AML710 CAD LECTURE 5

2D TRANSFORMATIONS (Contd.)

Sequence of operations, Matrix multiplication, concatenation, combination of operations

Types of Transformation

Affine Map: A map ϕ that maps E^3 into itself is called an affine Map if it leaves barycentric conditions invariant.

If $x = \sum \beta_j a_j, \quad x, a_j \in E^3$

Then, $\phi x = \sum \beta_j \phi a_j \quad \phi x, \phi a_j \in E^3$

Most of the transformations that are used to position or scale an object in CAD are affine maps. The name was given by L. Euler and studied systematically by A. Mobius

Euclidian Maps: Rigid body motions like rotation and translation where lengths and angles are unchanged are called Euclidian maps. This is a special case of affine maps

Transformation Groups and Symmetries

Affine transformations are classified (Felix Klein) as follows

Similarity Groups: Rigid motion and scaling

Eg: Congruent, similar

Symmetry Groups:Rotation, Reflection, Translation

1. No translation and no reflection – Circle group

Eg: a gear wheel

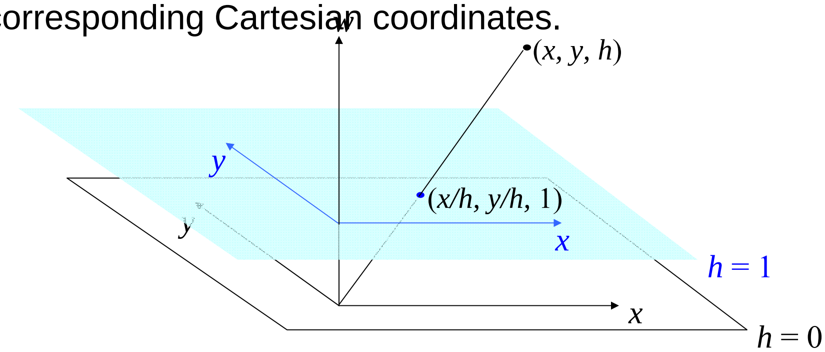
2. No translation and at least one reflection – Dihedral group

3. Translation along one direction

4. Translation along more than one direction

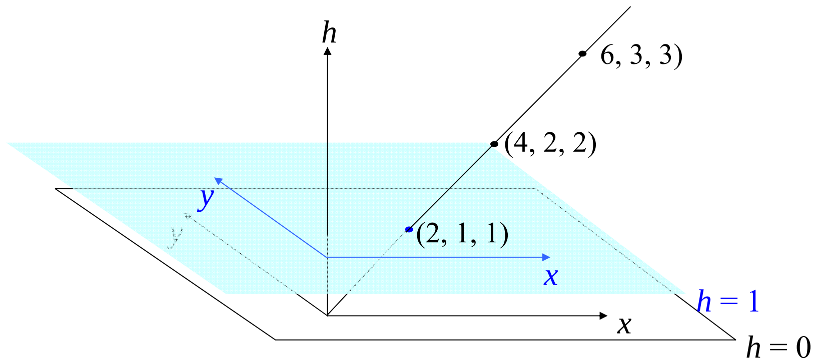
Homogeneous coordinates of vertices

- A point in homogeneous coordinates (x, y, h), $h \neq 0$, corresponds to the 2-D vertex $(x/h, y/h)$ in Cartesian coordinates.
- Conceive that the Cartesian coordinates axes lies on the plane of $h = 1$. The intersection of the plane and the line connecting the origin and (x, y, h) gives the corresponding Cartesian coordinates.



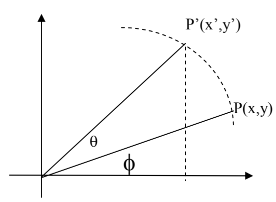
- For example, both the points (6, 9, 3) and (4, 6, 2) in the homogeneous coordinates corresponds to (2, 3) in the Cartesian coordinates.

  Conversely, the point (2, 1) of the Cartesian corresponds to (2, 1, 1), (4, 2, 2) or (6, 3, 3) of the homogeneous system



Rotation

Consider the following figure where a position vector p(x,y) which makes an angle ϕ to x-axis after transformation to p'(x',y') makes an angle $\phi+\theta$ degrees.



P= [x y] = [rcosϕ rsinϕ]

P'= [x' y'] = [rcos(ϕ+θ) rsin(ϕ+θ)]

= [r(cosϕcosθ - sinϕsinθ) r(cosϕsinθ+ sinϕcosθ)]

Using the definition of x=rcosϕ and y=rsinϕ, we can write

P'= [x' y'] = [x cosθ – y sinθ x sinθ +y cosθ] or the transformation matrix is

$$[X][T]=[x \quad y]\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

Some Common Cases of Rotation

Rotation of 90° counter clockwise about the origin

$$[T]=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}=\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Rotation of 180° counter clockwise about the origin

$$[T]=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

Rotation of 270° counter clockwise about the origin

$$[T]=\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}=\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

In all the above cases det[T]=1

Reflection- Special case of Rotation

Reflection is a special case of rotation of 180° about a line in xy plane passing through the origin. Eg about y=0 (x-axis)

$$[T]=\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

About x=0 (y-axis)

$$[T]=\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

About the lines x=y and x-y respectively are:

$$[T]=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad [T]=\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

Reflection- Properties

If two pure reflections about a line passing through the origin are applied successively the result is a pure rotation.

The determinant of a pure reflection matrix is -1

$$[T]=\begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}$$

Properties of transformation Matrices

Det[T]=? In case of a) rotation, b) reflection.

What is the geometrical interpretation of inverse of [T]?

Show that TT^{-1} = [I]

$$[T]=\begin{bmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{bmatrix}$$

$$[T]^{-1}=\begin{bmatrix}\cos(-\theta) & \sin(-\theta)\\ -\sin(-\theta) & \cos(-\theta)\end{bmatrix}=\begin{bmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{bmatrix}=T^T$$

Thus for a pure rotation (det[T]=1) the inverse of T is its transpose

Example

Show that the following transformation matrix gives a pure rotation

$$[T]=\begin{bmatrix}\dfrac{1-t^2}{1+t^2} & \dfrac{2t}{1+t^2}\\ \dfrac{-2t}{1+t^2} & \dfrac{1-t^2}{1+t^2}\end{bmatrix}$$

$$\det[T]=\left(\frac{1-t^2}{1+t^2}\right)\left(\frac{1-t^2}{1+t^2}\right)-\left(\frac{-2t}{1+t^2}\right)\left(\frac{2t}{1+t^2}\right)=1$$

Example

A A unit square is transformed by a 2×2 transformation matrix. The resulting position vectors are:

$$[x']=\begin{bmatrix}0&0\\2&3\\8&4\\6&1\end{bmatrix}$$

Determine the transformation matrix used.

Ans:

$$[x']=[X][T]=\begin{bmatrix}0&0\\1&0\\1&1\\0&1\end{bmatrix}\begin{bmatrix}a&b\\c&d\end{bmatrix}=\begin{bmatrix}0&0\\a&b\\a+c&b+d\\c&d\end{bmatrix}=\begin{bmatrix}0&0\\2&3\\8&4\\6&1\end{bmatrix};\therefore[T]=\begin{bmatrix}2&3\\6&1\end{bmatrix}$$

Example

Show that the shear transformation in x and y directions together is not the same as shear along x followed by shear along y?

Ans:

$$\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix} = \begin{bmatrix} 1+bc & b \\ c & 1 \end{bmatrix} \neq \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ c & 1 \end{bmatrix}\begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & b \\ c & 1+bc \end{bmatrix} \neq \begin{bmatrix} 1 & b \\ c & 1 \end{bmatrix}$$

Example Problem

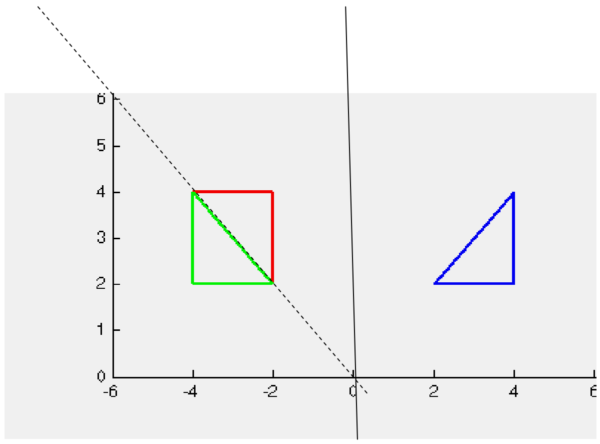
- **Consider a triangle whose vertices are (2 2), (4 2) and (4 4). Find the concatenated transformation matrix and the transformed vertices for rotatation of 90 about the origin followed by reflection through the line *y* = -*x*. Comment on the sequence of transformations.**

$$[X\}[T_1\}[T_2\} = [X\}[T\} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -4 & 2 \\ -4 & 4 \end{bmatrix}$$

For seeing the effect of changing the sequence of operations let us reverse the order, i.e, first reflection and then rotation.

$$[X\}[T_2\}[T_1\} = [X\}[T\} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 4 & 2 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 4 & -2 \\ 4 & -4 \end{bmatrix}$$

Example Problem: Solution



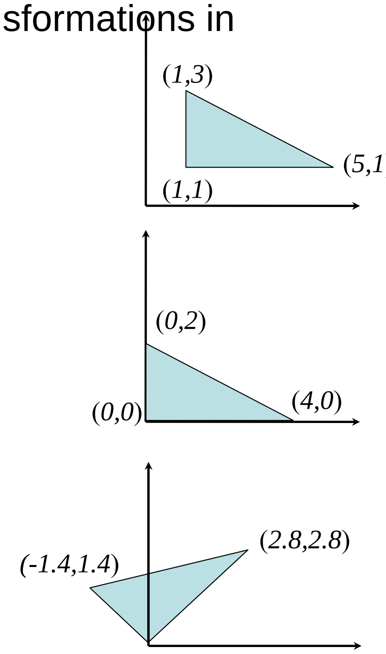
- Conduct a combination of transformations in sequence
  - Translate the right-angle vertex to the origin (*Tx* = -1, *Ty* = -1)

$$\begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}.$$

  - Rotate 45^o ($^{\pi}/_4$ radian)

    $\sin {}^{\pi}/_4 = \cos {}^{\pi}/_4 = 0.7071$

$$\begin{bmatrix} x'' \\ y'' \\ w'' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix}$$



$$\begin{bmatrix} x'' \\ y'' \\ w'' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x' \\ y' \\ w' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & T_x \\ 0 & 1 & T_y \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

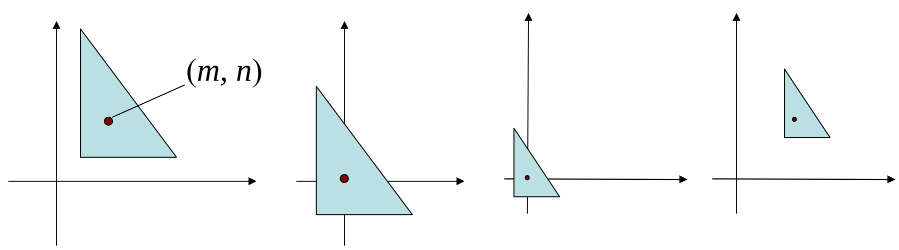
$$= \begin{bmatrix} \cos\theta & -\sin\theta & T_x\cos\theta - T_y\sin\theta \\ \sin\theta & \cos\theta & T_x\sin\theta + T_y\cos\theta \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ w \end{bmatrix}$$

- The computation of [ ] from [ ] [ ] is called ***matrix multiplication***. The general form is:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} e & f \\ g & h \end{bmatrix} = \begin{bmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{bmatrix}$$

- A sequence of transformations can be lumped in a single matrix via matrix multiplications

- Scaling relative to a fixed point



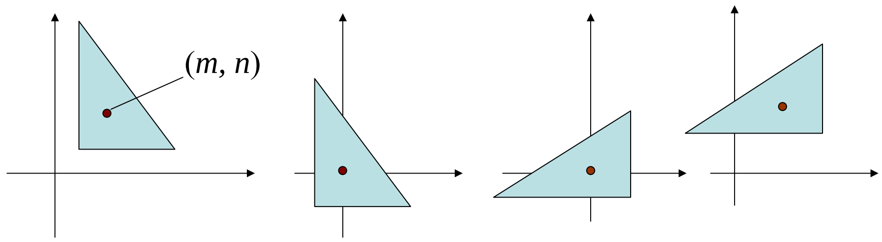
1. Translate the point *(-m, -n)* to the origin
2. Scale the object by Sx, Sy scale factors
3. Translate the point back *(m, n)*

$$[T] = [T_t]^{-1}[S][T_t]$$

$$[T] = \begin{bmatrix} 1 & 0 & m \\ 0 & 1 & n \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} Sx & 0 & 0 \\ 0 & Sy & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -m \\ 0 & 1 & -n \\ 0 & 0 & 1 \end{bmatrix}$$

Column Vectors

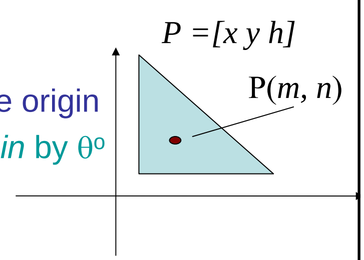
➢ Rotation about an arbitrary point



$$[T]=[T_t][R][T_t]^{-1}$$

Row Vectors $$[T]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -m & -n & 1\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ m & n & 1\end{bmatrix}$$

Rotation about an arbitrary point



1. Translate the point(-m, -*n*) to the origin
2. Rotate the object about the *origin* by θ°
3. Translate back (m,n)

$$[T]=[T_t][R][T_t]^{-1}$$

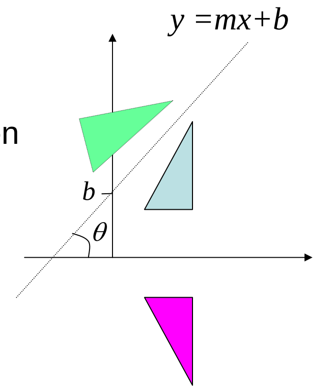
$$[T]=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ -m & -n & 1\end{bmatrix}\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ m & n & 1\end{bmatrix}$$

Ex: Center of an object (4,3) , rotate it about the center by 90° CC

$$[x \quad y \quad 1]\begin{bmatrix}0 & 1 & 0\\ -1 & 0 & 0\\ 7 & -1 & 1\end{bmatrix} \qquad OR \qquad \begin{bmatrix}0 & -1 & 7\\ 1 & 0 & -1\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}x\\ y\\ 1\end{bmatrix}$$

Reflection O(3)

Reflection is a special case of rotation SO(3) when angel of rotation is 180° about an axis in the plane

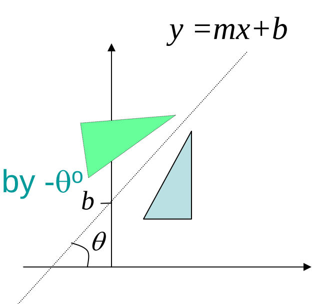


$$\theta = \tan^{-1}(m)$$

$$\sin(180) = 0, \cos(180) = -1$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Reflection about an arbitrary line



1. Translate (0, -*b*) so that the line passes through the origin
2. Rotate the line about the *x* axis by -θº
3. Reflect object about the *x* axis
4. Rotate back the line by θº
5. Translate back (0,b)

$$\theta = \tan^{-1}(m)$$

$$[T] = [T_t][T_r][R][T_r]^{-1}[T_t]^{-1}$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \\ -1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} & 0 \\ 1/\sqrt{5} & 2/\sqrt{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

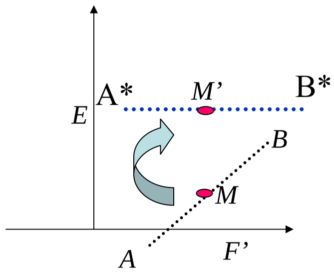
Alternate Method?

Ex: Reflect triangle (2,4),(4,6),(2,6) about line $y = \frac{1}{2}(x+4)$

$$\begin{bmatrix} 3/5 & 4/5 & -8/5 \\ 4/5 & -3/5 & 16/5 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 4 & 2 \\ 4 & 6 & 6 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 14/5 & 28/5 & 22/5 \\ 12/5 & 14/5 & 6/5 \\ 1 & 1 & 1 \end{bmatrix}$$

Mid-point Transformation

Consider a straight line between $A(x_1,y_1)$ and $B(x_2,y_2)$. Let us try to transform this line using a general a 2×2 transformation matrix



$$[T]=\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} A^* \\ B^* \end{bmatrix}=\begin{bmatrix} A \\ B \end{bmatrix}[T]=\begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} ax_1+cy_1 & bx_1+dy_1 \\ ax_2+cy_2 & bx_2+dy_2 \end{bmatrix}=\begin{bmatrix} x_1^* & y_1^* \\ x_2^* & y_2^* \end{bmatrix}$$

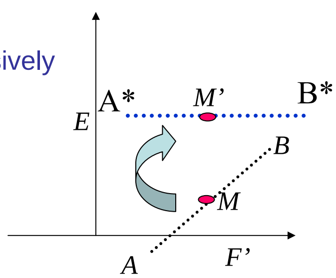
$$\begin{bmatrix} x_m^* & y_m^* \end{bmatrix}=\begin{bmatrix} \frac{x_1^*+x_2^*}{2} & \frac{y_1^*+y_2^*}{2} \end{bmatrix}=\begin{bmatrix} \frac{ax_1+cy_1+ax_2+cy_2}{2} & \frac{bx_1+dy_1+bx_2+dy_2}{2} \end{bmatrix}$$

$$=\begin{bmatrix} a\frac{x_1+x_2}{2}+c\frac{y_1+y_2}{2} & b\frac{x_1+x_2}{2}+d\frac{y_1+y_2}{2} \end{bmatrix} \qquad (1)$$

$$\begin{bmatrix} x'_m & y'_m \end{bmatrix}=\begin{bmatrix} \frac{x_1+x_2}{2} & \frac{y_1+y_2}{2} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}=\begin{bmatrix} a\frac{x_1+x_2}{2}+c\frac{y_1+y_2}{2} & b\frac{x_1+x_2}{2}+d\frac{y_1+y_2}{2} \end{bmatrix} \quad (2)$$

Mid-point Transformation

From (1) and (2) we can conclude that by recursively continuing this process, we will cover all points on the line AB and hence their images on A*B* by corresponding map.

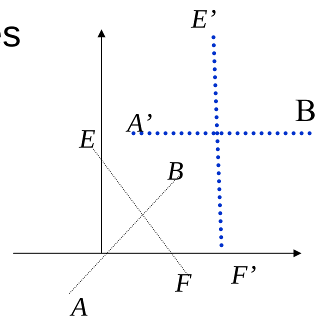


But a general proof of end points transformation actually transforms the whole line can be given usign the section formula. Any point **m** that divides the line AB in the ratio p:q can be given as:

$$\mathbf{m}=\frac{1}{p+q}(p\mathbf{b}+q\mathbf{a})$$

Needless to say that the same is applicable to A*B*. The special case of mid point transformation occurs when p:q=1.

Transformation of intersecting lines



When two intersecting lines are transformed using general 2x2 transformation matrix the resulting lines are also intersecting

Consider 2 intersection lines

$$y = m_1 x + b_1$$

$$y = m_2 x + b_2$$

The point of intersection:

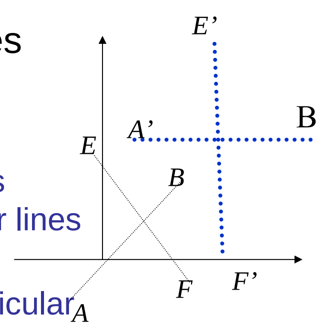
$$[X_i] = [x_i \quad y_i] = \begin{bmatrix} \dfrac{b_1 - b_2}{m_2 - m_1} & \dfrac{b_1 m_2 - b_2 m_1}{m_2 - m_1} \end{bmatrix}$$

Transformed point of intersection:

$$\Rightarrow [X_i] \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$[x_i' \quad y_i'] = \begin{bmatrix} \dfrac{a(b_1 - b_2) + c(b_1 m_2 - b_2 m_1)}{m_2 - m_1} & \dfrac{b(b_1 - b_2) + d(b_1 m_2 - b_2 m_1)}{m_2 - m_1} \end{bmatrix}$$

Transformation of intersecting lines



Observe these points

In the last slide

1. A pair of non-perpendicular lines get transformed to perpendicular lines
2. By inverse T^{-1} a pair of perpendicular lines can get transformed to non-perpendicular lines
3. Such a result can have disastrous effect on the resulting geometry

Transformation of two parallel lines

When two parallel lines are transformed using general 2x2 transformation matrix the resulting lines remain parallel

Consider 2 lines parallel to each other between the end points A(x1,y1),B(x2,y2) and CD. Since they are parallel the slope is the same given by $m = \frac{y_2 - y_1}{x_2 - x_1}$

$$[T][X] = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ y_1 & y_2 \end{bmatrix} = \begin{bmatrix} ax_1 + cy_1 & ax_2 + cy_2 \\ bx_1 + dy_1 & bx_2 + dy_2 \end{bmatrix}$$

$$m' = \frac{(bx_2 + dy_2) - (bx_1 + dy_1)}{(ax_2 + cy_2) - (ax_1 + cy_1)} = \frac{b(x_2 - x_1) + d(y_2 - y_1)}{a(x_2 - x_1) + c(y_2 - y_1)}$$

$$m' = \frac{b + d\frac{(y_2 - y)}{(x_2 - x_1)_1}}{a + c\frac{(y_2 - y_1)}{(x_2 - x_1)}} = \frac{b + dm}{a + cm}$$

Intersecting lines and rigid body transformations

When do perpendicular lines transform as perpendicular lines?

Consider the **scalar(dot)** and **vector** products of two vectors as follows

$$V_1 \cdot V_2 = V_{1x}V_{2x} + V_{1y}V_{2y} = |V_1||V_2|\cos\theta$$

$$V_1 \times V_2 = \left(V_{1x}V_{2x} - V_{1y}V_{2y}\right)\vec{k} = |V_1||V_2|\vec{k}\sin\theta$$

Let us now transform these vectors by general 2x2 T and find **dot** and **cross** products again

$$\begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} V_{1x} & V_{2x} \\ V_{1y} & V_{2x} \end{bmatrix} = \begin{bmatrix} aV_{1x} + cV_{1y} & aV_{2x} + cV_{2y} \\ bV_{1x} + dV_{1y} & bV_{2x} + dV_{2y} \end{bmatrix}$$

$$(a^2 + b^2)V_{1x}V_{2x} + (c^2 + d^2)V_{1y}V_{2y} + (ac + bd)(V_{1x}V_{2y} + V_{1y}V_{2x}) = |V_1'||V_2'|\cos\theta$$

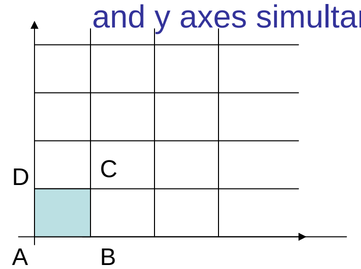
$$(ad - bc)(V_{1x}V_{2y} - V_{2x}V_{1y})\vec{k} = |V_1'||V_2'|\vec{k}\sin\theta$$

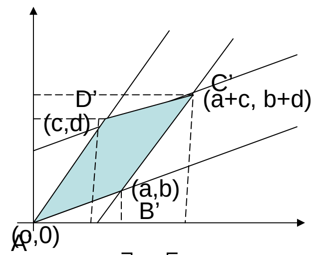
We require for magnitude and angle to remain unchanged

$$a^2 + b^2 = c^2 + d^2 = 1;\ ac + bd = 0;\ ad - bc = +1\ OR\ [T][T]^{-1} = [T][T]^T = [I]$$

Transformation of Unit square and area

Observe unit square being scaled and sheared along x and y axes simultaneously





$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix};\ and\ \ P' = \begin{bmatrix} a & c \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & a & a+c & c \\ 0 & b & b+d & d \end{bmatrix}$$

$$A_p = (a+c)(b+d) - \frac{1}{2}(ab) - \frac{1}{2}(cd)$$

$$-\frac{c}{2}(b+b+d) - \frac{b}{2}(c+a+c) = ad - bc$$

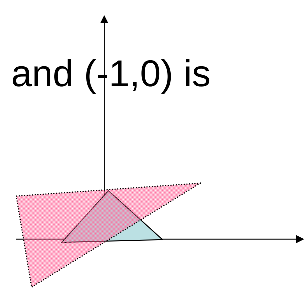
$$= \det[T]$$

$$\therefore A_p = A_s \det[T]$$

Area Scaling example

A triangle with vertices (1,0), (0,1) and (-1,0) is transformed by

$$[T] = \begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix}$$



Area $A_i = \frac{1}{2}(2)(1) = 1 \text{ sq.units}$

Vertices after transformation are

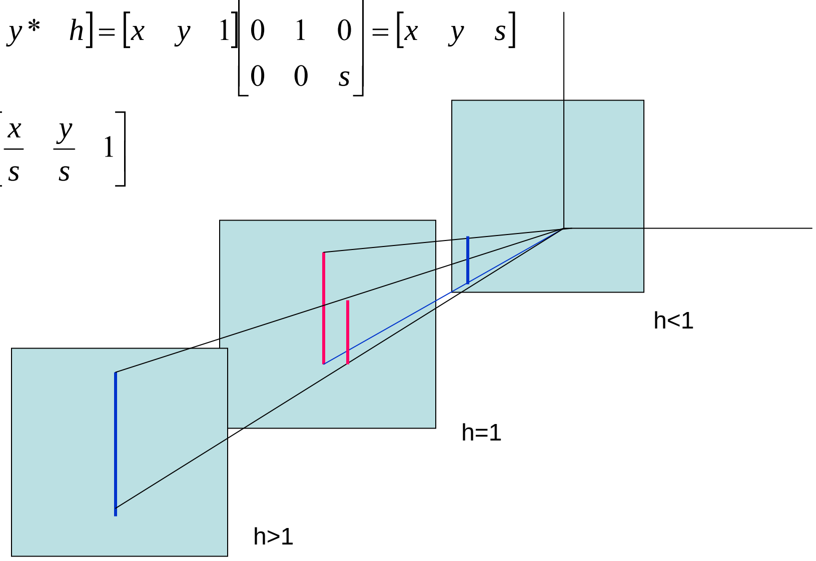
$$[X'] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 3 & 2 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ -1 & 2 \\ -3 & -2 \end{bmatrix}$$

Transformed area $A_t = A_i \frac{1}{2}(4)(4) = 8 \text{ sq.units}$

Geometrical Interpretation of Overall Scaling

$$[x^* \quad y^* \quad h] = [x \quad y \quad 1]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix} = [x \quad y \quad s]$$

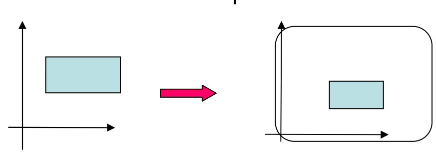
$$= \begin{bmatrix} \frac{x}{s} & \frac{y}{s} & 1 \end{bmatrix}$$



Viewing Transformations

- It is a transformation from world CS to Screen CS. The dimensions of the WCS may be in any chosen system of units SI or FPS etc the latter is measured in pixels.

1. Translate the MCS base point to origin
2. Apply the scaling necessary to fit into the screen limits (xmin,ymin and xmax ymax)
3. Move the base point back to its original position

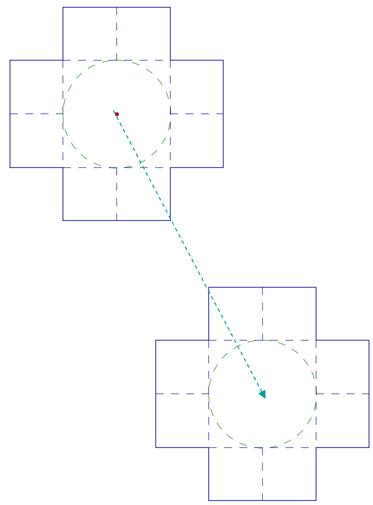


$$[T] = [T_t][S][T_t]^{-1}$$

$$[T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -x_{\min} & -y_{\min} & 1 \end{bmatrix}\begin{bmatrix} \frac{u_{\max}-u_{\min}}{x_{\max}-x_{\min}} & 0 & 0 \\ 0 & \frac{v_{\max}-v_{\min}}{y_{\max}-y_{\min}} & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ u_{\min} & v_{\min} & 1 \end{bmatrix}$$

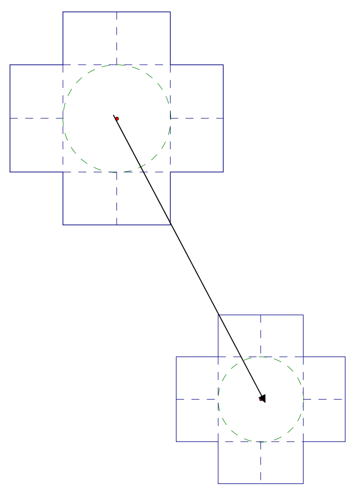
Translation

- What are invariant?
  - Lengths
  - Angles
- Which transformation group?
  - Congruent
- det[T]=1

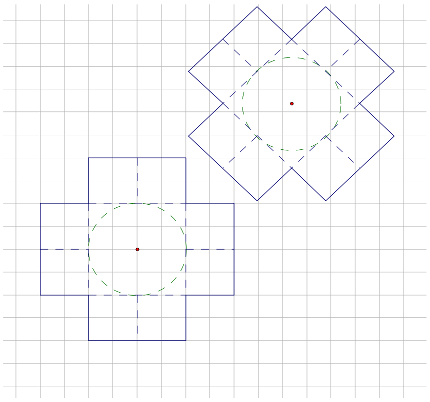


Translation + Dilation

- What are invariant?
  - Ratio of Lengths
  - Angles
- Which transformation group?
  - similarity
- det[T]=1



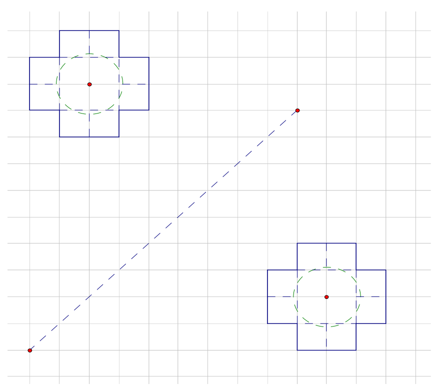
Translation + Rotation



- What are the invariants?
- Lengths
- Angles
- Which transformation group?
- Congruent
- det[T]=1

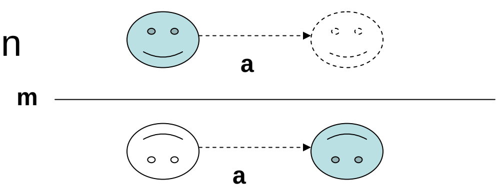
Reflection

- What are the invariants?
- Lengths
- Angles
- Which transformation group?
- Congruent
- det[T] = -1

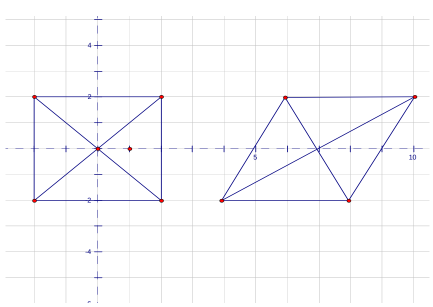


Reflection+Translation=Glide

- What are the invariants?
- Lengths
- Angles
- Which transformation group?
- Congruent
- det[T] = -1
- Glide= $R_m+T_a=T_a+R_m$



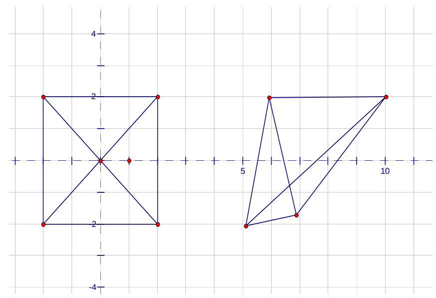
Shear Transformation



- What are invariant?
- parallelism
- Which transformation group?
- affine

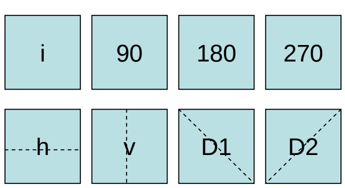
Projective Transformation

- What are invariant?
- Incidence
- Cross ratios of lengths
- Order of curves



Isometries of a Square

- How many isometries?
- identity
- Rotational
- Reflection



Cayle Table of isometries of a square

	i	90	180	270	H	V	D1	D2
I	i	90	180	270	H	V	D1	D2
90	90	180	270	i	D1	D2	V	H
180	180	270	i	90	V	h	D2	D1
270	270	i	90	180	D2	D1	H	V
H	H	D2	V	D1	i	180	270	90
V	V	D1	H	D2	180	i	90	270
D1	D1	H	D2	V	90	270	i	180
D2	D2	V	D1	H	270	90	180	i

Transformations Possible in different Geometries

Transformation / Geometries	Euclidian	Similarity	Affine	Projective
Rotation	✓	✓	✓	✓
Translation	✓	✓	✓	✓
Uniform Scaling	✗	✓	✓	✓
Non-uniform Scaling	✗	✗	✓	✓
Shear	✗	✗	✓	✓
Central Projection	✗	✗	✗	✓

Invariant Quantities in different Geometries

Transformation / Geometries	Euclidian	Similarity	Affine	Projective
Lengths	✓	✗	✗	✗
Angles	✓	✓	✗	✗
Ratios of Lengths	✓	✓	✗	✗
Parallelism	✓	✓	✓	✗
Incidence	✓	✓	✓	✓
Cross-ratios of Lengths	✓	✓	✓	✓