = 256$.

EXAMPLE 5 *Given 4 flags of different colours, how many different signals can be generated, if a signal requires the use of 2 flags one below the other?* **[NCERT]**

SOLUTION The total number of signals is equal to the number of ways of filling in 2 vacant places in succession by four flags of different colours. The upper cacant place can be filled in 4 different ways by any one of the 4 flags; following which, the lower vacant place can be filled in 3 different ways by any one of the remaining the different flags.

Hence, by the fundamental principle of multiplication, the required number of signals is $4 \times 3 = 12$.

EXAMPLE 6 *Find the number of different signals that can be generated by arranging at least 2 flags in order (one below the other) on a vertical staff, if five different flags are available.* **[NCERT]**

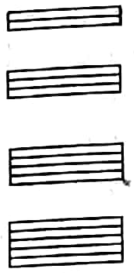
SOLUTION Since a signal may consist of either 2 flags, 3 flags, 4 flags or 5 flags. Therefore,

Total number of signals = Number of 2 flags signals

+ Nnumber of 3 flags signals

+ Number of 4 flags signals

+ Number of 5 flags signals



$= 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1$

$= 20 + 60 + 120 + 120 = 320$

EXAMPLE 7 *In a monthly test, the teacher decides that there will be three questions, one from each of Exercises 7, 8 and 9 of the text book. If there are 12 questions in Exercise 7, 18 in Exercise 8 and 9 in Exercise 9, in how many ways can three questions be selected ?*

SOLUTION There are 12 questions in exercise 7. So, one question from exercise 7 can be selected in 12 ways. Exercise 8 contains 18 questions. So, second question can be selected in 18 ways. There are 9 questions in exercise 9. So, third question can be selected in 9 ways. Hence, three questions can be selected in $12 \times 18 \times 9 = 1944$ ways.

EXAMPLE 8 *How many words (with or without meaning) of three distinct letters of the English alphabets are there ?*

SOLUTION Here we have to fill up three places by distinct letters of the English alphabets. Since there are 26 letters of the English alphabet, the first place can be filled by any of these letters. So, there are 26 ways of filling up the first place. Now, the second place can be filled up by any of the remaining 25 letters. So, there are 25 ways of filling up the second place. After filling up the first two places only 24 letters are left to fill up the third place. So, the third place can be filled in 24 ways.

Hence, the required number of words $= 26 \times 25 \times 24 = 15600$

EXAMPLE 9 *There are 6 multiple choice questions in an examination. How many sequence of answers are possible, if the first three questions have 4 choices each and the next three have 5 each ?*

SOLUTION Here we have to perform 6 jobs of answering 6 multiple choice questions. Each one of the first three questions can be answered in 4 ways and each one of the next three can be answered in 5 different ways.

So, the total number of different sequences $= 4 \times 4 \times 4 \times 5 \times 5 \times 5 = 8000$

EXAMPLE 10 *Find the total number of ways of answering 5 objective type questions, each question having 4 choices.*

SOLUTION Since each question can be answered in 4 ways. So, the total number of ways of answering 5 questions is $4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

EXAMPLE 11 *How many three-digit numbers can be formed without using the digits 0, 2, 3, 4, 5 and 6 ?*

SOLUTION We have to determine the total number of three digit numbers formed by using the digits 1, 7, 8, 9. Clearly, the repetition of digits is allowed.

A three digit number has three places viz. units's, ten's and hundred's. Unit's place can be filled by any of the digits 1, 7, 8, 9. So, unit's place can be filled in 4 ways. Similarly, each one of the ten's and hundred's place can be filled in 4 ways.

$\therefore$ Total number of required numbers $= 4 \times 4 \times 4 = 64$.

EXAMPLE 12 *How many numbers are there between 100 and 1000 in which all the digits are distinct ?*

SOLUTION A number between 100 and 1000 has three digits. So, we have to form all possible 3-digit numbers with distinct digits. We cannot have 0 at the hundred's place. So, the hundred's place can be filled with any of the 9 digits 1, 2, 3, ..., 9. So, there are 9 ways of filling the hundred's place.

Now, 9 digits are left including 0. So, ten's place can be filled with any of the remaining 9 digits in 9 ways. Now, the unit's place can be filled with in any of the remaining 8 digits. So, there are 8 ways of filling the unit's place.

Hence, the total number of required numbers $= 9 \times 9 \times 8 = 648$.

EXAMPLE 13 *How many numbers are there between 100 and 1000 such that every digit is either 2 or 9 ?*

SOLUTION Every number between 100 and 1000 consists of three digits. So, we have to determine the total number of three digit numbers such that every digit is either 2 or 9.

Clearly, each one of the unit's, ten's and hundred's place can be filled in 2 ways.

So, the total number of required numbers $= 2 \times 2 \times 2 = 8$.

EXAMPLE 14 *How many numbers are there between 100 and 1000 such that 7 is in the unit's place.*

SOLUTION Every number between 100 and 1000 is a three digit number. So, we have to form 3-digit numbers with 7 at the unit's place by using the digits 0, 1, 2, ..., 9. Clearly, repetition of digits is allowed. The hundred's place can be filled with any of the digits from 1 to 9 (zero cannot be there at hundred's place). So, hundred's place can be filled in 9 ways. Now, the ten's place can be filled with any of the digits from 0 to 9. So, ten's place can be filled in 10 ways. Since all the numbers have digit 7 at the unit's place, so, unit's place can be filled in only one way. Hence, by the fundamental principle of counting the total number of numbers between 100 and 1000 having 7 at the unit's place $= 9 \times 10 \times 1 = 90$.

EXAMPLE 15 *A gentleman has 6 friends to invite. In how many ways can he send invitation cards to them, if he has three servants to carry the cards ?*

SOLUTION Since a card can be sent by any one of the three servants, so the number of ways of sending the invitation card to the first friend = 3. Similarly, invitation cards can be sent to each of the six friends in 3 ways.

So, the required number of ways $= 3 \times 3 \times 3 \times 3 \times 3 \times 3 = 3^6 = 729$.

EXAMPLE 16 *How many three-digit numbers more than 600 can be formed by using the digits 2, 3, 4, 6, 7.*

SOLUTION Clearly, repetition of digits is allowed. Since a three-digit number greater than 600 will have 6 or 7 at hundred's place. So, hundred's place can be filled in 2 ways. Each of the ten's and one's place can be filled in 5 ways.

Hence, total number of required numbers $= 2 \times 5 \times 5 = 50$.

EXAMPLE 17 *How many numbers between 3000 and 4000 can be formed from the digits 3, 4, 5, 6, 7 and 8, no digit being repeated in any number ?*

SOLUTION Clearly, a number between 3000 and 4000 must have 3 at thousand's place. So, thousand's place can be filled in only one way. Now, hundred's place can be filled in 5 ways. Since repetition of digits is not allowed so ten's and one's places can be filled in 4 and 3 ways respectively.

So, total number of required numbers $= 1 \times 5 \times 4 \times 3 = 60$.

EXAMPLE 18 *How many numbers divisible by 5 and lying between 4000 and 5000 can be formed from the digits 4, 5, 6, 7 and 8.*

SOLUTION Clearly, a number between 4000 and 5000 must have 4 at thousand's place. Since the number is divisible by 5 it must have 5 at unit's place. Now, each of the remaining places (viz. hundred's and ten's) can be filled in 5 ways.

Hence, total number of required numbers $= 1 \times 5 \times 5 \times 1 = 25$.

EXAMPLE 19 *How many four-digit numbers can be formed using the digits 0, 1, 2, 3, 4, 5 if (i) repetition of digits is not allowed (ii) repetition of digits is allowed ?*

SOLUTION (i) In a four-digit number 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways. (viz. 1, 2, 3, 4, 5). Since repetition of digits is not allowed and 0 can be used at hundred's place, so hundred's place can be filled in 5 ways.

Now, any one of the remaining four digits can be used to fill up ten's place. So, ten's place can be filled in 4 ways. One's place can be filled from the remaining three digits in 3 ways.

Hence, the required number of numbers $= 5 \times 5 \times 4 \times 3 = 300$.

(ii) For a four-digit number we have to fill up four places and 0 cannot appear in the thousand's place. So, thousand's place can be filled in 5 ways. Since repetition of digits is allowed, so each of the remaining three places viz. hundred's, ten's and one's can be filled in 6 ways.

Hence, the required number of numbers $= 5 \times 6 \times 6 \times 6 = 1080$..

EXAMPLE 20 *How many numbers greater than 1000, but not greater than 4000 can be formed with the digits 0, 1, 2, 3, 4 if : (i) repetition of digits is allowed ? (ii) repetition of digits is not allowed ?*

SOLUTION (i) Every number between 1000 and 4000 is a four digit number. In thousand's place we can put either 1 or 2 or 3 but not 4. So, thousand's place can be filled in 3 ways. Since repetition of digits is allowed, so each of the hundred's, ten's and one's place can be filled in 5 ways. So, total number of numbers between 1000 and 4000, including 1000 and excluding 4000 is $3 \times 5 \times 5 \times 5 = 375$. But, we have to find the total number of numbers greater than 1000 but not greater than 4000.

Hence, required number of numbers $= 375 + 1$ (for 4000) $- 1$ (for 1000) $= 375$.

(ii) As discussed above thousand's place can be filled in 3 ways. Since repetition of digits is not allowed, so, hundred's place can be filled from the remaining digits in 4 ways. Now, three digits are left, so ten's place can be filled in 3 ways. One's place can be filled in 2 ways.

Hence, required number of numbers $= 3 \times 4 \times 3 \times 2 = 72$.

EXAMPLE 21 *How many three digit odd numbers can be formed by using the digits 1, 2, 3, 4, 5, 6 if:* **[NCERT]**

(i) the repetition of digits is not allowed ? (ii) the repetition of digits is allowed ?

SOLUTION For a number to be odd, we must have 1, 3 or 5 at the unit's place. So, there are 3 ways of filling the unit's place.

(i) Since the repetition of digits is not allowed, the ten's place can be filled with any of the remaining 5 digits in 5 ways. Now, four digits are left. So, hundred's place can be filled in 4 ways.

So, required number of numbers $= 3 \times 5 \times 4 = 60$

(ii) Since the repetition of digits is allowed, so each of the ten's and hundred's place can be filled in 6 ways.

Hence, required number of numbers $= 3 \times 6 \times 6 = 108$.

EXAMPLE 22 *How many 3-digit even numbers can be formed from the digits 1, 2, 3, 4, 5, 6 if the digits can be repeated?* **[NCERT]**

SOLUTION For a number to be even, we must have 2, 4 or 6 at the unit's place. So, there are 3 ways to fill in the unit's place. Since digits can be repeated, so each of the ten's and hundred's place can be filled in 6 ways.

EXAMPLE 23 *How many numbers of 3 digits can be formed with the digits 1, 2, 3, 4, 5 when digits may be repeated ?*

SOLUTION The unit's place can be filled in 5 ways. Since, the repetition of digits is allowed, therefore ten's place can be filled in 5 ways and hundred's place can also be filled in 5 ways.

Therefore, by the fundamental principle of counting, the required number of three digit numbers $= 5 \times 5 \times 5 = 125$.

EXAMPLE 24 *Find the number of numbers of 5 digits that can be formed with the digits 0, 1, 2, 3, 4 if the digits can be repeated in the same number.*

SOLUTION In a five digit number 0 cannot be put in ten thousand's place. So, the number of ways of filling up the ten thousand's place = 4.

Since the repetition of digits is allowed, therefore each of the other places can be filled in 5 ways.

So, the required number of numbers $= 4 \times 5 \times 5 \times 5 \times 5 = 2500$.

EXAMPLE 25 *How many 4-digit numbers are there, when a digit may be repeated any number of times ?*

SOLUTION In a four digit number 0 cannot be placed at thousand's place. So, thousand's place can be filled with any digit from 1 to 9. Thus, thousand's place can be filled in 9 ways.

Since repetition of digits is allowed, therefore each of the remaining 3 places can be filled in 10 ways by using the digits from 0 to 9.

Hence, the required number of numbers $= 9 \times 10 \times 10 \times 10 = 9000$.

EXAMPLE 26 *How many three-letter words can be formed using a, b, c, d, e if : (i) repetition is not allowed (ii) repetition is allowed ?*

SOLUTION (i) Clearly, the total number of three-letter words is equal to the number of ways of filling three places. First place can be filled in 5 ways. Now, four letters are left. So, the second place can be filled in 4 ways. Since the repetition of letters is not allowed, so the third place can be filled from any one of the remaining 3 digits in 3 ways.

Hence, total number of words $= 5 \times 4 \times 3 = 60$.

(ii) In this case repetition of letters is allowed, so each of the three places can be filled in 5 ways.

Hence, total number of words $= 5 \times 5 \times 5 = 125$.

EXAMPLE 27 *In how many ways can the following prizes be given away to a class of 30 students, first and second in Mathematics, first and second in Physics, first in Chemistry and first in English ?*

SOLUTION Here we have to give prizes in four subjects and the process of distributing prizes can be completed by giving prizes in the four subjects.

First and second prizes can be given in Mathematics in (30×29) ways.

First and second prizes can be given in Physics in (30×29) ways.

First prize can be given in Chemistry in 30 ways.

First prize can be given in English in 30 ways.

Hence, the number of ways to give prizes in all the four subjects

$$= (30 \times 29) \times (30 \times 29) \times 30 \times 30 = 6.8121 \times 10^8$$

EXAMPLE 28 *In how many ways 5 rings of different types can be worn in 4 fingers ?*

SOLUTION The first ring can be worn in any of the 4 fingers. So, there are 4 ways of wearing it. Similarly, each one of the other rings can be worn in 4 ways.

Hence, the requisite number of ways $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

EXAMPLE 29 *In how many ways can 5 letters be posted in 4 letter boxes ?*

SOLUTION Since each letter can be posted in any one of the four letter boxes. So, a letter can be posted in 4 ways. Since there are 5 letters and each letter can be posted in 4 ways. So, total number of ways in which all the five letters can be posted is $4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

LEVEL-II

EXAMPLE 30 *Five persons entered the lift cabin on the ground floor of an 8-floor house. Suppose each of them can leave the cabin independently at any floor beginning with the first. Find the total number of ways in which each of the five persons can leave the cabin (i) at any one of the 7 floors (ii) at different floors.*

SOLUTION Suppose A_1, A_2, A_3, A_4, A_5 are five persons.

(i) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Similarly, each of A_2, A_3, A_4, A_5 can leave the cabin in 7 ways. Thus, the total number of ways in which each of the five persons can leave the cabin at any of the seven floors is $7 \times 7 \times 7 \times 7 \times 7 = 7^5$.

(ii) A_1 can leave the cabin at any of the seven floors. So, A_1 can leave the cabin in 7 ways. Now, A_2 can leave the cabin at any of the remaining 6 floors. So, A_2 can leave the cabin in 6 ways. Similarly, A_3, A_4, and A_5 can leave the cabin in 5, 4 and 3 ways respectively. Thus, the total number of ways in which each of the five persons can leave the cabin at different floors is $7 \times 6 \times 5 \times 4 \times 3 = 2520$.

EXAMPLE 31 *A mint prepares metallic calendars specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of February calendars should it prepare to serve for all the possibilities in the future years ?*

SOLUTION The mint has to perform two jobs, viz.

(i) selecting the number of days in the February month (there can be 28 days or 29 days), and

(ii) selecting the first day of the February month.

The first job can be completed in 2 ways while the second can be performed in 7 ways by selecting any one of the seven days of a week.

Thus, the required number of plates $= 2 \times 7 = 14$.

EXAMPLE 32 *For a set of five true/false questions, no student has written all correct answers, and no two students have given the same sequence of answers. What is the maximum number of students in the class, for this to be possible ?*

SOLUTION Since a true/false type question can be answered in 2 ways either by marking it true or false. So, there are 2 ways of answering each of the 5 questions.

$\therefore$ Total number of different sequences of answers $= 2 \times 2 \times 2 \times 2 \times 2 = 2^5 = 32$.

Out of these 32 sequences of answers there is only one sequence of answering all the five questions correctly. But no student has written all the correct answers and different students have given different sequences of answers.

$\therefore$ Maximum number of students in the class

= Number of sequences except one sequence in which all answers are correct $= 32 - 1 = 31$

EXAMPLE 33 *How many numbers are there between 100 and 1000 such that at least one of their digits is 7?*

SOLUTION Clearly, a number between 100 and 1000 has 3-digits

$\therefore$ Total number of 3-digit numbers having at least one of their digits as 7

= (Total number of three-digit numbers) – (Total number of 3-digit numbers in which 7 does not appear at all)

Total number of three-digit numbers : We have to form three-digit numbers by using the digits 0, 1, 2, 3, ..., 9. Clearly, hundred's place can be filled in 9 ways and each of the ten's and one's place can be filled in 10 ways.

So, total number of 3-digit number $= 9 \times 10 \times 10 = 900$.

Total number of three-digit number in which 7 does not appear at all : Here we have to form three-digit numbers by using the digits 0 to 9, except 7. So, hundred's place can be filled in 8 ways and each of the ten's and one's place can be filled in 9 ways. So, total number of three-digit numbers in which 7 does not appear at all is $8 \times 9 \times 9$.

Hence, total number of 3-digit numbers having at least one of their digits as 7 is

$$9 \times 10 \times 10 - 8 \times 9 \times 9 = 252.$$

EXAMPLE 34 *How many numbers are there between 100 and 1000 which have exactly one of their digits as 7?*

SOLUTION A number between 100 and 1000 contains 3-digits. So, we have to form 3-digit numbers having exactly one of their digits as 7. Such type of numbers can be divided into three types:

(i) Those numbers that have 7 in the unit's place but not in any other place.
(ii) Those numbers that have 7 in the ten's place but not in any other place.
(iii) Those numbers that have 7 in the hundred's place but not in any other place.

Required number of numbers is the total number of these three types of numbers.

We shall now count these three types of numbers separately.

(i) *Those three-digit numbers that have 7 in the unit's place but not in any other place.*

The hundred's place can have any one of the digits from 0 to 9 except 0 and 7. So, hundred's place can be filled in 8 ways. The ten's place can have any one of the digits from 0 to 9 except 7. So, the number of ways the ten's place can be filled is 9. The unit's place has 7. So, it can be filled in only one way.

Thus, there are $8 \times 9 \times 1 = 72$ numbers of the first kind.

(ii) *Those three-digit numbers that have 7 in the ten's place but not in any other place.*

The number of ways to fill the hundred's place = 8

(by any one of the digits from 1, 2, 3, 4, 5, 6, 8, 9)

The number of ways to fill the ten's place = 1 (by 7 only)

The number of ways to fill the one's place = 9 (by any one of the digits 0, 1, 2, 3, 4, 5, 6, 8, 9)

Thus, there are $8 \times 1 \times 9 = 72$ numbers of the second kind.

(iii) *Those three-digit numbers that have 7 in the hundred's place but not at any other place.*

In this case, the hundred's place can be filled only in one way and each of the ten's and one's place can be filled in 9 ways.

So, there are $1 \times 9 \times 9 = 81$ numbers of the third kind.

Hence, the total number of required type of numbers $= 72 + 72 + 81 = 225$.

EXAMPLE 35 *A telegraph has 5 arms and each arm is capable of 4 distinct positions, including the position of rest. What is the total number of signals that can be made?*

SOLUTION Since each arm can be kept in 4 positions and a signal is possible when all the 5 arms are simultaneously placed in positions.

$\therefore$ Total number of ways of placing the arms $= 4 \times 4 \times 4 \times 4 \times 4 = 4^5$.

But, this includes one inadmissible case, when all the arms are in the position of rest and then no signal can be made.

Hence, required number of signals $= (4^5 - 1) = 1023$.

EXAMPLE 36 *In how many ways can 3 prizes be distributed among 4 boys, when (i) no boy gets more than one prize? (ii) a boy may get any number of prizes? (iii) no boy gets all the prizes?*

SOLUTION (i) The first prize can be given away in 4 ways as it may be given to any one of the 4 boys. The second prize can be given away in 3 ways, because the boy who got the first prize cannot receive the second prize. The third prize can be given away to anyone of the remaining 2 boys in 2 ways. So, the number of ways in which all the prizes can be given away $= 4 \times 3 \times 2 = 24$.

ALITER The total number of ways is the number of arrangements of 4 taken 3 at a time. So, the requisite number of ways $= {}^4P_3 = 4! = 24$.

(ii) The first prize can be given away in 4 ways as it may be given to anyone of the 4 boys. The second prize can also be given away in 4 ways, since it may be obtained by the boy who has already received a prize. Similarly, third prize can be given away in 4 ways.

Hence, the number of ways in which all the prizes can be given away $= 4 \times 4 \times 4 = 4^3 = 64$.

(iii) Since any one of the 4 boys may get all the prizes. So, the number of ways in which a boy gets all the 3 prizes is 4.

So, the number of ways in which a boy does not get all the prizes $= 64 - 4 = 60$.

EXAMPLE 37 *Find the total number of ways in which n distinct objects can be put into two different boxes.*

SOLUTION Let the two boxes be B_1 and B_2. We observe that there are two choices for each of the n objects. Therefore, by fundamental principle of counting

$$\text{Total number of ways} = \underbrace{2 \times 2 \times \dots \times 2}_{n-\text{times}} = 2^n$$

EXAMPLE 38 *Find the total number of ways in which n-distinct objects can be put into two different boxes so that no box remains empty.*

SOLUTION Each object can be put either in box B_1 (say) or in box B_2 (say). So, there are two choices for each of the n objects. Therefore, the number of choices for n distinct objects is $\underbrace{2 \times 2 \times \dots \times 2}_{n-\text{times}} = 2^n$. Two of these choices correspond to either the first or the second box being empty. Thus, there are $2^n - 2$ ways in which neither box is empty.

EXAMPLE 39 *By using the digits 0, 1, 2, 3, 4 and 5 (repetitions not allowed) numbers are formed by using any number of digits. Find the total number of non-zero numbers that can be formed.*

SOLUTION Required number of numbers

= Number of 1 digit number + No. of 2 digit numbers + ... + Number of 6 digit numbers

$= 5 + 5 \times 5 + 5 \times 5 \times 4 + 5 \times 5 \times 4 \times 3 + 5 \times 5 \times 4 \times 3 \times 2 + 5 \times 5 \times 4 \times 3 \times 2 \times 1$

$= 5 + 25 + 100 + 300 + 600 + 600 = 1630$.

EXERCISE 16.2

LEVEL-1

1. In a class there are 27 boys and 14 girls. The teacher wants to select 1 boy and 1 girl to represent the class in a function. In how many ways can the teacher make this selection?
2. A person wants to buy one fountain pen, one ball pen and one pencil from a stationery shop. If there are 10 fountain pen varieties, 12 ball pen varieties and 5 pencil varieties, in how many ways can he select these articles?
3. From Goa to Bombay there are two roots; air, and sea. From Bombay to Delhi there are three routes; air, rail and road. From Goa to Delhi via Bombay, how many kinds of routes are there?
4. A mint prepares metallic calenders specifying months, dates and days in the form of monthly sheets (one plate for each month). How many types of calendars should it prepare to serve for all the possibilities in future years?
5. There are four parcels and five post-offices. In how many different ways can the parcels be sent by registered post?

6. A coin is tossed five times and outcomes are recorded. How many possible outcomes are there?
7. In how many ways can an examinee answer a set of ten true/false type questions?
8. A letter lock consists of three rings each marked with 10 different letters. In how many ways it is possible to make an unsuccessful attempt to open the lock?
9. There are 6 multiple choice questions in an examination. How many sequences of answers are possible, if the first three questions have 4 choices each and the next three have 2 each?
10. There are 5 books on Mathematics and 6 books on Physics in a book shop. In how many ways can a student buy : (i) a Mathematics book and a Physics book (ii) either a Mathematics book or a Physics book?
11. Given 7 flags of different colours, how many different signals can be generated if a signal requires the use of two flags, one below the other? **[NCERT]**
12. A team consists of 6 boys and 4 girls and other has 5 boys and 3 girls. How many single matches can be arranged between the two teams when a boy plays against a boy and a girl plays against a girl?
13. Twelve students compete in a race. In how many ways first three prizes be given?
14. How many A.P.'s with 10 terms are there whose first term is in the set {1, 2, 3} and whose common difference is in the set {1, 2, 3, 4, 5}?
15. From among the 36 teachers in a college, one principal, one vice-principal and the teacher-incharge are to be appointed. In how many ways can this be done?
16. How many three-digit numbers are there with no digit repeated?
17. How many three-digit numbers are there?
18. How many three-digit odd numbers are there?
19. How many different five-digit number licence plates can be made if
    (i) first digit cannot be zero and the repetition of digits is not allowed,
    (ii) the first-digit cannot be zero, but the repetition of digits is allowed?
20. How many four-digit numbers can be formed with the digits 3, 5, 7, 8, 9 which are greater than 7000, if repetition of digits is not allowed?
21. How many four-digit numbers can be formed with the digits 3, 5, 7, 8, 9 which are greater than 8000, if repetition of digits is not allowed?
22. In how many ways can six persons be seated in a row?
23. How many 9-digit numbers of different digits can be formed?
24. How many odd numbers less than 1000 can be formed by using the digits 0, 3, 5, 7 when repetition of digits is not allowed?
25. How many 3-digit numbers are there, with distinct digits, with each digit odd?
26. How many different numbers of six digits each can be formed from the digits 4, 5, 6, 7, 8, 9 when repetition of digits is not allowed?
27. How many different numbers of six digits can be formed from the digits 3, 1, 7, 0, 9, 5 when repetition of digits is not allowed?
28. How many four digit different numbers, greater than 5000 can be formed with the digits 1, 2, 5, 9, 0 when repetition of digits is not allowed?
29. Serial numbers for an item produced in a factory are to be made using two letters followed by four digits (0 to 9). If the letters are to be taken from six letters of English alphabet

without repetition and the digits are also not repeated in a serial number, how many serial numbers are possible?

30. A number lock on a suitcase has 3 wheels each labelled with ten digits 0 to 9. If opening of the lock is a particular sequence of three digits with no repeats, how many such sequences will be possible? Also, find the number of unsuccessful attempts to open the lock.

31. A customer forgets a four-digit code for an Automatic Teller Machine (ATM) in a bank. However, he remembers that this code consists of digits 3, 5, 6 and 9. Find the largest possible number of trials necessary to obtain the correct code.

32. In how many ways can three jobs I, II and III be assigned to three persons A, B and C if one person is assigned only one job and all are capable of doing each job?

33. How many four digit natural numbers not exceeding 4321 can be formed with the digits 1, 2, 3 and 4, if the digits can repeat?

34. How many numbers of six digits can be formed from the digits 0, 1, 3, 5, 7 and 9 when no digit is repeated? How many of them are divisible by 10?

35. If three six faced die each marked with numbers 1 to 6 on six faces, are thrown find the total number of possible outcomes.

36. A coin is tossed three times and the outcomes are recorded. How many possible outcomes are there ? How many possible outcomes if the coin is tossed four times? Five times? n times?

37. How many numbers of four digits can be formed with the digits 1, 2, 3, 4, 5 if the digits can be repeated in the same number?

38. How many three digit numbers can be formed by using the digits 0, 1, 3, 5, 7 while each digit may be repeated any number of times?

39. How many natural numbers less than 1000 can be formed from the digits 0, 1, 2, 3, 4, 5 when a digit may be repeated any number of times?

40. How many five digit telephone numbers can be constructed using the digits 0 to 9. If each number starts with 67 and no digit appears more than once? [NCERT]

LEVEL-2

41. Find the number of ways in which 8 distinct toys can be distributed among 5 children.

42. Find the number of ways in which one can post 5 letters in 7 letter boxes.

43. Three dice are rolled. Find the number of possible outcomes in which at least one die shows 5.

44. Find the total number of ways in which 20 balls can be put into 5 boxes so that first box contains just one ball.

45. In how many ways can 5 different balls be distributed among three boxes?

46. In how many ways can 7 letters be posted in 4 letter boxes?

47. In how many ways can 4 prizes be distributed among 5 students, when
    (i) no student gets more than one prize?
    (ii) a student may get any number of prizes?
    (iii) no student gets all the prizes?

48. There are 10 lamps in a hall. Each one of them can be switched on independently. Find the number of ways in which the hall can be illuminated. [NCERT]

ANSWERS

378
512
648
720
720, 719
8, 16, 2^n
91
$2^{10}-1$.

2. 600 3. 6 4. 14 5. 625 6. 32 7. 1024 8. 999
10. (i) 30 (ii) 11 11. 42 12. 42 13. 1320 14. 15 15. 42840
17. 900 18. 450 19. (i) 27216 (ii) 90000 20. 72 21. 48
23. 9(9!) 24. 21 25. 60 26. 720 27. 600 28. 48 29. 151200
31. 24 32. 6 33. 229 34. 600, 120 35. 216
37. 625 38. 100 39. 215 40. 336 41. 5^8 42. 7^5
44. 20×4^{19} 45. 243 46. 4^7 47. (i) 5! (ii) 625 (iii) 620

HINTS TO NCERT & SELECTED PROBLEMS

1. No. of ways $= 27 \times 14$.
2. Required number of ways $= 10 \times 12 \times 5 = 600$.
3. No of routes $= 2 \times 3 = 6$.
4. Total number of calendars $= 7 \times 2 = 14$.
5. Since a parcel can be sent to any one of the five post offices. So, required number of ways $= 5 \times 5 \times 5 \times 5 = 5^4$.
6. Since toss of each coin can result in 2 ways. So, required no. of ways $= 2 \times 2 \times 2 \times 2 \times 2 = 2^5$.
8. Required no. of ways $= 10 \times 10 \times 10 - 1$.
9. Each one of the first three questions can be answered in 4 ways and each one of the next three questions can be answered in 2 ways. So, total no. of sequences of answers $= 4 \times 4 \times 4 \times 2 \times 2 \times 2$.
11. Required no. of signals $= 7 \times 6$.
12. A boy can be selected from the first team in 6 ways, and from the second in 5 ways. So, no. of single matches between the boys of two teams $= 6 \times 5 = 30$. Similarly, the no. of single matches between the girls of two teams $= 4 \times 3 = 12$. So, total number of matches $= 30 + 12 = 42$.
13. Required no. of ways $= 12 \times 11 \times 10$.
14. There are 3 ways to choose the first term and corresponding to each such way there are 5 ways of selecting the common difference. So, required no. of A.P.'s $= 3 \times 5$.
15. Required no. of ways $= 36 \times 35 \times 34$.
16. The total no. of required numbers $= 9 \times 9 \times 8$.
17. The total no. of required numbers $= 9 \times 10 \times 10$.
18. The total no. of required number $= 9 \times 10 \times 5$.
19. (i) Required no. of licence plates $= 9 \times 9 \times 8 \times 7 \times 6$
 (ii) Required no. of licence plates $= 9 \times 10 \times 10 \times 10 \times 10$.
20. Required no. of numbers $= 3 \times 4 \times 3 \times 2$.
21. Required no. of numbers $= 2 \times 4 \times 3 \times 2$.
22. Required no. of ways $= 6 \times 5 \times 4 \times 3 \times 2 \times 1$.
23. Required no. of numbers $= 9 \times 9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2$.
24. An odd number less than 1000 may be a one-digit number, two-digit number or a three-digit number. So, required no. of numbers is
 3 (one-digit nos.) + 2×3 (two-digit nos.) + $2 \times 2 \times 3$ (3-digit nos.).
25. Required no. of numbers $= 5 \times 4 \times 3$.

26. Required no. of numbers $= 6 \times 5 \times 4 \times 3 \times 2 \times 1$.

27. Required no. of numbers $= 5 \times 5 \times 4 \times 3 \times 2 \times 1$.

28. Required no. of numbers $= 2 \times 4 \times 3 \times 2$.

29. Here we have to perform 6 jobs. So, required number of serial numbers i

$$6 \times 5 \times 10 \times 9 \times 8 \times 7$$

30. Required number of sequences $= 10 \times 9 \times 8$.
Also, total number of unsuccessful attempts $= 10 \times 9 \times 8 - 1$

31. Number of trials $= 4 \times 3 \times 2 \times 1$

32. Required number of ways $= 3 \times 2 \times 1$

36. Since a toss of a coin can result in a head or a tail. Therefore, if a coin is tossed n-times, then the total number of outcomes is $\underbrace{2 \times 2 \times 2 \times \ldots \times 2}_{n-times} = 2^n$

41. Each toy can be distributed in 5 ways.
So, total number of ways $= 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 \times 5 = 5^8$

42. Each letter can be posted in any one of the 7 letter boxes.
So, required number of ways $= 7 \times 7 \times 7 \times 7 \times 7 = 7^5$

43. Required number of possible outcomes
= Total number of possible outcomes − Number of possible out- comes in which 5 does not appear on any dice.

$$= 6^3 - 5^3 = 216 - 125 = 91.$$

44. One ball can be put in first box in 20 ways because we can put any one of the twenty balls in the first box. Now, remaining 19 balls are to be put into remaining 4 boxes. This can be done in 4^{19} ways, because there are 4 choices for each ball. Hence, the required number of ways $= 20 \times 4^{19}$.

16.3 PERMUTATIONS

Each of the arrangements which can be made by taking some or all of a number of things is called a permutation.

For example, if there are three objects, then the permutations of these objects, taking two at a time, are

$$ab, \quad ba, \quad bc, \quad cb, \quad ac, \quad ca$$

So, the number of permutations of three different things taken two at a time is 6.

NOTE It should be noted that in permutations the order of arrangement is taken into account; when the order is changed, a different permutation is obtained.

ILLUSTRATION 1 *Write down all the permutations of the set of three letters A, B, C.*

SOLUTION The permutations of three letters A, B, C taking all at a time are :

$$ABC, \quad ACB, \quad BCA, \quad BAC, \quad CBA, \quad CAB.$$

Clearly, there are 6 permutations.

ILLUSTRATION 2 *Write down all the permutations of the vowels A, E, I, O, U in English alphabets taking three at a time, and starting with A.*

SOLUTION The permutations of vowels A, E, I, O, U taking three at a time, and starting with A are:

$$AEI, \ AIE, \ AEO, \ AOE, \ AEU, \ AUE, \ AIO, \ AOI, \ AIU, \ AUI, \ AOU, \ AUO$$

Clearly, there are 12 permutations.

ILLUSTRATION 3 *Write down all the permutations of letters A, B, C, D taking three at a time.*

SOLUTION The desired permutations are :

ABC	ABD	BCD	ACD
ACB	ADB	BDC	ADC
BCA	BDA	CBD	CAD
BAC	BAD	CDB	CDA
CAB	DAB	DCB	DAC
CBA	DBA	DBC	DCA

Clearly, there are 24 permutations. These permutations are obtained by first selecting three letters out of 4 and then arranging them in all possible ways.

A NOTATION *If n and r are positive integers such that $1 \le r \le n$, then the number of all permutations of n distinct things, taken r at a time is denoted by the symbol $P(n, r)$ or ${}^nP_r$.*

Thus,

$${}^nP_r \text{ or, } P(n, r) = \text{Total number of permutations of n distinct things, taken } r \text{ at a time.}$$

In illustration 3, we have seen that there are 24 permutations, on a set of 4 letters, taken 3 at a time. Therefore, as per our notation, we have ${}^4P_3 = 24$ or, $P(4, 3) = 24$.

THEOREM 1 *Let r and n be positive integers such that $1 \le r \le n$. Then the number of all permutations of n distinct things taken r at a time is given by $n(n-1)(n-2)(n-3)\ldots(n-(r-1))$.*

i.e. $\quad P(n, r) = {}^nP_r = n(n-1)(n-2)\ldots(n-(r-1))$.

PROOF The number of permutations of n distinct things, taken r at a time, is same as the number of ways in which we can fill up r-places when we have n different things at our disposal. The first place can be filled in n ways, for any one of the n things can be used to fill up the first place. Having filled it, there are $(n-1)$ things left and any one of these $(n-1)$ things can be used to fill up the second place. So, the second place can be filled in $(n-1)$ ways. Hence, by the fundamental principle of counting, the first two places can be filled in $n(n-1)$ ways. When the first two places are filled, there are $(n-2)$ places left, so that the third place can be filled from the remaining $(n-2)$ things in $(n-2)$ ways. Therefore, the first three places can be filled in $n(n-1)(n-2)$ ways. Continuing in this manner, we find that the first $(r-1)$ places can be filled in $n(n-1)(n-2)\ldots(n-(r-2))$ ways. After filling up first $(r-1)$ places, exactly $n-(r-1) = n-r+1$ things are left. So, the rth place can be filled in $(n-(r-1))$ ways. Hence, the r places can be filled in $n(n-1)(n-2)\ldots(n-(r-1))$ ways.

Hence, the total number of permutations of n distinct things, taken r at a time is

$$n(n-1)(n-2)(n-3)\ldots(n-(r-1)).$$

Thus, $\quad P(n, r) = n(n-1)(n-2)(n-3)\ldots(n-(r-1)).$

THEOREM 2 *Prove that:* $\quad P(n, r) = {}^nP_r = \dfrac{n!}{(n-r)!}$.

PROOF We have,

$$P(n, r) = n(n-1)(n-2)(n-3)\ldots(n-(r-1))$$

$$\Rightarrow \quad P(n, r) = \frac{n(n-1)(n-2)(n-3)\ldots(n-(r-1))(n-r)(n-(r+1))\ldots 3.2.1}{(n-r)(n-(r+1))\ldots 3.2.1}$$

$$\Rightarrow \quad P(n, r) = \frac{n!}{(n-r)!}$$

THEOREM 3 *The number of all permutations of n distinct things, taken all at a time is $n!$.*

PROOF The number of all permutations of n distinct things, taken all at a time is same as the number of ways of filling n places when we have n distinct things at our disposal.

Proceeding as in theorem 1, we have

$$P(n, n) = n(n-1)(n-2)(n-3)\dots(n-(n-1)) = n(n-1)(n-2)(n-3)\dots 3.2.1 = n!$$

THEOREM 4 *Prove that $0! = 1$.*

PROOF We have,

$$P(n, r) = \frac{n!}{(n-r)!}$$

$$\Rightarrow \quad P(n, n) = \frac{n!}{0!} \qquad \text{[Putting } r = n\text{]}$$

$$\Rightarrow \quad n! = \frac{n!}{0!} \qquad [\because P(n, n) = n! \text{ (See Theorem 3)}]$$

$$\Rightarrow \quad 0! = \frac{n!}{n!} = 1.$$

ILLUSTRATIVE EXAMPLES

LEVEL-1

Type I **PROBLEMS BASED UPON THE VALUE OF ^nP_r OR $P(n, r)$**

EXAMPLE 1 *Evaluate the following:*

(i) $^5P_3$ (ii) $P(15, 3)$ (iii) $P(5, 5)$

SOLUTION (i) $^5P_3 = \frac{5!}{(5-3)!}$ $\qquad \left[\because {}^nP_r = \frac{n!}{(n-r)!}\right]$

$$\Rightarrow \quad {}^5P_3 = \frac{5!}{2!} = \frac{5 \times 4 \times 3 \times 2!}{2!} = 60$$

(ii) $$P(15, 3) = \frac{15!}{(15-3)!} = \frac{15!}{12!} = \frac{15 \times 14 \times 13 \times 12!}{12!} = 2730$$

(iii) $$P(5, 5) = \frac{5!}{(5-5)!} = \frac{5!}{0!} = 5! = 120.$$

Type II **ON FINDING THE VALUE OF REQUIRED UN-KNOWN WHEN A RELATION CONNECTING $P(n, r)$ IS GIVEN**

EXAMPLE 2 *If $2. P(5, 3) = P(n, 4)$, find n.*

SOLUTION We have,

$$2. P(5, 3) = P(n, 4)$$

$$\Rightarrow \quad P(n, 4) = 2 . P(5, 3)$$

$$\Rightarrow \quad \frac{n!}{(n-4)!} = 2\left(\frac{5!}{(5-3)!}\right)$$

$$\Rightarrow \quad \frac{n(n-1)(n-2)(n-3)(n-4)!}{(n-4)!} = \frac{2(5!)}{2!}$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5!$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5 \times 4 \times 3 \times 2 \times 1$$

$$\Rightarrow \quad n(n-1)(n-2)(n-3) = 5(5-1)(5-2)(5-3)$$

$$\Rightarrow \quad n = 5 \qquad \text{[By comparing two sides]}$$

EXAMPLE 3 *If $P(n, 4) = 20 \times P(n, 2)$, find n.*

SOLUTION We have,

$$P(n, 4) = 20 \times P(n, 2)$$

$$\Rightarrow \quad \frac{n!}{(n-4)!} = 20 \times \frac{n!}{(n-2)!}$$

$$\Rightarrow \quad (n-2)! = 20 \times (n-4)!$$

$$\Rightarrow \quad (n-2)(n-3)(n-4)! = 20 \times (n-4)!$$

$$\Rightarrow \quad (n-2)(n-3) = 20$$

$$\Rightarrow \quad (n-2)(n-3) = 5 \times 4$$

$$\Rightarrow \quad n-3 = 4$$

$$\Rightarrow \quad n = 7$$

[By comparing two sides]

EXAMPLE 4 *If $P(5, r) = 2.\, P(6, r-1)$, find r.* **[NCERT]**

SOLUTION We have,

$$P(5, r) = 2\,.\,P(6, r-1)$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = 2.\frac{6!}{(6-(r-1))!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{2 \times 6 \times 5!}{(7-r)!}$$

$$\Rightarrow \quad \frac{5!}{(5-r)!} = \frac{12 \times 5!}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow \quad 1 = \frac{12}{(7-r)(6-r)}$$

$$\Rightarrow \quad (7-r)(6-r) = 12$$

$$\Rightarrow \quad (7-2)(6-r) = 4 \times 3$$

[By comparing]

$$\Rightarrow \quad 7-r = 4$$

$$\Rightarrow \quad r = 3$$

EXAMPLE 5 *If ${}^{10}P_r = 5040$, find the value of r.*

SOLUTION We have,

$${}^{10}P_r = 5040$$

$$\Rightarrow \quad \frac{10!}{(10-r)!} = 10 \times 504$$

$$\Rightarrow \quad \frac{10!}{(10-r)!} = 10 \times 9 \times 8 \times 7$$

$$\Rightarrow \quad \frac{10!}{(10-r)!} = \frac{10 \times 9 \times 8 \times 7 \times 6!}{6!}$$

$$\Rightarrow \quad \frac{10!}{(10-r)!} = \frac{10!}{6!} \Rightarrow (10-r)! = 6! \Rightarrow 10-r = 6 \Rightarrow r = 4.$$

EXAMPLE 6 *If $P(n-1, 3) : P(n, 4) = 1 : 9$, find n.*

SOLUTION We have,

$$P(n-1, 3) : P(n, 4) = 1 : 9$$

$$\Rightarrow \quad \frac{P(n-1, 3)}{P(n, 4)} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{\dfrac{(n-1)!}{(n-1-3)!}}{\dfrac{n!}{(n-4)!}} = \frac{1}{9}$$

$$\Rightarrow \quad \frac{(n-1)!}{(n-4)!} \times \frac{(n-4)!}{n!} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{n!} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{n \cdot (n-1)!} = \frac{1}{9} \Rightarrow n = 9$$

EXAMPLE 7 *If* ${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_r$, *find r.*

SOLUTION We have,

$${}^9P_5 + 5 \cdot {}^9P_4 = {}^{10}P_r$$

$$\Rightarrow \quad \frac{9!}{(9-5)!} + 5 \cdot \frac{9!}{(9-4)!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{9!}{4!} + 5 \cdot \frac{9!}{5!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{9!}{4!} + \frac{9!}{4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad 2 \times \frac{9!}{4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{5 \times 2 \times 9!}{5 \times 4!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{10 \times 9!}{5!} = \frac{10!}{(10-r)!}$$

$$\Rightarrow \quad \frac{10!}{5!} = \frac{10!}{(10-r)!} \Rightarrow (10-r)! = 5! \Rightarrow 10 - r = 5 \Rightarrow r = 5$$

EXAMPLE 8 *If* ${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30\,800 : 1$, *find r.*

SOLUTION We have,

$${}^{56}P_{r+6} : {}^{54}P_{r+3} = 30\,800 : 1$$

$$\Rightarrow \quad \frac{56!}{(56-r-6)!} : \frac{54!}{(54-r-3)!} = \frac{30800}{1}$$

$$\Rightarrow \quad \frac{56!}{(50-r)!} : \frac{54!}{(51-r)!} = 30800 : 1$$

$$\Rightarrow \quad \frac{56!}{(50-r)!} \times \frac{(51-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow \quad \frac{56 \times 55 \times 54!}{(50-r)!} \times \frac{(51-r) \times (50-r)!}{54!} = \frac{30800}{1}$$

$$\Rightarrow \quad 56 \times 55 \times (51-r) = 30800 \Rightarrow (51-r) = 10 \Rightarrow r = 41.$$

EXAMPLE 9 *If* ${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$, *find n.*

SOLUTION We have,

$${}^{2n+1}P_{n-1} : {}^{2n-1}P_n = 3 : 5$$

$$\frac{^{2n+1}P_{n-1}}{^{2n-1}P_n} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)!}{(n+2)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{(2n+1)(2n)(2n-1)!}{(n+2)(n+1)n(n-1)!} \times \frac{(n-1)!}{(2n-1)!} = \frac{3}{5}$$

$$\Rightarrow \quad \frac{2(2n+1)}{(n+2)(n+1)} = \frac{3}{5}$$

$$\Rightarrow \quad 10(2n+1) = 3(n+2)(n+1)$$

$$\Rightarrow \quad 3n^2 + 9n + 6 = 20n + 10$$

$$\Rightarrow \quad 3n^2 - 11n - 4 = 0 \Rightarrow (n-4)(3n+1) = 0 \Rightarrow n = 4 \qquad [\because n \neq -1/3]$$

EXAMPLE 10 *If* $^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$, *find* r.

SOLUTION We have,

$$^{22}P_{r+1} : {}^{20}P_{r+2} = 11:52$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} : \frac{20!}{(18-r)!} = 11:52$$

$$\Rightarrow \quad \frac{22!}{(21-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22 \times 21 \times 20!}{(21-r)(20-r)(19-r)\cdot(18-r)!} \times \frac{(18-r)!}{20!} = \frac{11}{52}$$

$$\Rightarrow \quad \frac{22 \times 21}{(21-r)(20-r)(19-r)} = \frac{11}{52}$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 2 \times 21 \times 52$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 2 \times 3 \times 7 \times 4 \times 13$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = 12 \times 13 \times 14$$

$$\Rightarrow \quad (21-r)(20-r)(19-r) = (21-7)(20-7)(19-7)$$

$$\Rightarrow \quad r = 7$$

Type III ON PROVING RESULTS RELATED TO $P(n, r)$ or ^nP_r

EXAMPLE 11 *Prove the following:*

(i) $P(n,n) = 2P(n, n-2)$

(ii) $P(n,n) = P(n, n-1)$

(iii) $P(n,r) = P(n-1,r) + r \,.\, P(n-1, r-1)$

(iv) $P(n,r) = n \cdot P(n-1, r-1)$

SOLUTION (i) $2P(n, n-2) = 2\dfrac{n!}{(n-(n-2))!} = 2\left(\dfrac{n!}{2!}\right) = n! = P(n,n)$

(ii) $P(n, n-1) = \dfrac{n!}{(n-(n-1))!} = \dfrac{n!}{1!} = n! = P(n,n)$

(iii)
$$P(n-1,r) + r \cdot P(n-1, r-1) = \frac{(n-1)!}{(n-1-r)!} + r \cdot \frac{(n-1)!}{((n-1)-(r-1))!}$$

$$= \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)!} = \frac{(n-1)!}{(n-r-1)!} + r\frac{(n-1)!}{(n-r)(n-r-1)!}$$

$$= \frac{(n-1)!}{(n-r-1)!}\left\{1 + \frac{r}{n-r}\right\} = \frac{(n-1)!}{(n-r-1)!}\left(\frac{n-r+r}{n-r}\right)$$

(iv) $n \cdot P(n-1, r-1) = n \dfrac{(n-1)!}{((n-1)-(r-1))!} = \dfrac{n!}{(n-r)!} = P(n, r)$

Type III PRACTICAL PROBLEMS ON PERMUTATIONS

<u>NOTE</u> ALITER 2 of each of the following examples should be done after studying permutations and combinations.

EXAMPLE 12 *In how many ways three different rings can be worn in four fingers with at most one in each finger?*

SOLUTION The total number of ways is same as the number of arrangements of 4 fingers, taken 3 at a time.

So, required number of ways $= {}^4P_3 = \dfrac{4!}{(4-3)!} = \dfrac{4!}{1!} = 4! = 24.$

<u>ALITER 1</u> Let R_1, R_2, R_3 be three rings. Since R_1 can be put in any one of the four fingers. So, there are four ways in which R_1 can be worn. Now, R_2 can be worn in any one of the remaining three fingers in 3 ways. In the remaining 2 fingers ring R_3 can be worn in 2 ways. So, by the fundamental principle of counting the total number of ways in which three different rings can be worn in four fingers is $4 \times 3 \times 2 = 24$.

<u>ALITER 2</u> Out of 4 fingers, 3 fingers can be chosen in ${}^4C_3$ ways. Now, three rings can be worn in the selected three fingers in 3! ways. Hence, three rings can be worn in four fingers in ${}^4C_3 \times 3! = 24$ ways.

EXAMPLE 13 *Seven athletes are participating in a race. In how many ways can the first three prizes be won?*

SOLUTION The total number of ways in which first three prizes can be won is the number of arrangements of seven different things taken 3 at a time.

So, required number of ways $= {}^7P_3 = \dfrac{7!}{(7-3)!} = \dfrac{7!}{4!} = \dfrac{7 \times 6 \times 5 \times 4!}{4!} = 210.$

<u>ALITER 1</u> First prize can be won in seven ways. Second prize can be won by any one of the remaining six athletes in 6 ways. Now, five athletes are left. So, third prize can be won by any one of the remaining 5 athletes in 5 ways.

Hence, by the fundamental principle of counting, the required number of ways $= 7 \times 6 \times 5 = 210$.

<u>ALITER 2</u> Out of 7 athletes, 3 can be chosen for prize in ${}^7C_3$ ways. Now, three prizes can be given to three chosen athletes in 3! ways.

$\therefore$ Numbers of ways in which 3 prizes can be won $= {}^7C_3 \times 3! = 210$

EXAMPLE 14 *How many different signals can be made by 5 flags from 8 flags of different colours?*

SOLUTION The total number of signals is the number of arrangements of 8 flags by taking 5 flags at a time.

Hence, required number of signals $= {}^8P_5 = \dfrac{8!}{(8-5)!} = \dfrac{8!}{3!} = \dfrac{8 \times 7 \times 6 \times 5 \times 4 \times 3!}{3!} = 6720$

EXAMPLE 15 *In how many ways can 6 persons stand in a queue?*

SOLUTION The number of ways in which 6 persons can stand in a queue is same as the number of arrangements of 6 different things taken all at a time.

Hence, the required number of ways $= {}^6P_6 = 6! = 720.$

EXAMPLE 16 *It is required to seat 8 men and 4 women in a row so that the women occupy the even places. How many such arrangements are possible?*

SOLUTION In all 12 persons are to be seated in a row and in the row of 12 positions there are exactly 6 even places viz second, fourth, sixth, eighth, tenth and twelfth. It is given that four women are to occupy 4 places out of these six even places. This can be done in ${}^6P_4$ ways (ways of arranging 6 women in 4 positions). The remaining 8 positions can be filled by the 8 men in ${}^8P_8$ ways. So, by the fundamental principle of counting, the number of seating arrangements as required, is ${}^6P_4 \times {}^8P_8 = 360 \times 40320 = 14515200$.

ALITER 1 In all 12 persons are to be seated in a row and in the row of 12 positions there are exactly 6 even places viz. 2nd, 4th, 6th, 8th and 12th. It is given that 4 women are to occupy any 4 places out of these six positions. This can be done in ${}^6C_4 \times 4!$ ways. The remaining 8 positions are to be occupied by 8 men. This can be done in ${}^8C_8 \times 8!$ ways.

Hence, total number of seating arrangements $= ({}^6C_4 \times 4!) \times ({}^8C_8 \times 8!)$

$$= 360 \times 40320 = 14515200.$$

EXAMPLE 17 *Three men have 4 coats, 5 waist coats and 6 caps. In how many ways can they wear them?*

SOLUTION The total number of ways in which three men can wear 4 coats is the number of arrangements of 4 different coats taken 3 at a time. So, three men can wear 4 coats in ${}^4P_3$ ways. Similarly, 5 waist coats and 6 caps can be worn by three men in ${}^5P_3$ and ${}^6P_3$ ways respectively. Hence, by the fundamental principle of counting, the required number of ways as desired

$$= {}^4P_3 \times {}^5P_3 \times {}^6P_3 = (4!) \times (5 \times 4 \times 3) \times (6 \times 5 \times 4) = 172800$$

EXAMPLE 18 *How many different signals can be given using any number of flags from 5 flags of different colours?*

SOLUTION The signals can be made by using at a time one or two or three or four or five flags. The total number of signals when r flags are used at a time from 5 flags is equal to the number of arrangements of 5, taking r at a time i.e. ${}^5P_r$. Since r can take values 1, 2, 3, 4, 5. Hence, by the fundamental principle of addition, the total number of signals

$$= {}^5P_1 + {}^5P_2 + {}^5P_3 + {}^5P_4 + {}^5P_5$$

$$= 5 + 5 \times 4 + 5 \times 4 \times 3 + 5 \times 4 \times 3 \times 2 + 5 \times 4 \times 3 \times 2 \times 1 = 5 + 20 + 60 + 120 + 120 = 325$$

EXAMPLE 19 *How many numbers lying between 100 and 1000 can be formed with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed?*

SOLUTION Every number lying between 100 and 1000 is a three digit number. Therefore, we have to find the number of permutations of five digits 1, 2, 3, 4, 5 taken three at a time.

Hence, the required number of numbers $= {}^5P_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5 \times 4 \times 3 = 60$

EXAMPLE 20 *How many four digit numbers are there with distinct digits?*

SOLUTION The total number of arrangements of ten digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 taking 4 at a time is ${}^{10}P_4$. But, these arrangements also include those numbers which have 0 at thousand's place. Such numbers are not four digit numbers. When 0 is fixed at thousand's place, we have to arrange remaining 9 digits by taking 3 at a time. The number of such arrangements is ${}^9P_3$.

So, the total number of numbers having 0 at thousand's place $= {}^9P_3$.

Hence, the total number of four digit numbers $= {}^{10}P_4 - {}^9P_3 = 5040 - 504 = 4536$.

EXAMPLE 21 *In how many ways 7 pictures can be hung from 5 picture nails on a wall ?*

SOLUTION The number of ways in which 7 pictures can be hung from 5 picture nails on a wall is same as the number of arrangements of 7 things, taking 5 at a time.

Hence, the required number $= {}^7P_5 = \dfrac{7!}{(7-5)!} = \dfrac{7!}{2!} = 2520.$

EXAMPLE 22 *Determine the number of natural numbers smaller than 10^4, in the decimal notation of which all the digits are distinct.*

SOLUTION The required natural numbers consist of 4 digits, 3 digits, 2 digits and one digit.

Total number of 4 digit natural numbers with distinct digits $= {}^{10}P_4 - {}^9P_3$

Total number of 3 digit natural numbers with distinct digits $= {}^{10}P_3 - {}^9P_2$

Total number of 2 digit natural numbers with distinct digits $= {}^{10}P_2 - {}^9P_1$

Total number of one digit natural numbers = 9

Hence, the required number of natural numbers $= ({}^{10}P_4 - {}^9P_3) + ({}^{10}P_3 - {}^9P_2) + ({}^{10}P_2 - {}^9P_1) + 9$

$= 9 \times 9 \times 8 \times 7 + 9 \times 9 \times 8 + 9 \times 9 + 9 = 5274.$

EXAMPLE 23 *How many words, with or without meaning, can be formed using all the letters of the word EQUATION, using each letter exactly once.*

SOLUTION There are eight letters in the word 'EQUATION'. So, the total number of words is equal to the number of arrangements of these letters, taken all at a time. The number of such arrangements is ${}^8P_8 = 8!$ Hence, the total number of words $= 8!$

EXAMPLE 24 *How many 4-letter words, with or without meaning, can be formed out of the letters of the word, 'LOGARITHMS', if repetition of letters is not allowed?*

SOLUTION There are 10 letters in the word 'LOGARITHMS'.

So, the number of 4 - letter word = Number of arrangements of 10 letters, taken 4 at a time

$= {}^{10}P_4 = 5040.$

LEVEL-2

EXAMPLE 25 *Prove that if $r \le s \le n$, then $P(n, s)$ is divisible by $P(n, r)$.*

SOLUTION Let $s = r + k$ where $0 \le k \le s - r$. Then,

$$P(n, s) = \frac{n!}{(n-s)!} = n(n-1)(n-2)\ldots(n-(s-1))$$

$$\Rightarrow \quad P(n, s) = n(n-1)(n-2)\ldots\{n-(r+k-1)\}$$

$$\Rightarrow \quad P(n, s) = n(n-1)(n-2)\ldots\{n-(r-1)\}(n-r)\{n-(r+1)\}\ldots\{n-(r+k-1)\}$$

$$\Rightarrow \quad P(n, s) = \{n(n-1)(n-2)\ldots n-(r-1)\}\{(n-r)(n-(r+1))\ldots(n-(r+k-1))\}$$

$$\Rightarrow \quad P(n, s) = P(n, r)\cdot\{(n-r)(n-(r+1))\ldots(n-(r+k-1))\}$$

$$\left[\because\ P(n, r) = \frac{n!}{(n-r)!} = n(n-1)(n-2)\ldots(n-(r-1))\right]$$

$$\Rightarrow \quad P(n, s) = P(n, r)\cdot\{(n-r)(n-(r+1))\ldots(n-(r+k-1))\}$$

$\Rightarrow$ $P(n, s)$ is divisible by $P(n, r)$.

EXAMPLE 26 *If P_m stands for ${}^mP_m$, then prove that:*

$$1 + 1\cdot P_1 + 2\cdot P_2 + 3\cdot P_3 + \ldots + n\cdot P_n = (n+1)!$$

SOLUTION We have, $P_m = {}^mP_m = m!$

So, $\quad 1 + 1\cdot P_1 + 2\cdot P_2 + 3\cdot P_3 + \ldots + n\cdot P_n$

$$= 1 + 1 + 2\cdot 2! + 3\cdot 3! + 4\cdot 4! + \ldots + n\cdot n!$$

$$= 1 + \sum_{r=1}^{n} r\cdot r!$$

$$= 1 + \sum_{r=1}^{n} \{(r+1) - 1\}\, r!$$

$$= 1 + \sum_{r=1}^{n} [(r+1)\, r! - r!]$$

$$= 1 + \sum_{r=1}^{n} [(r+1)! - r!]$$

$$= 1 + [(2! - 1!) + (3! - 2!) + (4! - 3!) + \ldots + ((n+1)! - n!)] = 1 + ((n+1)! - 1!) = (n+1)!$$

EXAMPLE 27 *In an examination hall there are four rows of chairs. Each row has 8 chairs one behind the other. There are two classes sitting for the examination with 16 students in each class. It is desired that in each row, all students belong to the same class and that no two adjacent rows are allotted to the same class. In how many ways can these 32 students be seated?*

SOLUTION Let the two classes be C_1 and C_2 and the four rows be R_1, R_2, R_3, R_4. There are 16 students in each class. So, there are 32 students. According to the given conditions there are two different ways in which 32 students can be seated:

	R_1	R_2	R_3	R_4
I	C_1	C_2	C_1	C_2
II	C_2	C_1	C_2	C_1

Since the seating arrangement can be completed by using any one of these two ways. So, by the fundamental principle of addition,

Total number of seating arrangements = No. of arrangement in I case + No. of arrangements in II case.

In case I, 16 students of class C_1 can be seated in R_1 and R_3 in ${}^{16}P_8 \times 8! = 16!$ ways. And 16 students of class C_2 can be seated in R_2 and R_4 in ${}^{16}P_8 \times 8! = 16!$ ways

$\therefore$ Number of seating arrangements in case $I = 16! \times 16!$

Similarly, Number of seating arrangements in case $II = 16! \times 16!$

Hence, Total number of seating arrangements = $(16! \times 16!) + (16! \times 16!) = 2\,(16! \times 16!)$

EXAMPLE 28 *Ten different letters of an alphabet are given. Words with five letters are formed from these given letters. Determine the number of words which have at least one letter repeated.*

SOLUTION The number of 5-letter words which can be formed from 10 letters when one or more of its letters is repeated $= 10 \times 10 \times 10 \times 10 \times 10 = 10^5$.

The number of 5-letter words which can be formed when none of their letters is repeated
= Number of arrangements of 10 letters by taking 5 at a time $= {}^{10}P_5 = 30240$

Hence, the number of 5-letter words which have at least one of their letters repeated is $10^5 - 30240 = 69760$.

EXAMPLE 29 *Find the sum of all the numbers that can be formed with the digits 2, 3, 4, 5 taken all at a time.*

SOLUTION The total number of numbers formed with the digits 2, 3, 4, 5 taken all at a time
= Number of arrangement of 4 digits, taken all at a time $= {}^4P_4 = 4! = 24$.

To find the sum of these 24 numbers, we will find the sum of digits at unit's, ten's, hundred's and thousand's places in all these numbers.

Consider the digits in the unit's places in all these numbers. Each of the digits 2, 3, 4, 5 occurs in $3!(=6)$ times in the unit's place.

So, total for the digits in the unit's place in all the numbers $= (2+3+4+5)\times 3! = 84$.

Since each of the digits 2, 3, 4, 5 occurs $3!$ times in any one of the remaining places.

So, the sum of the digits in the ten's, hundred's and thousand's places in all the numbers

$$= (2+3+4+5)\times 3! = 84.$$

Hence, the sum of all the numbers $= 84\,(10^0 + 10^1 + 10^2 + 10^3) = 93324$.

EXERCISE 16.2

LEVEL-1

1. Evaluate each of the following:
   (i) ${}^8P_3$ (ii) ${}^{10}P_4$ (iii) ${}^6P_6$ (iv) $P(6, 4)$ **[NCERT]**
2. If $P(5, r) = P(6, r-1)$, find r.
3. If $5\,P(4, n) = 6\,.\,P(5, n-1)$, find n.
4. If $P(n, 5) = 20\,.\,P(n, 3)$, find n
5. If ${}^nP_4 = 360$, find the value of n.
6. If $P(9, r) = 3024$, find r.
7. If $P(11, r) = P(12, r-1)$ find r.
8. If $P(n, 4) = 12\,.\,P(n, 2)$, find n.
9. If $P(n-1, 3) : P(n, 4) = 1:9$, find n. **[NCERT]**
10. If $P(2n-1, n) : P(2n+1, n-1) = 22:7$ find n.
11. If $P(n, 5) : P(n, 3) = 2:1$, find n.
12. Prove that: $1\,.\,P(1, 1) + 2\,.\,P(2, 2) + 3\,.\,P(3, 3) + \ldots + n\,.\,P(n, n) = P(n+1, n+1) - 1$.
13. If $P(15, r-1) : P(16, r-2) = 3:4$, find r.
14. If ${}^{n+5}P_{n+1} = \dfrac{11\,(n-1)}{2}\,{}^{n+3}P_n$, find n.
15. In how many ways can five children stand in a queue?
16. From among the 36 teachers in a school, one principal and one vice-principal are to be appointed. In how many ways can this be done?
17. Four letters E, K, S and V, one in each, were purchased from a plastic warehouse. How many ordered pairs of letters, to be used as initials, can be formed from them?
18. Four books, one each in Chemistry, Physics, Biology and Mathematics, are to be arranged in a shelf. In how many ways can this be done?
19. Find the number of different 4-letter words, with or without meanings, that can be formed from the letters of the word 'NUMBER'.
20. How many three-digit numbers are there, with distinct digits, with each digit odd?
21. How many words, with or without meaning, can be formed by using all the letters of the word 'DELHI', using each letter exactly once?
22. How many words, with or without meaning, can be formed by using the letters of the word 'TRIANGLE'?

23. There are two works each of 3 volumes and two works each of 2 volumes; In how many ways can the 10 books be placed on a shelf so that the volumes of the same work are not separated?
24. There are 6 items in column A and 6 items in column B. A student is asked to match each item in column A with an item in column B. How many possible, correct or incorrect, answers are there to this question?
25. How many three-digit numbers are there, with no digit repeated?
26. How many 6-digit telephone numbers can be constructed with digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 if each number starts with 35 and no digit appears more than once?
27. In how many ways can 6 boys and 5 girls be arranged for a group photograph if the girls are to sit on chairs in a row and the boys are to stand in a row behind them?
28. If a denotes the number of permutations of $(x+2)$ things taken all at a time, b the number of permutations of x things taken 11 at a time and c the number of permutations of $x-11$ things taken all at a time such that $a = 182\,bc$, find the value of x.
29. How many 3-digit numbers can be formed by using the digits 1 to 9 if no digit is repeated? **[NCERT]**
30. How many 3-digit even numbers can be made using the digits 1, 2, 3, 4, 5, 6, 7, if no digits is repeated? **[NCERT]**
31. Find the numbers of 4-digit numbers that can be formed using the digits 1, 2, 3, 4, 5, if no digit is repreated? Howmany of these will be even? **[NCERT]**
32. All the letters of the word 'EAMCOT' are arranged in different possible ways. Find the number of arrangements in which no two vowels are adjacent to each other.

ANSWERS

1. (i) 336	(ii) 5040	(iii) 720	(iv) 360
2. 4	3. 3	4. 8	5. 6
6. 4	7. 9	8. 6	9. 9
10. 10	11. 5	13. 14	14. 6, 7
15. 120	16. 1260	17. 12	18. 24
19. 360	20. 60	21. 120	22. 8!
23. 3456	24. 720	25. 648	26. 1680
27. 86400	28. 12	29. 504	30. 90
31. 120, 48	32. 144		

HINTS TO NCERT & SELECTED PROBLEMS

2. We have

$$P(5, r) = P(6, r-1)$$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{6!}{\{6-(r-1)\}!}$$

$$\Rightarrow \frac{5!}{(5-r)!} = \frac{6 \times 5!}{(7-r)!}$$

$$\Rightarrow \frac{1}{(5-r)!} = \frac{6}{(7-r)(6-r)(5-r)!}$$

$$\Rightarrow 1 = \frac{6}{(7-r)(6-r)} \Rightarrow (7-r)(6-r) = 3 \times 2 \Rightarrow 7-r = 3 \Rightarrow r = 4$$

9. $P(n-1,3) : P(n,4) = 1:9 \Rightarrow \frac{P(n-1,3)}{P(n,4)} = \frac{1}{9} \Rightarrow \frac{(n-1)!}{(n-4)!} \times \frac{(n-4)!}{n!} = \frac{1}{9} \Rightarrow \frac{1}{n} = \frac{1}{9} \Rightarrow n = 9.$

15. The total no. of ways = No. of arrangements of 5 things, taken all at a time = $^5P_5$.

16. Total no of ways = No. of arrangements of 36 things taken two at a time = $^{36}P_2$.

17. The total no. of ordered pairs = No. of arrangements of 4 letters, taken two at a time = $^4P_2$.

18. No. of ways = No. of arrangements of 4 books, taken all at a time = $^4P_4$.

19. Total no. of words = No. of arrangements of 6 letters, taken 4 at a time = $^6P_4$.

20. Required number of numbers = Number of arrangements of digits 1, 3, 5, 7, 9 by taking 3 at a time = $^5P_3$.

23. Let $\frac{W_{11}, W_{12}, W_{13}}{W_1}$; $\frac{W_{21}, W_{22}, W_{23}}{W_2}$; $\frac{W_{31}, W_{32}}{W_3}$; $\frac{W_{41}, W_{42}}{W_4}$ be 4 works. These 4 works can be arranged in 4! ways. Now, volumes of each work can be arranged in the following ways:
$W_1 \to 3!$ ways; $W_2 \to 3!$ ways, $W_3 \to 2!$ ways, $W_4 \to 2!$ ways.
Hence, total no. of ways to arrange all books $= 4!(3! \times 3! \times 2! \times 2!) = 3456$.

24. Each answer to the given question is an arrangement of the 6 items of column B keeping the order of items in column A fixed. Hence, the total number of answers = Number of arrangements of 6 items in column $B = {^6P_6} = 6!$.

25. Total number of three digit numbers with distinct digits = $^{10}P_3 - {^9P_2}$.

26. Required number of telephone numbers = $^8P_4$.

27. Five girls can sit on chairs in a row in $^5P_5 = 5!$ ways. Also, 6 boys can stand behind them in a row in $^6P_6 = 6!$ ways. Hence, the total number of ways $= 5! \times 6!$

31. The total number of 4 digit numbers formed by using the digits 1, 2, 3, 4, 5 is same as the number of arrangements of 5 digits taken 4 at a time.
So, required number of numbers $= {^5P_4} = \frac{5!}{(5-4)!} = 120$

An even number will have 2 or 4 at its unit's place. So, unit's place can be filled in 2 ways and the remaining three places (tens, hundreds and thousands) can be filled with remaining 4 digits in $^4P_3$ ways. Hence, total number of 4 digit even numbers formed by using the given digits is $^4P_3 \times 2 = 48$.

16.4 PERMUTATIONS UNDER CERTAIN CONDITIONS

In this section, we shall discuss permutations where either repetitions of items are allowed or distinction between some of the items are ignored or a particular item occurs in every arrangement etc. Such type of permutations are known as permutations under certain conditions as discussed below.

THEOREM 1 *Prove that the number of all permutations of n different objects taken r at a time, when a particular object is to be always included in each arrangement, is* $r \cdot {^{n-1}P_{r-1}}$.

PROOF Here we have to find the number of ways in which r places can be filled with n given objects such that a particular object occurs in each arrangement. Suppose the particular object is placed at the first place. Then, the remaining $(n-1)$ places can be filled with remaining $(r-1)$ objects in ${}^{n-1}P_{r-1}$ ways. Similarly, by fixing the particular object at the second, third, fourth, ..., rth places, we find that the number of permutations in each case is ${}^{n-1}P_{r-1}$.

Hence, by the fundamental principle of addition,

The required number of permutations $= {}^{n-1}P_{r-1} + {}^{n-1}P_{r-1} + \dots + {}^{n-1}P_{r-1} = r \cdot {}^{n-1}P_{r-1}$.

Q.E.D.

THEOREM 2 *Prove that the number of permutations of n distinct objects taken r at a time, when a particular object is never taken in each arrangement, is* ${}^{n-1}P_r$.

PROOF Since one particular object out of n given objects is never taken. So, we have to determine the number of ways in which r places can be filled with $(n-1)$ distinct objects.

Clearly, the number of such arrangement is ${}^{n-1}P_r$.

Q.E.D.

THEOREM 3 *Prove that the number of permutations of n different objects taken r at a time in which two specified objects always occur together is* $2!\,(r-1)\,{}^{n-2}P_{r-2}$.

PROOF First let us leave out the two specified objects. Then the number of permutations of the remaining $(n-2)$ objects, taken $(r-2)$at a time, is ${}^{n-2}P_{r-2}$. Now, we consider two specified objects temporarily as a single object and add it to each of these ${}^{n-2}P_{r-2}$ permutations which can be done in $(r-1)$ ways. Thus, the number of permutations becomes $(r-1)\,{}^{n-2}P_{r-2}$. But two specified things can be put together in 2! ways.

Hence, the required number of permutations is $2!\,.(r-1)\,.\,{}^{n-2}P_{r-2}$.

Q.E.D.

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *In how many ways can the letters of the word PENCIL be arranged so that (i) N is always next to E ? (ii) N and E are always together ?*

SOLUTION (i) Let us keep EN together and consider it as one letter. Now, we have 5 letters which can be arranged in a row in ${}^5P_5 = 5! = 120$ ways. Hence, the total number of ways in which N is always next to E is 120.

(ii) Keeping E and N together and considering it as one letter, we have 5 letters which can be arranged in ${}^5P_5 = 5!$ ways. But, E and N can be put together 2! ways (viz. EN, NE).

Hence, the total number of ways $= 5! \times 2! = 240$.

EXAMPLE 2 *How many different words can be formed with the letters of the word EQUATION so that*

(i) *the words begin with E ?* (ii) *the words begin with E and end with N ?*

(iii) *the words begin and end with a consonant ?*

SOLUTION Clearly, the given word contains 8 letters out of which 5 are vowels and 3 consonants.

(i) Since all words must begin with E. So, we fix E at the first place. Now, remaining 7 letters can be arranged in ${}^7P_7 = 7!$ ways.

So, total number of words $= 7!$

(ii) Since all words must begin with E and end with N. So, we fix E at the first place and N at the last place. Now, remaining 6 letters can be arranged in ${}^6P_6 = 6!$ ways.

Hence, the required number of words $= {}^6P_6 = 6!$

(iii) There are 3 consonants and all words should begin and end with a consonant. So, first and last places can be filled with 3 consonants in ${}^3P_2$ ways. Now, the remaining 6 places are to be filled up with the remaining 6 letters in ${}^6P_6$ ways.

Hence, the required number of words $= {}^3P_2 \times {}^6P_6 = 6 \times 720 = 4320$

EXAMPLE 3 *How many words can be formed from the letters of the word, 'TRIANGLE'? How many of these will begin with T and end with E?*

SOLUTION There are 8 letters in the word 'TRIANGLE'. The total number of words formed with these 8 letters is the number of arrangements of 8 items, taken all at a time, which is equal to ${}^8P_8 = 8! = 40320$. If we fix up T in the beginning and E at the end, then the remaining 6 letters can be arranged in ${}^6P_6 = 6!$ ways.

So, the total number of words which begin with T and end with $E = 6! = 720$.

EXAMPLE 4 *How many words can be formed with the letters of the word 'ORDINATE' so that vowels occupy odd places?*

SOLUTION There are 4 vowels and 4 consonants in the word 'ORDINATE'. We have to arrange 8 letters in a row such that vowels occupy odd places. There are 4 odd places viz. 1, 3, 5, 7. Four vowels can be arranged in these 4 odd places in 4! ways. Remaining 4 even places viz. 2, 4, 6, 8 are to be occupied by the 4 consonants. This can be done in 4! ways. Hence, the total number of words in which vowels occupy odd places $= 4! \times 4! = 576$.

EXAMPLE 5 *In how many ways 5 boys and 3 girls can be seated in a row so that no two girls are together?*

SOLUTION The 5 boys can be seated in a row in ${}^5P_5 = 5!$ ways. In each of these arrangements 6 places are created, shown by the cross-marks, as given below:

$$\times B \times B \times B \times B \times B \times$$

Since no two girls are to sit together, so we may arrange 3 girls in 6 places. This can be done in ${}^6P_3$ ways i.e. 3 girls can be seated in ${}^6P_3$ ways.

Hence, the total number of seating arrangements $= {}^5P_5 \times {}^6P_3 = 5! \times 6 \times 5 \times 4 = 14400$.

EXAMPLE 6 *In how many ways can the letters of the word 'DELHI' be arranged so that the vowels occupy only even places?*

SOLUTION There are 5 distinct letters in the word 'DELHI'. We wish to find the total number of arrangements of these 5 letters so that vowels occupy only even places. There are two vowels E and I and 2 even places viz 2nd and 4th. These two vowels can be arranged in the two even places in 2! ways. The remaining three letters (D, L, H) can be arranged in 3 places (viz Ist 3rd, 5th) in 3! ways. Hence, by the fundamental principle of counting the total number of arrangements $= 3! \times 2! = 12$.

EXAMPLE 7 *How many words can be formed from the letters of the word 'DAUGHTER' so that*

(i) *the vowels always come together?* (ii) *the vowels never come together?* **[NCERT]**

SOLUTION There are 8 letters in the word 'DAUGHTER', including 3 vowels (A, U, E) and 5 consonants (D, G, H, T, R).

(i) Considering three vowels as one letter, we have 6 letters which can be arranged in ${}^6P_6 = 6!$ ways. But, corresponding each way of these arrangements, the vowels A, U, E can be put together in 3! ways.

Hence, required number of words $= 6! \times 3! = 720 \times 6 = 4320$

(ii) The total number of words formed by using all the eight letters of the word 'DAUGHTER' is $^8P_8 = 8! = 40320$.

So, the total number of words in which vowels are never together

$=$ Total number of words $-$ Number of words in which vowels are always together

$= 40320 - 4320 = 36000$

EXAMPLE 8 *In how many ways can 9 examination papers be arranged so that the best and the worst papers are never together?*

SOLUTION The number of arrangements in which the best and the worst papers never come together can be obtained by subtracting from the total number of arrangements, the number of arrangements in which the best and worst come together.

The total number of arrangements of 9 papers $= {}^9P_9 = 9!$

Considering the best and the worst papers as one paper, we have 8 papers which can be arranged in $^8P_8 = 8!$ ways. But, the best and worst papers can be put together in $2!$ ways. So, the number of permutations in which the best and the worst papers can be put together $= (2! \times 8!)$.

Hence, the number of ways in which the best and the worst papers never come together $= 9! - 2! \times 8! = 9 \times 8! - 2 \times 8! = 7 \times 8! = 282240$.

EXAMPLE 9 *In how many ways can 5 children be arranged in a row such that*

(i) *two of them, Ram and Shyam, are always together?*

(ii) *two of them, Ram and Shyam, are never together?*

SOLUTION There are five children including Ram and Shyam.

(i) Considering Ram and Shyam as one child, there are four children. They can be arranged in a row in 4! ways. But Ram and Shyam can be arranged together in 2! ways.

Hence, the required number of arrangements $= 4! \times 2! = 48$.

(ii) Total number of arrangements of 5 children in a row $= 5! = 120$.

$\therefore$ Total number of arrangements in which Ram and Shyam are never together

$=$ Total number of arrangements $-$ Number of arrangements in which Ram and Shyam are together

$= 120 - 48 = 72$.

EXAMPLE 10 *A code word is to consist of two distinct English alphabets followed by two distinct numbers from 1 to 9. For example, CA 23 is a code word. How many such code words are there? How many of them end with an even integer?*

SOLUTION There are 26 English alphabets. So, first two places in the code word can be filled in $^{26}P_2$ ways. In last two places we have to use two distinct numbers from 1 to 9. So, last two places can be filled in $^9P_2$ ways. Hence, by the fundamental principle of counting, the total number of code words $= {}^{26}P_2 \times {}^9P_2 = 650 \times 72 = 46800$.

Number of code words ending with an even integer.

In this case, the code word can have any of the numbers 2, 4, 6, 8 at the extreme right position. So, the extreme right position can be filled in 4 ways. Now, next left position can be filled with any one of the remaining 8 digits in 8 ways and the two extreme left positions can be filled by two English alphabets in $^{26}P_2$ ways.

Hence, the total number of code words which end with an even integer $= 4 \times 8 \times {}^{26}P_2$

$= 4 \times 8 \times 650 = 20800$.

EXAMPLE 11 *The Principal wants to arrange 5 students on the platform such that the boy 'SALIM' occupies the second position and such that the girl, 'SITA' is always adjacent to the girl 'RITA'. How many such arrangements are possible ?*

SOLUTION Since SALIM occupies the second position and the two girls RITA and SITA are always adjacent to each other. So, none of these two girls can occupy the first seat. Thus, first seat can be occupied by any one of the remaining two students in 2 ways. Second seat can be occupied by SALIM in only one way.

Now, in the remaining three seats SITA and RITA can be seated in the following four ways:

	I	II	III	IV	V
1.	×	SALIM	SITA	RITA	×
2.	×	SALIM	RITA	SITA	×
3.	×	SALIM	×	SITA	RITA
4.	×	SALIM	×	RITA	SITA

Now, only one seat is left which can be occupied by the 5th student in one way.

Hence, the number of required type of arrangements $= 2 \times 4 \times 1 = 8$.

EXAMPLE 12 *How many numbers between 400 and 1000 can be formed with the digits 0, 2, 3, 4, 5, 6 if no digit is repeated in the same number ?*

SOLUTION Number between 400 and 1000 consist of three digits with digit at hundred's place greater than or equal to 4. Hundred's place can be filled, by using the digits 4, 5, 6 in 3 ways. Now, ten's and unit's places can be filled by the remaining 5 digits in ${}^5P_2$ ways.

Hence, the required number of numbers $= 3 \times {}^5P_2 = 3 \times \frac{5!}{3!} = 3 \times 20 = 60$.

EXAMPLE 13 *In a class of 10 students there are 3 girls A, B, C. In how many different ways can they be arranged in a row such that no two of the three girls are consecutive.*

SOLUTION There are 7 boys and 3 girls. Seven boys can be arranged in a row in ${}^7P_7 = 7!$ ways.

Now, we have 8 places in which we can arrange 3 girls in ${}^8P_3$ ways.

Hence, by the fundamental principle of counting, the number of arrangements $= 7! \times {}^8P_3$

$= 7! \times 336.$

LEVEL-2

EXAMPLE 14 *When a group photograph is taken, all the seven teachers should be in the first row and all the twenty students should be in the second row. If the two corners of the second row are reserved for the two tallest students, interchangeable only between them, and if the middle seat of the front row is reserved for the Principal, how many arrangements are possible?*

SOLUTION Since the middle seat of the front row is reserved for the Principal, the remaining 6 teachers can be arranged in the front row in ${}^6P_6 = 6!$ ways.

The two corners of the second row are reserved for the two tallest students. They can occupy these two places in 2! ways. The remaining 18 seats may be occupied by the remaining 18 students in 18! ways.

Hence, by the fundamental principle of counting, the total number of arrangements

$= 6! \times (18! \times 2!) = 18! \times 1440.$

EXAMPLE 15 *How many even numbers are there with three digits such that if 5 is one of the digits, then 7 is the next digit?*

SOLUTION We have to determine the total number of even numbers formed by using the given condition. So, at unit's place we can use one of the digits 0, 2, 4, 6, 8. If 5 is at ten's place then, as per the given condition, 7 should be at unit's place. In such a case the number will not be an even number. So, 5 cannot be at ten's and one's places. Hence, 5 can be only at hundred's place.

Now two cases arise.

CASE I *When 5 is at hundred's place:*

If 5 is at hundred's place, then 7 will be at ten's place. So, unit's place can be filled in 5 ways by using the digits 0, 2, 4, 6, 8.

So, total number of even numbers $= 1 \times 1 \times 5 = 5$.

CASE II *When 5 is not at hundred's place:*

In this case, hundred's place can be filled in 8 ways (0 and 5 cannot be used at hundred's place). In ten's place we can use any one of the ten digits except 5. So, ten's place can be filled in 9 ways. At unit's place we have to use one of the even digits 0, 2, 4, 6, 8. So, units place can be filled in 5 ways.

So, total number of even numbers $= 8 \times 9 \times 5 = 360$

Hence, the total number of required even numbers $= 360 + 5 = 365$.

EXAMPLE 16 *How many four digit numbers divisible by 4 can be made with the digits 1, 2, 3, 4, 5 if the repetition of digits is not allowed ?*

SOLUTION Recall that a number is divisible by 4 if the number formed by the last two digits is divisible by 4. The digits at unit's and ten's places can be arranged as follows:

Th	H	T	O
×	×	1	2
×	×	2	4
×	×	3	2
×	×	5	2

Now, corresponding each such way the remaining three digits at thousand's and hundred's places can be arranged in ${}^3P_2$ ways.

Hence, the required number of numbers $= {}^3P_2 \times 4 = 3! \times 4 = 24$.

EXAMPLE 17 *Find the number of ways in which 5 boys and 5 girls be seated in a row so that*

(i) *No two girls may sit together.* (ii) *All thgirls sit together and all the boys sit together.*

(iii) *All the girls are never together.*

SOLUTION (i) 5 boys can be seated in a row in ${}^5P_5 = 5!$ ways. Now, in the 6 gaps 5 girls can be arranged in ${}^6P_5$ ways.

Hence, the number of ways in which no two girls sit together $= 5! \times {}^6P_5 = 5! \times 6!$

(ii) The two groups of girls and boys can be arranged in 2! ways. 5 girls can be arranged among themselves in 5! ways. Similarly, 5 boys can be arranged among themselves in 5! ways. Hence, by the fundamental principle of counting, the total number of requisite seating arrangements $= 2!(5! \times 5!) = 2(5!)^2$.

(iii) The total number of ways in which all the girls are never together
$=$ Total number of arrangements $-$ Total number of arrangements in which all the girls are always together

$= 10! - 5! \times 6!$

EXAMPLE 18 *Five boys and five girls form a line with the boys and girls alternating. Find the number of ways of making the line.*

SOLUTION 5 boys can be arranged in a line in ${}^5P_5 = 5!$ ways. Since the boys and girls are alternating. So, corresponding each of the 5! ways of arrangements of 5 boys we obtain 5 places marked by cross as shown below:

(i) $B_1 \times B_2 \times B_3 \times B_4 \times B_5 \times$ (ii) $\times B_1 \times B_2 \times B_3 \times B_4 \times B_5$.

Clearly, 5 girls can be arranged in 5 places marked by cross in (5! + 5!) ways.

Hence, the total number of ways of making the line $= 5! \times (5! + 5!) = 2\,(5!)^2$

EXAMPLE 19 *In how many ways three girls and nine boys can be seated in two vans, each having numbered seats, 3 in the front and 4 at the back? How many seating arrangements are possible if 3 girls sit together in a back row on adjacent seats ?*

SOLUTION Total number of persons = 3 girls + 9 boys = 12.

Total number of numbered seats $= 2 \times 3 + 4 \times 2 = 14$

So, total number of ways in which 12 persons can be seated on 14 seats

= Number of arrangements or 14 seats by taking 12 at a time $= {}^{14}P_{12}$.

Three girls can be seated together in a back row on adjacent seats in the following ways:

1, 2, 3 or 2, 3, 4 of first van

and, 1, 2, 3 or 2, 3, 4 of second one.

In each way the three girls can interchange among themselves in 3! ways. So, the total number of ways in which three girls can be seated together in a back row on adjacent seats $= 4 \times 3!$

Now, 9 boys are to be seated on remaining 11 seats, which can be done in ${}^{11}P_9$ ways.

Hence, by the fundamental principle of counting, the total number of seating arrangements is ${}^{11}P_9 \times 4 \times 3!$.

EXAMPLE 20 *A tea party is arranged for 16 persons along two sides of a long table with 8 chairs on each side. Four persons wish to sit on one particular and two on the other side. In how many ways can they be seated?*

SOLUTION Let the two sides be A and B. Assume that four persons wish to sit on side A. Four persons who wish to sit on side A can be accommodated on eight chairs in ${}^8P_4$ ways and two persons who wish to sit on side B can be accommodated on 8 chairs in ${}^8P_2$ ways. Now, 10 persons are left, who can sit on 10 chairs on both the sides of the table in 10! ways.

Hence, the total number of ways in which 16 persons can be seated $= {}^8P_4 \times {}^8P_2 \times 10!$

EXERCISE 16.4

LEVEL-1

1. In how many ways can the letters of the word 'FAILURE' be arranged so that the consonants may occupy only odd positions?
2. In how many ways can the letters of the word 'STRANGE' be arranged so that
   (i) the vowels come together ? (ii) the vowels never come together? and
   (iii) the vowels occupy only the odd places?
3. How many words can be formed from the letters of the word 'SUNDAY'? How many of these begin with D?

4. How many words can be formed out of the letters of the word, 'ORIENTAL', so that the vowels always occupy the odd places ?

5. How many different words can be formed with the letters of word 'SUNDAY'? How many of the words begin with N? How many begin with N and end in Y?

6. How many different words can be formed from the letters of the word 'GANESHPURI' ? In how many of these words:
   (i) the letter G always occupies the first place?
   (ii) the letters P and I respectively occupy first and last place?
   (iii) the vowels are always together?
   (iv) the vowels always occupy even places?

7. How many permutations can be formed by the letters of the word, 'VOWELS', when
   (i) there is no restriction on letters?
   (ii) each word begins with E?
   (iii) each word begins with O and ends with L?
   (iv) all vowels come together?
   (v) all consonants come together?

8. How many words can be formed out of the letters of the word 'ARTICLE', so that vowels occupy even places?

9. In how many ways can a lawn tennis mixed double be made up from seven married couples if no husband and wife play in the same set?

10. m men and n women are to be seated in a row so that no two women sit together. If $m > n$ then show that the number of ways in which they can be seated as $\dfrac{m!\,(m+1)!}{(m-n+1)!}$.

11. How many words (with or without dictionary meaning) can be made from the letters in the word MONDAY, assuming that no letter is repeated, if
   (i) 4 letters are used at a time?
   (ii) all letters are used at a time?
   (iii) all letters are used but first is vowel?

12. How many three letter words can be made using the letters of the word 'ORIENTAL'?

ANSWERS

1. 576 2. (i) 1440 (ii) 3600 (iii) 1440 3. 720, 120 4. 576
5. 720, 120, 24 6. 10! (i) 9! (ii) 8! (iii) $7! \times 4!$ (iv) $5! \times 6!$
7. (i) 720 (ii) 120 (iii) 24 (iv) 240 (v) 144 8. 144 9. 840
11. (i) 360 (ii) 720 (iii) 240 12. 336

16.5 PERMUTATIONS OF OBJECTS NOT ALL DISTINCT

So far we were discusing permutations of distinct objects (things) by taking some or all at a time. In this section, we intend to discuss the permutations of a given number of objects when objects are not all different. For example, the number of arrangements of the letters of the word MISSISSIPPI, the number of six digit numbers formed by using the digits 1, 1, 2, 3, 3, 4 etc. The following theorem is very helpful to determine the number of such arrangements.

THEOREM *The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p + q = n$ is $\dfrac{n!}{p!\,q!}$.*

PROOF Let the required number of permutations be x. Consider one of these x permutations. Now, replace p alike things in this permutation by p distinct things which are also different from others. These p different things may be permuted among themselves in $p!$ ways without changing the positions of other things. Similarly, if we replace q alike things by q distinct things, which are also different from others, then they can be permuted among themselves in $q!$ ways.

Thus, if both the replacements are done simultaneously, then we find that each one of the x permutations give rise to $p! \times q!$ permutations. Therefore, x permutations give rise to $x \times p! \times q!$ permutations. Now, each of these $x \times p! \times q!$ permutations, is a permutation of n different things, taken all at a time.

$\therefore \quad x \times p! \times q! =$ Number of permutations of n different things taken all at a time $= n!$

Hence, $\quad x = \dfrac{n!}{p!\,q!}$

Q.E.D.

<u>REMARK 1</u> *The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ...; p_r are alike of rth kind such that $p_1 + p_2 + ... + p_r = n$, is*

$$\frac{n!}{p_1!\,p_2!\,p_3!\ldots p_r!}$$

<u>REMARK 2</u> *The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is $\dfrac{n!}{p!\,q!}$.*

<u>REMARK 3</u> *Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \ldots p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is $\dfrac{(p_1 + p_2 + ... + p_r)!}{p_1!\,p_2!\,p_3!\ldots p_r!}$.*

ILLUSTRATIVE EXAMPLES

LEVEL-1

EXAMPLE 1 *How many different words can be formed with the letters of the word 'MISSISSIPPI' ? In how many of these permutations four I's do not come together?* **[NCERT]**

SOLUTION There are 11 letters in the given word, of which 4 are S's, 4 are I's and 2 are P's. So, total number of words is the number of arrangements of 11 things, of which 4 are similar of one kind, 4 are similar of second kind and 2 are similar of third kind i.e. $\dfrac{11!}{4!\,4!\,2!}$.

Hence,the total number of words $= \dfrac{11!}{4!\,4!\,2!} = 34650$.

Considering 4 I's as one letter, we have 8 letters of which 4 are S's and 2 are P's. These 8 letters can be arranged in $\dfrac{8!}{4!\,2!}$ ways.

$\therefore \quad$ Number of words in which 4 I's come together $= \dfrac{8!}{4!\,2!} = 840$.

Hence, number of words in which 4 I's do not come together $= 34650 - 840 = 33810$.

EXAMPLE 2 *How many permutations of the letters of the word 'APPLE' are there?*

SOLUTION Here there are 5 letters, two of which are of the same kind. The others are each of its own kind. So, the required number of permutations is $\dfrac{5!}{2!1!1!1!} = \dfrac{120}{2} = 60$.

EXAMPLE 3 *How many words can be formed using the letter A thrice, the letter B twice and the letter C thrice ?*

SOLUTION We are given 8 letters viz. A, A, A, B, B, C, C, C. Clearly, there are 8 letters of which three are of one kind, two are of second kind and three are of third kind.

So, the total number of permutations $= \dfrac{8!}{3!\,2!\,3!} = 560$.

Hence, the requisite number of words $= 560$.

EXAMPLE 4 *Find the number of different permutations of the letters of the word BANANA ?*

SOLUTION Clearly, there are six letters in the word 'BANANA' of which three are alike of one kind (3 A's), two are alike of second kind (2 N's) and one of its own kind.

$\therefore$ Total number of their permutations $= \dfrac{6!}{3!\,2!\,1!} = 60.$

Hence, the requisite number of words = 60

EXAMPLE 5 (i) *How many different words can be formed with the letters of the word HARYANA?*

(ii) *How many of these begin with H and end with N?*

(iii) *In how many of these H and N are together?*

SOLUTION (i) There are 7 letters in the word 'HARYANA' of which 3 are A's and remaining all are each of its own kind.

So, total number of words $= \dfrac{7!}{3!1!1!1!1!} = \dfrac{7!}{3!} = 840.$

(ii) After fixing H in first place and N in last place, we have 5 letters out of which three are alike i.e. A's and remaining all are each of its own kind.

So, total number of words $= \dfrac{5!}{3!} = 20.$

(iii) Considering H and N together we have $7 - 2 + 1 = 6$ letters out of which three are alike i.e. A's and others are each of its own kind. These six letters can be arranged in $\dfrac{6!}{3!}$ ways. But H and N can be arranged amongst themselves in 2! ways.

Hence, the requisite number of words $= \dfrac{6!}{3!} \times 2! = 120 \times 2 = 240.$

EXAMPLE 6 *How many different words can be formed by using all the letters of the word 'ALLAHABAD' ?* **[NCERT]**

(i) *In how many of them vowels occupy the even positions ?*

(ii) *In how many of them both L do not come together ?*

SOLUTION There are 9 letters in the word 'ALLAHABAD' out of which 4 are A's, 2 are L's and the rest are all distinct.

So, the requisite number of words $= \dfrac{9!}{4!\,2!} = 7560.$

(i) There are 4 vowels and all are alike i.e. 4 A's. Also, there are 4 even places viz 2nd, 4th, 6th and 8th. So, these 4 even places can be occupied by 4 vowels in $\dfrac{4!}{4!} = 1$ way. Now, we are left with 5 places in which 5 letters, of which two are alike (2 L's) and other distinct, can be arranged in $\dfrac{5!}{2!}$ ways.

Hence, the total number of words in which vowels occupy the even places $= \dfrac{5!}{2!} \times \dfrac{4!}{4!} = \dfrac{5!}{2!} = 60.$

(ii) Considering both L together and treating them as one letter we have 8 letters out of which A repeats 4 times and others are distinct. These 8 letters can be arranged in $\dfrac{8!}{4!}$ ways.

So, the number of words in which both L come together $= \dfrac{8!}{4!} = 1680.$

Hence, the number of words in which both L do not come together

= Total no. of words – No. of words in which both L come together = 7560 – 1680 = 5880.

EXAMPLE 7 *Find the number of arrangements of the letters of the word INDEPENDENCE. In how many of these arrangements* **[NCERT]**

(i) *do the words start with P?* (ii) *do all the vowels always occur together?*

(iii) *do all the vowels never occur together?* (iv) *do the words begin with I and end in P?*

SOLUTION In the word 'INDEPENDENCE' there are 12 letters of which 3 are N's, 4 are E's and 2 are D's. Therefore,

Total number of arrangements $= \frac{12!}{3!\,4!\,2!} = 1663200$

(i) After fixing the letter P at the extreme left position, there are 11 letters consisting of 3 N's, 4E's and 2D's. These 11 letters can be arranged in $\frac{11!}{3!\,4!\,2!} = 138600$

$\therefore$ Number of words beginning with P $= \frac{11!}{3!\,4!\,2!} = 138600$

(ii) There are 5 vowels in the given word of which 4 are E's and one I. These vowels can be put together in $\frac{5!}{4!\,1!}$ ways. Considering these 5 vowels as one letter there are 8 letters (taking 7 remaining letters) which can be arranged in $\frac{8!}{3!\,2!}$ ways (as there are 3 N's and 2D's). Since corresponding to each arrangement of 5 vowels there are $\frac{8!}{3!\,2!}$ ways of arranging remaining 7 letters and one letter formed by 5 vowels.

Hence, by fundamental principle of multiplication, the required number of arrangements is $\frac{8!}{3!\,2!} \times \frac{5!}{4!\,1!} = 16800$

(iii) The required number of arrangements

= The total number of arrangements − The number of arrangements in which all the vowels occur together

$= 1663200 - 16800 = 1646400$

(iv) Let us fix I at the extreme left end and P at the extreme right end. Now, we are left with 10 letters of which 3 are N's, 4 are E's and 2 are D's. These ten letters can be arranged in $\frac{10!}{4!\,3!\,2!}$ ways.

Hence, required number of arrangements $= \frac{10!}{4!\,3!\,2!} = 12600.$

EXAMPLE 8 *In how many ways can the letters of the word PERMUTATIONS be arranged if (i) the words start with P end with S (ii) vowels are all together.*

SOLUTION (i) There are 12 letters in the given word of which 2 are T's and the remaining are distinct. Remaining 10 letters between P and S can be arranged in $\frac{10!}{2!}$ ways.

$\therefore$ Total number of words starting with P and ending in S $= \frac{10!}{2!} = 1814400$

(ii) There are 5 vowels in the given word. These vowels can be put together in 5! ways. Considering these 5 vowels as one letter, we have 8 letters (7 remaining letters and one letter formed by 5 vowels) of which 2 are T's. These 8 letters can be arranged in $\frac{8!}{2!}$ ways.

Hence, by the fundamental principle of multiplication, required number of words is $5! \times \frac{8!}{2!} = 2419200.$

EXAMPLE 9 *How many numbers greater than a million can be formed with the digits 2, 3, 0, 3, 4, 2, 3 ?*

SOLUTION Any number greater than a million will contain all the seven digits.
Now, we have to arrange these seven digits, out of which 2 occur twice, 3 occurs twice and the rest are distinct.

The number of such arrangements $= \dfrac{7!}{2! \times 3!} = 420.$

These arrangements also include those numbers which contain 0 at the million's place.
Keeping 0 fixed at the millionth place, we have 6 digits out of which 2 occurs twice, 3 occurs thrice and the rest are distinct. These 6 digits can be arranged in $\dfrac{6!}{2! \times 3!} = 60$ ways.

Hence, the number of required numbers = 420 – 60 = 360.

EXAMPLE 10 *There are six periods in each working day of a school. In how many ways can one arrange 5 subjects such that each subject is allowed at least one period?*

SOLUTION Since each subject is allowed at least one period. So, we first select one subject for the left out period. This can be done in ${}^5C_1$ ways. Now, six subject can be arranged in $\dfrac{6!}{2!}$ ways.

Hence, the total number of arrangements $= {}^5C_1 \times \dfrac{6!}{2!} = 1800$

LEVEL-2

EXAMPLE 11 *How many arrangements can be made with the letters of the word 'MATHEMATICS'? In how many of them vowels are together ?*

SOLUTION There are 11 letters in the word 'MATHEMATICS' of which two are M's, two are A's, two are T's and all other are distinct. So,

Required number of arrangements $= \dfrac{11!}{2! \times 2! \times 2!} = 4989600$

There are 4 vowels viz. A, E, A, I. Considering these four vowels as one letter we have 8 letters (M, T, H, M, T, C, S and one letter obtained by combining all vowels), out of which M occurs twice, T occurs twice and the rest all different. These 8 letters can be arranged in $\dfrac{8!}{2! \times 2!}$ ways.

But, the four vowels (A, E, A, I) can be put together in $\dfrac{4!}{2!}$ ways.

Hence, the total number of arrangements in which vowels are always together $= \dfrac{8!}{2! \times 2!} \times \dfrac{4!}{2!}$

$= 10080 \times 12 = 120960.$

EXAMPLE 12 *If all the letters of the word 'AGAIN' be arranged as in a dictionary, what is the fiftieth word?* **[NCERT]**

SOLUTION In dictionary the words at each stage are arranged in alphabetical order. Starting with the letter A, and arranging the other four letters GAIN, we obtain 4! = 24 words.
Thus, there are 24 words which start with A. These are the first 24 words.
Then, starting with G, and arranging the other four letters A, A, I, N in different ways, we obtain $\dfrac{4!}{2!} = \dfrac{24}{2} = 12$ words. Thus, there are 12 words, which start with G.

Now, we start with I. The remaining 4 letters A, G, A, N can be arranged in $\dfrac{4!}{2!} = 12$ ways. So, there are 12 words, which start with I.

Thus, we have so far constructed 48 words. The 49th word is NAAGI and hence the 50th word is NAAIG.

EXAMPLE 13 *The letters of the word 'RANDOM' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'RANDOM'.*

SOLUTION In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we must therefore consider the words beginning with A, D, M, N, O, R in order. A will occur in the first place as often as there are ways of arranging the remaining 5 letters all at a time i.e. A will occur 5! times. Similarly, D, M, N, O will occur in the first place the same number of times.

$\therefore$ Number of words starting with A = 5! = 120
Number of words starting with D = 5! = 120
Number of words starting with M = 5! = 120
Number of words starting with N = 5! = 120
Number of words starting with O = 5! = 120

Number of words beginning with R is 5!, but one of these words is the word RANDOM. So, we first find the number of words beginning with RAD and RAM.

No. of words starting with RAD = 3! = 6
No. of words starting with RAM = 3! = 6

Now, the words beginning with 'RAN' must follow. There are 3! words beginning with RAN. One of these words is the word RANDOM itself.

The first word beginning with RAN is the word RANDMO and the next word is RANDOM.

$\therefore$ Rank of RANDOM $= 5 \times 120 + 2 \times 6 + 2 = 614.$

EXAMPLE 14 *If the different permutations of the word, 'EXAMINATION' are listed as in a dictionary, how many items are there in the list before the first word starting with E?* **[NCERT]**

SOLUTION In a dictionary the words at each stage are arranged in alphabetical order. In the given problem we have to find the total number of words starting with A, because the very next word will start with E.

For finding the number of words starting with A, we have to find the number of arrangements of the remaining 10 letters, EXMINATION, of which there are 2 I's, 2 N's and the others each of its own kind.

The number of such arrangements $= \dfrac{10!}{2!\,2!} = 907200.$

Hence, the required number of items = 907200.

EXERCISE 16.5

LEVEL-1

1. Find the number of words formed by permuting all the letters of the following words:
   (i) INDEPENDENCE (ii) INTERMEDIATE (iii) ARRANGE
   (iv) INDIA (v) PAKISTAN (vi) RUSSIA
   (vii) SERIES (viii) EXERCISES (ix) CONSTANTINOPLE
2. In how many ways can the letters of the word 'ALGEBRA' be arranged without changing the relative order of the vowels and consonants?
3. How many words can be formed with the letters of the word 'UNIVERSITY', the vowels remaining together?
4. Find the total number of arrangements of the letters in the expression $a^3 b^2 c^4$ when written at full length.

5. How many words can be formed with the letters of the word 'PARALLEL' so that all L's do not come together?
6. How many words can be formed by arranging the letters of the word 'MUMBAI' so that all M's come together?
7. How many numbers can be formed with the digits 1, 2, 3, 4, 3, 2, 1 so that the odd digits always occupy the odd places?
8. How many different signals can be made from 4 red, 2 white and 3 green flags by arranging all of them vertically on a flagstaff?
9. How many number of four digits can be formed with the digits 1, 3, 3, 0?
10. In how many ways can the letters of the word 'ARRANGE' be arranged so that the two R's are never together?
11. How many different numbers, greater than 50000 can be formed with the digits 0, 1, 1, 5, 9.
12. How many words can be formed from the letters of the word 'SERIES' which start with S and end with S?
13. How many permutations of the letters of the word 'MADHUBANI' do not begin with M but end with I?
14. Find the number of numbers, greater than a million, that can be formed with the digits 2, 3, 0, 3, 4, 2, 3.
15. There are three copies each of 4 different books. In how many ways can they be arranged in a shelf?
16. How many different arrangements can be made by using all the letters in the word 'MATHEMATICS'. How many of them begin with C? How many of them begin with T?
17. A biologist studying the genetic code is interested to know the number of possible arrangements of 12 molecules in a chain. The chain contains 4 different molecules represented by the initials *A* (for Adenine), *C* (for Cytosine), *G* (for Guanine) and *T* (for Thymine) and 3 molecules of each kind. How many different such arrangements are possible?
18. In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same colour are indistinguishable? [NCERT]
19. How many numbers greater than 1000000 can be formed by using the digits 1, 2, 0, 2, 4, 2, 4? [NCERT]
20. In how many ways can the letters of the word ASSASSINATION be arranged so that all the S's are together? [NCERT]
21. Find the total number of permutations of the letters of the word 'INSTITUTE'. [NCERT]

LEVEL-2

22. The letters of the word 'SURITI' are written in all possible orders and these words are written out as in a dictionary. Find the rank of the word 'SURITI'.
23. If the letters of the word 'LATE' be permuted and the words so formed be arranged as in a dictionary, find the rank of the word LATE.
24. If the letters of the word 'MOTHER' are written in all possible orders and these words are written out as in a dictionary, find the rank of the word 'MOTHER'.
25. If the permutations of a, b, c, d, e taken all together be written down in alphabetical order as in dictionary and numbered, find the rank of the permutation debac.
26. Find the total number of ways in which six '+' and four '–' signs can be arranged in a line such that no two '–' signs occur together.

27. In how many ways can the letters of the word "INTERMEDIATE" be arranged so that:
   (i) the vowels always occupy even places?
   (ii) the relative order of vowels and consonants do not alter?

28. The letters of the word 'ZENITH' are written in all possible orders. How many words are possible if all these words are written out as in a dictionary ? What is the rank of the word 'ZENITH'?

ANSWERS

1. (i) 1663200	(ii) 19958400	(iii) 1260	(iv) 60
(v) 20160	(vi) 360	(vii) 180	(viii) 30240
(ix) $\frac{14!}{24}$	2. 72	3. 60480	4. 1260
5. 3000	6. 120	7. 18	8. 1260
9. 9	10. 900	11. 24	12. 12
13. 17640	14. 360	15. $12!/(3!)^4$	16. $\frac{11!}{2!\,2!\,2!}, \frac{10!}{2!\,2!\,2!}, \frac{10!}{2!\,2!}$
17. 369600	18. 1260	19. 360	20. 151200
21. $\frac{9!}{2!\,3!}$	22. 236	23. 14	24. 309
25. 93	26. 35	27. (i) 21600	(ii) 21600
28. 616			

HINTS TO NCERT & SELECTED PROBLEMS

2. The consonants can be arranged among themselves in 4! ways and the vowels among themselves in $\frac{3!}{2!}$ ways. Hence, the required number of arrangements $= 4! \times \frac{3!}{2!} = 72$.

4. There are 3 a's, 2 b's and 4 c's. So, the total number of arrangements $= \frac{9!}{3!\,2!\,4!} = 1260$.

7. There are 4 odd digits 1, 1, 3, 3 and 4 odd places. So, odd digits can be arranged in odd places in $\frac{4!}{2!\,2!}$ ways. The remaining 3 even digits 2, 2, 4 can be arranged in 3 even places in $\frac{3!}{2!}$ ways. Hence, the requisite number of numbers $= \frac{4!}{2!\,2!} \times \frac{3!}{2!} = 18$.

8. We have to arrange 9 flags, out of which 4 are of one kind, 2 are of another kind and 3 are of third kind. So, total number of signals $= \frac{9!}{4!\,2!\,3!}$.

9. Required number of numbers $= \frac{4!}{2!} - \frac{3!}{2!}$.

11. Numbers greater than 50000 will have either 5 or 9 in the first place and will consist of 5 digits.
   Number of numbers of with digit 5 at first place $= \frac{4!}{2!}$
   Number of numbers with digit 9 at first place $= \frac{4!}{2!}$
   Hence, the required number of numbers $= \frac{4!}{2!} + \frac{4!}{2!} = 24$.

18. Required number of ways $= \frac{(4+3+2)!}{4!\,3!\,2!} = \frac{9!}{4!\,3!\,2!} = 1260$

19. Number of numbers greater than 1000000 that can be formed by using the digits 1, 2, 0, 2, 4, 2, 4.

= Number of numbers formed by given digits – Number of numbers having 0 as left most digit

$= \frac{7!}{3!\,2!} - \frac{6!}{3!\,2!} = \frac{7!-6!}{3!\,2!} = \frac{6 \times 6!}{3!\,2!} = 360$

20. Considering all S as one letter there are 10 letters containing 3A's, 2I's, 2N's, 1T, 1O which can be arranged in $\frac{10!}{3!\,2!\,2!} = 151200$ ways.

21. There are 9 letters in the word INSTITUTE containing 2I's, 3T's, 1N, 1S, 1U and 1E. These letters can be arranged in $\frac{9!}{2!\,3!} = 21040$ ways.

26. Six '+' signs can be arranged in a row in $\frac{6!}{6!} = 1$ way. Now, we are left with seven places in which four different things can be arranged in ${}^7P_4$ ways but as all the four '–' signs are identical, therefore, four '–' signs can be arranged in $\frac{{}^7P_4}{4!} = 35$ ways.

Hence, the required number of ways $= 1 \times 35 = 35$.

VERY SHORT ANSWER QUESTIONS (VSAQs)

Answer each of the following questions in one word or one sentence or as per exact requirement of the question:

1. In how many ways can 4 letters be posted in 5 letter boxes?
2. Write the number of 5 digit numbers that can be formed using digits 0, 1 and 2.
3. In how many ways 4 women draw water from 4 taps, if no tap remains unused?
4. Write the total number of possible outcomes in a throw of 3 dice in which at least one of the dice shows an even number.
5. Write the number of arrangements of the letters of the word BANANA in which two N's come together.
6. Write the number of ways in which 7 men and 7 women can sit on a round table such that no two women sit together.
7. Write the number of words that can be formed out of the letters of the word 'COMMITTEE'.
8. Write the number of all possible words that can be formed using the letters of the word 'MATHEMATICS'.
9. Write the number of ways in which 6 men and 5 women can dine at a round table if no two women sit together.
10. Write the number of ways in which 5 boys and 3 girls can be seated in a row so that each girl is between 2 boys.
11. Write the remainder obtained when $1! + 2! + 3! + \ldots + 200!$ is divided by 14.
12. Write the number of numbers that can be formed using all for digits 1, 2, 3, 4.

1. 5^4 2. 2×3^4 3. 41 4. 180 5. 20 6. $7! \times 6!$ 7. $\frac{9!}{(2!)^3}$ 8. $\frac{11!}{2!\,2!\,2!}$

9. $6! \times 5!$ 10. 2880 11. 5 12. 24

MULTIPLE CHOICE QUESTIONS (MCQs)

Mark the correct alternative in each of the following:

1. The number of permutations of n different things taking r at a time when 3 particular things are to be included is
   (a) ${}^{n-3}P_{r-3}$ (b) ${}^{n-3}P_r$ (c) ${}^{n}P_{r-3}$ (d) $r!\,{}^{n-3}C_{r-3}$
2. The number of five-digit telephone numbers having at least one of their digits repeated is
   (a) 90000 (b) 100000 (c) 30240 (d) 69760
3. The number of words that can be formed out of the letters of the word "ARTICLE" so that vowels occupy even places is
   (a) 574 (b) 36 (c) 754 (d) 144
4. How many numbers greater than 10 lacs be formed from 2, 3, 0, 3, 4, 2, 3 ?
   (a) 420 (b) 360 (c) 400 (d) 300
5. The number of different signals which can be given from 6 flags of different colours taking one or more at a time, is
   (a) 1958 (b) 1956 (c) 16 (d) 64
6. The number of words from the letters of the word 'BHARAT' in which B and H will never come together, is
   (a) 360 (b) 240 (c) 120 (d) none of these
7. The number of six letter words that can be formed using the letters of the word "ASSIST" in which S's alternate with other letters is
   (a) 12 (b) 24 (c) 18 (d) none of these
8. The number of arrangements of the word "DELHI" in which E precedes I is
   (a) 30 (b) 60 (c) 120 (d) 59
9. The number of ways in which the letters of the word 'CONSTANT' can be arranged without changing the relative positions of the vowels and consonants is
   (a) 360 (b) 256 (c) 444 (d) none of these
10. The number of ways to arrange the letters of the word CHEESE are
    (a) 120 (b) 240 (c) 720 (d) 6
11. Number of all four digit numbers having different digits formed of the digits 1, 2, 3, 4 and 5 and divisible by 4 is
    (a) 24 (b) 30 (c) 125 (d) 100
12. If the letters of the word KRISNA are arranged in all possible ways and these words are written out as in a dictionary, then the rank of the word KRISNA is
    (a) 324 (b) 341 (c) 359 (d) none of these
13. If in a group of n distinct objects, the number of arrangements of 4 objects is 12 times the number of arrangements of 2 objects, then the number of objects is
    (a) 10 (b) 8 (c) 6 (d) none of these

14. The number of ways in which 6 men can be arranged in a row so that three particular men are consecutive, is

(a) $4! \times 3!$ (b) $4!$ (c) $3! \times 3!$ (d) none of these

15. A 5-digit number divisible by 3 is to be formed using the digits 0, 1, 2, 3, 4 and 5 without repetition. The total number of ways in which this can be done is

(a) 216 (b) 600 (c) 240 (d) 3125

16. The product of r consecutive positive integers is divisible by

(a) $r!$ (b) $r! + 1$ (c) $(r+1)!$ (d) none of these

17. If ${}^{k+5}P_{k+1} = \dfrac{11(k-1)}{2} \cdot {}^{k+3}P_k$, then the values of k are

(a) 7 and 11 (b) 6 and 7 (c) 2 and 11 (d) 2 and 6

18. The number of arrangements of the letters of the word BHARAT taking 3 at a time is

(a) 72 (b) 120 (c) 14 (d) none of these.

19. The number of words that can be made by re-arranging the letters of the word APURBA so that vowels and consonants are alternate is

(a) 18 (b) 35 (c) 36 (d) none of these

20. The number of different ways in which 8 persons can stand in a row so that between two particular persons A and B there are always two persons, is

(a) $60 \times 5!$ (b) $15 \times 4! \times 5!$ (c) $4! \times 5!$ (d) none of these

21. The number of ways in which the letters of the word ARTICLE can be arranged so that even places are always occupied by consonants is

(a) 576 (b) ${}^4C_3 \times 4!$ (c) $2 \times 4!$ (d) none of these

22. In a room there are 12 bulbs of the same wattage, each having a separate switch. The number of ways to light the room with different amounts of illumination is

(a) $12^2 - 1$ (b) 2^{12} (c) $2^{12} - 1$ (d) none of these

ANSWERS

1. (d) 2. (d) 3. (d) 4. (b) 5. (b) 6. (b) 7. (a) 8. (b) 9. (a)
10. (a) 11. (a) 12. (a) 13. (c) 14. (a) 15. (a) 16. (a) 17. (b) 18. (a)
19. (c) 20. (a) 21. (a) 22. (c)

SUMMARY

1. The continued product of first n natural numbers is called the "n factorial" and is denoted by $\lfloor n$ or $n!$.
Thus, $n! = 1 \times 2 \times 3 \times 4 \times \ldots \times (n-1) \times n$
Factorials of proper fractions and negative integers are not defined.

2. $\dfrac{(2n)!}{n!} = 1 \cdot 3 \cdot 5 \ldots (2n-1)\, 2^n$

3. $n! + 1$ is not divisible by any natural number between 2 and n.

4. Let p be a prime number and n be a natural number, if $E_p(n)$ denotes the exponent of p in n, then

$$E_p(n!) = \left[\frac{n}{p}\right] + \left[\frac{n}{p^2}\right] + \ldots + \left[\frac{n}{p^s}\right]$$

where s is the largest positive integer such that $p^s \leq n < p^{s+1}$ and $[x]$ denotes the greatest integer less than or equal to x.

5. If n is a natural number and r is a positive integer such that $0 \leq r \leq n$, then ${}^nP_r = \frac{n!}{(n-r)!}$.

6. (i) *(Fundamental Principle of Multiplication)*: If there are two jobs such that one of them can be completed in m ways, and when it has been completed in any one of these m ways, second job can be completed in n ways; then the two jobs in succession can be completed in $m \times n$ ways.

(ii) *(Fundamental Principle of Addition)* If there are two jobs such that they can be performed independently in m and n ways respectively, then either of the two jobs can be performed in (ways.

7. (i) Let r and n be positive integers such that $1 \leq r \leq n$. Then, the number of all permutations of n distinct items or objects taken r at a time is

$$n(n-1)(n-2)(n-3)\ldots(n-(r-1))$$

(ii) The number of all permutations (arrangements) of n distinct objects taken all at a time is $n!$.

(iii) The number of mutually distinguishable permutations of n things, taken all at a time, of which p are alike of one kind, q alike of second such that $p+q=n$, is $\frac{n!}{p!\,q!}$.

(iv) The number of permutations of n things, of which p_1 are alike of one kind; p_2 are alike of second kind; p_3 are alike of third kind; ...; pr are alike of rth kind such that $p_1 + p_2 + \ldots + p_r = n$, is $\frac{n!}{p_1!\,p_2!\,p_3!\ldots p_r!}$.

(v) The number of permutations of n things, of which p are alike of one kind, q are alike of second kind and remaining all are distinct, is $\frac{n!}{p!\,q!}$.

(vi) Suppose there are r things to be arranged, allowing repetitions. Let further $p_1, p_2, \ldots p_r$ be the integers such that the first object occurs exactly p_1 times, the second occurs exactly p_2 times, etc. Then the total number of permutations of these r objects to the above condition is $\frac{(p_1 + p_2 + \ldots + p_r)!}{p_1!\,p_2!\,p_3!\ldots p_r!}$