Chapter 13

BINOMIAL THEOREM. POSITIVE INTEGRAL INDEX

161. It may be shewn by actual multiplication that

$$(x+a)(x+b)(x+c)(x+d)$$
$$= x^4 + (a+b+c+d)x^3 + (ab+ac+ad+bc+bd+cd)x^2 + (abc+abd+acd+bcd)x + abcd \quad \ldots(1)$$

We may, however, write down this result by inspection; for the complete product consists of the sum of a number of partial products each of which is formed by multiplying together four letters, *one* being taken from *each* of the four factors. If we examine the way in which the various partial products are formed, we see that

(1) the term x^4 is formed by taking the letter x out of each of the factors.

(2) the terms involving x^3 are formed by taking the letter x out of *any three* factors, in every way possible, and *one* of the letters a, b, c, d out of the remaining factor.

(3) the terms involving x^2 are formed by taking the letter x out of *any two* factors, in every way possible, and two of the letters a, b, c, d out of the remaining factors.

(4) the terms involving x are formed by taking the letter x out of *any one* factor, and *three* of the letters a, b, c, d out of the remaining factors.

(5) the term independent of x is the product of all the letters a, b, c, d.

Example 1. $(x-2)(x+3)(x-5)(x+9)$

$$= x^4 + (-2+3-5+9)x^3 + (-6+10-18-15+27-45)x^2 + (30-54+90-135)x + 270$$
$$= x^4 + 5x^3 - 47x^2 - 69x + 270.$$

Example 2. Find the coefficient of x^3 in the product

$$(x-3)(x+5)(x-1)(x+2)(x-8).$$

The terms involving x^3 are formed by multiplying together the x in *any three* of the factors, and *two* of the numerical quantities out of the two remaining factors; hence the coefficient is equal to the sum of the products of the quantities $-3, 5, -1, 2, -8$ taken two at a time.

Thus the required coefficient

$$= -15 + 3 - 6 + 24 - 5 + 10 - 40 - 2 + 8 - 16$$
$$= -39.$$

162. If in equation (1) of the preceding article we suppose $b = c = d = a$, we obtain

$$(x + a)^4 = x^4 + 4ax^3 + 6a^2x^2 + 4a^3x + a^4.$$

The method here exemplified of deducing a particular case from a more general result is one of frequent occurrence in Mathematics; for it often happens that it is more easy to prove a general proposition than it is to prove a particular case of it.

We shall in the next article employ the same method to prove a formula known as the Binomial Theorem, by which any binomial of the form $x + a$ can be raised to any assigned positive integral power.

163. *To find the expansion of $(x + a)^n$ when n is a positive integer.*

Consider the expression

$$(x + a)(x + b)(x + c) \dots (x + k),$$

the number of factors being n.

The expansion of this expression is the continued product of the n factors, $x + a, x + b, x + c, \dots, x + k$, and every term in the expansion is of n dimensions, being a product formed by multiplying together n letters, *one* taken from each of these n factors.

The highest power of x is x^n, and is formed by taking the letter x from *each* of the n factors.

The terms involving x^{n-1} are formed by taking the letter x from any $(n - 1)$ of the factors and *one* of the letters $a, b, c, \dots k$ from the remaining factor; thus the coefficient of x^{n-1} in the final product is the sum of the letters $a, b, c, \dots k$; denote it by S_1.

The terms involving x^{n-2} are formed by taking the letter x from *any* $n - 2$ of the factors, and two of the letters $a, b, c, \dots k$ from the two remaining factors; thus the coefficient of x^{n-2} in the final product is the sum of the products of the letters $a, b, c, \dots k$ taken two at a time; denote it by S_2.

And, generally, the terms involving x^{n-r} are formed by taking the letter x from *any* $n - r$ of the factors, and r of the letters $a, b, c, \dots k$ from the r remaining factors; thus the coefficient of x^{n-r} in the final product is the sum of the products of the letters $a, b, c, \dots k$ taken r at a time; denote it by S_r.

The last term in the product is $abc \dots k$; denote it by S_n.

Hence $(x+a)(x+b)(x+c)\dots(x+k)$

$$= x^n + S_1 x^{n-1} + S_2 x^{n-2} + \dots + S_r x^{n-r} + \dots + S_{n-1} x + S_n.$$

In S_1 the *number of terms* is n; in S_2 *the number of terms* is the same as the number of combinations of n things 2 at a time; that is ${}^nC_2$; in S_3 *the number of terms is* ${}^nC_3$; and so on.

Now suppose $b, c, \dots k$, each equal to a; then S_1 becomes ${}^nC_1\, a$; S_2 becomes ${}^nC_2\, a^2$; S_3 becomes ${}^nC_3\, a^3$; and so on; thus

$$(x+a)^n = x^n + {}^nC_1\, ax^{n-1} + {}^nC_2\, a^2x^{n-2} + {}^nC_3\, a^3x^{n-3} + \dots + {}^nC_n\, a^n;$$

substituting for ${}^nC_1, {}^nC_2, \dots$ we obtain

$$(x+a)^n = x^n + nax^{n-1} + \frac{n(n-1)}{1\cdot 2} a^2x^{n-2} + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} a^3x^{n-3} + \dots + a^n,$$

the series containing $n+1$ terms.

This is the *Binomial Theorem,* and the expression on the right is said to be the expansion of $(x+a)^n$.

164. The Binomial Theorem may also be proved as follows :

By induction we can find the product of the n factors $x+a, x+b, x+c \dots x+k$ as explained in Art. 158, Ex. 2; We can then deduce the expansion of $(x+a)^n$ as in Art. 163.

165. The coefficients in the expansion of $(x+a)^n$ are very conveniently expressed by the symbols ${}^nC_1, {}^nC_2, {}^nC_3, \dots {}^nC_n$. We shall, however, sometimes further abbreviate them by omitting n, and writing $C_1, C_2, C_3, \dots C_n$. With this notation we have

$$(x+a)^n = x^n + C_1\, ax^{n-1} + C_2\, a^2x^{n-2} + C_3\, a^3x^{n-3} + \dots + C_n\, a^n.$$

If we write $-a$ in the place of a, we obtain

$$(x-a)^n = x^n + C_1(-a)\, x^{n-1} + C_2(-a)^2\, x^{n-2} + C_3(-a)^3\, x^{n-3} + \dots + C_n(-a)^n$$

$$= x^n - C_1\, ax^{n-1} + C_2\, a^2x^{n-2} - C_3\, a^3x^{n-3} + \dots + (-1)^n C_n\, a^n.$$

Thus the terms in the expansion of $(x+a)^n$ and $(x-a)^n$ are numerically the same, but in $(x-a)^n$ they are alternately positive and negative, and the last term is positive or negative according as n is even or odd.

Example 1. Find the expansion of $(x+y)^6$.

By the formula,

$$(x+y)^6 = x^6 + {}^6C_1\, x^5y + {}^6C_2\, x^4y^2 + {}^6C_3\, x^3y^3 + {}^6C_4\, x^2y^4 + {}^6C_5\, xy^5 + {}^6C_6\, y^6$$
$$= x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6,$$

on calculating the values of ${}^6C_1, {}^6C_2, {}^6C_3, \ldots$.

Example 2. Find the expansion of $(a-2x)^7$

$$(a-2x)^7 = a^7 - {}^7C_1\, a^6\,(2x) + {}^7C_2\, a^5\,(2x)^2 - {}^7C_3\, a^4\,(2x)^3 + \ldots \text{ to 8 terms.}$$

Now remembering that ${}^nC_r = {}^nC_{n-r}$, after calculating the coefficients up to ${}^7C_3$, the rest may be written down at once; for ${}^7C_4 = {}^7C_3$; ${}^7C_5 = {}^7C_2$, and so on.

Hence $(a-2x)^7 = a^7 - 7a^6\,(2x) + \dfrac{7\cdot6}{1\cdot2}\,a^5\,(2x)^2 - \dfrac{7\cdot6\cdot5}{1\cdot2\cdot3}\,a^4\,(2x)^3 + \ldots$

$$= a^7 - 7a^6\,(2x) + 21a^5\,(2x)^2 - 35a^4\,(2x)^3 + 35a^3\,(2x)^4$$
$$- 21a^2\,(2x)^5 + 7a\,(2x)^6 - (2x)^7$$
$$= a^7 - 14a^6x + 84a^5x^2 - 280a^4x^3 + 560a^3x^4 - 672\,a^2x^5 + 448ax^6 - 128x^7.$$

Example 3. Find the value of

$$(a+\sqrt{a^2-1})^7 + (a-\sqrt{a^2-1})^7.$$

We have here the sum of two expansions whose terms are numerically the same; but in the second expansion the second, fourth sixth, and eighth terms are negative, and therefore destroy the corresponding terms of the first expansion. Hence the value

$$= 2\,\{a^7 + 21a^5\,(a^2-1) + 35a^3\,(a^2-1)^2 + 7a\,(a^2-1)^3\}$$
$$= 2a\,(64a^6 - 112a^4 + 56a^2 - 7).$$

166. In the expansion of $(x+a)^n$, the coefficient of the second term is ${}^nC_1$; of the third term is ${}^nC_2$; of the fourth term is ${}^nC_3$; and so on; the suffix in each term being one less than the number of the term to which it applies; hence ${}^nC_r$ is the coefficient of the $(r+1)^{\text{th}}$ term. This is called the **general term,** because by giving to r different numerical values any of the coefficients may be found from ${}^nC_r$; and by giving to x and a their appropriate indices any assigned term may be obtained.

Thus the $(r+1)^{\text{th}}$ term may be written

$$^nC_r x^{n-r} a^r, \text{ or } \quad \frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} x^{n-r} a^r.$$

In applying this formula to any particular case, it should be observed that *the index of* a *is the same as the suffix of* C, *and that the sum of the indices of* x *and* a is n

Example 1. Find the fifth term of $(a+2x^3)^{17}$.

The required term $= {}^{17}C_4\, a^{13}\, (2x^3)^4$

$$= \frac{17 \cdot 16 \cdot 15 \cdot 14}{1 \cdot 2 \cdot 3 \cdot 4} \times 16a^{13}\, x^{12}$$

$$= 38080a^{13}\, x^{12}.$$

Exmple 2. Find the fourteenth term of $(3-a)^{15}$.

The required term $= {}^{15}C_{13}\, (3)^2\, (-a)^{13}$

$$= {}^{15}C_2 \times (-9a^{13}) \qquad \text{[Art. 145.]}$$

$$= -945a^{13}.$$

167. The simplest form of the binomial theorem is the expansion of $(1+x)^n$. This is obtained from the general formula of Art. 163, by writing 1 in the place of x, and x in the place of a. Thus

$$(1+x)^n = 1 + {}^nC_1\, x + {}^nC_2\, x^2 + \ldots + {}^nC_r\, x^r + \ldots {}^nC_n\, x^n.$$

$$= 1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \ldots x^n;$$

the general term being

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!} x^r.$$

The expansion of a binomial may always be made to depend upon the case in which the first term is unity; thus

$$(x+y)^n = \left\{ x\left(1+\frac{y}{x}\right)\right\}^n$$

$$= x^n (1+z)^n, \text{ where } z = \frac{y}{x}.$$

Example 1. Find the coefficient of x^{16} in the expansion of $(x^2-2x)^{10}$.

We have $\quad (x^2-2x)^{10} = x^{20}\left(1-\frac{2}{x}\right)^{10};$

and, since x^{20} multiplies every term in the expansion of $\left(1-\frac{2}{x}\right)^{10}$, we have in this expansion to seek the coefficient of the term which contains $\frac{1}{x^4}$.

Hence the required coefficient $= {}^{10}C_4\, (-2)^4$

$$=\frac{10\cdot9\cdot8\cdot7}{1\cdot2\cdot3\cdot4}\times16$$

$$=3360.$$

In some cases the following method is simpler.

Example 2. Find the coefficient of x^r in the expansion of $\left(x^2+\frac{1}{x^3}\right)^n$.

Suppose that x^r occurs in the $(p+1)^{\text{th}}$ term.

The $(p+1)^{\text{th}}$ term $= {}^nC_p\,(x^2)^{n-p}\left(\frac{1}{x^3}\right)^p$

$$= {}^nC_p\,x^{2n-5p}.$$

But this term contains x^r, and therefore $2n-5p=r$, or $p=\frac{2n-r}{5}$.

Thus the required coefficient $= {}^nC_p = {}^nC_{\frac{2n-r}{5}}$

$$=\frac{n!}{\left(\frac{1}{5}(2n-r)\right)!\left(\frac{1}{5}(3n+r)\right)!}.$$

Unless $\frac{2n-r}{5}$ is a positive integer there will be no term containing x^r in the expansion.

168. In Art. 163 we deduced the expansion of $(x+a)^n$ from the product of n factors $(x+a)(x+b)\ldots(x+k)$, and the method of proof there given is valuable in consequence of the wide generality of the results obtained. But the following shorter proof of the Binomial Theorem should be noticed.

It will be seen in Chap. 15. that a similar method is used to obtain the general term of the expansion of

$$(a+b+c+\ldots)^n.$$

169. *To prove the Binomial theorem.*

The expansion of $(x+a)^n$ is the product of n factors, each equal to $x+a$, and every term in the expansion is of n dimensions, being a product formed by multiplying together n letters, *one* taken from each of the n factors. Thus each term involving $x^{n-r}a^r$ is obtained by taking a out of *any* r of the factors, and x out of the remaining $n-r$ factors. Therefore the number of terms which involve $x^{n-r}a^r$ must be equal to the number of ways in which r things can be selected out of n; that is, the coefficient of $x^{n-r}a^r$ is ${}^nC_r$, and by giving to r the values 0., 1, 2, 3, …, n in succession we obtain the coefficients of all the terms. Hence

$$(x+a)^n = x^n + {}^nC_1 x^{n-1} a + {}^nC_2 x^{n-2} a^2 + \ldots + {}^nC_r x^{n-r} a^r + \ldots + a^n$$

since ${}^nC_0$ and ${}^nC_n$ are each equal to unity.

EXAMPLES XIII. a.

Expand the following binomials :

1. $(x-3)^5$.
2. $(3x+2y)^4$.
3. $(2x-y)^5$.
4. $(1-3a^2)^6$.
5. $(x^2+x)^5$.
6. $(1-xy)^7$.
7. $\left(2-\frac{3x^2}{2}\right)^4$.
8. $\left(3a-\frac{2}{3}\right)^6$.
9. $\left(1+\frac{x}{2}\right)^7$.
10. $\left(\frac{2}{3}x-\frac{3}{2x}\right)^6$.
11. $\left(\frac{1}{2}+a\right)^8$.
12. $\left(1-\frac{1}{x}\right)^{10}$.

Write down and simplify :

13. The 4^{th} term of $(x-5)^{13}$.
14. The 10^{th} term of $(1-2x)^{12}$.
15. The 12^{th} term of $(2x-1)^{13}$.
16. The 28^{th} term of $(5x+8y)^{30}$.
17. The 4^{th} term of $\left(\frac{a}{3}+9b\right)^{10}$.
18. The 5^{th} term of $\left(2a-\frac{b}{3}\right)^8$.
19. The 7^{th} term of $\left(\frac{4x}{5}-\frac{5}{2x}\right)^9$.
20. The 5^{th} term of $\left(\frac{x^{3/2}}{a^{1/2}}-\frac{y^{5/2}}{b^{3/2}}\right)^8$.

Find the value of

21. $(x+\sqrt{2})^4+(x-\sqrt{2})^4$.
22. $(\sqrt{x^2-a^2}+x)^5-(\sqrt{x^2-a^2}-x)^5$.
23. $(\sqrt{2}+1)^6-(\sqrt{2}-1)^6$.
24. $(2-\sqrt{1-x})^6+(2+\sqrt{1-x})^6$.
25. Find the middle term of $\left(\frac{a}{x}+\frac{x}{a}\right)^{10}$.
26. Find the middle term of $\left(1-\frac{x^2}{2}\right)^{14}$.
27. Find the coefficient of x^{18} in $\left(x^2+\frac{3a}{x}\right)^{15}$.
28. Find the coefficient of x^{18} in $(ax^4-bx)^9$.
29. Find the coefficients of x^{32} and x^{-17} in $\left(x^4-\frac{1}{x^3}\right)^{15}$.
30. Find the two middle terms of $\left(3a-\frac{a^3}{6}\right)^9$.

31. Find the term independent of x in $\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$.

32. Find the 13^{th} term of $\left(9x - \frac{1}{3\sqrt{x}}\right)^{18}$

33. If x^r occurs in the expansion of $\left(x + \frac{1}{x}\right)^n$, find its coefficient.

34. Find the term independent of x in $\left(x - \frac{1}{x^2}\right)^{3n}$

35. If x^p occurs in the expansion of $\left(x^2 + \frac{1}{x}\right)^{2n}$, prove that its coefficient is $\dfrac{2n\,!}{\left(\frac{1}{3}(4n-p)\right)!\left(\frac{1}{3}(2n+p)\right)!}$

170. *In the expansion of $(1+x)^n$ the coefficients of terms equidistant from the beginning and end are equal.*

The coefficient of the $(r+1)^{th}$ term from the beginning is ${}^nC_r$.

The $(r+1)^{th}$ term from the end has $n+1-(r+1)$, or $n-r$ terms before it; therefore counting from the beginning it is the $(n-r+1)^{th}$ term, and its coefficient is ${}^nC_{n-r}$, which has been shewn to be equal to ${}^nC_r$.[Art. 145.] Hence the proposition follows.

171. *To find the greatest coefficient in the expansion of $(1+x)^n$.*

The coefficient of the general term of $(1+x)^n$ is ${}^nC_r$; and we have only to find for what value of r this is greatest.

By Art. 154, when n is even, the greatest coefficient is ${}^nC_{\frac{n}{2}}$; and when n is odd, it is ${}^nC_{\frac{n-1}{2}}$, or ${}^nC_{\frac{n+1}{2}}$; these two coefficients being equal.

172. *To find the greatest term in the expansion of $(x+a)^n$.*

We have $(x+a)^n = x^n\left(1+\frac{a}{x}\right)^n$;

therefore, since x^n multiplies every term in $\left(1+\frac{a}{x}\right)^n$, it will be sufficient to find the greatest term in this latter expansion.

Let the r^{th} and $(r+1)^{th}$ be any two consecutive terms. The $(r+1)^{th}$ term is obtained by multiplying the r^{th} term by $\frac{n-r+1}{r} \cdot \frac{a}{x}$; that is, by $\left(\frac{n+1}{r}-1\right)\frac{a}{x}$. [Art. 166.]

The factor $\frac{n+1}{r}-1$ decreases as r increases; hence the $(r+1)^{th}$ term is not always greater than the r^{th} term, but only until $\left(\frac{n+1}{r}-1\right)\frac{a}{x}$ becomes equal to 1, or less than 1.

Now,
$$\left(\frac{n+1}{r}-1\right)\frac{a}{x}>1,$$
so long as
$$\frac{n+1}{r}-1>\frac{x}{a};$$
that is,
$$\frac{n+1}{r}>\frac{x}{a}+1,$$
or
$$\frac{n+1}{\frac{x}{a}+1}>r \qquad \ldots(1)$$

If $\frac{n+1}{\frac{x}{a}+1}$ be an integer, denote it by p; then if $r=p$ the multiplying factor becomes 1, and the $(p+1)^{th}$ term is equal to the p^{th}; and these are greater than any other term.

If $\frac{n+1}{\frac{x}{a}+1}$ be not an integer, denote its integral part by q; then the greatest value of r consistent with (1) is q; hence the $(q+1)^{th}$ term is the greatest.

Since we are only concerned with the *numerically greatest term,* the investigation will be the same for $(x-a)^n$; therefore in any numerical example it is unnecessary to consider the sign of the second term of the binomial. Also it will be found best to work each example independently of the general formula.

Example 1. If $x=\frac{1}{3}$, find the greatest term in the expansion of $(1+4x)^8$.

Denote the r^{th} and $(r+1)^{th}$ terms by T_r and T_{r+1} respectively; then

$$T_{r+1}=\frac{8-r+1}{r}\cdot 4x\times T_r$$

$$= \frac{9-r}{r} \times \frac{4}{3} \times T_r;$$

hence $T_{r+1} > T_r$,

so long as $\frac{9-r}{r} \times \frac{4}{3} > 1;$

that is $36 - 4r > 3r$,

or $36 > 7r$.

The greatest value of r consistent with this is 5; hence the greatest term is the sixth, and its value

$$= {}^8C_5 \times \left(\frac{4}{3}\right)^5 = {}^8C_3 \times \left(\frac{4}{3}\right)^5 = \frac{57344}{243}.$$

Example 2. Find the greatest term in the expansion of $(3-2x)^9$ when $x = 1$.

$$(3-2x)^9 = 3^9\left(1 - \frac{2x}{3}\right)^9;$$

thus it will be sufficient to consider the expansion of

$$\left(1 - \frac{2x}{3}\right)^9.$$

Here, $T_{r+1} = \frac{9-r+1}{r} \cdot \frac{2x}{3} \times T_r$, *numerically*,

$$= \frac{10-r}{r} \times \frac{2}{3} \times T_r;$$

hence $T_{r+1} > T_r$,

so long as $\frac{10-r}{r} \times \frac{2}{3} > 1;$

that is, $20 > 5r$.

Hence for all values of r up to 3, we have $T_{r+1} > T_r$; but if $r = 4$, then $T_{r+1} = T_r$, and these are the greatest terms. Thus the 4^{th} and 5^{th} terms are numerically equal and greater than any other term, and their value

$$= 3^9 \times {}^9C_3 \times \left(\frac{2}{3}\right)^3 = 3^6 \times 84 \times 8 = 489888.$$

173. *To find the sum of the coefficients in the expansion of* $(1+x)^n$.

In the identity $(1+x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + \ldots + C_nx^n$,

Put $x = 1$; thus

$$2^n = 1 + C_1 + C_2 + C_3 + \ldots + C_n$$

= sum of the coefficients.

Cor. $C_1 + C_2 + C_3 + \ldots + C_n = 2^n - 1;$

that is "total number of combinations of n things" is $2^n - 1$. [Art. 153.]

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174. *To prove that in the expansion of $(1+x)^n$, the sum of the coefficients of the odd terms is equal to the sum of the coefficients of the even terms.*

In the identity $(1+x)^n = 1 + C_1x + C_2x^2 + C_3x^3 + \dots + C_nx^n$,

put $x = -1$; thus

$$0 = 1 - C_1 + C_2 - C_3 + C_4 - C_5 + \dots;$$

$$\therefore \quad 1 + C_2 + C_4 + \dots = C_1 + C_3 + C_5 + \dots$$

$$= \frac{1}{2}\text{ (sum of all the coefficients)}$$

$$= 2^{n-1}.$$

175. The Binomial Theorem may also be applied to expand expressions which contain more than two terms.

Example. Find the expansion of $(x^2 + 2x - 1)^3$.

Regarding $2x - 1$ as a single term, the expansion

$$= (x^2)^3 + 3(x^2)^2(2x-1) + 3x^2(2x-1)^2 + (2x-1)^3$$

$$= x^6 + 6x^5 + 9x^4 - 4x^3 - 9x^2 + 6x - 1, \text{ on reduction.}$$

176. The following example is instructive.

Example. If $(1+x)^n = c_0 + c_1x + c_2x^2 + \dots + c_nx^n$,

find the value of $\quad c_0 + 2c_1 + 3c_2 + 4c_3 + \dots (n+1)\,c_n \quad \dots(1),$

and $\quad c_1^2 + 2c_2^2 + 3c_3^2 + \dots + nc_n^2 \quad \dots(2).$

The series (1) $= (c_0 + c_1 + c_2 + \dots + c_n) + (c_1 + 2c_2 + 3c_3 + \dots + nc_n)$

$$= 2^n + n\left\{1 + (n-1) + \frac{(n-1)(n-2)}{1\cdot 2} + \dots + 1\right\}$$

$$= 2^n + n(1+1)^{n-1}$$

$$= 2^n + n\cdot 2^{n-1}$$

To find the value of the series (2), we proceed thus :

$$c_1x + 2c_2x^2 + 3c_3x^3 + \dots + nc_nx^n$$

$$= nx\left\{1 + (n-1)x + \frac{(n-1)(n-2)}{1\cdot 2}x^2 + \dots + x^{n-1}\right\}$$

$$= nx(1+x)^{n-1};$$

hence, by changing x into $\frac{1}{x}$, we have

$$\frac{c_1}{x} + \frac{2c_2}{x^2} + \frac{3c_3}{x^3} + \dots + \frac{nc_n}{x^n} = \frac{n}{x}\left(1 + \frac{1}{x}\right)^{n-1} \quad \dots(3),$$

Also

$$c_0 + c_1x + c_2x^2 + \dots + c_nx^n = (1+x)^n \quad \dots(4).$$

If we multiply together the two series on the left-hand sides of (3) and (4), we see that in the product the term independent of x is the series (2); hence the series (2) = term independent of x in

$$\frac{n}{x}(1+x)^n\left(1+\frac{1}{x}\right)^{n-1}$$

$$= \text{term independent of } x \text{ in } \frac{n}{x^n}(1+x)^{2n-1}$$

$$= \text{coefficient of } x^n \text{ in } n\,(1+x)^{2n-1}$$

$$= n \times {}^{2n-1}C_n$$

$$= \frac{(2n-1)\,!}{(n-1)\,!\,(n-1)\,!}.$$

EXAMPLES XIII. b.

In the following expansions find which is the greatest term :

1. $(x-y)^{30}$ when $x = 11$, $y = 4$.
2. $(2x-3y)^{28}$ when $x = 9$, $y = 4$.
3. $(2a+b)^{14}$ when $a = 4$, $b = 5$.
4. $(3+2x)^{15}$ when $x = \frac{5}{2}$.

In the following expansions find the value of the greatest term :

5. $(1+x)^n$ when $x = \frac{2}{3}$, $n = 6$.
6. $(a+x)^n$ when $a = \frac{1}{2}$, $x = \frac{1}{3}$, $n = 9$.
7. Shew that the coefficient of the middle term of $(1+x)^{2n}$ is equal to the sum of the coefficients of the middle terms of $(1+x)^{2n-1}$.
8. If A be the sum of the odd terms and B the sum of the even terms in the expansion of $(x+a)^n$, prove that $A^2 - B^2 = (x^2 - a^2)^n$.
9. The 2$^{\text{nd}}$, 3$^{\text{rd}}$, 4$^{\text{th}}$ terms in the expansion of $(x+y)^n$ are 240, 720, 1080 respectively; find x, y, n.
10. Find the expansion of $(1+2x-x^2)^4$.
11. Find the expansion of $(3x^2-2ax+3a^2)^3$.
12. Find the r^{th} term from the end in $(x+a)^n$.
13. Find the $(p+2)^{\text{th}}$ term from the end in $\left(x-\frac{1}{x}\right)^{2n+1}$
14. In the expansion of $(1+x)^{43}$ the coefficients of the $(2r+1)^{\text{th}}$ and the $(r+2)^{\text{th}}$ terms are equal; find r.

15. Find the relation between r and n in order that the coefficients of the $3r^{th}$ and $(r+2)^{th}$ terms of $(1+x)^{2n}$ may be equal.

16. Shew that the middle term in the expansion of $(1+x)^{2n}$ is
$$\frac{1 \cdot 3 \cdot 5 \ldots (2n-1)}{n\,!} 2^n x^n.$$

If $c_0, c_1, c_2, \ldots c_n$ denote the coefficients in the expansion of $(1+x)^n$, prove that

17. $c_1 + 2c_2 + 3c_3 + \ldots + nc_n = n.\ 2^{n-1}$.

18. $c_0 + \dfrac{c_1}{2} + \dfrac{c_2}{3} + \ldots + \dfrac{c_n}{n+1} = \dfrac{2^{n+1}-1}{n+1}$.

19. $\dfrac{c_1}{c_0} + \dfrac{2c_2}{c_1} + \dfrac{3c_3}{c_2} + \ldots + \dfrac{{}^n c_n}{c_{n-1}} = \dfrac{n\,(n+1)}{2}$

20. $(c_0 + c_1)(c_1 + c_2) \ldots (c_{n-1} + c_n) = \dfrac{c_1 c_2 \ldots c_n (n+1)^n}{n\,!}$.

21. $2c_0 + \dfrac{2^2 c_1}{2} + \dfrac{2^3 c_2}{3} + \dfrac{2^4 c_3}{4} + \ldots + \dfrac{2^{n+1} c_n}{n+1} = \dfrac{3^{n+1}-1}{n+1}$.

22. $c_0^2 + c_1^2 + c_2^2 + \ldots + c_n^2 = \dfrac{2n\,!}{n\,!\,n\,!}$.

23. $c_0 c_r + c_1 c_{r+1} + c_2 c_{r+2} + \ldots + c_{n-r} c_n = \dfrac{2n\,!}{(n-r)\,!\,(n+r)\,!}$.

■■■

Chapter 14

BINOMIAL THEOREM. ANY INDEX.

177. In the last chapter we investigated the Bionomial Theorem when the index was any positive integer; we shall now consider whether the formulae there obtained hold in the case of negative and fractional values of the index.

Since, by Art. 167, every binomial may be reduced to one common type, it will be sufficient to confine our attention to binomials of the form $(1+x)^n$.

By actual evolution, we have

$$(1+x)^{1/2} = \sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \ldots;$$

and by actual division,

$$(1-x)^{-2} = \frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \ldots;$$

[Compare Ex. 1, Art. 60.]

and in each of these series the number of terms is unlimited.

In these cases we have by independent processes obtained an expansion for each of the expressions $(1+x)^{1/2}$ and $(1+x)^{-2}$. We shall presently prove that they are only particular cases of the general formula for the expansion of $(1+x)^n$, where n is any rational quantity.

This formula was discovered by Newton.

178. Suppose we have two expressions arranged in ascending powers of x, such as

$$1 + mx + \frac{m(m-1)}{1 \cdot 2}x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3}x^3 + \ldots \qquad \ldots(1)$$

and $$1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}x^3 + \ldots \qquad \ldots(2)$$

The product of these two expressions will be a series in ascending powers of x; denote it by

$$1 + Ax + Bx^2 + Cx^3 + Dx^4 + \ldots;$$

then it is clear that $A, B, C, \ldots$ are functions of m and n, and therefore the actual values of $A, B, C, \ldots$ in any particular case will depend upon the values of m and n in that case. But the way in which the coefficients of the powers of x in (1) and (2) combine to give $A, B, C, \ldots$ is quite independent of m and n; in other words, *whatever values m and n may have, $A, B, C, \ldots$ preserve the same invariable form*. If therefore we can determine the form of $A, B, C, \ldots$ for any value of m and n, we conclude that $A, B, C, \ldots$ will have the same form *for all values of m and n.*

The principle here explained is often referred to as an example of "the permanence of equivalent forms;" in the present case we have only to recognise the fact that in *any algebraical product* the *form* of the result will be the same whether the quantities involved are whole numbers, or fractions; positive, or negative.

We shall make use of this principle in the general proof of the Binomial Theorem for any index. The proof which we give is due to Euler.

179. *To prove the Binomial Theorem when the index is a positive fraction.*

Whatever be the value of m, *positive or negative, integral or fractional,* let the symbol $f(m)$ stand for the series

$$1 + mx + \frac{m(m-1)}{1 \cdot 2} x^2 + \frac{m(m-1)(m-2)}{1 \cdot 2 \cdot 3} x^3 + \ldots;$$

then $f(n)$ will stand for the series

$$1 + nx + \frac{n(n-1)}{1 \cdot 2} x^2 + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^3 + \ldots$$

If we multiply these two series together the product will be another series in ascending powers of x, whose *coefficients will be unaltered in form whatever* m *and* n *may be.*

To determine this *invariable form of the product* we may give to m and n any values that are most convenient; for this purpose suppose that m and n are positive integers. In this case $f(m)$ is the expanded form of $(1+x)^m$, and $f(n)$ is the expanded form of $(1+x)^n$; and therefore

$$f(m) \times f(n) = (1+x)^m \times (1+x)^n = (1+x)^{m+n},$$

but when m and n are positive integers the expansion of $(1+x)^{m+n}$ is

$$1 + (m+n)x + \frac{(m+n)(m+n-1)}{1 \cdot 2} x^2 + \ldots$$

This then is the *form* of the product of $f(m) \times f(n)$ *in all cases,* whatever the values of m and n may be; and in agreement with our previous notation it may be denoted by $f(m+n)$; therefore for all values of m and n.

$$f(m) \times f(n) = f(m+n).$$

Also
$$f(m) \times f(n) \times f(p) = f(m+n) \times f(p)$$
$$= f(m+n+p) \text{ similarly.}$$

Proceeding in this way we may shew that

$$f(m) \times f(n) \times f(p) \ldots \text{ to } k \text{ factors} = f(m+n+p+\ldots \text{ to } k \text{ terms}).$$

Let each of these quantities $m, n, p, \ldots$ be equal to $\frac{h}{k}$, where h and k are positive integers;

$$\therefore \qquad \left\{f\left(\frac{h}{k}\right)\right\}^k = f(h);$$

but since h is a positive integer, $f(h) = (1+x)^h$;

$$\therefore \qquad (1+x)^h = \left\{f\left(\frac{h}{k}\right)\right\}^h;$$

$$\therefore \qquad (1+x)^{h/k} = f\left(\frac{h}{k}\right);$$

but $f\left(\frac{h}{k}\right)$ stands for the series

$$1 + \frac{h}{k}x + \frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1 \cdot 2}x^2 + \ldots;$$

$$\therefore \qquad (1+x)^{h/k} = 1 + \frac{h}{k}x + \frac{\frac{h}{k}\left(\frac{h}{k}-1\right)}{1 \cdot 2}x^2 + \ldots,$$

which proves the Binomial Theorem for any positive fractional index.

180. *To prove the Binomial Theorem when the index is any negative quantity.*

It has been proved that

$$f(m) \times f(n) = f(m+n)$$

for all values of m and n. Replacing m by $-n$ (where n is positive), we have

$$f(-n) \times f(n) = f(-n+n)$$
$$= f(0)$$
$$= 1.$$

Since all terms of the series except the first vanish;

$$\therefore \qquad \frac{1}{f(n)} = f(-n);$$

but $f(n) = (1+x)^n$, for *any positive* value of n;

$$\frac{1}{(1+x)^n} = f(-n),$$

$\therefore$

$$(1+x)^{-n} = f(-n).$$

or

But $f(-n)$ stands for the series

$$1 + (-n)\,x + \frac{(-n)(-n-1)}{1 \cdot 2}x^2 + \ldots;$$

$$\therefore \quad (1+x)^{-n} = 1 + (-n)\,x + \frac{(-n)(-n-1)}{1 \cdot 2}x^2 + \ldots;$$

which proves the Binomial Theorem for any negative index. Hence the theorem is completely established.

181. The proof contained in the two preceding articles may not appear wholly satisfactory, and will probably present some difficulties to the student. There is only one point to which we shall now refer.

In the expression for $f(m)$ the number of terms is finite when m is a positive integer, and unlimited in all other cases. See Art. 182. It is therefore necessary to enquire in what sense we are to regard the statement that $f(m) \times f(n) = f(m+n)$. It will be seen in Chapter 21., that when $x < 1$, each of the series $f(m)$, $f(n)$, $f(m+n)$ is *convergent*, and $f(m+n)$ is the true arithmetical equivalent of $f(m) \times f(n)$. But when $x > 1$, all these series are divergent, and we only assert that if we multiply the series denoted by $f(m)$ by the series denoted by $f(n)$, the first r terms of the product will agree with the first r terms of $f(m+n)$, whatever finite value r may have. [See Art. 308.]

Example 1. Expand $(1-x)^{3/2}$ to four terms.

$$(1-x)^{\frac{3}{2}} = 1 + \frac{3}{2}(-x) + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)}{1 \cdot 2}(-x)^2 + \frac{\frac{3}{2}\left(\frac{3}{2}-1\right)\left(\frac{3}{2}-2\right)}{1 \cdot 2 \cdot 3}(-x)^3 + \ldots$$

$$= 1 - \frac{3}{2}x + \frac{3}{8}x^2 + \frac{1}{16}x^3 + \ldots$$

Example 2. Expand $(2+3x)^{-4}$ to four terms.

$$(2+3x)^{-4} = 2^{-4}\left(1 + \frac{3x}{2}\right)^{-4}$$

$$= \frac{1}{2^4}\left[1 + (-4)\left(\frac{3x}{2}\right) + \frac{(-4)(-5)}{1 \cdot 2}\left(\frac{3x}{2}\right)^2 + \frac{(-4)(-5)(-6)}{1 \cdot 2 \cdot 3}\left(\frac{3x}{2}\right)^3 + \ldots\right]$$

$$= \frac{1}{16}\left(1 - 6x + \frac{45}{2}x^2 - \frac{135}{2}x^3 + \ldots\right)$$

182. In finding the general term we must now use the formula.

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

written in full; for the symbol ${}^nC_r$ can no longer be employed when n is fractional or negative.

Also the coefficient of the general term can never vanish unless one of the factors of its numerator is zero; the series will therefore stop at the r^{th} term, when $n-r+1$ is zero; that is, when $r=n+1$; but since r is a positive integer this equality can never hold except when the index n is positive and integral. Thus the expansion by the Binomial Theorem extends to $n+1$ terms when n is a positive integer, and to an infinite number of terms in all other cases.

Example 1. Find the general term in the expansion of $(1+x)^{\frac{1}{2}}$.

$$\text{The } (r+1)^{\text{th}} \text{ term} = \frac{\frac{1}{2}\left(\frac{1}{2}-1\right)\left(\frac{1}{2}-2\right)\ldots\left(\frac{1}{2}-r+1\right)}{r!}x^r$$

$$= \frac{1(-1)(-3)(-5)\ldots(-2r+3)}{2^r\, r!}x^r.$$

The number of factors in the numerator is r, and $r-1$ of these are negative; therefore, by taking -1 out of each of these negative factors, we may write the above expression

$$(-1)^{r-1}\frac{1\cdot3\cdot5\ldots(2r-3)}{2^r\, r!}x^r.$$

Example 2. Find the general term in the expansion of $(1-nx)^{1/n}$.

$$\text{The } (r+1)^{\text{th}} \text{ term} = \frac{\frac{1}{n}\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-2\right)\ldots\left(\frac{1}{n}-r+1\right)}{r!}(-nx)^r$$

$$= \frac{1(1-n)(1-2n)\ldots(1-\overline{r-1}\cdot n)}{n^r\, r!}(-1)^r n^r x^r$$

$$= (-1)^r\frac{1(1-n)(1-2n)\ldots(1-\overline{r-1}\cdot n)}{r!}x^r$$

$$= (-1)^r(-1)^{r-1}\frac{(n-1)(2n-1)\ldots(\overline{r-1}\,.\,n-1)}{r!}x^r$$

$$= -\frac{(n-1)(2n-1)\ldots(\overline{r-1}\cdot n-1)}{r!}x^r,$$

since $\qquad (-1)^r(-1)^{r-1} = (-1)^{2r-1} = -1.$

Example 3. Find the general term in the expansion of $(1-x)^{-3}$.

$$\text{The } (r+1)^{\text{th}} \text{ term} = \frac{(-3)(-4)(-5)\ldots(-3-r+1)}{r!}(-x)^r$$

$$= (-1)^r \frac{3 \cdot 4 \cdot 5 \ldots (r+2)}{r!} (-1)^r x^r$$

$$= (-1)^{2r} \frac{3 \cdot 4 \cdot 5 \ldots (r+2)}{1 \cdot 2 \cdot 3 \ldots r} x^r$$

$$= \frac{(r+1)(r+2)}{1 \cdot 2} x^r,$$

by removing like factors from the numerator and denominator.

EXAMPLES XIV. a.

Expand to 4 terms the following expressions :

1. $(1+x)^{\frac{1}{2}}$ 2. $(1+x)^{\frac{3}{2}}$ 3. $(1-x)^{\frac{2}{5}}$

4. $(1+x^2)^{-2}$ 5. $(1-3x)^{\frac{1}{3}}$ 6. $(1-3x)^{-\frac{1}{3}}$

7. $(1+2x)^{-\frac{1}{2}}$ 8. $\left(1+\frac{x}{3}\right)^{-3}$ 9. $\left(1+\frac{2x}{3}\right)^{\frac{3}{2}}$

10. $\left(1+\frac{1}{2}a\right)^{-4}$ 11. $(2+x)^{-3}$ 12. $(9+2x)^{\frac{1}{2}}$

13. $(8+12a)^{\frac{2}{3}}$ 14. $(9-6x)^{-\frac{3}{2}}$ 15. $(4a-8x)^{-\frac{1}{2}}$

Write down and simplify :

16. The 8^{th} term of $(1+2x)^{-\frac{1}{2}}$.
17. The 11^{th} term of $(1-2x^3)^{\frac{11}{2}}$.
18. The 10^{th} term of $(1+3a^2)^{\frac{16}{3}}$.
19. The 5^{th} term of $(3a-2b)^{-1}$.
20. The $(r+1)^{\text{th}}$ term of $(1-x)^{-2}$.
21. The $(r+1)^{\text{th}}$ term of $(1-x)^{-4}$.
22. The $(r+1)^{\text{th}}$ term of $(1+x)^{\frac{1}{2}}$.
23. The $(r+1)^{\text{th}}$ term of $(1+x)^{\frac{11}{3}}$.
24. The 14^{th} term of $(2^{10}-2^7x)^{13/2}$.
25. The 7^{th} term of $(3^8+6^4x)^{\frac{11}{4}}$.

183. If we expand $(1-x)^{-2}$ by the Binomial Theorem, we obtain

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots;$$

but, by referring to Art. 60, we see that this result is only true when x is less than 1. This leads us to enquire whether we are always justified in assuming the truth of the statement

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \ldots,$$

and, if not, under what conditions the expansion of $(1+x)^n$ may be used as its true equivalent.

Suppose, for instance, that $n = -1$; then we have

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + x^4 + \ldots \qquad \ldots(1)$$

in this equation put $x = 2$; we then obtain

$$(-1)^{-1} = 1 + 2 + 2^2 + 2^3 + 2^4 + \ldots$$

This contradictory result is sufficient to shew that we cannot take

$$1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \ldots$$

as the true arithmetical equivalent of $(1+x)^n$ in all cases.

Now from the formula for the sum of a geometrical progression, we know that the sum of the first r terms of the series (1)

$$= \frac{1-x^r}{1-x} = \frac{1}{1-x} - \frac{x^r}{1-x};$$

and, when x is numerically less than 1, by taking r sufficiently large we can make $\frac{x^r}{1-x}$ as small as we please; that is, by taking a sufficient number of terms the sum can be made to differ as little as we please from $\frac{1}{1-x}$. But when x is numerically greater than 1, the value of $\frac{x^r}{1-x}$ increases with r, and therefore no such approximation to the value of $\frac{1}{1-x}$ is obtained by taking any number of terms of the series

$$1 + x + x^2 + x^3 + \ldots$$

It will be seen in the chapter on Convergency and Divergency of Series that the expansion by the Binomial Theorem of $(1+x)^n$ in ascending powers of x is always arithmetically intelligible when x is less than 1.

But if x is greater than 1, then since the general term of the series

$$1 + nx + \frac{n(n-1)}{1 \cdot 2}x^2 + \ldots$$

contains x^r, it can be made greater than any finite quantity by taking r sufficiently large; in which case there is no limit to the value of the above series; and therefore the expansion of $(1+x)^n$ as an infinite series in ascending powers of x has no meaning arithmetically intelligible when x is greater than 1.

184. We may remark that we can always expand $(x+y)^n$ by the Binomial Theorem; for we may write the expression in either of the two following forms :

$$x^n\left(1+\frac{y}{x}\right)^n,\ y^n\left(1+\frac{x}{y}\right)^n;$$

and we obtain the expansion from the first or second of these according as x is greater or less than y.

185. *To find in its simplest form the general term in the expansion of* $(1-x)^{-n}$.

The $(r+1)^{\text{th}}$ term

$$=\frac{(-n)(-n-1)(-n-2)\ldots(-n-r+1)}{r!}(-x)^r$$

$$=(-1)^r\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}(-1)^r x^r$$

$$=(-1)^{2r}\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r$$

$$=\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}x^r.$$

From this it appears that every term in the expansion of $(1-x)^{-n}$ is positive.

Although the general term in the expansion of any binomial may always be found as explained in Art. 182, it will be found more expeditious in practice to use the above form of the general term in all cases where the index is negative, retaining the form

$$\frac{n(n-1)(n-2)\ldots(n-r+1)}{r!}x^r$$

only in the case of positive indices.

Example **:** Find the general term in the expansion of $\frac{1}{\sqrt[3]{1-3x}}$.

$$\frac{1}{\sqrt[3]{1-3x}}=(1-3x)^{-\frac{1}{3}}.$$

The $(r+1)^{\text{th}}$ term

$$= \frac{\frac{1}{3}\left(\frac{1}{3}+1\right)\left(\frac{1}{3}+2\right)\ldots\left(\frac{1}{3}+r-1\right)}{r!}(3x)^r$$

$$= \frac{1\cdot4\cdot7\ldots(3r-2)}{3^r\, r!}3^r x^r$$

$$= \frac{1\cdot4\cdot7\ldots(3r-2)}{r!}x^r.$$

If the given expression had been $(1+3x)^{-\frac{1}{3}}$ we should have used the same formula for the general term, replacing $3x$ by $-3x$.

186. The following expansions should be remembered :

$$(1-x)^{-1} = 1 + x + x^2 + x^3 + \ldots + x^r + \ldots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \ldots + (r+1)x^r + \ldots$$

$$(1-x)^{-3} = 1 + 3x + 6x^2 + 10x^3 + \ldots + \frac{(r+1)(r+2)}{1\cdot2}x^r + \ldots$$

187. The general investigation of the greatest term in the expansion of $(1+x)^n$, when n is unrestricted in value, will be found in Art. 189; but the student will have no difficulty in applying to any numerical example the method explained in Art. 172.

Example : Find the greatest term in the expansion of $(1+x)^{-n}$ when $x = \frac{2}{3}$, and $n = 20$.

We have $\qquad T_{r+1} = \frac{n+r-1}{r}\cdot x \times T_r$, *numerically,*

$$= \frac{19+r}{r} \times \frac{2}{3} \times T_r;$$

$\therefore \qquad T_{r+1} > T_r,$

So long as $\qquad \frac{2(19+r)}{3r} > 1;$

that is, $\qquad 38 > r.$

Hence for all values of r up to 37, we have $T_{r+1} > T_r$; but if $r = 38$ then $T_{r+1} = T_r$, and these are the greatest terms. Thus the 38^{th} and 39^{th} terms are equal numerically and greater than any other term.

188. Some useful applications of the Binomial Theorem are explained in the following examples.

Example 1. Find the first three terms in the expansion of

$$(1+3x)^{\frac{1}{2}}(1-2x)^{-\frac{1}{3}}.$$

Expanding the two binomials as far as the term containing x^2, we have

$$\left(1+\frac{3}{2}x-\frac{9}{8}x^2-\dots\right)\left(1+\frac{2}{3}x+\frac{8}{9}x^2+\dots\right)$$
$$=1+x\left(\frac{3}{2}+\frac{2}{3}\right)+x^2\left(\frac{8}{9}+\frac{3}{2}\cdot\frac{2}{3}-\frac{9}{8}\right)\dots$$
$$=1+\frac{13}{6}x+\frac{55}{72}x^2.$$

If in this Example $x = .002$, so that $x^2 = .000004$, we see that the third term is a decimal fraction beginning with 5 ciphers. If therefore we were required to find the numerical value of the given expression correct to 5 places of decimals it would be sufficient to substitute .002 for x in $1+\frac{13}{6}x$, neglecting the term involving x^2.

Example 2. When x is so small that its square and higher powers may be neglected, find the value of

$$\frac{\left(1+\frac{2}{3}x\right)^{-5}+\sqrt{4+2x}}{\sqrt{(4+x)^3}}$$

Since x^2 and the higher powers may be neglected, it will be sufficient to retain the first two terms in the expansion of each binomial.

Therefore the expression $$=\frac{\left(1+\frac{2}{3}x\right)^{-5}+2\left(1+\frac{x}{2}\right)^{\frac{1}{2}}}{8\left(1+\frac{x}{4}\right)^{\frac{3}{2}}}$$

$$=\frac{\left(1-\frac{10}{3}x\right)+2\left(1+\frac{1}{4}x\right)}{8\left(1+\frac{3}{8}x\right)}$$
$$=\frac{1}{8}\left(3-\frac{17}{6}x\right)\left(1+\frac{3}{8}x\right)^{-1}$$
$$=\frac{1}{8}\left(3-\frac{17}{6}x\right)\left(1-\frac{3}{8}x\right)$$
$$=\frac{1}{8}\left(3-\frac{95}{24}x\right),$$

the term involving x^2 being neglected.

Example 3. Find the value of $\frac{1}{\sqrt{47}}$ to four places of decimals.

$$\frac{1}{\sqrt{47}}=(47)^{-\frac{1}{2}}=(7^2-2)^{-\frac{1}{2}}=\frac{1}{7}\left(1-\frac{2}{7^2}\right)^{-\frac{1}{2}}$$

$$=\frac{1}{7}\left(1+\frac{1}{7^2}+\frac{3}{2}\cdot\frac{1}{7^4}+\frac{5}{2}\cdot\frac{1}{7^6}+\ldots\right)$$
$$=\frac{1}{7}+\frac{1}{7^3}+\frac{3}{2}\cdot\frac{1}{7^5}+\frac{5}{2}\cdot\frac{1}{7^7}+\ldots$$

To obtain the values of the several terms we proceed as follows :

$$\begin{array}{l} 7\,)\,1 \\ \overline{7\,)\,0.142857} \quad \ldots\ldots\ldots = \frac{1}{7}, \\ \overline{7\,)\,0.020408} \\ 7\,)\,0.002915 \ldots\ldots\ldots = \frac{1}{7^3}, \\ \overline{7\,)\,0.000416} \\ \quad 0.000059 \ldots\ldots\ldots = \frac{1}{7^5}; \end{array}$$

and we can see that the term $\frac{5}{2}\cdot\frac{1}{7^7}$ is a decimal fraction beginning with 5 ciphers.

$$\therefore \quad \frac{1}{\sqrt{47}} = 0{\cdot}142857 + 0{\cdot}002915 + 0{\cdot}000088$$
$$= 0{\cdot}14586,$$

and this result is correct to at least four places of decimals.

Example 4. Find the cube root of 126 to 5 places of decimals.

$$(126)^{1/3} = (5^3+1)^{1/3}$$
$$= 5\left(1+\frac{1}{5^3}\right)^{\frac{1}{3}}$$
$$= 5\left(1+\frac{1}{3}\cdot\frac{1}{5^3}-\frac{1}{9}\cdot\frac{1}{5^6}+\frac{5}{81}\cdot\frac{1}{5^9}-\ldots\right)$$
$$= 5+\frac{1}{3}\cdot\frac{1}{5^2}-\frac{1}{9}\cdot\frac{1}{5^5}+\frac{1}{81}\cdot\frac{1}{5^7}-\ldots$$
$$= 5+\frac{1}{3}\cdot\frac{2^2}{10^2}-\frac{1}{9}\cdot\frac{2^5}{10^5}+\frac{1}{81}\cdot\frac{2^7}{10^7}-\ldots$$
$$= 5+\frac{0.04}{3}-\frac{0.00032}{9}+\frac{0.0000128}{81}-\ldots$$
$$= 5+0.013333\ldots-0.000035\ldots+\ldots$$
$= 5.01329$, to five places of decimals.

EXAMPLES XIV. b.

Find the $(r+1)^{\text{th}}$ term in each of the following expansions :

1. $(1+x)^{-1/2}$. **2.** $(1-x)^{-5}$. **3.** $(1+3x)^{1/3}$.

4. $(1+x)^{-\frac{2}{3}}$.
5. $(1+x^2)^{-3}$.
6. $(1-2x)^{-\frac{3}{2}}$.
7. $(a+bx)^{-1}$.
8. $(2-x)^{-2}$.
9. $\sqrt[3]{(a^3-x^3)^2}$.
10. $\frac{1}{\sqrt{1+2x}}$.
11. $\frac{1}{\sqrt[3]{(1-3x)^2}}$.
12. $\frac{1}{\sqrt[n]{a^n-nx}}$.

Find the greatest term in each of the following expansions :

13. $(1+x)^{-7}$ when $x=\frac{4}{15}$.
14. $(1+x)^{\frac{21}{2}}$ when $x=\frac{2}{3}$.
15. $(1-7x)^{-\frac{11}{4}}$ when $x=\frac{1}{8}$.
16. $(2x+5y)^{12}$ when $x=8$ and $y=3$.
17. $(5-4x)^{-7}$ when $x=\frac{1}{2}$.
18. $(3x^2+4y^3)^{-n}$ when $x=9, y=2, n=15$.

Find to five places of decimals the value of

19. $\sqrt{98}$.
20. $\sqrt[3]{998}$.
21. $\sqrt[3]{1003}$.
22. $\sqrt[4]{2400}$.
23. $\frac{1}{\sqrt[3]{128}}$.
24. $\left(1\frac{1}{250}\right)^{\frac{1}{3}}$.
25. $(630)^{-\frac{3}{4}}$.
26. $\sqrt[5]{3128}$.

If x be so small that its square and higher powers may be neglected, find the value of

27. $(1-7x)^{\frac{1}{3}}(1+2x)^{-\frac{3}{4}}$.
28. $\sqrt{4-x}\cdot\left(3-\frac{x}{2}\right)^{-1}$
29. $\frac{(8+3x)^{\frac{2}{3}}}{(2+3x)\sqrt{4-5x}}$
30. $\frac{\left(1+\frac{2}{3}x\right)^{-5}\times(4+3x)^{\frac{1}{2}}}{(4+x)^{\frac{3}{2}}}$.
31. $\frac{\sqrt[4]{1-\frac{3}{5}x}+\left(1+\frac{5}{6}x\right)^{-6}}{\sqrt[3]{1+2x}+\sqrt[5]{1-\frac{x}{2}}}$
32. $\frac{\sqrt[3]{8+3x}-\sqrt[5]{1-x}}{(1+5x)^{\frac{3}{5}}+\left(4+\frac{x}{2}\right)^{\frac{1}{2}}}$.

33. Prove that the coefficient of x^r in the expansion of $(1-4x)^{-\frac{1}{2}}$ is $\frac{(2r)!}{(r!)^2}$.

34. Prove that $(1+x)^n = 2^n \left\{ 1 - n\frac{1-x}{1+x} + \frac{n(n+1)}{1 \cdot 2}\left(\frac{1-x}{1+x}\right)^2 \ldots \right\}$.

35. Find the first three terms in the expansion of

$$\frac{1}{(1+x)^2\sqrt{1+4x}}.$$

36. Find the first three terms in the expansion of

$$\frac{(1+x)^{\frac{3}{4}} + \sqrt{1+5x}}{(1-x)^2}.$$

37. Shew that the n^{th} coefficient in the expansion of $(1-x)^{-n}$ is double of the $(n-1)^{\text{th}}$.

189. *To find the numerically greatest term in the expansion of* $(1+x)^n$, *for any rational value of* n.

Since we are only concerned with the *numerical* value of the greatest term, we shall consider x throughout as positive.

Case I: Let n be a positive integer.

The $(r+1)^{\text{th}}$ term is obtained by multiplying the r^{th} term by $\frac{n-r+1}{r} \cdot x$; that is, by $\left(\frac{n+1}{r} - 1\right)x$; and therefore the terms continue to increase so long as

$$\left(\frac{n+1}{r} - 1\right)x > 1;$$

that is,

$$\frac{(n+1)\,x}{r} > 1 + x,$$

or

$$\frac{(n+1)\,x}{1+x} > r.$$

If $\frac{(n+1)\,x}{1+x}$ be an integer, denote it by p; then if $r = p$, the multiplying factor is 1, and the $(p+1)^{\text{th}}$ term is equal to the p^{th}, and these are greater than any other term.

If $\frac{(n+1)\,x}{1+x}$ be not an integer, denote its integral part by q; then the greatest value of r is q, and the $(q+1)^{\text{th}}$ term is the greatest.

Case II: Let n be a positive fraction.

As before, the $(r+1)^{\text{th}}$ term is obtained by multiplying the r^{th} term by $\left(\frac{n+1}{r} - 1\right)x$.

(1) If x be greater than unity, by increasing r the above multiplier can be made as near as we please to $-x$; so that after a certain term each term is nearly x times the preceding term numerically, and thus the terms increase continually, and there is no greatest term.

(2) If x be less than unity we see that the multiplying factor continues positive, and decreases until $r > n+1$, and from this point it becomes negative but always remains less than 1 numerically; therefore there will be a greatest term.

As before, the multiplying factor will be greater than 1

so long as $$\frac{(n+1)\,x}{1+x} > r.$$

If $\frac{(n+1)\,x}{1+x}$ be an integer, denote it by p; then, as in Case I., the $(p+1)^{\text{th}}$ term is equal to the p^{th}, and these are greater than any other term.

If $\frac{(n+1)\,x}{1+x}$ be not an integer, let q be its integral part; then the $(q+1)^{\text{th}}$ term is the greatest.

Case III: Let n be negative.

Let $n = -m$, so that m is positive; then the numerical value of the multiplying factor is $\frac{m+r-1}{r} \cdot x$; that is

$$\left(\frac{m-1}{r}+1\right)x.$$

(1) If α be greater than unity we may shew, as in Case II., that there is no greatest term.

(2) If x be less than unity, the multiplying factor will be greater than 1, so long as

$$\left(\frac{m-1}{r}+1\right)x > 1;$$

that is $$\frac{(m-1)\,x}{r} > 1-x,$$

or $$\frac{(m-1)\,x}{1-x} > r.$$

If $\frac{(m-1)\,x}{1-x}$ be a positive integer, denote it by p; then the $(p+1)^{\text{th}}$ term is equal to the p^{th} term, and these are greater than any other term.

If $\frac{(m-1)\,x}{1-x}$ be positive but not an integer, let q be its integral part; then the $(q+1)^{\text{th}}$ term is the greatest.

If $\frac{(m-1)x}{1-x}$ be negative, then m is less than unity; and by writing the multiplying factor in the form $\left(1-\frac{1-m}{r}\right)x$, we see that it is always less than 1 : hence each term is less than the preceding, and consequently the first term is the greatest.

190. *To find the number of homogeneous products of r dimensions that can be formed out of the n letters a, b., c, … and their powers.*

By division, or by the Binomial Theorem, we have

$$\frac{1}{1-ax}=1+ax+a^2x^2+a^3x^3+\ldots,$$

$$\frac{1}{1-bx}=1+bx+b^2x^2+b^3x^3+\ldots,$$

$$\frac{1}{1-cx}=1+cx+c^2x^2+c^3x^3+\ldots,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots$$

Hence, by multiplication,

$$\frac{1}{1-ax}\cdot\frac{1}{1-bx}\cdot\frac{1}{1-cx}\cdot\ldots$$

$$=(1+ax+a^2x^2+\ldots)(1+bx+b^2x^2+\ldots)(1+cx+c^2x^2+\ldots)\ldots$$

$$=1+x(a+b+c+\ldots)+x^2(a^2+ab+ac+b^2+bc+c^2+\ldots)+\ldots$$

$$=1+S_1x+S_2x^2+S_3x^3+\ldots \text{ suppose;}$$

where $S_1, S_2, S_3, \ldots$ are the *sums* of the homogeneous products of *one, two, three,* … dimensions that can be formed of $a, b, c, \ldots$ and their powers.

To obtain the *number* of these products, put $a, b, c, \ldots$ each equal to 1; each term in $S_1, S_2, S_3, \ldots$ now becomes 1, and the values of $S_1, S_2, S_3, \ldots$ so obtained give the number of the homogeneous products of *one, two, three,* … dimensions.

Also $\frac{1}{1-ax}\cdot\frac{1}{1-bx}\cdot\frac{1}{1-cx}\ldots$

becomes $\frac{1}{(1-x)^n}$ or $(1-x)^{-n}$.

Hence $S_r=$ coefficient of x^r in the expansion of $(1-x)^{-n}$

$$=\frac{n(n+1)(n+2)\ldots(n+r-1)}{r!}$$

$$= \frac{(n+r-1)!}{r!(n-1)!}$$

191. *To find the number of terms in the expansion of any multinomial when the index is a positive integer.*

In the expansion of $(a_1 + a_2 + a_3 + \dots + a_r)^n$,

every term is of n dimensions; therefore the number of terms is the same as the number of homogeneous products of n dimensions that can be formed out of the r quantities $a_1, a_2, \dots, a_r$, and their powers; and therefore by the preceding article is equal to

$$\frac{(r+n-1)!}{n!(r-1)!}$$

192. From the result of Art. 190 we may deduce a theorem relating to the number of combinations of n things.

Consider n letters $a, b, c, d, \dots$; then if we were to write down all the homogeneous products of r dimensions which can be formed of these letters and their powers, every such product would represent one of the combinations, r at a time, of the n letters, when any one of the letters might occur once, twice, thrice, ... up to r times.

Therefore the number of combinations of n things r at a time when repetitions are allowed is equal to the number of homogeneous products of r dimensions which can be formed out of n letters, and therefore equal to $\frac{(n+r-1)!}{r!(n-1)!}$, or ${}^{n+r-1}C_r$.

That is, the number of combinations of n things r at a time when repetitions are allowed is equal to the number of combinations of $n+r-1$ things r at a time when repetitions are excluded.

193. We shall conclude this chapter with a few miscellaneous examples.

Example 1. Find the coefficient of x^r in the expansion of $\frac{(1-2x)^2}{(1+x)^3}$.

The expression $= (1 - 4x + 4x^2)(1 + p_1x + p_2x^2 + \dots + p_rx^r + \dots)$ suppose.

The coefficient of x^r will be obtained by multiplying p_r, p_{r-1}, p_{r-2} by $1, -4, 4$ respectively, and adding the results; hence the required coefficient $= p_r - 4p_{r-1} + 4p_{r-2}$.

But $$p_r = (-1)^r \frac{(r+1)(r+2)}{2}.$$ [Ex. 3, Art. 182.]

Hence the required coefficient

$$= (-1)^r \frac{(r+1)(r+2)}{2} - 4(-1)^{r-1}\frac{r(r+1)}{2} + 4(-1)^{r-2}\frac{(r-1)r}{2}$$

$$= \frac{(-1)^r}{2}[(r+1)(r+2)+4r(r+1)+4r(r-1)]$$

$$= \frac{(-1)^r}{2}(9r^2+3r+2).$$

Example 2. Find the value of the series

$$2+\frac{5}{2!3}+\frac{5\cdot7}{3!3^2}+\frac{5\cdot7\cdot9}{4!3^3}+\dots$$

$$\text{The expression} = 2+\frac{3\cdot5}{2!}\cdot\frac{1}{3^2}+\frac{3\cdot5\cdot7}{3!}\cdot\frac{1}{3^3}+\frac{3\cdot5\cdot7\cdot9}{4!}\cdot\frac{1}{3^4}+\dots$$

$$=2+\frac{\frac{3}{2}\cdot\frac{5}{2}}{2!}\cdot\frac{2^2}{3^2}+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{3!}\cdot\frac{2^3}{3^3}+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2}}{4!}\cdot\frac{2^4}{3^4}+\dots$$

$$=1+\frac{\frac{3}{2}}{1}\cdot\frac{2}{3}+\frac{\frac{3}{2}\cdot\frac{5}{2}}{2!}\cdot\left(\frac{2}{3}\right)^2+\frac{\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}}{3!}\cdot\left(\frac{2}{3}\right)^3+\dots$$

$$=\left(1-\frac{2}{3}\right)^{-\frac{3}{2}}=\left(\frac{1}{3}\right)^{-\frac{3}{2}}$$

$$=3^{\frac{3}{2}}=3\sqrt{3}.$$

Example 3. If n is any positive integer, shew that the integral part of $(3+\sqrt{7})^n$ is an odd number.

Suppose I to denote the integral and f the fractional part of $(3+\sqrt{7})^n$.

Then

$$I+f=3^n+C_1 3^{n-1}\sqrt{7}+C_2 3^{n-2}7+C_3 3^{n-3}(\sqrt{7})^3+\dots \quad \dots(1)$$

Now $3-\sqrt{7}$ is positive and less than 1, therefore $(3-\sqrt{7})^n$ is a proper fraction; denote it by f';

$$\therefore \quad f'=3^n-C_1 3^{n-1}\sqrt{7}+C_2 3^{n-2}7-C_3 3^{n-3}(\sqrt{7})^3+\dots \quad \dots(2).$$

Add together (1) and (2); the irrational terms disappear, and we have

$$I+f+f'=2(3^n+C_2 3^{n-2}7+\dots)$$

$$= \text{an even integer.}$$

But since f and f' are proper fractions their sum must be 1;

$$\therefore \quad I = \text{an odd integer.}$$

EXAMPLES XIV. c.

Find the coefficient of

1. x^{100} in the expansion of $\dfrac{3-5x}{(1-x)^2}$.

2. a^{12} in the expansion of $\dfrac{4+2a-a^2}{(1+a)^3}$.

3. x^n in the expansion of $\dfrac{3x^2-2}{x+x^2}$.

4. Find the coefficient of x^n in the expansion of $\dfrac{2+x+x^2}{(1+x)^3}$.

5. Prove that

$$1-\frac{1}{2}\cdot\frac{1}{2}+\frac{1\cdot3}{2\cdot4}\cdot\frac{1}{2^2}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\frac{1}{2^3}+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\cdot\frac{1}{2^4}-\ldots=\sqrt{\frac{2}{3}}.$$

6. Prove that

$$\sqrt{8}=1+\frac{3}{4}+\frac{3\cdot5}{4\cdot8}+\frac{3\cdot5\cdot7}{4\cdot8\cdot12}+\ldots$$

7. Prove that

$$1+\frac{2n}{3}+\frac{2n\,(2n+2)}{3\cdot6}+\frac{2n\,(2n+2)\,(2n+4)}{3\cdot6\cdot9}+\ldots$$
$$=2^n\left\{1+\frac{n}{3}+\frac{n\,(n+1)}{3\cdot6}+\frac{n\,(n+1)\,(n+2)}{3\cdot6\cdot9}+\ldots\right\}.$$

8. Prove that

$$7^n\left\{1+\frac{n}{7}+\frac{n\,(n-1)}{7\cdot14}+\frac{n\,(n-1)\,(n-2)}{7\cdot14\cdot21}+\ldots\right\}$$
$$=4^n\left\{1+\frac{n}{2}+\frac{n\,(n+1)}{2\cdot4}+\frac{n\,(n+1)\,(n+2)}{2\cdot4\cdot6}+\ldots\right\}.$$

9. Prove that approximately, when x is very small,

$$\frac{3\left(x+\frac{4}{9}\right)^{1/2}\left(1-\frac{3}{4}x^2\right)^{1/3}}{2\left(1+\frac{9}{16}x\right)^2}=1-\frac{307}{256}x^2.$$

10. Shew that the integral part of $(5+2\sqrt{6})^n$ is odd, if n be a positive integer.

11. Shew that the integral part of $(8+3\sqrt{7})^n$ is odd, if n be a positive integer.

12. Find the coefficient of x^n in the expansion of $(1-2x+3x^2-4x^3+\ldots)^{-n}$.

13. Shew that the middle term of $\left(x+\frac{1}{x}\right)^{4n}$ is equal to the coefficient of x^n in the expansion of $(1-4x)^{-\left(n+\frac{1}{2}\right)}$

14. Prove that the expansion of $(1-x^3)^n$ may be put into the form

$$(1-x)^{3n} + 3nx(1-x)^{3n-2} + \frac{3n(3n-3)}{1 \cdot 2}x^2(1-x)^{3n-4} + \dots$$

15. Prove that the coefficient of x^n in the expansion $\frac{1}{1+x+x^2}$ is 1, 0, – 1 according as n is of the form $3m$, $3m-1$, or $3m+1$.

16. In the expansion of $(a+b+c)^3$ find (1) the number of terms, (2) the sum of the coefficients of the terms.

17. Prove that if n be an even integer,

$$\frac{1}{1!\,(n-1)!} + \frac{1}{3!\,(n-3)!} + \frac{1}{5!\,(n-5)!} + \dots + \frac{1}{(n-1)!\,1!} = \frac{2^{n-1}}{n!}.$$

18. If $c_0, c_1, c_2, \dots, c_n$ are the coefficients in the expansion of $(1+x)^n$, when n is a positive integer, prove that

(1) $c_0 - c_1 + c_2 - c_3 + \dots + (-1)^r c_r = (-1)^r \frac{n!-1}{r!\,(n-r-1)!}$.

(2) $c_0 - 2c_1 + 3c_2 - 4c_3 + \dots + (-1)^n (n+1) c_n = 0$.

(3) $c_0^2 - c_1^2 + c_2^2 - c_3^2 + \dots + (-1)^n c_n^2 = 0$, or $(-1)^{\frac{n}{2}} c_{\frac{n}{2}}$,

according as n is odd or even.

19. If s_n denote the sum of the first n natural numbers, prove that

(1) $(1-x)^{-3} = s_1 + s_2 x + s_3 x^2 + \dots + s_n x^{n-1} + \dots$

(2) $2(s_1 s_{2n} + s_2 s_{2n-1} + \dots + s_n s_{n+1}) = \frac{2n+4}{5!\,(2n-1)!}$.

20. If $q_n = \frac{1 \cdot 3 \cdot 5 \cdot 7 \dots (2n-1)}{2 \cdot 4 \cdot 6 \cdot 8 \dots 2n}$, prove that

(1) $q_{2n+1} + q_1 q_{2n} + q_2 q_{2n-1} + \dots + q_{n-1} q_{n+2} + q_n q_{n+1} = \frac{1}{2}$.

(2) $2\{q_{2n} - q_1 q_{2n-1} + q_2 q_{2n-2} + \dots + (-1)^{n-1} q_{n-1} q_{n+1}\}$
$= q_n + (-1)^{n-1} q_n^2$.

21. Find the sum of the products, two at a time, of the coefficients in the expansion of $(1+x)^n$, when n is a positive integer.

22. If $(7+4\sqrt{3})^n = p + \beta$, where n and p are positive integers, and β a proper fraction, shew that $(1-\beta)(p+\beta) = 1$.

23. If $c_0, c_1, c_2, \ldots c_n$ are the coefficients in the expansion of $(1+x)^n$, where n is a positive integer, shew that

$$c_1 - \frac{c_2}{2} + \frac{c_3}{3} - \ldots + \frac{(-1)^{n-1} c_n}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}.$$

■■■

Chapter 15

MULTINOMIAL THEOREM

194. We have already seen in Art. 175, how we may apply the Binomial Theorem to obtain the expansion of a multinomial expression. In the present chapter our object is not so much to obtain the complete expansion of a multinomial as to find the coefficient of any assigned term.

***Example*:** Find the coefficient of $a^4b^2c^3d^5$ in the expansion of

$$(a+b+c+d)^{14}.$$

The expansion is the product of 14 factors each equal to $a+b+c+d$, and every term in the expansion is of 14 dimensions, being a product formed by taking one letter out of each of these factors. Thus to form the term $a^4b^2c^3d^5$, we take a out of any *four* of the fourteen factors, b out of any *two* of the remaining ten, c out of any *three* of the remaining eight. But the number of ways in which this can be done is clearly equal to the number of ways of arranging 14 letters when four of them must be a, two b, three c, and five d; that is, equal to

$$\frac{14!}{4!\,2!\,3!\,5!}. \qquad \text{[Art. 151.]}$$

This is therefore the number of times in which the term $a^4b^2c^3d^5$ appears in the final product, and consequently the coefficient required is 2522520.

195. *To find the coefficient of any assigned term in the expansion of* $(a+b+c+d+\ldots)^p$, *where p is a positive integer.*

The expansion is the product of p factors each equal to $a+b+c+d+\ldots$, and every term in the expansion is formed by taking one letter out of each of these p factors; and therefore the number of ways in which any term $a^\alpha b^\beta c^\gamma d^\delta \ldots$ will appear in the final product is equal to the number of ways of arranging p letters when α of them must be a, β must be b, γ must be c; and so on. That is,

the coefficient of $\quad a^\alpha b^\beta c^\gamma d^\delta \ldots$ is $\dfrac{p!}{\alpha!\,\beta!\,\gamma!\,\delta!\ldots}$,

where $\qquad \alpha+\beta+\gamma+\delta+\ldots=p.$

Cor: In the expansion of

$$(a+bx+cx^2+dx^3+\ldots)^p,$$

the term involving $a^\alpha b^\beta c^\gamma d^\delta \ldots$ is

$$\frac{p!}{\alpha!\beta!\gamma!\delta!\ldots} a^\alpha (bx)^\beta (cx^2)^\gamma (dx^3)^\delta \ldots,$$

or

$$\frac{p!}{\alpha!\beta!\gamma!\delta!\ldots} a^\alpha b^\beta c^\gamma d^\delta \ldots x^{\beta+2\gamma+3\delta+} \ldots,$$

where

$$\alpha+\beta+\gamma+\delta+\ldots=p.$$

This may be called *the general term* of the expansion.

Example. Find the coefficient of x^5 in the expansion of $(a+bx+cx^2)^9$.

The general term of the expansion is

$$\frac{9!}{\alpha!\beta!\gamma!} a^\alpha b^\beta c^\gamma x^{\beta+2\gamma} \qquad \ldots(1)$$

where

$$\alpha+\beta+\gamma=9.$$

We have to obtain by trial all the positive integral values of β and γ which satisfy the equation $\beta+2\gamma=5$; the values of α can then be found from the equation

$$\alpha+\beta+\gamma=9.$$

Putting $\gamma=2$, we have $\beta=1$, and $\alpha=6$;
putting $\gamma=1$, we have $\beta=3$, and $\alpha=5$;
putting $\gamma=0$, we have $\beta=5$, and $\alpha=4$.

The required coefficient will be the sum of the corresponding values of the expression (1).

Therefore the coefficient required

$$=\frac{9!}{6!2!}a^6bc^2+\frac{9!}{5!3!}a^5b^3c+\frac{9!}{4!5!}a^4b^5$$

$$=252a^6bc^2+504a^5b^3c+126a^4b^5.$$

196. *To find the general term in the expansion of*

$$(a+bx+cx^2+dx^3+\ldots)^n,$$

where n is any rational quantity.

By the Binomial Theorem, the general term is

$$\frac{n(n-1)(n-2)\ldots(n-p+1)}{p!} a^{n-p} (bx+cx^2+dx^3+\ldots),$$

where p is a positive integer.

And, by Art. 195, the general term of the expansion of

$$(bx+cx^2+dx^3+\ldots)^p$$

is

$$\frac{p!}{\beta!\gamma!\delta!\ldots} b^\beta c^\gamma d^\delta \ldots x^{\beta+2\gamma+3\delta+\ldots},$$

where $\beta, \gamma, \delta \ldots$ are positive integers whose sum is p.

Hence the general term in the expansion of the given expression is

$$\frac{n(n-1)(n-2)\ldots(n-p+1)}{\beta!\,\gamma!\,\delta!\ldots} a^{n-p} b^{\beta} c^{\gamma} d^{\delta} \ldots x^{\beta+2\gamma+3\delta+\ldots}$$

where $\beta+\gamma+\delta+\ldots=p$.

197. Since $(a+bx+cx^2+dx^3+\ldots)^n$ may be written in the form

$$a^n\left(1+\frac{b}{a}x+\frac{c}{a}x^2+\frac{d}{a}x^3+\ldots\right)^n,$$

it will be sufficient to consider the case in which the first term of the multinomial is unity.

Thus the general term of

$$(1+bx+cx^2+dx^3+\ldots)^n$$

is
$$\frac{n(n-1)(n-2)\ldots(n-p+1)}{\beta!\,\gamma!\,\delta!} b^{\beta} c^{\gamma} d^{\delta} \ldots x^{\beta+2\gamma+3\delta+\ldots}$$

where $\beta+\gamma+\delta+\ldots=p$.

Example: Find the coefficient of x^3 in the expansion of

$$(1-3x-2x^2+6x^3)^{\frac{2}{3}}.$$

The general term is

$$\frac{\frac{2}{3}\left(\frac{2}{3}-1\right)\left(\frac{2}{3}-2\right)\ldots\left(\frac{2}{3}-p+1\right)}{\beta!\,\gamma!\,\delta!}(-3)^{\beta}(-2)^{\gamma}(6)^{\delta}x^{\beta+2\gamma+3\delta} \quad \ldots(1)$$

We have to obtain by trial all the positive integral values of β, γ, δ which satisfy the equation $\beta+2\gamma+3\delta=3$; and then p is found from the equation $p=\beta+\gamma+\delta$. The required coefficient will be the sum of the corresponding values of the expression (1).

In finding $\beta, \gamma, \delta, \ldots$ it will be best to commence by giving to δ successive integral values beginning with the greatest admissible. In the present case the values are found to be

$$\delta=1,\quad \gamma=0,\quad \beta=0,\quad p=1;$$
$$\delta=0,\quad \gamma=1,\quad \beta=1,\quad p=2;$$
$$\delta=0,\quad \gamma=0,\quad \beta=3,\quad p=3.$$

Substituting these values in (1) the required coefficient

$$=\left(\frac{2}{3}\right)(6)+\left(\frac{2}{3}\right)\cdot\left(-\frac{1}{3}\right)(-3)(-2)+\frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}(-3)^3$$

$$=4-\frac{4}{3}-\frac{4}{3}=\frac{4}{3}.$$

198. Sometimes it is more expeditious to use the Binomial Theorem.

Example. Find the coefficient of x^4 in the expansion of $(1-2x+3x^2)^{-3}$.

The required coefficient is found by picking out the coefficient of x^4 from the first few terms of the expansion of $(1-2x+3x^2)^{-3}$ by the Binomial Theorem; that is, from

$$1+3(2x-3x^2)+6(2x-3x^2)^2+10(2x-3x^2)^3+15(2x-3x^2)^4;$$

we stop at this term for all the other terms involve powers of x higher than x^4.

The required coefficient $= 6\cdot 9 + 10\cdot 3\,(2)^2(-3) + 15\,(2)^4$
$= -66.$

EXAMPLES. XV.

Find the coefficient of

1. $a^2b^3c^4d$ in the expansion of $(a-b-c+d)^{10}$.
2. a^2b^5d in the expansion of $(a+b-c-d)^8$.
3. a^3b^3c in the expansion of $(2a+b+3c)^7$.
4. $x^2y^3z^4$ in the expansion of $(ax-by+cz)^9$.
5. x^3 in the expansion of $(1+3x-2x^2)^3$.
6. x^4 in the expansion of $(1+2x+3x^2)^{10}$.
7. x^6 in the expansion of $(1+2x-x^2)^5$.
8. x^8 in the expansion of $(1-2x+3x^2-4x^3)^4$;

Find the coefficient of

9. x^{23} in the expansion of $(1-2x+3x^2-x^4-x^5)^5$.
10. x^5 in the expansion of $(1-2x+3x^2)^{-\frac{1}{2}}$.
11. x^3 in the expansion of $(1-2x+3x^2-4x^3)^{\frac{1}{2}}$.
12. x^8 in the expansion of $\left(1-\dfrac{x^2}{3}+\dfrac{x^4}{9}\right)^{-2}$.
13. x^4 in the expansion of $(2-4x+3x^2)^{-2}$.
14. x^6 in the expansion of $(1+4x^2+10x^4+20x^6)^{-\frac{3}{4}}$.
15. x^{12} in the expansion of $(3-15x^3+18x^6)^{-1}$.
16. Expand $(1-2x-2x^2)^{\frac{1}{4}}$ as far as x^2.
17. Expand $(1+3x^2-6x^3)^{-\frac{2}{3}}$ as far as x^5.

18. Expand $(8 - 9x^3 + 18x^4)^{\frac{4}{3}}$ as far as x^8.

19. If $(1 + x + x^2 + \ldots + x^p)^n = a_0 + a_1x + a_2x^2 + \ldots + a_{np}x^{np}$,

prove that

(1) $a_0 + a_1 + a_2 + \ldots + a_{np} = (p+1)^n$.

(2) $a_1 + 2a_2 + 3a_3 + \ldots + np \cdot a_{np} = \frac{1}{2} np\,(p+1)^n$.

20. If $a_0, a_1, a_2, a_3, \ldots$ are the coefficients in order of the expansion of $(1 + x + x^2)^n$; prove that

$$a_0^2 - a_1^2 + a_2^2 - a_3^2 + \ldots + (-1)^{n-1} a_{n-1}^2 = \frac{1}{2} a_n \{1 - (-1)^n a_n\}.$$

21. If the expansion of $(1 + x + x^2)^n$

be $a_0 + a_1x + a_2x^2 + \ldots + a_rx^r + \ldots + a_{2n}x^{2n}$,

shew that

$$a_0 + a_3 + a_6 + \ldots = a_1 + a_4 + a_7 + \ldots = a_2 + a_5 + a_8 + \ldots = 3^{n-1}.$$