EXERCISE–III (A)

If $C_0, C_1, C_2, \ldots\ldots, C_n$ are the combinatorial coefficients in the expansion of $(1+x)^n$, $n \in N$, then prove the following :

Q.1 $C_0^2 + C_1^2 + C_2^2 + \ldots\ldots + C_n^2 = \dfrac{(2n)!}{n!\, n!}$

Q.2 $C_0 C_1 + C_1 C_2 + C_2 C_3 + \ldots. + C_{n-1} C_n = \dfrac{(2n)!}{(n+1)!\,(n-1)!}$

Q.3 $C_1 + 2C_2 + 3C_3 + \ldots\ldots + n \cdot C_n = n \cdot 2^{n-1}$

Q.4 $C_0 + 2C_1 + 3C_2 + \ldots\ldots + (n+1)C_n = (n+2)2^{n-1}$

Q.5 $C_0 + 3C_1 + 5C_2 + \ldots\ldots + (2n+1)C_n = (n+1)\,2^n$

Q.6 $(C_0+C_1)(C_1+C_2)(C_2+C_3)\ \ldots\ldots\ (C_{n-1}+C_n) = \dfrac{C_0 . C_1 . C_2 \ldots . C_{n-1}(n+1)^n}{n!}$

Q.7 $\dfrac{C_1}{C_0} + \dfrac{2C_2}{C_1} + \dfrac{3C_3}{C_2} + \ldots\ldots + \dfrac{n.C_n}{C_{n-1}} = \dfrac{n(n+1)}{2}$

Q 8. $C_0 + \dfrac{C_1}{2} + \dfrac{C_2}{3} + \ldots\ldots + \dfrac{C_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$

Q.9 $2 . C_0 + \dfrac{2^2 . C_1}{2} + \dfrac{2^3 . C_2}{3} + \dfrac{2^4 . C_3}{4} + \ldots\ldots \dfrac{2^{n+1} . C_n}{n+1} = \dfrac{3^{n+1}-1}{n+1}$

Q.10 $C_0C_r + C_1C_{r+1} + C_2C_{r+2} + \ldots . + C_{n-r}\,C_n = \dfrac{2n!}{(n-r)!\,(n+r)!}$

Q.11 $C_0 - \dfrac{C_1}{2} + \dfrac{C_2}{3} - \ldots\ldots + (-1)^n \dfrac{C_n}{n+1} = \dfrac{1}{n+1}$

Q.12 $C_0 - C_1 + C_2 - C_3 + \ldots . + (-1)^r . C_r = \dfrac{(-1)^r\,(n-1)!}{r!\,.\,(n-r-1)!}$

Q.13 $C_0 - 2C_1 + 3C_2 - 4C_3 + \ldots . + (-1)^n\,(n+1)\,C_n = 0$

Q.14 $C_0^2 - C_1^2 + C_2^2 - C_3^2 + \ldots\ldots + (-1)^n\,C_n^2 = 0$ or $(-1)^{n/2}\,C_{n/2}$ according as n is odd or even.

Q.15 If n is an integer greater than 1, show that ;
$a - {}^nC_1(a-1) + {}^nC_2(a-2) - \ldots\ldots + (-1)^n\,(a-n) = 0$

Q.16 $(n-1)^2 . C_1 + (n-3)^2 . C_3 + (n-5)^2 . C_5 + \ldots\ldots = n\,(n+1)2^{n-3}$

Q.17 $1 . C_0^2 + 3 .. C_1^2 + 5 . C_2^2 + \ldots\ldots + (2n+1)\,C_n^2 = \dfrac{(n+1)\,(2n)!}{n!\,n!}$

Q.18 If $a_0, a_1, a_2, \ldots\ldots$ be the coefficients in the expansion of $(1+x+x^2)^n$ in ascending powers of x, then prove that : (i) $a_0\,a_1 - a_1\,a_2 + a_2\,a_3 - \ldots . = 0$.
(ii) $a_0a_2 - a_1a_3 + a_2a_4 - \ldots\ldots + a_{2n-2}\,a_{2n} = a_{n+1}$ or a_{n-1}.
(iii) $E_1 = E_2 = E_3 = 3^{n-1}$; where $E_1 = a_0 + a_3 + a_6 + \ldots\ldots$; $E_2 = a_1 + a_4 + a_7 + \ldots\ldots$ & $E_3 = a_2 + a_5 + a_8 + \ldots\ldots$

Q.19 Prove that : $\displaystyle\sum_{r=0}^{n-2}\left({}^nC_r . {}^nC_{r+2}\right) = \dfrac{(2n)!}{(n-2)!\,(n+2)!}$

Q.20 If $(1+x)^n = C_0 + C_1x + C_2x^2 + \ldots . + C_nx^n$, then show that the sum of the products of the C_i's taken two at a time, represented by $\underset{0 \le i < j \le n}{\Sigma\Sigma}\,C_i\,C_j$ is equal to $2^{2n-1} - \dfrac{2n!}{2(n!)^2}$.

Q.21 $\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} + \ldots\ldots + \sqrt{C_n} \le 2^{n-1} + \dfrac{n-1}{2}$

Q.22 $\sqrt{C_1} + \sqrt{C_2} + \sqrt{C_3} + \ldots\ldots + \sqrt{C_n} \le \left[n\left(2^n-1\right)\right]^{1/2}$ for $n \ge 2$.