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# MATHEMATICAL HANDBOOK

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Containing the Chief Formulas of Algebra, Trigonometry, Circular  
and Hyperbolic Functions, Differential and Integral Calculus, and  
Analytical Geometry, Together With Mathematical Tables

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by  
Edwin P. Seaver

*Forgotten Books*



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Functions, Differential and Integral Calculus,  
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*by*

**Edwin P. Seaver**

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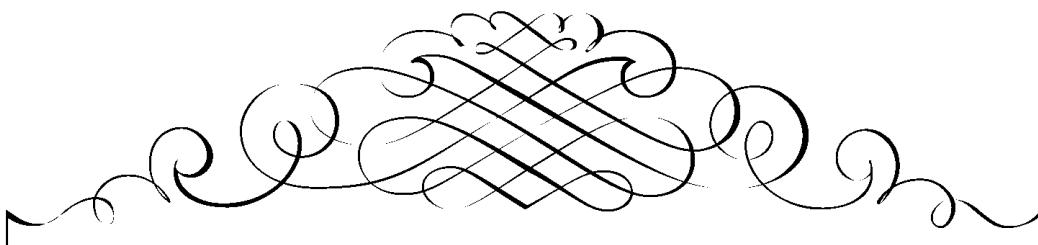
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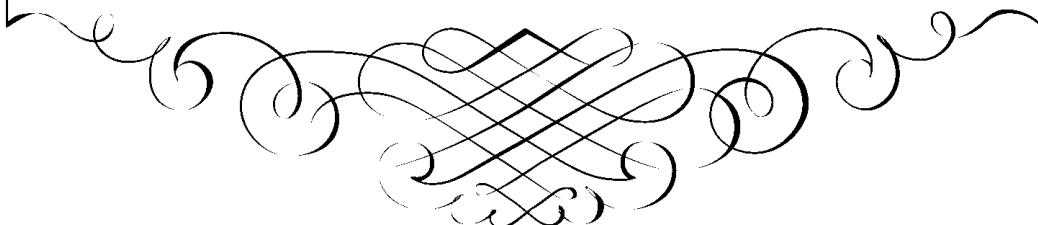


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*"In changing the base metals into gold and silver by the projection of the Stone, it follows (by an accelerated process) the method of nature, and therefore is natural."*

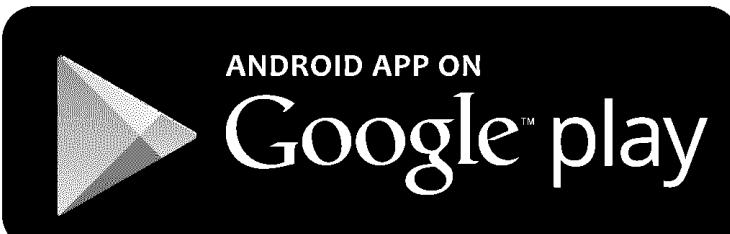
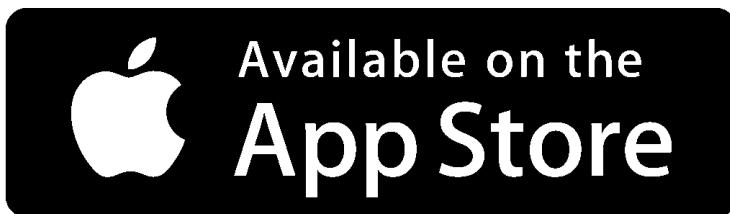
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# MATHEMATICAL HANDBOOK

CONTAINING

THE CHIEF FORMULAS OF ALGEBRA, TRIGONOMETRY,  
CIRCULAR AND HYPERBOLIC FUNCTIONS,  
DIFFERENTIAL AND INTEGRAL  
CALCULUS, AND ANALYTIC  
GEOMETRY

TOGETHER WITH

*MATHEMATICAL TABLES*

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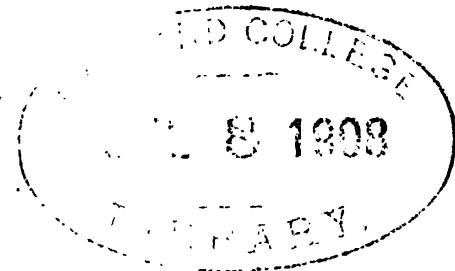
EDWIN P. SEAVER, A.M., LL.B.

*FORMERLY ASSISTANT PROFESSOR OF MATHEMATICS IN  
HARVARD UNIVERSITY*



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65

## PREFACE.

THE uses which this book may serve hardly need to be pointed out. Some years ago the writer composed the part relating to Trigonometry and used it as a syllabus for instruction in his college classes. It served its purpose and soon went out of print. But a stray copy of it found its way to the table of a well-known civil engineer, to whom it proved constantly useful, and by whom it was often referred to as "his memory." This engineer has suggested a revision and republication of the original book with important enlargements. Accordingly there have been added Sections on Algebra, the Differential and Integral Calculus, and Analytic Geometry. The subject of Hyperbolic Functions, which now receives much more attention than formerly, has been more fully treated. Tables have been added, which include not only those universally used, but also some — like those of the Hyperbolic Functions, of the Natural Logarithms of Numbers, and that of the Velocity of Falling Bodies ( $v = 2\sqrt{gh}$ ) — that have been hitherto not readily accessible.

Of course no efforts have been spared to secure correctness in the printing of the formulas and the tables; but persons experienced in such work need not be reminded of the improbability that the first edition of a book of this kind should be absolutely free from error. The writer and the publishers can only add, that notice of any errors that may be detected will be thankfully received, and the necessary corrections will be promptly made and published. Also, suggestions of desirable additions to the book and of other improvements are invited with a view to their use in possible future editions.

E. P. S.

JUNE, 1907.



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# SECTION I.

## ALGEBRA.

---

### The General Laws of Common Algebra.

#### 1. *The Law of Association.*

$$a + b + c = a + (b + c),$$

$$a + b - c = a + (b - c),$$

$$a - b + c = a - (b - c),$$

$$a - b - c = a - (b + c),$$

$$abc = a(bc) = (ab)c,$$

$$a \times b \div c = a \times (b \div c),$$

$$a \div b \div c = a \div (b \times c),$$

$$a \div b \times c = a \div (b \div c),$$

wherein the concurrence of *like* signs gives the *direct* sign + or  $\times$ ; and the concurrence of *unlike* signs gives the *indirect* sign - or  $\div$ . Thus,

$$\begin{array}{ll} + (+c) = +c, & \times (\times c) = \times c, \\ - (-c) = +c, & \div (\div c) = \times c, \\ + (-c) = -c, & \times (\div c) = \div c, \\ - (+c) = -c, & \div (\times c) = \div c. \end{array}$$

#### 2. *The Law of Commutation.*

$$\begin{aligned} a + b &= b + a, \\ a - b &= -b + a, \end{aligned}$$

$$ab = ba,$$

$$a \times b \times c = a \times c \times b,$$

$$a \times b \div c = a \div c \times b,$$

$$a \div b \div c = a \div c \div b,$$

$$a \div b \times c = a \times c \div b.$$

**3. The Law of Distribution.**

For multiplication,

$$a(b + c) = ab + ac,$$

$$(\pm a \pm b) \times (\pm c \pm d) = + (\pm a) \times (\pm c) + (\pm a) \times (\pm d) \\ + (\pm b) \times (\pm c) + (\pm b) \times (\pm d) = \pm ac \pm ad \pm bc \pm bd,$$

wherein the signs of each partial product are determined by the following rule:

If a partial product has factors with like signs, it must have the sign +; if factors with unlike signs, it must have the sign -. Thus,

$$+ (+a) \times (+c) = + ac, \quad + (+a) \times (-c) = - ac, \\ + (-a) \times (-c) = + ac, \quad + (-a) \times (+c) = - ac.$$

For division,

$$(\pm a \pm b) \div (\pm c) = + (\pm a) \div (\pm c) + (\pm b) \div (\pm c),$$

with the following rule for signs:

If the dividend and divisor of a partial quotient have like signs, the partial quotient must have the sign +; if they have unlike signs, it must have the sign -. Thus,

$$+ (+a) \div (+c) = + (a \div c), \quad + (+a) \div (-c) = - (a \div c), \\ + (-a) \div (-c) = + (a \div c), \quad + (-a) \div (+c) = - (a \div c).$$

Otherwise expressed, this law is

$$\frac{\pm a \pm b}{\pm c} = + \frac{\pm a}{\pm c} + \frac{\pm b}{\pm c},$$

with the same rule for signs; that is,

$$+ \left( \frac{+a}{+c} \right) = + \frac{a}{c}, \quad + \left( \frac{+a}{-c} \right) = - \frac{a}{c},$$

$$+ \left( \frac{-a}{-c} \right) = + \frac{a}{c}, \quad + \left( \frac{-a}{+c} \right) = - \frac{a}{c}.$$

The divisor cannot be distributed.

## Definitions and Laws of the Symbols

$0, 1$ , and  $\infty$ .

4.  $0 = +a - a = -a + a, \quad 1 = \times a \div a = \div a \times a,$   
 $\pm b + 0 = \pm b - 0, \quad \frac{\times}{\div} a \times 1 = \frac{\times}{\div} a \div 1,$   
 $+0 = -0. \quad \times 1 = \div 1.$
- $0 \times (\pm b) = (\pm b) \times 0 = 0, \quad \infty \times (\pm b) = (\pm b) \times \infty = \pm \infty,$   
 $0 \div (\pm b) = 0, \quad \infty \div (\pm b) = \pm \infty,$   
 $+b \div 0 = +\infty, \quad \left. \begin{array}{l} \\ \end{array} \right\} \quad (\pm b) \div (\pm \infty) = 0.$   
 $-b \div 0 = -\infty.$

5. Using  $A$  and  $B$  to represent any two algebraic expressions of quantity,

- If  $A \times B = 0$ , either  $A = 0$  or  $B = 0$ ,  
or both  $A = 0$  and  $B = 0$ .
- If  $A \div B = 0$  and  $B$  is not 0, then  $A = 0$ .
- If  $A \div B = 0$  and  $A$  is not 0, then  $B = \infty$ .

The forms  $0 \times \infty$ ,  $0 \div 0$  or  $\frac{0}{0}$ ,  $\infty \div \infty$  or  $\frac{\infty}{\infty}$ , and  $\infty - \infty$  require special investigations to determine their values in the particular cases in which they arise. See pages 109, 110.

## Fractions and Ratios.

6. Equivalent forms of notation,

$$a \div b = \frac{a}{b} = a : b = a/b.$$

7. Addition of fractions,

$$\frac{a}{b} + \frac{c}{d} = \frac{ad + bc}{bd}.$$

8. Subtraction of fractions,

$$\frac{a}{b} - \frac{c}{d} = \frac{ad - bc}{bd}.$$

9. Multiplication of fractions,

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}.$$

## 10. Division of fractions,

$$\left. \begin{aligned} \frac{a}{b} \div \frac{c}{d} &= \frac{a}{b} \times \frac{d}{c} \\ &= \frac{a \div c}{b \div d} \end{aligned} \right\} = \frac{ad}{bc}.$$

## Proportions.

11. If  $a : b = c : d$ , then  $ad = bc$ .12. If  $ad = bc$ , then  $a : b = c : d$ ,  
 $b : a = d : c$ ,  
 $a : c = b : d$ ,  
 $c : a = d : b$ ;

also,

$$\begin{aligned} a \pm c : b \pm d &= a : b = c : d, \\ a + c : a - c &= b + d : b - d, \\ a + b : a - b &= c + d : c - d, \\ a \pm b : a &= c \pm d : c, \\ a \pm b : b &= c \pm d : d; \end{aligned}$$

also,

$$\begin{aligned} ma : mb &= nc : nd, \\ \frac{a}{m} : \frac{b}{m} &= \frac{c}{n} : \frac{d}{n}, \\ ma : nb &= mc : nd, \\ \frac{a}{m} : \frac{b}{n} &= \frac{c}{m} : \frac{d}{n}. \end{aligned}$$

13. If  $a : b = c : d$ , and  $p : q = r : s$ ,  
then  $ap : bq = cr : ds$ .14. If  $a : b = c : d$ ,  
then  $a^2 : b^2 = c^2 : d^2$ ,  
 $a^n : b^n = c^n : d^n$ ,  
 $\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{c} : \sqrt[n]{d}$ .15. If  $a : b = c : x$ , then  $x = \frac{bc}{a}$ .16. If  $a : b = x : d$ , then  $x = \frac{ad}{b}$ .

17. If

$$\frac{a}{A} = \frac{b}{B} = \frac{c}{C} = \dots$$

then

$$\begin{aligned}\frac{a}{A} &= \frac{a + b + c + \dots}{A + B + C + \dots}, \\ &= \frac{pa + qb + rc + \dots}{pA + qB + rC + \dots}.\end{aligned}$$

18. If  $a : b = b : c$ , then  $b = \sqrt{ac}$ , one geometric mean between  $a$  and  $c$ .

19. If  $a : b = b : c = c : d$ , then  $b = \sqrt[3]{a^2d}$  and  $c = \sqrt[3]{ad^2}$ , two geometric means between  $a$  and  $d$ .

20. The reciprocal of  $a$  is  $\frac{1}{a} = a^{-1}$ ,

of  $\frac{a}{b}$  is  $\frac{b}{a} = \left(\frac{a}{b}\right)^{-1}$ ,

of  $\frac{1}{a}$  is  $a = \left(\frac{1}{a}\right)^{-1}$ .

21. If  $a : b = \frac{1}{p} : \frac{1}{q}$ , then  $p$  and  $q$  are *inversely or reciprocally proportional* to  $a$  and  $b$ ; and the proportion may be written

$$a : b = q : p,$$

or  $a : b = p^{-1} : q^{-1}$ .

22. If  $x$  varies as  $y$  *directly*, then

$$x_1 : x_2 = y_1 : y_2$$

wherein  $x_1, y_1$  and  $x_2, y_2$  denote simultaneous or corresponding values of the variables  $x$  and  $y$ .

23. If  $x$  varies as  $y$  *inversely*, then

$$x_1 : x_2 = \frac{1}{y_1} : \frac{1}{y_2}$$

or

$$x_1 : x_2 = y_2 : y_1.$$

For example, the force of gravitation,  $g$ , varies inversely as the square of the distance,  $d^2$ , that is

$$g_1 : g_2 = \frac{1}{d_1^2} : \frac{1}{d_2^2} = d_2^2 : d_1^2.$$

## Powers.

24.  $(+a)^n = +a^n$ .      31.  $a^m \times a^{-n} = a^m \div a^n$ .  
 25.  $(-a)^{2n} = +a^{2n}$ .      32.  $a^m \div a^{-n} = a^m \times a^n$ .  
 26.  $(-a)^{2n+1} = -a^{2n+1}$ .      33.  $a^0 \div a^{-n} = a^n = \frac{1}{a^{-n}}$ .  
 27.  $a^m \times a^n = a^{m+n}$ .      34.  $(a^m)^n = a^{mn} = (a^n)^m$ .  
 28.  $a^m \div a^n = a^{m-n}$ .      35.  $(ab)^m = a^m b^m$ .  
 29.  $a^m \div a^m = a^0 = 1$ .  
 30.  $a^0 \div a^n = a^{-n} = \frac{1}{a^n}$ .      36.  $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$ .  
 37.  $\left(\frac{a}{b}\right)^{-n} = \frac{a^{-n}}{b^{-n}} = \frac{b^n}{a^n} = \left(\frac{b}{a}\right)^n$ .  
 38.  $0^n = 0$ .      41.  $\infty^{-n} = 0$ .  
 39.  $0^{-n} = \infty$ .      42.  $a^{+0} = 1$ .  
 40.  $\infty^n = \infty$ .      43.  $a^{-0} = 1$ .  
 44. If  $a > 1$ , then  $a^\infty = \infty$ , and  $a^{-\infty} = 0$ .  
 45. If  $a < 1$ , then  $a^\infty = 0$ , and  $a^{-\infty} = \infty$ .  
 46.  $\log 0 = -\infty$ .  
 47.  $\log 1 = 0$ .  
 48.  $\log \infty = \infty$ .

The forms  $0^0$ ,  $1^\infty$ ,  $\infty^0$  require special investigation. See page 109.

## Products and Factors.

49.  $a^2 - b^2 = (a - b)(a + b)$ .  
 50.  $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ .  
 51.  $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$ .  
 52.  $a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \dots + b^{n-1})$ , always.  
 53.  $a^n - b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \dots - b^{n-1})$ , if  $n$  be even.  
 54.  $a^n + b^n = (a + b)(a^{n-1} - a^{n-2}b + a^{n-3}b^2 - \dots + b^{n-1})$ , if  $n$  be odd.  
 55.  $(x + a)(x + b) = x^2 + (a + b)x + ab$ .

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$$71. bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + a^3 + b^3 + c^3 \\ = (a + b + c)(a^2 + b^2 + c^2).$$

$$72. bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + 3abc \\ = (a + b + c)(bc + ca + ab).$$

$$73. bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b + 2abc \\ = (b + c)(c + a)(a + b).$$

$$74. bc^2 + b^2c + ca^2 + c^2a + ab^2 + a^2b - 2abc - a^3 - b^3 - c^3 \\ = (b + c - a)(c + a - b)(a + b - c).$$

$$75. bc^2 - b^2c + ca^2 - c^2a + ab^2 - a^2b = (b - c)(c - a)(a - b).$$

$$76. 2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4 \\ = (a + b + c)(b + c - a)(c + a - b)(a + b - c).$$

$$77. a^3 + 2a^2b + 2ab^2 + b^3 = (a + b)(a^2 + ab + b^2).$$

### The Binomial Theorem.

$$78. (a + b)^n = \\ a^n + \frac{n}{1} a^{n-1} b + \frac{n(n-1)}{1 \times 2} a^{n-2} b^2 + \frac{n(n-1)(n-2)}{1 \times 2 \times 3} a^{n-3} b^3 + \dots$$

wherein  $n$  may be positive or negative, integral or fractional. When  $n$  is a positive integer, the right hand member has  $n + 1$  terms; when  $n$  is negative or fractional, the number of terms is infinite.

79. The general expression for the  $(r+1)^{\text{th}}$  term is

$$\frac{n(n-1)(n-2)(n-3)\dots(n-r+1)}{1 \times 2 \times 3 \times \dots r} a^{n-r} b^r,$$

$$\text{or } \frac{n(n-1)(n-2)\dots(3 \times 2 \times 1)}{1 \times 2 \times 3 \times \dots r \times (n-r)(n-r-1) \times \dots 2 \times 1} a^{n-r} b^r,$$

or, using the factorial notation,

$$\frac{n!}{r!(n-r)!} a^{n-r} b^r;$$

and the formula may be written

$$(a + b)^n = \sum_{r=0}^{r=n} \frac{n!}{r!(n-r)!} a^{n-r} b^r. \quad [\text{N. B. } 0! = 1.]$$

**80.** The coefficients of the several terms in the expansion of the  $n^{\text{th}}$  power of a binomial are conveniently designated by  $C_0, C_1, C_2, \dots$ , etc. These are functions of  $n$  as follows:

$$C_0 = n^0 = 1,$$

$$C_1 = \frac{n}{1}, \quad \text{and in general}$$

$$C_2 = \frac{n(n-1)}{1 \times 2}, \quad C_r = \frac{n(n-1)(n-2) \dots (n-r+1)}{1 \times 2 \times 3 \times \dots r}$$

$$C_3 = \frac{n(n-1)(n-2)}{1 \times 2 \times 3}, \quad = \frac{n!}{r!(n-r)!}.$$

$$C_4 = \frac{n(n-1)(n-2)(n-3)}{1 \times 2 \times 3 \times 4}.$$

Then

$$81. (a+b)^n = C_0 a^n + C_1 a^{n-1} b + C_2 a^{n-2} b^2 + C_3 a^{n-3} b^3 + \dots$$

Also,

$$82. (a-b)^n = C_0 a^n - C_1 a^{n-1} b + C_2 a^{n-2} b^2 - C_3 a^{n-3} b^3 + \dots$$

The numerical values of  $C_1, C_2, C_3, \dots$  for each power of the binomial from the first ( $n=1$ ) to the twentieth ( $n=20$ ) power may be found in a table on page 244.

The numerical values of factorials from  $n=0$  to  $n=20$  may be found in a table on the same page.

### Inequalities.

**83.** The value of the fraction

$$\frac{a_1 + a_2 + a_3 + \dots + a_n}{b_1 + b_2 + b_3 + \dots + b_n}$$

is less than the greatest and greater than the least of the fractions  $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots, \frac{a_n}{b_n}$  provided the denominators of the latter are all positive.

**84.** The arithmetical mean of two numbers is greater than the geometrical, and the geometrical is greater than the harmonical. That is,

$$\frac{a+b}{2} > \sqrt{ab} > \frac{2ab}{a+b}.$$

Also,

$$85. \quad \frac{a_1 + a_2 + \dots + a_n}{n} > \sqrt[n]{a_1 a_2 \dots a_n}.$$

The arithmetical mean of the powers is greater than the power of the arithmetical mean, that is,

$$86. \quad \frac{a^m + b^m}{2} > \left(\frac{a+b}{2}\right)^m,$$

and, in general,

$$87. \quad \frac{a_1^m + a_2^m + \dots + a_n^m}{n} > \left(\frac{a_1 + a_2 + \dots + a_n}{n}\right)^m,$$

excepting when  $m$  is a positive proper fraction.

**88.** If  $a, b, c$ , be positive quantities,

$$a^a b^b > \left(\frac{a+b}{2}\right)^{a+b},$$

$$a^a b^b c^c > \left(\frac{a+b+c}{3}\right)^{a+b+c}.$$

**89.** If  $m > n > a$ ,

$$\left(\frac{m+a}{m-a}\right)^m < \left(\frac{n+a}{n-a}\right)^n.$$

### Roots.

$$90. \quad a^{\frac{1}{m}} = \sqrt[m]{a}.$$

$$91. \quad \sqrt[m]{a} \times \sqrt[n]{a} = a^{\frac{1}{m}} \times a^{\frac{1}{n}} = a^{\frac{m+n}{mn}} = \sqrt[mn]{a^{m+n}}.$$

$$92. \quad \sqrt[n]{a} \div \sqrt[m]{a} = a^{\frac{1}{n}} \div a^{\frac{1}{m}} = a^{\frac{m-n}{mn}} = \sqrt[mn]{a^{m-n}}.$$

$$93. \quad \left(a^{\frac{1}{m}}\right)^n = a^{\frac{n}{m}} = \left(\sqrt[m]{a}\right)^n = \sqrt[m]{a^n}.$$

$$94. \quad \left(a^{\frac{1}{m}}\right)^m = \left(\sqrt[m]{a}\right)^m \quad \sqrt[m]{a^m} = a.$$

$$95. \left( a^m \right)^{\frac{1}{n}} = a^{\frac{1}{mn}} = \sqrt[n]{\sqrt[m]{a}} = \sqrt[m]{\sqrt[n]{a}} = \sqrt[mn]{a}.$$

$$96. \sqrt[n]{a^{mn}} = a^{\frac{mn}{n}} = a^m.$$

$$97. \sqrt[n]{a^m} = \sqrt[kn]{a^{km}} = \sqrt[k]{a^{\frac{m}{k}}} = a^{\frac{km}{kn}} = a^{\frac{m}{n}}.$$

$$98. \sqrt[m]{ab} = \sqrt[m]{a} \times \sqrt[m]{b}.$$

$$99. (ab)^{\frac{1}{m}} = a^{\frac{1}{m}} b^{\frac{1}{m}}.$$

$$100. \sqrt[m]{\left(\frac{a}{b}\right)} = \frac{\sqrt[m]{a}}{\sqrt[m]{b}}.$$

$$101. \left(\frac{a}{b}\right)^{\frac{1}{m}} = \frac{a^{\frac{1}{m}}}{b^{\frac{1}{m}}}.$$

$$102. \sqrt[m]{\frac{1}{a}} = \frac{1}{\sqrt[m]{a}} = a^{-\frac{1}{m}}.$$

Let  $A$  represent a positive number, and  $a$  the arithmetical value of its indicated root. Then,

$$103. \begin{cases} \sqrt[2n]{+A} = \pm a, \\ \sqrt[2n]{-A} = \pm ia, \\ \text{wherein } i = \sqrt{-1}. \end{cases} \quad \begin{cases} \sqrt[2n+1]{+A} = +a, \\ \sqrt[2n+1]{-A} = -a, \end{cases}$$

### Surds.

104. If a number partly rational and partly surd is equal to another number also partly rational and partly surd, the two rational parts are equal and the two surd parts are equal. Thus if

$$a + \sqrt[n]{b} = x + \sqrt[n]{y},$$

wherein  $a$  and  $x$  are rational, and  $\sqrt[n]{b}$  and  $\sqrt[n]{y}$  are surds, then

$$a = x, \text{ and } b = y.$$

$$105. \text{ If } \sqrt{a + \sqrt{b}} = \sqrt{x} + \sqrt{y},$$

$$\text{then } \sqrt{a - \sqrt{b}} = \sqrt{x} - \sqrt{y}.$$

$$106. \sqrt{a + \sqrt{b}} = \sqrt{a + b + 2\sqrt{ab}}.$$

$$107. \sqrt{a - \sqrt{b}} = \sqrt{a + b - 2\sqrt{ab}}.$$

$$108. (a \pm \sqrt{b})^2 = a^2 + b \pm 2a\sqrt{b}.$$

$$109. (a + \sqrt{b})(a - \sqrt{b}) = a^2 - b.$$

**The imaginary unit,  $i$ , and its powers.**

**110.** By definition,

$$i = + \sqrt{-1}, \quad i^2 = -1, \quad i^3 = -\sqrt{-1}, \quad i^4 = +1.$$

**111.** Then,

$$\begin{aligned} i^5 &= i^9 = i^{4n+1} = i; \\ i^6 &= i^{10} = i^{4n+2} = i^2 = -1; \end{aligned}$$

**112.** Also,

$$\frac{1}{i} = i^{-1} = i^3 = -i;$$

$$\begin{aligned} i^7 &= i^{11} = i^{4n+3} = i^3 = -i; \\ i^8 &= i^{12} = i^{4n} = i^4 = +1. \end{aligned}$$

$$i^0 = +1.$$

**Complex Numbers.**

**113.** A complex number is a collection of units partly real and partly imaginary. In its simplest form it is written  $a \pm bi$  or  $a \pm ib$ , wherein  $a$  denotes the number of real units and  $b$  the number of imaginary units in the collection. Both  $a$  and  $b$  are real coefficients, the first of 1 the second of  $i$ .

**114.** If two complex numbers are equal their real parts are equal and their imaginary parts are equal.

Thus if  $A + iB = a + ib$ , then  $A = a$  and  $B = b$ .

**115.** The two complex numbers  $a + ib$  and  $a - ib$  are *conjugates* the one of the other, and

$$(a + ib)(a - ib) = a^2 + b^2.$$

**Product and quotient of two complex numbers.**

**116.**  $(a + ib)(c + id) = ac - bd + i(bc + ad).$

**117.**  $\frac{a + ib}{c + id} = \frac{ac + bd}{c^2 + d^2} + i \frac{(bc - ad)}{c^2 + d^2}.$

118. Every complex number can be brought into the form

$$a \pm ib = r(\cos \theta \pm i \sin \theta)$$

wherein  $r = \sqrt{a^2 + b^2}$  = the *modulus*,

$$\theta = \tan^{-1} \frac{b}{a} = \cos^{-1} \frac{a}{r} = \sin^{-1} \frac{b}{r} = \text{the argument.}$$

119. The product of two complex numbers is found by multiplying their moduli and adding their arguments. Thus,

$$r_1(\cos \theta_1 \pm i \sin \theta_1) \times r_2(\cos \theta_2 \pm i \sin \theta_2) = \\ r_1 r_2 [\cos(\theta_1 + \theta_2) \pm i \sin(\theta_1 + \theta_2)]$$

120. The quotient is found by dividing one modulus by the other and subtracting the argument of the divisor from that of the dividend. Thus,

$$\frac{r_1(\cos \theta_1 \pm i \sin \theta_1)}{r_2(\cos \theta_2 \pm i \sin \theta_2)} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) \pm i \sin(\theta_1 - \theta_2)].$$

121. Powers of a complex number.

$$[r(\cos \theta + i \sin \theta)]^m = r^m (\cos m\theta + i \sin m\theta).$$

122. Roots of a complex number.

$$\left[ r(\cos \theta + i \sin \theta) \right]^{\frac{m}{n}} = r^{\frac{m}{n}} \left[ \cos \frac{m\theta + 2k\pi}{n} + i \sin \frac{m\theta + 2k\pi}{n} \right].$$

Relations of conjugates.

123.  $(a + ib)(a - ib) = r^2.$

124.  $\begin{cases} a + ib = r(\cos \theta + i \sin \theta) = re^{i\theta}. \\ a - ib = r(\cos \theta - i \sin \theta) = re^{-i\theta}. \end{cases}$

125.  $(\cos \theta + i \sin \theta)(\cos \theta - i \sin \theta) = 1.$

Roots of 1 and of  $-1$ .

126.  $\sqrt[n]{1} = \cos \frac{2k\pi}{n} \pm i \sin \frac{2k\pi}{n}.$

127.  $\sqrt[n]{-1} = \cos \frac{(2k+1)\pi}{n} \pm i \sin \frac{(2k+1)\pi}{n}.$

## Logarithms.

The relation between a number,  $x$ , and its logarithm,  $u$ , is expressed by the equations

$$128. \quad x = a^u, \quad u = \log_a x$$

wherein  $a$  is the base of the system of logarithms intended.

The relations between logarithms of the same number in systems having different bases are thus shown.

If  $x = a^u$ , then  $u = \log_a x$ ;

and if  $x = b^v$ , then  $v = \log_b x$ .

Whence  $\begin{aligned} 1 &= a^u \div b^v, \\ 0 &= u - v \log_a b, \\ 0 &= u \log_b a - v, \end{aligned}$

$$129. \quad \log_a x = \log_b x \times \log_a b,$$

$$130. \quad \log_b x = \log_a x \times \log_b a,$$

$$131. \quad \log_a b \times \log_b a = 1.$$

The two systems of logarithms most in use are the Natural System, founded upon the Exponential Base,  $e$ , and the Common System, founded upon the base 10. Logarithms of the former system are often called hyperbolic logarithms or Naperian logarithms, and those of the latter Briggs' logarithms or denary logarithms.

Writing 10 for  $a$  and  $e$  for  $b$ , the foregoing equations become

$$132. \quad \log_{10} x = \log_e x \times \log_{10} e,$$

$$133. \quad \log_e x = \log_{10} x \times \log_e 10,$$

$$134. \quad \log_{10} e \times \log_e 10 = 1.$$

The Exponential Base,  $e$ , is the limit of  $\left(1 + \frac{1}{m}\right)^m$  as  $m$  approaches  $\infty$ .

$$135. \quad e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \dots$$

$$= 2.718\ 281\ 828\ 459\dots$$

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Imaginary numbers have imaginary logarithms,

$$145. \log i = \frac{1}{2}\pi i, \quad 146. i^i = e^{-\frac{1}{2}\pi} = 0.20788\dots$$

Rules for the practical use of logarithms are based on the following principles:

$$147. \log(xy) = \log x + \log y.$$

$$148. \log\left(\frac{x}{y}\right) = \log x - \log y.$$

$$149. \log(x^n) = n \log x.$$

$$150. \log \sqrt[n]{x} = \frac{1}{n} \log x.$$

$$151. \log \text{base} = 1, \log 1 = 0, \log 0 = -\infty.$$

152.

$$\left\{ \begin{array}{ll} \text{If} & 1 < x < +\infty, \\ \text{then,} & 0 < \log x < +\infty. \\ \text{If} & 0 < x < +1, \\ \text{then,} & -\infty < \log x < 0. \end{array} \right\} \dots \left\{ \begin{array}{l} \text{That is, if } x \text{ is positive} \\ \text{and greater than 1, its} \\ \text{logarithm is positive;} \\ \text{if} \\ \text{positive and less than 1,} \\ \text{its logarithm is negative.} \end{array} \right\}$$

### Permutations and Combinations.

153. The number of *permutations* (sometimes called arrangements) of  $n$  things taken all at a time is

$$n(n-1)(n-2)\dots 2 \times 1, \text{ or } n!$$

154. The number of permutations of  $n$  things taken  $r$  at a time may be denoted by the symbol  $P(n, r)$ .

$$P(n, r) = n(n-1)(n-2)\dots \text{to } r \text{ factors,}$$

$$= \frac{n!}{(n-r)!}.$$

155. The number of *combinations* of  $n$  things taken  $r$  at a time may be denoted by the symbol  $C(n, r)$ .

$$C(n, r) = \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \times 2 \times 3 \dots r} = \frac{P(n, r)}{r!}$$

$$= \frac{n!}{r!(n-r)!} = C(n, n-r).$$

Comparing 155 with 79 it may be seen that

156.  $C(n, r) =$  the binomial coefficient of the  $(r + 1)^{\text{th}}$  term of the development of  $(a + b)^n$ .

Numerical values of  $C(n, r)$  up to  $n = 20$  are found in the table on page 244.

### Determinants.

If there be  $n^2$  quantities whose symbols are arrayed in the form of a square of  $n$  rows and  $n$  columns, this array is the symbol of a determinant. The  $n^2$  quantities forming the array are the elements of the determinant. The determinant itself is the algebraic sum of all the products that can be formed of  $n$  elements taken one from each column and each row in all possible ways, one half of these products being written with the positive sign, the other half with the negative.

157. An array of four elements, the symbol of a determinant of the second order, gives  $2!(=2)$  terms, thus:

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1b_2 - a_2b_1.$$

158. An array of nine elements, the symbol of a determinant of the third order, gives  $3!(=6)$  terms, thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1b_2c_3 - a_1b_3c_2 + a_2b_3c_1 - a_2b_1c_3 + a_3b_1c_2 - a_3b_2c_1.$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix}$$

NOTE.— If the determinant array of the third order be written with the first two rows repeated as shown in the margin, then the positive terms of its development can be found by reading the three diagonal rows from the left downwards, and the negative terms by reading the three diagonal rows from the left upwards.

**159.** An array of sixteen elements, the symbol of a determinant of the fourth order, gives  $4! (=24)$  terms, thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 b_2 c_3 d_4 - a_1 b_2 c_4 d_3 + a_3 b_1 c_2 d_4 - a_3 b_1 c_4 d_2 \\ + a_1 b_3 c_4 d_2 - a_1 b_3 c_2 d_4 + a_3 b_2 c_4 d_1 - a_3 b_2 c_1 d_4 \\ + a_1 b_4 c_2 d_3 - a_1 b_4 c_3 d_2 + a_3 b_4 c_1 d_2 - a_3 b_4 c_2 d_1 \\ - a_2 b_1 c_3 d_4 + a_2 b_1 c_4 d_3 - a_4 b_1 c_2 d_3 + a_4 b_1 c_3 d_2 \\ - a_2 b_3 c_4 d_1 + a_2 b_3 c_1 d_4 - a_4 b_2 c_3 d_1 + a_4 b_2 c_1 d_3 \\ - a_2 b_4 c_1 d_3 + a_2 b_4 c_3 d_1 - a_4 b_3 c_1 d_2 + a_4 b_3 c_2 d_1$$

An array of twenty-five elements, the symbol of a determinant of the fifth order, gives  $5! (=120)$  terms. In general, a determinant of the  $n^{\text{th}}$  order consists of  $n!$  terms.

**160.** If the row and the column in which a given element stands be stricken out, the determinant formed of the remaining elements is the *minor determinant* relative to the given element.

If each element in one column of the major determinant be multiplied by its relative minor determinant and the positive sign be given to each element taken from an odd numbered row and the negative sign to each element taken from an even numbered row, the algebraic sum of the results is equal to the major determinant. Thus,

**161.**

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix} \\ = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1).$$

**162.**

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}$$

Thus can a determinant of any order, the  $n^{\text{th}}$ , be made to depend on  $n$  determinants of the  $(n - 1)^{\text{th}}$  order, and

each of these again on  $n - 1$  determinants of the  $(n - 2)^{\text{th}}$  order, and so on, the ultimate result being that the original determinant depends on a series of determinants of the third or of the second order, which last are easily computed directly. This method of reduction makes easy the computation of the value of any determinant with numerical elements.

**163.** In any determinant the columns can be made rows and the rows columns without changing its value. Thus,

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}; \quad \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}.$$

**164.** If, in any determinant, two columns or two rows change places with each other, the new determinant so formed is equal to the first one with the opposite sign.

Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = - \begin{vmatrix} b_1 & a_1 & c_1 \\ b_2 & a_2 & c_2 \\ b_3 & a_3 & c_3 \end{vmatrix} = - \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix} = \text{etc.}$$

**165.** If the elements in two columns or in two rows are equal or proportional each to each, the value of the determinant is 0.

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0; \quad \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = 0; \quad \begin{vmatrix} a_1 & na_1 & b_1 \\ a_2 & na_2 & b_2 \\ a_3 & na_3 & b_3 \end{vmatrix} = n \begin{vmatrix} a_1 & a_1 & b_1 \\ a_2 & a_2 & b_2 \\ a_3 & a_3 & b_3 \end{vmatrix} = n0 = 0.$$

**166.** A determinant is multiplied or divided by a number by multiplying or dividing all the elements in one column or in one row by that number.

$$p \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} pa_1 & pb_1 & pc_1 \\ pa_2 & pb_2 & pc_2 \\ pa_3 & pb_3 & pc_3 \end{vmatrix} = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

**167.** A determinant can be split into two or more determinants. Thus,

$$\begin{vmatrix} a_1 + p_1 + q_1 & b_1 & c_1 \\ a_2 + p_2 + q_2 & b_2 & c_2 \\ a_3 + p_3 + q_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} p_1 & b_1 & c_1 \\ p_2 & b_2 & c_2 \\ p_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} q_1 & b_1 & c_1 \\ q_2 & b_2 & c_2 \\ q_3 & b_3 & c_3 \end{vmatrix}.$$

**168.** A determinant is not changed in value when the elements of one column or row are each increased or diminished by  $n$  times the corresponding elements of a parallel column or row. Thus,

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 \pm nb_1 & b_1 & c_1 \\ a_2 \pm nb_2 & b_2 & c_2 \\ a_3 \pm nb_3 & b_3 & c_3 \end{vmatrix}.$$

*The Solution of Equations of the First Degree by Determinants.*

**169.** The solution of

$$\left. \begin{array}{l} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{array} \right\} \text{ is } \begin{cases} x = \frac{|k_1 b_1|}{|k_2 b_2|} \div \frac{|a_1 b_1|}{|a_2 b_2|} \\ y = \frac{|a_1 k_1|}{|a_2 k_2|} \div \frac{|a_1 b_1|}{|a_2 b_2|} \end{cases}$$

**170.** The solution of

$$\left. \begin{array}{l} a_1x + b_1y + c_1z = k_1 \\ a_2x + b_2y + c_2z = k_2 \\ a_3x + b_3y + c_3z = k_3 \end{array} \right\}, \text{ by putting } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = D,$$

$$\text{is } x = \frac{\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}}{D}; y = \frac{\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}}{D}; z = \frac{\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}}{D}.$$

The same method applies to a set of  $n$  equations with  $n$  unknown quantities. Observe that the denominator of the value of each unknown quantity is the determinant formed of the coefficients of all the unknown quantities, while the numerator is the same determinant with the column of the coefficients of that unknown quantity replaced by the column of the absolute terms *on the right-hand side of the equations.*

171. If  $\begin{aligned} a_1x + b_1y + c_1z &= 0 \}, \\ a_2x + b_2y + c_2z &= 0 \end{aligned}$ .

then  $x : y : z = \left| \begin{array}{c} b_1c_1 \\ b_2c_2 \end{array} \right| : \left| \begin{array}{c} c_1a_1 \\ c_2a_2 \end{array} \right| : \left| \begin{array}{c} a_1b_1 \\ a_2b_2 \end{array} \right|,$   
 $= (b_1c_2 - b_2c_1) : (c_1a_2 - c_2a_1) : (a_1b_2 - a_2b_1).$

### Quadratic Equations.

172. Solution of  $Ax^2 + Bx + C = 0$ ,

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

173. Solution of  $x^2 + px + q = 0$ ,

$$x = -\frac{1}{2}p \pm \frac{1}{2}\sqrt{p^2 - 4q}.$$

174. The roots are

real when  $B^2 > 4AC$ , or  $p^2 > 4q$ ,

imaginary when  $B^2 < 4AC$ , or  $p^2 < 4q$ ,

real and equal when  $B^2 = 4AC$ , or  $p^2 = 4q$ .

175. In all cases, the sum of the roots  $= -p$ ,

and the product of the roots  $= q$ .

176. If  $x_1$  and  $x_2$  denote the roots, then the equation

$$x^2 - (x_1 + x_2)x + x_1x_2 = 0,$$

which may also be written

$$(x - x_1)(x - x_2) = 0,$$

is identical with  $x^2 + px + q = 0$ .

177. To find two numbers whose sum and product are known, form a quadratic equation, putting the negative of the given sum for  $p$  and the given product for  $q$ , and solve.

178. Any rational integral equation of the  $n^{\text{th}}$  degree in  $x$ , may be written in the form

$$x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \dots + p_n = 0.$$

**179.** If  $x_1, x_2, x_3, \dots, x_n$  be the  $n$  roots of this equation, the first member is divisible by each of the factors  $x - x_1$ ,  $x - x_2$ , etc., and is the product of them all, thus,

$$(x - x_1)(x - x_2)(x - x_3) \dots (x - x_n) = 0.$$

**180.** Any equation the first member of which can be separated into factors in the form just given is solved by putting each factor equal to zero. For example, the equation  $x^3 + 3x^2 - 4x - 12 = 0$ , by separating the first member into factors, becomes

$$(x + 3)(x + 2)(x - 2) = 0,$$

the roots of which are  $-3, -2$ , and  $+2$ .

Again,  $x^3 + 4x^2 + 4x + 3 = 0$ , being reducible to the form

$$(x + 3)(x^2 + x + 1) = 0,$$

is solved by putting  $x + 3 = 0$  and  $x^2 + x + 1 = 0$ .

Many cubic equations and equations of higher degrees are easily solved by this method, if the first member is separable into factors of a degree not higher than the second.

**181.** If the factors of the first member of 179 be multiplied together, the result is

$$\begin{aligned} x^n - (x_1 + x_2 + \dots + x_n)x^{n-1} + (x_1x_2 + x_2x_3 + \dots)x^{n-2} \\ - (x_1x_2x_3 + x_2x_3x_4 + \dots)x^{n-3} + \dots \pm x_1x_2x_3 \dots x_n = 0, \end{aligned}$$

an equation which is identical with

$$x^n + p_1x^{n-1} + p_2x^{n-2} + \dots + p_n = 0.$$

Therefore,

- |             |  |
|-------------|--|
| <b>182.</b> | $\left. \begin{array}{l} -p_1 = \text{the sum of the roots,} \\ p_2 = \text{the sum of all the products of the} \\ \quad \text{roots taken two by two,} \\ -p_3 = \text{the sum of all the products of the} \\ \quad \text{roots taken three by three,} \\ \dots \dots \dots \dots \dots \dots \dots \\ (-1)^k p_k = \text{the sum of all the products of the} \\ \quad \text{roots taken } k \text{ by } k, \\ \dots \dots \dots \dots \dots \dots \dots \\ (-1)^n p_n = \text{the product of all the roots.} \end{array} \right\}$ |
|-------------|--|

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Then the roots are

$$186. \quad \begin{cases} y_1 = \mp 2\sqrt{\frac{1}{3}p} \operatorname{Cosh} \frac{1}{3}\varphi, \\ y_2 = \pm \sqrt{\frac{1}{3}p} \operatorname{Cosh} \frac{1}{3}\varphi + i\sqrt{p} \operatorname{Sinh} \frac{1}{3}\varphi, \\ y_3 = \pm \sqrt{\frac{1}{3}p} \operatorname{Cosh} \frac{1}{3}\varphi - i\sqrt{p} \operatorname{Sinh} \frac{1}{3}\varphi. \end{cases}$$

(iii) When  $y^3 - py \pm q = 0$ ,  $p$  and  $q$  being positive and  $(\frac{1}{3}p)^3 > (\frac{1}{2}q)^2$ , compute the value of the angle  $\varphi$  from

$$\cos \varphi = \frac{\frac{1}{2}q}{\frac{1}{3}p\sqrt{\frac{1}{3}p}}.$$

Then the roots are

$$187. \quad \begin{cases} y_1 = \mp 2\sqrt{\frac{1}{3}p} \cos \frac{1}{3}\varphi, \\ y_2 = \mp 2\sqrt{\frac{1}{3}p} \cos (\frac{1}{3}\varphi + 120^\circ), \\ y_3 = \mp 2\sqrt{\frac{1}{3}p} \cos (\frac{1}{3}\varphi + 240^\circ). \end{cases}$$

(iv) When  $y^3 - py \pm q = 0$ ,  $p$  and  $q$  being positive,

and

$$(\frac{1}{3}p)^3 = (\frac{1}{2}q)^2,$$

$$188. \text{ the roots are } \begin{cases} y_1 = \mp 2\sqrt{\frac{1}{3}p}, \\ y_2 = y_3 = \pm \sqrt{\frac{1}{3}p}. \end{cases}$$

### Series.

#### Arithmetic Series.

189. The  $n^{\text{th}}$  term of the series

$$a, a + d, a + 2d, a + 3d, \dots$$

is

$$a + (n - 1)d;$$

and the sum of  $n$  terms is

$$190. \quad S = \frac{n}{2}[2a + (n - 1)d].$$

#### Geometric Series.

191. The  $n^{\text{th}}$  term of the series  $a, ar, ar^2, ar^3, \dots$  is  $ar^{n-1}$   
and the sum of  $n$  terms is

$$192. \quad S = a \left( \frac{r^n - 1}{r - 1} \right).$$

If the number of terms,  $n$ , be infinite and the ratio,  $r$ , be a proper fraction, the series is convergent, and

$$193. \quad S = \frac{a}{1 - r}.$$

### *Harmonic Series.*

194. The terms  $a, b, c, d, \dots$ , etc., form a harmonic series if their reciprocals  $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}, \frac{1}{d}, \dots$ , etc., form an arithmetic series, that is, when the relation subsisting between any three consecutive terms is

$$\frac{a}{c} = \frac{a - b}{b - c}.$$

195. The  $n^{\text{th}}$  term in a harmonic series is

$$\frac{ab}{(n-1)a - (n-2)b}.$$

196. The arithmetic mean between  $a$  and  $b$  =  $\frac{a+b}{2}$ .

197. The geometric mean between  $a$  and  $b$  =  $\sqrt{ab}$ .

198. The harmonic mean between  $a$  and  $b$  =  $\frac{2ab}{a+b}$ .

199. A series partly arithmetic and partly geometric is represented by

$$a, (a+d)r, (a+2d)r^2, (a+3d)r^3, \text{ etc.}$$

The sum of  $n$  terms of this series,

$$S = \frac{a - [a + (n-1)d]r^n}{1 - r} + \frac{rd(1 - r^{n-1})}{(1 - r)^2}.$$

200.  $1 + 2 + 3 + 4 + 5 + \dots + (n-1) + n = \frac{n(n+1)}{2}$ .

201.  $p + (p+1) + (p+2) + \dots + (q-1) + q$

$$= \frac{(q+p)(q-p+1)}{2}.$$

202.  $2 + 4 + 6 + 8 + \dots + (2n-2) + 2n = n(n+1)$ .

203.  $1 + 3 + 5 + 7 + \dots + (2n-3) + (2n-1) = n^2$ .

$$204. \quad 1^2 + 2^2 + 3^2 + 4^2 + \dots + n^2 = \frac{n(n+1)(2n+1)}{1.2.3}.$$

$$205. \quad 1^3 + 2^3 + 3^3 + 4^3 + \dots + n^3 = \left[ \frac{n(n+1)}{2} \right]^2.$$

### Binomial Series.

**206.**

$$(1 \pm x)^n = 1 \pm nx + \frac{n(n-1)}{2!} x^2 \pm \frac{n(n-1)(n-2)}{3!} x^3 + \dots$$

Convergent if  $x^2 < 1$ .

**207.**

$$(1 \pm x)^{-n} = 1 \mp nx + \frac{n(n+1)}{2!} x^2 \mp \frac{n(n+1)(n+2)}{3!} x^3 + \dots$$

Convergent if  $x^2 < 1$ .

$$208. \quad (a - bx)^{-1} = \frac{1}{a} \left( 1 + \frac{bx}{a} + \frac{b^2 x^2}{a^2} + \frac{b^3 x^3}{a^3} + \dots \right)$$

Convergent if  $b^2 x^2 < a^2$ .

$$209. \quad (1 \pm x)^{-1} = 1 \mp x + x^2 \mp x^3 + x^4 \mp x^5 + \dots$$

Convergent if  $x^2 < 1$ .

$$210. \quad (1 \pm x)^{-2} = 1 \mp 2x + 3x^2 \mp 4x^3 + 5x^4 \mp 6x^5 + \dots$$

Convergent if  $x^2 < 1$ .

$$211. \quad (1 \pm x)^{\frac{1}{2}} = 1 \pm \frac{1}{2}x - \frac{1.1}{2.4}x^2 \pm \frac{1.1.3}{2.4.6}x^3 - \frac{1.1.3.5}{2.4.6.8}x^4 \pm \dots$$

Convergent if  $x^2 < 1$ .

$$212. \quad (1 \pm x)^{-\frac{1}{2}} = 1 \mp \frac{1}{2}x + \frac{1.3}{2.4}x^2 \mp \frac{1.3.5}{2.4.6}x^3 + \frac{1.3.5.7}{2.4.6.8}x^4 \mp \dots$$

Convergent if  $x^2 < 1$ .

$$213. \quad (1 \pm x)^{\frac{1}{3}} = 1 \pm \frac{1}{3}x - \frac{1.2}{3.6}x^2 \pm \frac{1.2.5}{3.6.9}x^3 - \frac{1.2.5.8}{3.6.9.12}x^4 \pm \dots$$

Convergent if  $x^2 < 1$ .

$$214. \quad (1 \pm x)^{-\frac{1}{3}} = 1 \mp \frac{1}{3}x + \frac{1.4}{3.6}x^2 \mp \frac{1.4.7}{3.6.9}x^3 + \frac{1.4.7.10}{3.6.9.12}x^4 \mp \dots$$

Convergent if  $x^2 < 1$ .

### Exponential and Logarithmic Series.

**215.**  $e = 1 + \frac{1}{1} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \dots$

= Limit of  $\left(1 + \frac{1}{m}\right)^m$  for  $m = \infty$ .

**216.**  $e^x = 1 + \frac{x}{1} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$

[ $-\infty < x < +\infty$ ].

**217.**  $a^x = 1 + cx + \frac{c^2 x^2}{2!} + \frac{c^3 x^3}{3!} + \frac{c^4 x^4}{4!} + \dots$

[ $-\infty < x < +\infty$ ].

wherein  $c = \log_e a$ .

**218.**  $a^x = 1 + \frac{\log_e a}{1} x + \frac{(\log_e a)^2}{2!} x^2 + \frac{(\log_e a)^3}{3!} x^3 + \dots$

[ $-\infty < x < +\infty$ ].

**219.**  $\log_e (1 \pm x) = \pm x - \frac{1}{2}x^2 \pm \frac{1}{3}x^3 - \frac{1}{4}x^4 \pm \frac{1}{5}x^5 - \dots$

[ $x^2 < 1$ ].

**220.**  $\frac{1}{2} \log_e \frac{1+x}{1-x} = x + \frac{1}{3}x^3 + \frac{1}{5}x^5 + \frac{1}{7}x^7 + \dots$  [ $x^2 < 1$ ].

**221.**  $\frac{1}{2} \log_e \frac{x+1}{x-1} = x^{-1} + \frac{1}{3}x^{-3} + \frac{1}{5}x^{-5} + \frac{1}{7}x^{-7} + \dots$  [ $x^2 > 1$ ].

**222.**  $\log_e x$

$$= 2 \left[ \frac{x-1}{x+1} + \frac{1}{3} \left( \frac{x-1}{x+1} \right)^3 + \frac{1}{5} \left( \frac{x-1}{x+1} \right)^5 + \frac{1}{7} \left( \frac{x-1}{x+1} \right)^7 + \dots \right],$$

[ $0 < x < +\infty$ ].

**223.**  $\log_e (a+x)$

$$= \log_e a + 2 \left[ \frac{x}{2a+x} + \frac{1}{3} \left( \frac{x}{2a+x} \right)^3 + \frac{1}{5} \left( \frac{x}{2a+x} \right)^5 + \dots \right],$$

[ $0 < a < +\infty$ ,  $-a < x < +\infty$ ].

**224.**  $\log \left( \frac{n+1}{n} \right) = \log (n+1) - \log n$

$$= 2 \left\{ \frac{1}{2n+1} + \frac{1}{3(2n+1)^3} + \frac{1}{5(2n+1)^5} + \dots \right\}.$$

[ $0 < n < +\infty$ ].

**225.**  $\log(x + \sqrt{1 + x^2})$

$$= x - \frac{1x^3}{2.3} + \frac{1.3x^5}{2.4.5} - \frac{1.3.5x^7}{2.4.6.7} + \dots \quad [x^3 < 1.]$$

See Formulas 754 and 1036.

**226.**  $\log_e x = (x - 1) - \frac{1}{2}(x - 1)^2 + \frac{1}{3}(x - 1)^3 - \dots$

$$[0 < x < 2.]$$

**227.**  $\log_e x = \frac{x - 1}{x} + \frac{1}{2}\left(\frac{x - 1}{x}\right)^2 + \frac{1}{3}\left(\frac{x - 1}{x}\right)^3 + \dots$

$$[x > \frac{1}{2}].$$

### Interest and Annuities.

Let  $r$  be the rate, that is,

$r$  = interest on one dollar for one year,

$n$  = the number of years,

$P$  = the principal,

$A$  = the amount in  $n$  years.

Then,

**228.** At simple interest,  $A = P(1 + nr)$ .

**229.** At compound interest,  $A = P(1 + r)^n$ .

**230.** If interest be compounded  $q$  times a year,

$$A = P \left(1 + \frac{r}{q}\right)^{nq}.$$

If  $A$  be an amount of money payable  $n$  years hence, and  $P$  the present worth of  $A$ , then

**231.** At simple interest,  $P = \frac{A}{1 + nr}$ .

**232.** At compound interest,  $P = \frac{A}{(1 + r)^n}$ .

**233.** Discount =  $A - P$ .\*

\* This is *true discount*, so-called to distinguish it from commercial discount, which, for commercial convenience, is based on a simpler rule.

234. The amount of an annuity of one dollar in  $n$  years at simple interest . . . . .  $\left\{ = n + \frac{n(n-1)}{2}r \right.$
235. Present value of such an annuity . . . . .  $\left\{ = \frac{n + \frac{1}{2}n(n-1)r}{1+nr} \right.$
236. Amount at compound interest . . . . .  $\left\{ = \frac{(1+r)^n - 1}{(1+r) - 1} \right.$
237. Present value . . . . .  $\left\{ = \frac{1 - (1+r)^{-n}}{(1+r) - 1} \right.$
238. Amount when the payments of interest are made  $q$  times a year . . . . .  $\left\{ = \frac{\left(1 + \frac{r}{q}\right)^{nq} - 1}{\left(1 + \frac{r}{q}\right)^q - 1} \right.$
239. Present value . . . . .  $\left\{ = \frac{1 - \left(1 + \frac{r}{q}\right)^{-nq}}{\left(1 + \frac{r}{q}\right)^q - 1} \right.$
240. Amount when payments of the annuity are made  $m$  times a year . . . . .  $\left\{ = \frac{(1+r)^n - 1}{m[(1+r)^m - 1]} \right.$
241. Present value . . . . .  $\left\{ = \frac{1 - (1+r)^{-n}}{m[(1+r)^m - 1]} \right.$
242. Amount when the interest is paid  $q$  times and the annuity  $m$  times a year . . . . .  $\left\{ = \frac{\left(1 + \frac{r}{q}\right)^{nq} - 1}{m\left[\left(1 + \frac{r}{q}\right)^{\frac{q}{m}} - 1\right]} \right.$
243. Present value . . . . .  $\left\{ = \frac{1 - \left(1 + \frac{r}{q}\right)^{-nq}}{m\left[\left(1 + \frac{r}{q}\right)^{\frac{q}{m}} - 1\right]} \right.$

### Probabilities.

If there are  $a$  ways in which an event can happen, and  $b$  ways in which it must fail to happen, the *chances* (or odds) in favor of the event are said to be as  $a$  to  $b$ , and the chances (or odds) against it as  $b$  to  $a$ .

The *probability* of an event is the ratio of the number of favorable chances to the total number of chances, both favorable and unfavorable. In the case above stated,

$$244. \quad \left\{ \begin{array}{l} \frac{a}{a+b} = \text{the probability that the event may happen.} \\ \frac{b}{a+b} = \text{the probability that the event may fail to happen.} \end{array} \right.$$

The sum of these two probabilities is 1; and since the event is *certain* either to happen or fail to happen,

$$245. \quad \text{Certainty} = 1.$$

If  $p$  be the probability of an event, the probability that that the event may fail is  $1 - p$ .

If  $E_1$  and  $E_2$  are two possible and *independent* events, and  $p_1$  and  $p_2$  are their respective probabilities, then

$$246. \quad p_1 p_2 = \text{the probability that both } E_1 \text{ and } E_2 \text{ may happen.}$$

$$247. \quad 1 - p_1 p_2 = \text{the probability that not both } E_1 \text{ and } E_2 \text{ may happen.}$$

$$248. \quad (1 - p_1) p_2 = \text{the probability that } E_1 \text{ may fail and } E_2 \text{ happen.}$$

$$249. \quad p_1 (1 - p_2) = \text{the probability that } E_1 \text{ may happen and } E_2 \text{ fail.}$$

$$250. \quad p_1 + p_2 - 2p_1 p_2 = \text{the probability that one event may happen, and the other fail.}$$

$$251. \quad (1 - p_1) (1 - p_2) = \text{the probability that both events may fail.}$$

The value of  $p$  may be determined, approximately at least, by observation of a large number of cases. Thus the experience of life assurance companies shows that out of 69,517 persons living in their fifty-first year 55,973 were living in their sixty-first year. Therefore the probability that an assured person at the age of fifty may live ten years is the ratio of these numbers, 0.805.

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And from the definitions together with the equation

$$h^2 = a^2 + b^2$$

follow the further relations,

310.  $\sin^2 A + \cos^2 A = 1,$

311.  $\sec^2 A = 1 + \tan^2 A,$

312.  $\csc^2 A = 1 + \operatorname{ctn}^2 A.$

Also, since

$$B = 90^\circ - A,$$

313. 
$$\begin{cases} \sin (90^\circ - A) = \cos A, \\ \cos (90^\circ - A) = \sin A, \\ \tan (90^\circ - A) = \operatorname{ctn} A, \\ \operatorname{ctn} (90^\circ - A) = \tan A, \\ \sec (90^\circ - A) = \csc A, \\ \csc (90^\circ - A) = \sec A. \end{cases}$$

Therefore, if the values of all the functions of each angle from  $0^\circ$  to  $45^\circ$  are given (as in the table on page 249), the values of the functions of all angles from  $90^\circ$  to  $45^\circ$  are given also.

The functions of acute angles as above defined, when computed and tabulated, are sufficient for the solution of right triangles in all cases. They are also sufficient for the solution of an oblique triangle, if the latter be converted into the sum or the difference of two right triangles by drawing a perpendicular from a vertex to the opposite side or to the opposite side extended. For methods of solution, see pages 60–66,

*General Definitions of Angle, its Measures, and its Functions.*

314. An angle is any amount of turning in a fixed plane by which a straight line may be changed from one direction to any other direction in that plane.

If the turning amount to less than a quarter of a revolution the angle is a geometric acute angle; if to more than a

quarter and less than a half of a revolution, it is a geometric obtuse angle; if to more than a half and less than a whole revolution, it is a so-called convex angle.

The turning may amount to more than one whole revolution or to more than any number of whole revolutions however great. Moreover, the turning may be one way, positive, or the other way, negative. Therefore the general value of an angle is expressed by

$$315. \quad \pm A \pm k 360^\circ, \quad \text{or } \pm \alpha \pm 2k\pi,$$

wherein  $k$  is any integer or 0.

Angles are measured in *degrees*, *minutes*, and *seconds*; or in units of arc-measure, called *radians*. The arc-measure is the ratio of the arc to the radius, the arc being the whole arc described by any point of the turning line, and the radius the distance of that point from the centre of revolution. The arc-measure of one whole revolution is the circumference of a circle divided by its radius, or  $2\pi$ . The infinite range of value which an angle,  $A$ , or its arc-measure,  $\alpha$ , takes may be thus expressed,

$$\text{in degrees,} \quad -\infty^\circ < A < +\infty^\circ.$$

$$\text{in radians,} \quad -\infty < \alpha < +\infty.$$

The two measures of an angle are thus related,

$$316. \quad \left\{ \begin{array}{l} 1 \text{ Radian} = 57^\circ 17' 44'' .806 \\ 180^\circ = \pi = 3.14159265 \text{ radians.} \end{array} \right.$$

A table for converting either kind of measure into the other is given on the next page.

As a matter of notation in the following pages, capital italic letters will, in general, indicate that the angles are to be expressed in degrees, minutes and seconds, while Greek letters or small italics will indicate that they are to be expressed in arc-measure or radians. It is, however, in many formulas a matter of indifference which notation is used.

317.

## TABLE

(a) For finding the Length of the Arc measuring any given Angle in a Circle of which the Radius is 1.

Angle in Degrees	Arc in Radians	Angle in Degrees	Arc in Radians
$90^\circ$	$\frac{1}{2}\pi = 1.57079633$	$270^\circ$	$\frac{3}{2}\pi = 4.71238898$
$180^\circ$	$\pi = 3.14159265$	$360^\circ$	$2\pi = 6.28318531$

Angle	Degrees	Minutes	Seconds
1	0.0174533	0.0002909	0.0000048
2	0.0349066	0.0005818	0.0000097
3	0.0523599	0.0008727	0.0000145
4	0.0698132	0.0011636	0.0000194
5	0.0872665	0.0014544	0.0000242
6	0.1047198	0.0017453	0.0000291
7	0.1221730	0.0020362	0.0000339
8	0.1396263	0.0023271	0.0000388
9	0.1570796	0.0026180	0.0000436

(b) For finding the Angle measured by any given Length of Arc in a Circle of which the Radius is 1.

Arc	Angle	Arc	Angle
1	$57^\circ 17' 44.8''$	4	$229^\circ 10' 59.2''$
2	$114^\circ 35' 29.6''$	5	$286^\circ 28' 44.0''$
3	$171^\circ 53' 14.4''$	6	$343^\circ 46' 28.8''$

	Tenths	Hundredths	Thousandths	Ten-thousandths	Hundred-thousandths
1	$5^\circ 43' 46.5''$	$0^\circ 34' 22.6''$	$3' 26.3''$	$0' 20.6''$	$2.1''$
2	$11^\circ 27' 33.0''$	$1^\circ 8' 45.3''$	$6' 52.5''$	$0' 41.3''$	$4.1''$
3	$17^\circ 11' 19.4''$	$1^\circ 43' 7.9''$	$10' 18.8''$	$1' 1.9''$	$6.2''$
4	$22^\circ 55' 5.9''$	$2^\circ 17' 30.6''$	$13' 45.1''$	$1' 22.5''$	$8.3''$
5	$28^\circ 38' 52.4''$	$2^\circ 51' 53.2''$	$17' 11.3''$	$1' 43.1''$	$10.3''$
6	$34^\circ 22' 38.9''$	$3^\circ 26' 15.9''$	$20' 37.6''$	$2' 3.8''$	$12.4''$
7	$40^\circ 6' 25.4''$	$4^\circ 0' 38.5''$	$24' 3.9''$	$2' 24.4''$	$14.4''$
8	$45^\circ 50' 11.8''$	$4^\circ 35' 1.2''$	$27' 30.1''$	$2' 45.0''$	$16.5''$
9	$51^\circ 33' 58.3''$	$5^\circ 9' 23.8''$	$30' 56.4''$	$3' 5.6''$	$18.6''$

*Functions of the General Angle Defined.*

Drawing rectangular axes,  $xx'$  horizontal and  $yy'$  vertical, intersecting in  $o$ , and the line  $OP$  in any required direction, let  $ox$  be the initial side and  $OP$  the terminal side of any angle whatever (defined as in 314).

It is evident that  $OP$  may fall in any one of the four quadrants, the first  $xoy$ , the second  $yox'$ , the third  $x'o'y'$ , or the fourth  $y'ox$ .

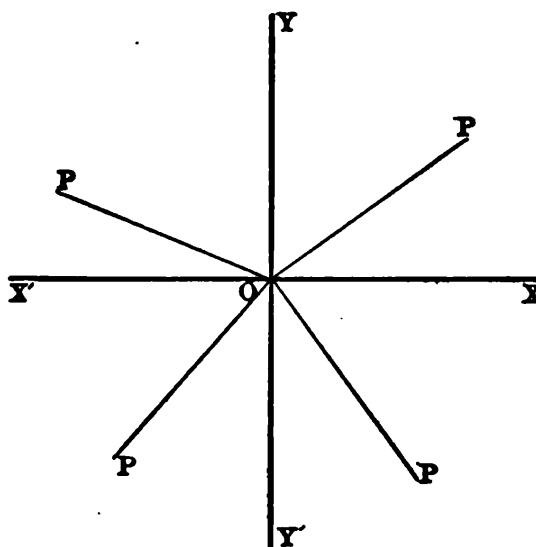


FIGURE 1.

Let the coördinates of the point  $P$  in any situation be

$x$  = the *abscissa*, or distance of  $P$  to the right or left of the vertical axis,

$y$  = the *ordinate*, or distance of  $P$  above or below the horizontal axis.

Let  $r$  = the *distance* of  $P$  from  $o$ .

Then are the six functions of  $A$  (any angle whatever) defined as follows:

$$318. \quad \begin{cases} \sin A = \frac{\text{ordinate}}{\text{distance}} = \frac{y}{r}, & \operatorname{ctn} A = \frac{\text{abscissa}}{\text{ordinate}} = \frac{x}{y}, \\ \cos A = \frac{\text{abscissa}}{\text{distance}} = \frac{x}{r}, & \sec A = \frac{\text{distance}}{\text{abscissa}} = \frac{r}{x}, \\ \tan A = \frac{\text{ordinate}}{\text{abscissa}} = \frac{y}{x} & \csc A = \frac{\text{distance}}{\text{ordinate}} = \frac{r}{y}. \end{cases}$$

An angle is said to be an angle of the first, second, third, or fourth quadrant according as its terminal side falls in the first, second, third, or fourth quadrant.

The functions of angles of different quadrants have positive or negative values dependent on the values of the coordinates used in the definitions 318. The *abscissa* is positive or negative according as  $P$  is to the right or left of the vertical axis; the *ordinate* is positive or negative according as  $P$  is above or below the horizontal axis; the *distance*  $OP$  is positive in all situations.

Hence the values of the functions of angles of the several quadrants are positive or negative as indicated below.

### 319.

Quadrant	sin	cos	tan	cotn	sec	csc
I.	+	+	+	+	+	+
II.	+	-	-	-	-	+
III.	-	-	+	+	-	-
IV.	-	+	-	-	+	-

As an angle increases from  $0^\circ$  to  $90^\circ$ ,  $180^\circ$ ,  $270^\circ$ ,  $360^\circ$ , etc., its functions vary, some increasing, some decreasing, but all reaching maximum or minimum numerical values for the cardinal values of the angle above mentioned. When the function passes through the value 0 or  $\infty$  it changes its sign, as is indicated in the following table.

### 320.

### Cardinal Values.

Angle	Arc	Sin	Cos	Tan	Ctn	Sec	Csc
$0^\circ$	$0$	$\pm 0$	$+1$	$\pm 0$	$\pm\infty$	$+1$	$\pm\infty$
$90^\circ$	$\frac{1}{2}\pi$	$+1$ incr.	$\pm 0$ decr.	$\pm\infty$ incr.	$\pm 0$ decr.	$\pm\infty$ incr.	$+1$ decr.
$180^\circ$	$\pi$	$\pm 0$	$-1$	$\mp 0$	$\pm\infty$	$-1$	$\pm\infty$
$270^\circ$	$\frac{3}{2}\pi$	$-1$ incr.	$\mp 0$ incr.	$\pm\infty$ incr.	$\pm 0$ decr.	$\mp 0$ decr.	$-1$ decr.
$360^\circ$	$2\pi$	$\pm 0$	$+1$	$\mp 0$	$\pm\infty$	$+1$	$\pm\infty$

*Fundamental Relations Generalized.*

From the definitions 318 follow at once the relations

$$321. \quad \begin{cases} \sin A \csc A = 1, & \tan A = \frac{\sin A}{\cos A}, \\ \cos A \sec A = 1, & \\ \tan A \ctn A = 1, & \ctn A = \frac{\cos A}{\sin A}; \end{cases}$$

and from the equation  $x^2 + y^2 = r^2$  follow the relations

$$322. \quad \begin{cases} \sin^2 A + \cos^2 A = 1, \\ 1 + \tan^2 A = \sec^2 A, \\ 1 + \ctn^2 A = \csc^2 A, \end{cases}$$

which are identical with 302-312, as they should be; but these are applicable to angles (or arcs) of all magnitudes positive or negative, while those relate only to positive acute angles.

From 322 result six radical forms,

$$323. \quad \begin{cases} \sin A = \pm \sqrt{1 - \cos^2 A}, \\ \cos A = \pm \sqrt{1 - \sin^2 A}, \\ \tan A = \pm \sqrt{\sec^2 A - 1}, \\ \ctn A = \pm \sqrt{\csc^2 A - 1}, \\ \sec A = \pm \sqrt{1 + \tan^2 A}, \\ \csc A = \pm \sqrt{1 + \ctn^2 A}. \end{cases}$$

The interpretation of the double signs of these radicals is found in the fact that to a given value of any one function belong two angles between  $0^\circ$  and  $360^\circ$ ; and the other five functions of these two angles are numerically equal each to each; but four of them have opposite algebraic signs. These four are the ones which are given by the quadratic solutions. The fifth is the reciprocal of the given one, and, like that, has the same value for the two angles.

*Anti-Functions.*

If  $\sin A = x$ , or  $\sin \alpha = x$ , then  $A$  is the angle the sine of which is  $x$ , or  $\alpha$  is the arc the sine of which is  $x$ , a relation usually expressed by the notation

$$A = \sin^{-1} x, \quad \text{or} \quad \alpha = \sin^{-1} x,$$

and read "  $A$  (or  $\alpha$ ) is the *anti-sine* of  $x$ ."

Some writers use the notation  $\text{arc-sin } x$  instead of  $\sin^{-1} x$ .

Similarly, if  $\tan B = y, \quad B = \tan^{-1} y$   
 $\sec C = z, \quad C = \sec^{-1} z.$

The value of  $\sin^{-1} x$  is not only  $A$ , as given above, but any one of the infinite series of angles included in the general expression  $A \pm k 360^\circ$ , or  $\alpha \pm 2k\pi$ .

Hence,  $k$  being any integer, including 0,

$$324. \quad \begin{cases} \sin^{-1} x = A \pm k 360^\circ = \alpha \pm 2k\pi, \\ \tan^{-1} y = B \pm k 360^\circ = \beta \pm 2k\pi, \\ \sec^{-1} z = C \pm k 360^\circ = \gamma \pm 2k\pi. \end{cases}$$

Also,

$$325. \quad \begin{cases} \cos^{-1} x = (90^\circ - A) \pm k 360^\circ = (\frac{1}{2}\pi - \alpha) \pm 2k\pi, \\ \ctn^{-1} y = (90^\circ - B) \pm k 360^\circ = (\frac{1}{2}\pi - \beta) \pm 2k\pi, \\ \csc^{-1} z = (90^\circ - C) \pm k 360^\circ = (\frac{1}{2}\pi - \gamma) \pm 2k\pi. \end{cases}$$

Whence

$$326. \quad \begin{cases} \sin^{-1} x + \cos^{-1} x = 90^\circ \pm k 360^\circ = \frac{1}{2}\pi \pm 2k\pi. \\ \tan^{-1} y + \ctn^{-1} y = 90^\circ \pm k 360^\circ = \frac{1}{2}\pi \pm 2k\pi. \\ \sec^{-1} z + \csc^{-1} z = 90^\circ \pm k 360^\circ = \frac{1}{2}\pi \pm 2k\pi. \end{cases}$$

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Formulas expressing each function in terms of each of the others.

$$328. \sin A = \pm \sqrt{1 - \cos^2 A} = \frac{\tan A}{\pm \sqrt{1 + \tan^2 A}}$$

$$= \frac{1}{\pm \sqrt{1 + \operatorname{ctn}^2 A}} = \frac{\pm \sqrt{\sec^2 A - 1}}{\sec A} = \frac{1}{\csc A}.$$

$$329. \cos A = \pm \sqrt{1 - \sin^2 A} = \frac{1}{\pm \sqrt{1 + \tan^2 A}}$$

$$= \frac{\operatorname{ctn} A}{\pm \sqrt{1 + \operatorname{ctn}^2 A}} = \frac{1}{\sec A} = \frac{\pm \sqrt{\csc^2 A - 1}}{\csc A}.$$

$$330. \tan A = \frac{\sin A}{\pm \sqrt{1 - \sin^2 A}} = \frac{\pm \sqrt{1 - \cos^2 A}}{\cos A} = \frac{1}{\operatorname{ctn} A},$$

$$= \pm \sqrt{\sec^2 A - 1} = \frac{1}{\pm \sqrt{\csc^2 A - 1}}.$$

$$331. \operatorname{ctn} A = \frac{\pm \sqrt{1 - \sin^2 A}}{\sin A} = \frac{\cos A}{\pm \sqrt{1 - \cos^2 A}} = \frac{1}{\tan A},$$

$$= \frac{1}{\pm \sqrt{\sec^2 A - 1}} = \pm \sqrt{\csc^2 A - 1}.$$

$$332. \sec A = \frac{1}{\pm \sqrt{1 - \sin^2 A}} = \frac{1}{\cos A} = \pm \sqrt{1 + \tan^2 A},$$

$$= \frac{\pm \sqrt{1 + \operatorname{ctn}^2 A}}{\operatorname{ctn} A} = \frac{\csc A}{\pm \sqrt{\csc^2 A - 1}}.$$

$$333. \csc A = \frac{1}{\sin A} = \frac{1}{\pm \sqrt{1 - \cos^2 A}} = \frac{\pm \sqrt{1 + \tan^2 A}}{\tan A},$$

$$= \pm \sqrt{1 + \operatorname{ctn}^2 A} = \frac{\sec A}{\pm \sqrt{\sec^2 A - 1}}.$$

*Positive and Negative Lines.*

If the distance from a point  $A$  to any other point  $B$  on a straight line be reckoned as positive, then the distance from  $B$  to  $A$  must be reckoned as negative; so that it is always true that

$$334. \quad AB + BA = 0.$$

Let three points  $A$ ,  $B$ , and  $C$  be arranged in any order on a straight line. Then the algebraic sum of the distances  $AB$  and  $BC$  is always  $AC$ , that is,

$$335. \quad AB + BC = AC,$$

which by adding  $CA$  to each member becomes

$$336. \quad AB + BC + CA = 0.$$

The same principle applies to any number of points, arranged in any order whatever on a straight line, and their distances, that is

$$337. \quad AB + BC + CD + \dots + MN + NA = 0.$$

*Projections.*

The projections of a line  $AB$  upon the axes of  $x$  and  $y$  are

$$338. \quad \begin{cases} AB \cos A = \text{projection on the axis of } x, \\ AB \sin A = \text{projection on the axis of } y, \end{cases}$$

wherein  $A$  denotes the angle between the positive direction of the axis of  $x$  and the positive direction of the projected line.

The sum of the projections of the sides of any closed polygon, taken in order around the polygon, upon any chosen line is equal to 0.

In the case of a triangle  $ABC$  placed anyhow in the plane of the axes  $ox$  and  $oy$ , if  $A'$ ,  $B'$ ,  $C'$  be the projections of the points  $A$ ,  $B$ ,  $C$  on the axis of  $x$  and  $A''$ ,  $B''$ ,  $C''$ , the projections of the same points on the axis of  $y$ , then, whatever the order in which the projections fall on either axis,

$$339. \quad \begin{cases} A'B' + B'C' + C'A' = 0. \\ A''B'' + B''C'' + C''A'' = 0. \end{cases}$$

If the axes be rectangular and ABC a right-angled triangle, these equations give the formulas for the sine and the cosine of the sum and of the difference of two angles.

*Positive and Negative Angles.*

If the angle AOB be reckoned as positive, then the angle BOA must be reckoned as negative ; so that it is always true that

$$340. \quad AOB + BOA = 0, \text{ or } = \pm k 360^\circ, \text{ or } = \pm 2k\pi.$$

Also, whatever the order of the lines radiating from O,

$$341. \quad AOB + BOC = AOC, \text{ or } = AOC \pm k 360^\circ, \\ \text{or } = AOC \pm 2k\pi.$$

$$342. \quad AOB + BOC + COD + \dots + MON + NOA = 0, \text{ or } = \pm k 360^\circ, \text{ or } = \pm 2k\pi.$$

$$343. \quad AOB + BOC + COD + \dots + MON + NOA = 0, \\ \text{or } = \pm k 360^\circ, \text{ or } = \pm 2k\pi.$$

*Functions of the Sum and of the Difference of Two Angles.*

$$344. \quad \sin (A + B) = \sin A \cos B + \cos A \sin B.$$

$$345. \quad \sin (A - B) = \sin A \cos B - \cos A \sin B.$$

$$346. \quad \cos (A + B) = \cos A \cos B - \sin A \sin B.$$

$$347. \quad \cos (A - B) = \cos A \cos B + \sin A \sin B.$$

$$348. \quad \tan (A + B) = \frac{\tan A + \tan B}{1 - \tan A \tan B}.$$

$$349. \quad \tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

$$350. \quad \operatorname{ctn} (A + B) = \frac{\operatorname{ctn} B \operatorname{ctn} A - 1}{\operatorname{ctn} B + \operatorname{ctn} A}.$$

$$351. \quad \operatorname{ctn} (A - B) = \frac{\operatorname{ctn} B \operatorname{ctn} A + 1}{\operatorname{ctn} B - \operatorname{ctn} A}.$$

*Functions of the Sum of Three Angles.*

**352.**  $\sin(A + B + C)$

$$= -\sin A \sin B \sin C + \sin A \cos B \cos C \\ + \cos A \sin B \cos C + \cos A \cos B \sin C.$$

**353.**  $\cos(A + B + C)$

$$= \cos A \cos B \cos C - \cos A \sin B \sin C \\ - \sin A \cos B \sin C - \sin A \sin B \cos C.$$

*Functions of a Negative Angle.*

**354.** 
$$\begin{cases} \sin(-A) = -\sin A, & \operatorname{ctn}(-A) = -\operatorname{ctn} A, \\ \cos(-A) = \cos A, & \sec(-A) = \sec A, \\ \tan(-A) = -\tan A, & \csc(-A) = -\csc A. \end{cases}$$

*Functions of  $A + 90^\circ$ .*

**355.** 
$$\begin{cases} \sin(A + 90^\circ) = \cos A, \\ \cos(A + 90^\circ) = -\sin A, \\ \tan(A + 90^\circ) = -\operatorname{ctn} A, \\ \text{etc.} \end{cases}$$

*Functions of  $A - 90^\circ$ .*

**356.** 
$$\begin{cases} \sin(A - 90^\circ) = -\cos A, \\ \cos(A - 90^\circ) = \sin A, \\ \tan(A - 90^\circ) = -\operatorname{ctn} A, \\ \text{etc.} \end{cases}$$

*Functions of  $90^\circ - A$ .*

**357** 
$$\begin{cases} \sin(90^\circ - A) = \cos A, \\ \cos(90^\circ - A) = \sin A, \\ \tan(90^\circ - A) = \operatorname{ctn} A, \\ \text{etc.} \end{cases}$$

*Functions of  $A + 180^\circ$ .*

**358.** 
$$\begin{cases} \sin(A + 180^\circ) = -\sin A, \\ \cos(A + 180^\circ) = -\cos A, \\ \tan(A + 180^\circ) = \tan A, \\ \text{etc.} \end{cases}$$

*Functions of  $A - 180^\circ$ .*

$$359. \quad \begin{cases} \sin(A - 180^\circ) = -\sin A, \\ \cos(A - 180^\circ) = -\cos A, \\ \tan(A - 180^\circ) = \tan A, \\ \text{etc.} \end{cases}$$

*Functions of  $180^\circ - A$ .*

$$360. \quad \begin{cases} \sin(180^\circ - A) = \sin A, \\ \cos(180^\circ - A) = -\cos A, \\ \tan(180^\circ - A) = -\tan A, \\ \text{etc.} \end{cases}$$

*Functions of  $A + 270^\circ$ .*

$$361. \quad \begin{cases} \sin(A + 270^\circ) = -\cos A, \\ \cos(A + 270^\circ) = \sin A, \\ \tan(A + 270^\circ) = -\operatorname{ctn} A, \\ \text{etc.} \end{cases}$$

*Functions of  $A - 270^\circ$ .*

$$362. \quad \begin{cases} \sin(A - 270^\circ) = \cos A, \\ \cos(A - 270^\circ) = -\sin A, \\ \tan(A - 270^\circ) = -\operatorname{ctn} A, \\ \text{etc.} \end{cases}$$

*Functions of  $270^\circ - A$ .*

$$363. \quad \begin{cases} \sin(270^\circ - A) = -\cos A, \\ \cos(270^\circ - A) = -\sin A, \\ \tan(270^\circ - A) = \operatorname{ctn} A, \\ \text{etc.} \end{cases}$$

*Functions of  $A \pm 360^\circ$ .*

$$364. \quad \begin{cases} \sin(A \pm 360^\circ) = \sin A, \\ \cos(A \pm 360^\circ) = \cos A, \\ \tan(A \pm 360^\circ) = \tan A, \\ \text{etc.} \end{cases}$$

*Functions of  $360^\circ - A$ .*

365.  $\begin{cases} \sin(360^\circ - A) = -\sin A, \\ \cos(360^\circ - A) = \cos A, \\ \tan(360^\circ - A) = -\tan A, \\ \text{etc.} \end{cases}$

*Solution of equations*  $\sin A = a$ ,  $\cos A = a$ , and  $\tan A = a$ .

If  $A$  is to be found from a given value  $a$  of its sine, that is, if the equation  $\sin A = a$  is to be solved for  $A$ , all the values of  $A$  are given by the formula

366.  $\sin^{-1} a = k 180^\circ + (-1)^k A,$

wherein  $k$  is 0 or any integer.

In the same way, all the values of  $A$  obtainable from the equation  $\cos A = a$  are given by

367.  $\cos^{-1} a = k 360^\circ \pm A.$

And all values of  $A$  obtainable from  $\tan A = a$  are given by

368.  $\tan^{-1} a = k 180^\circ + A.$

*Sums and Products of Functions.*

369.  $\sin A \cos B = \frac{1}{2} \sin(A + B) + \frac{1}{2} \sin(A - B).$

370.  $\cos A \sin B = \frac{1}{2} \sin(A + B) - \frac{1}{2} \sin(A - B).$

371.  $\sin A \sin B = \frac{1}{2} \cos(A - B) - \frac{1}{2} \cos(A + B).$

372.  $\cos A \cos B = \frac{1}{2} \cos(A - B) + \frac{1}{2} \cos(A + B).$

373.  $\sin(A + B) \sin(A - B) = \sin^2 A - \sin^2 B,$   
 $= \cos^2 B - \cos^2 A.$

374.  $\cos(A + B) \cos(A - B) = \cos^2 A - \sin^2 B,$   
 $= \cos^2 B - \sin^2 A.$

375.  $\sin A + \sin B = 2 \sin \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$

376.  $\sin A - \sin B = 2 \cos \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B).$

377.  $\cos A + \cos B = 2 \cos \frac{1}{2}(A + B) \cos \frac{1}{2}(A - B).$

378.  $\cos A - \cos B = -2 \sin \frac{1}{2}(A + B) \sin \frac{1}{2}(A - B).$

$$379. \frac{\sin A + \sin B}{\sin A - \sin B} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)}.$$

$$380. \frac{\cos A - \cos B}{\cos A + \cos B} = -\tan \frac{1}{2}(A+B) \tan \frac{1}{2}(A-B).$$

$$381. \frac{\sin A \pm \sin B}{\cos A + \cos B} = \tan \frac{1}{2}(A \pm B).$$

$$382. \frac{\sin A \mp \sin B}{\cos A - \cos B} = -\operatorname{ctn} \frac{1}{2}(A \pm B).$$

$$383. \frac{\sin(A \mp B)}{\sin A \pm \sin B} = \frac{\sin \frac{1}{2}(A \mp B)}{\sin \frac{1}{2}(A \pm B)}.$$

$$384. \frac{\sin(A \mp B)}{\sin A \mp \sin B} = \frac{\cos \frac{1}{2}(A \mp B)}{\cos \frac{1}{2}(A \pm B)}.$$

$$385. \tan A \pm \tan B = \frac{\sin(A \pm B)}{\cos A \cos B}.$$

$$386. \operatorname{ctn} B \pm \operatorname{ctn} A = \frac{\sin(A \pm B)}{\sin A \sin B}.$$

$$387. \operatorname{ctn} B \pm \tan A = \frac{\cos(A \mp B)}{\cos A \sin B}.$$

$$388. \frac{\sin(A \pm B)}{\sin(A \mp B)} = \frac{\tan A \pm \tan B}{\tan A \mp \tan B} = \frac{\operatorname{ctn} B \pm \operatorname{ctn} A}{\operatorname{ctn} B \mp \operatorname{ctn} A}.$$

$$389. \frac{\cos(A \pm B)}{\cos(A \mp B)} = \frac{1 \mp \tan A \tan B}{1 \pm \tan A \tan B} = \frac{\operatorname{ctn} B \mp \tan A}{\operatorname{ctn} B \pm \tan A}.$$

$$390. \frac{\cos(A \mp B)}{\sin(A \pm B)} = \frac{\operatorname{ctn} B \pm \tan A}{\operatorname{ctn} B \tan A \pm 1}.$$

$$391. \cos^2 A + \cos^2 B - 2 \cos A \cos B \cos(A+B) \\ = \sin^2(A+B).$$

$$392. \frac{\tan A + \tan B}{\operatorname{ctn} A + \operatorname{ctn} B} = \tan A \tan B.$$

$$393. \sec^2 A + \csc^2 A = \sec^2 A \csc^2 A.$$

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*Functions of Half an Angle.*

$$409. \quad 1 = \sin^2 \frac{1}{2}A + \cos^2 \frac{1}{2}A.$$

$$410. \quad \sin A = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A.$$

$$411. \quad \cos A = \cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A.$$

$$412. \quad 1 + \sin A = (\sin \frac{1}{2}A + \cos \frac{1}{2}A)^2.$$

$$413. \quad 1 - \sin A = (\sin \frac{1}{2}A - \cos \frac{1}{2}A)^2.$$

$$414. \quad 1 + \cos A = 2 \cos^2 \frac{1}{2}A.$$

$$415. \quad 1 - \cos A = 2 \sin^2 \frac{1}{2}A.$$

$$416. \quad \sin \frac{1}{2}A = \sqrt{\frac{1}{2}(1 - \cos A)}, \\ = \frac{1}{2}\sqrt{1 + \sin A} - \frac{1}{2}\sqrt{1 - \sin A}.$$

$$417. \quad \cos \frac{1}{2}A = \sqrt{\frac{1}{2}(1 + \cos A)}, \\ = \frac{1}{2}\sqrt{1 + \sin A} + \frac{1}{2}\sqrt{1 - \sin A}.$$

$$418. \quad \tan \frac{1}{2}A = \sqrt{\frac{1 - \cos A}{1 + \cos A}} = \frac{1 - \cos A}{\sin A} = \frac{\sin A}{1 + \cos A}, \\ = \frac{1 + \sin A - \cos A}{1 + \sin A + \cos A}.$$

$$419. \quad \operatorname{ctn} \frac{1}{2}A = \sqrt{\frac{1 + \cos A}{1 - \cos A}} = \frac{1 + \cos A}{\sin A} = \frac{\sin A}{1 - \cos A}, \\ = \frac{1 + \sin A + \cos A}{1 + \sin A - \cos A}.$$

$$420. \quad \sec \frac{1}{2}A = \sqrt{\frac{2 \sec A}{\sec A + 1}}.$$

$$421. \quad \csc \frac{1}{2}A = \sqrt{\frac{2 \sec A}{\sec A - 1}}.$$

$$422. \quad 1 + \sin A = 2 \sin^2 (45^\circ + \frac{1}{2}A) = 2 \cos^2 (45^\circ - \frac{1}{2}A).$$

$$423. \quad 1 - \sin A = 2 \sin^2 (45^\circ - \frac{1}{2}A) = 2 \cos^2 (45^\circ + \frac{1}{2}A).$$

$$424. \quad \tan (45^\circ \pm A) = \operatorname{ctn} (45^\circ \mp A) = \frac{1 \pm \tan A}{1 \mp \tan A}, \\ = \sqrt{\frac{1 \pm \sin 2A}{1 \mp \sin 2A}} = \frac{\cos A \pm \sin A}{\cos A \mp \sin A}.$$

$$425. \tan(45^\circ \pm \frac{1}{2}A) = \operatorname{ctn}(45^\circ \mp \frac{1}{2}A) = \sqrt{\frac{1 \pm \sin A}{1 \mp \sin A}},$$

$$= \frac{1 \pm \sin A}{\cos A} = \sec A \pm \tan A$$

$$= \frac{\cos A}{1 \mp \sin A}.$$

$$426. \tan(A - 45^\circ) = \frac{\tan A - 1}{\tan A + 1}.$$

$$427. \sin(45^\circ + A) = \cos(45^\circ - A) = \frac{\sin A + \cos A}{\sqrt{2}}.$$

$$428. \cos(45^\circ + A) = \sin(45^\circ - A) = \frac{\cos A - \sin A}{\sqrt{2}}.$$

$$429. \tan(45^\circ + A) + \tan(45^\circ - A) = 2 \sec 2A.$$

$$430. \tan(45^\circ + A) - \tan(45^\circ - A) = 2 \tan 2A.$$

$$431. \tan(45^\circ + A) \tan(45^\circ - A) = 1.$$

$$432. \sin(30^\circ + A) + \sin(30^\circ - A) = \cos A.$$

$$433. \sin(30^\circ + A) - \sin(30^\circ - A) = \sqrt{3} \sin A.$$

$$434. \cos(30^\circ + A) + \cos(30^\circ - A) = \sqrt{3} \cos A.$$

$$435. \cos(30^\circ + A) - \cos(30^\circ - A) = -\sin A.$$

*Expressions Equivalent to sin A.*

$$436. \sin A = \sqrt{1 - \cos^2 A} = \sqrt{(1 + \cos A)(1 - \cos A)},$$

$$= \cos A \tan A = \frac{\cos A}{\operatorname{ctn} A} = \frac{\tan A}{\sec A} = \frac{1}{\csc A},$$

$$= \frac{\tan A}{\sqrt{1 + \tan^2 A}} = \frac{1}{\sqrt{1 + \operatorname{ctn}^2 A}} = \frac{\sqrt{\sec^2 A - 1}}{\sec A},$$

$$= \sqrt{\frac{1}{2}(1 - \cos 2A)} = 2 \sin \frac{1}{2}A \cos \frac{1}{2}A,$$

$$= \frac{2 \tan \frac{1}{2}A}{1 + \tan^2 \frac{1}{2}A} \cdot \frac{1}{\operatorname{ctn} \frac{1}{2}A - \operatorname{ctn} A},$$

$$= \frac{1}{\tan \frac{1}{2}A + \operatorname{ctn} A} = \frac{2}{\tan \frac{1}{2}A + \operatorname{ctn} \frac{1}{2}A},$$

$$= 2 \sin^2(45^\circ + \frac{1}{2}A) - 1,$$

$$= 1 - 2 \sin^2(45^\circ - \frac{1}{2}A) = \frac{1 - \tan^2(45^\circ - \frac{1}{2}A)}{1 + \tan^2(45^\circ - \frac{1}{2}A)}.$$

*Expressions Equivalent to cos A.*

$$\begin{aligned}
 437. \cos A &= \sqrt{1 - \sin^2 A} = \sqrt{(1 + \sin A)(1 - \sin A)}, \\
 &= \sin A \operatorname{ctn} A = \frac{\sin A}{\tan A} = \frac{\operatorname{ctn} A}{\csc A} = \frac{1}{\sec A}, \\
 &= \frac{\operatorname{ctn} A}{\sqrt{1 + \operatorname{ctn}^2 A}} = \frac{1}{\sqrt{1 + \tan^2 A}} = \frac{\sqrt{\csc^2 A - 1}}{\csc A}, \\
 &= \sqrt{\frac{1}{2}(1 + \cos 2A)} = \frac{\sin 2A}{2 \sin A}, \\
 &= \cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A, \\
 &= 1 - 2 \sin^2 \frac{1}{2}A = 2 \cos^2 \frac{1}{2}A - 1 = \frac{1 - \tan^2 \frac{1}{2}A}{1 + \tan^2 \frac{1}{2}A}, \\
 &= \frac{\operatorname{ctn}^2 \frac{1}{2}A - 1}{\operatorname{ctn}^2 \frac{1}{2}A + 1} = \frac{\operatorname{ctn} \frac{1}{2}A - \tan \frac{1}{2}A}{\operatorname{ctn} \frac{1}{2}A + \tan \frac{1}{2}A}, \\
 &= \frac{1}{\tan A \operatorname{ctn} \frac{1}{2}A - 1} = \frac{1}{1 + \tan A \tan \frac{1}{2}A}, \\
 &= \frac{2}{\tan(45^\circ + \frac{1}{2}A) + \operatorname{ctn}(45^\circ + \frac{1}{2}A)}, \\
 &= 2 \cos(45^\circ + \frac{1}{2}A) \cos(45^\circ - \frac{1}{2}A), \\
 &= \cos^4 \frac{1}{2}A - \sin^4 \frac{1}{2}A.
 \end{aligned}$$

*Expressions Equivalent to tan A.*

$$\begin{aligned}
 438. \tan A &= \frac{\sin A}{\cos A} = \frac{1}{\operatorname{ctn} A} = \sqrt{\sec^2 A - 1}, \\
 &= \frac{1}{\sqrt{\csc^2 A - 1}} = \frac{\sin A}{\sqrt{1 - \sin^2 A}} = \frac{\sqrt{1 - \cos^2 A}}{\cos A}, \\
 &= \sqrt{\frac{1 - \cos 2A}{1 + \cos 2A}} = \frac{\sin 2A}{1 + \cos 2A} = \frac{1 - \cos 2A}{\sin 2A}, \\
 &= \csc 2A - \operatorname{ctn} 2A = \operatorname{ctn} A - 2 \operatorname{ctn} 2A, \\
 &= \frac{2 \tan \frac{1}{2}A}{1 - \tan^2 \frac{1}{2}A} = \frac{2 \operatorname{ctn} \frac{1}{2}A}{\operatorname{ctn}^2 \frac{1}{2}A - 1}, \\
 &= \frac{2}{\operatorname{ctn} \frac{1}{2}A - \tan \frac{1}{2}A}, \\
 &= \frac{\tan(45^\circ + \frac{1}{2}A) - \tan(45^\circ - \frac{1}{2}A)}{2}, \\
 &= \frac{\tan(45^\circ + A) - 1}{\tan(45^\circ + A) + 1} = \frac{1 - \tan(45^\circ - A)}{1 + \tan(45^\circ - A)}.
 \end{aligned}$$

Expressions equivalent to  $\operatorname{ctn} A$ ,  $\sec A$ , and  $\csc A$  are the reciprocals of those above given for  $\tan A$ ,  $\cos A$ , and  $\sin A$ , respectively,

*Functions of Periodic Values of the Arc or Angle.*

In the following equations,  $k$  is any integer positive, negative or 0.

$$439. \begin{cases} \sin k\pi = 0, & \sin \frac{2k+1}{2}\pi = (-1)^k, \\ \cos k\pi = (-1)^k, & \cos \frac{2k+1}{2}\pi = 0, \\ \tan k\pi = 0, & \tan \frac{2k+1}{2}\pi = \infty. \end{cases}$$

$$440. \sin \varphi = \pm \sin (2k\pi \pm \varphi) = \mp \sin [(2k+1)\pi \pm \varphi], \\ = \mp \cos \left( \frac{4k+1}{2}\pi \pm \varphi \right) = \pm \cos \left( \frac{4k-1}{2}\pi \pm \varphi \right).$$

$$441. \csc \varphi = \pm \csc (2k\pi \pm \varphi) = \text{etc.}$$

$$442. \cos \varphi = \cos (2k\pi \pm \varphi) = -\cos [(2k+1)\pi \pm \varphi], \\ = \sin \left( \frac{4k+1}{2}\pi \pm \varphi \right) = -\sin \left( \frac{4k-1}{2}\pi \pm \varphi \right).$$

$$443. \sec \varphi = \sec (2k\pi \pm \varphi) = \text{etc.}$$

$$444. \tan \varphi = \pm \tan (k\pi \pm \varphi) = \mp \operatorname{ctn} \left( \frac{2k+1}{2}\pi \pm \varphi \right).$$

$$445. \operatorname{ctn} \varphi = \pm \operatorname{ctn} (k\pi \pm \varphi) = \text{etc.}$$

The formulas 440, 442, and 444 give the only solutions of the equations.

$$\begin{aligned} \sin \varphi &= \pm \sin \alpha, \\ \sin \varphi &= \pm \cos \alpha, \end{aligned}$$

$$\begin{aligned} \tan \varphi &= \pm \tan \alpha, \\ \tan \varphi &= \pm \operatorname{ctn} \alpha, \end{aligned}$$

and of the equivalent equations.

$$\begin{aligned} \csc \varphi &= \pm \csc \alpha, \\ \csc \varphi &= \pm \sec \alpha, \end{aligned}$$

$$\begin{aligned} \operatorname{ctn} \varphi &= \pm \operatorname{ctn} \alpha, \\ \operatorname{ctn} \varphi &= \pm \tan \alpha. \end{aligned}$$

If *any two* of the six elementary functions (not being reciprocals of each other) have equal values for  $\varphi$  and  $\alpha$  the only solution is

$$\varphi = 2k\pi + \alpha.$$

### *Inverse Circular Functions.*

$$\begin{aligned}
 446. \quad & \sin^{-1} x = \cos^{-1} \sqrt{1 - x^2} = \tan^{-1} \frac{x}{\sqrt{1 - x^2}}, \\
 & = \operatorname{ctn}^{-1} \frac{1}{x} \sqrt{1 - x^2} = \sec^{-1} \frac{1}{\sqrt{1 - x^2}} = \csc^{-1} \frac{1}{x}, \\
 & = 2 \sin^{-1} \sqrt{\frac{1}{2}(1 - \sqrt{1 - x^2})} = \frac{1}{2} \sin^{-1}(2x \sqrt{1 - x^2}), \\
 & = 2 \tan^{-1} \frac{1 - \sqrt{1 - x^2}}{x} = \frac{1}{2} \tan^{-1} \frac{2x \sqrt{1 - x^2}}{1 - 2x^2}, \\
 & = \frac{1}{2}\pi - \cos^{-1} x = \frac{1}{2}\pi - \sin^{-1} \sqrt{1 - x^2} \\
 & = -\sin^{-1}(-x), \\
 & = \frac{1}{4}\pi + \frac{1}{2} \sin^{-1}(2x^2 - 1) = \frac{1}{2} \cos^{-1}(1 - 2x^2).
 \end{aligned}$$

$$\begin{aligned}
 447. \quad & \cos^{-1} x = \sin^{-1} \sqrt{1 - x^2} = \tan^{-1} \frac{1}{x} \sqrt{1 - x^2} \\
 & = \operatorname{ctn}^{-1} \frac{x}{\sqrt{1 - x^2}} = \sec^{-1} \frac{1}{x}, \\
 & = \csc^{-1} \frac{1}{\sqrt{1 - x^2}}, \quad = 2 \cos^{-1} \sqrt{\frac{1}{2}(1 + x)}, \\
 & = \frac{1}{2} \cos^{-1}(2x^2 - 1) = 2 \tan^{-1} \sqrt{\frac{1 - x}{1 + x}} \\
 & = 2 \operatorname{ctn}^{-1} \sqrt{\frac{1 + x}{1 - x}}, \\
 & = \frac{1}{2} \tan^{-1} \left( \frac{2x \sqrt{1 - x^2}}{2x^2 - 1} \right), \\
 & = \frac{1}{2}\pi - \sin^{-1} x = \pi - \cos^{-1}(-x), \\
 & = \frac{1}{2}\pi - \cos^{-1} \sqrt{1 - x^2}.
 \end{aligned}$$

$$\begin{aligned}
 448. \quad & \tan^{-1} x = \sin^{-1} \frac{x}{\sqrt{1+x^2}} \quad \cos^{-1} \frac{1}{\sqrt{1+x^2}} = \operatorname{ctn}^{-1} \frac{1}{x}, \\
 & = \sec^{-1} \sqrt{1+x^2} = \csc^{-1} \frac{1}{x} \sqrt{1+x^2} \\
 & = \frac{1}{2}\pi - \tan^{-1} \frac{1}{x} = -\tan^{-1}(-x), \\
 & = \frac{1}{2} \tan^{-1} \frac{2x}{1-x^2} = \frac{1}{2}\pi - \operatorname{ctn}^{-1} x, \\
 & = \frac{1}{2} \sin^{-1} \frac{2x}{1+x^2} = \frac{1}{2} \cos^{-1} \frac{1-x^2}{1+x^2}, \\
 & = 2 \cos^{-1} \sqrt{\frac{1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}} \\
 & = 2 \sin^{-1} \sqrt{\frac{-1+\sqrt{1+x^2}}{2\sqrt{1+x^2}}}. \\
 & = 2 \tan^{-1} \left( \frac{-1+\sqrt{1+x^2}}{x} \right)
 \end{aligned}$$

$$\begin{aligned}
 449. \quad & \sin^{-1} x + \sin^{-1} y \\
 & = \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}), \\
 & = \pi - \sin^{-1} (x\sqrt{1-y^2} + y\sqrt{1-x^2}). \\
 450. \quad & \sin^{-1} x - \sin^{-1} y \\
 & = \sin^{-1} (x\sqrt{1-y^2} - y\sqrt{1-x^2}). \\
 451. \quad & \cos^{-1} x \pm \cos^{-1} y \\
 & = \cos^{-1} (xy \mp \sqrt{(1-x^2)(1-y^2)}).
 \end{aligned}
 \quad \left. \begin{array}{l} \\ \\ \\ \end{array} \right\} x^2 + y^2 \leq 1.$$

$$\begin{aligned}
 452. \quad & \tan^{-1} x + \tan^{-1} y = \tan^{-1} \frac{x+y}{1-xy}, \quad [xy \leq 1. \\
 453. \quad & \tan^{-1} x + \tan^{-1} y = \pi - \tan^{-1} \frac{x+y}{xy-1}, \quad [xy \geq 1. \\
 454. \quad & \tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{x-y}{1+xy}.
 \end{aligned}$$

*Relations of Circular, Exponential, and Logarithmic Functions.*

455.  $e^{ix} = \cos x + i \sin x.$

456.  $e^{-ix} = \cos x - i \sin x.$

457.  $(\cos x + i \sin x)(\cos x - i \sin x) = 1.$

458.  $e^{2ix} = \frac{1 + i \tan x}{1 - i \tan x}.$

459.  $\cos x = \frac{1}{2}(e^{ix} + e^{-ix}).$

460.  $\sin x = -\frac{1}{2}i(e^{ix} - e^{-ix}), \text{ or } i \sin x = \frac{1}{2}(e^{ix} - e^{-ix}).$

461.  $\tan x = -i \frac{e^{2ix} - 1}{e^{2ix} + 1}, \quad \text{or } i \tan x = \frac{e^{2ix} - 1}{e^{2ix} + 1}.$

462.  $\cos^{-1} x = -i \log_e(x + i \sqrt{1 - x^2}).$

463.  $\sin^{-1} x = -i \log_e(ix + \sqrt{1 - x^2}).$

464.  $\tan^{-1} x = -\frac{1}{2}i \log_e \frac{1 + ix}{1 - ix} = \frac{1}{2}i \log_e \frac{1 - ix}{1 + ix} = \frac{1}{2}i \log_e \frac{i+x}{i-x}.$

465.  $\cos ix = \frac{1}{2}(e^x + e^{-x}) = \text{Cosh } x.$

466.  $\sin ix = \frac{1}{2}i(e^x - e^{-x}) = i \text{ Sinh } x.$

467.  $\tan ix = i \frac{(e^x - e^{-x})}{(e^x + e^{-x})} = i \text{ Tanh } x.$

} See 714.

468.  $e^{x+iy} = e^x(\cos y + i \sin y).$

469.  $a^{x+iy} = a^x[\cos(y \log_e a) + i \sin(y \log_e a)].$

From  $2 \cos u = e^{iu} + e^{-iu},$

$2i \sin u = e^{iu} - e^{-iu},$

are obtained,

470.  $2^{m-1} \cos^m u = \cos mu + C_1 \cos(m-2)u + C_2 \cos(m-4)u + C_3 \cos(m-6)u + \dots$

471.  $(-1)^{\frac{m}{2}} 2^{m-1} \sin^m u = \cos mu - C_1 \cos(m-2)u + C_2 \cos(m-4)u - C_3 \cos(m-6)u + \dots$   
when  $m$  is even,

472.  $(-1)^{\frac{m-1}{2}} 2^{m-1} \sin^m u = \sin mu - C_1 \sin(m-2)u + C_2 \sin(m-4)u - C_3 \sin(m-6)u + \dots$   
when  $m$  is odd,

wherein  $C_1, C_2, C_3 \dots$  are the Binomial Coefficients.

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## PLANE TRIANGLES.

*General Properties.*

Formulas expressing the general properties of triangles usually occur in sets of three, of which only one needs to be printed. The others are obtained from it by a *cyclic change of letters*, that is, by changing  $a$  to  $b$ ,  $b$  to  $c$ , and  $c$  to  $a$ , also  $A$  to  $B$ ,  $B$  to  $C$ , and  $C$  to  $A$ , always in this fixed order. This process applied to the first equation of 501 gives the second, applied to the second gives the third, and applied to the third gives the first again. And so in all cases.

$$501. \begin{cases} a = b \cos C + c \cos B, \\ b = c \cos A + a \cos C, \\ c = a \cos B + b \cos A. \end{cases}$$

All the relations between the six parts of a plane triangle are implicitly contained in, and can, by algebraic transformations, be derived from the three equations, 501.

$$502. A + B + C = 180^\circ.$$

$$503. \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}.$$

$$504. \frac{a+b}{a-b} = \frac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} \dots \textcircled{3}.*$$

$$505. \frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)} \dots \textcircled{3}.$$

$$506. \frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)} \dots \textcircled{3}.$$

$$507. a^2 = b^2 + c^2 - 2bc \cos A \dots \textcircled{3}.$$

$$508. a^2 + b^2 + c^2 = 2bc \cos A + 2ca \cos B + 2ab \cos C.$$

\* The symbol  $\textcircled{3}$  indicates that there is a full set of three equations of which only one is printed, the other two being obtainable from this one by a cyclic change of letters.

If  $d$  denote the diagonal of a parallelogram drawn from the point where the sides  $a$  and  $b$  meet making an angle  $C$  with each other,

$$509. \quad d^2 = a^2 + b^2 + 2ab \cos C.$$

$$510. \text{ Let } s = \frac{1}{2}(a + b + c),$$

$$\text{whence, } s - a = \frac{1}{2}(-a + b + c),$$

$$s - b = \frac{1}{2}(a - b + c),$$

$$s - c = \frac{1}{2}(a + b - c).$$

$$511. \quad \sin \frac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{bc}} \dots \textcircled{3}.$$

$$512. \quad \cos \frac{1}{2}A = \sqrt{\frac{s(s - a)}{bc}} \dots \textcircled{3}.$$

$$513. \quad \tan \frac{1}{2}A = \sqrt{\frac{(s - b)(s - c)}{s(s - a)}} \dots \textcircled{3}.$$

Let  $r$  denote the radius of the inscribed circle.

$$514. \quad r = \sqrt{\frac{(s - a)(s - b)(s - c)}{s}},$$

$$= s \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C,$$

$$= \frac{a \sin \frac{1}{2}B \sin \frac{1}{2}C}{\cos \frac{1}{2}A} \dots \textcircled{3}.$$

$$515. \quad \tan \frac{1}{2}A = \frac{r}{s - a} \dots \textcircled{3}.$$

Let  $T$  denote the area of the triangle.

$$516. \quad T = sr = \sqrt{s(s - a)(s - b)(s - c)},$$

$$= \frac{1}{4} \sqrt{2b^2c^2 + 2c^2a^2 + 2a^2b^2 - a^4 - b^4 - c^4},$$

$$= \frac{2abc}{a + b + c} \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C,$$

$$= \frac{1}{4}(a + b + c)^2 \tan \frac{1}{2}A \tan \frac{1}{2}B \tan \frac{1}{2}C,$$

$$= \frac{c^2 \sin A \sin B}{2 \sin(A + B)} = \frac{1}{2}ab \sin C \dots \textcircled{3}.$$

$$517. \quad \sin A = \frac{2T}{bc} = \frac{2sr}{bc} \dots \textcircled{3}.$$

Let  $p_a$ ,  $p_b$ ,  $p_c$  denote perpendiculars from the vertices upon the sides  $a$ ,  $b$ , and  $c$  respectively. Then

$$518. \quad p_a = b \sin C = c \sin B = \frac{bc \sin A}{a} \dots \textcircled{3},$$

$$= \frac{\sin B \sin C}{\sin A} a = \frac{2T}{a} \dots \textcircled{3}.$$

Let  $r_a$ ,  $r_b$ , and  $r_c$  denote the radii of the three escribed circles touching externally the sides  $a$ ,  $b$ , and  $c$  respectively.

$$519. \quad r_a = s \tan \frac{1}{2}A = \frac{T}{s-a} = \frac{sr}{s-a} = \frac{a \cos \frac{1}{2}B \cos \frac{1}{2}C}{\cos \frac{1}{2}A} \quad \textcircled{3}.$$

$$520. \quad T = \sqrt{r r_a r_b r_c}.$$

$$521. \quad r_a r_b r_c = abc \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C.$$

$$522. \quad r = \sqrt{r_a r_b} + \sqrt{r_b r_c} + \sqrt{r_c r_a}.$$

$$523. \quad \tan \frac{1}{2}A = \sqrt{\frac{r r_a}{r_b r_c}} \dots \textcircled{3}.$$

$$524. \quad \frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r} = \frac{1}{p_a} + \frac{1}{p_b} + \frac{1}{p_c}.$$

Let  $R$  denote the radius of the circumscribed circle.

$$525. \quad R = \frac{1}{2}a \csc A$$

$$= \frac{1}{4}\sqrt{(b+c)^2 \sec^2 \frac{1}{2}A + (b-c)^2 \csc^2 \frac{1}{2}A} \dots \textcircled{3},$$

$$= \frac{1}{4}s \sec \frac{1}{2}A \sec \frac{1}{2}B \sec \frac{1}{2}C,$$

$$= \frac{1}{4}(r_a + r_b + r_c - r) = \frac{abc}{4T}.$$

$$526. \quad 2Rr = \frac{abc}{a+b+c}.$$

$$527. \quad R + r = \frac{1}{2}(a \ctn A + b \ctn B + c \ctn C),$$

= sum of the perpendiculars to the sides from the centre of the circumscribing circle.

$$528. \quad \sqrt{R^2 - 2Rr} = \text{distance between the centres of the inscribed and circumscribed circles.}$$

If  $O$  be the centre of the inscribed circle its distance from the vertex  $A$  of the triangle is

$$529. \quad OA = \frac{2bc}{a+b+c} \cos \frac{1}{2}A \dots \textcircled{3}.$$

$$530. \quad \cos A + \cos B + \cos C = 1 + \frac{2a \sin B \sin C}{a+b+c} \dots \textcircled{3}.$$

$$531. \quad a \cos A + b \cos B + c \cos C = 4R \sin A \sin B \sin C, \\ = 2a \sin B \sin C \dots \textcircled{3}.$$

If  $A + B + C = 180^\circ$ , then follow 532–540,

$$532. \quad \sin A + \sin B + \sin C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \cos \frac{1}{2}C,$$

$$533. \quad \sin A + \sin B - \sin C = 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \cos \frac{1}{2}C,$$

$$534. \quad \cos A + \cos B + \cos C = 4 \sin \frac{1}{2}A \sin \frac{1}{2}B \sin \frac{1}{2}C + 1,$$

$$535. \quad \cos A + \cos B - \cos C = 4 \cos \frac{1}{2}A \cos \frac{1}{2}B \sin \frac{1}{2}C - 1,$$

$$536. \quad \tan A + \tan B + \tan C = \tan A \tan B \tan C,$$

$$537. \quad \operatorname{ctn} \frac{1}{2}A + \operatorname{ctn} \frac{1}{2}B + \operatorname{ctn} \frac{1}{2}C = \operatorname{ctn} \frac{1}{2}A \operatorname{ctn} \frac{1}{2}B \operatorname{ctn} \frac{1}{2}C,$$

$$538. \quad \sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C,$$

$$539. \quad \cos 2A + \cos 2B + \cos 2C = -4 \cos A \cos B \cos C - 1,$$

$$540. \quad \operatorname{ctn} A \operatorname{ctn} B + \operatorname{ctn} B \operatorname{ctn} C + \operatorname{ctn} C \operatorname{ctn} A = 1.$$


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If  $A + B + C = 90^\circ$ ,

$$541. \quad \tan A \tan B + \tan B \tan C + \tan C \tan A = 1.$$

### *The Quadrilateral Inscribed in a Circle.*

$$542. \quad \text{Angles, } A + C = 180^\circ, \quad \text{Sides, } AB = a, \quad CD = c, \\ B + D = 180^\circ. \quad BC = b, \quad DA = d.$$

$$543. \quad \cos B = \frac{a^2 + b^2 - c^2 - d^2}{2(ab + cd)}.$$

$$544. \quad \text{Diagonal, } \overline{AC}^2 = \frac{(ac + bd)(ad + bc)}{ab + cd}.$$

$$\text{Put } s = \frac{1}{2}(a + b + c + d),$$

$$545. \quad Q = \sqrt{(s-a)(s-b)(s-c)(s-d)} \\ = \text{area of the quadrilateral.}$$

546.  $\sin A = \sin C = \frac{2Q}{bc + da}.$

547.  $\sin B = \sin D = \frac{2Q}{ab + cd}.$

Radius of the circumscribed circle,

548.  $R = \frac{1}{4Q} \sqrt{(ab + cd)(ac + bd)(ad + bc)}.$

*Solution of Right Triangles.*

549. CASE I. Given an angle and the hypotenuse,  $A$  and  $h$ , to find  $B$ ,  $a$ , and  $b$ .

$$B = 90^\circ - A.$$

$$a = h \sin A.$$

$$\log a = \log h + \log \sin A.$$

$$b = h \cos A.$$

$$\log b = \log h + \log \cos A.$$

TEST. The computed values of  $a$  and  $b$  should satisfy

$$2 \log a = \log (h + b) + \log (h - b),$$

$$2 \log b = \log (h + a) + \log (h - a).$$

550. CASE II. Given an angle and the leg opposite,  $A$  and  $a$ , to find  $B$ ,  $h$ , and  $b$ .

$$B = 90^\circ - A.$$

$$h = a \csc A.$$

$$\log h = \log a + \log \csc A.$$

$$b = a \ctn A.$$

$$\log b = \log a + \log \ctn A.$$

TEST. The computed values of  $h$  and  $b$  should satisfy

$$2 \log a = \log (h + b) + \log (h - b),$$

$$2 \log b = \log (h + a) + \log (h - a).$$

551. CASE III. Given an angle and the leg adjacent,  $A$  and  $b$ , to find  $B$ ,  $h$ , and  $a$ .

$$B = 90^\circ - A.$$

$$h = b \sec A.$$

$$\log h = \log b + \log \sec A.$$

$$a = b \tan A.$$

$$\log a = \log b + \log \tan A.$$

TEST. The computed values of  $h$  and  $a$  should satisfy

$$2 \log a = \log (h + b) + \log (h - b),$$

$$2 \log b = \log (h + a) + \log (h - a).$$

**552. CASE IV.** Given the hypotenuse and a leg,  $h$  and  $a$ , to find  $A$ ,  $B$ , and  $b$ .

$$\sin A = \cos B = \frac{a}{h}, \quad \log \sin A = \log \cos B \\ = \log a + \text{co-log } h - 10.$$

$$\text{whence } A = \sin^{-1} \frac{a}{h},$$

$$B = \cos^{-1} \frac{a}{h}.$$

$$b = \sqrt{(h+a)(h-a)}. \quad \log b = \frac{1}{2} [\log(h+a) + \log(h-a)].$$

**TEST.** The computed values of  $A$  and  $b$  should satisfy

$$\log b + \log \tan A = \log a.$$

**553. CASE V.** Given the two legs,  $a$  and  $b$ , to find  $A$ ,  $B$ , and  $h$ .

$$\tan A = \frac{a}{b}. \quad \log \tan A = \log \operatorname{ctn} B \\ = \log a + \text{co-log } b - 10.$$

$$\operatorname{ctn} B = \frac{a}{b}.$$

$$A = \tan^{-1} \frac{a}{b}.$$

$$B = \operatorname{ctn}^{-1} \frac{a}{b}.$$

$$h = a \csc A, \quad \log h = \log a + \log \csc A. \\ = \sqrt{a^2 + b^2}.$$

**TEST.** The computed value of  $h$  should satisfy

$$2 \log a = \log(h+b) + \log(h-b), \\ 2 \log b = \log(h+a) + \log(h-a).$$

### *Special Formulas for Plane Right Triangles.*

$$554. \quad h - b = 2h \sin^2 \frac{1}{2}A,$$

gives  $h - b$  with great accuracy when  $A$  is small, or  $A$  with great accuracy when  $h$  and  $b$  are nearly equal.

$$555. \quad \tan \frac{1}{2}A = \sqrt{\frac{h-b}{h+b}} = \frac{h-b}{a}.$$

$$556. \tan(45^\circ \pm A) = \frac{b \pm a}{b \mp a}$$

$$557. \sin(B - A) = \frac{(b + a)(b - a)}{h^2}, \cos(B - A) = \frac{2ab}{h^2}.$$

$$558. \tan(B - A) = \frac{(b + a)(b - a)}{2ab},$$

a formula which gives  $B - A$  with great accuracy when  $a$  and  $b$  are given nearly equal, or  $b - a$  with great accuracy when  $A$  and  $B$  are given nearly equal.

For the use of  $S$  and  $T$ , and a table of the values of these functions up to  $2^\circ$ , see page 83.

### *Solution of Plane Oblique Triangles.*

An oblique triangle can be solved in two ways, (*i*) by converting it into the sum or the difference of two right triangles, formed by drawing a perpendicular from any vertex, and solving these right triangles by methods above given (549–553), or (*ii*) by substituting the given parts in general formulas, and working out the required parts. Either method serves well to test the accuracy of the results obtained by the other. The outlines of solutions by both methods are given in the following formulas.

To preserve a uniformity of notation let the perpendiculars from the vertices  $A$ ,  $B$ , and  $C$  upon the sides  $a$ ,  $b$ , and  $c$  of the triangle be marked  $AP$ ,  $BQ$ ,  $CR$  respectively; so that always

$$559. a = BP + PC, b = CQ + QA, c = AR + RB.$$

These equations hold as well when the perpendicular falls without as when it falls within the triangle, if regard be had to the principle stated in 335.

**CASE I.** Given two angles and a side,  $A$ ,  $B$ , and  $c$ , to find  $a$ ,  $b$ , and  $C$ .

$$C = 180^\circ - (A + B).$$

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**564. SECOND METHOD.** By the general formulas.

$$\frac{1}{2}(A + B) = 90^\circ - \frac{1}{2}C.$$

$$\tan \frac{1}{2}(A - B) = \frac{a - b}{a + b} \tan \frac{1}{2}(A + B).$$

$$A = \frac{1}{2}(A + B) + \frac{1}{2}(A - B).$$

$$B = \frac{1}{2}(A + B) - \frac{1}{2}(A - B).$$

$$c = \frac{a \sin C}{\sin A} = \frac{b \sin C}{\sin B} = 2R \sin C.$$

Or, as a test,  $c$  may be computed by either of the equations

$$c = (a + b) \frac{\cos \frac{1}{2}(A + B)}{\cos \frac{1}{2}(A - B)} = (a - b) \frac{\sin \frac{1}{2}(A + B)}{\sin \frac{1}{2}(A - B)}.$$

**CASE III.** Given two sides and the angle opposite one of them  $a$ ,  $b$ , and  $A$ , to find  $B$ ,  $C$ , and  $c$ .

**565. FIRST METHOD.** By drawing a perpendicular from the point where the given sides meet.

Perpendicular CR.

$$CR = b \sin A.$$

$$AR = b \cos A.$$

$$RB = \sqrt{(a + CR)(a - CR)}.$$

$c = AR \pm RB$ , a double solution.

$$c_1 = AR + RB \}$$

$$c_2 = AR - RB \}$$

$$\text{Angle ACR} = 90^\circ - A.$$

$$\text{Angle RCB} = \sin^{-1} \frac{RB}{a} = \cos^{-1} \frac{CR}{a}.$$

$$C_1 = ACR + RCB \}.$$

$$C_2 = ACR - RCB \}$$

$$B_1 = 180^\circ - (A + C_1) = A + C_2 \}.$$

$$B_2 = 180^\circ - (A + C_2) = A + C_1 \}$$

If  $CR = b \sin A < a$ , there is no real solution.

**566. SECOND METHOD.** By the general formulas.

$$\sin B = \frac{b \sin A}{a}, \text{ giving two values of } B, B_1 < 90^\circ < B_2.$$

$$C_1 = B_2 - A, \quad C_2 = B_1 - A.$$

$$c_1 = \frac{a \sin C_1}{\sin A}, \quad c_2 = \frac{a \sin C_2}{\sin A}.$$

If  $\sin B$  comes out greater than 1, there is no real solution.

If  $C_2$  comes out negative the second solution is inadmissible.

**CASE IV.** Given the three sides  $a$ ,  $b$ , and  $c$ , to find the angles  $A$ ,  $B$ , and  $C$ .

**567. FIRST METHOD.** By drawing a perpendicular to the longest side, or to the side that is not less than either of the other two.

Let  $c > b > a$ . Or, let  $c$  be not less than either  $b$  or  $a$ .

Perpendicular CR.

$$AR + RB = c.$$

$$AR - RB = \frac{(b + a)(b - a)}{c}.$$

$$\cos A = \frac{AR}{b}, \text{ or } A = \cos^{-1} \frac{AR}{b}.$$

$$\cos B = \frac{RB}{a}, \text{ or } B = \cos^{-1} \frac{RB}{a}.$$

$$C = 180^\circ - (A + B).$$

TEST, by drawing another perpendicular, which *may* fall outside the triangle.

Perpendicular AP (which is supposed to fall outside the triangle),

$$BP - CP = a.$$

$$BP + CP = \frac{(c + b)(c - b)}{a}.$$

$$\cos B = \frac{BP}{c}, \text{ or } B = \cos^{-1} \frac{BP}{c}.$$

$$\cos C = \frac{-CP}{b}, \text{ or } C = \cos^{-1} \left( \frac{-CP}{b} \right) > 90^\circ.$$

$$A = 180^\circ - (B + C).$$

**568. SECOND METHOD.** By the general formulas.

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

$$\tan \frac{1}{2}A = \frac{r}{s-a}, \quad \tan \frac{1}{2}B = \frac{r}{s-b}, \quad \tan \frac{1}{2}C = \frac{r}{s-c}.$$

TEST.  $A + B + C = 180^\circ$ .

The half-angles may also be computed by

$$\sin \frac{1}{2}A = \sqrt{\frac{(s-b)(s-c)}{bc}} \dots \textcircled{3}.$$

or by  $\cos \frac{1}{2}A = \sqrt{\frac{s(s-a)}{bc}} \dots \textcircled{3}.$

Special Formulas for the case of two nearly equal sides or angles,

$$569. \quad a - b = \frac{2a \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{\sin A} \dots \textcircled{3}.$$

$$570. \quad \sin \frac{1}{2}(A-B) = \frac{(a-b) \sin A}{2a \cos \frac{1}{2}(A+B)} \dots \textcircled{3}.$$

For the use of  $S$  and  $T$  and a table of the values of these functions up to  $2^\circ$ , see page 83.

## SPHERICAL TRIANGLES.

*General Properties.*

Let  $a, b, c$ , denote the sides and  $A, B, C$ , the angles of a spherical triangle; and let  $a', b', c'$ , denote the sides and  $A', B', C'$ , the angles of its polar triangle. Then

$$\begin{array}{ll} \text{601.} & a + A' = 180^\circ. \quad A + a' = 180^\circ. \\ & b + B' = 180^\circ. \quad B + b' = 180^\circ. \\ & c + C' = 180^\circ. \quad C + c' = 180^\circ. \end{array}$$

The perpendicular great-circle arcs  $AP, BQ, CR$ , drawn in the spherical triangle coincide with the perpendicular great-circle arcs  $A'P', B'Q', C'R'$ , similarly drawn in its polar triangle.

The fundamental equations, from which are or can be derived all other general equations relating to spherical triangles, are these three,

$$\begin{array}{l} \text{602. } \left\{ \begin{array}{l} \cos a = \cos b \cos c + \sin b \sin c \cos A, \\ \cos b = \cos c \cos a + \sin c \sin a \cos B, \\ \cos c = \cos a \cos b + \sin a \sin b \cos C. \end{array} \right. \end{array}$$

$$\begin{array}{l} \text{603. } \frac{\sin a}{\sin A} \quad \frac{\sin b}{\sin B} \quad \frac{\sin c}{\sin C} = M = \text{The Modulus.} \end{array}$$

$$\begin{array}{l} \text{604. } \left\{ \begin{array}{l} \cos A = -\cos B \cos C + \sin B \sin C \cos a, \\ \cos B = -\cos C \cos A + \sin C \sin A \cos b, \\ \cos C = -\cos A \cos B + \sin A \sin B \cos c. \end{array} \right. \end{array}$$

$$\begin{array}{l} \text{605. } \left\{ \begin{array}{l} \operatorname{ctn} a \sin b = \cos b \cos C + \sin C \operatorname{ctn} A, \\ \operatorname{ctn} b \sin c = \cos c \cos A + \sin A \operatorname{ctn} B, \\ \operatorname{ctn} c \sin a = \cos a \cos B + \sin B \operatorname{ctn} C, \\ \operatorname{ctn} a \sin c = \cos c \cos B + \sin B \operatorname{ctn} A, \\ \operatorname{ctn} b \sin a = \cos a \cos C + \sin C \operatorname{ctn} B, \\ \operatorname{ctn} c \sin b = \cos b \cos A + \sin A \operatorname{ctn} C. \end{array} \right. \end{array}$$

$$606. \left\{ \begin{array}{ll} \sin a \cos B & \sin c \cos b - \cos c \sin b \cos A, \\ \sin b \cos C & \sin a \cos c - \cos a \sin c \cos B, \\ \sin c \cos A & \sin b \cos a - \cos b \sin a \cos C, \\ \\ \sin a \cos C & \sin b \cos c - \cos b \sin c \cos A, \\ \sin b \cos A & \sin c \cos a - \cos c \sin a \cos B, \\ \sin c \cos B & = \sin a \cos b - \cos a \sin b \cos C. \end{array} \right.$$

$$607. \left\{ \begin{array}{ll} \sin A \cos b & = \sin C \cos B + \cos C \sin B \cos a, \\ \sin B \cos c & = \sin A \cos C + \cos A \sin C \cos b, \\ \sin C \cos a & = \sin B \cos A + \cos B \sin A \cos c, \\ \\ \sin A \cos c & = \sin B \cos C + \cos B \sin C \cos a, \\ \sin B \cos a & = \sin C \cos A + \cos C \sin A \cos b, \\ \sin C \cos b & = \sin A \cos B + \cos A \sin B \cos c. \end{array} \right.$$

$$608. \cos a = \cos(b+c) + 2 \sin b \sin c \cos^2 \frac{1}{2}A \dots (3), \\ = \cos(b-c) - 2 \sin b \sin c \sin^2 \frac{1}{2}A \dots (3).$$

$$609. \cos A = -\cos(B+C) - 2 \sin B \sin C \sin^2 \frac{1}{2}a \dots (3) \\ = -\cos(B-C) + 2 \sin B \sin C \cos^2 \frac{1}{2}a \dots (3)$$

$$610. s = \frac{1}{2}(a+b+c).$$

$$611. \sin \frac{1}{2}A = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin b \sin c}} \dots (3).$$

$$612. \cos \frac{1}{2}A = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} \dots (3).$$

$$613. \tan \frac{1}{2}A = \sqrt{\frac{\sin(s-b) \sin(s-c)}{\sin s \sin(s-a)}} \dots (3).$$

$$614. S = \frac{1}{2}(A+B+C).$$

Spherical Excess of a Spherical Triangle =  $2E$ .

$$615. 2E = A + B + C - 180^\circ = 2S - 180^\circ.$$

$$616. \cos \frac{1}{2}a = \sqrt{\frac{\sin(B-E) \sin(C-F)}{\sin B \sin C}} \dots (3), \\ = \sqrt{\frac{\cos(S-B) \cos(S-C)}{\sin B \sin C}} \dots (3).$$

$$617. \sin \frac{1}{2}a = \sqrt{\frac{\sin E \sin (A-E)}{\sin B \sin C}} \dots \textcircled{3},$$

$$- \sqrt{\frac{-\cos S \cos (S-A)}{\sin B \sin C}} \dots \textcircled{3}.$$

$$618. \operatorname{ctn} \frac{1}{2}a = \sqrt{\frac{\sin (B-E) \sin (C-E)}{\sin E \sin (A-E)}} \dots \textcircled{3},$$

$$- \sqrt{\frac{\cos (S-B) \cos (S-C)}{-\cos S \cos (S-A)}} \dots \textcircled{3}.$$

$$619. \tan \frac{1}{2}E = \sqrt{\tan \frac{1}{2}s \tan \frac{1}{2}(s-a) \tan \frac{1}{2}(s-b) \tan \frac{1}{2}(s-c)}.$$

Let  $r$  denote the polar radius of the circle inscribed in the spherical triangle.

$$620. \tan r = \sqrt{\frac{\sin (s-a) \sin (s-b) \sin (s-c)}{\sin s}}.$$

Let  $R$  denote the polar radius of the circle circumscribed about the spherical triangle.

$$621. \operatorname{ctn} R = \sqrt{\frac{\sin (A-E) \sin (B-E) \sin (C-E)}{\sin E}}$$

$$622. \tan \frac{1}{2}A = \frac{\tan r}{\sin (s-a)} \dots \textcircled{3}.$$

$$623. \operatorname{ctn} \frac{1}{2}a = \frac{\operatorname{ctn} R}{\sin (A-E)} \dots \textcircled{3}.$$

$$624. \operatorname{ctn} R = \sqrt{\frac{\cos (S-A) \cos (S-B) \cos (S-C)}{-\cos S}}.$$

$$625. \operatorname{ctn} \frac{1}{2}a = \frac{\operatorname{ctn} R}{\cos (s-A)} \dots \textcircled{3}.$$

$$626. 2 \tan r \sin s = \sin A \sin b \sin c \dots \textcircled{3}.$$

$$627. 2 \operatorname{ctn} R \sin E = \sin a \sin B \sin C \dots \textcircled{3}.$$

$$628. M = \text{the modulus} = \frac{\tan r \sin s}{\operatorname{ctn} R \sin E}$$

$$629. \quad \tan E = \frac{\tan \frac{1}{2}a \tan \frac{1}{2}b \sin C}{1 + \tan \frac{1}{2}a \tan \frac{1}{2}b \cos C} \dots \textcircled{3},$$

$$= \frac{2 \tan r \sin s}{1 + \cos a + \cos b + \cos c}.$$

Let  $p_a, p_b, p_c$ , denote the perpendicular great-circle arcs upon the sides  $a, b, c$ , respectively from the opposite vertices.

$$630. \quad \sin p_a = \sin b \sin C = \sin c \sin B \textcircled{3},$$

$$= \frac{\sin b \sin c \sin A}{\sin a} = \frac{\sin B \sin C \sin a}{\sin A} \textcircled{3},$$

$$= \frac{2 \tan r \sin s}{\sin a} = \frac{2 \operatorname{ctn} R \sin E}{\sin A} \dots \textcircled{3}.$$

Let  $r_a, r_b, r_c$ , denote the radii of the circles inscribed in the three triangles formed by prolonging the sides of the original triangle till they meet in points diametrically opposite its vertices.

Let  $R_a, R_b, R_c$  denote the radii of the circles circumscribed about these same triangles.

$$631. \quad \tan r_a = \sin s \tan \frac{1}{2}A = \frac{\sin s \tan r}{\sin(s - a)} \dots \textcircled{3}.$$

$$632. \quad \operatorname{ctn} R_a = \sin E \operatorname{ctn} \frac{1}{2}a = \frac{\sin E \operatorname{ctn} R}{\sin(A - E)} \dots \textcircled{3}.$$

### *Cagnoli's Equation.*

633.

$$\sin a \sin b + \cos a \cos b \cos C = \sin A \sin B - \cos A \cos B \cos c.$$

Either member of this equation has the same value in a given spherical triangle and in its polar triangle.

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*The Solution of Spherical Right Triangles.*

**638. CASE I.** Given  $A$  and  $h$ , to find  $a$ ,  $b$ , and  $B$ .

For solution,

$$\begin{aligned}\sin a &= \sin A \sin h, \\ \tan b &= \cos A \tan h, \\ \operatorname{ctn} B &= \tan A \cos h.\end{aligned}$$

For a test,

$$\sin a = \tan b \operatorname{ctn} B.$$

Two values of each computed part,

$$\begin{aligned}a_2 &= 180^\circ - a_1, \\ b_2 &= 180^\circ + b_1, \\ B_2 &= 180^\circ + B_1.\end{aligned}$$

For deciding which of the computed values belong together,

$$\begin{aligned}\tan A &= \frac{\tan a}{\sin b}, \\ \cos A &= \cos a \sin B.\end{aligned}$$

**639. CASE II.** Given  $A$  and  $a$ , to find  $h$ ,  $b$ , and  $B$ .

For solution,

$$\begin{aligned}\sin h &= \csc A \sin a, \\ \sin b &= \operatorname{ctn} A \tan a, \\ \sin B &= \cos A \sec a.\end{aligned}$$

For a test,

$$\sin b = \sin B \sin h.$$

No solution if

$$\pm (a - 90^\circ) < \pm (A - 90^\circ);$$

that is, if  $\sin a > \sin A$ .

Two values of each computed part,

$$\begin{aligned}h_2 &= 180^\circ - h_1, \\ b_2 &= 180^\circ - b_1, \text{ or } 540^\circ - b_1, \\ B_2 &= 180^\circ - B_1, \text{ or } 540^\circ - B_1.\end{aligned}$$

For deciding which of the computed values belong together,

$$\begin{aligned}\cos a &= \frac{\cos h}{\cos b}, \\ \tan a &= \tan h \cos B.\end{aligned}$$

**640. CASE III.** Given  $A$  and  $b$ , to find  $a$ ,  $h$ , and  $B$ .

For solution,

$$\begin{aligned}\tan a &= \tan A \sin b, \\ \operatorname{ctn} h &= \cos A \operatorname{ctn} b, \\ \cos B &= \sin A \cos b.\end{aligned}$$

For a test,

$$\tan a \operatorname{ctn} h = \cos B.$$

Two values of each computed part,

$$\begin{aligned}a_2 &= 180^\circ + a_1, \\ h_2 &= 180^\circ + h_1, \\ B_2 &= 360^\circ - B_1.\end{aligned}$$

For deciding which of the computed values belong together,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} \quad \sin h.$$

**641. CASE IV.** Given  $h$  and  $a$ , to find  $A$ ,  $b$ , and  $B$ .

For solution,

$$\begin{aligned}\sin A &= \csc h \sin a, \\ \cos b &= \cos h \sec a, \\ \cos B &= \operatorname{ctn} h \tan a.\end{aligned}$$

For a test,

$$\sin A \cos b = \cos B.$$

Two values of each computed part,

$$\begin{aligned}A_2 &= 180^\circ - A_1, \\ b_2 &= 360^\circ - b_1, \\ B_2 &= 360^\circ - B_1.\end{aligned}$$

For deciding which of the computed values belong together,

$$\begin{aligned}\tan a &= \sin b \tan A, \\ \cos a &= \frac{\cos A}{\sin B}.\end{aligned}$$

No solution if  $a < 90^\circ$  and either  $h < a$  or  $h > 180^\circ - a$ ;  
or if  $a > 90^\circ$  and either  $h > a$  or  $h < 180^\circ - a$ .

**642. CASE V.** Given  $a$  and  $b$ , to find  $h$ ,  $A$ , and  $B$ .

For solution,

$$\begin{aligned}\cos h &= \cos a \cos b, \\ \operatorname{ctn} A &= \operatorname{ctn} a \sin b, \\ \operatorname{ctn} B &= \sin a \operatorname{ctn} b.\end{aligned}$$

For a test,

$$\operatorname{ctn} A \operatorname{ctn} B = \cos h.$$

Two values of each computed part,

$$\begin{aligned}h_2 &= 360^\circ - h_1, \\ A_2 &= 180^\circ + A_1, \\ B_2 &= 180^\circ + B_1.\end{aligned}$$

For deciding which of the computed values belong together,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \sin h.$$

**643. CASE VI.** Given  $A$  and  $B$ , to find  $h$ ,  $a$ , and  $b$ .

For solution,

$$\begin{aligned}\cos h &= \operatorname{ctn} A \operatorname{ctn} B, \\ \cos a &= \cos A \csc B, \\ \cos b &= \csc A \cos B.\end{aligned}$$

For a test,

$$\cos a \cos b = \cos h.$$

Two values of each computed part,

$$\begin{aligned}h_2 &= 360^\circ - h_1, \\ a_2 &= 360^\circ - a_1, \\ b_2 &= 360^\circ - b_1.\end{aligned}$$

For deciding which of the computed values belong together,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \sin h.$$

*Solution of Spherical Oblique Triangles.*

**644. CASE I.** Given two sides and the included angle,  $a$ ,  $b$ , and  $C$ , to find  $A$ ,  $B$ , and  $c$ .

Two triangles always possible.

$$\begin{aligned}A_2 &= 180^\circ + A_1, \\B_2 &= 180^\circ + B_1, \\c_2 &= 360^\circ - c_1.\end{aligned}$$

Which of the computed values belong together as parts of one triangle and which as parts of the other is decided by means of the equations

$$\frac{\sin a}{\sin A} - \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = M.$$

The computed values of  $A$ ,  $B$ , and  $c$ , which, with the given values of  $a$ ,  $b$ , and  $C$ , make each of these ratios positive, belong to one triangle, and those which make each ratio negative belong to the other triangle. In other words, one triangle has a positive, the other a negative modulus.

**645. FIRST METHOD.** By 605 and 602,

$$\operatorname{ctn} A = \frac{\cos a \sin b - \sin a \cos b \cos C}{\sin a \sin C},$$

$$\operatorname{ctn} B = \frac{\sin a \cos b - \cos a \sin b \cos C}{\sin b \sin C},$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

**646. SECOND METHOD.** Based on Gauss's Equations [634], which, by taking the logarithm of each term, may be written thus:

$$(i) \log \sin \frac{1}{2}(A+B) + \log \cos \frac{1}{2}c = \log \cos \frac{1}{2}(a-b) + \log \cos \frac{1}{2}C,$$

$$(ii) \log \sin \frac{1}{2}(A-B) + \log \sin \frac{1}{2}c = \log \sin \frac{1}{2}(a-b) + \log \cos \frac{1}{2}C,$$

$$(iii) \log \cos \frac{1}{2}(A+B) + \log \cos \frac{1}{2}c = \log \cos \frac{1}{2}(a+b) + \log \sin \frac{1}{2}C,$$

$$(iv) \log \cos \frac{1}{2}(A-B) + \log \sin \frac{1}{2}c = \log \sin \frac{1}{2}(a+b) + \log \sin \frac{1}{2}C.$$

**647. THIRD METHOD.** Based on Napier's Analogies [635], which, by taking the logarithm of each term, may be written thus:

- (iv)  $\log \tan \frac{1}{2}(A + B) = \log \sec \frac{1}{2}(a + b) + \log \cos \frac{1}{2}(a - b) + \log \operatorname{ctn} \frac{1}{2}C,$
- (iii)  $\log \tan \frac{1}{2}(A - B) = \log \csc \frac{1}{2}(a + b) + \log \sin \frac{1}{2}(a - b) + \log \operatorname{ctn} \frac{1}{2}C,$
- (ii)  $\log \tan \frac{1}{2}c = \log \cos \frac{1}{2}(A + B) + \log \sec \frac{1}{2}(A - B) + \log \tan \frac{1}{2}(a + b),$
- (i)  $\log \tan \frac{1}{2}c = \log \sin \frac{1}{2}(A + B) + \log \csc \frac{1}{2}(A - B) + \log \tan \frac{1}{2}(a - b).$

**648. FOURTH METHOD.** By drawing from  $B$  or from  $A$  a great-circle arc perpendicular to the opposite side, the required triangle is converted into the sum or the difference of two spherical right triangles.

Perpendicular $BQ$ . $\tan CQ = \tan a \cos C.$ $QA = b - CQ,$ or $QA = 360^\circ + b - CQ.$ $\operatorname{ctn} A = \operatorname{ctn} C \csc CQ \sin QA.$ $\cos c = \cos a \sec CQ \cos QA.$	Perpendicular $AP$ . $\tan CP = \tan b \cos C.$ $PB = a - CP,$ or $PB = 360^\circ + a - CP.$ $\operatorname{ctn} B = \operatorname{ctn} C \csc CP \sin PB.$ $\cos c = \cos b \sec CP \cos PB.$
---	---

This method is equivalent to that of some writers who introduce an auxiliary angle,  $\varphi$  or  $\chi$ , defined by the equation  $\tan \varphi = \tan a \cos C$  or  $\tan \chi = \tan b \cos C$ .

**649. CASE II.** Given two angles and the included side  $A$ ,  $B$ , and  $c$  to find  $a$ ,  $b$ , and  $C$ . Two triangles always possible.

$$\begin{aligned} a_2 &= 180^\circ + a_1, \\ b_2 &= 180^\circ + b_1, \\ C_2 &= 360^\circ - C_1. \end{aligned}$$

Which values belong together is decided by means of the equations

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = M.$$

The computed values of  $a$ ,  $b$ , and  $C$ , which, with the given values of  $A$ ,  $B$ , and  $c$ , make each of these ratios positive belong to one triangle, and those which make each ratio negative belong to the other triangle. One triangle has a positive, the other a negative modulus.

FIRST METHOD. By 605 and 604,

$$\operatorname{ctn} a = \frac{\cos A \sin B + \sin A \cos B \cos c}{\sin A \sin c},$$

$$\operatorname{ctn} b = \frac{\sin A \cos B + \cos A \sin B \cos c}{\sin B \sin c},$$

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

650. SECOND METHOD.—Based on Gauss's Equations (634), which, by taking the logarithm of each term, may be written thus:

$$(i) \log \cos \frac{1}{2}(a-b) + \log \cos \frac{1}{2}C = \\ \log \sin \frac{1}{2}(A+B) + \log \cos \frac{1}{2}c,$$

$$(ii) \log \sin \frac{1}{2}(a-b) + \log \cos \frac{1}{2}C = \\ \log \sin \frac{1}{2}(A-B) + \log \sin \frac{1}{2}c,$$

$$(iii) \log \cos \frac{1}{2}(a+b) + \log \sin \frac{1}{2}C = \\ \log \cos \frac{1}{2}(A+B) + \log \cos \frac{1}{2}c,$$

$$(iv) \log \sin \frac{1}{2}(a+b) + \log \sin \frac{1}{2}C = \\ \log \cos \frac{1}{2}(A-B) + \log \sin \frac{1}{2}c.$$

651. THIRD METHOD. Based on Napier's Analogies (635), which, by taking the logarithm of each term, may be written thus:

$$(i) \log \tan \frac{1}{2}(a-b) = \\ \log \csc \frac{1}{2}(A+B) + \log \sin \frac{1}{2}(A-B) + \log \tan \frac{1}{2}c,$$

$$(ii) \log \tan \frac{1}{2}(a+b) = \\ \log \sec \frac{1}{2}(A+B) + \log \cos \frac{1}{2}(A-B) + \log \tan \frac{1}{2}c,$$

$$(iii) \log \operatorname{ctn} \frac{1}{2}C = \\ \log \csc \frac{1}{2}(a-b) + \log \sin \frac{1}{2}(a+b) + \log \tan \frac{1}{2}(A-B),$$

$$(iv) \log \operatorname{ctn} \frac{1}{2}C = \\ \log \sec \frac{1}{2}(a-b) + \log \cos \frac{1}{2}(a+b) + \log \tan \frac{1}{2}(A+B).$$

**652. FOURTH METHOD.** By drawing from  $B$  or from  $A$  a great-circle arc perpendicular to the opposite side, the required triangle is converted into the sum or the difference of two spherical right triangles.

Perpendicular  $BQ$ .

$$\begin{aligned} \text{ctn } ABQ &= \tan A \cos c, \\ QBC &= B - ABQ \\ \text{or } QBC &= 360^\circ + B - ABQ \end{aligned} \quad \left. \begin{array}{l} \text{,} \\ \text{,} \end{array} \right\}, \quad \begin{aligned} \text{ctn } a &= \text{ctn } c \sec ABQ \cos QBC, \\ \cos C &= \cos A \csc ABQ \sin QBC. \end{aligned}$$

$$\begin{aligned} \text{ctn } BAP &= \tan B \cos c, \\ PAC &= A - BAP \\ \text{or } PAC &= 360^\circ + A - BAP \end{aligned} \quad \left. \begin{array}{l} \text{,} \\ \text{,} \end{array} \right\}, \quad \begin{aligned} \text{ctn } b &= \text{ctn } c \sec BAP \cos PAC, \\ \cos C &= \cos B \csc BAP \sin PAC. \end{aligned}$$

**653. CASE III.** Given two sides and the angle opposite one of them  $a$ ,  $b$ , and  $A$ , to find  $B$ ,  $C$ , and  $c$ .

Two triangles,

$$\begin{array}{ll} \text{both possible when} & \sin a > \sin b \sin A, \\ \text{both impossible when} & \sin a < \sin b \sin A, \\ \text{identical when} & \sin a = \sin b \sin A. \end{array}$$

The value of the modulus is given by  $M = \frac{\sin a}{\sin A}$ . Hence the computed parts must be so taken as to give both triangles the *same modulus*, which will be positive or negative according to the given values of  $a$  and  $A$ .

**654. FIRST METHOD.** By 602, 606, and 607,

$$\sin B = \frac{\sin b \sin A}{\sin a},$$

$$B_1 + B_2 = 180^\circ, \text{ or } B_1 + B_2 = 540^\circ.$$

$$\sin c = \frac{\cos^2 a - \cos^2 b}{\cos a \sin b \cos A - \sin a \cos b \cos B},$$

$$\cos c = \frac{\sin b \cos b \cos A - \sin a \cos a \cos B}{\cos a \sin b \cos A - \sin a \cos b \cos B},$$

$$\sin C = \frac{\cos^2 A - \cos^2 B}{\cos A \sin B \cos a - \sin A \cos B \cos b},$$

$$\cos C = \frac{\sin A \cos A \cos b - \sin B \cos B \cos a}{\cos A \sin B \cos a - \sin A \cos B \cos b}.$$

Since the formulas give the values of both  $\sin c$  and  $\cos c$ , there can result but one value of  $c$  between  $0^\circ$  and  $360^\circ$  for each value of  $B$  substituted. The same remark applies to  $C$ .

**655. SECOND METHOD.** The two values of  $B$  having been computed as in the First Method,  $c$  and  $C$  are found by substituting the values of  $B$  separately in Gauss's Equations (634), which may be written as follows:

- (i)  $\log \cos \frac{1}{2}C - \log \cos \frac{1}{2}c =$   
 $\log \sin \frac{1}{2}(A + B) - \log \cos \frac{1}{2}(a - b),$
- (ii)  $\log \cos \frac{1}{2}C - \log \sin \frac{1}{2}c =$   
 $\log \sin \frac{1}{2}(A - B) - \log \sin \frac{1}{2}(a - b),$
- (iii)  $\log \sin \frac{1}{2}C - \log \cos \frac{1}{2}c =$   
 $\log \cos \frac{1}{2}(A + B) - \log \cos \frac{1}{2}(a + b),$
- (iv)  $\log \sin \frac{1}{2}C - \log \sin \frac{1}{2}c =$   
 $\log \cos \frac{1}{2}(A - B) - \log \sin \frac{1}{2}(a + b).$

**656. THIRD METHOD.** The two values of  $B$ , having been computed as in the First Method, are separately substituted in Napier's Analogies (635), which may be written thus:

- (i)  $\log \tan \frac{1}{2}c =$   
 $\log \sin \frac{1}{2}(A + B) + \log \csc \frac{1}{2}(A - B) + \log \tan \frac{1}{2}(a - b),$
- (ii)  $\log \tan \frac{1}{2}c =$   
 $\log \cos \frac{1}{2}(A + B) + \log \sec \frac{1}{2}(A - B) + \log \tan \frac{1}{2}(a + b),$
- (iii)  $\log \ctn \frac{1}{2}C =$   
 $\log \sin \frac{1}{2}(a + b) + \log \csc \frac{1}{2}(a - b) + \log \tan \frac{1}{2}(A - B),$
- (iv)  $\log \ctn \frac{1}{2}C =$   
 $\log \cos \frac{1}{2}(a + b) + \log \sec \frac{1}{2}(a - b) + \log \tan \frac{1}{2}(A + B).$

**657. FOURTH METHOD.** By drawing a great-circle arc  $CR$  from  $C$  perpendicular to the opposite side, two right triangles  $CRA$  and  $CRB$  are formed which may be solved as follows:

$$\begin{aligned}\sin B &= \csc a \sin b \sin A, \\ \tan ACR &= \tan b \cos A, \\ \cos CRB &= \cos ACR \cos a \sec b, \\ c_1 &= ACR + CRB \\ c_2 &= ACR - CRB \\ \tan ACR &= \sec b \ctn A, \\ \cos RCB &= \cos ACR \ctn a \tan b, \\ C_1 &= ACR + RCB \\ C_2 &= ACR - RCB\end{aligned}$$

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**662. FOURTH METHOD.** By drawing a great-circle arc  $CR$  from  $C$  perpendicular to the opposite side, two right triangles are formed, which may be solved as follows:

$$\sin b = \csc A \sin B \sin a,$$

$$\operatorname{ctn} BCR = \tan B \cos a,$$

$$\sin RCA = \sin BCR \cos A \sec B,$$

which gives supplementary values of  $RCA$ ,

$$C_1 = BCR + RCA_1 \quad \left. \right\} .$$

$$C_2 = BCR + RCA_2 \quad \left. \right\}$$

$$\operatorname{ctn} BR = \sec B \operatorname{ctn} a,$$

$$\sin RA = \sin BR \operatorname{ctn} A \tan B,$$

which gives supplementary values of  $RA$ .

$$c_1 = BR + RA_1 \quad \left. \right\} .$$

$$c_2 = BR + RA_2 \quad \left. \right\}$$

If the values of  $\sin RCA$  and  $\sin RA$  are negative, then

$$RCA_1 + RCA_2 = 540^\circ \text{ and } RA_1 + RA_2 = 540^\circ.$$

**663. CASE V.** Given the three sides,  $a$ ,  $b$ , and  $c$ , to find the three angles  $A$ ,  $B$ , and  $C$ .

Two triangles always possible.

$$A_1 + A_2 = B_1 + B_2 = C_1 + C_2 = 360^\circ.$$

Which of the computed values of the angles belong to one triangle and which to the other is decided by means of the equations

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = M.$$

One triangle has a positive, the other a negative modulus.

**664. FIRST METHOD.** By 602,

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c},$$

$$\cos B = \frac{\cos b - \cos c \cos a}{\sin c \sin a},$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b}.$$

**665. SECOND METHOD.** The halves of the angles can be computed from their sines (611), or from their cosines (612), or from their tangents (613). The most convenient form of computation from the tangents is the following derived from 620 and 622:

$$\tan r = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}},$$

$$\tan \frac{1}{2}A = \frac{\tan r}{\sin(s-a)}, \quad \tan \frac{1}{2}B = \frac{\tan r}{\sin(s-b)}, \quad \tan \frac{1}{2}C = \frac{\tan r}{\sin(s-c)},$$

The right-hand members of these equations, being square roots, are to be taken as both positive and negative. Hence each half angle has two values, which doubled are the two values of the whole angle.

**666. CASE VI.** Given the three angles  $A$ ,  $B$ , and  $C$ , to find the three sides,  $a$ ,  $b$ , and  $c$ .

Two triangles always possible.

$$a_1 + a_2 = b_1 + b_2 = c_1 + c_2 = 360^\circ.$$

Which of the computed values of the sides belong to one triangle and which to the other is decided by means of the equations

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C} = M.$$

One triangle has a positive, the other a negative modulus.

**667. FIRST METHOD.** By 604,

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C},$$

$$\cos b = \frac{\cos B + \cos C \cos A}{\sin C \sin A},$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B}.$$

**668. SECOND METHOD.** The halves of the sides can be computed from their sines (617), or from their cosines (616), or from their cotangents (618). The most conven-

ient form of computation from the cotangents is the following derived from 624 and 625:

$$\operatorname{ctn} R = \sqrt{\frac{\cos(S - A) \cos(S - B) \cos(S - C)}{-\cos S}},$$

$$\operatorname{ctn} \frac{1}{2}a = \frac{\operatorname{ctn} R}{\cos(S - A)}, \operatorname{ctn} \frac{1}{2}b = \frac{\operatorname{ctn} R}{\cos(S - B)}, \operatorname{ctn} \frac{1}{2}c = \frac{\operatorname{ctn} R}{\cos(S - C)}.$$

The right-hand members of these equations, being square roots, are to be taken as both positive and negative. Hence each half side has two values, which doubled are the two values of the whole side.

### *Special Formulas for Spherical Right Triangles.*

For more accurate results than the general formulas give in extreme cases, the following equations may be used.

See also 676-681.

When  $A$  is near  $90^\circ$ ,

$$669. \quad \tan(45^\circ - \frac{1}{2}A) = \sqrt{\frac{\tan \frac{1}{2}(h-a)}{\tan \frac{1}{2}(h+a)}} \\ = \sqrt{\tan \frac{1}{2}(B+b) \tan \frac{1}{2}(B-b)}.$$

When  $A$  is near  $0^\circ$  or near  $180^\circ$ ,

$$670. \quad \tan \frac{1}{2}A = \sqrt{\frac{\sin(h-b)}{\sin(h+b)}}.$$

When  $a$  is near  $90^\circ$ ,

$$671. \quad \tan(45^\circ - \frac{1}{2}a) = \sqrt{\frac{\sin(B-b)}{\sin(B+b)}}.$$

When  $a$  is near  $0^\circ$  or near  $180^\circ$ ,

$$672. \quad \tan \frac{1}{2}a = \sqrt{\tan \frac{1}{2}(h+b) \tan \frac{1}{2}(h-b)}.$$

When  $h$  is near  $90^\circ$ ,

$$673. \quad \tan(45^\circ - \frac{1}{2}h) = \sqrt{\frac{\tan \frac{1}{2}(A-a)}{\tan \frac{1}{2}(A+a)}}.$$

When  $h$  is near  $0^\circ$ ,

$$674. \quad \sin \frac{1}{2}h = \sqrt{\frac{-\cos(A+B)}{2 \sin A \sin B}}.$$

When  $h$  is near  $180^\circ$ ,

$$675. \quad \cos \frac{1}{2}h = \sqrt{\frac{\cos(A-B)}{2 \sin A \sin B}}.$$

*Accurate Computation of Angles near 0° and near 90°.*676. *Values of S and T.* $x''$  = the angle expressed in seconds. $(90^\circ - x)''$  = the complement of the angle expressed in seconds.

$x''.$	S.	$\log \sin x.$	$x''.$	T.	$\log \tan x.$
0	4.68 557	8.06 740	200	4.68 557	6.98 660
2409	4.68 556	8.21 920	1726	4.68 558	7.92 263
3417	4.68 555	8.30 776	2432	4.68 559	8.07 156
4190	4.68 554	8.37 038	2976	4.68 560	8.15 924
4840	4.68 553	8.41 904	3434	4.68 561	8.22 142
5414	4.68 552	8.45 872	3838	4.68 562	8.26 973
5932	4.68 551	8.49 223	4204	4.68 563	8.30 930
6408	4.68 550	8.52 125	4540	4.68 564	8.34 270
6851	4.68 549	8.54 321	4853	4.68 565	8.37 167
7206			5146	4.68 566	8.39 713
			5424	4.68 567	8.41 999
			5689	4.68 568	8.44 072
			5941	4.68 569	8.45 955
			6184	4.68 570	8.47 697
			6417	4.68 571	8.49 305
			6642	4.68 572	8.50 802
			6859	4.68 573	8.52 200
			7070	4.68 574	8.53 516
			7202	4.68 575	8.54 321
1. When the angle $x$ is between $0^\circ$ and $2^\circ$ , $\log \sin x = \log x'' + S.$ $\log \tan x = \log x'' + T.$ $\log \operatorname{ctn} x = 20 - \log \tan x.$					
2. When the angle $x$ is between $88^\circ$ and $90^\circ$ , $\log \cos x = \log (90^\circ - x)'' + S.$ $\log \operatorname{ctn} x = \log (90^\circ - x)'' + T.$ $\log \tan x = 20 - \log \operatorname{ctn} x.$					
3. When the logarithm of the function is less than 8.54321, $\log x'' = \log \sin x - S.$ $\log x'' = \log \tan x - T.$ $\log (90^\circ - x)'' = \log \cos x - S.$ $\log (90^\circ - x)'' = \log \operatorname{ctn} x - T.$					
4. When the logarithm of the function is greater than 11.45678, $\log x'' = 20 - \log \operatorname{ctn} x - T.$ $\log (90^\circ - x)'' = 20 - \log \tan x - T.$					

When tables of the values of S and T are not within reach, the following equations may be used to compute very closely any needed values:

$$677. \quad \begin{cases} S = 4.6855749 - \frac{1}{3}(10 - \log \cos x) \\ T = 4.6855749 + \frac{2}{3}(10 - \log \cos x) \end{cases}$$

Or, the formulas may be written as follows:

$$678. \log \sin x = \log x'' + 4.6855749 - \frac{1}{3}(10 - \log \cos x).$$

$$679. \log \tan x = \log x'' + 4.6855749 + \frac{2}{3}(10 - \log \cos x).$$

$$680. \log x'' = \log \sin x + 5.3144251 + \frac{1}{3}(10 - \log \cos x) - 10.$$

$$681. \log x'' = \log \tan x + 5.3144251 - \frac{2}{3}(10 - \log \cos x) - 10.$$

## SECTION III.

### HYPERBOLIC FUNCTIONS.

#### *Definitions.*

Connected with the equilateral hyperbola are six functions which, from their close resemblance to the six trigonometric or circular functions, have been named the hyperbolic sine, the hyperbolic cosine, and so on. They are derived geometrically from certain properties of central conics, or they may be denoted algebraically through their relations to the exponential  $e^u$ . Thus in the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

$OA$  being the major semi-axis, or any radius vector, and  $u$  double the measure of the hyperbolic sector  $AOP$ , the hyperbolic sine and cosine are denoted by the equations

701. 
$$\left\{ \begin{array}{l} \text{hyp. sine of } u = \frac{y}{b}, \\ \text{hyp. cosine of } u = \frac{x}{a}. \end{array} \right.$$

The quotient of these is defined as the hyperbolic tangent. The hyperbolic cotangent, secant, and cosecant are denoted as the reciprocals of the hyperbolic tangent, cosine, and sine respectively.

Or, the defining equations may be

702. 
$$\left\{ \begin{array}{ll} \text{hyp. cosine of } u &= \frac{1}{2} (e^u + e^{-u}) \\ \text{hyp. sine of } u &= \frac{1}{2} (e^u - e^{-u}) \\ \text{etc.} &\quad \text{etc.} \end{array} \right.$$

which are the analogues of

$$\text{circular cos } u = \frac{1}{2} (e^{iu} + e^{-iu})$$

$$\text{circular sin } u = \frac{1}{2i} (e^{iu} - e^{-iu}).$$

etc.                    etc.

As to notation, it is convenient to use the abbreviations of the names of the six circular functions, sin, cos, tan, ctn, sec, csc, marking the distinction by using a capital initial letter and adding the letter *h*; thus,

Sinh, Cosh, Tanh, Ctnh, Sech, CsCh.\*

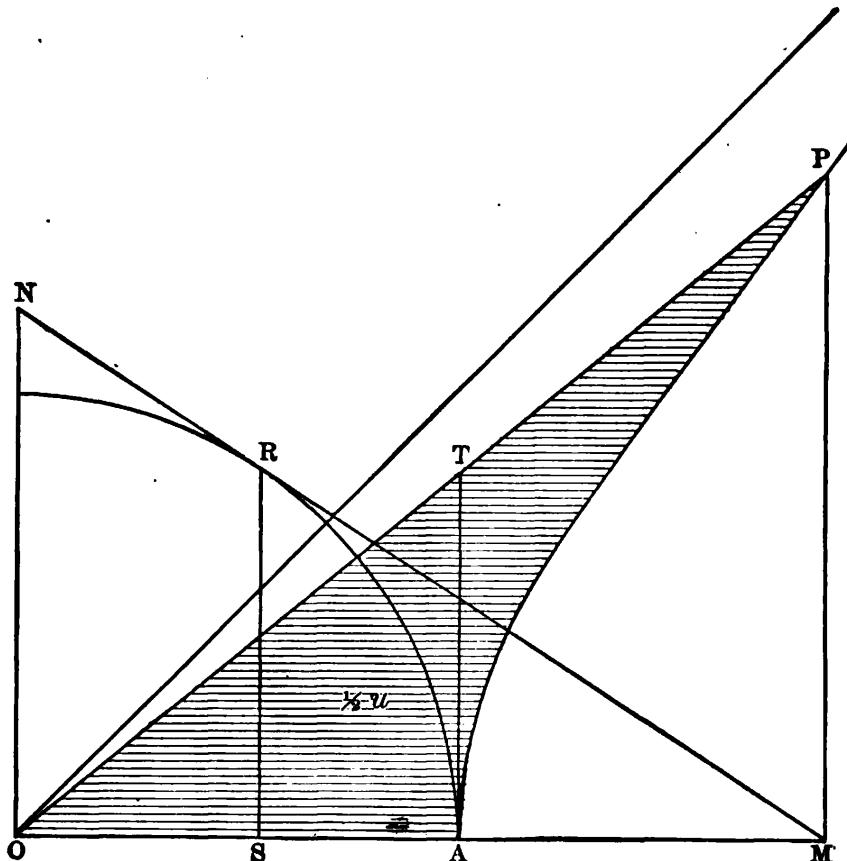


FIGURE 2.

$$\text{Hyperbolic Sector } AOP = \frac{1}{2}u.$$

$$\text{Circular Sector } AOR = \frac{1}{2}\theta.$$

$$\text{Radius } OA = 1.$$

In the *equilateral hyperbola*,  $x^2 - y^2 = 1$ , Figure 2, the hyperbolic functions of  $u$  (double the area of the sector  $AOP$ ) are represented by, and are numerically equal to the lines drawn as follows:

703. 
$$\begin{cases} \text{Sinh } u = \text{the ordinate of } P = MP, \\ \text{Cosh } u = \text{the abscissa of } P = OM, \\ \text{Tanh } u = \frac{MP}{OM} = \frac{AT}{OA} = \frac{AT}{1} = AT. \end{cases}$$

\* Some writers merely add the *h*, but the capital initial serves well to catch the attention.

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$$712. \quad \begin{aligned} \cosh u &= \sqrt{\sinh^2 u + 1} = \frac{1}{\sqrt{1 - \tanh^2 u}}, \\ &= \frac{\operatorname{Ctnh} u}{\sqrt{\operatorname{Ctnh}^2 u - 1}} = \frac{\sqrt{\operatorname{Csch}^2 u + 1}}{\operatorname{Csch} u} = \frac{1}{\operatorname{Sech} u}. \end{aligned}$$

$$713. \quad \begin{aligned} \tanh u &= \frac{\sinh u}{\sqrt{\sinh^2 u + 1}} = \frac{\sqrt{\cosh^2 u - 1}}{\cosh u}, \\ &= \sqrt{1 - \operatorname{Sech}^2 u} = \frac{1}{\sqrt{\operatorname{Csch}^2 u + 1}} = \frac{1}{\operatorname{Ctnh} u}. \end{aligned}$$

*Relations between the hyperbolic and the circular functions of the same variable.*

In the exponential equivalents of the circular functions

$$\begin{aligned} \cos x &= \frac{1}{2} (e^{ix} + e^{-ix}) \\ i \sin x &= \frac{1}{2} (e^{ix} - e^{-ix}) \end{aligned}$$

and in those of the hyperbolic functions

$$\begin{aligned} \cosh x &= \frac{1}{2} (e^x + e^{-x}) \\ \sinh x &= \frac{1}{2} (e^x - e^{-x}), \end{aligned}$$

by writing  $iu$  in place of  $x$ , there are found the following relations:

$$714. \quad \begin{cases} \sin iu = i \sinh u, & \sinh iu = i \sin u, \\ \cos iu = \cosh u, & \cosh iu = \cos u, \\ \tan iu = i \tanh u, & \tanh iu = i \tan u, \\ \operatorname{ctn} iu = -i \operatorname{Ctnh} u, & \operatorname{Ctnh} iu = -i \operatorname{ctn} u, \\ \sec iu = \operatorname{Sech} u, & \operatorname{Sech} iu = \sec u, \\ \csc iu = -i \operatorname{Csch} u, & \operatorname{Csch} iu = -i \operatorname{csc} u. \end{cases}$$

$$715. \quad \begin{cases} \cos y + i \sin y & \cosh iy + \sinh iy = e^y, \\ \cos y - i \sin y & \cosh iy - \sinh iy = e^{-y}, \\ \cos iy + i \sin iy & \cosh y - \sinh y = e^{-y}, \\ \cos iy - i \sin iy & \cosh y + \sinh y = e^y. \end{cases}$$

*Hyperbolic functions of  $-u$ .*

$$716. \quad \begin{aligned} \operatorname{Sinh} (-u) &= -\operatorname{Sinh} u, & \operatorname{Ctnh} (-u) &= -\operatorname{Ctnh} u, \\ \cosh (-u) &= +\cosh u, & \operatorname{Sech} (-u) &= +\operatorname{Sech} u, \\ \tanh (-u) &= -\tanh u, & \operatorname{Csch} (-u) &= -\operatorname{Csch} u. \end{aligned}$$

*Variations and Cardinal Values.*

As the radius vector  $OP$  swings from the axis  $OA$  to the upper right-hand asymptote, the variable,  $u$ , passes through all positive values from 0 to  $+\infty$ . At the same time  $\text{Sinh } u$  passes from 0 to  $+\infty$ ;  $\text{Cosh } u$  from 1 to  $+\infty$ , and  $\text{Tanh } u$  from 0 to 1. On the other hand, as  $OP$  swings from the axis  $OA$  to the lower right-hand asymptote,  $u$  passes through all negative values from 0 to  $-\infty$ .  $\text{Sinh } u$  is likewise negative, passing from 0 to  $-\infty$ ;  $\text{Cosh } u$  is positive, passing from 1 to  $+\infty$ ; and  $\text{Tanh } u$  is negative, passing from 0 to -1.

For all real values of  $u$ , therefore,

$$\begin{aligned} & -\infty \leq \text{Sinh } u < +\infty, \\ 717. \quad & +1 < \text{Cosh } u < +\infty, \\ & -1 < \text{Tanh } u \leq +1. \end{aligned}$$

Cardinal values,

718.

$$\begin{cases} \text{Sinh } 0 = 0, & \text{Cosh } 0 = 1, & \text{Tanh } 0 = 0, \\ \text{Sinh } \infty = \infty, & \text{Cosh } \infty = \infty, & \text{Tanh } \infty = 1, \\ \text{Sinh } (-\infty) = -\infty, & \text{Cosh } (-\infty) = \infty, & \text{Tanh } (-\infty) = -1. \end{cases}$$

*Relations of Hyperbolic to Trigonometric Formulas.*

By means of the relations expressed in 714, any general formula of trigonometry can be changed into an analogous formula involving the hyperbolic functions. Thus, by putting  $x = iu$  and reducing,

$$\cos^2 x + \sin^2 x = 1 \text{ becomes } \text{Cosh}^2 u - \text{Sinh}^2 u = 1.$$

$$1 + \tan^2 x = \sec^2 x \text{ becomes } 1 - \text{Tanh}^2 u = \text{Sech}^2 u.$$

$$\operatorname{ctn}^2 x + 1 = \csc^2 x \text{ becomes } \text{Ctnh}^2 u - 1 = \text{Csch}^2 u.$$

*The Addition and Subtraction Formulas.*

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y,$$

by putting  $x = iu$  and  $y = iv$ , become

$$719. \quad \text{Sinh}(u \pm v) = \text{Sinh } u \text{Cosh } v \pm \text{Cosh } u \text{Sinh } v.$$

$$720. \quad \text{Cosh}(u \pm v) = \text{Cosh } u \text{Cosh } v \pm \text{Sinh } u \text{Sinh } v.$$

From these are deduced

$$721. \quad \text{Sinh } u + \text{Sinh } v = 2 \text{ Sinh } \frac{1}{2} (u + v) \text{ Cosh } \frac{1}{2} (u - v).$$

$$722. \quad \text{Sinh } u - \text{Sinh } v = 2 \text{ Cosh } \frac{1}{2} (u + v) \text{ Sinh } \frac{1}{2} (u - v).$$

$$723. \quad \text{Cosh } u + \text{Cosh } v = 2 \text{ Cosh } \frac{1}{2} (u + v) \text{ Cosh } \frac{1}{2} (u - v).$$

$$724. \quad \text{Cosh } u - \text{Cosh } v = 2 \text{ Sinh } \frac{1}{2} (u + v) \text{ Sinh } \frac{1}{2} (u - v).$$

$$725. \quad \text{Sinh } 2u = 2 \text{ Sinh } u \text{ Cosh } u.$$

$$726. \quad \text{Cosh } 2u = \text{Cosh}^2 u + \text{Sinh}^2 u,$$

$$= 2 \text{ Sinh}^2 u + 1,$$

$$= 2 \text{ Cosh}^2 u - 1.$$

$$727. \quad \text{Sinh } 2u = \frac{2 \text{ Tanh } u}{1 - \text{Tanh}^2 u}.$$

$$728. \quad \text{Cosh } 2u = \frac{1 + \text{Tanh}^2 u}{1 - \text{Tanh}^2 u}.$$

$$729. \quad \text{Tanh } 2u = \frac{2 \text{ Tanh } u}{1 + \text{Tanh}^2 u}.$$

$$730. \quad \text{Sinh } \frac{u}{2} = \sqrt{\frac{1}{2} (\text{Cosh } u - 1)}.$$

$$731. \quad \text{Cosh } \frac{u}{2} = \sqrt{\frac{1}{2} (\text{Cosh } u + 1)}.$$

$$732. \quad \text{Tanh } \frac{u}{2} = \sqrt{\frac{\text{Cosh } u - 1}{\text{Cosh } u + 1}} = \frac{\text{Cosh } u - 1}{\text{Sinh } u} = \frac{\text{Sinh } u}{\text{Cosh } u + 1}.$$

$$733. \quad \text{Sinh}^{-1} x \pm \text{Sinh}^{-1} y = \text{Sinh}^{-1} [x\sqrt{1 + y^2} \pm y\sqrt{1 + x^2}].$$

$$734. \quad \text{Cosh}^{-1} x \pm \text{Cosh}^{-1} y = \text{Cosh}^{-1} [xy \pm \sqrt{x^2 - 1}\sqrt{y^2 - 1}].$$

### *Hyperbolic Functions of a Complex Variable, $u = x + iy$ .*

$$735. \quad \text{Sinh } (x + iy) = \text{Sinh } x \cos y + i \text{ Cosh } x \sin y.$$

$$736. \quad \text{Cosh } (x + iy) = \text{Cosh } x \cos y + i \text{ Sinh } x \sin y.$$

$$737. \quad \text{Sinh } (x \pm iy) = \pm i \sin (y \mp ix).$$

$$738. \quad \text{Cosh } (x \pm iy) = \cos (y \mp ix).$$

$$739. \quad \sin (x \pm iy) = \pm i \text{ Sinh } (y \mp ix).$$

$$740. \quad \cos (x \pm iy) = \text{Cosh } (y \mp ix).$$

*Periodicity.*

The hyperbolic functions, like the circular, are periodic. The period of  $\text{Sinh } u$  and of  $\text{Cosh } u$  is  $2\pi i$ ; that of  $\text{Tanh } u$  and of  $\text{Ctnh } u$  is  $\pi i$ . Thus,

- 741.  $\text{Sinh } (u + 2\pi i) = \text{Sinh } u.$
- 742.  $\text{Cosh } (u + 2\pi i) = \text{Cosh } u.$
- 743.  $\text{Sinh } k\pi i = 0.$
- 744.  $\text{Cosh } k\pi i = (-1)^k.$
- 745.  $\text{Tanh } k\pi i = 0.$
- 746.  $\text{Sinh } (u + \pi i) = -\text{Sinh } u.$
- 747.  $\text{Cosh } (u + \pi i) = -\text{Cosh } u.$
- 748.  $\text{Tanh } (u + \pi i) = \text{Tanh } u.$
- 749.  $\text{Ctnh } (u + \pi i) = \text{Ctnh } u.$
- 750.  $\text{Sinh } (2k + 1)\frac{1}{2}\pi i = \pm i.$
- 751.  $\text{Cosh } (2k + 1)\frac{1}{2}\pi i = 0.$
- 752.  $\text{Sinh } (u + \frac{1}{2}\pi i) = i \text{Cosh } u.$
- 753.  $\text{Cosh } (u + \frac{1}{2}\pi i) = i \text{Sinh } u.$

*Hyperbolic Anti-Functions expressed as Logarithms.*

- 754.  $\text{Sinh}^{-1} u = \log_e(u + \sqrt{u^2 + 1}).$
- 755.  $\text{Cosh}^{-1} u = \log_e(u + \sqrt{u^2 - 1}).$
- 756.  $\text{Tanh}^{-1} u = \frac{1}{2} \log_e \frac{1+u}{1-u}.$
- 757.  $\text{Ctnh}^{-1} u = \frac{1}{2} \log_e \frac{u+1}{u-1}.$
- 758.  $\text{Sech}^{-1} u = \log_e \frac{1+\sqrt{1-u^2}}{u}.$
- 759.  $\text{Csch}^{-1} u = \log_e \frac{1+\sqrt{1+u^2}}{u}.$

*The Gudermanian Function and Angle.*

If a straight line, Figure 2, be drawn from the foot of the ordinate of  $P$  tangent to the circle which has the axis of the equilateral hyperbola for its diameter its length

$$760. \quad MR = \sqrt{\overline{OM}^2 - \overline{OR}^2} = \sqrt{\cosh^2 u - 1} = \sinh u = MP.$$

The angle  $\angle AOR$  is the Gudermanian Angle. Its radian measure,  $\theta$ , is numerically equal to double the area of the circular sector  $AOR$ , the radius  $OA$  being unity. The circular sector  $AOR$  and the hyperbolic sector  $AOP$  are functions the one of the other. If  $\theta$  and  $u$  are double their respective measures, then  $\theta$  is the gudermanian of  $u$  and  $u$  is the anti-gudermanian of  $\theta$ ; and the relation is expressed by

$$761. \quad \theta = gd u, \quad u = gd^{-1} \theta.$$

The six hyperbolic functions of  $u$  severally have equivalents among the six circular functions of the Gudermanian Angle, thus:

$$762. \quad \begin{cases} \cosh u = \sec \theta = OM \\ \sinh u = \tan \theta = MP = RM \\ \tanh u = \sin \theta = AT = SR \\ \operatorname{Ctnh} u = \csc \theta = ON \\ \operatorname{Sech} u = \cos \theta = OS \\ \operatorname{Csch} u = \operatorname{ctn} \theta = RN \end{cases}$$

Also,

$$763. \quad \tanh \frac{1}{2}u = \tan \frac{1}{2}\theta.$$

$$764. \quad gd 0 = 0, \quad gd \infty = \frac{1}{2}\pi, \quad gd (-\infty) = -\frac{1}{2}\pi.$$

$$765. \quad u = gd^{-1} \theta = \log_e \tan (\frac{1}{4}\pi + \frac{1}{2}\theta).$$

## SECTION IV.

### DIFFERENTIAL AND INTEGRAL CALCULUS.

#### *Limits.*

**801.**  $\lim_{x \underset{\approx}{\rightarrow} \infty} \left( \frac{a}{x} + b \right) = b.$ \*

**802.**  $\lim_{\omega \underset{\approx}{\rightarrow} \infty} \left( 1 + \frac{1}{\omega} \right)^{\omega} = e.$

**803.**  $\lim_{\partial \underset{\approx}{\rightarrow} 0} (1 + \partial)^{\frac{1}{\partial}} = e.$

**804.**  $\lim_{\partial \underset{\approx}{\rightarrow} 0} \frac{\log (1 + \partial)}{\partial} = \log e.$

**805.**  $\lim_{\theta \underset{\approx}{\rightarrow} 0} \frac{a^{\theta} - 1}{\theta} = \frac{1}{\log_a e} \log_e a.$

**806.**  $\lim_{\partial \underset{\approx}{\rightarrow} 0} \frac{(1 + \partial)^m - 1}{\partial} = m.$

**807.**  $\lim_{\theta \underset{\approx}{\rightarrow} 0} \frac{\sin \theta}{\theta} = 1. \quad \lim_{\theta \underset{\approx}{\rightarrow} 0} \frac{\tan \theta}{\theta} = 1.$

#### *Definitions and Notation.*

**808.** If  $y = f(x)$ , then  $y + \Delta y = f(x + \Delta x)$ .

Here  $\Delta x$  and  $\Delta y$  denote the finite differences, or increments, of the independent variable  $x$  and the dependent variable  $y$ , respectively.

The value of  $\Delta x$  is arbitrary (may be any we please) and in particular may be made to vary and to approach 0 as nearly as we please. The value of  $\Delta y$  depends on that of  $\Delta x$ . If  $\Delta x$  approaches 0,  $\Delta y$  does the same, and both reach their limit 0 at the same time.

\*The sign  $\underset{\approx}{\rightarrow}$  is read "approaches infinitely near to."

The definite value, or *limit*, which the ratio

**809.** 
$$\frac{\Delta y}{\Delta x} = \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

approaches when  $\Delta x$  and  $\Delta y$  simultaneously approach 0 is the *differential coefficient* of the function with respect to  $x$ .

It is denoted by  $\frac{dy}{dx}$  or  $\frac{d}{dx} f(x)$ .

The differential coefficient is also called *differential quotient*, *derivative*, or *derived function*. It is also denoted by

$$D_x y, D_x f(x), \text{ or } f'(x).$$

The symbols  $\frac{d}{dx}$  and  $D_x$  are to be regarded as symbols of operation, that of differentiation, upon  $f(x)$ ; the result of which operation is a new function of  $x$ , often denoted by  $f'(x)$ .

The symbols  $\frac{d}{dx}$  and  $D_x$ ,  $\frac{d}{dy}$  and  $D_y$ , etc., when applied to functions  $u$ ,  $v$ , etc., are obedient to

The Distributive Law,

**810.** 
$$\frac{d}{dx} (u + v) = \frac{du}{dx} + \frac{dv}{dx},$$

or  $D_x (u + v) = D_x u + D_x v$ .

The Commutative Law,

**811.** 
$$\frac{d}{dx} \left( \frac{du}{dy} \right) = \frac{d}{dy} \left( \frac{du}{dx} \right), \text{ or } D_x (D_y u) = D_y (D_x u).$$

The Law of Indices,

**812.** 
$$\frac{d^m}{dx^m} \frac{d^n}{dx^n} u = \frac{d^{m+n}}{dx^{m+n}} u, \text{ or } D_x^m D_x^n u = D_x^{m+n} u.$$

The symbols  $dx$  and  $df(x)$  denote the *differentials* of  $x$  and  $f(x)$  respectively. They express the *rates* at which  $x$  and  $f(x)$  are changing their values at any given instant,

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$$822. \frac{d(uvw\ldots)}{uvw\ldots} = \frac{du}{u} + \frac{dv}{v} + \frac{dw}{w} + \dots$$

$$823. d\left(\frac{u}{v}\right) = \frac{vdu - udv}{v^2}.$$

$$824. d \log_a u = \log_a e \cdot \frac{du}{u}.$$

$$825. d \log_e u = \frac{du}{u}. \quad 826. \int \frac{du}{u} = \log_e u + C.$$

$$827. da^u = \log_e a \cdot a^u \cdot du.$$

$$828. de^u = e^u du. \quad 829. \int e^u du = e^u + C.$$

$$830. du^v = vu^{v-1} \cdot du + \log_e u \cdot u^v \cdot dv.$$

If  $u = F(y)$  and  $y = f(x)$ , that is,  $u = F(f(x))$ , then

$$831. \frac{du}{dx} = \frac{du}{dy} \cdot \frac{dy}{dx} = F'(y) \cdot f'(x) = D_y u \cdot D_x y.$$

Or

$$832. du = \frac{du}{dy} \cdot \frac{dy}{dx} \cdot dx = F'(y) \cdot f'(x) \cdot dx = D_y u \cdot D_x y \cdot dx.$$

$$833. \int u \cdot dy = \int u \cdot f'(x) dx.$$

*Differentials and Integrals of the Simpler Functions of x.*

$$834. d(a+x) = dx. \quad 835. \int dx = x + C.$$

$$836. d(ax) = adx. \quad 837. \int adx = a \int dx = ax + C.$$

$$838. d(x^n) = nx^{n-1} \cdot dx. \quad 839. \int x^m \cdot dx = \frac{x^{m+1}}{m+1} + C.$$

N.B. When  $m = -1$ , use the next formula.

$$840. d(\log_e x) = \frac{dx}{x}. \quad 841. \int \frac{dx}{x} = \log_e x + C = \log_e cx.$$

$$842. d\sqrt{x} = \frac{dx}{2\sqrt{x}}. \quad 843. \int \frac{dx}{\sqrt{x}} = 2\sqrt{x} + C.$$

844.  $d(a + bx)^n = nb(a + bx)^{n-1}.$

845.  $\int (a + bx)^m dx = \frac{(a + bx)^{m+1}}{(m + 1)b} + C.$

846.  $d \log(a + bx) = \frac{bdx}{a + bx}.$

847.  $\int \frac{dx}{a + bx} = \frac{1}{b} \log_e(a + bx) + C.$

848.  $d\left(\frac{1}{x}\right) = -\frac{dx}{x^2}$ . 849.  $\int \frac{dx}{x^2} = -\frac{1}{x} + C.$

850.  $d\left(\frac{1}{(a + bx)^2}\right) = -\frac{bdx}{(a + bx)^3}.$

851.  $\int \frac{dx}{(a + bx)^2} = -\frac{1}{b(a + bx)} + C.$

852.  $da^x = \log_e a \cdot a^x \cdot dx.$  853.  $\int a^x \cdot dx = \frac{a^x}{\log_e a} + C.$

854.  $de^x = e^x \cdot dx.$  855.  $\int e^x \cdot dx = e^x + C.$

856.  $de^{ax} = ae^{ax} \cdot dx.$  857.  $\int e^{ax} \cdot dx = \frac{e^{ax}}{a} + C.$

858.  $d \log_{10} x = \frac{1}{\log_e 10} \cdot \frac{dx}{x} = \log_{10} e \cdot \frac{dx}{x} = \frac{M dx}{x}.$

859.  $dx^x = x^x (1 + \log_e x) dx.$

860.  $d \sin x = \cos x \cdot dx.$  861.  $\int \cos x \cdot dx = \sin x + C.$

862.  $d \text{covers } x^* = -\cos x \cdot dx.$

863.  $\int \cos x \cdot dx = -\text{covers } x + C.$

864.  $d \cos x = -\sin x \cdot dx.$  865.  $\int \sin x \cdot dx = -\cos x + C.$

866.  $d \text{vers } x^* = \sin x \cdot dx.$  867.  $\int \sin x \cdot dx = \text{vers } x + C.$

\* The versed sine of  $x$  is the difference between 1 and the cosine of  $x$ . The covered sine of  $x$  is the difference between 1 and the sine of  $x$ .

$\text{vers } x = 1 - \cos x.$

$\text{covers } x = 1 - \sin x.$

$$868. \quad d \tan x = \sec^2 x \cdot dx. \quad 869. \quad \int \sec^2 x \cdot dx = \tan x + C.$$

$$870. \quad d \tan x = \frac{2dx}{1 + \cos 2x}.$$

$$871. \quad \int \frac{dx}{1 + \cos x} = \tan \frac{x}{2} + C. \\ = \csc x - \ctn x + C.$$

$$872. \quad \int \frac{dx}{1 + \sin x} = \tan \left( \frac{\pi}{4} - \frac{x}{2} \right) + C.$$

$$873. \quad d \ctn x = -\csc^2 x \cdot dx.$$

$$874. \quad \int \csc^2 x \cdot dx = -\ctn x + C.$$

$$875. \quad d \ctn x = \frac{-2dx}{1 - \cos 2x}.$$

$$876. \quad \int \frac{dx}{1 - \cos x} = -\ctn \frac{x}{2} + C. \\ = -\ctn x - \csc x + C.$$

$$877. \quad \int \frac{dx}{1 - \sin x} = \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) + C.$$

$$878. \quad d \sec x = \sec x \tan x \cdot dx.$$

$$879. \quad \int \sec x \tan x \cdot dx = \sec x + C.$$

$$880. \quad d \csc x = -\csc x \ctn x \cdot dx.$$

$$881. \quad \int \csc x \ctn x \cdot dx = -\csc x + C.$$

$$882. \quad d \log_e \sin x = \ctn x.$$

$$883. \quad \int \ctn x \cdot dx = \log_e \sin x + C.$$

$$884. \quad d \log_e \cos x = -\tan x.$$

$$885. \quad \int \tan x \cdot dx = -\log_e \cos x + C.$$

$$886. \quad d \log_e \tan x = \sec x \csc x \cdot dx.$$

$$887. \quad \int \sec x \csc x \cdot dx = \log_e \tan x + C. \\ = -\log_e \ctn x + C.$$

888.  $d \log_e \operatorname{ctn} x = - \sec x \csc x . dx.$

889.

$$\int \sec x . dx = \frac{1}{2} \log_e \left( \frac{1 + \sin x}{1 - \sin x} \right) + C = \log_e \tan \left( \frac{\pi}{4} + \frac{x}{2} \right) + C.$$

$$= g d^{-1} x \quad = \log_e (\sec x + \tan x) + C.$$

890.

$$\int \csc x . dx = \frac{1}{2} \log_e \left( \frac{1 - \cos x}{1 + \cos x} \right) + C = \log_e \tan \frac{x}{2} + C.$$

$$= - \operatorname{Cosh}^{-1} (\csc x) \quad = \log_e (\csc x - \operatorname{ctn} x) + C.$$

891.  $d \operatorname{Sinh} x = \operatorname{Cosh} x . dx.$

892.  $\int \operatorname{Cosh} x . dx = \operatorname{Sinh} x + C.$

893.  $d \operatorname{Cosh} x = \operatorname{Sinh} x . dx.$

894.  $\int \operatorname{Sinh} x . dx = \operatorname{Cosh} x + C.$

895.  $d \operatorname{Tanh} x = \operatorname{Sech}^2 x . dx.$

896.  $\int \operatorname{Sech}^2 x . dx = \operatorname{Tanh} x + C.$

897.  $d \operatorname{Ctn} x = - \operatorname{Csch}^2 x . dx.$

898.  $\int \operatorname{Csch}^2 x . dx = - \operatorname{Ctn} x + C.$

899.  $d \operatorname{Sech} x = - \operatorname{Sech} x \cdot \operatorname{Tanh} x . dx.$

900.  $\int \operatorname{Sech} x \cdot \operatorname{Tanh} x . dx = - \operatorname{Sech} x + C.$

901.  $d \operatorname{Csch} x = - \operatorname{Csch} x \cdot \operatorname{Ctnh} x . dx.$

902.  $\int \operatorname{Csch} x \cdot \operatorname{Ctnh} x . dx = - \operatorname{Csch} x + C.$

903.  $d \log_e \operatorname{Sinh} x = \operatorname{Ctnh} x . dx.$

904.  $\int \operatorname{Ctnh} x . dx = \log_e \operatorname{Sinh} x + C.$

905.  $d \log \operatorname{Cosh} x = \operatorname{Tanh} x \cdot dx.$

906.  $\int \operatorname{Tanh} x \cdot dx = \log \operatorname{Cosh} x + C.$

907.  $d \log \operatorname{Tanh} x = \operatorname{Sech} x \operatorname{Csch} x \cdot dx.$

908.  $\int \operatorname{Sech} x \operatorname{Csch} x \cdot dx = \log \operatorname{Tanh} x + C,$   
 $= -\log \operatorname{Ctnh} x + C.$

909.  $d \log \operatorname{Ctnh} x = -\operatorname{Sech} x \operatorname{Csch} x \cdot dx.$

910.  $\int \operatorname{Sech} x \cdot dx = 2 \tan^{-1} e^x + C,$   
 $= gdx.$

911.  $\int \operatorname{Csch} x \cdot dx = \log \operatorname{Tanh} \frac{x}{2} + C$   
 $= -\operatorname{Sinh}^{-1} (\operatorname{Csch} x).$

912.  $d \sin^{-1} x = \frac{dx}{\sqrt{1-x^2}}.$

913.  $\int \frac{dx}{\sqrt{1-x^2}} = \sin^{-1} x + C,$   
 $= -\cos^{-1} x + C.$

914.  $d \cos^{-1} x = -\frac{dx}{\sqrt{1-x^2}}.$

915.  $d \tan^{-1} x = \frac{dx}{1+x^2}.$

916.  $\int \frac{dx}{1+x^2} = \tan^{-1} x + C,$   
 $= -\operatorname{ctn}^{-1} x + C.$

917.  $d \operatorname{ctn}^{-1} x = -\frac{dx}{1+x^2}.$

918.  $d \sec^{-1} x = \frac{dx}{x \sqrt{x^2-1}}.$

919.  $\int \frac{dx}{x \sqrt{x^2-1}} = \sec^{-1} x + C,$   
 $= -\csc^{-1} x + C.$

920.  $d \csc^{-1} x = -\frac{dx}{x \sqrt{x^2-1}}.$

$$921. \quad d \operatorname{vers}^{-1} x = \frac{dx}{\sqrt{2x - x^2}}.$$

$$922. \quad \int \frac{dx}{\sqrt{2x - x^2}} = \operatorname{vers}^{-1} x + C,$$

$$= - \operatorname{covers}^{-1} x + C.$$

$$923. \quad d \operatorname{covers}^{-1} = - \frac{dx}{\sqrt{2x - x^2}}.$$

$$924. \quad d \operatorname{Sinh}^{-1} x = \frac{dx}{\sqrt{x^2 + 1}}.$$

$$925. \quad \int \frac{dx}{\sqrt{x^2 + 1}} = \operatorname{Sinh}^{-1} x + C,$$

$$= \log_e (x + \sqrt{x^2 + 1}) + C,$$

$$= \operatorname{Cosh}^{-1} \sqrt{x^2 + 1}.$$

$$926. \quad d \operatorname{Cosh}^{-1} x = \frac{dx}{\sqrt{x^2 - 1}}.$$

$$927. \quad \int \frac{dx}{\sqrt{x^2 - 1}} = \operatorname{Cosh}^{-1} x + C,$$

$$= \log_e (x + \sqrt{x^2 - 1}) + C,$$

$$= \operatorname{Sinh}^{-1} \sqrt{x^2 - 1}.$$

$$928. \quad d \operatorname{Tanh}^{-1} x = \frac{dx}{1 - x^2}.$$

$$929. \quad \int \frac{dx}{1 - x^2} = \operatorname{Tanh}^{-1} x + C,$$

$$= \frac{1}{2} \log_e \left( \frac{1+x}{1-x} \right) + C. \quad x < 1.$$

$$930. \quad d \operatorname{Ctnh}^{-1} x = - \frac{dx}{x^2 - 1}.$$

$$931. \quad \int \frac{dx}{x^2 - 1} = - \operatorname{Ctnh}^{-1} x + C,$$

$$= \frac{1}{2} \log_e \left( \frac{x-1}{x+1} \right) + C. \quad x > 1.$$

$$932. \ d \operatorname{Sech}^{-1} x = - \frac{dx}{x \sqrt{1 - x^2}}.$$

$$933. \ \int \frac{dx}{x \sqrt{1 - x^2}} = - \operatorname{Sech}^{-1} x.$$

$$= - \log_e \left( \frac{1}{x} + \frac{1}{x} \sqrt{1 - x^2} \right)$$

$$934. \ d \operatorname{Csch}^{-1} x = - \frac{dx}{x \sqrt{1 + x^2}}.$$

$$935. \ \int \frac{dx}{x \sqrt{1 + x^2}} = - \operatorname{Csch}^{-1} x.$$

$$= - \log_e \left( \frac{1}{x} + \frac{1}{x} \sqrt{1 + x^2} \right)$$

*Additional Integrals of Simple Form.*

$$936. \ \int \sin ax \cdot dx = - \frac{\cos ax}{a} + C.$$

$$937. \ \int \cos ax \cdot dx = \frac{\sin ax}{a} + C.$$

$$938. \ \int \sin^2 x \cdot dx = - \frac{1}{4} \sin 2x + \frac{1}{2}x + C.$$

$$939. \ \int \cos^2 x \cdot dx = \frac{1}{4} \sin 2x + \frac{1}{2}x + C.$$

$$940. \ \int \sin x \cos x \cdot dx = \frac{1}{2} \sin^2 x + C = - \frac{1}{4} \cos 2x + C.$$

$$941. \ \int \tan^2 x \cdot dx = \tan x - x + C.$$

$$942. \ \int \operatorname{ctn}^2 x \cdot dx = - \operatorname{ctn} x - x + C.$$

$$943. \ \int x \sin x \cdot dx = \sin x - x \cos x + C.$$

$$944. \ \int x \cos x \cdot dx = \cos x + x \sin x + C.$$

$$945. \ \int x \operatorname{Sinh} x \cdot dx = x \operatorname{Cosh} x - \operatorname{Sinh} x + C.$$

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$$962. \int \operatorname{Cosh}^{-1} x \cdot dx = x \operatorname{Cosh}^{-1} x - \sqrt{x^2 - 1} + C.$$

$$963. \int \operatorname{Tanh}^{-1} x \cdot dx = x \operatorname{Tanh}^{-1} x + \frac{1}{2} \log_e (1 - x^2) + C.$$

$$964. \int \operatorname{Ctnh}^{-1} x \cdot dx = x \operatorname{Ctnh}^{-1} x + \frac{1}{2} \log_e (x^2 - 1) + C.$$

*Successive Differentiation.*

If the derivative  $f'(x)$  be differentiated, the result,  $f''(x)$ , is called the *second* derivative; and if this be differentiated, the result,  $f'''(x)$ , is called the *third* derivative; and so on, the result of  $n$  successive differentiations being denoted by  $f^{(n)}(x)$ .

The symbols of operation are

$$\frac{d}{dx} \text{ of } \frac{d}{dx} f(x), \text{ or } \frac{d^2}{dx^2} f(x); D \text{ of } Df(x), \text{ or } D^2f(x);$$

$$\frac{d}{dx} \text{ of } \frac{d^2}{dx^2} f(x), \text{ or } \frac{d^3}{dx^3} f(x); D \text{ of } D^2f(x), \text{ or } D^3f(x);$$

$$\text{etc.} \quad \text{etc.} \quad \text{etc.} \quad \text{etc.}$$

$$1001. \quad \frac{d^n}{dx^n} f(x) = D^n f(x) = f^{(n)}(x).$$

All three forms of notation are in use.

*Successive Differentiation of a Product of Functions.*

1002.  $D(uv) = DuD^0v + D^0uDv$ , wherein  $D^0v = v$  and  $D^0u = u$  are written for the sake of symmetry.

$$1003. \quad D^2(uv) = D^2uD^0v + 2DuDv + D^0uD^2v.$$

$$1004. \quad D^3(uv) = D^3uD^0v + 3D^2uDv + 3DuD^2v + D^0uD^3v.$$

. . . . . . . . . . . . . . .

$$1005. \quad D^n(uv) = D^nuD^0v + \frac{n}{1} D^{n-1}uDv$$

$$+ \frac{n(n-1)}{1 \cdot 2} D^{n-2}uD^2v + \dots + D^0uD^nv.$$

The form of development is that of the Binomial Theorem.

*Taylor's Theorem.*

$$1006. \quad f(x+h) = f(x) + \frac{h}{1!} f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \dots$$

$$\dots + \frac{h^{n-2}}{(n-2)!} f^{(n-2)}(x) + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(x) + R_n.$$

$$R_n = \frac{h^n}{n!} f^{(n)}(x + \theta h), \text{ wherein } 0 < \theta < +1.$$

Or

$$R_n = \frac{(1-\theta)^{n-1} h^n}{(n-1)!} f^{(n)}(x + \theta h), \text{ wherein } 0 < \theta < +1.$$

*Maclaurin's Theorem.*

$$1007. \quad f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\dots + \frac{x^{n-2}}{(n-2)!} f^{(n-2)}(0) + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n.$$

$$R_n = \frac{x^n}{n!} f^{(n)}(\theta x), \text{ wherein } 0 < \theta < +1.$$

$$R_n = \frac{(1-\theta)^{n-1} x^n}{(n-1)!} f^{(n)}(\theta x), \text{ wherein } 0 < \theta < +1.$$

N.B. — Taylor's and Maclaurin's Theorems are valid only when  $f(x)$  and its first  $n$  successive derivatives are finite and continuous for all values of  $x$  in the interval from  $x$  to  $x+h$ , for Taylor's, and in the interval from 0 to  $x$ , for Maclaurin's Theorem.

*Taylor's Theorem for a Function of Two Variables.*

$$f(x, y) = u.$$

1008.

$$f(x+h, y+k) = u + (hD_x u + kD_y u)$$

$$+ \frac{1}{2!} (h^2 D_x^2 u + 2hk D_x D_y u + k^2 D_y^2 u)$$

$$+ \frac{1}{3!} (h^3 D_x^3 u + 3h^2 k D_x^2 D_y u + 3hk^2 D_x D_y^2 u + k^3 D_y^3 u) + \dots$$

the general term being  $\frac{1}{n!} (hD_x + kD_y)^n u$ .

The symbolic form of Taylor's Theorem for one variable is,

$$1009. \quad f(x + h) = e^{hD_x} f(x),$$

for two variables,

$$1010. \quad f(x + h, y + k) = e^{hD_x + kD_y} f(x, y).$$

*Circular and Hyperbolic Functions expressed  
in Series.*

$$1011. \quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad [x^2 < \infty.$$

$$1012. \quad \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad [x^2 < \infty.$$

$$1013. \quad \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \dots \quad [x^2 < \frac{1}{4}\pi^2.$$

$$1014. \quad \operatorname{ctn} x = \frac{1}{x} - \frac{x}{3} - \frac{x^3}{45} - \frac{2x^5}{945} - \frac{x^7}{4725} - \dots \quad [x^2 < \pi^2.$$

$$1015. \quad \sec x = 1 + \frac{x^2}{2} + \frac{5x^4}{24} + \frac{61x^6}{720} + \dots \quad [x^2 < \frac{1}{4}\pi^2.$$

$$1016. \quad \csc x = \frac{1}{x} + \frac{x}{6} + \frac{7x^3}{360} + \frac{31x^5}{15120} + \dots \quad [x^2 < \pi^2.$$

$$1017. \quad \sin^{-1} x = x + \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \times 3}{2 \times 4} \cdot \frac{x^5}{5} + \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \cdot \frac{x^7}{7} + \dots \\ [x^2 < 1.$$

$$1018. \quad \cos^{-1} x = \frac{1}{2}\pi - \sin^{-1} x.$$

$$1019. \quad \tan^{-1} x = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots \quad [x^2 < 1.$$

$$1020. \quad \operatorname{ctn}^{-1} x = \frac{1}{2}\pi - \tan^{-1} x.$$

$$1021. \quad \tan^{-1} x = \frac{1}{2}\pi - x^{-1} + \frac{1}{3}x^{-3} - \frac{1}{5}x^{-5} + \frac{1}{7}x^{-7} - \dots \\ [x^2 > 1.$$

$$1022. \quad \sec^{-1} x =$$

$$\frac{1}{2}\pi - x^{-1} - \frac{1}{2} \cdot \frac{x^{-3}}{3} - \frac{1 \times 3}{2 \times 4} \cdot \frac{x^{-5}}{5} - \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \cdot \frac{x^{-7}}{7} + \dots \quad [x^2 > 1.$$

$$1023. \quad \csc^{-1} x = \frac{1}{2}\pi - \sec^{-1} x.$$

*Bernouilli's and Euler's Numbers.*

Some of the series above given take more symmetrical forms when certain coefficients  $B_1, B_2, B_3, \dots$  are used.

**1024.** The coefficients of odd orders, known as Bernouilli's Numbers, are

$$B_1 = \frac{1}{6} \quad B_{11} = \frac{691}{2730}$$

$$B_3 = \frac{1}{30} \quad B_{13} = \frac{7}{6}$$

$$B_5 = \frac{1}{42} \quad B_{15} = \frac{3617}{510}$$

$$B_7 = \frac{1}{30} \quad B_{17} = \frac{43867}{798}$$

$$B_9 = \frac{5}{66} \quad \text{etc.} \quad \text{etc.}$$

**1025.** The coefficients of even orders, known as Euler's Numbers, are

$$B_2 = 1 \quad B_{10} = 50521$$

$$B_4 = 5 \quad B_{12} = 2702765$$

$$B_6 = 61 \quad \text{etc.} \quad \text{etc.}$$

$$B_8 = 1385$$

Thus,

**1026.**  $\tan x =$

$$\frac{2^2(2^2 - 1)}{2!} B_1 x + \frac{2^4(2^4 - 1)}{4!} B_3 x^3 + \frac{2^6(2^6 - 1)}{6!} B_5 x^5 + \dots$$

$[x^2 < \frac{1}{4}\pi^2]$

$$\text{ctn } x = \frac{1}{x} - \frac{2^2}{2!} B_1 x - \frac{2^4}{4!} B_3 x^3 - \frac{2^6}{6!} B_5 x^5 - \frac{2^8}{8!} B_7 x^7 - \dots$$

$[x^2 < \pi^2]$

$$\sec x = 1 + \frac{B_2}{2!} x^2 + \frac{B_4}{4!} x^4 + \frac{B_6}{6!} x^6 + \dots \quad [x^2 < \frac{1}{4}\pi^2]$$

**1029.**  $\csc x =$

$$\frac{1}{x} + \frac{2(2^2 - 1)}{2!} B_1 x + \frac{2(2^4 - 1)}{4!} B_3 x^3 + \frac{2(2^6 - 1)}{6!} B_5 x^5 + \dots$$

$[x^2 < \pi^2]$

$$\text{Sinh } x = \frac{1}{2}(e^x - e^{-x}) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots$$

$[x^2 < \infty]$

$$\text{Cosh } x = \frac{1}{2}(e^x + e^{-x}) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$[x^2 < \infty]$

1032.  $\operatorname{Tanh} x =$

$$\begin{aligned} & \frac{2^2 (2^2 - 1)}{2!} B_1 x - \frac{2^4 (2^4 - 1)}{4!} B_3 x^3 + \frac{2^6 (2^6 - 1)}{6!} B_5 x^5 - \dots \\ &= x - \frac{x^3}{3} + \frac{2x^5}{15} - \frac{17x^7}{315} + \frac{62x^9}{2835} \dots [x^2 < \frac{1}{4}\pi^2]. \end{aligned}$$

1033.

$$\begin{aligned} \operatorname{Ctnh} x &= \frac{1}{x} + \frac{2^2}{2!} B_1 x - \frac{2^4}{4!} B_3 x^3 + \frac{2^6}{6!} B_5 x^5 - \frac{2^8}{8!} B_7 x^7 + \dots \\ &= \frac{1}{x} + \frac{x}{3} - \frac{x^3}{45} + \frac{2x^5}{945} - \frac{x^7}{4725} + \dots [x^2 = \pi^2]. \end{aligned}$$

1034.  $\operatorname{Sech} x = 1 - \frac{B_2}{2!} x^2 + \frac{B_4}{4!} x^4 - \frac{B_6}{6!} x^6 + \frac{B_8}{8!} x^8 - \dots$   
 $= 1 - \frac{x^2}{2} + \frac{5x^4}{24} - \frac{61x^6}{720} + \frac{1385x^8}{40320} - \dots$

$$[x^2 < \frac{1}{4}\pi^2].$$

1035.  $\operatorname{Csch} x =$

$$\begin{aligned} & \frac{1}{x} - \frac{2(2^2 - 1)}{2!} B_1 x + \frac{2(2^3 - 1)}{4!} B_3 x^3 - \frac{2(2^5 - 1)}{6!} B_5 x^5 + \dots \\ &= \frac{1}{x} - \frac{x}{6} + \frac{7x^3}{360} - \frac{31x^5}{15120} + \dots [x^2 < \pi^2]. \end{aligned}$$

1036.  $\operatorname{Sinh}^{-1} x = x - \frac{1}{2} \cdot \frac{x^3}{3} + \frac{1 \times 3}{2 \times 4} \cdot \frac{x^5}{5} - \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \cdot \frac{x}{7} + \dots$   
 $[x^2 < 1].$

1037.  $\operatorname{Tanh}^{-1} x = x + \frac{1}{3} x^3 + \frac{1}{5} x^5 + \frac{1}{7} x^7 + \dots [x^2 < 1].$   
 $= \frac{1}{2} \log_e \frac{1+x}{1-x}.$

1038.  $\operatorname{Ctnh}^{-1} x = x^{-1} + \frac{1}{3} x^{-3} + \frac{1}{5} x^{-5} + \frac{1}{7} x^{-7} + \dots$   
 $= \frac{1}{2} \log_e \frac{x+1}{x-1} [x^2 > 1].$

1039.  $\operatorname{Csch}^{-1} x =$

$$x^{-1} - \frac{1}{2} \cdot \frac{x^{-3}}{3} + \frac{1 \times 3}{2 \times 4} \cdot \frac{x^{-5}}{5} - \frac{1 \times 3 \times 5}{2 \times 4 \times 6} \cdot \frac{x^{-7}}{7} + \dots [x^2 > 1].$$

**1040.**  $\frac{x}{e^x - 1} = 1 - \frac{x}{2} + \frac{B_1 x^2}{2!} - \frac{B_3 x^4}{4!} + \frac{B_5 x^6}{6!} - \dots$  [ $x < 2\pi$ .]

**1041.**  $\frac{x}{e^x + 1} =$

$$\frac{x}{2} - \frac{2^2 - 1}{2!} B_1 x^2 + \frac{2^4 - 1}{4!} B_3 x^4 - \frac{2^6 - 1}{6!} B_5 x^6 + \dots$$

**1042.**  $\frac{1}{2} \cdot \frac{e^x - 1}{e^x + 1} = \frac{2^2 - 1}{2!} B_1 x - \frac{2^4 - 1}{4!} B_3 x^3 + \frac{2^6 - 1}{6!} B_5 x^5 - \dots$

*Evaluation of Indeterminate Forms.*

**1051.**  $\frac{0}{0}$ . When the fraction  $\frac{f(x)}{F(x)}$  for any given value of the variable,  $x = a$ , takes the form  $\frac{0}{0}$ , its true value can be found by differentiating both numerator and denominator and in the result,  $\frac{f'(x)}{F'(x)}$ , substituting  $a$  for  $x$ . If

$\frac{f'(x)}{F'(x)}$  also takes the form  $\frac{0}{0}$ , differentiate again, and, if necessary, again and again until a result is found which does not take the form  $\frac{0}{0}$  for  $x = a$ . This last result will give the true value sought.

**1052.**  $\frac{\infty}{\infty}$ . When the fraction assumes this form, the true value is found in the same way as in the case of  $\frac{0}{0}$ .

**1053.**  $0 \cdot \infty$ . When, for  $x = a$ ,  $f(x) = 0$  and  $\varphi(x) = \infty$ , so that the product  $f(x) \cdot \varphi(x)$  takes the form  $0 \cdot \infty$ , the true value of the product may be found by putting  $\varphi(x) = \frac{1}{F(x)}$  and finding that of  $\frac{f(x)}{F(x)}$ , which takes the form  $\frac{0}{0}$ .

**1054.**  $0^0, 1^\infty, \infty^0$ . When, for  $x = a$ , the expression  $\varphi(x)^{f(x)}$  takes either of these forms, its logarithm may be taken and dealt with as in the last case. Thus, putting

$$\varphi(x)^{f(x)} = y, \quad \log_e y = f(x) \cdot \log_e (\varphi(x)),$$

The true value of the exponent,  $f(x) \cdot \log_e(\varphi(x))$ , having been found, that of  $y$ , or that of the given function, follows.

**1055.**  $\infty - \infty$ . When  $f(x) - F(x)$  takes the form  $\infty - \infty$  it can be changed to a form which becomes  $\frac{0}{0}$ , and dealt with as in the first case.

Thus,

$$f(x) - F(x) = \frac{\frac{1}{F(x)} - \frac{1}{f(x)}}{\frac{1}{f(x) \cdot F'(x)}} = \frac{\frac{1}{\infty} - \frac{1}{\infty}}{\frac{1}{\infty \cdot \infty}} = \frac{0 - 0}{0} = \frac{0}{0}.$$

### Partial Differential Coefficients.

A differential coefficient of a function of two or more variables,  $f(x, y, z \dots)$ , obtained on the supposition that all the variables save one are, for the time being, constants, is a *partial differential coefficient*.

It is written

when  $x$  alone is supposed to vary,  $\frac{\partial}{\partial x} f(x, y, z \dots)$ ,

when  $y$  alone is supposed to vary,  $\frac{\partial}{\partial y} f(x, y, z \dots)$ ,

when  $z$  alone is supposed to vary,  $\frac{\partial}{\partial z} f(x, y, z \dots)$ ,

the form of the letter  $\partial$  indicating that the differentiation is partial.

Partial differential coefficients are also written

$D_x f(x, y, z \dots)$   $D_y f(x, y, z \dots)$   $D_z f(x, y, z \dots)$  etc.

the subscript letters  $x, y, z$ , etc., indicating the variables with reference to which the partial differentiation takes place.

Often the single letter  $f$  is used as an abbreviation for  $f(x, y, z \dots)$ , and  $df$  for the total differential of the same.

Thus

$$\text{1056. } df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz + \dots$$

or

$$\text{1057. } df = D_x f \cdot dx + D_y f \cdot dy + D_z f \cdot dz + \dots$$

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$$1070. \quad \frac{d^2y}{dx^2} = \frac{\frac{dx}{dt} \cdot \frac{d^2y}{dt^2} - \frac{dy}{dt} \cdot \frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3},$$

or  $D_x^2y = \frac{D_t x \cdot D_t^2y - D_t y \cdot D_t^2x}{(D_t x)^3}.$

1071.

$$\frac{d^3y}{dx^3} = \frac{\left(\frac{dx}{dt}\right)^2 \frac{d^3y}{dt^3} - 3 \frac{dx}{dt} \cdot \frac{d^2x}{dt^2} \cdot \frac{d^2y}{dt^2} + 3 \frac{dy}{dt} \cdot \left(\frac{d^2x}{dt^2}\right)^2 - \frac{dx}{dt} \cdot \frac{dy}{dt} \cdot \frac{d^3x}{dt^3}}{\left(\frac{dx}{dt}\right)^5}.$$

or

$$D_x^3y =$$

$$\frac{(D_t x)^2 D_t^3y - 3 D_t x \cdot D_t^2x \cdot D_t^2y + 3 D_t y (D_t^2x)^2 - D_t x \cdot D_t y \cdot D_t^3x}{(D_t x)^5}.$$

### *Change of Independent Variable.*

In formulas 1069, 1070 and 1071, put  $t = y$ . The result is the following three formulas for changing the independent variable from  $x$  to  $y$ , that is when  $y = f(x)$  becomes  $x = \phi(y)$ .

$$1072. \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}, \text{ or } D_x y = \frac{1}{D_y x}.$$

$$1073. \quad \frac{d^2y}{dx^2} = -\frac{\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}, \text{ or } D_x^2y = -\frac{D_y^2x}{(D_y x)^3},$$

$$1074. \quad \frac{d^3y}{dx^3} = \frac{3 \left(\frac{d^2x}{dy^2}\right)^2 - \frac{dx}{dy} \cdot \frac{d^3x}{dy^3}}{\left(\frac{dx}{dy}\right)^5},$$

or  $D_x^3y = \frac{3 (D_y^2x)^2 - D_y x \cdot D_y^3x}{(D_y x)^5}$

To find  $D_x y$ ,  $D_x^2 y$ , etc., when  $y$  is an implicit function of  $x$ ,  $f(x, y) = 0$ .

$$\frac{\partial f}{\partial x} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dx} = 0, \text{ whence } \frac{dy}{dx} = -\frac{\partial f}{\partial x} \div \frac{\partial f}{\partial y}$$

1075. Or

$$D_x f + D_y f \cdot D_x y = 0, \text{ whence } D_x y = -D_x f \div D_y f.$$

1076.  $D_x^2 f + 2D_x D_y f \cdot (D_x y) + D_y^2 f \cdot (D_x y)^2 + D_y f \cdot D_x^2 y = 0$ , which, by substituting the value of  $D_x y$  found in 1075, gives  $D_x^2 y$ .

$$\begin{aligned} 1077. \quad & D_x^3 f + 3D_x^2 D_y f \cdot (D_x y) + 3D_x D_y^2 f \cdot (D_x y)^2 \\ & + D_y^3 f \cdot (D_x y)^3 + 3D_x D_y f \cdot (D_x^2 y) \\ & + 3D_y^2 f \cdot (D_x y)(D_x^2 y) + D_y f \cdot D_x^3 y = 0, \end{aligned}$$

which, by substituting the values of  $D_x y$  and  $D_x^2 y$  found in 1075 and 1076, gives  $D_x^3 y^3$ .

### *Maxima and Minima.*

1078. *Functions of one variable.* The values of  $x$  which give a maximum or a minimum value to  $y = f(x)$  are found by solving the equation  $f'(x) = 0$ . Any root,  $a$ , being substituted for  $x$ ,

if  $f''(a) < 0$ ,  $f(a)$  is a maximum,

if  $f''(a) > 0$ ,  $f(a)$  is a minimum.

If  $f''(a) = 0$  and  $f'''(a)$  is not 0,  $f(a)$  is neither a maximum nor a minimum; but if  $f'''(a)$  is also 0,  $f(a)$  is a maximum or a minimum according as  $f''''(a) \gtrless 0$ .

And in general, if the first derivative which does not vanish for  $x = a$  is of an odd order  $f(a)$  is neither a maximum nor a minimum; but if it is of an even order, the  $2n^{\text{th}}$ , then  $f(a)$  is a maximum or a minimum according as  $f^{(2n)}(a) > 0$ .

**1079.** *Implicit functions.*  $f(x, y) = 0$ . The values of  $x$  which give a maximum or a minimum value to  $y$  must satisfy the equations

$$\frac{\partial f(x, y)}{\partial x} = 0 \text{ and } f(x, y) = 0,$$

as well as fulfill the condition  $\frac{\partial f(x, y)}{\partial y} > 0$ .

$y$  is a maximum when  $\left[ -\frac{\partial^2 f}{\partial x^2} \right] \div \left[ \frac{\partial f}{\partial y} \right] < 0$ ,

$y$  is a minimum when  $\left[ -\frac{\partial^2 f}{\partial x^2} \right] \div \left[ \frac{\partial f}{\partial y} \right] > 0$ .

**1080.** *Function of two independent variables.*

$u = f(x, y)$  will be a maximum or a minimum for values of  $x$  and  $y$  that satisfy the two equations

$$\frac{\partial}{\partial x} f(x, y) = 0, \quad \frac{\partial}{\partial y} f(x, y) = 0,$$

and also the condition

$$\frac{\partial^2 f}{\partial x^2} \cdot \frac{\partial^2 f}{\partial y^2} - \left( \frac{\partial^2 f}{\partial x \partial y} \right)^2 > 0.$$

A maximum } requires  $\frac{\partial^2 f}{\partial x^2}$  and  $\frac{\partial^2 f}{\partial y^2}$  both to be } negative  
A minimum } positive.

### *Integration of Rational Algebraic Functions.*

**1081.** Rational algebraic functions are either of the form

$$Ax^m + Bx^{m-1} + Cx^{m-2} + \dots + Kx + L,$$

or of the form

$$\frac{Ax^m + Bx^{m-1} + Cx^{m-2} + \dots + Kx + L}{x^n + ax^{n-1} + bx^{n-2} + \dots + kx + l}.$$

In the former case the function is integrated by integrating each term separately. Formulas 816 and 839. In the latter case, if the degree  $m$  of the numerator is not less

than the degree  $n$  of the denominator, the numerator is divided by the denominator until a remainder appears which is of a degree at least one less than  $n$ . Thus the function to be integrated takes the form

$$A_0x^d + A_1x^{d-1} + A_2x^{d-2} + \dots + A_d + \frac{f(x)}{F(x)}.$$

where  $d = m - n$ , and the degree of  $f(x)$  is not higher than  $n - 1$ .

The first part of this expression is integrated, as before, by integrating its separate terms; and there remains to be integrated the fraction  $\frac{f(x)}{F(x)}$ , the degree of the numerator being less than that of the denominator. Such a fraction is called a *rational proper fraction*.

### *Integration of Rational Proper Fractions.*

A rational proper fraction is of the general form

$$108 \quad \frac{f(x)}{F(x)} = \frac{Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \dots + Kx + L}{x^n + ax^{n-1} + bx^{n-2} + \dots + kx + l}.$$

This is to be separated into partial fractions the sum of which shall be equal to  $\frac{f(x)}{F(x)}$ . For this purpose  $F(x)$  is separated into factors by solving the equation  $F(x) = 0$  to find the roots  $x_1, x_2, x_3, \dots, x_n$ , and then putting

$$F(x) = (x - x_1)(x - x_2)(x - x_3)\dots(x - x_n).$$

1083. Now assume

$$\frac{f(x)}{F(x)} = \frac{A_1}{x - x_1} + \frac{A_2}{x - x_2} + \dots + \frac{A_n}{x - x_n}$$

where the constants  $A_1, A_2, \dots, A_n$  are to be determined. This is done by clearing the equation of fractions, which gives the identity

$$f(x) = A_1f_1(x) + A_2f_2(x) + \dots + A_nf_n(x),$$

and equating the coefficients of like powers of  $x$  on each side of this identity.

Thus the function to be integrated becomes the sum of  $n$  partial fractions each of the form  $\frac{A}{x-a}$ , which is integrable by 847.

$$\int \frac{Adx}{x-a} = A \log_e (x-a) + C.$$

Thus far it has been assumed that the  $n$  roots of the equation  $F(x) = 0$  are all real and different.

**1084.** If  $F(x) = 0$  have equal roots, say  $p$  roots each equal to  $x_1$ ,  $q$  roots each equal to  $x_2$ , and so on, then

$$F(x) = (x - x_1)^p (x - x_2)^q (x - x_3)^r \dots$$

$$\text{where } p + q + r + \dots = n.$$

The  $n$  assumed partial fractions are

$$\begin{aligned} \frac{f(x)}{F(x)} &= \frac{A_1}{(x-x_1)^p} + \frac{A_2}{(x-x_1)^{p-1}} + \dots + \frac{A_p}{x-x_1} \\ &+ \frac{B_1}{(x-x_2)^q} + \frac{B_2}{(x-x_2)^{q-1}} + \dots + \frac{B_q}{x-x_2} \\ &+ \dots \quad \dots \quad \dots \end{aligned}$$

The values of the numerators are found, as before, by clearing of fractions and equating the coefficients of like powers of  $x$ .

Thus the function to be integrated becomes the sum of  $n$  partial fractions, some of the form  $\frac{A}{x-a}$ , already integrated, and the rest of the form  $\frac{A}{(x-a)^k}$ , which is integrable by 845.

$$\int \frac{A}{(x-a)^k} = \frac{-A}{(k-1)(x-a)^{k-1}} + C.$$

**1085.** If  $F(x) = 0$  have imaginary roots, these come in pairs; that is, if  $x_1$  is of the form  $p + iq$  some other root,  $x_2$ , say, is of the conjugate form  $p - iq$ . The corresponding partial fractions being added give a fraction of the form  $\frac{Px + Q}{(x-p)^2 + q^2}$ , which is free from imaginaries and is used in place of the two partial fractions that were added to form it.

If there be two or more, say  $l$ , equal imaginary roots and their conjugates, the corresponding partial fractions to be assumed are

$$\frac{P_1x + Q_1}{[(x - p)^2 + q^2]^1} + \frac{P_2x + Q_2}{[(x - p)^2 + q^2]^{l-1}} + \dots + \frac{P_lx + Q_l}{(x - p)^2 + q^2}.$$

And similarly for any other set of equal imaginary roots. The values of the  $P$ 's and  $Q$ 's, as well as of any other assumed constants in the numerators of other partial fractions, are found, as before, by clearing the equation of fractions and equating coefficients of like powers of  $x$ . Thus the function to be integrated is broken up into partial fractions, some, it may be, of forms already integrated and

the others of the forms  $\frac{Px + Q}{(x - p)^2 + q^2}$  and  $\frac{Px + Q}{[(x - p)^2 + q^2]^l}$ ,

which are integrable by the next ten formulas.

**1086.**

$$\int \frac{A + Bx}{a + bx + cx^2} dx = A \int \frac{dx}{a + bx + cx^2} + B \int \frac{x dx}{a + bx + cx^2}.$$

$$\text{1087. } \int \frac{dx}{a + bx + cx^2} = \frac{2}{\sqrt{4ac - b^2}} \tan^{-1} \frac{b + 2cx}{\sqrt{4ac - b^2}} + C,$$

when  $4ac - b^2 > 0$ .

$$\text{1088. } \int \frac{dx}{a + bx + cx^2} = \frac{-2}{b + 2cx} + C,$$

when  $4ac - b^2 = 0$ .

$$\text{1089. } \int \frac{dx}{a + bx + cx^2} = \frac{1}{\sqrt{b^2 - 4ac}} \log_e \left( \frac{\sqrt{(b^2 - 4ac)} - b - 2cx}{\sqrt{(b^2 - 4ac)} + b + 2cx} \right) + C,$$

when  $4ac - b^2 < 0$ .

$$\text{1090. } \int \frac{dx}{a + bx + cx^2} = \frac{1}{\sqrt{b^2 - 4ac}} \log_e \left( \frac{b + 2cx - \sqrt{b^2 - 4ac}}{b + 2cx + \sqrt{b^2 - 4ac}} \right) + C,$$

which is merely another form of 1089, to be used when  $\sqrt{b^2 - 4ac}$  is less than  $b + 2cx$ .

$$1091. \int \frac{dx}{a + bx + cx^2} = \frac{-2}{\sqrt{b^2 - 4ac}} \operatorname{Tanh}^{-1} \frac{b + 2cx}{\sqrt{b^2 - 4ac}} + C,$$

which is analogous to 1087, and to be used when  $4ac - b^2 < 0$ , and  $b + 2cx < \sqrt{b^2 - 4ac}$ . If  $b + 2cx > \sqrt{b^2 - 4ac}$ ,  $\operatorname{Ctnh}^{-1}$  may be written in place of  $\operatorname{Tanh}^{-1}$ .

$$1092. \int \frac{x \, dx}{a + bx + cx^2} =$$

$$\frac{1}{2c} \log_e (a + bx + cx^2) - \frac{b}{2c} \int \frac{dx}{a + bx + cx^2}.$$

$$1093. \int \frac{x \, dx}{(a + bx + cx^2)^p} =$$

$$\frac{-1}{2c(p-1)(a+bx+cx^2)^{p-1}} - \frac{b}{2c} \int \frac{dx}{(a+bx+cx^2)^p}.$$

$$1094. \int \frac{dx}{(a + bx + cx^2)^p} =$$

$$\frac{1}{4ac - b^2} \cdot \frac{b + 2cx}{(p-1)(a+bx+cx^2)^{p-1}}$$

$$+ \frac{(2p-3)2c}{(4ac-b^2)(p-1)} \int \frac{dx}{(a+bx+cx^2)^{p-1}}.$$

The repeated application of this reduction formula makes the integral in the first member depend ultimately on

$$\int \frac{dx}{a + bx + cx^2}, \text{ for which refer above to 1087-1091.}$$

Formula 1086 by substitution from 1092 becomes

$$1095. \int \frac{A + Bx}{a + bx + cx^2} =$$

$$\frac{B}{2c} \log_e (a + bx + cx^2) + \frac{2Ac - Bb}{2c} \int \frac{dx}{a + bx + cx^2}.$$

This completes the list of formulas necessary for the integration of rational proper fractions by breaking them into partial fractions.

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1104. For the removal of a radical of the form

$\sqrt{a^2 + b^2x^2}$ , put  $\frac{\sqrt{a^2 + b^2x^2} - bx}{a} = y$ , then

$$x = \frac{a}{b} \cdot \frac{1 - y^2}{2y},$$

$$\sqrt{a^2 + b^2x^2} = \frac{a(1 + y^2)}{2y},$$

$$dx = -\frac{a}{b} \cdot \frac{1 + y^2}{2y^2} dy.$$

1105. For the removal of a radical of the form  $\sqrt{a^2 - b^2x^2}$ , put

$$\sqrt{\frac{a - bx}{a + bx}} = y, \text{ then } x = \frac{a}{b} \cdot \frac{1 - y^2}{1 + y^2},$$

$$\sqrt{a^2 - b^2x^2} = \frac{2ay}{1 + y^2},$$

$$dx = -\frac{4a}{b} \cdot \frac{y dy}{(1 + y^2)^2}.$$

For the removal of a radical of the form  $\sqrt{a + bx \pm cx^2}$ , where  $c$  in itself is always positive, there are two devices, one for the upper sign and one for the lower.

1106. (i) Put, for abbreviation,  $\sqrt{4ac - b^2} = h$ , and then put

$$\frac{2\sqrt{c}\sqrt{a + bx + cx^2} - (b + 2cx)}{h} = y, \text{ which gives}$$

$$x = \frac{h(1 - y^2) - 2by}{4cy},$$

$$\sqrt{a + bx + cx^2} = \frac{h(1 + y^2)}{4\sqrt{c} \cdot y},$$

1107. (ii) Put, for abbreviation,  $\sqrt{4ac + b^2} = k$ , and then put

$$\sqrt{\frac{k + b - 2cx}{k - b + 2cx}} = y, \text{ which gives}$$

$$x = \frac{k + b - (k - b) y^2}{2c(1 + y^2)}$$

$$\sqrt{a + bx - cx^2} = \frac{ky}{\sqrt{c.(1 + y^2)}}$$

$$dx = - \frac{2kydy}{c(1 + y^2)^2}.$$

$$1108. \int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1} \frac{x}{a} + C, \text{ or } = - \cos^{-1} \frac{x}{a} + C.$$

$$1109. \int \frac{x^2 dx}{\sqrt{a^2 - x^2}} = - \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C.$$

$$1110. \int \sqrt{a^2 - x^2} dx = \frac{x}{2} \sqrt{a^2 - x^2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} + C.$$

$$1111. \int \frac{dx}{\sqrt{x^2 + a^2}} = \operatorname{Sinh}^{-1} \frac{x}{a} + C. \\ = \log_e (x + \sqrt{x^2 + a^2}) + C.$$

$$1112. \int \frac{x^2 dx}{\sqrt{x^2 + a^2}} = \frac{x}{2} \sqrt{x^2 + a^2} - \frac{a^2}{2} \operatorname{Sinh}^{-1} \frac{x}{a} + C. \\ = \frac{x}{2} \sqrt{x^2 + a^2} - \frac{a^2}{2} \log_e (x + \sqrt{x^2 + a^2}) + C.$$

$$1113. \int \sqrt{x^2 + a^2} dx = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \operatorname{Sinh}^{-1} \frac{x}{a} + C \\ = \frac{x}{2} \sqrt{x^2 + a^2} + \frac{a^2}{2} \log_e (x + \sqrt{x^2 + a^2}) + C.$$

$$1114. \int \frac{dx}{\sqrt{x^2 - a^2}} = \operatorname{Cosh}^{-1} \frac{x}{a} + C. \\ = \log_e (x + \sqrt{x^2 - a^2}) + C.$$

$$\text{III5. } \int \frac{x^2 dx}{\sqrt{x^2 - a^2}} = \frac{x}{2} \sqrt{x^2 - a^2} + \frac{a^2}{2} \operatorname{Cosh}^{-1} \frac{x}{a} + C.$$

$$= \frac{x}{2} \sqrt{x^2 - a^2} + \frac{a^2}{2} \log_e (x + \sqrt{x^2 - a^2}) + C.$$

$$\text{III6. } \int \sqrt{x^2 - a^2} dx = \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \operatorname{Cosh}^{-1} \frac{x}{a}$$

$$= \frac{x}{2} \sqrt{x^2 - a^2} - \frac{a^2}{2} \log_e (x + \sqrt{x^2 - a^2}) + C.$$

$$\text{III7. } \int \frac{dx}{\sqrt{2ax - x^2}} = \operatorname{vers}^{-1} \frac{x}{a} + C = \sin^{-1} \frac{x - a}{a} + C.$$

$$\text{III8. } \int \frac{x dx}{\sqrt{2ax - x^2}} = -\sqrt{2ax - x^2} + a \operatorname{vers}^{-1} \frac{x}{a} + C$$

$$= -\sqrt{2ax - x^2} + a \sin^{-1} \frac{x - a}{a} + C.$$

$$\text{III9. } \int \sqrt{2ax - x^2} dx =$$

$$\frac{x - a}{2} \sqrt{2ax - x^2} + \frac{a^2}{2} \sin^{-1} \left( \frac{x - a}{a} \right) + C.$$

**III20.**

$$\int \frac{dx}{\sqrt{a + bx + cx^2}} = \frac{1}{\sqrt{c}} \log_e \left( \frac{b + 2cx}{2} + \sqrt{c} \sqrt{a + bx + cx^2} \right) + C,$$

when  $c > 0$ .

$$= \frac{1}{\sqrt{c}} \operatorname{Sinh}^{-1} \left( \frac{b + 2cx}{\sqrt{4ac - b^2}} \right) + C,$$

when  $4ac - b^2 > 0, c > 0$ .

$$= \frac{1}{\sqrt{c}} \operatorname{Cosh}^{-1} \left( \frac{b + 2cx}{\sqrt{b^2 - 4ac}} \right) + C,$$

when  $4ac - b^2 < 0, c > 0$ .

$$= \frac{-1}{\sqrt{-c}} \sin^{-1} \left( \frac{b + 2cx}{\sqrt{b^2 - 4ac}} \right) + C, \text{ when } c < 0:$$

**III21.**

$$\int \sqrt{\frac{b+2cx}{4c}} dx = \frac{b+2cx}{4c} \sqrt{X} + \frac{4ac - b^2}{8c} \int \frac{dx}{\sqrt{X}},$$

wherein  $\sqrt{X} = \sqrt{a + bx + cx^2}$ .

1122.

$$\int \frac{(A + Bx) dx}{\sqrt{a + bx + cx^2}} = \frac{B\sqrt{X}}{c} + \frac{2Ac - Bb}{2c} \int \frac{dx}{\sqrt{X}},$$

wherein  $\sqrt{X} = \sqrt{a + bx + cx^2}$ .

1123.

$$\int \frac{x^m dx}{\sqrt{a + bx + cx^2}} = \frac{x^{m-1} \sqrt{X}}{mc} - \frac{(2m-1)b}{2mc} \int \frac{x^{m-1} dx}{\sqrt{X}} \\ - \frac{(m-1)a}{mc} \int \frac{x^{m-2} dx}{\sqrt{X}},$$

wherein  $\sqrt{X} = \sqrt{a + bx + cx^2}$ .1124.  $\int \frac{dx}{(x-p)^m \sqrt{a+bx+cx^2}}$ , by the substitution of

$y$  for  $\frac{1}{x-p}$ , is changed to the form of the first member of  
the preceding formula.

1125.

$$\int \frac{dx}{(a-x)\sqrt{x-b}} = \frac{2}{\sqrt{a-b}} \operatorname{Tanh}^{-1} \sqrt{\frac{x-b}{a-b}} + C$$

or

$$-\frac{2}{\sqrt{a-b}} \operatorname{Ctnh}^{-1} \sqrt{\frac{x-b}{a-b}} + C$$

or

$$-\frac{-2}{\sqrt{b-a}} \tan^{-1} \sqrt{\frac{x-b}{b-a}} + C$$

The real  
form to be  
taken.

1126.

$$\int \frac{dx}{(a-x)\sqrt{b-x}} = \frac{2}{\sqrt{b-a}} \operatorname{Tanh}^{-1} \sqrt{\frac{b-x}{b-a}} + C$$

or

$$-\frac{2}{\sqrt{b-a}} \operatorname{Ctnh}^{-1} \sqrt{\frac{b-x}{b-a}} + C$$

or

$$-\frac{-2}{\sqrt{a-b}} \tan^{-1} \sqrt{\frac{b-x}{a-b}} + C$$

The real  
form to be  
taken.

1127.  $\int \frac{(A + Bx) dx}{(f + gx + hx^2) \sqrt{a + bx + cx^2}}$ , by putting  $\frac{py + q}{y + 1}$

in place of  $x$ , the values of  $p$  and  $q$  being determined from the equations

$$2cpq + b(p + q) + 2a = 0$$

$$2hpq + g(p + q) + 2f = 0$$

and then putting  $z$  in place of  $y^2$ , is reduced to the forms

$\int \frac{dz}{(f_1 + h_1 z) \sqrt{a_1 + c_1 z}}$  and  $\int \frac{dz}{(f_2 + h_2 z) \sqrt{a_1 z + c_1 z^2}}$ , which are

integrable by 1102, 1096, and 1098.

1128.  $\int F(x) \frac{f(x) dx}{\sqrt{a + bx + cx^2}}$  can be integrated by the

separation of  $\frac{f(x)}{F(x)}$  into a series of terms and partial fractions, multiplying each by  $\frac{dx}{\sqrt{a + bx + cx^2}}$ , and integrating

the separate terms. The partial fractions will be of the form 1124 or, when a pair of imaginary roots of the equation  $F(x) = 0$  occurs, of the form 1127.

1129.  $\int \frac{(A + Bx) dx}{(a + bx + cx^2)^{\frac{3}{2}}} = \frac{(4aB - 2bA) + (2bB - 4cA)x}{(b^2 - 4ac) \sqrt{a + bx + cx^2}} + C.$

*Reduction formulas for integral powers of the trigonometric functions.*

1130.  $\int \sin^m x \cdot \cos^n x \cdot dx$

$$= -\frac{\sin^{m-1} x \cdot \cos^{n+1} x}{m+n} + \frac{m-1}{m+n} \int \sin^{m-2} x \cdot \cos^n x \cdot dx,$$

1131.  $= \frac{\sin^{m+1} x \cos^{n-1} x}{m+n} + \frac{n-1}{m+n} \int \sin^m x \cdot \cos^{n-2} x \cdot dx.$

1132.  $\int \frac{\sin^m x}{\cos^n x} dx$

$$= \frac{\sin^{m+1} x}{(n-1) \cos^{n-1} x} + \frac{n-m-2}{n-1} \int \frac{\sin^m x}{\cos^{n-2} x} dx.$$

$$\text{1133.} \quad = -\frac{\sin^{m-1} x}{(m-n) \cos^{n-1} x} + \frac{m-1}{m-n} \int \frac{\sin^{m-2} x}{\cos^n x} dx.$$

$$\text{1134.} \quad = \frac{\sin^{m-1} x}{(n-1) \cos^{n-1} x} - \frac{m-1}{n-1} \int \frac{\sin^{m-2} x}{\cos^{n-2} x} dx.$$

$$\text{1135.} \quad \int \frac{\cos^n x}{\sin^m x} dx \\ = -\frac{\cos^{n+1} x}{(m-1) \sin^{m-1} x} + \frac{m-n-2}{m-1} \int \frac{\cos^n x}{\sin^{m-2} x} dx.$$

$$\text{1136.} \quad = -\frac{\cos^{n-1} x}{(m-1) \sin^{m-1} x} - \frac{n-1}{m-1} \int \frac{\cos^{n-2} x}{\sin^{m-2} x} dx.$$

$$\text{1137.} \quad \int \frac{dx}{\sin^m x \cos^n x} \\ = \frac{1}{(n-1) \sin^{m-1} x \cdot \cos^{n-1} x} + \frac{m+n-2}{n-1} \int \frac{dx}{\sin^m x \cdot \cos^{n-2} x}.$$

$$\text{1138.} \quad = \frac{-1}{(m-1) \sin^{m-1} x \cos^{n-1} x} + \frac{m+n-2}{m-1} \int \frac{dx}{\sin^{m-2} x \cos^n x}.$$

The result of repeated applications of one or of two (alternately) of the foregoing formulas (1130-1138) is one of the following integrable forms:

$$\int dx = x,$$

$$\int \cos x \cdot dx = \sin x, \quad \int \frac{\cos x}{\sin x} dx = \log_e \sin x,$$

$$\int \sin x \cdot dx = -\cos x, \quad \int \sin x \cos x \cdot dx = -\frac{1}{2} \cos 2x,$$

$$\int \frac{\sin x}{\cos x} dx = -\log_e \cos x, \quad \int \frac{dx}{\sin x \cos x} = \log_e \tan x,$$

$$\int \frac{dx}{\sin x} = \log_e \tan \frac{x}{2}, \quad \int \frac{dx}{\cos x} = \log_e \tan \left( \frac{\pi}{4} + \frac{x}{2} \right).$$

For  $\int \sin^n x \cdot dx$ , use 1130, putting  $n = 0$ .

$\int \cos^n x \cdot dx$ , use 1131, putting  $m = 0$ .

$\int \tan^n x \cdot dx$ , use 1134, putting  $n = m$ .

$\int \operatorname{ctn}^n x \cdot dx$ , use 1136, putting  $m = n$ .

$\int \sec^n x \cdot dx$ , use 1137, putting  $m = 0$ .

$\int \csc^n x \cdot dx$ , use 1138, putting  $n = 0$ .

### Miscellaneous Integrals.

$$\begin{aligned} 1139. \quad & \int \frac{dx}{a \pm b \sin x} \\ &= \frac{2 \sec \beta}{a} \tan^{-1} \left( \sec \beta \tan \frac{x}{2} \pm \tan \beta \right) + C, \\ & \text{when } a > b, \text{ and } b = a \sin \beta. \end{aligned}$$

$$\begin{aligned} 1140. \quad & \int \frac{dx}{a \pm b \sin x} = \pm \frac{\sec \alpha}{b} \log_e \frac{\sin \frac{1}{2}(\alpha \pm x)}{\cos \frac{1}{2}(x \mp \alpha)} + C, \\ & \text{when } b > a, \text{ and } a = b \sin \alpha. \end{aligned}$$

$$\begin{aligned} 1141. \quad & \int \frac{dx}{a + b \cos x} = \frac{2}{\sqrt{a^2 - b^2}} \tan^{-1} \left( \sqrt{\frac{a-b}{a+b}} \tan \frac{x}{2} \right) + \\ & \text{when } a^2 > b^2. \end{aligned}$$

$$\begin{aligned} 1142. \quad & \int \frac{dx}{a + b \cos x} \\ &= \frac{2}{\sqrt{b^2 - a^2}} \operatorname{Tanh}^{-1} \left( \sqrt{\frac{b-a}{b+a}} \tan \frac{x}{2} \right) + C, \\ & \text{when } b^2 > a^2. \end{aligned}$$

$$1143. \quad \int \frac{\cos x \, dx}{a + b \cos x} = \frac{x}{b} - \frac{a}{b} \int \frac{dx}{a + b \cos x} + C.$$

$$1144. \quad \int \frac{\sin x \, dx}{a + b \cos x} = -\frac{1}{b} \log_e (a + b \cos x) + C.$$

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*Definite Integrals.*

$$1157. \int_a^b = - \int_b^a.$$

$$1158. \int_a^c = \int_a^b + \int_b^c.$$

$$1159. \int_a^c - \int_a^b = \int_b^c.$$

$$1160. \int_0^\infty \frac{dx}{a + bx^2} = \frac{\pi}{2\sqrt{ab}}.$$

$$1161. \int_0^{\sqrt{a/b}} \frac{dx}{a + bx^2} = \int_0^\infty \frac{dx}{a + bx^2} = \frac{\pi}{4\sqrt{ab}}.$$

$$1162. \int_0^{\sqrt{a/b}} \frac{dx}{\sqrt{a - bx^2}} = \frac{\pi}{2\sqrt{b}}.$$

$$1163. \int_0^\infty \frac{\sin bx}{x} dx = \frac{\pi}{2}, \quad \text{if } b \text{ is positive; } = -\frac{\pi}{2}, \quad \text{if } b \text{ is negative; } = 0, \text{ if } b = 0.$$

$$1164. \int_0^\infty \frac{\cos bx}{x} dx = \infty.$$

$$1165. \int_0^\infty \frac{\sin x dx}{\sqrt{x}} = \int_0^\infty \frac{\cos x dx}{\sqrt{x}} = \sqrt{\frac{1}{2}\pi}.$$

$$1166. \int_0^\pi \sin^2 mx dx = \int_0^\pi \cos^2 mx dx = \frac{1}{2}\pi.$$

$$1167. \int_0^{\frac{1}{2}\pi} \sin^{2n+1} x dx = \int_0^{\frac{1}{2}\pi} \cos^{2n+1} x dx, \\ = \frac{2 \times 4 \times 6 \times \dots \times 2n}{3 \times 5 \times 7 \times \dots \times (2n+1)}.$$

$$1168. \int_0^{\frac{1}{2}\pi} \sin^{2n} x dx = \int_0^{\frac{1}{2}\pi} \cos^{2n} x dx, \\ = \frac{1 \times 3 \times 5 \times \dots \times (2n-1)}{2 \times 4 \times 6 \times \dots \times 2n} \cdot \frac{\pi}{2}$$

} wherein  $n$  is a positive integral number.

$$1169. \int_0^{\infty} e^{-x} dx = 1. \quad 1170. \int_0^{\infty} e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}.$$

$$1171. \int_0^{\infty} x^n e^{-ax} dx = \frac{n!}{a^n}, \quad [\text{wherein } a \text{ is positive, and } n \text{ is a positive integer.}]$$

$$1172. \int_0^{\infty} \frac{x^{n-1} dx}{x+1} = \frac{\pi}{\sin n\pi}, \quad [0 < n < 1.]$$

*Approximate Integration.*

To find the area of the plane space included between an arc of the curve  $y = f(x)$ , the axis of  $x$ , and two fixed ordinates, suppose the given space to be divided into  $n$  narrow strips by a series of equidistant ordinates,  $h$  being their common distance apart. The larger the number of strips the narrower each strip, and the closer the approximation to the required area made by any of the following methods.

(i) Each strip regarded as a rectangle of base  $h$  and height equal to the ordinate on the left-hand side of it. The areas of the strips are

$$hy_0, hy_1, hy_2, \dots hy_{n-1},$$

and their sum,  $A$ , is approximately the area of the given space.

$$1181. A = h(y_0 + y_1 + y_2 + \dots + y_{n-1}).$$

(ii) Each strip regarded as a trapezoid the areas of which are

$$h \frac{y_0 + y_1}{2}, h \frac{y_1 + y_2}{2}, h \frac{y_2 + y_3}{2}, \dots h \frac{y_{n-1} + y_n}{2}$$

and their sum

$$1182. A = h(\frac{1}{2}y_0 + y_1 + y_2 + \dots + y_{n-1} + \frac{1}{2}y_n).$$

This approximation is closer than that given by 1181.

(iii) A still closer approximation is obtained by regarding the curve passing through the ends of each set of three

successive ordinates as being approximately the arc of a parabola having its axis parallel to the ordinates. For this purpose, make  $n$  an even number.

The area of the two strips lying between the ordinates  $y_0, y_1, y_2$  is

$$\frac{1}{3}h(y_0 + 4y_1 + y_2).$$

The area of the next two strips lying between the ordinates  $y_2, y_3, y_4$  is

$$\frac{1}{3}h(y_2 + 4y_3 + y_4).$$

And so on, the area of the last two strips being

$$\frac{1}{3}h(y_{n-2} + 4y_{n-1} + y_n).$$

The sum of these  $\frac{1}{3}n$  double strips is

$$\text{1183. } A = \frac{1}{3}h[y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) \\ + 2(y_2 + y_4 + y_6 + \dots + y_{n-2}) + y_n].$$

This formula is known under the name of *Simpson's Rule*.

Simpson's Rule may be used for computing the value of the definite integral

$$\int_a^b f(x) dx,$$

provided  $f(x)$  and its derivatives  $f'(x)$ ,  $f''(x)$ , and  $f'''(x)$  remain finite and continuous in the interval from  $x = a$  to  $x = b$ , and provided further that  $f'''(x)$  undergo no change of sign in the same interval. Under these conditions

$$\text{1184. } \int_a^b f(x) dx = \frac{1}{3}h[y_0 + 4(y_1 + y_3 + y_5 + \dots + y_{n-1}) \\ + 2(y_2 + y_4 + y_6 + \dots + y_{n-2}) + y_n] \\ + \frac{1}{288} \rho h^4 [f'''(b) - f'''(a)].$$

The last term represents  $E$ , the error of Simpson's Rule, that is

$$\text{1185. } E = \frac{1}{288} \rho h^4 [f'''(b) - f'''(a)].$$

In this expression  $\rho$  is no further known than that it is a positive or negative proper fraction,

$$-1 < \rho < +1;$$

therefore the limits of the error are found by substituting  $-1$  and  $+1$  for  $\rho$  in the above expression.

**1186.** Limits of error =  $\pm \frac{h^4}{288} [f''(b) - f''(a)].$

### *Differential Equations of the First Order.*

A differential equation of the first order involves only the first derivatives of the dependent variable.

**1201.** Given,  $f(x) dx + F(y) dy = 0.$

Solution,  $\int f(x) dx + \int F(y) dy = C.$

### *Separation of the Variables.*

**1202.** Given,  $f(x) \cdot \varphi(y) dx + F(x) \cdot \psi(y) dy = 0.$

Solution,  $\int \frac{f(x)}{F(x)} dx + \int \frac{\psi(y)}{\varphi(y)} dy = C.$

**1203.** Given,  $f(x, y) dx + \varphi(x, y) dy = 0.$

This form is integrable if

$$D_y f(x, y) = D_x \varphi(x, y);$$

and the solution is either

$$\int f(x, y) dx + \int \left[ \varphi(x, y) - \int D_y f(x, y) dx \right] dy = C,$$

or  $\int \varphi(x, y) dy + \int \left[ f(x, y) - \int D_x \varphi(x, y) dy \right] dx = C.$

**1204.** But if the condition of integrability above given is not satisfied, the equation may become integrable when multiplied by an integrating factor,  $M$ , which must satisfy the partial differential equation

$$f(x, y) D_y M - \varphi(x, y) D_x M = M [D_x \varphi(x, y) - D_y f(x, y)].$$

**1205.** In many cases  $M$  can be found from the equations

$$\frac{dy}{f(x, y)} = \frac{dx}{\varphi(x, y)} = \frac{dM}{M[D_x \varphi(x, y) - D_y f(x, y)]}.$$

*Homogeneous Differential Equations.*

**1206.** If a differential equation of the first order, homogeneous as to  $x$  and  $y$ , can be reduced to the general form

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right),$$

it can be solved by putting

$$\frac{y}{x} = t, \quad y = xt, \quad dy = xdt + tdx.$$

The solution is

$$\log_e x = \int \frac{dt}{f(t) - t} + C.$$

*Linear Differential Equations.*

A linear differential equation is one in which the dependent variable and its derivatives occur in the first degree only. The general form of such an equation of the first order is

$$\frac{dy}{dx} + Py = Q,$$

wherein  $P$  and  $Q$  are functions of  $x$  or constants.

**1207.** The method of solution is as follows.

First, determine  $y$  by integrating the equation

$$\frac{dy}{dx} + Py = 0, \quad \text{that is, } \frac{dy}{y} = -Pdx,$$

which gives

$$y = Ce^{-\int Pdx}, \quad \text{or } y e^{\int Pdx} = C.$$

Next, differentiate  $y e^{\int Pdx} = C$ , obtaining

$$e^{\int Pdx} (dy + Py dx) = 0;$$

which shows that  $e^{\int Pdx}$  is an integrating factor of the given equation.

Finally, multiply the given equation by this integrating factor, obtaining

$$e^{\int P dx} (dy + Py dx) = e^{\int P dx} Q dx.$$

which integrated gives

$$ye^{\int P dx} = \int e^{\int P dx} Q dx + C,$$

or  $y = e^{-\int P dx} \left[ \int e^{\int P dx} Q dx + C \right].$

### 1208. A differential equation of the general form

$$\frac{dy}{dx} + Py = Qy^n,$$

wherein  $P$  and  $Q$  are functions of  $x$  or constants, can be reduced to the linear form by dividing through by  $y^n$ , multiplying through by  $(-n+1)$ , and, in the result

$$(-n+1)y^{-n}\frac{dy}{dx} + (-n+1)Py^{-n+1} = (-n+1)Q,$$

putting  $z$  for  $y^{-n+1}$ . The result is thus reduced to

$$\frac{dz}{dx} + (1-n)Pz = (1-n)Q,$$

which is linear as to  $z$ .

### *Differential Equations of the Second Order.*

1209. Given  $\frac{d^2y}{dx^2} = f(x).$

Solutions

$$(1) \quad y = \int dx \int f(x) dx + C_1x + C_2.$$

$$(2) \quad y = x \int f(x) dx - \int xf(x) dx + C_1x + C_2.$$

**1210.** Given

$$\frac{d^2y}{dx^2} = f(y).$$

Solution

$$x = \int \frac{dy}{\sqrt{C_1 + 2 \int f(y) dy}} + C_2$$

**1211.** Given

$$\frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}\right).$$

By putting  $\frac{dy}{dx} = z$ ,  $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ , there are derived two equations,

$$x = \int \frac{dz}{f(z)} + C_1, \quad y = \int \frac{z dz}{f(z)} + C_2,$$

from which, by the elimination of  $z$ , the required solution results.

**1212.** Given

$$\frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}, x\right).$$

By putting  $\frac{dy}{dx} = z$ ,  $\frac{d^2y}{dx^2} = \frac{dz}{dx}$ , in the given equation, the

result is a differential equation of the first order in  $z$ , namely  $\frac{dz}{dx} = f(z, x)$ , which may be integrable.

If the result of integrating it be  $z = \varphi(x)$ , the solution of the given equation is then obtained by integrating  $dy = \varphi(x) dx$ .

**1213.** Given  $\frac{d^2y}{dx^2} = f\left(\frac{dy}{dx}, y\right).$

By putting  $\frac{dy}{dx} = z$ ,  $\frac{d^2y}{dx^2} = z \frac{dz}{dy}$ , in the given equation,

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(iii) If some, say  $p$ , of the roots are equal,

$$r_1 = r_2 = r_3 = \dots = r_p,$$

the general solution then becomes

$$\begin{aligned} 1216. \quad y = e^{r_1 x} & (C_1 + C_2 x + C_3 x^2 + \dots + C_p x^{p-1}) \\ & + C_{p+1} e^{r_{p+1} x} + \dots + C_n e^{r_n x}. \end{aligned}$$

1217. Given

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \dots + a_{n-1} \frac{dy}{dx} + a_n y = f(x).$$

The roots  $r_1, r_2, r_3, \dots, r_n$  are first determined as in the last problem.

The general solution is

$$y = z_1 e^{r_1 x} + z_2 e^{r_2 x} + z_3 e^{r_3 x} + \dots + z_n e^{r_n x},$$

wherein  $z_1, z_2, z_3, \dots, z_n$  are the results of solving the  $n$  equations

$$e^{r_1 x} \frac{dz_1}{dx} + e^{r_2 x} \frac{dz_2}{dx} + e^{r_3 x} \frac{dz_3}{dx} + \dots + e^{r_n x} \frac{dz_n}{dx} = 0.$$

$$r_1 e^{r_1 x} \frac{dz_1}{dx} + r_2 e^{r_2 x} \frac{dz_2}{dx} + r_3 e^{r_3 x} \frac{dz_3}{dx} + \dots + r_n e^{r_n x} \frac{dz_n}{dx} = 0.$$

$$r_1^2 e^{r_1 x} \frac{dz_1}{dx} + r_2^2 e^{r_2 x} \frac{dz_2}{dx} + r_3^2 e^{r_3 x} \frac{dz_3}{dx} + \dots + r_n^2 e^{r_n x} \frac{dz_n}{dx} = 0$$

$$\begin{matrix} \cdot & \cdot \\ \cdot & \cdot \end{matrix}$$

$$r_1^{n-1} e^{r_1 x} \frac{dz_1}{dx} + r_2^{n-1} e^{r_2 x} \frac{dz_2}{dx} + \dots + r_n^{n-1} e^{r_n x} \frac{dz_n}{dx} = f(x).$$

If the given equation had a constant  $A$  instead of  $f(x)$  for its right-hand member, its solution would be

$$1218. \quad y = C_1 e^{r_1 x} + C_2 e^{r_2 x} + \dots + C_n e^{r_n x} + \frac{A}{x}.$$

# SECTION V.

## ANALYTIC GEOMETRY.

*The Point and the Straight Line in a Plane.*

N.B. Rectangular coördinates are assumed unless otherwise stated.

The coördinates of a point  $(x, y)$  dividing the distance between two points  $(x_1, y_1)$  and  $(x_2, y_2)$  in a given ratio  $l:m$  are

$$1301. \quad x = \frac{lx_2 + mx_1}{l + m}, \quad y = \frac{ly_2 + my_1}{l + m}.$$

The point of bisection is given by

$$1302. \quad x = \frac{x_2 + x_1}{2}, \quad y = \frac{y_2 + y_1}{2}.$$

The point of external division is given by

$$1303. \quad x = \frac{lx_2 - mx_1}{l - m}, \quad y = \frac{ly_2 - my_1}{l - m}.$$

The distance between two points  $(x_1, y_1)$  and  $(x_2, y_2)$  is

$$1304. \quad d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}.$$

1305. If the axes be oblique making an angle  $\omega$  the distance is given by the equation

$$d^2 = (x_1 - x_2)^2 + (y_1 - y_2)^2 + 2(x_1 - x_2)(y_1 - y_2) \cos \omega.$$

If the angle which the line  $d$  makes with the axis of  $x$  is  $\alpha$ , then

$$1306. \quad \cos \alpha = \frac{x_1 - x_2}{d}, \quad \sin \alpha = \frac{y_1 - y_2}{d}.$$

The distance of any point  $(x, y)$  from the origin is

$$1307. \quad d = \sqrt{x^2 + y^2}.$$

If the axes be oblique, making an angle  $\omega$ ,

$$1308. \quad d = \sqrt{x^2 + y^2 + 2xy \cos \omega}.$$

The general equation of the straight line is

$$1309. \quad Ax + By + C = 0.$$

If  $A = 0$ , the straight line is parallel to the axis of  $x$ .

If  $B = 0$ , the straight line is parallel to the axis of  $y$ .

If  $C = 0$ , the straight line passes through the origin.

If  $a$  and  $b$  be the intercepts which the straight line makes on the axes of  $x$  and  $y$ , the equation may be written

$$1310. \quad \frac{x}{a} + \frac{y}{b} = 1. \quad \left[ a = \frac{-C}{A}, \quad b = \frac{-C}{B} \right].$$

If the straight line makes an angle  $\beta$  with the positive direction of the axis of  $x$  and makes an intercept  $b$  on the axis of  $y$ , its equation is

$$1311. \quad y = mx + b,$$

wherein  $m = \tan \beta$ . The coefficient  $m$  is often called the *slope* of the line.

$$\left[ m = \frac{-A}{B} = -\frac{b}{a} = \tan \beta. \right]$$

If the straight line passes through a given point  $(x_1, y_1)$  and has a given inclination to the axis of  $x$ , its equation is

$$1312. \quad y - y_1 = m(x - x_1).$$

If the straight line passes through two given points  $(x_1, y_1)$  and  $(x_2, y_2)$  its equation is

$$1313. \quad \frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

$$\left[ m = \tan \beta = \frac{y_2 - y_1}{x_2 - x_1}. \right]$$

If a perpendicular from the origin upon a straight line be  $p$  in length and make an angle  $\alpha$  with the axis of  $x$  the equation of the straight line takes the *normal form*,

$$1314. \quad x \cos \alpha + y \sin \alpha - p = 0.$$

An equation of the form  $Ax + By + C = 0$  is reduced to the normal form by putting

$$1315. \quad \frac{A}{\pm\sqrt{A^2+B^2}} = \cos \alpha, \quad \frac{B}{\pm\sqrt{A^2+B^2}} = \sin \alpha,$$

$$\frac{-C}{\pm\sqrt{A^2+B^2}} = p.$$

N.B. The sign of the radical is to be so chosen as to give  $p$  a positive value.

The length of the perpendicular drawn from any point  $(x_1, y_1)$  to the straight line  $x \cos \alpha + y \sin \alpha - p = 0$  is

$$1316. \quad l = \pm (x_1 \cos \alpha + y_1 \sin \alpha - p),$$

wherein the positive sign is to be taken when the point  $(x_1, y_1)$  and the origin are on opposite sides of the line, and the negative sign when they are on the same side of the line.

If the equation of the given straight line be of the form  $Ax + By + C = 0$ , the length of the perpendicular is given in the form

$$1317. \quad l = \frac{Ax_1 + By_1 + C}{\sqrt{A^2 + B^2}}.$$

All points for which the numerator of this fraction has the same sign as  $C$  are on the same side of the given line as the origin.

The equation

$$1318. \quad (Ax + By + C) + k(A'x + B'y + C') = 0$$

wherein  $k$  is arbitrary, represents all straight lines drawn through the intersection of the two straight lines

$$Ax + By + C = 0 \text{ and } A'x + B'y + C' = 0$$

The bisectors of the supplementary angles between

$$Ax + By + C = 0 \text{ and } A'x + B'y + C' = 0$$

are represented by the equation

$$1319. \quad \frac{Ax + By + C}{\sqrt{A^2 + B^2}} \mp \frac{A'x + B'y + C'}{\sqrt{A'^2 + B'^2}} = 0.$$

Or, if the equations of the given straight lines are in the normal form 1314, the equations of the bisectors are

$$(x \cos \alpha + y \sin \alpha - p) \pm (x \cos \alpha' + y \sin \alpha' - p') = 0.$$

The condition that three straight lines pass through one and the same point is that the determinant formed by eliminating  $x$  and  $y$  from their three equations should vanish, that is

$$1320. \quad \begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix} = 0.$$

1321. The condition that three points  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$  lie on one straight line is

$$y_1(x_2 - x_3) + y_2(x_3 - x_1) + y_3(x_1 - x_2) = 0,$$

that is, that the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0.$$

The angle  $\theta$  which two straight lines

$$Ax + By + C = 0 \text{ and } A'x + B'y + C' = 0$$

make with each other is given by either of the equations

$$1322. \quad \cos \theta = \frac{AA' + BB'}{\pm \sqrt{(A^2 + B^2)} \sqrt{(A'^2 + B'^2)}}.$$

$$1323. \quad \sin \theta = \frac{A'B - AB'}{\pm \sqrt{A^2 + B^2} \sqrt{A'^2 + B'^2}}.$$

$$1324. \quad \tan \theta = \frac{A'B - AB'}{AA' + BB'}.$$

The lines are perpendicular to each other ( $\theta = 90^\circ$ ) when

$$1325. \quad AA' + BB' = 0;$$

they are parallel ( $\theta = 0^\circ$ ) when

$$1326. \quad \frac{A}{A'} = \frac{B}{B'}.$$

If the equations of the straight lines are given in the explicit form

$$y = mx + b \text{ and } y = m'x + b'$$

the angle which they make with each other is given by the equation

1327.  $\tan \theta = \frac{m - m'}{1 + m'm}.$

1328. The lines are parallel when  $m' = m$ .

1329. They are perpendicular when  $m'm = -1$ .

The equation of a straight line which cuts a given straight line  $y = mx + b$  in the point  $(x_1, y_1)$  and makes with it an angle  $\theta$  is

1330.  $y - y_1 = \frac{m + \tan \theta}{1 - m \tan \theta} (x - x_1).$

The equation of a straight line passing through the point  $(x_1, y_1)$  and perpendicular to the given line  $y = mx + b$  is

1331.  $y - y_1 = -\frac{1}{m} (x - x_1).$

Or, if the given straight line is represented by the equation  $Ax + By + C = 0$ , the equation of the perpendicular through  $(x_1, y_1)$  is

1332.  $A(y - y_1) = B(x - x_1).$

Two straight lines through the point  $(x_1, y_1)$  and making an angle  $\theta$  with the line  $y = mx + b$

1333.  $\frac{y - y_1}{x - x_1} = \frac{m - \tan \theta}{1 + m \tan \theta}, \quad \frac{y - y_1}{x - x_1} = \frac{m + \tan \theta}{1 - m \tan \theta}.$

1334. Double the area of the triangle the vertices of which are the three points  $(x_1, y_1), (x_2, y_2), (x_3, y_3)$  is the determinant

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}.$$

The condition that these three points lie in one straight line is that this determinant vanish. Hence the equation of a straight line passing through two given points may be written

$$1335. \quad \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

A system of polar coördinates of a point in a plane consists of

$r$  = the distance of the point from the origin.

$\varphi$  = the angle which  $r$  makes with the positive direction of the axis of  $x$ .

These polar coördinates are related to rectangular coördinates as follows:

$$1336. \quad r = \sqrt{x^2 + y^2}, \quad x = r \cos \varphi, \quad y = r \sin \varphi, \quad \varphi = \tan^{-1} \frac{y}{x}.$$

The distance,  $d$ , between two points  $(r_1, \varphi_1)$  and  $(r_2, \varphi_2)$  is given by the equation

$$1337. \quad d^2 = r_1^2 + r_2^2 - 2r_1r_2 \cos(\varphi_1 - \varphi_2).$$

The polar equation of a straight line is

$$1338. \quad r \cos(\varphi - \alpha) = p,$$

wherein  $p$  is the perpendicular from the origin upon the straight line and  $\alpha$  is the angle which  $p$  makes with the fixed axis.

The polar equation of a straight line passing through two points  $(r_1, \varphi_1)$  and  $(r_2, \varphi_2)$  is

$$1339.$$

$$rr_1 \sin(\varphi - \varphi_1) + r_1r_2 \sin(\varphi_1 - \varphi_2) + r_2r \sin(\varphi_2 - \varphi) = 0.$$

### Transformation of Coördinates.

If  $x$  and  $y$  be the coördinates of any point, and  $x'$  and  $y'$  the coordinates of the same point with reference to a new set of axes, transformation of an equation to the new axes

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**1344.** From oblique axes to rectangular, when the axis of  $y'$  is parallel to that of  $y$ , that is  $\frac{y'}{y} = 0$ ,  $\frac{x'}{y} = 90^\circ$  or  $270^\circ$ .

$$(x - x_0) \sin \frac{x}{y} = x',$$

$$(y - y_0) \sin \frac{y}{x} = x' \sin \frac{x'}{x} + y' \cos \frac{x'}{x}.$$

**1345.** From rectangular axes to oblique.

$$x - x_0 = x' \cos \frac{x'}{x} + y' \cos \frac{y'}{x},$$

$$y - y_0 = x' \sin \frac{x'}{x} + y' \sin \frac{y'}{x}.$$

**1346.** From rectangular axes to oblique, when the axis of  $x'$  is parallel to that of  $x$ , that is  $\frac{x'}{x} = 0$ .

$$x - x_0 = x' + y' \cos \frac{y'}{x},$$

$$y - y_0 = y' \sin \frac{y'}{x}.$$

**1347.** Rectangular axes to oblique, when the axis of  $y'$  is parallel to that of  $y$ , that is  $\frac{y'}{x} = 90^\circ$ .

$$x - x_0 = x' \cos \frac{x'}{x},$$

$$y - y_0 = x' \sin \frac{x'}{x} + y'.$$

**1348.** Rectangular axes to rectangular.

$$x - x_0 = x' \cos \frac{x'}{x} - y' \sin \frac{x'}{x},$$

$$y - y_0 = x' \sin \frac{x'}{x} + y' \cos \frac{x'}{x}.$$

By putting the angle  $\frac{x'}{x} = \frac{y'}{y} = \theta$ , these equations may be written in the form more commonly used,

**1349.**  $x - x_0 = x' \cos \theta - y' \sin \theta,$   
 $y - y_0 = x' \sin \theta + y' \cos \theta.$

The general equation of the second degree in two variables is  
**1350.**  $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$

The Discriminant is

$$\text{1351. } \Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2.$$

**1352.** The derivative relative to  $c$  of the discriminant is

$$C = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2.$$

**1353.** If  $C$  be *not* 0, the curve represented is a *central* curve, either an ellipse (including the circle) or a hyperbola, according as  $C$  is positive or negative.

**1354.** If  $C = 0$ , the curve represented is either a parabola or a pair of parallel straight lines.

**1355.** If  $\Delta = 0$ , the equation represents a pair of straight lines.

### *Transformation of the general equation.*

If  $C$  be *not* 0, and therefore the curve be *central*, the general equation can be transformed to parallel axes through the center.

The coördinates  $x_0, y_0$  of the center are found by solving the equations

$$\begin{aligned} ax_0 + hy_0 + g &= 0, \\ hx_0 + by_0 + f &= 0. \end{aligned}$$

These equations express the condition that the new  $f$  and  $g$  in the transformed equation should each be equal to 0.

**1356.** The coördinates of the center are

$$x_0 = \frac{hf - bg}{ab - h^2}, \quad y_0 = \frac{hg - af}{ab - h^2}.$$

The general equation is thus freed of terms of the first degree and takes the form

$$\text{1357. } ax^2 + 2hxy + by^2 + c' = 0,$$

wherein  $c'$  (which is the result of substituting the coördinates of the center for  $x$  and  $y$  in the general equation) =  $\frac{\Delta}{\Delta}$ .

Further, the term in  $xy$  can be made to disappear by turning the axes on the center so as to make the new  $h$  vanish. This can be done by virtue of the relations that exist between the old and the new coefficients when an equation is transformed from one set of rectangular axes to another.

These relations in the present case are

$$\begin{aligned} 1358. \quad a + b &= a' + b' \\ ab - h^2 &= a'b' - h'^2 \end{aligned}$$

By making  $h' = 0$ , the new coefficients,  $a'$ ,  $b'$ , become known; for their sum and their product are known. They are the two roots of a quadratic equation, which may be called *the discriminating quadratic*:

$$1359. \quad k^2 - (a + b)k + ab - h^2 = 0.$$

The two values of  $k$  found by solving this equation are the  $a'$  and  $b'$  sought.

The angle  $\theta$  through which the axes are turned to make the term in  $xy$  disappear is given by the equation

$$1360. \quad \tan 2\theta = \frac{2h}{a - b}.$$

The angle  $\theta$  is the inclination of the principal axis of the conic to the axis of  $x$ .

The general equation now becomes

$$1361. \quad a'x^2 + b'y^2 + c' = 0.$$

This will assume different forms according to the signs of the roots of the discriminating quadratic.

(i) Let both roots be positive. The curve makes real intercepts on each axis, is a closed curve, and if  $a$  and  $b$  be these intercepts, the equation may be written in the form

$$1362. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad \text{An Ellipse.}$$

If  $a = b$  the ellipse is a Circle.

(ii) Let one root be negative. Then

$$1363. \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \quad \text{A Hyperbola.}$$

(iii) Let both roots be negative. Then

$$1364. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1, \quad \text{No real locus.}$$

(iv) When  $c' = \frac{\Delta}{C}$  vanishes in consequence of  $\Delta = 0$ ,

$$1365. \quad \frac{x^2}{a^2} : \frac{y^2}{b^2} = 0. \quad \begin{array}{l} \text{An infinitely small ellipse at the} \\ \text{origin. Or, a pair of imaginary} \\ \text{straight lines intersecting at the} \\ \text{origin.} \end{array}$$

$$1366. \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0. \quad \begin{array}{l} \text{A pair of real straight lines in-} \\ \text{tersecting at the origin, being the} \\ \text{asymptotes of the corresponding} \\ \text{hyperbola.} \end{array}$$

If  $C = 0$ , that is if  $ab - h^2 = 0$ , the first three terms of the general equation form a perfect square. The coordinates of the center (see 1356) become infinite, so that transformation to parallel axes through the center is impossible.

By turning the axes without changing the origin through the angle  $\theta$ , given by  $\tan 2\theta = \frac{2h}{a - b}$ , the term in  $xy$  can be made to disappear; and, since one root of the discriminating quadratic either  $a'$  or  $b'$  now becomes 0, the general equation also loses the term in  $x^2$  or in  $y^2$ , and so takes the form

$$1367. \quad b'y^2 + 2g'x + 2f'y + c' = 0.$$

This can be transformed to parallel axes through a new origin so as to make  $c'$  and  $f'$  disappear, so that finally the general equation becomes

$$1368. \quad y^2 = px, \quad \text{A Parabola.}$$

If, in the general equation,  $ab - h^2 = 0$ , and also either  $hf - bg = 0$  or  $hg - af = 0$ , the equation takes the form of a quadratic in  $y$  or in  $x$ ,

$$1369. \quad b'y^2 + 2f'y + c' = 0,$$

which can be separated into factors of the form

$$1370. \quad b'(y - y_1)(y - y_2) = 0$$

wherein  $y_1, y_2$  are the roots of the quadratic. The equation under these conditions represents a pair of parallel straight lines, real, coincident, or imaginary.

*SPECIAL FORMULAS.**The Circle.*

The general equation of the circle is

$$1371. \quad x^2 + y^2 + Ax + By + C = 0.$$

If the coördinates of the center be  $a, b$  and the radius  $r$  the equation is

$$1372. \quad (x - a)^2 + (y - b)^2 = r^2.$$

If the center be at the origin, the equation, called the *central equation*, is

$$1373. \quad x^2 + y^2 = r^2.$$

If the circumference of the circle pass through the origin, and the center be in the positive part of the axis of  $x$ , the equation is

$$1374. \quad x^2 + y^2 = 2rx.$$

The equation of the circle referred to oblique axes is

$$1375. \quad (x - a)^2 + (y - b)^2 + 2(x - a)(y - b) \cos \omega = r^2.$$

The polar equation of the circle in general form is

$$1376. \quad \rho^2 + (A \cos \varphi + B \sin \varphi) \rho + C = 0.$$

The polar equation of a circle, when the fixed axis passes through the center and  $d$  is the distance of the center from the pole, is

$$1377. \quad \rho^2 - 2d\rho \cos \varphi + d^2 - r^2 = 0.$$

If the fixed axis does not pass through the center, but makes an angle  $\alpha$  with the straight line that does pass through the center, the polar equation of the circle takes the form

$$1378. \quad \rho^2 - 2d\rho \cos(\varphi - \alpha) + d^2 - r^2 = 0.$$

If the pole is on the circle, the equation becomes

$$1379. \quad \rho = 2r \cos \varphi.$$

## Two straight lines

$$y - y' = m_1 (x - x') \quad \text{and} \quad y - y' = m_2 (x - x')$$

passing through  $(x', y')$  are tangents to the circle  $x^2 + y^2 = r^2$  if  $m_1$  and  $m_2$  are the two roots of the quadratic equation

$$1380. \quad (x'^2 - r^2)m^2 - 2x'y'm + y'^2 - r^2 = 0.$$

The equation of a circle being  $(x - a)^2 + (y - b)^2 = r^2$ , that of a tangent at  $(x', y')$  and that of the polar to  $(x', y')$  is

$$1381. \quad (x - a)(x' - a) + (y - b)(y' - b) = r^2,$$

$$\text{or} \quad (x - a) \cos \alpha + (y - b) \sin \alpha = r$$

wherein  $\alpha$  is the angle between the axis of  $x$  and the radius to or through the point  $(x', y')$ .

The equation of the chord joining two points  $(x_1, y_1)$ ,  $(x_2, y_2)$  on the circle  $x^2 + y^2 = r^2$  is

$$1382. \quad x(x_1 + x_2) + y(y_1 + y_2) = x_1x_2 + y_1y_2 + r^2,$$

$$\text{or} \quad x \cos \frac{1}{2}(\theta_1 + \theta_2) + y \sin \frac{1}{2}(\theta_1 + \theta_2) = r \cos \frac{1}{2}(\theta_1 - \theta_2),$$

$$\text{wherein} \quad r \cos \theta_1 = x_1, \quad r \sin \theta_1 = y_1, \text{ etc.}$$

The equation of a circle passing through three given points  $(x_1, y_1)$ ,  $(x_2, y_2)$ ,  $(x_3, y_3)$  is

$$1383. \quad (x^2 + y^2) \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} - (x_1^2 + y_1^2) \begin{vmatrix} x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \\ x & y & 1 \end{vmatrix}$$

$$+ (x_2^2 + y_2^2) \begin{vmatrix} x_3 & y_3 & 1 \\ x & y & 1 \\ x_1 & y_1 & 1 \end{vmatrix} - (x_3^2 + y_3^2) \begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0.$$

1384. Centers of similitude for two circles lie in the straight line passing through their centers and divide it internally and externally in the ratio of their radii. Apply 1301 and 1303.

If there be given two circles

$$\begin{aligned}x^2 + y^2 + Ax + By + C = 0 \\x^2 + y^2 + A'x + B'y + C' = 0\end{aligned}$$

the equation

$$1385. \quad (A - A')x + (B - B')y + C - C' = 0$$

represents their common chord, if they intersect, or their radical axis, whether they intersect or not.

1386. Tangents to two circles drawn from any point of their radical axis are equal.

### *Conic Sections.*

The locus of a point which moves so that its distance  $\rho$  from a fixed point bears a constant ratio  $e$  to its distance  $x$  from a fixed straight line is a conic section and its equation is

$$1387. \quad \rho = ex.$$

The fixed point is the *focus* and the fixed straight line is the *directrix* of the curve. Taking the directrix as the axis of  $y$  and a line perpendicular thereto passing through the focus as the axis of  $x$ , and denoting the distance from the focus to the directrix by  $d$ , the equation of the locus in rectangular coordinates is

$$1388. \quad \left\{ \begin{array}{l} (x - d)^2 + y^2 = e^2 x^2, \\ \text{or} \quad (1 - e^2) x^2 + y^2 - 2dx + d^2 = 0. \end{array} \right.$$

This represents an ellipse when  $e < 1$ , a parabola when  $e = 1$ , and a hyperbola when  $e > 1$ .

The ratio  $e$  is called the *eccentricity*. For the circle  $e = 0$ , the focus is the center, but the directrix is infinitely distant.

If the origin be moved to the focus the equation becomes

$$1389. \quad x^2 + y^2 = e^2 (x + d)^2.$$

If the origin be moved to the point where the curve cuts the axis of  $x$  between the focus and the directrix the equation takes the form

$$1390. \quad (1 - e^2) x^2 + y^2 - 2edx = 0.$$

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If two lines  $r'$  and  $r''$ , called *focal radii*, be drawn one from each focus to any one point on the ellipse,

$$1404. \quad r' = a - ex \text{ and } r'' = a + ex,$$

$$\text{whence,} \quad r' + r'' = 2a,$$

which last equation expresses the fact that the sum of the distances of any point on the ellipse to the two foci is constant, and equal to the axis major.

### *The Hyperbola.*

The general equation  $x^2 + y^2 = e^2 (x + d)^2$ , when the origin is changed from the focus to the center, becomes for the Hyperbola

$$1405. \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

wherein

$$1406. \quad a = \text{half the transverse axis} = \frac{de}{e^2 - 1},$$

$$1407. \quad b = \text{half the conjugate axis} = \frac{de}{\sqrt{e^2 - 1}}.$$

$$1408. \quad c = \text{distance from center to focus} = \frac{de^2}{e^2 - 1}.$$

$$1409. \quad b^2 = a^2 (e^2 - 1). \quad 1410. \quad e^2 = \frac{a^2 + b^2}{a^2}.$$

$$1411. \quad c = ae. \quad 1412. \quad a^2 + b^2 = c^2.$$

1413. The distance from the center to the directrix

$$= c - d = \frac{a}{e} = \frac{a^2}{c}.$$

1414. The distances from the directrix to the vertices of the hyperbola are  $\frac{d}{e+1}$  and  $\frac{d}{e-1}$ .

1415. The distances from the focus to the vertices are

$$\frac{de}{e-1} \text{ and } \frac{de}{e+1}.$$

1416. The latus rectum,  $p = 2de = 2a (e^2 - 1) = \frac{2b^2}{a}$ .

**1417.** The conjugate axis is a mean proportional between the transverse axis and the latus rectum.

The focal radii of any point on the hyperbola are

$$\text{1418. } r' = ex - a \text{ and } r'' = ex + a,$$

whence

$$r'' - r' = 2a,$$

which expresses the fact that the difference of the distances of any point on the hyperbola from the foci is constant and equal to the transverse axis.

The equation of the asymptotes to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ is}$$

$$\text{1419. } \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0,$$

$$\text{or } \frac{x}{a} + \frac{y}{b} = 0 \text{ and } \frac{x}{a} - \frac{y}{b} = 0.$$

The equation of the hyperbola referred to its asymptotes as axes is

$$\text{1420. } xy = \frac{1}{4} (a^2 + b^2).$$

The equation of the tangent at  $(x'y')$  or of the polar of the point  $(x'y')$  wherever situated is

$$\text{1421. } yx' + xy' = \frac{1}{2} (a^2 + b^2).$$

The equation of the equilateral hyperbola, when referred to the principal axes, is

$$\text{1422. } x^2 - y^2 = a^2,$$

and when referred to the asymptotes, it is

$$\text{1423. } xy = \frac{1}{2} a^2.$$

The eccentricity of the equilateral hyperbola =  $\sqrt{2}$ .

**1424.** The equation  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$  represents a hyperbola

which is conjugate to the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ . It has

foci in the axis of  $y$ ; eccentricity  $e = \frac{\sqrt{a^2 + b^2}}{b}$ ; directrices

parallel to the axis of  $x$  and at the distance  $\pm \frac{b}{e}$  therefrom;

and the same asymptotes as the hyperbola to which it is conjugate.

**1425.** Two conics  $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$  and  $\frac{x^2}{a'^2} \pm \frac{y^2}{b'^2} = 1$

are confocal if

$$a^2 - a'^2 = b^2 - b'^2;$$

or if, one being an ellipse and the other a hyperbola,

$$a^2 - a'^2 = b^2 + b'^2.$$

### *The Parabola.*

The general equation  $x^2 + y^2 = e^2 (x + d)^2$ , when the origin is changed from the focus to the vertex of the curve, becomes for the Parabola, since  $e = 1$ ,

**1426.**  $y^2 = 2dx,$

which is also written

**1427.**  $y^2 = 4ax$  and  $y^2 = px.$

**1428.** The distance from the directrix to the vertex of the parabola, which is equal to the distance from the vertex to the focus }  $= a = \frac{1}{2}d = \frac{1}{4}p.$

**1429.** The latus rectum or parameter  $= p = 2d = 4a.$

The equation of a parabola referred to a tangent and the diameter through the point of contact is

**1430.**  $y^2 = 4a'x,$

wherein  $a'$  is the distance from the focus to the point of contact.

The equation of each conic section referred to the transverse axis ( $x$ ) and the tangent at the vertex ( $y$ ).

$$\left\{ \begin{array}{ll} y^2 = px - \frac{b^2}{a^2}x^2, & \text{Ellipse,} \\ y^2 = px, & \text{Parabola,} \\ y^2 = px + \frac{b^2}{a^2}x^2, & \text{Hyperbola.} \end{array} \right.$$

The polar equation of the Ellipse or of the Hyperbola, the pole being at the center and the axis on the positive part of the transverse axis, is

$$1432. \frac{1}{\rho^2} = \frac{\cos^2 \varphi}{a^2} \pm \frac{\sin^2 \varphi}{b^2}, \text{ or } \rho^2 = \pm \frac{b^2}{1 - e^2 \cos^2 \varphi}.$$

The polar equation of any conic section, the pole being at the focus and the axis on the transverse axis, is

$$1433. \rho = \frac{de}{1 \pm e \cos \varphi},$$

wherein the upper sign is to be used when the axis points towards the *nearest vertex* of the curve, and the lower sign when the axis points the other way.

For the Parabola, the polar equation above given may be changed to the form

$$1434. \rho \cos^2 \frac{1}{2}\varphi = a, \text{ or } \rho \sin^2 \frac{1}{2}\varphi = a.$$

The equation

$$1435. xx' + yy' = r^2, \text{ for the Circle,}$$

$$1436. \frac{xx'}{a^2} + \frac{yy'}{b^2} = 1, \text{ for the Ellipse,}$$

$$1437. \frac{xx'}{a^2} - \frac{yy'}{b^2} = 1, \text{ for the Hyperbola,}$$

$$1438. yy' = 2a(x + x'), \text{ for the Parabola,}$$

represents a straight line which is

(i) a tangent to the curve at  $(x', y')$ , when  $(x', y')$  is on the curve.

(ii) the polar of the point  $(x', y')$  relatively to the curve, wherever  $(x', y')$  may be.

(iii) the chord passing through the points of contact of a pair of tangents drawn to the curve from  $(x', y')$ , when  $(x', y')$  is outside the curve.

(iv) the locus of the intersection of pairs of tangents drawn at the ends of all chords passing through  $(x', y')$ , wherever  $(x', y')$  may be.

The equation of the tangent to a conic in terms of its slope  $m$  (= the trigonometric tangent of the angle  $\tau$  which the tangent line makes with the axis of  $x$ ) is

$$1439. \quad y = mx \pm r \sqrt{m^2 + 1} \text{ for the Circle,}$$

$$1440. \quad y = mx \pm \sqrt{a^2 m^2 + b^2} \text{ for the Ellipse,}$$

$$1441. \quad y = mx \pm \sqrt{a^2 m^2 - b^2} \text{ for the Hyperbola,}$$

$$1442. \quad y = mx + \frac{a}{m} \text{ for the Parabola.}$$

The condition that the ellipse or hyperbola  $\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$  be touched by the straight line  $y = mx + g$  is that

$$1443. \quad g^2 = a^2 m^2 \pm b^2.$$

The condition that the straight line  $Ax + By + C = 0$  touch the ellipse  $b^2 x^2 + a^2 y^2 - a^2 b^2 = 0$  is that

$$1444. \quad A^2 a^2 + B^2 b^2 = C^2.$$

The equation of the tangent to an ellipse may be written in the normal form,

$$1445. \quad x \cos \alpha + y \sin \alpha = p = \sqrt{a^2 \cos^2 \alpha + b^2 \sin^2 \alpha} \\ = a \sqrt{1 - e^2 \sin^2 \alpha}$$

wherein  $\alpha$  denotes the inclination of  $p$  to the axis of  $x$ .

Two straight lines,

$$y - y' = m_1 (x - x') \text{ and } y - y' = m_2 (x - x'),$$

passing through the point  $(x', y')$  are tangents to the ellipse or hyperbola

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1,$$

if  $m_1$  and  $m_2$  are the two roots of the quadratic equation

$$1446. \quad (x'^2 - a^2) m^2 - 2x'y'm + y'^2 \pm b^2 = 0.$$

Two straight lines,

$$y = m_1 x + \frac{a}{m_1} \text{ and } y = m_2 x + \frac{a}{m_2},$$

passing through the point  $(x', y')$  are tangents to the parabola  $y^2 = 4ax$ ,

if  $m_1$  and  $m_2$  are the two roots of the quadratic equation

$$1447. \quad x'm^2 - y'm + a = 0.$$

The normal line is perpendicular to the tangent and passes through the point of contact  $(x', y')$ . Its equation is

$$1448. \quad y = \frac{y'}{x'} x \text{ for the Circle,}$$

$$1449. \quad y - y' = -\frac{y'}{2a}(x - x') \text{ for the Parabola,}$$

$$1450. \quad y - y' = \frac{a^2 y'}{b^2 x'} (x - x') \text{ for the Ellipse,}$$

$$1451. \quad y - y' = -\frac{a^2 y'}{b^2 x'} (x - x') \text{ for the Hyperbola.}$$

The equation of the normal to a conic in terms of its slope  $n$  (= the trigonometric tangent of the angle  $\nu$  which the normal line makes with the axis of  $x$ ) is

$$1452. \quad y = nx - \frac{n(\sqrt{a^2 \pm b^2 n^2})}{\sqrt{a^2 \pm b^2 n^2}} \text{ for the Ellipse and Hyperbola,}$$

wherein  $n = \tan \nu = \frac{a^2 y'}{\pm b^2 x'}$ .

$$1453. \quad y = nx - an(2 + n^2) \text{ for the Parabola,}$$

wherein  $n = \tan \nu = -\frac{y'}{2a}$ .

$$1454. \quad \text{The equation } \frac{y - y'}{x - x'} = \frac{a^2 y'}{\pm b^2 x'} \text{ represents a straight}$$

line passing through the pole  $(x', y')$  perpendicular to the polar. It becomes the normal when  $(x', y')$  is on the curve the polar then becoming a tangent.

*Diameters.*

The locus of the middle points of a system of chords all parallel to the line  $y = mx$  is a diameter, the equation of which is

$$1455. \quad y = -\frac{1}{m}x \text{ for the Circle,}$$

$$1456. \quad y = \frac{2a}{m} \text{ for the Parabola,}$$

$$1457. \quad y = -\frac{b^2}{a^2m}x \text{ for the Ellipse,}$$

$$1458. \quad y = \frac{b^2}{a^2m}x \text{ for the Hyperbola.}$$

The equation of the diameter which bisects the system of chords inclined at the angle  $\theta$  to the axis of  $x$  is

$$1459. \quad \frac{x \cos \theta}{a^2} + \frac{y \sin \theta}{b^2} = 0 \text{ for the Ellipse,}$$

$$1460. \quad \frac{x \cos \theta}{a^2} - \frac{y \sin \theta}{b^2} = 0 \text{ for the Hyperbola.}$$

1461. In the ellipse or hyperbola the relation

$$m_1 m_2 = \frac{\mp b^2}{a^2}$$

expresses these facts:

- (i) that  $y = m_2 x$  bisects all the chords represented by  $y = m_1 x + c$ , wherein  $c$  is an arbitrary constant;
- (ii) that  $y = m_1 x$  and  $y = m_2 x$  are conjugate diameters, each bisecting all chords drawn parallel to the other;
- (iii) that  $y - y' = m_1 (x - x')$  and  $y + y' = m_2 (x + x')$ , being supplemental chords drawn from the point  $(x', y')$  are parallel respectively to the conjugate diameters  $y = m_1 x$  and  $y = m_2 x$ ;
- (iv) that the tangent  $y - y' = m_2 (x - x')$  at  $(x', y')$ , the end of a diameter  $y = m_1 x$ , is parallel to the conjugate of that diameter,  $y = m_2 x$ , and to the system of chords  $y = m_2 x + c$ .

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and the equations of these curves referred to a pair of conjugate diameters as oblique coordinate axes are

$$1472. \quad \frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1.$$

The equations

$$1473. \quad x = a \cos u \text{ and } y = b \sin u$$

represent an ellipse; and  $u$  is the *eccentric angle* for any point  $(x, y)$  on the ellipse. In Astronomy,  $u$  is called the *eccentric anomaly*.

The equations

$$1474. \quad x = a \sec u \text{ and } y = b \tan u$$

represent a hyperbola, and  $u$  is the eccentric angle for any point  $(x, y)$  on the hyperbola.

The equation of a tangent to the ellipse at a point for which the eccentric angle is  $u$  is

$$1475. \quad \frac{x}{a} \cos u + \frac{y}{b} \sin u = 1.$$

The equation of a tangent to the hyperbola at a point for which the eccentric angle is  $u$  is

$$1476. \quad \frac{x}{a} \sec u - \frac{y}{b} \tan u = 1.$$

1477. Two diameters are conjugate if  $u_1$  and  $u_2$ , the eccentric angles of their ends, satisfy the condition

$$u_1 \mp u_2 = \frac{\pi}{2}.$$

## GENERAL PROPERTIES OF PLANE CURVES.

*Tangents, Normals, Curvature, Evolutes, Involutes, Areas, Length of Arc, Envelopes, Pedal Curves, Trajectories.*

### *Tangents and Normals.*

If the tangent to a curve,  $y = f(x)$ , make an angle  $\tau$  with the axis of  $x$ , then

$$1501. \quad \tan \tau = \frac{dy}{dx} = f'(x) = y'.$$

Also,

$$1502. \cos \tau = \frac{1}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{1}{\sqrt{1 + y'^2}},$$

$$1503. \sin \tau = \frac{\frac{dy}{dx}}{\sqrt{1 + \left(\frac{dy}{dx}\right)^2}} = \frac{y'}{\sqrt{1 + y'^2}},$$

Or,

$$1504. \cos \tau = \frac{dx}{\sqrt{dx^2 + dy^2}},$$

$$1505. \sin \tau = \frac{dy}{\sqrt{dx^2 + dy^2}},$$

Or, since the differential of the arc of the curve

$$1506. ds = \sqrt{dx^2 + dy^2},$$

$$1507. \cos \tau = \frac{dx}{ds}, \quad \sin \tau = \frac{dy}{ds}.$$

If the equation of a plane curve be given in the explicit form  $y = f(x)$ , the equation of the tangent at any point  $(x, y)$  is

$$1508. Y - y = \frac{dy}{dx} (X - x), \text{ or } Y - y = f'(x) \cdot (X - x),$$

wherein  $X$  and  $Y$  denote the running coördinates of the tangent, and  $x$  and  $y$  those of the curve.

The equation of the normal at  $(x, y)$  is

$$1509. Y - y = - \frac{dx}{dy} (X - x)$$

wherein  $X$  and  $Y$  denote the running coördinates of the normal, and  $x$  and  $y$  those of the curve.

If the equation of the curve be given in the implicit form  $F(x, y) = 0$ , the equation of the tangent is

$$1510. D_x F \cdot (X - x) + D_y F \cdot (Y - y) = 0,$$

and the equation of the normal is

$$1511. D_y F \cdot (X - x) - D_x F \cdot (Y - y) = 0.$$

*Asymptotes.*

The equation of the tangent. 1508, may be written in the form

$$1512. \quad Y = Xf'(x) + f(x) - xf'(x).$$

If, when  $x$  increases to  $\infty$ ,  $f'(x)$  approaches a definite limit  $A$  and  $f(x) - xf'(x)$  also approaches a definite limit  $B$ , then the tangent becomes an *Asymptote*, of which the equation is

$$1513. \quad Y = AX + B.$$

*Applications to Particular Curves.*

The equation of the tangent to the circle  $x^2 + y^2 = r^2$  at the point  $(x, y)$  is

$$1514. \quad Xx + Yy = r^2.$$

The equation of the normal at the same point is

$$1515. \quad Xy - Yx = 0.$$

The equation of a pair of tangents from the point  $(x_1, y_1)$  to the circle  $x^2 + y^2 = r^2$  is

$$1516. \quad (x_1^2 + y_1^2 - r^2)(X^2 + Y^2 - r^2) = (Xx_1 + Yy_1 - r^2)^2.$$

The equation of the tangent to the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

at the point  $(xy)$  is

$$1517. \quad \frac{Xx}{a^2} + \frac{Yy}{b^2} - 1.$$

The equation of the normal at the same point is

$$1518. \quad \frac{X - x}{b^2x} - \frac{Y - y}{a^2y} = 0, \text{ or } \frac{a^2X}{x} - \frac{b^2Y}{y} = c^2.$$

The equation of the tangent to the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \text{ at the point } (x, y) \text{ is}$$

$$1519. \quad \frac{Xx}{a^2} - \frac{Yy}{b^2} = 1.$$

The equation of the normal at the same point is

$$1520. \quad \frac{X-x}{b^2x} + \frac{Y-y}{a^2y} = 0, \text{ or } \frac{a^2X}{x} + \frac{b^2Y}{y} = c^2.$$

The equation of the tangent to the parabola  $y^2 = 2dx$  at the point  $(x, y)$  is

$$1521. \quad Yy = d(X+x), \text{ or } \frac{Y}{\frac{1}{2}y} - \frac{X}{x} = 1.$$

The equation of the normal at the same point is

$$1522. \quad Y - y = -\frac{y}{d}(X - x).$$

The general equation of the tangent to the conic

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

at the point  $(x, y)$  is

1523.

$$aXx + h(Xy + Yx) + bYy + g(X + x) + f(Y + y) + c = 0.$$

*The tangent* is the name given to that part of the tangent line which extends from T, where it cuts the axis of  $x$ , to P, where it touches the curve = TP.

$$1525. \quad TP = y \frac{dx}{dy} \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = y \csc \tau = y \frac{ds}{dy}$$

$$= \frac{y}{y'} \sqrt{1 + y'^2},$$

wherein  $y'$  is written for  $f'(x)$ , derived from the equation

$$y = f(x).$$

*The subtangent* is the projection of the tangent on the axis of  $x$ .

$$1526. \quad TP' = y \frac{dx}{dy} = y \operatorname{ctn} \tau = \frac{y}{y'}.$$

The normal is the name given to that part of the normal line which extends from N, where it cuts the axis of  $x$ , to P, where it cuts the curve = NP.

$$1527. \quad NP = y \sqrt{1 - \left( \frac{dy}{dx} \right)^2} = y \sec \tau = y \frac{ds}{dx} = y \sqrt{1 + y^2}.$$

The subnormal is the projection of the normal on the axis of  $x$ .

$$1528. \quad NP' = y \frac{dx}{dy} = y \tan \tau = yy'.$$

Circle  $x^2 + y^2 = r^2$ ,

$$1529. \quad \text{Tangent} = \frac{xy}{x}. \quad 1530. \quad \text{Normal} = r.$$

$$1531. \quad \text{Subtangent} = \frac{-r^2}{x} + x. \quad 1532. \quad \text{Subnormal} = x.$$

Ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ .

$$1533. \quad \text{Tangent} = \frac{ay}{bx} \sqrt{a^2 - c^2 x^2},$$

$$1534. \quad \text{Normal} = \frac{b}{a} \sqrt{a^2 - c^2 x^2}.$$

$$1535. \quad \text{Subtangent} = \frac{-a^2}{x} - x.$$

$$1536. \quad \text{Subnormal} = \frac{-b^2 x}{a^2}.$$

Hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ ,

$$1537. \quad \text{Tangent} = \frac{ay}{bx} \sqrt{c^2 x^2 - a^2},$$

$$1538. \quad \text{Normal} = \frac{b}{a} \sqrt{c^2 x^2 - a^2},$$

$$1539. \quad \text{Subtangent} = \frac{a^2}{x} - x, \quad 1540. \quad \text{Subnormal} = \frac{b^2 x}{a^2}.$$

Parabola  $y^2 = 4ax$ ,

$$1541. \quad \text{Subtangent} = 2x,$$

$$1542. \quad \text{Subnormal} = 2a \text{ (constant)}.$$

*Polar Tangents and Normals.*

If  $r$  denote the distance from the origin to  $P$ , the point where the tangent touches the curve, and  $\varphi$  denote the angle which  $r$  makes with the axis of  $x$ ; and if through the origin a straight line be drawn perpendicular to  $r$  and meeting the tangent and normal lines in  $T_0$  and  $N_0$  respectively; then

$$1543. \text{ Polar Tangent} = PT_0 = r \sqrt{1 + r^2 \left( \frac{d\varphi}{dr} \right)^2} = \frac{rds}{dr}.$$

$$1544. \text{ Polar Subtangent} = OT_0 = \frac{r^2 d\varphi}{dr} = \frac{r^2}{r'}.$$

$$1545. \text{ Polar Normal} = PN_0 = \sqrt{r^2 + \left( \frac{dr}{d\varphi} \right)^2} = \frac{ds}{d\varphi}$$

$$= \sqrt{r^2 + r'^2}.$$

$$1546. \text{ Polar Subnormal} = ON_0 = \frac{dr}{d\varphi} = r'.$$

$$1547. ds^2 = dr^2 + (rd\varphi)^2, \text{ or } ds = d\varphi \sqrt{r^2 + r'^2}.$$

In these and following formulas,  $r'$ ,  $r''$  are written for  $f'(\varphi)$ ,  $f''(\varphi)$ , or for  $\frac{dr}{d\varphi}$ ,  $\frac{d^2r}{d\varphi^2}$ , derived from the equation  $r = f(\varphi)$ .

The angle between the tangent and the radius vector is  $\tau - \varphi$ , and

$$1548. \quad \operatorname{ctn}(\tau - \varphi) = \frac{r'}{r}.$$

*Curvature.*

The radius of curvature, or radius of the osculating circle,

$$1549. \rho = \frac{ds}{d\tau} = \frac{\left( \frac{ds}{dx} \right)^3}{\frac{d^2y}{dx^2}} = \frac{\left[ 1 + \left( \frac{dy}{dx} \right)^2 \right]^{\frac{3}{2}}}{\frac{d^2y}{dx^2}} = \frac{(1 + y'^2)^{\frac{3}{2}}}{y''},$$

wherein  $y''$  is written for  $f''(x)$ .

Also

**I550.**  $\rho = \frac{n^3}{y^3 y''}$ , wherein  $n$  = the normal.

The radius of curvature in polar coördinates is

**I551.**  $\rho = \frac{(r^2 + r'^2)^{\frac{3}{2}}}{r^2 + 2r'^2 - rr''}.$

The coördinates  $X_0$  and  $Y_0$  of the centre of curvature are

**I552.**  $X_0 = x - \frac{1 + y'^2}{y''} y', \quad Y_0 = y + \frac{1 + y'^2}{y''}.$

### *The Evolute.*

When  $y$ ,  $y'$ , and  $y''$  are all explicit functions of  $x$ , the elimination of  $x$  from the two equations I552 gives an equation,

**I553.**  $F(X, Y) = 0,$

which represents the locus of all the centres of curvature of the given curve. This locus is the *evolute* of the given curve.

The Evolute of the parabola  $y^2 = 2dx$  is

**I554.**  $Y^2 = \frac{8}{27} \left( \frac{X-d}{d} \right)^3$ , a semi-cubic Parabola.

The Evolute of the ellipse  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  is

**I555.**  $\left( \frac{aX}{a^2 - b^2} \right)^{\frac{2}{3}} + \left( \frac{bY}{a^2 - b^2} \right)^{\frac{2}{3}} = 1.$

The Evolute of the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$  is

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In the hyperbola  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ , the area included between the vertex, the axis of  $x$ , the ordinate  $y$ , and the intercepted arc of the curve, is

$$\begin{aligned} 1562. \quad A &= \int_a^{a+x} \frac{b}{a} \sqrt{x^2 - a^2} \cdot dx = \frac{xy}{2} - \frac{ab}{2} \log_e \left( \frac{x}{a} + \frac{y}{b} \right), \\ &= \frac{xy}{2} - \frac{ab}{2} \operatorname{Cosh}^{-1} \frac{x}{a}. \end{aligned}$$

In the equilateral hyperbola  $x^2 - y^2 = 1$ , the last term of the right-hand member becomes  $\frac{1}{2} \operatorname{Cosh}^{-1} x$ , that is half the variable area  $u$ , of which the hyperbolic sine, cosine, tangent, etc., are defined as functions. See pp. 85-87.

### *Lengths of Arcs.*

The length of the arc of a curve between two points corresponding to the two abscissas  $x_0$  and  $x$ ,

$$1563. \quad s = \int_{x_0}^x \sqrt{dx^2 + dy^2} = \int_{x_0}^x \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \cdot dx.$$

Or, in polar coördinates,

$$1564. \quad s = \int_{\phi_0}^{\phi} \sqrt{r^2 + \left(\frac{dr}{d\varphi}\right)^2} \cdot d\varphi = \int_{r_0}^r \sqrt{1 + r^2 \left(\frac{d\varphi}{dr}\right)^2} \cdot dr.$$

### *Envelopes.*

An envelope of a curve is the locus of the ultimate intersections of all curves of the series resulting from a continuous variation of the parameter of the curve.

1565. The equation of the envelope is obtained by eliminating the parameter  $a$  between the equation of the curve

$$F(x, y, a) = 0 \quad \text{and} \quad D_a F(x, y, a) = 0.$$

The envelope of circles described on the central radii of the ellipse  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1$  is

$$1566. \quad (x^2 + y^2)^2 = a^2 x^2 + b^2 y^2.$$

*Pedal Curves.*

A *pedal curve* is the locus of the foot of a perpendicular drawn from the pole to a moving straight line which always touches a given curve.

The pedal curve to a conic is

$$1567. \quad (x^2 + y^2 + dx)^2 = a^2 x^2 \pm b^2 y^2,$$

wherein  $a$  and  $b$  are the semi-axes of the conic, and  $d$  is the distance from the centre of the conic to the pole. The axes are rectangular with the origin at the pole, and the axis of  $x$  coincides with the transverse axis of the conic.

*Special Cases.* (i) Let the pole be at the centre ( $d = 0$ ) and the conic an equilateral hyperbola ( $a^2 = b^2$ ). Then the equation becomes

$$1568. \quad (x^2 + y^2)^2 = a^2 (x^2 - y^2), \text{ a Lemniscate,}$$

the polar equation of which is

$$r = a \sqrt{\cos 2\varphi}.$$

(ii) Let the pole be at a vertex of the conic ( $d = -a$ ) and the conic a circle ( $a^2 = b^2$ ).

Then the equation becomes

$$1569. \quad (x^2 + y^2 - ax)^2 = a^2 (x^2 + y^2), \text{ a Cardioid,}$$

the polar equation of which is

$$r = a (1 + \cos \varphi).$$

(iii) Let the pole be at the distance  $2a$  from the centre ( $d = -2a$ ) and the conic a circle ( $a^2 = b^2$ ). Then the equation becomes

$$1570. \quad (x^2 + y^2 - 2ax)^2 = a^2 (x^2 + y^2), \text{ a Limaçon,}$$

the polar equation of which is

$$r = a (1 + 2 \cos \varphi).$$

*Trajectories.*

A *trajectory* is a curve which cuts according to a given law each curve of a series obtained by varying the parameter of  $F(x, y, p) = 0$ .

**1571.** The differential equation of the trajectory which cuts at a constant angle  $\beta$  the series of curves  $F(x, y, p)$  is obtained by eliminating  $p$  from the equations

$$F(X, Y, p) = 0 \text{ and } \tan \beta = \frac{D_X F + D_Y F \cdot D_X Y}{D_Y F - D_X F \cdot D_X Y}.$$

wherein  $X$  and  $Y$  are the running coördinates of the trajectory sought.

If the trajectory is to cut at right angles,  $\tan \beta = \infty$ , and the second equation becomes

$$D_Y F - D_X F \cdot D_X Y = 0,$$

whence

**1572.**  $\frac{dY}{dX} = \frac{D_Y F}{D_X F}$ , the differential equation of the orthogonal trajectory. The integration of this equation gives a series of orthogonal trajectories.

### *The Cycloid.*

A cycloid is the curve generated by a point in the circumference of a circle which rolls without sliding on a fixed straight line. Taking the fixed straight line, called the base of the cycloid, as the axis of  $x$ , and as origin one of the points where the generating point falls on the axis of  $x$ , the equations of the cycloid are

**1573.**  $x = a\omega - a \sin \omega$  and  $y = a - a \cos \omega$ ,  
which by the elimination of  $\omega$  give

$$\text{1574. } x = a \cos^{-1} \frac{a - y}{a} \pm \sqrt{(2a - y)y},$$

wherein  $a$  denotes the radius of the rolling circle, and  $\omega$  the measure of its turning around its centre.

Then

$$\text{1575. } dx = a(1 - \cos \omega) d\omega, \quad dy = a \sin \omega d\omega$$

$$\tan \tau = \frac{dy}{dx} = \frac{\sin \omega}{1 - \cos \omega} = \cot \frac{1}{2}\omega,$$

whence

$$\text{1576. } \tau = 90^\circ - \frac{1}{2}\omega.$$

Hence, if from the ends of the vertical diameter of the rolling circle in any position straight lines be drawn through the generating point, these lines will be tangent and normal to the cycloid.

$$1577. \text{ Normal} = 2a \sin \frac{1}{2}\omega = \sqrt{2ay}.$$

$$\text{Radius of curvature, } \rho = 4a \sin \frac{1}{2}\omega = 2\sqrt{2ay}.$$

The evolute of a cycloid is composed of two halves of an equal cycloid so placed below the original cycloid that their summits are at the two ends of its base.

The area between the curve and its base from the origin to the ordinate  $y$  is

$$1578. \quad A = a^2 (\frac{3}{2}\omega - 2 \sin \omega + \frac{1}{2} \sin 2\omega), \\ = \frac{3}{2}ax - \frac{1}{2}y \sqrt{(2a - y)y}.$$

The area between the whole curve and its base =  $3\pi a^2$ .

The arc of the cycloid from the origin to the top of the ordinate  $y$ ,

$$1579. \quad s = 4a (1 - \cos \frac{1}{2}\omega) = 4a - 2\sqrt{2a(2a - y)}.$$

The whole arc =  $8a$ .

A point rigidly connected with the rolling circle at a distance  $p$  from its centre ( $p$  being greater or less than the radius  $a$ ) generates a curve of which the equations are

$$1580. \quad x = aw - p \sin \omega \text{ and } y = a - p \cos \omega.$$

If  $p > a$ , the curve is a *prolate cycloid*.

If  $p < a$ , the curve is a *curtate cycloid*.

### *Epicycloid and Hypocycloid.*

A point on the circumference of a circle which rolls, without sliding, on the circumference of a fixed circle generates an epicycloid or a hypocycloid according as the rolling circle is outside or inside the fixed circle.

Let  $a$  be the radius of the fixed circle,  $b$  that of the rolling circle; let  $A$  be the point on the fixed circle where the generating point  $P$ , at the start, touches the fixed circle; and let  $B$  be the point where the two circles touch at any subse-

quent time. The two arcs  $AB$  and  $PB$  are by definition of the curve equal; and, if  $\alpha$  and  $\beta$  be the angles which they subtend each in its own circle,

$$1581. \quad a\alpha = AB = PB = b\beta,$$

whence

$$\beta = \frac{a}{b} \alpha.$$

The equations, with the foregoing notation, and with the origin at the centre of the fixed circle, are for the

Epicycloid

$$1582. \quad \begin{cases} x = (a + b) \cos \alpha - b \cos \left( \frac{a + b}{b} \alpha \right), \\ y = (a + b) \sin \alpha - b \sin \left( \frac{a + b}{b} \alpha \right). \end{cases}$$

Hypocycloid

$$1583. \quad \begin{cases} x = (a - b) \cos \alpha + b \cos \left( \frac{a - b}{b} \alpha \right), \\ y = (a - b) \sin \alpha - b \sin \left( \frac{a - b}{b} \alpha \right). \end{cases}$$

In the Epicycloid,

$$1584. \quad \tan \tau = \frac{dy}{dx} = \tan \left( \frac{a + 2b}{2b} \alpha \right), \quad \tau = \frac{a + 2b}{2b} \alpha.$$

In the Hypocycloid,

$$1585. \quad \tan \tau = \frac{dy}{dx} = -\tan \left( \frac{a - 2b}{2b} \alpha \right), \quad \tau = 180^\circ - \frac{a - 2b}{2b} \alpha.$$

In the Epicycloid,

$$1586. \quad \rho = \frac{4b(a + b)}{a + 2b} \sin \left( \frac{a}{2b} \alpha \right).$$

In the Hypocycloid,

$$1587. \quad \rho = \frac{4b(a - b)}{a - 2b} \sin \left( \frac{a}{2b} \alpha \right).$$

The length of one loop is

$$1588. \quad \text{for the Epicycloid, } \frac{8(a + b)b}{a};$$

$$1589. \quad \text{for the Hypocycloid, } \frac{8(a - b)b}{a}.$$

The area between one loop and the subtending arc of the fixed circle is

$$1590. \text{ for the Epicycloid, } \frac{\pi b^2}{a} (3a + 2b),$$

$$1591. \text{ for the Hypocycloid, } \frac{\pi b^2}{a} (3a - 2b).$$

1593. *Special Forms.* When  $a$  and  $b$  bear a commensurable ratio to each other,  $\alpha$  may be eliminated from the equations of the epicycloid and hypocycloid and the result will be an algebraic equation in  $x$  and  $y$ .

1594. (i) When  $b = \infty$ ,  $a$  remaining finite, the rolling circle becomes a straight line, any point of which generates the Involute of a Circle. See 1613.

1595. (ii) When  $b = \frac{1}{2}a$ , the hypocycloid becomes a straight line; and any point rigidly connected with the rolling circle, but not on its circumference, generates an ellipse.

1596. (iii) When  $b = \frac{1}{4}a$ , the hypocycloid takes the form of a four-pointed star, its equation being  $x^4 + y^4 = a^4$ .

1597. (iv) When  $b = a$ , the epicycloid becomes the Cardioid, the equation of which is, if the origin be moved to the point where the cardioid touches the fixed circle,

$$(y^2 + x^2 - 2ax)^2 = 4a^2 (y^2 + x^2)$$

or, in polar coördinates,

$$r = 2a (1 + \cos \varphi).$$

### *The Epitrochoid and the Hypotrochoid.*

Any point rigidly connected with but not on the circumference of the rolling circle, and at a distance  $p$  from its centre, generates a curve, of which the equations are

$$1598. \quad \begin{cases} x = (a \pm b) \cos \alpha \mp p \cos \left( \frac{a \pm b}{b} \alpha \right), \\ y = (a \pm b) \sin \alpha - p \sin \left( \frac{a \pm b}{b} \alpha \right). \end{cases}$$

**1599.** When the rolling circle is outside the fixed circle the curve is an Epitrochoid, for which the above equations are to be read with the *upper* signs.

**1600.** When the rolling circle is inside the fixed circle the curve is a Hypotrochoid, for which the above equations are to be read with the *lower* signs.

**1601.** When  $a = 2b$  the hypotrochoid becomes the ellipse

$$\frac{x^2}{(b+p)^2} + \frac{y^2}{(b-p)^2} = 1.$$

*The Catenary.* The curve of the hanging chain.

Equations,

$$\begin{cases} y = \frac{h}{2} \left( e^{\frac{x}{h}} + e^{-\frac{x}{h}} \right) = h \operatorname{Cos}^{-1} \frac{x}{h}. \\ x = h \log_e \left( \frac{y}{h} \pm \sqrt{\frac{y^2}{h^2} - 1} \right) = h \operatorname{Cosh}^{-1} \frac{y}{h}. \end{cases}$$

The axis of  $x$  is horizontal, the axis of  $y$  vertical, passing through the lowest point of the curve, and the origin at the distance  $h$  below the lowest point of the curve. The slope of the tangent is

$$\text{1603. } \tan \tau = \frac{dy}{dx} = \frac{1}{2} \left( e^{\frac{x}{h}} - e^{-\frac{x}{h}} \right) = \operatorname{Sinh} \frac{x}{h} = \sqrt{\frac{y^2}{h^2} - 1}.$$

$$\cos \tau = \frac{h}{y}.$$

By making  $\tau$  the independent variable, the following equations of the catenary are obtained:

**1604.**

$$\begin{cases} x = h \log_e \frac{1 + \sin \tau}{\cos \tau} = h \log_e \tan \left( \frac{\pi}{4} + \frac{\tau}{2} \right) = h \operatorname{Sinh}^{-1} (\tan \tau). \\ y = h \sec \tau. \end{cases}$$

$$\text{1605. The radius of curvature, } \rho = \frac{y^2}{h} = h \sec^2 \tau.$$

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Let  $A$  be the point where, at the start, the generating point is on the circumference of the fixed circle,  $T$  the point where the moving line in any other position touches the fixed circle, and  $P$  the generating point.

Then is the tangent  $PT$  always equal to the arc  $AT$ .

The radius of curvature,  $\rho = PT$ . The measure of the arc  $AT$  is  $r_0\theta$ , wherein  $r_0$  denotes the radius of the fixed circle.

The rectangular coördinates of  $P$ , the origin being at the centre of the fixed circle, are

$$1614. \quad \begin{cases} x = r_0 (\cos \theta + \theta \sin \theta), \\ y = r_0 (\sin \theta - \theta \cos \theta). \end{cases}$$

The polar equation is

$$1615. \quad \varphi = \sqrt{\frac{r^2}{r_0^2} - 1} - \tan^{-1} \sqrt{\frac{r^2}{r_0^2} - 1}.$$

$$1616. \quad \text{The length of the curve } AP, s = \frac{\rho^3}{2r_0} = \frac{1}{2}r_0\theta^2.$$

$$1617. \quad \text{The area } AOP = \frac{1}{8}r_0^2\theta^3.$$

1618. Since  $\rho = r_0\theta$ , the curve becomes a straight line when  $\theta$  has increased to infinity.

### *Parabolic Curves.*

The family of Parabolic Curves has for its general equation

$$1619. \quad y^n = ax^n.$$

Parabolic Spirals,

$$1620. \quad r^m = a\varphi^n.$$

The Parabolic Spiral, a particular case,

$$1621. \quad r^2 = a\varphi.$$

The Parabola of the  $n^{\text{th}}$  degree,

$$1622. \quad a^{n-1}y = x^n.$$

$$1623. \quad \text{The Cubic Parabola, } a^2y = x^3.$$

$$1624. \quad \text{The Semi-cubic Parabola, } a^{\frac{1}{2}}y = x^{\frac{3}{2}}.$$

**1625.** The Parabola referred to tangents at ends of the latus rectum,

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = a^{\frac{1}{2}}.$$

*The Spiral of Archimedes* is generated by a point moving at a uniform rate in a straight line while the straight line itself is turning at a uniform rate around a fixed point or pole.

**1626.** The polar equation is  $r = a\varphi = \frac{r_0}{2\pi} \varphi$ , wherein  $r_0$  is the value of  $r$  corresponding to  $\varphi = 2\pi$ .

**1627.** The polar subtangent  $= \frac{r^2}{a}$ .

**1628.** The polar subnormal  $= a$ , constant.

**1629.** The radius of curvature,  $\rho = \frac{(a^2 + r^2)^{\frac{3}{2}}}{2a^2 + r^2}$ .

**1630.** Length of the curve,  $s = \frac{r_0}{4\pi} (\varphi \sqrt{1 + \varphi^2} + \text{Sinh}^{-1} \varphi)$ .

The family of *Hyperbolic Curves* has the general equation

**1631.**  $y^m x^n = a$ .

Hyperbolic Spirals,

**1632.**  $r^m \varphi^n = a$ .

**1633.** The Lituus, a particular case,  $r^2 \varphi = a$ .

*The Hyperbolic Spiral.* Draw a diameter through a series of concentric circles, and on each circle beginning at this diameter measure off an arc of fixed length  $a$ , all in the same direction; the other ends of these arcs will be points in the hyperbolic spiral. Its equation is

**1634.**  $r\varphi = a$ .

This curve is also called the Reciprocal Spiral, since

$$r = \frac{a}{\varphi}.$$

**1635.** A straight line drawn parallel to the diameter used in the construction and at a distance  $a$  therefrom is the asymptote of the spiral.

**1636.** The origin is an asymptotic point around which the spiral must make an infinite number of turns to reach it.

**1637.** The polar subtangent =  $-a$ , constant.

**1638.** The polar subnormal =  $-\frac{r^2}{a}$ .

Compare 1627, 1628.

**1639.** The radius of curvature,  $\rho = \frac{r}{\cos^3 \theta}$ , wherein  $\theta$  is the angle between  $r$  and the tangent.

### *The Logarithmic Curve.*

$$\text{1640. } y = ae^{\frac{x}{n}} \text{ or } x = n \log_e \frac{y}{a},$$

wherein  $n$  = the constant subtangent, and  $a$  the intercept on the axis of  $y$ .

The most general form of the equation of this curve is

$$\text{1641. } y = ae^{\frac{x}{n}} \left( \cos \frac{2r\pi}{n} + i \sin \frac{2r\pi}{n} \right).$$

*The Logarithmic Spiral* is the locus of the equation

$$\text{1642. } r = ae^{m\phi}.$$

It is also called the Equiangular Spiral, since the tangent and radius vector form a constant angle.

The origin is an asymptotic point around which, for negative values of  $\varphi$ , the spiral must make an infinite number of turns to reach it.

The tangent and radius vector form a constant angle  $\alpha$  with each other, and  $\operatorname{ctn} \alpha = m$ . The value of  $a$  is found by putting  $\varphi = 0$ .

**1643.** The polar normal =  $r \sqrt{1 + m^2} = r \csc \alpha$ .

**1644.** The polar subnormal =  $r \operatorname{ctn} \alpha = rm$ .

**1645.** The radius of curvature =  $r \sqrt{1 + m^2} = r \csc \alpha$ .

The evolute is another spiral which will coincide with the original spiral by turning it through an angle

$$\frac{\pi}{2} - \frac{1}{m} \log_e m.$$

The area passed over by  $r$  from  $\varphi = -\infty$  (the pole) to any positive value of  $\varphi$ ,

1646.  $A = \frac{r^3}{4m}.$

1647. The length of the arc from the pole to any point,

$$s = r \sec \alpha.$$

*The Lemniscate.*

1648.  $(x^2 + y^2)^2 = a^2 (x^2 - y^2), r = a \sqrt{\cos 2\varphi}.$

This curve is a particular case of the Cassinian Oval, see 1662.

*The Cissoid.*

1649.  $y^2 (2a - x) = x^3, r = \frac{2a \sin^2 \varphi}{\cos \varphi}.$

The area between the curve and its asymptote =  $3\pi a^2$ .

*Descartes' Folium.*

1650.  $x^3 + y^3 = 3axy, r = \frac{3a \sin \varphi \cos \varphi}{\sin^3 \varphi + \cos^3 \varphi}.$

The line  $x + y + a = 0$  is an asymptote to this curve.

*Quadrifolium.*

1651.  $(x^2 + y^2)^3 = 4a^2 x^2 y^2, r = a \sin 2\varphi.$

*The Witch of Agnesi, or The Versiera.*

1652.  $y = \frac{8a^3}{x^2 + 4a^2}.$

*The Conchoid.*

If a radiant from a fixed point  $O$  cut a fixed straight line, the directrix, in  $R$  and a constant length  $RP = b$  be measured from  $R$  either way along the radiant, the locus of  $P$  is a *conchoid*.

Let  $a$  be the length of the perpendicular  $OB$  from  $O$  to the directrix. Then with  $B$  for origin of rectangular coordinates, the equation is

1653.  $x^2 y^2 = (a + y)^2 (b^2 - y^2),$

or, with  $O$  for pole,

$$r = a \sec \varphi \pm b.$$

*The Limaçon.* By changing the directrix of the conchoid into a fixed circle on  $OB = a$  as diameter, the curve becomes a limaçon. With  $OB$  for initial line and axis of  $x$ , the equations of this curve are

1654.  $r = a \cos \varphi \pm b$ ,  $(x^2 + y^2 - ax)^2 = b^2(x^2 + y^2)$ ,  
wherein  $a = OB$ ,  $b = RP$ .

When  $b > a$ ,  $O$  is a conjugate point.

When  $b < a$ ,  $O$  is a node.

When  $a = 2b$ , this curve has been called the Trisectrix.

*The Folium,*

1655.  $y^2 = \frac{x^2(3a - x)}{3(a + x)}$ .

*The Logocyclic Curve,*

1656.  $y^2 = \frac{x(x - r)^2}{2a - x}$ ,  $r = a(\sec \varphi \pm \tan \varphi)$ .

*The Cubic Trisectrix,*

1657.  $y^2 = \frac{x^2(3a - x)}{a - x}$ ,  $r = 2a \frac{\sin 3\varphi}{\sin 2\varphi}$ ,  
or  $r = 4a \cos \varphi - a \sec \varphi$ .

*The Quadratrix* is the locus of the intersection of the radius of a circle and an ordinate both of which move uniformly so that  $x:a = \varphi:\frac{1}{2}\pi$ . The equation is

1658.  $y = x \tan\left(\frac{a - x}{a} \cdot \frac{\pi}{2}\right)$ .

*Cartesian Ovals.* The sum or difference of certain fixed multiples of the distances of a point  $P$  on the curve from two fixed points  $A$  and  $B$ , called the foci, is constant.

Let  $r_1$  and  $r_2$  be the focal radii and  $c$  the distance between the foci.

The equations of the inner and outer ovals respectively are

1659.  $mr_1 + lr_2 = nc$ , 1660.  $mr_1 - lr_2 = nc$ ,

wherein

$$n > m > l.$$

*The Ovals of Cassini* are defined by the property that the product of the focal radii is constant

$$1662. \quad r_1 r_2 = b^2 \text{ or } [y^2 + (c+x)^2] \times [y^2 + (c-x)^2] = b^4.$$

The polar equation is

$$1663. \quad r^4 - 2c^2r^2 \cos 2\varphi = b^4 - c^4,$$

the pole being half way between the foci, whose distance apart =  $2c$ .

The equation

$$1664. \quad r = a (\sec \varphi + n \cos \varphi)$$

represents a family of curves to which belong the Cissoid and the Cubic Trisectrix.

#### *Miscellaneous Polar Equations.*

$$1665. \quad r = a \sin \varphi \text{ (a circle).} \quad 1666. \quad r = a \sin \frac{1}{2}\varphi.$$

$$1667. \quad r = a \sin 2\varphi. \quad 1668. \quad r \cos \varphi = a \cos 2\varphi.$$

$$1669. \quad r = a \sin 3\varphi. \quad 1670. \quad r = a \cos \varphi + b.$$

1671.  $r = a \sin n\varphi$  ( $n$  loops when  $n$  is odd,  $2n$  loops when  $n$  is even).

$$1672. \quad r = \sin^3 \frac{1}{3}\varphi. \quad 1673. \quad r = a + b \csc \varphi.$$

$$1674. \quad r = a \tan \varphi. \quad 1675. \quad r = a \sec^2 \frac{1}{2}\varphi \text{ (a parabola).}$$

$$1676. \quad r = a (\sec 2\varphi + \tan 2\varphi).$$

#### *Miscellaneous Rectangular Equations.*

$$1677. \quad y^2 = \frac{x^3 - a^3}{x - 2a}. \quad 1678. \quad y^2 = x \log (1 + x).$$

$$1679. \quad y^2 = \frac{x^3}{x - a}. \quad 1680. \quad a^2 y^2 = a^2 x^2 - x^4.$$

$$1681. \quad y^2 = x^2 \frac{a + x}{a - x}. \quad 1682. \quad a y^2 = (x - a)^2 (x - b).$$

$$1683. \quad y^2 = \pm \frac{x^4}{a^2 - x^2}. \quad 1684. \quad a y^2 = x^3 \pm b x^2.$$

1685.  $y^2 = x(x + a)^2.$

1686.  $x^4 - a^2xy + b^2y^2 = 0.$

1687.  $y^2 = \frac{x^4}{a^2} - \frac{x^6}{a^4}.$

1688.  $a^3y^2 - 2abx^2y - x^5 = 0.$

1689.  $y^3 \pm x^3 = 2ax^2.$

1690.  $x^3 + 2x^2 + 2xy - y^2 + 5x - 2y = 0.$

1691.  $ay^3 + axy^2 - x^4 = 0.$  1692.  $(y^2 - x^2)^2 = x^5.$

1693.  $ay^3 - 2ax^2y - x^4 = 0.$  1694.  $y = \frac{.x^3}{a^2 + x^2}.$

1695.  $ax^3 + by^3 - c = 0.$  1696.  $a^3y - a^2x^2 + x^4 = 0.$

1697.  $x^3 + y^3 = a^3.$  1698.  $a^2y - 3bx^2 + x^3 = 0.$

1699.  $\left(\frac{x}{a}\right)^3 + \left(\frac{y}{b}\right)^3 = 1.$  1700.  $y = x^4 - 4x^3 - 18x^2.$

1701.  $\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^3 = 1.$  1702.  $y = \log \sec x.$

1703.  $x^2(x + y) = a^2(x - y).$

1704.  $\left(\frac{x}{a}\right)^{2n+1} + \left(\frac{y}{b}\right)^{2n+1} = 1.$  1705.  $\left(\frac{x}{a}\right)^{2n} + \left(\frac{y}{b}\right)^{2n} = 1.$

1706.  $y(y^2 - 1) = (x^2 - \frac{3}{4})(x^2 - \frac{1}{4}).$

1707.  $y^4 - 96a^2y^2 + 100a^2x^2 - x^4 = 0.$

1708.  $y^4 + x^4 = a^2(y^2 - x^2).$

1709.  $x^2y^2 - 2axy - b^2(y^2 + b^2) + 2a^2b^2 = 0.$

1710.  $x^5 - 2a^3xy + y^5 = 0.$

1711. *The Probability Curve,  $y = e^{-x^2}.$*

*The Point, the Straight Line, and the Plane in Space.*

The coördinates of a point  $(x, y, z)$  dividing the distance between two points  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$  in a given ratio  $l : m$  are

1802.  $x = \frac{lx_2 + mx_1}{l + m}, \quad y = \frac{ly_2 + my_1}{l + m}, \quad z = \frac{lz_2 + mz_1}{l + m}.$

The point of external division is given by changing  $m$  to  $-m.$

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wherein  $\alpha, \beta, \gamma$ , are the direction angles of the perpendicular to be found,  $\alpha_1, \beta_1, \gamma_1$  and  $\alpha_2, \beta_2, \gamma_2$ , the direction angles of the given lines, and  $\theta$  the angle between them. If  $\theta = 90^\circ$ ,  $\sin \theta = 1$ ; and the above equations then give the direction cosines of one of three mutually perpendicular lines in terms of the direction cosines of the other two.

### *Transformation of Coördinates.*

*Parallel Transformation.* The coördinates in the old system being  $x, y, z$ ; those in the new  $x', y', z'$ ; and the coordinates of the new origin relatively to the old axes being  $x_0, y_0, z_0$ ; the transformation is made by substituting for  $x, y, z$  their equivalents.

$$1812. \quad x = x_0 + x', \quad y = y_0 + y', \quad z = z_0 + z'.$$

*Turning rectangular axes around the origin still keeping them rectangular.* Let the angles made by the new axes of  $x, y, z$  with the old axes be  $\alpha_1, \beta_1, \gamma_1; \alpha_2, \beta_2, \gamma_2; \alpha_3, \beta_3, \gamma_3$ , respectively; and for shortness put  $\cos \alpha_1 = a_1, \cos \beta_1 = b_1$ , etc.

1813.

$$\begin{cases} x = a_1x' + a_2y' + a_3z' \\ y = b_1x' + b_2y' + b_3z' \\ z = c_1x' + c_2y' + c_3z' \end{cases}$$

1814.

$$\begin{cases} x' = a_1x + b_1y + c_1z \\ y' = a_2x + b_2y + c_2z \\ z' = a_3x + b_3y + c_3z \end{cases}$$

The following relations between these nine direction cosines hold good,

1815.

$$\begin{cases} a_1^2 + a_2^2 + a_3^2 = 1 \\ b_1^2 + b_2^2 + b_3^2 = 1 \\ c_1^2 + c_2^2 + c_3^2 = 1. \end{cases}$$

1816.

$$\begin{cases} a_1^2 + b_1^2 + c^2 = 1 \\ a_2^2 + b_2^2 + c_2^2 = 1 \\ a_3^2 + b_3^2 + c_3^2 = 1. \end{cases}$$

1817.

$$\begin{cases} a_1b_1 + a_2b_2 + a_3b_3 = 0 \\ b_1c_1 + b_2c_2 + b_3c_3 = 0 \\ c_1a_1 + c_2a_2 + c_3a_3 = 0. \end{cases}$$

1818.

$$\begin{cases} a_1a_2 + b_1b_2 + c_1c_2 = 0 \\ a_2a_3 + b_2b_3 + c_2c_3 = 0 \\ a_3a_1 + b_3b_1 + c_3c_1 = 0. \end{cases}$$

1819.

$$\begin{cases} a_1 = b_2 c_3 - b_3 c_2 \\ b_1 = c_2 a_3 - c_3 a_2 \\ c_1 = a_2 b_3 - a_3 b_2. \end{cases}$$

1820.

$$\begin{cases} a_2 = c_1 b_3 - c_3 b_1 \\ b_2 = a_1 c_3 - a_3 c_1 \\ c_2 = b_1 a_3 - b_3 a_1. \end{cases}$$

1821.

$$\begin{cases} a_3 = b_1 c_2 - b_2 c_1 \\ b_3 = c_1 a_2 - c_2 a_1 \\ c_3 = a_1 b_2 - a_2 b_1. \end{cases}$$

Another system of polar coördinates, used in Astronomy, consists of

$r$  = the radius vector,

$\theta$  = the angle which  $r$  makes with its projection on the plane of  $xy$ ,

$\varphi$  = the angle which this projection of  $r$  makes with the axis of  $x$ .

1822. These polar coördinates are related to rectangular coordinates as follows:

$$x = r \cos \theta \cos \varphi, \quad y = r \cos \theta \sin \varphi, \quad z = r \sin \theta.$$

$$r^2 = x^2 + y^2 + z^2.$$

*The general form of the equation of a plane is*

$$1823. \quad Ax + By + Cz + D = 0.$$

If  $A = 0$ , the plane is parallel to the axis of  $x$ ;

$B = 0$ , the plane is parallel to the axis of  $y$ ;

$C = 0$ , the plane is parallel to the axis of  $z$ .

If  $D = 0$ , the plane passes through the origin.

If  $A = 0$  and  $B = 0$ , the plane is parallel to the  $xy$ -plane.

$B = 0$  and  $C = 0$ , the plane is parallel to the  $yz$ -plane.

$C = 0$  and  $A = 0$ , the plane is parallel to the  $zx$ -plane.

If the plane makes intercepts  $a, b, c$  on the axes of  $x, y, z$  respectively, the equation is

$$1824. \quad \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1.$$

If a perpendicular from the origin upon a plane be  $p$  in length and make the angles  $\alpha, \beta, \gamma$  with the axes of  $x, y, z$ , the equation of the plane takes the *normal form*

$$1825. \quad x \cos \alpha + y \cos \beta + z \cos \gamma = p.$$

An equation of the form  $Ax + By + Cz + D = 0$  is reduced to the normal form by putting

$$1826.$$

$$\begin{cases} \pm \frac{A}{\sqrt{A^2 + B^2 + C^2}} = \cos \alpha, & \pm \frac{C}{\sqrt{A^2 + B^2 + C^2}} = \cos \gamma, \\ \pm \frac{B}{\sqrt{A^2 + B^2 + C^2}} = \cos \beta, & \pm \frac{-D}{\sqrt{A^2 + B^2 + C^2}} = p. \end{cases}$$

N.B. The sign of the radical is to be so chosen as to give  $p$  a positive value.

The angle  $\theta$  between two planes

$$\begin{aligned} Ax + By + Cz + D &= 0 \\ A'x + B'y + C'z + D' &= 0 \end{aligned}$$

is given by either of the equations

$$1827. \cos \theta = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2} \sqrt{A'^2 + B'^2 + C'^2}},$$

$$1828. \sin^2 \theta = \frac{(BC' - B'C)^2 + (CA' - C'A)^2 + (AB' - A'B)^2}{(A^2 + B^2 + C^2)(A'^2 + B'^2 + C'^2)}.$$

The condition that the planes should be perpendicular to each other is

$$1829. \quad AA' + BB' + CC' = 0;$$

and the condition that they should be parallel is

$$1830. \quad \frac{A}{A'} = \frac{B}{B'} = \frac{C}{C'}.$$

The length of a perpendicular from a given point  $(x_1, y_1, z_1)$  to a given plane,  $x \cos \alpha + y \cos \beta + z \cos \gamma = p$ , is

$$1831. \quad l = \pm (x_1 \cos \alpha + y_1 \cos \beta + z_1 \cos \gamma - p),$$

wherein the positive sign is to be taken when the origin and the point  $(x_1, y_1, z_1)$  are on opposite sides of the plane, and the negative sign when they are on the same side of the plane.

If the equation of the given plane is of the form

$$Ax + By + Cz + D = 0,$$

the length of the perpendicular is given in the form

$$1832. \quad l = \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}.$$

All points for which the numerator of this fraction has the same sign as  $D$  are on the same side of the plane as the origin.

The equation

1833.  $(Ax + By + Cz + D) + k(A'x + B'y + C'z + D') = 0$ , wherein  $k$  is arbitrary, represents all planes passing through the intersection of the two planes

$$\begin{aligned} Ax + By + Cz + D &= 0 \\ A'x + B'y + C'z + D' &= 0. \end{aligned}$$

1834. If  $k$  be given alternately one of the two values

$$\pm \frac{\sqrt{A^2 + B^2 + C^2}}{\sqrt{A'^2 + B'^2 + C'^2}},$$

the resulting equations will represent the two planes perpendicular to each other which bisect the supplemental angles between the two given planes.

Or, if the equations of the given planes are in the normal form 1825, the equations of the two bisecting planes are

$$1835. \quad (x \cos \alpha + y \cos \beta + z \cos \gamma - p)$$

$$\pm (x \cos \alpha' + y \cos \beta' + z \cos \gamma' - p') = 0.$$

The condition that four planes should pass through one and the same point is that the determinant formed by eliminating  $x, y, z$  from their four equations should vanish; that is

$$1836. \quad \begin{vmatrix} A & B & C & D \\ A' & B' & C' & D' \\ A'' & B'' & C'' & D'' \\ A''' & B''' & C''' & D''' \end{vmatrix} = 0.$$

1837. This is also the condition that two straight lines in space intersect, each line being represented by two of these equations.

1838. Six times the volume of the tetrahedron whose vertices are  $(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3, z_3), (x_4, y_4, z_4)$

is the determinant

$$\begin{vmatrix} x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \\ x_4 & y_4 & z_4 & 1 \end{vmatrix}.$$

The condition that four points lie all in one plane is that this determinant vanish.

Hence the equation of a plane passing through three given points may be written

$$1839. \quad \begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0.$$

A straight line in space is determined by two equations of the general form

$$1840. \quad Ax + By + Cz + D = 0$$

$$A'x + B'y + C'z + D' = 0.$$

Each of these equations represents a plane, and the straight line represented is the intersection of these two planes.

By eliminating  $y$  and  $x$  alternately between these equations, they assume the explicit form commonly used,

$$1841. \quad x = mz + a, \quad y = nz + b.$$

The first is the equation of the *projection* of the straight line on the plane of  $xz$ , and the second that of its projection on the plane of  $yz$ . Or, regarding these equations as representing planes, the first represents a plane parallel to the axis of  $y$ , the second a plane parallel to the axis of  $x$ , the straight line they represent being the intersection of these two planes.

If the straight line passes through a given point  $(x_1, y_1, z_1)$  and forms angles  $\alpha, \beta, \gamma$ , with the axes of  $x, y$ , and  $z$ , its equations are

$$1842. \quad \frac{x - x_1}{\cos \alpha} = \frac{y - y_1}{\cos \beta} = \frac{z - z_1}{\cos \gamma}.$$

If the straight line passes through two given points,  $(x_1, y_1, z_1)$  and  $(x_2, y_2, z_2)$ , its equations are

$$1843. \quad \frac{x - x_1}{x_1 - x_2} = \frac{y - y_1}{y_1 - y_2} = \frac{z - z_1}{z_1 - z_2}.$$

The equations of a straight line passing through the point  $(x_1, y_1, z_1)$  perpendicular to the plane

$$Ax + By + Cz + D = 0 \text{ are}$$

$$1844. \quad \frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C}.$$

1845. If the equations of a straight line are brought into the form

$$\frac{x - x_1}{A} = \frac{y - y_1}{B} = \frac{z - z_1}{C},$$

the direction cosines of the line are  $A, B$ , and  $C$  each divided by  $\sqrt{A^2 + B^2 + C^2}$ .

The angle  $\theta$  between two lines

$$\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n} \text{ and } \frac{x-a}{l'} = \frac{y-b}{m'} = \frac{z-c}{n'}$$

is given by the equation

$$1846. \quad \cos \theta = \frac{ll' + mm' + nn'}{\sqrt{l^2 + m^2 + n^2} \sqrt{l'^2 + m'^2 + n'^2}}.$$

The lines are perpendicular to each other if

$$1847. \quad ll' + mm' + nn' = 0.$$

The angle  $\theta$  between the plane  $Ax + By + Cz + D = 0$

and the line  $\frac{x-a}{l} = \frac{y-b}{m} = \frac{z-c}{n}$  is given by the equation

$$1848. \quad \sin \theta = \frac{Al + Bm + Cn}{\sqrt{l^2 + m^2 + n^2} \sqrt{A^2 + B^2 + C^2}}.$$

The line is parallel to the plane if

$$1849. \quad Al + Bm + Cn = 0.$$

1850. The conditions that the straight line

$$x = mz + a, \quad y = nz + b$$

lie wholly in the plane

$$Ax + By + Cz + D = 0$$

are

$$Aa + Bb + D = 0$$

$$Am + Bn + C = 0.$$

If there be two straight lines given by the equations

$$\frac{x - x_1}{a_1} = \frac{y - y_1}{b_1} = \frac{z - z_1}{c_1} \text{ and } \frac{x - x_2}{a_2} = \frac{y - y_2}{b_2} = \frac{z - z_2}{c_2}$$

wherein  $a_1, b_1, c_1, a_2, b_2, c_2$ , are the direction cosines of the two lines, the equations of planes passing through each line parallel to the other are

$$1851. \quad (x - x_1)(b_1c_2 - b_2c_1) + (y - y_1)(c_1a_2 - c_2a_1) \\ + (z - z_1)(a_1b_2 - a_2b_1) = 0$$

$$(x - x_2)(b_1c_2 - b_2c_1) + (y - y_2)(c_1a_2 - c_2a_1) \\ + (z - z_2)(a_1b_2 - a_2b_1) = 0.$$

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*Transformation of the General Equation.*

If  $D$  be not 0, and therefore the surface be *central*, the general equation can be transformed to parallel axes through the centre.

**1859.** The coördinates  $x_0, y_0, z_0$  of the centre are found by solving the equations

$$\begin{aligned} ax_0 + hy_0 + gz_0 + l &= 0, \\ hx_0 + by_0 + fz_0 + m &= 0, \\ gx_0 + fy_0 + cz_0 + n &= 0. \end{aligned}$$

These equations express the condition that the new  $l, m, n$  in the transformed equation should each be equal to 0.

Thus the equation is freed of terms of the first degree and becomes

**1860.**  $ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + d' = 0,$

wherein

$$d' = \frac{\Delta}{D}.$$

**1861.** Further, the terms in  $yz, zx$ , and  $xy$  can be made to disappear by turning the axes on the centre so as to make the new  $f, g$ , and  $h$  vanish. This can be done by virtue of the relations that exist between the old and new coefficients when an equation is transformed from one set of rectangular axes to another with the same origin.

These relations are

$$\begin{aligned} a + b + c &= a' + b' + c', \\ bc + ca + ab - f^2 - g^2 - h^2 &= b'c' + c'a' + a'b' - f'^2 - g'^2 - h'^2, \\ abc + 2fgh - af^2 - bg^2 - ch^2 &= a'b'c' + 2f'g'h' - a'f'^2 - b'g'^2 - c'h'^2. \end{aligned}$$

By making  $f', g'$ , and  $h'$  each = 0, these equations enable us to form a cubic equation whose roots are  $a', b'$ , and  $c'$  (for their sum, their product and the sum of their products taken two by two have all become known).

This cubic equation is known as the *discriminating cubic*, and it is written

$$1862. \quad k^3 - (a + b + c)k^2 + (bc + ca + ab - f^2 - g^2 - h^2)k - (abc + 2fgh - af^2 - bg^2 - ch^2) = 0.$$

The three values of  $k$ , which are all real, found by solving this cubic, are the  $a'$ ,  $b'$ , and  $c'$  sought.

The general equation now becomes

$$1863. \quad a'x^2 + b'y^2 + c'z^2 + d' = 0.$$

This will assume different forms according to the nature of the roots of the discriminating cubic.

(i) Let all three roots be positive. The surface then makes a real intercept on each axis, is a closed surface, and if  $a$ ,  $b$ , and  $c$  denote these intercepts, the equation may be written in the form

$$1864. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad \text{An Ellipsoid.}$$

If  $a = b = c$ , the surface is that of a Sphere.

If  $a = b > c$ , the surface is that of an Oblate Spheroid.

If  $a = b < c$ , the surface is that of a Prolate Spheroid.

(ii) Let one root be negative. Then

$$1865. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad \text{A Hyperboloid of one sheet.}$$

If  $a = b$ , it is a surface of revolution.

(iii) Let two of the roots be negative. Then

$$1866. \quad \frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1, \quad \text{A Hyperboloid of two sheets.}$$

If  $b = c$ , it is a surface of revolution.

(iv) Let all three roots be negative. Then

$$1867. \quad \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1, \quad \text{No real locus.}$$

(v) When  $d' = \Delta$  vanishes in consequence of  $\Delta = 0$ ,

1868.  $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$ , An infinitely small ellipsoid at the origin.

1869.  $\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0$ , A Cone, asymptotic to the corresponding hyperboloid.

If  $D = 0$ , the surface represented by the general equation is *not* central, and transformation to parallel axes through the centre is impossible; but the axes may be turned, without changing the origin, so as to make the terms in  $yz$ ,  $zx$ , and  $xy$  disappear.

In doing this, observe that the last term of the discriminating cubic  $= D = 0$ , and therefore one of the roots  $a'$ ,  $b'$ ,  $c'$  is equal to 0. Say  $c' = 0$ . Thus the equation becomes

$$1870. \quad a'x^2 \pm b'y^2 + 2l'x + 2m'y + 2n'z + d = 0.$$

Transformation to parallel axes through a new origin can make the coefficients of  $x$  and  $y$  vanish but not that of  $z$ ; and the equation becomes

$$1871. \quad a'x^2 \pm b'y^2 + 2n'z + d' = 0,$$

which represents the following surfaces.

(i) If  $n' = 0$ ,

$$1872. \quad a'x^2 + b'y^2 + d' = 0, \quad \text{Elliptic Cylinder.}$$

If  $a' = b'$ , the cylinder is circular.

$$1873. \quad a'x^2 - b'y^2 + d' = 0, \quad \text{Hyperbolic Cylinder.}$$

The axis of these cylinders is the coördinate axis of  $z$ .

If also  $d' = 0$ ,

$$1874. \quad a'x^2 - b'y^2 = 0, \quad \text{A pair of planes}$$

intersecting in the axis of  $z$  and asymptotic to the hyperbolic cylinder  $a'x^2 - b'y^2 + d' = 0$ .

(ii) If  $n'$  be not = 0, a change of origin will make  $d'$  disappear and the equation will take one or the other of the two following forms

$$1875. \quad a'x^2 + b'y^2 + 2n'z = 0, \quad \text{An Elliptic Paraboloid.}$$

If  $a' = b'$ , the paraboloid is one of revolution.

**1876.**  $a'x^2 - b'y^2 + 2n'z = 0$ , *A Hyperbolic Paraboloid.*

(iii) If also  $b' = 0$ , that is, if two roots of the discriminating cubic are 0, the equation takes the form

$$a'x^2 + 2m'y + 2n'z + d = 0,$$

which, by a change of axes in the plane  $yz$ , becomes

**1877.**  $a'x^2 + 2m'y + d = 0$ , *A Parabolic Cylinder.*

(iv) If  $c' = 0$ ,  $b' = 0$ ,  $m' = 0$ , and  $n' = 0$ , the equation becomes

**1878.**  $a'x^2 + d = 0$ , *A pair of parallel planes.*

Every homogeneous equation of the second degree in three variables,

**1879.**  $Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fzx = 0$ ,

represents a cone.

Every equation of the form

**1880.**  $x^2 + y^2 + z^2 + Ax + By + Cz + D = 0$

represents a sphere.

If  $r$  be the radius and  $a, b, c$  the coördinates of the centre of the sphere, its equation takes the form

**1881.**  $(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$ .

And if the origin be at the centre, its equation takes the form

**1882.**  $x^2 + y^2 + z^2 = r^2$ .

### *Curved Surfaces.*

The general equation of a curved surface is

$$F(x, y, z) = 0,$$

which, by solving for  $z$ , becomes

**1883.**  $z = f(x, y)$ .

Equation of the plane tangent at the point  $(x, y, z)$ ,

**1884.**  $(X - x) D_x F + (Y - y) D_y F + (Z - z) D_z F = 0$ .

Equations of the normal at the point  $(x, y, z)$ ,

$$1885. \quad \frac{X - x}{D_x F} = \frac{Y - y}{D_y F} = \frac{Z - z}{D_z F}.$$

The angles  $\alpha, \beta, \gamma$ , which the normal makes with the coördinate axes are determined by

$$1886. \quad \cos \alpha = \frac{D_x F}{R}, \cos \beta = \frac{D_y F}{R}, \cos \gamma = \frac{D_z F}{R},$$

wherein

$$R = \sqrt{(D_x F)^2 + (D_y F)^2 + (D_z F)^2}.$$

Applying the foregoing formulas to the general equation of quadrics 1853, we have

$$F(x, y, z) = ax^2 + by^2 + cz^2 + d + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz = 0,$$

which for shortness shall be written  $u = 0$ .

Then,

$$1887. \quad \begin{cases} D_x F = D_x u = 2(ax + hy + gz + l) \\ D_y F = D_y u = 2(hx + by + fz + m) \\ D_z F = D_z u = 2(gx + fy + cz + n), \end{cases}$$

and the equation of the plane tangent at the point  $(x, y, z)$  becomes

$$1888. \quad \left. \begin{array}{l} (X - x)(ax + hy + gz + l) \\ + (Y - y)(hx + by + fz + m) \\ + (Z - z)(gx + fy + cz + n) \end{array} \right\} = 0,$$

or, writing at full length and reducing,

$$1889. \quad \left. \begin{array}{l} X(ax + hy + gz + l) \\ + Y(hx + by + fz + m) \\ + Z(gx + fy + cz + n) \\ + lx + my + nz + d \end{array} \right\} = 0.$$

The equations of the normal at the point  $(x, y, z)$  become

$$1890. \quad \frac{X - x}{ax + hy + gz + l} = \frac{Y - y}{hx + by + fz + m} = \frac{Z - z}{gx + fy + cz + n},$$

and the direction cosines of the normal are

$$1891. \quad \frac{ax + hy + gz + l}{R}, \quad \frac{hx + by + fz + m}{R}, \quad \frac{gx + fy + cz + n}{R},$$

wherein  $R =$

$$\sqrt{(ax + hy + gz + l)^2 + (hx + by + fz + m)^2 + (gx + fy + cz + n)^2}.$$

The equation

$$1892. \quad \frac{xx'}{a^2} + \frac{yy'}{b^2} + \frac{zz'}{c^2} = 1,$$

represents the *polar plane* of the point  $(x', y', z')$  relatively to the ellipsoid

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1;$$

or, if the point  $(x', y', z')$  be on the surface, it represents the *tangent plane* at that point. By changing the sign of  $c^2$  or of both  $b^2$  and  $c^2$  the same statement and formulas apply to the hyperboloid of one sheet or of two.

The length of a perpendicular,  $p$ , from the origin upon the tangent plane is given by

$$1893. \quad \frac{1}{p^2} = \frac{x'^2}{a^4} + \frac{y'^2}{b^4} + \frac{z'^2}{c^4}.$$

The angles  $\alpha, \beta, \gamma$ , which this perpendicular makes with the coordinate axes are given by

$$1894. \quad \cos \alpha = \frac{px'}{a^2}, \quad \cos \beta = \frac{py'}{b^2}, \quad \cos \gamma = \frac{pz'}{c^2}.$$

Whence,

$$1895. \quad p^2 = a^2 \cos^2 \alpha + b^2 \cos^2 \beta + c^2 \cos^2 \gamma.$$

The condition that the plane

$$Ax + By + Cz + D = 0$$

should touch the surface

$$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1$$

is that

$$1896. \quad A^2a^2 + B^2b^2 + C^2c^2 = D^2.$$

The equations of the normal at the point  $(x', y', z')$  are

$$1897. \quad \frac{a^2}{x'}(x - x') = \frac{b^2}{y'}(y - y') = \frac{c^2}{z'}(z - z').$$

The equation

$$1898. \quad \frac{xx'}{a^2} \pm \frac{yy'}{b^2} \pm \frac{zz'}{c^2} = 0$$

represents the *diametral plane* which is conjugate to the diameter drawn from the point  $(x', y', z')$  on the surface, and is parallel to the plane tangent at that point.

If two diameters are conjugate to each other their direction cosines are thus related,

$$1899. \quad \frac{\cos \alpha \cos \alpha'}{a^2} + \frac{\cos \beta \cos \beta'}{b^2} + \frac{\cos \gamma \cos \gamma'}{c^2} = 0.$$

### *Curves of Double Curvature.*

A curve of double curvature is represented by two equations

$$1900. \quad F_1(x, y, z) = 0, \quad F_2(x, y, z) = 0.$$

The curve is the intersection of the two surfaces which these two equations taken separately represent.

This curve is also represented by the equations of its projections on two of the coordinate planes, thus,

$$1901. \quad y = f_1(x) \text{ and } z = f_2(x),$$

which are the equations of two cylindrical surfaces the elements of which are parallel to the axes of  $z$  and  $y$  respectively.

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Let the angle between two consecutive tangents be denoted by  $d\tau$ ; then

$$1909. \quad d\tau = \frac{R}{ds^2},$$

and the radius of curvature in the osculating plane, or the so-called *first curvature* of the curve, is

$$1910. \quad \rho_1 = \frac{ds}{d\tau} = \frac{ds^3}{R}.$$

The coördinates  $x_0, y_0, z_0$ , of the centre of the first curvature are

$$1911. \quad x_0 = x + \rho_1^2 \frac{d \frac{dx}{ds}}{ds},$$

$$y_0 = y + \rho_1^2 \frac{d \frac{dy}{ds}}{ds},$$

$$z_0 = z + \rho_1^2 \frac{d \frac{dz}{ds}}{ds}.$$

Let the angle between two consecutive osculating planes be denoted by  $d\theta$ ; then

$$1912. \quad d\theta = \sqrt{(d \cos \lambda)^2 + (d \cos \mu)^2 + (d \cos \nu)^2}$$

$$= \frac{Ad^3x + Bd^3y + Cd^3z}{R^2} ds,$$

and the radius of the so-called *second curvature*, or *torsion* of the curve, is

$$1913. \quad \rho_2 = \frac{ds}{d\theta}.$$

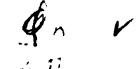
The curve is a plane curve (no torsion) when for all points  $d\theta = 0$ , that is, when

$$1914. \quad Ad^3x + Bd^3y + Cd^3z = \begin{vmatrix} dx & dy & dz \\ d^2x & d^2y & d^2z \\ d^3x & d^3y & d^3z \end{vmatrix} = 0.$$

*The Helix.*

The helix is the curve formed by the thread of a screw, and may be defined as the locus of a point which moves with uniform velocity in the circumference of a circle while the circle itself moves with uniform velocity so as to generate a right circular cylinder.

Let the radius of the circle, also of the cylinder, =  $r$ ; let the height through which the generating point rises during one passage around the circle (or cylinder) =  $h$  (= the interval between the threads of the screw); let  $\alpha$  be the angle which the tangent to the curve makes with the plane of the generating circle, that is with the  $xy$  plane; let the axis of the cylinder be the axis of  $z$ ; and let  $\varphi$  be the angle between the axis of  $x$  and the projection of the radius vector on the  $xy$  plane. Then

✓ 1915.  $\tan \alpha = \frac{h}{2\pi r}$ ,  $x = r \cos \varphi$ ,  $y = r \sin \varphi$ ,  $z = r \varphi \tan \alpha$ , = 

and the equations of the helix may be written

1916.  $x = r \cos \left( \frac{z}{r \tan \alpha} \right)$  and  $y = r \sin \left( \frac{z}{r \tan \alpha} \right)$ .

The equations of the tangent to the helix are

1917. 
$$-\frac{X - x}{\sin \varphi} = \frac{Y - y}{\cos \varphi} = \frac{Z - z}{\tan \alpha}.$$

If the surface of the cylinder be spread out on a plane the helix will become a straight line in that plane.

The radii of curvature are

1918.  $\rho_1 = r (1 + \tan^2 \alpha) = \frac{r}{\cos^2 \alpha},$

$\frac{1}{\rho_1}$  = the *First Curvature*.

1919.  $\rho_2 = r \frac{1 + \tan^2 \alpha}{\tan \alpha} = \frac{r}{\sin \alpha \cos \alpha},$

$\frac{1}{\rho_2}$  = the *Second Curvature.*

Length of an arc of the curve

1920.  $s = \frac{r}{\cos \alpha} \varphi = r \varphi \sqrt{1 + \tan^2 \alpha}.$

# TABLES



TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal
1	1	1	1.0000000	1.0000000	1.0000000	1.000000000
2	4	8	1.4142136	1.2599210	1.5874011	0.500000000
3	9	27	1.7320508	1.4422496	2.0800837	0.333333333
4	16	64	2.0000000	1.5874011	2.5198421	0.250000000
5	25	125	2.2360680	1.7099759	2.9240177	0.200000000
6	36	216	2.4494897	1.8171206	3.3019272	0.166666667
7	49	343	2.6457513	1.9129312	3.6593057	0.142857143
8	64	512	2.8284271	2.0000000	4.0000000	0.125000000
9	81	729	3.0000000	2.0800837	4.3267487	0.111111111
10	100	1000	3.1622777	2.1544347	4.6415888	0.100000000
11	121	1331	3.3166248	2.2239801	4.9460874	0.090909091
12	144	1728	3.4641016	2.2894286	5.2414828	0.083333333
13	169	2197	3.6055513	2.3513347	5.5287748	0.076923077
14	196	2744	3.7416574	2.4101422	5.8087857	0.071428571
15	225	3375	3.8729833	2.4662121	6.0822020	0.066666667
16	256	4096	4.0000000	2.5198421	6.3496042	0.062500000
17	289	4913	4.1231056	2.5712816	6.6114890	0.058823529
18	324	5832	4.2426407	2.6207414	6.8682855	0.055555556
19	361	6859	4.3588989	2.6684016	7.1203674	0.052631579
20	400	8000	4.4721360	2.7144177	7.3680630	0.050000000
21	441	9261	4.5825757	2.7589243	7.6116626	0.047619048
22	484	10648	4.6904158	2.8020393	7.8514244	0.045454545
23	529	12167	4.7958315	2.8438670	8.0875794	0.043478261
24	576	13824	4.8989795	2.8844991	8.3203353	0.041666667
25	625	15625	5.0000000	2.9240177	8.5498797	0.040000000
26	676	17576	5.0990195	2.9624960	8.7763830	0.038461538
27	729	19683	5.1961524	3.0000000	9.0000000	0.037037037
28	784	21952	5.2915026	3.0365889	9.2208726	0.035714286
29	841	24389	5.3851648	3.0723168	9.4391307	0.034482759
30	900	27000	5.4772256	3.1072325	9.6548938	0.033333333
31	961	29791	5.5677644	3.1413806	9.8682724	0.032258065
32	1024	32768	5.6568542	3.1748021	10.0793684	0.031250000
33	1089	35937	5.7445626	3.2075343	10.2882765	0.030303030
34	1156	39304	5.8309519	3.2396118	10.4950847	0.029411765
35	1225	42875	5.9160798	3.2710663	10.6998748	0.028571429
36	1296	46656	6.0000000	3.3019272	10.9027235	0.027777778
37	1369	50653	6.0827625	3.3322218	11.1037025	0.027027027
38	1444	54872	6.1644140	3.3619754	11.3028786	0.026315789
39	1521	59319	6.2449980	3.3912114	11.5003151	0.025641026
40	1600	64000	6.3245553	3.4109519	11.6960709	0.025000000
41	1681	68921	6.4031242	3.4482172	11.8902022	0.024390244
42	1764	74088	6.4807407	3.4760266	12.0827612	0.023809524
43	1849	79507	6.5574385	3.5033981	12.2737980	0.023255814
44	1936	85184	6.6332496	3.5303483	12.4633594	0.022727273
45	2025	91125	6.7082039	3.5568933	12.6514900	0.022222222
46	2116	97336	6.7823300	3.5830479	12.8382321	0.021739130
47	2209	103823	6.8556546	3.6088261	13.0236256	0.021276600
48	2304	110592	6.9282032	3.6342411	13.2077090	0.020833333
49	2401	117649	7.0000000	3.6593057	13.3905183	0.020408163
50	2500	125000	7.0710678	3.6840314	13.5720880	0.020000000

TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal
51	2601	132651	7.1414284	3.7084298	13.7524514	0.019607843
52	2704	140608	7.2111026	3.7325111	13.9316395	0.019230769
53	2809	148877	7.2801099	3.7562858	14.1096827	0.018867925
54	2916	157464	7.3484692	3.7797631	14.2866095	0.018518519
55	3025	166375	7.4161985	3.8029525	14.4624474	0.018181818
56	3136	175616	7.4833148	3.8258624	14.6372228	0.017857143
57	3249	185193	7.5498344	3.8485011	14.8109610	0.017543860
58	3364	195112	7.6157731	3.8708766	14.9836859	0.017241379
59	3481	205379	7.6811457	3.8929965	15.1554212	0.016949153
60	3600	216000	7.7459667	3.9148676	15.3261887	0.016666667
61	3721	226981	7.8102497	3.9364972	15.4960101	0.016393443
62	3844	238328	7.8740079	3.9578915	15.6649060	0.016129032
63	3969	250047	7.9372539	3.9790571	15.8328962	0.015873016
64	4096	262144	8.0000000	4.0000000	16.0000000	0.015625000
65	4225	274625	8.0622577	4.0207256	16.1662356	0.015384615
66	4356	287496	8.1240384	4.0412401	16.3316209	0.015151515
67	4489	300763	8.1853528	4.0615480	16.4961730	0.014925373
68	4624	314432	8.2462113	4.0816551	16.6599083	0.014705882
69	4761	328509	8.3066239	4.1015661	16.8228430	0.014492754
70	4900	343000	8.3666003	4.1212853	16.9849925	0.014285714
71	5041	357911	8.4261498	4.1408178	17.1463716	0.014084507
72	5184	373248	8.4852814	4.1601676	17.3069948	0.013888889
73	5329	389017	8.5440037	4.1793392	17.4668762	0.013698630
74	5476	405224	8.6023253	4.1983364	17.6260290	0.013513514
75	5625	421875	8.6602540	4.2171633	17.7844665	0.013333333
76	5776	438976	8.7177979	4.2358236	17.9422014	0.013157895
77	5929	456533	8.7749644	4.2543210	18.0992460	0.012987013
78	6084	474552	8.8317609	4.2726586	18.2556122	0.012820513
79	6241	493039	8.8881944	4.2908404	18.4113116	0.012658228
80	6400	512000	8.9442719	4.3088695	18.5663553	0.012500000
81	6561	531441	9.0000000	4.3267487	18.7207544	0.012345679
82	6724	551368	9.0553851	4.3444815	18.8745194	0.012195122
83	6889	571787	9.1104336	4.3620707	19.0276606	0.012048193
84	7056	592704	9.1651514	4.3795191	19.1801879	0.011904762
85	7225	614125	9.2195445	4.3968296	19.3321111	0.011764706
86	7396	636056	9.2736185	4.4140049	19.4834398	0.011627907
87	7569	658503	9.3273791	4.4310476	19.6341830	0.011494253
88	7744	681472	9.3808315	4.4479602	19.7843498	0.011363636
89	7921	704969	9.4339811	4.4647451	19.9339487	0.011235955
90	8100	729000	9.4868330	4.4814047	20.0829885	0.011111111
91	8281	753571	9.5393920	4.4979414	20.2314773	0.010989011
92	8464	778688	9.5916630	4.5143574	20.3794231	0.010869565
93	8649	804357	9.6436508	4.5306549	20.5268337	0.010752688
94	8836	830584	9.6953597	4.5468359	20.6737171	0.010638298
95	9025	857375	9.7467943	4.5629026	20.8200805	0.010526316
96	9216	884736	9.7979590	4.5788570	20.9659311	0.010416667
97	9409	912673	9.8488578	4.5947009	21.1112763	0.010309278
98	9604	941192	9.8994949	4.6104363	21.2561227	0.010204082
99	9801	970299	9.9498744	4.6260650	21.4004775	0.010101010
100	10000	10000000	10.0000000	4.6415888	21.5443469	0.010000000

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TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
151	22801	3442951	12.2882057	5.3250740	28.35641	6622517
152	23104	3511808	12.3288280	5.3368033	28.48147	6578947
153	23409	3581577	12.3693169	5.3484812	28.60625	6535948
154	23716	3652264	12.4096736	5.3601084	28.73076	6493506
155	24025	3723875	12.4498996	5.3716854	28.85500	6451613
156	24336	3796416	12.4899960	5.3832126	28.97898	6410256
157	24649	3869893	12.5299641	5.3946907	29.10269	6369427
158	24964	3944312	12.5698051	5.4061202	29.22614	6329114
159	25281	4019679	12.6095202	5.4175015	29.34932	6289308
160	25600	4096000	12.6491106	5.4288352	29.47225	6250000
161	25921	4173281	12.6885775	5.4401218	29.59493	6211180
162	26244	4251528	12.7279221	5.4513618	29.71735	6172840
163	26569	4330747	12.7671453	5.4625556	29.83951	6134969
164	26896	4410944	12.8062485	5.4737037	29.96143	6097561
165	27225	4492125	12.8452326	5.4848066	30.08310	6060606
166	27556	4574296	12.8840987	5.4958647	30.20453	6024096
167	27889	4657463	12.9228480	5.5068784	30.32571	5988024
168	28224	4741632	12.9614814	5.5178484	30.44665	5952381
169	28561	4826809	13.0000000	5.5287748	30.56735	5917160
170	28900	4913000	13.0384048	5.5396583	30.68781	5882353
171	29241	5000211	13.0766968	5.5504991	30.80804	5847953
172	29584	5088448	13.1148770	5.5612978	30.92803	5813953
173	29929	5177717	13.1529464	5.5720546	31.04779	5780347
174	30276	5268024	13.1909060	5.5827702	31.16732	5747126
175	30625	5359375	13.2287566	5.5934447	31.28662	5714286
176	30976	5451776	13.2664992	5.6040787	31.40570	5681818
177	31329	5545233	13.3041347	5.6146724	31.52455	5649718
178	31684	5639752	13.3416641	5.6252263	31.64317	5617978
179	32041	5735339	13.3790882	5.6357408	31.76157	5586592
180	32400	5832000	13.4164079	5.6462162	31.87976	5555556
181	32761	5929741	13.4536240	5.6566528	31.99772	5524862
182	33124	6028568	13.4907376	5.6670511	32.11547	5494505
183	33489	6128487	13.5277493	5.6774114	32.23300	5464481
184	33856	6229504	13.5646600	5.6877340	32.35032	5434783
185	34225	6331625	13.6014705	5.6980192	32.46742	5405405
186	34596	6434856	13.6381817	5.7082675	32.58432	5376344
187	34969	6539203	13.6747943	5.7184791	32.70100	5347594
188	35344	6644672	13.7113092	5.7286543	32.81748	5319149
189	35721	6751269	13.7477271	5.7387936	32.93375	5291005
190	36100	6859000	13.7840488	5.7488971	33.04982	5263158
191	36481	6967871	13.8202750	5.7589652	33.16568	5235602
192	36864	7077888	13.8564065	5.7689982	33.28134	5208333
193	37249	7189057	13.8924440	5.7789966	33.39680	5181347
194	37636	7301384	13.9283883	5.7889604	33.51206	5154639
195	38025	7414875	13.9642400	5.7988900	33.62713	5128205
196	38416	7529536	14.0000000	5.8087857	33.74199	5102041
197	38809	7645373	14.0356688	5.8186479	33.85666	5076142
198	39204	7762392	14.0712473	5.8284767	33.97114	5050505
199	39601	7880599	14.1067360	5.8382725	34.08543	5025126
200	40000	8000000	14.1421356	5.8480355	34.19952	5000000

TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
201	40401	8120601	14.1774469	5.8577660	34.31342	4975124
202	40804	8242408	14.2126704	5.8674643	34.42714	4950495
203	41209	8365427	14.2478068	5.8771307	34.54066	4926108
204	41616	8489664	14.2828569	5.8867653	34.65400	4901961
205	42025	8615125	14.3178211	5.8963685	34.76716	4878049
206	42436	8741816	14.3527001	5.9059406	34.88013	4854369
207	42849	8869743	14.3874946	5.9154817	34.99292	4830918
208	43264	8998912	14.4222051	5.9249921	35.10553	4807692
209	43681	9129329	14.4568323	5.9344721	35.21796	4784689
210	44100	9261000	14.4913767	5.9439220	35.33021	4761905
211	44521	9393931	14.5258390	5.9533418	35.44228	4739336
212	44944	9528128	14.5602198	5.9627320	35.55417	4716981
213	45369	9663597	14.5945195	5.9720926	35.66589	4694836
214	45796	9800344	14.6287388	5.9814240	35.77743	4672897
215	46225	9938375	14.6628783	5.9907264	35.88880	4651163
216	46656	10077696	14.6969385	6.0000000	36.00000	4629630
217	47089	10218313	14.7309199	6.0092450	36.11103	4608295
218	47524	10360232	14.7648231	6.0184617	36.22188	4587156
219	47961	10503459	14.7986486	6.0276502	36.33257	4566210
220	48400	10648000	14.8323970	6.0368107	36.44308	4545455
221	48841	10793861	14.8660687	6.0459435	36.55343	4524887
222	49284	10941048	14.8996644	6.0550489	36.66362	4504505
223	49729	11089567	14.9331845	6.0641270	36.77364	4484305
224	50176	11239424	14.9666295	6.0731779	36.88349	4464286
225	50625	11390625	15.0000000	6.0822020	36.99318	4444444
226	51076	11543176	15.0332964	6.0911994	37.10271	4424779
227	51529	11697083	15.0665192	6.1001702	37.21208	4405286
228	51984	11852352	15.0996689	6.1091147	37.32128	4385965
229	52441	12008989	15.1327460	6.1180332	37.43033	4366812
230	52900	12167000	15.1657509	6.1269257	37.53922	4347826
231	53361	12326391	15.1986842	6.1357924	37.64795	4329004
232	53824	12487168	15.2315462	6.1446337	37.75652	4310345
233	54289	12649337	15.2643375	6.1534495	37.86494	4291845
234	54756	12812904	15.2970585	6.1622401	37.97320	4273504
235	55225	12977875	15.3297097	6.1710058	38.08131	4255319
236	55696	13144256	15.3622915	6.1797466	38.18927	4237288
237	56169	13312053	15.3948043	6.1884628	38.29707	4219409
238	56644	13481272	15.4272486	6.1971544	38.40472	4201681
239	57121	13651919	15.4596248	6.2058218	38.51223	4184100
240	57600	13824000	15.4919334	6.2144650	38.61958	4166667
241	58081	13997521	15.5241747	6.2230843	38.72678	4149378
242	58564	14172488	15.5563492	6.2316797	38.83383	4132231
243	59049	14348907	15.584573	6.2402515	38.94074	4115226
244	59536	14526784	15.6204994	6.2487998	39.04750	4098361
245	60025	14706125	15.6524758	6.2573248	39.15411	4081633
246	60516	14886936	15.6843871	6.2658266	39.26058	4065041
247	61009	15069223	15.7162336	6.2743054	39.36691	4048583
248	61504	15252992	15.7480157	6.2827613	39.47309	4032258
249	62001	15438249	15.7797338	6.2911946	39.57913	4016064
250	62500	15625000	15.8113883	6.2996053	39.68503	4000000

TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
251	63001	15813251	15.8429795	6.3079935	39.79078	3984064
252	63504	16003008	15.8745079	6.3163596	39.89640	3968254
253	64009	16194277	15.9059737	6.3247035	40.00187	3952569
254	64516	16387064	15.9373775	6.3330256	40.10721	3937008
255	65025	16581375	15.9687194	6.3413257	40.21241	3921569
256	65536	16777216	16.0000000	6.3496042	40.31747	3906250
257	66049	16974593	16.0312195	6.3578611	40.42240	3891051
258	66564	17173512	16.0623784	6.3660968	40.52719	3875969
259	67081	17373979	16.0934769	6.3743111	40.63184	3861004
260	67600	17576000	16.1245155	6.3825043	40.73636	3846154
261	68121	17779581	16.1554944	6.3906765	40.84075	3831418
262	68644	17984728	16.1864141	6.3988279	40.94500	3816794
263	69169	18191447	16.2172747	6.4069585	41.04912	3802281
264	69696	18399744	16.2480768	6.4150687	41.15311	3787879
265	70225	18609625	16.2788206	6.4231583	41.25696	3773585
266	70756	18821096	16.3095064	6.4312276	41.36069	3759398
267	71289	19034163	16.3401346	6.4392767	41.46428	3745318
268	71824	19248832	16.3707055	6.4473057	41.56775	3731343
269	72361	19465109	16.4012195	6.4553148	41.67109	3717472
270	72900	19683000	16.4316767	6.4633041	41.77430	3703704
271	73441	19902511	16.4620776	6.4712736	41.87738	3690037
272	73984	20123648	16.4924225	6.4792236	41.98034	3676471
273	74529	20346417	16.5227116	6.4871541	42.08317	3663004
274	75076	20570824	16.5529454	6.4950653	42.18587	3649635
275	75625	20796875	16.5831240	6.5029572	42.28845	3636364
276	76176	21024576	16.6132477	6.5108300	42.39091	3623188
277	76729	21253933	16.6433170	6.5186839	42.49324	3610108
278	77284	21484952	16.6733320	6.5265189	42.59545	3597122
279	77841	21717639	16.7032931	6.5343351	42.69753	3584229
280	78400	21952000	16.7332005	6.5421326	42.79950	3571429
281	78961	22188041	16.7630546	6.5499116	42.90134	3558719
282	79524	22425768	16.7928556	6.5576722	43.00306	3546099
283	80089	22665187	16.8226038	6.5654144	43.10467	3533569
284	80656	22906304	16.8522995	6.5731385	43.20615	3521127
285	81225	23149125	16.8819430	6.5808443	43.30751	3508772
286	81796	23393656	16.9115345	6.5885323	43.40876	3496503
287	82369	23639903	16.9410743	6.5962023	43.50988	3484321
288	82944	23887872	16.9705627	6.6038545	43.61089	3472222
289	83521	24137569	17.0000000	6.6114890	43.71179	3460208
290	84100	24389000	17.0293864	6.6191060	43.81256	3448276
291	84681	24642171	17.0587221	6.6267054	43.91322	3436426
292	85264	24897088	17.0880075	6.6342874	44.01377	3424658
293	85849	25153757	17.1172428	6.6418522	44.11420	3412969
294	86436	25412184	17.1464282	6.6493998	44.21452	3401361
295	87025	25672375	17.1755640	6.6569302	44.31472	3389831
296	87616	25934336	17.2046505	6.6644437	44.41481	3378378
297	88209	26198073	17.2336879	6.6719403	44.51479	3367003
298	88804	26463592	17.2626765	6.6794200	44.61465	3355705
299	89401	26730899	17.2916165	6.6868831	44.71441	3344482
300	90000	27000000	17.3205081	6.6943295	44.81405	3333333

TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
301	90601	27270901	17.3493516	6.7017593	44.91358	3322259
302	91204	27543608	17.3781472	6.7091729	45.01300	3311258
303	91809	27818127	17.4068952	6.165700	45.11231	3300330
304	92416	28094464	17.4355958	6.139508	45.21151	3289474
305	93025	28372625	17.4642492	6.313155	45.31061	3278689
306	93636	28652616	17.4928557	6.386641	45.40959	3267974
307	94249	28934443	17.5214155	6.459967	45.50847	3257329
308	94864	29218112	17.5499288	6.533134	45.60725	3246753
309	95481	29503629	17.5783958	6.606143	45.70591	3236246
310	96100	29791000	17.6068169	6.678095	45.80446	3225806
311	96721	30080231	17.6351921	6.751690	45.90291	3215434
312	97344	30371328	17.6635217	6.824229	46.00126	3205128
313	97969	30664297	17.6918060	6.896613	46.09950	3194888
314	98596	30959144	17.7200451	6.7908844	46.19764	3184713
315	99225	31255875	17.7482393	6.8040921	46.29566	3174603
316	99856	31554496	17.7763888	6.8112847	46.39360	3164557
317	00489	31855013	17.8044938	6.8184620	46.49142	3154574
318	01124	32157432	17.8325545	6.8256242	46.58915	3144654
319	01761	32461759	17.8605711	6.8327714	46.68677	3134796
320	02400	32768000	17.8885438	6.8399037	46.78428	3125000
321	103041	33076161	17.9164729	6.8470213	46.88170	3115265
322	03684	33386248	17.9443584	6.8541240	46.97902	3105590
323	04329	33698267	17.9722008	6.8612120	47.07623	3095975
324	04976	34012224	18.0000000	6.8682855	47.17335	3086420
325	05625	34328125	18.0277564	6.8753443	47.27036	3076923
326	06276	34645976	18.0554701	6.8823888	47.36727	3067485
327	06929	34965783	18.0831413	6.8894188	47.46409	3058104
328	07584	35287552	18.1107703	6.8964345	47.56081	3048780
329	08241	35611289	18.1383571	6.9034359	47.65743	3039514
330	08900	35937000	18.1659021	6.9104232	47.75395	3030303
331	09561	36264691	18.1934054	6.9173964	47.85037	3021148
332	110224	36594368	18.2208672	6.9243556	47.94670	3012048
333	0889	36926037	18.2482876	6.9313008	48.04293	3003003
334	1556	37259704	18.2756669	6.9382321	48.13906	2994012
335	2225	37595375	18.3030052	6.9451496	48.23510	2985075
336	2896	37933056	18.3303028	6.9520533	48.33104	2976190
337	3509	38272753	18.3575598	6.9589434	48.42689	2967359
338	4244	38614472	18.3847763	6.9658198	48.52265	2958580
339	4921	38958219	18.4119526	6.9726826	48.61830	2949853
340	115600	39304000	18.4390889	6.9795321	48.71387	2941176
341	116281	39651821	18.4661853	6.9863681	48.80934	2932551
342	116964	40001688	18.4932420	6.9931906	48.90473	2923977
343	7649	40353607	18.5202592	7.0000000	49.00000	2915452
344	8336	40707584	18.5472370	7.0067962	49.09519	2906977
345	9025	41063625	18.5741756	7.0135791	49.19029	2898551
346	9716	41421736	18.6010752	7.0203490	49.28530	2890173
347	20409	41781923	18.6279360	7.0271058	49.38022	2881844
348	21104	42144192	18.6547581	7.0338497	49.47504	2873563
349	21801	42508549	18.6815417	7.0405806	49.56978	2865330
350	22500	42875000	18.7082869	7.0472987	49.66442	2857143

TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
351	123201	43243551	18.7349940	7.0540041	49.75898	2849003
352	123904	43614208	18.7616630	7.0606967	49.85344	2840909
353	124609	43986977	18.7882942	7.0673767	49.94781	2832861
354	125316	44361864	18.8148877	7.0740440	50.04209	2824859
355	126025	44738875	18.8414437	7.0806988	50.13629	2816901
356	126736	45118016	18.8679623	7.0873411	50.23040	2808989
357	127449	45499293	18.8944436	7.0939709	50.32442	2801120
358	128164	45882712	18.9208879	7.1005885	50.41836	2793296
359	128881	46268279	18.9472953	7.1071937	50.51220	2785515
360	129600	46656000	18.9736660	7.1137866	50.60596	2777778
361	130321	47045881	19.0000000	7.1203674	50.69963	2770083
362	131044	47437928	19.0262976	7.1269360	50.79322	2762431
363	131769	47832147	19.0525589	7.1334925	50.88672	2754821
364	132496	48228544	19.0787840	7.1400370	50.98013	2747253
365	133225	48627125	19.1049732	7.1465695	51.07346	2739726
366	133956	49027896	19.1311265	7.1530901	51.16670	2732240
367	134689	49430863	19.1572441	7.1595988	51.25986	2724796
368	135424	49836032	19.1833261	7.1660957	51.35293	2717391
369	136161	50243409	19.2093727	7.1725809	51.44592	2710027
370	136900	50653000	19.2353841	7.1790544	51.53882	2702703
371	137641	51064811	19.2613603	7.1855162	51.63164	2695418
372	138384	51478848	19.2873015	7.1919663	51.72438	2688172
373	139129	51895117	19.3132079	7.1984050	51.81703	2680965
374	139876	52313624	19.3390796	7.2048322	51.90961	2673797
375	140625	52734375	19.3649167	7.2112479	52.00210	2666667
376	141376	53157376	19.3907194	7.2176522	52.09450	2659574
377	142129	53582633	19.4164878	7.2240450	52.18683	2652520
378	142884	54010152	19.4422221	7.2304268	52.27907	2645503
379	143641	54439939	19.4679223	7.2367972	52.37123	2638522
380	144400	54872000	19.4935887	7.2431565	52.46332	2631579
381	145161	55306341	19.5192213	7.2495045	52.55532	2624672
382	145924	55742968	19.5448203	7.2558415	52.64724	2617801
383	146689	56181887	19.5703858	7.2621675	52.73908	2610966
384	147456	56623104	19.5959179	7.2684824	52.83084	2604167
385	148225	57066625	19.6214169	7.2747864	52.92252	2597403
386	148996	57512456	19.6468827	7.2810794	53.01412	2590674
387	149769	57960603	19.6723156	7.2873617	53.10564	2583979
388	150544	58411072	19.6977156	7.2936330	53.19708	2577320
389	151321	58863869	19.7230829	7.2998936	53.28845	2570694
390	152100	59319000	19.7484177	7.3061436	53.37973	2564103
391	152881	59776471	19.7737199	7.3123828	53.47094	2557545
392	153664	60236288	19.7989899	7.3186114	53.56207	2551020
393	154449	60698457	19.8242276	7.3248295	53.65313	2544529
394	155236	61162984	19.8494332	7.3310369	53.74410	2538071
395	156025	61629875	19.8746069	7.3372339	53.83500	2531646
396	156816	62099136	19.8997487	7.3434205	53.92582	2525253
397	157609	62570773	19.9248588	7.3495966	54.01657	2518892
398	158404	63044792	19.9499373	7.3557624	54.10724	2512563
399	159201	63521199	19.9749844	7.3619178	54.19783	2506266
400	160000	64000000	20.0000000	7.3680630	54.28835	2500000

TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
401	160801	64481201	20. 0249844	7. 3741979	54. 37880	2493766
402	161604	64964808	20. 0499377	7. 3803227	54. 46916	2487562
403	162409	65450827	20. 0748599	7. 3864373	54. 55946	2481390
404	163216	65939264	20. 0997512	7. 3925418	54. 64967	2475248
405	164025	66430125	20. 1246118	7. 3986363	54. 73982	2469136
406	164836	66923416	20. 1494417	7. 4047206	54. 82989	2463054
407	165649	67419143	20. 1742410	7. 4107950	54. 91988	2457002
408	166464	67917312	20. 1990099	7. 4168595	55. 00981	2450980
409	167281	68417929	20. 2237484	7. 4229142	55. 09965	2444988
410	168100	68921000	20. 2484567	7. 4289589	55. 18943	2439024
411	168921	69426531	20. 2731349	7. 4349938	55. 27913	2433090
412	169744	69934528	20. 2977831	7. 4410189	55. 36876	2427184
413	170569	70444997	20. 3224014	7. 4470342	55. 45832	2421308
414	171396	70957944	20. 3469809	7. 4530399	55. 54780	2415459
415	172225	71473375	20. 3715488	7. 4590359	55. 63722	2409639
416	173056	71991296	20. 3960781	7. 4650223	55. 72656	2403846
417	173889	72511713	20. 4205779	7. 4709991	55. 81583	2398082
418	174724	73034632	20. 4450483	7. 4769664	55. 90503	2392344
419	175561	73560059	20. 4694895	7. 4829242	55. 99415	2386635
420	176400	74088000	20. 4939015	7. 4888724	56. 08321	2380952
421	177241	74618461	20. 5182845	7. 4948113	56. 17220	2375297
422	178084	75151448	20. 5426386	7. 5007406	56. 26111	2369668
423	178929	75686967	20. 5669638	7. 5066607	56. 34996	2364066
424	179776	76225024	20. 5912603	7. 5125715	56. 43873	2358491
425	180625	76765625	20. 6155281	7. 5184730	56. 52744	2352941
426	181476	77308776	20. 6397674	7. 5243652	56. 61607	2347418
427	182329	77854483	20. 6639783	7. 5302482	56. 70464	2341920
428	183184	78402752	20. 6881609	7. 5361221	56. 79314	2336449
429	184041	78953589	20. 7123152	7. 5419867	56. 88156	2331002
430	184900	79507000	20. 7364414	7. 5478423	56. 96992	2325581
431	185761	80062991	20. 7605395	7. 5536888	57. 05821	2320186
432	186624	80621568	20. 7846097	7. 5595263	57. 14644	2314815
433	187489	81182737	20. 8086520	7. 5653548	57. 23459	2309469
434	188356	81746504	20. 8326667	7. 5711743	57. 32268	2304147
435	189225	82312875	20. 8566536	7. 5769849	57. 41070	2298851
436	190096	82881856	20. 8806130	7. 5827865	57. 49865	2293578
437	190969	83453453	20. 9045450	7. 5885793	57. 58654	2288330
438	191844	84027672	20. 9284495	7. 5943633	57. 67435	2283105
439	192721	84604519	20. 9523268	7. 6001385	57. 76211	2277904
440	193600	85184000	20. 9761770	7. 6059049	57. 84979	2272727
441	194481	85766121	21. 0000000	7. 6116626	57. 93741	2267574
442	195364	86350888	21. 0237960	7. 6174116	58. 02496	2262443
443	196249	86938307	21. 0475652	7. 6231519	58. 11245	2257336
444	197136	87528384	21. 0713075	7. 6288837	58. 19987	2252252
445	198025	88121125	21. 0950231	7. 6346067	58. 28722	2247191
446	198916	88716536	21. 1187121	7. 6403213	58. 37451	2242152
447	199809	89314623	21. 1423745	7. 6460272	58. 46173	2237136
448	200704	89915392	21. 1660105	7. 6517247	58. 54889	2232143
449	201601	90518849	21. 1896201	7. 6574138	58. 63599	2227171
450	202500	91125000	21. 2132034	7. 6630943	58. 72301	2222222

TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
451	203401	91733851	21. 2367606	7. 6687665	58. 80998	2217295
452	204304	92345408	21. 2602916	7. 6744303	58. 89688	2212389
453	205209	92959677	21. 2837967	7. 6800857	58. 98372	2207506
454	206116	93576664	21. 3072758	7. 6857328	59. 07049	2202643
455	207025	94196375	21. 3307290	7. 6913717	59. 15720	2197802
456	207936	94818816	21. 3541565	7. 6970023	59. 24384	2192982
457	208849	95443993	21. 3775583	7. 7026246	59. 33043	2188184
458	209764	96071912	21. 4009346	7. 7082388	59. 41695	2183406
459	210681	96702579	21. 4242853	7. 7138448	59. 50340	2178649
460	211600	97336000	21. 4476106	7. 7194426	59. 58979	2173913
461	212521	97972181	21. 4709106	7. 7250325	59. 67611	2169197
462	213444	98611128	21. 4941853	7. 7306141	59. 76239	2164502
463	214369	99252847	21. 5174348	7. 7361877	59. 84860	2159827
464	215296	99897344	21. 5406592	7. 7417532	59. 93474	2155172
465	216225	100544625	21. 5638587	7. 7473109	60. 02083	2150538
466	217156	101194696	21. 5870331	7. 7528606	60. 10685	2145923
467	218089	101847563	21. 6101828	7. 7584023	60. 19281	2141328
468	219024	102503232	21. 6333077	7. 7639361	60. 27870	2136752
469	219961	103161709	21. 6564078	7. 7694620	60. 36454	2132196
470	220900	103823000	21. 6794834	7. 7749801	60. 45032	2127660
471	221841	104487111	21. 7025344	7. 7804904	60. 53603	2123142
472	222784	105154048	21. 7255610	7. 7859928	60. 62168	2118644
473	223729	105823817	21. 7485632	7. 7914875	60. 70728	2114165
474	224676	106496424	21. 7715411	7. 7969745	60. 79281	2109705
475	225625	107171875	21. 7944947	7. 8024538	60. 87828	2105263
476	226576	107850176	21. 8174242	7. 8079254	60. 96370	2100840
477	227529	108531333	21. 8403297	7. 8133892	61. 04905	2096436
478	228484	109215352	21. 8632111	7. 8188456	61. 13435	2092050
479	229441	109902239	21. 8860686	7. 8242942	61. 21958	2087683
480	230400	110592000	21. 9089023	7. 8297353	61. 30475	2083333
481	231361	111284641	21. 9317122	7. 8351688	61. 38987	2079002
482	232324	111980168	21. 9544984	7. 8405949	61. 47493	2074689
483	233289	112678587	21. 9772610	7. 8460134	61. 55993	2070393
484	234256	113379904	22. 0000000	7. 8514244	61. 64487	2066116
485	235225	114084125	22. 0227155	7. 8568281	61. 72975	2061856
486	236196	114791256	22. 0454077	7. 8622242	61. 81457	2057613
487	237169	115501303	22. 0680765	7. 8676130	61. 89933	2053388
488	238144	116214272	22. 0907270	7. 8729944	61. 98404	2049180
489	239121	116930169	22. 1133444	7. 8783684	62. 06869	2044990
490	240100	117649000	22. 1359436	7. 8837352	62. 15328	2040816
491	241081	118370771	22. 1585198	7. 8890946	62. 23781	2036660
492	242064	119095488	22. 1810730	7. 8944468	62. 32229	2032520
493	243049	119823157	22. 2036033	7. 8997917	62. 40671	2028398
494	244036	120553784	22. 2261108	7. 9051294	62. 49107	2024291
495	245025	121287375	22. 2485955	7. 9104599	62. 57538	2020202
496	246016	122023936	22. 2710575	7. 9157832	62. 65962	2016129
497	247009	122763473	22. 2934968	7. 9210994	62. 74382	2012072
498	248004	123505992	22. 3159136	7. 9264085	62. 82795	2008032
499	249001	124251499	22. 3383079	7. 9317104	62. 91203	2004008
500	250000	125000000	22. 3606798	7. 9370053	62. 99605	2000000

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TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
551	303601	167284151	23.4733892	8.1981753	67.21008	1814882
552	304704	168196608	23.4946802	8.2031319	67.29137	1811594
553	305809	169112377	23.5159520	8.2080825	67.37262	1808318
554	306916	170031464	23.5372046	8.2130271	67.45381	1805054
555	308025	170953875	23.5584380	8.2179657	67.53496	1801802
556	309136	171879616	23.5796522	8.2228985	67.61606	1798561
557	310249	172808693	23.6008474	8.2278254	67.69711	1795332
558	311364	173741112	23.6220236	8.2327463	67.77811	1792115
559	312481	174676879	23.6431808	8.2376614	67.85907	1788909
560	313600	175616000	23.6643191	8.2425706	67.93997	1785714
561	314721	176558481	23.6854386	8.2474740	68.02083	1782531
562	315844	177504328	23.7065392	8.2523715	68.10164	1779359
563	316969	178453547	23.7276210	8.2572633	68.18240	1776199
564	318096	179406144	23.7486842	8.2621492	68.26311	1773050
565	319225	180362125	23.7697286	8.2670294	68.34378	1769912
566	320356	181321496	23.7907545	8.2719039	68.42439	1766784
567	321489	182284263	23.8117618	8.2767726	68.50496	1763668
568	322624	183250432	23.8327506	8.2816355	68.58549	1760563
569	323761	184220009	23.8537209	8.2864928	68.66596	1757469
570	324900	185193000	23.8746728	8.2913444	68.74639	1754386
571	326041	186169411	23.8956063	8.2961903	68.82677	1751313
572	327184	187149248	23.9165215	8.3010304	68.90711	1748252
573	328329	188132517	23.9374184	8.3058651	68.98740	1745201
574	329476	189119224	23.9582971	8.3106941	69.06764	1742160
575	330625	190109375	23.9791576	8.3155175	69.14783	1739130
576	331776	191102976	24.0000000	8.3203353	69.22798	1736111
577	332929	192100033	24.0208243	8.3251475	69.30808	1733102
578	334084	193100552	24.0416306	8.3299542	69.38814	1730104
579	335241	194104539	24.0624188	8.3347553	69.46815	1727116
580	336400	195112000	24.0831891	8.3395509	69.54811	1724138
581	337561	196122941	24.1039416	8.3443410	69.62803	1721170
582	338724	197137368	24.1246762	8.3491256	69.70790	1718213
583	339889	198155287	24.1453929	8.3539047	69.78772	1715266
584	341056	199176704	24.1660919	8.3586784	69.86750	1712329
585	342225	200201625	24.1867732	8.3634466	69.94724	1709402
586	343396	201230056	24.2074369	8.3682095	70.02693	1706485
587	344569	202262003	24.2280829	8.3729668	70.10657	1703578
588	345744	203297472	24.2487113	8.3777188	70.18617	1700680
589	346921	204336469	24.2693222	8.3824653	70.26572	1697793
590	348100	205379000	24.2899156	8.3872065	70.34523	1694915
591	349281	206425071	24.3104916	8.3919423	70.42470	1692047
592	350464	207474688	24.3310501	8.3966729	70.50412	1689189
593	351649	208527857	24.3515913	8.4013981	70.58349	1686341
594	352836	209584584	24.3721152	8.4061180	70.66282	1683502
595	354025	210644875	24.3926218	8.4108326	70.74210	1680672
596	355216	211708736	24.4131112	8.4155419	70.82135	1677852
597	356409	212776173	24.4335834	8.4202460	70.90054	1675042
598	357604	213847192	24.4540385	8.4249448	70.97969	1672241
599	358801	214921799	24.4744765	8.4296383	71.05880	1669449
600	360000	216000000	24.4948974	8.4343267	71.13787	1666667

TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
601	361201	217081801	24. 5153013	8. 4390098	71. 21689	1663894
602	362404	218167208	24. 5356883	8. 4436877	71. 29586	1661130
603	363609	219256227	24. 5560583	8. 4483605	71. 37480	1658375
604	364816	220348864	24. 5764115	8. 4530281	71. 45368	1655629
605	366025	221445125	24. 5967478	8. 4576906	71. 53253	1652893
606	367236	222545016	24. 6170673	8. 4623479	71. 61133	1650165
607	368449	223648543	24. 6373700	8. 4670000	71. 69009	1647446
608	369664	224755712	24. 6576560	8. 4716471	71. 76881	1644737
609	370881	225866529	24. 6779254	8. 4762892	71. 84748	1642036
610	372100	226981000	24. 6981781	8. 4809261	71. 92611	1639344
611	373321	228099131	24. 7184142	8. 4855579	72. 00469	1636661
612	374544	229220928	24. 7386338	8. 4901848	72. 08324	1633987
613	375769	230346397	24. 7588368	8. 4948065	72. 16174	1631321
614	376996	231475544	24. 7790234	8. 4994233	72. 24020	1628664
615	378225	232608375	24. 7991935	8. 5040350	72. 31861	1626016
616	379456	233744896	24. 8193473	8. 5086417	72. 39698	1623377
617	380689	234885113	24. 8394847	8. 5132435	72. 47531	1620746
618	381924	236029032	24. 8596058	8. 5178403	72. 55360	1618123
619	383161	237176659	24. 8797106	8. 5224321	72. 63185	1615509
620	384400	238328000	24. 8997992	8. 5270189	72. 71005	1612903
621	385641	239483061	24. 9198716	8. 5316009	72. 78821	1610306
622	386884	240641848	24. 9399278	8. 5361780	72. 86633	1607717
623	388129	241804367	24. 9599679	8. 5407501	72. 94441	1605136
624	389376	242970624	24. 9799920	8. 5453173	73. 02245	1602564
625	390625	244140625	25. 0000000	8. 5498797	73. 10044	1600000
626	391876	245314376	25. 0199920	8. 5544372	73. 17840	1597444
627	393129	246491883	25. 0399681	8. 5589899	73. 25631	1594896
628	394384	247673152	25. 0599282	8. 5635377	73. 33418	1592357
629	395641	248858189	25. 0798724	8. 5680807	73. 41201	1589825
630	396900	250047000	25. 0998008	8. 5726189	73. 48979	1587302
631	398161	251239591	25. 1197134	8. 5771523	73. 56754	1584786
632	399424	252435968	25. 1396102	8. 5816809	73. 64525	1582278
633	400689	253636137	25. 1594913	8. 5862047	73. 72291	1579779
634	401956	254840104	25. 1793566	8. 5907238	73. 80053	1577287
635	403225	256047875	25. 1992063	8. 5952380	73. 87812	1574803
636	404496	257259456	25. 2190404	8. 5997476	73. 95566	1572327
637	405769	258474853	25. 2388589	8. 6042525	74. 03316	1569859
638	407044	259694072	25. 2586619	8. 6087526	74. 11062	1567398
639	408321	260917119	25. 2784493	8. 6132480	74. 18804	1564945
640	409600	262144000	25. 2982213	8. 6177388	74. 26542	1562500
641	410881	263374721	25. 3179778	8. 6222248	74. 34276	1560062
642	412164	264609288	25. 3377189	8. 6267063	74. 42006	1557632
643	413449	265847707	25. 3574447	8. 6311830	74. 49732	1555210
644	414736	267089984	25. 3771551	8. 6356551	74. 57454	1552795
645	416025	268336125	25. 3968502	8. 6401226	74. 65172	1550388
646	417316	269586136	25. 4165301	8. 6445855	74. 72886	1547988
647	418609	270840023	25. 4361947	8. 6490437	74. 80596	1545595
648	419904	272097792	25. 4558441	8. 6534974	74. 88302	1543210
649	421201	273359449	25. 4754784	8. 6579465	74. 96004	1540832
650	422500	274625000	25. 4950976	8. 6623911	75. 03702	1538462

TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
651	423801	275894451	25. 5147016	8. 6668310	75. 11396	1536098
652	425104	277167808	25. 5342907	8. 6712665	75. 19086	1533742
653	426409	278445077	25. 5538647	8. 6756974	75. 26772	1531394
654	427716	279726264	25. 5734237	8. 6801237	75. 34455	1529052
655	429025	281011375	25. 5929678	8. 6845456	75. 42133	1526718
656	430336	282300416	25. 6124969	8. 6889630	75. 49808	1524390
657	431649	283593393	25. 6320112	8. 6933759	75. 57478	1522070
658	432964	284890312	25. 6515107	8. 6977843	75. 65145	1519757
659	434281	286191179	25. 6709953	8. 7021882	75. 72808	1517451
660	435600	287496000	25. 6904652	8. 7065877	75. 80467	1515152
661	436921	288804781	25. 7099203	8. 7109827	75. 88122	1512859
662	438244	290117528	25. 7293607	8. 7153734	75. 95773	1510574
663	439569	291434247	25. 7487864	8. 7197596	76. 03421	1508296
664	440896	292754944	25. 7681975	8. 7241414	76. 11064	1506024
665	442225	294079625	25. 7875939	8. 7285187	76. 18704	1503759
666	443556	295408296	25. 8069758	8. 7328918	76. 26340	1501502
667	444889	296740963	25. 8263431	8. 7372604	76. 33972	1499250
668	446224	298077632	25. 8456960	8. 7416246	76. 41600	1497006
669	447561	299418309	25. 8650343	8. 7459846	76. 49225	1494768
670	448900	300763000	25. 8843582	8. 7503401	76. 56845	1492537
671	450241	302111711	25. 9036677	8. 7546913	76. 64462	1490313
672	451584	303464448	25. 9229628	8. 7590383	76. 72075	1488095
673	452929	304821217	25. 9422435	8. 7633809	76. 79684	1485884
674	454276	306182024	25. 9615100	8. 7677192	76. 87290	1483680
675	455625	307546875	25. 9807621	8. 7720532	76. 94892	1481481
676	456976	308915776	26. 0000000	8. 7763830	77. 02490	1479290
677	458329	310288733	26. 0192237	8. 7807084	77. 10084	1477105
678	459684	311665752	26. 0384331	8. 7850296	77. 17675	1474926
679	461041	313046839	26. 0576284	8. 7893466	77. 25261	1472754
680	462400	314432000	26. 0768096	8. 7936593	77. 32844	1470588
681	463761	315821241	26. 0959767	8. 7979679	77. 40424	1468429
682	465124	317214568	26. 1151297	8. 8022721	77. 47999	1466276
683	466489	318611987	26. 1342687	8. 8065722	77. 55571	1464129
684	467856	320013504	26. 1533937	8. 8108681	77. 63140	1461988
685	469225	321419125	26. 1725047	8. 8151598	77. 70704	1459854
686	470596	322828856	26. 1916017	8. 8194474	77. 78265	1457726
687	471969	324242703	26. 2106848	8. 8237307	77. 85822	1455604
688	473344	325660672	26. 2297541	8. 8280099	77. 93376	1453488
689	474721	327082769	26. 2488095	8. 8322850	78. 00926	1451379
690	476100	328509000	26. 2678511	8. 8365559	78. 08472	1449275
691	477481	329939371	26. 2868789	8. 8408227	78. 16015	1447178
692	478864	331373888	26. 3058929	8. 8450854	78. 23554	1445087
693	480249	332812557	26. 3248932	8. 8493440	78. 31089	1443001
694	481636	334255384	26. 3438797	8. 8535985	78. 38621	1440922
695	483025	335702375	26. 3628527	8. 8578489	78. 46149	1438849
696	484416	337153536	26. 3818119	8. 8620952	78. 53673	1436782
697	485809	338608873	26. 4007576	8. 8663375	78. 61194	1434720
698	487204	340068392	26. 4196896	8. 8705757	78. 68711	1432665
699	488601	341532099	26. 4386081	8. 8748099	78. 76225	1430615
700	490000	343000000	26. 4575131	8. 8790400	78. 83735	1428571

TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
701	491401	344472101	26.4764046	8.8832661	78.91242	1426534
702	492804	345948408	26.4952826	8.8874882	78.98745	1424501
703	494209	347428927	26.5141472	8.8917063	79.06244	1422475
704	495616	348913664	26.5329983	8.8959204	79.13740	1420455
705	497025	350402625	26.5518361	8.9001304	79.21232	1418440
706	498436	351895816	26.5706605	8.9043366	79.28721	1416431
707	499849	353393243	26.5894716	8.9085387	79.36206	1414427
708	501264	354894912	26.6082694	8.9127369	79.43688	1412429
709	502681	356400829	26.6270539	8.9169311	79.51166	1410437
710	504100	357911000	26.6458252	8.9211214	79.58641	1408451
711	505521	359425431	26.6645833	8.9253078	79.66112	1406470
712	506944	360944128	26.6833281	8.9294902	79.73580	1404494
713	508369	362467097	26.7020598	8.9336687	79.81044	1402525
714	509796	363994344	26.7207784	8.9378433	79.88504	1400560
715	511225	365525875	26.7394839	8.9420140	79.95962	1398601
716	512656	367061696	26.7581763	8.9461809	80.03415	1396648
717	514089	368601813	26.7768557	8.9503438	80.10865	1394700
718	515524	370146232	26.7955220	8.9545029	80.18312	1392758
719	516961	371694959	26.8141754	8.9586581	80.25756	1390821
720	518400	373248000	26.8328157	8.9628095	80.33195	1388889
721	519841	374805361	26.8514432	8.9669570	80.40632	1386963
722	521284	376367048	26.8700577	8.9711007	80.48065	1385042
723	522729	377933067	26.8886593	8.9752406	80.55494	1383126
724	524176	379503424	26.9072481	8.9793766	80.62920	1381215
725	525625	381078125	26.9258240	8.9835089	80.70343	1379310
726	527076	382657176	26.9443872	8.9876373	80.77763	1377410
727	528529	384240583	26.9629375	8.9917620	80.85178	1375516
728	529984	385828352	26.9814751	8.9958829	80.92591	1373626
729	531441	387420489	27.0000000	9.0000000	81.00000	1371742
730	532900	389017000	27.0185122	9.0041134	81.07406	1369863
731	534361	390617891	27.0370117	9.0082229	81.14808	1367989
732	535824	392223168	27.0554985	9.0123288	81.22207	1366120
733	537289	393832837	27.0739727	9.0164309	81.29603	1364256
734	538756	395446904	27.0924344	9.0205293	81.36995	1362398
735	540225	397065375	27.1108834	9.0246239	81.44384	1360544
736	541696	398688256	27.1293199	9.0287149	81.51769	1358696
737	543169	400315553	27.1477439	9.0328021	81.59151	1356852
738	544644	401947272	27.1661554	9.0368857	81.66530	1355014
739	546121	403583419	27.1845544	9.0409655	81.73906	1353180
740	547600	405224000	27.2029410	9.0450417	81.81278	1351351
741	549081	406869021	27.2213152	9.0491142	81.88647	1349528
742	550564	408518488	27.2396769	9.0531831	81.96012	1347709
743	552049	410172407	27.2580263	9.0572482	82.03375	1345895
744	553536	411830784	27.2763634	9.0613098	82.10734	1344086
745	555025	413493625	27.2946881	9.0653677	82.18089	1342282
746	556516	415160936	27.3130006	9.0694220	82.25442	1340483
747	558009	416832723	27.3313007	9.0734726	82.32791	1338688
748	559504	418508992	27.3495887	9.0775197	82.40136	1336898
749	561001	420189749	27.3678644	9.0815631	82.47479	1335113
750	562500	421875000	27.3861279	9.0856030	82.54818	1333333

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No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
751	564001	423564751	27. 4043792	9. 0896392	82. 62154	1331558
752	565504	425259008	27. 4226184	9. 0936719	82. 69487	1329787
753	567009	426957777	27. 4408455	9. 0977010	82. 76816	1328021
754	568516	428661064	27. 4590604	9. 1017265	82. 84143	1326260
755	570025	430368875	27. 4772633	9. 1057485	82. 91466	1324503
756	571536	432081216	27. 4954542	9. 1097669	82. 98785	1322751
757	573049	433798093	27. 5136330	9. 1137818	83. 06102	1321004
758	574564	435519512	27. 5317998	9. 1177931	83. 13415	1319261
759	576081	437245479	27. 5499546	9. 1218010	83. 20725	1317523
760	577600	438976000	27. 5680975	9. 1258053	83. 28032	1315789
761	579121	440711081	27. 5862284	9. 1298061	83. 35336	1314060
762	580644	442450728	27. 6043475	9. 1338034	83. 42636	1312336
763	582169	444194947	27. 6224546	9. 1377071	83. 49934	1310616
764	583696	445943744	27. 6405499	9. 1417874	83. 57228	1308901
765	585225	447697125	27. 6586334	9. 1457742	83. 64519	1307190
766	586756	449455096	27. 6767050	9. 1497576	83. 71806	1305483
767	588289	451217663	27. 6947648	9. 1537375	83. 79091	1303781
768	589824	452984832	27. 7128129	9. 1577139	83. 86372	1302083
769	591361	454756609	27. 7308492	9. 1616869	83. 93651	1300390
770	592900	456533000	27. 7488739	9. 1656565	84. 00926	1298701
771	594441	458314011	27. 7668868	9. 1696225	84. 08198	1297017
772	595984	460099648	27. 7848880	9. 1735852	84. 15467	1295337
773	597529	461889017	27. 8028775	9. 1775445	84. 22732	1293661
774	599076	463684824	27. 8208555	9. 1815003	84. 29995	1291990
775	600625	465484375	27. 8388218	9. 1854527	84. 37254	1290323
776	602176	467288576	27. 8567766	9. 1894018	84. 44511	1288660
777	603729	469097433	27. 8747197	9. 1933474	84. 51764	1287001
778	605284	470910952	27. 8926514	9. 1972897	84. 59014	1285347
779	606841	472729139	27. 9105715	9. 2012286	84. 66261	1283697
780	608400	474552000	27. 9284801	9. 2051641	84. 73505	1282051
781	609961	476379541	27. 9463772	9. 2090962	84. 80745	1280410
782	611524	478211768	27. 9642629	9. 2130250	84. 87983	1278772
783	613089	480048687	27. 9821372	9. 2169505	84. 95218	1277139
784	614656	481890304	28. 0000000	9. 2208726	85. 02449	1275510
785	616225	483736625	28. 0178515	9. 2247914	85. 09678	1273885
786	617796	485587656	28. 0356915	9. 2287068	85. 16903	1272265
787	619369	487443403	28. 0535203	9. 2326189	85. 24125	1270648
788	620944	489303872	28. 0713377	9. 2365277	85. 31344	1269036
789	622521	491169069	28. 0891438	9. 2404333	85. 38561	1267427
790	624100	493039000	28. 1069386	9. 2443355	85. 45774	1265823
791	625681	494913671	28. 1247222	9. 2482344	85. 52984	1264223
792	627264	496793088	28. 1424946	9. 2521300	85. 60191	1262626
793	628849	498677257	28. 1602557	9. 2560224	85. 67395	1261034
794	630436	500566184	28. 1780056	9. 2599114	85. 74596	1259446
795	632025	502459875	28. 1957444	9. 2637973	85. 81794	1257862
796	633616	504358336	28. 2134720	9. 2676798	85. 88989	1256281
797	635209	506261573	28. 2311884	9. 2715592	85. 96181	1254705
798	636804	508169592	28. 2488938	9. 2754352	86. 03370	1253133
799	638401	510082399	28. 2665881	9. 2793081	86. 10556	1251564
800	640000	512000000	28. 2842712	9. 2831777	86. 17739	1250000

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No.,	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
801	641601	513922401	28. 3019434	9. 2870440	86. 24919	1248439
802	643204	515849608	28. 3196045	9. 2909072	86. 32096	1246883
803	644809	517781627	28. 3372546	9. 2947671	86. 39270	1245330
804	646416	519718464	28. 3548938	9. 2986239	86. 46441	1243781
805	648025	521660125	28. 3725219	9. 3024775	86. 53609	1242236
806	649636	523606616	28. 3901391	9. 3063278	86. 60774	1240695
807	651249	525557943	28. 4077454	9. 3101750	86. 67936	1239157
808	652864	527514112	28. 4253408	9. 3140190	86. 75095	1237624
809	654481	529475129	28. 4429253	9. 3178599	86. 82251	1236094
810	656100	531441000	28. 4604989	9. 3216975	86. 89404	1234568
811	657721	533411731	28. 4780617	9. 3255320	86. 96555	1233046
812	659344	535387328	28. 4956137	9. 3293634	87. 03702	1231527
813	660969	537367797	28. 5131549	9. 3331916	87. 10847	1230012
814	662596	539353144	28. 5306852	9. 3370167	87. 17988	1228501
815	664225	541343375	28. 5482048	9. 3408386	87. 25127	1226994
816	665856	543338496	28. 5657137	9. 3446575	87. 32262	1225490
817	667489	545338513	28. 5832119	9. 3484731	87. 39395	1223990
818	669124	547343432	28. 6006993	9. 3522857	87. 46525	1222494
819	670761	549353259	28. 6181760	9. 3560952	87. 53652	1221001
820	672400	551368000	28. 6356421	9. 3599016	87. 60776	1219512
821	674041	553387661	28. 6530976	9. 3637049	87. 67897	1218027
822	675684	555412248	28. 6705424	9. 3675051	87. 75015	1216545
823	677329	557441767	28. 6879766	9. 3713022	87. 82131	1215067
824	678976	559476224	28. 7054002	9. 3750963	87. 89243	1213592
825	680625	561515625	28. 7228132	9. 3788873	87. 96353	1212121
826	682276	563559976	28. 7402157	9. 3826752	88. 03459	1210654
827	683929	565609283	28. 7576077	9. 3864600	88. 10563	1209190
828	685584	567663552	28. 7749891	9. 3902419	88. 17664	1207729
829	687241	569722789	28. 7923601	9. 3940206	88. 24762	1206273
830	688900	571787000	28. 8097206	9. 3977964	88. 31858	1204819
831	690561	573856191	28. 8270706	9. 4015691	88. 38950	1203369
832	692224	575930368	28. 8444102	9. 4053387	88. 46040	1201923
833	693889	578009537	28. 8617394	9. 4091054	88. 53126	1200480
834	695556	580093704	28. 8790582	9. 4128690	88. 60210	1199041
835	697225	582182875	28. 8963666	9. 4166297	88. 67291	1197605
836	698896	584277056	28. 9136646	9. 4203873	88. 74370	1196172
837	700569	586376253	28. 9309523	9. 4241420	88. 81445	1194743
838	702244	588480472	28. 9482297	9. 4278936	88. 88518	1193317
839	703921	590589719	28. 9654967	9. 4316423	88. 95588	1191895
840	705600	592704000	28. 9827535	9. 4353880	89. 02655	1190476
841	707281	594823321	29. 0000000	9. 4391307	89. 09719	1189061
842	708964	596947688	29. 0172363	9. 4428704	89. 16780	1187648
843	710649	599077107	29. 0344623	9. 4466072	89. 23839	1186240
844	712336	601211584	29. 0516781	9. 4503410	89. 30895	1184834
845	714025	603351125	29. 0688837	9. 4540719	89. 37948	1183432
846	715716	605495736	29. 0860791	9. 4577999	89. 44998	1182033
847	717409	607645423	29. 1032644	9. 4615249	89. 52045	1180638
848	719104	609800192	29. 1204396	9. 4652470	89. 59090	1179245
849	720801	611960049	29. 1376046	9. 4689661	89. 66132	1177856
850	722500	614125000	29. 1547595	9. 4726824	89. 73171	1176471

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No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
851	724201	616295051	29. 1719043	9. 4763957	89. 80208	1175088
852	725904	618470208	29. 1890390	9. 4801061	89. 87241	1173709
853	727609	620650477	29. 2061637	9. 4838136	89. 94272	1172333
854	729316	622835864	29. 2232784	9. 4875182	90. 01300	1170960
855	731025	625026375	29. 2403830	9. 4912200	90. 08326	1169591
856	732736	627222016	29. 2574777	9. 4949188	90. 15348	1168224
857	734449	629422793	29. 2745623	9. 4986147	90. 22368	1166861
858	736164	631628712	29. 2916370	9. 5023078	90. 29385	1165501
859	737881	633839779	29. 3087018	9. 5059980	90. 36400	1164144
860	739600	636056000	29. 3257566	9. 5096854	90. 43412	1162791
861	741321	638277381	29. 3428015	9. 5133699	90. 50421	1161440
862	743044	640503928	29. 3598365	9. 5170515	90. 57427	1160093
863	744769	642735647	29. 3768616	9. 5207303	90. 64431	1158749
864	746496	644972544	29. 3938769	9. 5244063	90. 71432	1157407
865	748225	647214625	29. 4108823	9. 5280794	90. 78430	1156069
866	749956	649461896	29. 4278779	9. 5317497	90. 85425	1154734
867	751689	651714363	29. 4448637	9. 5354172	90. 92418	1153403
868	753424	653972032	29. 4618397	9. 5390818	90. 99408	1152074
869	755161	656234909	29. 4788059	9. 5427437	91. 06396	1150748
870	756900	658503000	29. 4957624	9. 5464027	91. 13380	1149425
871	758641	660776311	29. 5127091	9. 5500589	91. 20363	1148106
872	760384	663054848	29. 5296461	9. 5537123	91. 27342	1146789
873	762129	665338617	29. 5465734	9. 5573630	91. 34319	1145475
874	763876	667627624	29. 5634910	9. 5610108	91. 41293	1144165
875	765625	669921875	29. 5803989	9. 5646559	91. 48264	1142857
876	767376	672221376	29. 5972972	9. 5682982	91. 55233	1141553
877	769129	674526133	29. 6141858	9. 5719377	91. 62199	1140251
878	770884	676836152	29. 6310648	9. 5755745	91. 69163	1138952
879	772641	679151439	29. 6479342	9. 5792085	91. 76124	1137656
880	774400	681472000	29. 6647939	9. 5828397	91. 83082	1136364
881	776161	683797841	29. 6816442	9. 5864682	91. 90037	1135074
882	777924	686128968	29. 6984848	9. 5900939	91. 96990	1133787
883	779689	688465387	29. 7153159	9. 5937169	92. 03940	1132503
884	781456	690807104	29. 7321375	9. 5973373	92. 10888	1131222
885	783225	693154125	29. 7489496	9. 6009548	92. 17833	1129944
886	784996	695506456	29. 7657521	9. 6045696	92. 24776	1128668
887	786769	697864103	29. 7825452	9. 6081817	92. 31715	1127396
888	788544	700227072	29. 7993289	9. 6117911	92. 38653	1126126
889	790321	702595369	29. 8161030	9. 6153977	92. 45587	1124859
890	792100	704969000	29. 8328678	9. 6190017	92. 52519	1123596
891	793881	707347971	29. 8496231	9. 6226030	92. 59449	1122334
892	795664	709732288	29. 8663690	9. 6262016	92. 66376	1121076
893	797449	712121957	29. 8831056	9. 6297975	92. 73300	1119821
894	799236	714516984	29. 8998328	9. 6333907	92. 80222	1118568
895	801025	716917375	29. 9165506	9. 6369812	92. 87141	1117318
896	802816	719323136	29. 9332591	9. 6405690	92. 94057	1116071
897	804609	721734273	29. 9499583	9. 6441542	93. 00971	1114827
898	806404	724150792	29. 9666481	9. 6477367	93. 07882	1113586
899	808201	726572699	29. 9833287	9. 6513166	93. 14791	1112347
900	810000	729000000	30. 0000000	9. 6548938	93. 21698	1111111

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TABLE I.

No.	Square	Cube	Square Root	Cube Root	Cube Root of the Square	Reciprocal 0.00
951	904401	860085351	30. 8382879	9. 8339238	96. 70606	1051525
952	906304	862801408	30. 8544972	9. 8373695	96. 77384	1050420
953	908209	865523177	30. 8706981	9. 8408127	96. 84159	1049318
954	910116	868250664	30. 8868904	9. 8442536	96. 90933	1048218
955	912025	870983875	30. 9030743	9. 8476920	96. 97704	1047120
956	913936	873722816	30. 9192497	9. 8511280	97. 04472	1046025
957	915849	876467493	30. 9354166	9. 8545617	97. 11239	1044932
958	917764	879217912	30. 9515751	9. 8579929	97. 18002	1043841
959	919681	881974079	30. 9677251	9. 8614218	97. 24764	1042753
960	921600	884736000	30. 9838668	9. 8648483	97. 31523	1041667
961	923521	887503681	31. 0000000	9. 8682724	97. 38280	1040583
962	925444	890277128	31. 0161248	9. 8716941	97. 45035	1039501
963	927369	893056347	31. 0322413	9. 8751135	97. 51787	1038422
964	929296	895841344	31. 0483494	9. 8785305	97. 58536	1037344
965	931225	898632125	31. 0644491	9. 8819451	97. 65284	1036269
966	933156	901428696	31. 0805405	9. 8853574	97. 72029	1035197
967	935089	904231063	31. 0966236	9. 8887673	97. 78772	1034126
968	937024	907039232	31. 1126984	9. 8921749	97. 85512	1033058
969	938961	909853209	31. 1287648	9. 8955801	97. 92251	1031992
970	940900	912673000	31. 1448230	9. 8989830	97. 98987	1030928
971	942841	915498611	31. 1608729	9. 9023835	98. 05720	1029866
972	944784	918330048	31. 1769145	9. 9057817	98. 12451	1028807
973	946729	921167317	31. 1929479	9. 9091776	98. 19180	1027749
974	948676	924010424	31. 2089731	9. 9125712	98. 25907	1026694
975	950625	926859375	31. 2249900	9. 9159624	98. 32631	1025641
976	952576	929714176	31. 2409987	9. 9193513	98. 39354	1024590
977	954529	932574833	31. 2569992	9. 9227379	98. 46073	1023541
978	956484	935441352	31. 2729915	9. 9261222	98. 52790	1022495
979	958441	938313739	31. 2889757	9. 9295042	98. 59505	1021450
980	960400	941192000	31. 3049517	9. 9328839	98. 66218	1020408
981	962361	944076141	31. 3209195	9. 9362613	98. 72929	1019368
982	964324	946966168	31. 3368792	9. 9396363	98. 79637	1018330
983	966289	949862087	31. 3528308	9. 9430092	98. 86343	1017294
984	968256	952763904	31. 3687743	9. 9463797	98. 93047	1016260
985	970225	955671625	31. 3847097	9. 9497479	98. 99748	1015228
986	972196	958585256	31. 4006369	9. 9531138	99. 06448	1014199
987	974169	961504803	31. 4165561	9. 9564775	99. 13144	1013171
988	976144	964430272	31. 4324673	9. 9598389	99. 19839	1012146
989	978121	967361669	31. 4483704	9. 9631981	99. 26532	1011122
990	980100	970299000	31. 4642654	9. 9665549	99. 33222	1010101
991	982081	973242271	31. 4801525	9. 9699095	99. 39910	1009082
992	984064	976191488	31. 4960315	9. 9732619	99. 46595	1008065
993	986049	979146657	31. 5119025	9. 9766120	99. 53279	1007049
994	988036	982107784	31. 5277655	9. 9799599	99. 59960	1006036
995	990025	985074875	31. 5436206	9. 9833055	99. 66639	1005025
996	992016	988047936	31. 5594677	9. 9866488	99. 73316	1004016
997	994009	991026973	31. 5753068	9. 9899900	99. 79990	1003009
998	996004	994011992	31. 5911380	9. 9933289	99. 86662	1002004
999	998001	997002999	31. 6069613	9. 9966656	99. 93332	1001001
1000	1000000	10000000000	31. 6227766	10. 0000000	100. 00000	1000000

COMMON LOGARITHMS  
OF  
NUMBERS

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FROM 100 TO 999

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
100	00 000	043	087	130	173	217	260	303	346	389	
101	432	475	518	561	604	647	689	732	775	817	
102	860	903	945	988	*030	*072	*115	*157	*199	*242	44 43 42
103	01 284	326	368	410	452	494	536	578	620	662	1 4.4 4.3 4.2 2 8.8 8.6 8.4
104	793	745	787	828	870	912	953	995	*036	*078	3 13.2 12.0 12.6
105	02 119	160	202	243	284	325	366	407	449	490	4 17.6 17.2 16.8 5 22.0 21.5 21.0
106	531	572	612	653	694	735	776	816	857	898	6 26.4 25.8 25.2
107	938	979	*019	*060	*100	*141	*181	*222	*262	*302	7 30.8 30.1 29.4
108	03 342	383	423	463	503	543	583	623	663	703	8 35.2 34.4 33.6 9 39.6 38.7 37.8
109	743	782	822	862	902	941	981	*021	*060	*100	
110	04 139	170	218	258	297	336	376	415	454	493	
111	532	571	610	650	689	727	766	805	844	883	
112	922	961	999	*038	*077	*115	*154	*192	*231	*269	41 40 39
113	05 308	346	385	423	461	500	538	576	614	652	1 4.1 4.0 3.9 2 8.2 8.0 7.8
114	690	729	767	805	843	881	918	956	994	*032	3 12.3 12.0 11.7 4 16.4 16.0 15.6
115	06 070	108	145	183	221	258	296	333	371	408	5 20.5 20.0 19.5
116	446	483	521	558	595	633	670	707	744	781	6 24.6 24.0 23.4
117	819	856	893	930	967	*004	*041	*078	*115	*151	7 28.7 28.0 27.3 8 32.8 32.0 31.2
118	07 188	225	262	298	335	372	408	445	482	518	9 30.0 30.0 35.1
119	555	591	628	664	700	737	773	809	846	882	
120	918	954	990	*027	*063	*099	*135	*171	*207	*243	
121	08 279	314	350	386	422	458	493	529	565	600	38 37 36
122	636	672	707	743	778	814	849	884	920	955	1 3.8 3.7 3.6 2 7.6 7.4 7.2
123	991	*026	*061	*096	*132	*167	*202	*237	*272	*307	3 11.4 11.1 10.8 4 15.2 14.8 14.4
124	09 342	377	412	447	482	517	552	587	621	656	5 19.0 18.5 18.0
125	691	726	760	795	830	864	899	934	968	*003	6 22.8 22.2 21.6 7 26.6 25.0 25.2
126	10 037	072	106	140	175	209	243	278	312	346	
127	380	415	449	483	517	551	585	619	653	687	8 30.4 29.6 28.8
128	721	755	789	823	857	890	924	958	992	*025	9 34.2 33.3 32.4
129	11 059	093	126	160	193	227	261	294	327	361	
130	394	428	461	494	528	561	594	628	661	694	
131	727	760	793	826	860	893	926	959	992	*024	35 34 33
132	12 057	090	123	156	189	222	254	287	320	352	1 3.5 3.4 3.3 2 7.0 6.8 6.6
133	385	418	450	483	516	548	581	613	646	678	3 10.5 10.2 9.9
134	710	743	775	808	840	872	905	937	969	*001	4 14.0 13.6 13.2
135	13 033	066	098	130	162	194	226	258	290	322	5 17.5 17.0 16.5 6 21.0 20.4 19.8
136	354	386	418	450	481	513	545	577	609	640	7 24.5 23.8 23.1
137	672	704	735	767	799	830	862	893	925	956	8 28.0 27.2 26.4
138	988	*019	*051	*082	*114	*145	*176	*208	*239	*270	9 31.5 30.6 29.7
139	14 301	333	364	395	426	457	489	520	551	582	
140	613	644	675	706	737	768	799	829	860	891	
141	922	953	983	*014	*045	*076	*106	*137	*168	*198	32 31 30
142	15 229	250	290	320	351	381	412	442	473	503	1 3.2 3.1 3.0 2 6.4 6.2 6.0
143	534	564	594	625	655	685	715	746	776	806	3 9.6 9.3 9.0
144	836	866	897	927	957	987	*017	*047	*077	*107	4 12.8 12.4 12.0
145	16 137	167	197	227	256	286	316	346	376	406	5 16.0 15.5 15.0 6 19.2 18.6 18.0
146	435	465	495	524	554	584	613	643	673	702	7 22.4 21.7 21.0 8 25.6 24.8 24.0
147	732	761	791	820	850	870	900	930	967	997	9 28.8 27.9 27.0
148	17 026	056	085	114	143	173	202	231	260	289	
149	319	348	377	406	435	464	493	522	551	580	

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
150	17	609	638	667	696	725	754	782	811	840	869
151		898	926	955	984	*013	*041	*070	*099	*127	*156
152	18	184	213	241	270	298	327	355	384	412	441
153		469	498	526	554	583	611	639	667	696	724
154		752	780	808	837	865	893	921	949	977	*005
155	19	033	061	089	117	145	173	201	229	257	285
156		312	340	368	396	424	451	479	507	535	562
157		590	618	645	673	700	728	756	783	811	838
158		866	893	921	948	976	*003	*030	*058	*085	*112
159	20	140	167	194	222	249	276	303	330	358	385
160		412	439	466	493	520	548	575	602	629	656
161		683	710	737	763	790	817	844	871	898	925
162		952	978	*005	*032	*059	*085	*112	*139	*165	*192
163	21	219	245	272	299	325	352	378	405	431	458
164		484	511	537	564	590	617	643	669	696	722
165		748	775	801	827	854	880	906	932	958	985
166	22	011	037	063	089	115	141	167	194	220	246
167		272	298	324	350	376	401	427	453	479	505
168		531	557	583	608	634	660	686	712	737	763
169		789	814	840	866	891	917	943	968	994	*019
170	23	045	070	096	121	147	172	198	223	249	274
171		300	325	350	376	401	426	452	477	502	528
172		553	578	603	629	654	679	704	729	754	779
173		805	830	855	880	905	930	955	980	*005	*030
174	24	055	080	105	130	155	180	204	229	254	279
175		304	329	353	378	403	428	452	477	502	527
176		551	576	601	625	650	674	699	724	748	773
177		797	822	846	871	895	920	944	969	993	*018
178	25	042	066	091	115	139	164	188	212	237	261
179		285	310	334	358	382	406	431	455	479	503
180		527	551	575	600	624	648	672	696	720	744
181		768	792	816	840	864	888	912	935	959	983
182	26	007	031	055	079	102	126	150	174	198	221
183		245	269	293	316	340	364	387	411	435	458
184		482	505	529	553	576	600	623	647	670	694
185		717	741	764	788	811	834	858	881	905	928
186		951	975	998	*021	*045	*068	*091	*114	*138	*161
187	27	184	207	231	254	277	300	323	346	370	393
188		416	439	462	485	508	531	554	577	600	623
189		646	669	692	715	738	761	784	807	830	852
190		875	898	921	944	967	989	*012	*035	*058	*081
191	28	103	126	149	171	194	217	240	262	285	307
192		330	353	375	398	421	443	466	488	511	533
193		556	578	601	623	646	668	691	713	735	758
194		780	803	825	847	870	892	914	937	959	981
195	29	003	026	048	070	092	115	137	159	181	203
196		226	248	270	292	314	336	358	380	403	425
197		447	469	491	513	535	557	579	601	623	645
198		667	688	710	732	754	776	798	820	842	863
199		885	907	929	951	973	994	*016	*038	*060	*081

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
200	30	103	125	146	168	190	211	233	255	276	298
201	320	341	363	384	406	428	449	471	492	514	
202	535	557	578	600	621	643	664	685	707	728	
203	750	771	792	814	835	856	878	899	920	942	1 2.2 2.1
204	963	984	*006	*027	*048	*069	*091	*112	*133	*154	2 4.4 4.2
205	31	175	197	218	239	260	281	302	323	345	366 4 8.8 8.4
206	387	408	429	450	471	492	513	534	555	576	5 11.0 10.5
207	597	618	639	660	681	702	723	744	765	785	6 13.2 12.6
208	806	827	848	869	890	911	931	952	973	994	7 15.4 14.7
209	32	015	035	056	077	098	118	139	160	181	201 8 17.6 16.8
210	222	243	263	284	305	325	346	366	387	408	9 19.8 18.9
211	428	449	469	490	510	531	552	572	593	613	
212	634	654	675	695	715	736	756	777	797	818	1 20
213	838	858	879	899	919	940	960	980	*001	*021	2 4.0
214	33	041	062	082	102	122	143	163	183	203	3 6.0
215	244	264	284	304	325	345	365	385	405	425	4 8.0
216	445	465	486	506	526	546	566	586	606	626	5 10.0
217	646	666	686	706	726	746	766	786	806	826	6 12.0
218	846	866	885	905	925	945	965	985	*005	*025	7 14.0
219	34	044	064	084	104	124	143	163	183	203	8 16.0
220	242	262	282	301	321	341	361	380	400	420	9 18.0
221	439	459	479	498	518	537	557	577	596	616	
222	635	655	674	694	713	733	753	772	792	811	1 1.0
223	830	850	869	889	908	928	947	967	986	*005	2 3.8
224	35	025	044	064	083	102	122	141	160	180	199 3 5.7
225	218	238	257	276	295	315	334	353	372	392	4 7.6
226	411	430	449	468	488	507	526	545	564	583	5 9.5
227	603	622	641	660	679	698	717	736	755	774	6 11.4
228	793	813	832	851	870	889	908	927	946	965	7 13.3
229	984	*003	*021	*040	*059	*078	*097	*116	*135	*154	8 15.2
230	36	173	192	211	229	248	267	286	305	324	342 9 17.1
231	361	380	399	418	436	455	474	493	511	530	550 10 18
232	549	568	586	605	624	642	661	680	698	717	736 11 1.8
233	736	754	773	791	810	829	847	866	884	903	922 12 3.6
234	922	940	959	977	996	*014	*033	*051	*070	*088	4 5.4
235	37	107	125	144	162	181	199	218	236	254	273 5 7.2
236	291	310	328	346	365	383	401	420	438	457	476 6 9.0
237	475	493	511	530	548	566	585	603	621	639	658 7 10.8
238	658	676	694	712	731	749	767	785	803	822	8 12.6
239	840	858	876	894	912	931	949	967	985	*003	9 14.4
240	38	021	039	057	075	093	112	130	148	166	184 10 16.2
241	202	220	238	256	274	292	310	328	346	364	
242	382	399	417	435	453	471	489	507	525	543	1 1.7
243	561	578	596	614	632	650	668	686	703	721	2 3.4
244	739	757	775	792	810	828	846	863	881	899	3 5.1
245	917	934	952	970	987	*005	*023	*041	*058	*076	4 6.8
246	39	094	111	129	146	164	182	199	217	235	252 5 8.5
247	270	287	305	322	340	358	375	393	410	428	6 10.2
248	445	463	480	498	515	533	550	568	585	602	7 11.9
249	620	637	655	672	690	707	724	742	759	777	8 13.6

TABLE II — LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
250	39	794	811	829	846	863	881	898	915	933	950
251	967	985	*002	*019	*037	*054	*071	*088	*106	*123	
252	40	140	157	175	192	209	226	243	261	278	295
253	312	329	346	364	381	398	415	432	449	466	1 1.8
254	483	500	518	535	552	569	586	603	620	637	2 3.6
255	654	671	688	705	722	739	756	773	790	807	3 5.4
256	824	841	858	875	892	909	926	943	960	976	4 7.2
257	993	*010	*027	*044	*061	*078	*095	*111	*128	*145	5 9.0
258	41	162	179	196	212	229	246	263	280	296	313
259	330	347	363	380	397	414	430	447	464	481	6 10.8
260	497	514	531	547	564	581	597	614	631	647	7 12.6
261	664	681	697	714	731	747	764	780	797	814	8 14.4
262	830	847	863	880	896	913	929	946	963	979	9 16.2
263	996	*012	*029	*045	*062	*078	*095	*111	*127	*144	1 1.7
264	42	160	177	193	210	226	243	259	275	292	308
265	325	341	357	374	390	406	423	439	455	472	4 6.8
266	488	504	521	537	553	570	586	602	619	635	5 8.5
267	651	667	684	700	716	732	749	765	781	797	6 10.2
268	813	830	846	862	878	894	911	927	943	959	7 11.9
269	975	991	*008	*024	*040	*056	*072	*088	*104	*120	8 13.6
270	43	136	152	169	185	201	217	233	249	265	281
271	297	313	329	345	361	377	393	409	425	441	1 1.6
272	457	473	489	505	521	537	553	569	584	600	2 3.2
273	616	632	648	664	680	696	712	727	743	759	3 4.8
274	775	791	807	823	838	854	870	886	902	917	4 6.4
275	933	949	965	981	996	*012	*028	*044	*059	*075	5 8.0
276	44	091	107	122	138	154	170	185	201	217	232
277	248	264	279	295	311	326	342	358	373	389	6 9.6
278	404	420	436	451	467	483	498	514	529	545	7 11.2
279	560	576	592	607	623	638	654	669	685	700	8 12.8
280	716	731	747	762	778	793	809	824	840	855	9 14.4
281	871	886	902	917	932	948	963	979	994	*010	1 1.5
282	45	025	040	056	071	086	102	117	133	148	163
283	179	194	209	225	240	255	271	286	301	317	2 3.0
284	332	347	362	378	393	408	423	439	454	469	3 4.5
285	484	500	515	530	545	561	576	591	606	621	4 6.0
286	637	652	667	682	697	712	728	743	758	773	5 7.5
287	788	803	818	834	849	864	879	894	909	924	6 9.0
288	939	954	969	984	*000	*015	*030	*045	*060	*075	7 10.5
289	46	090	105	120	135	150	165	180	195	210	225
290	240	255	270	285	300	315	330	345	359	374	8 12.0
291	389	404	419	434	449	464	479	494	509	523	9 13.5
292	538	553	568	583	598	613	627	642	657	672	1 1.4
293	687	702	716	731	746	761	776	790	805	820	2 2.8
294	835	850	864	879	894	909	923	938	953	967	3 4.2
295	982	997	*012	*026	*041	*056	*070	*085	*100	*114	4 5.6
296	47	129	144	159	173	188	202	217	232	246	261
297	276	290	305	319	334	349	363	378	392	407	5 7.0
298	422	436	451	465	480	494	509	524	538	553	6 8.4
299	567	582	596	611	625	640	654	669	683	698	7 9.8

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
300	47	712	727	741	756	770	784	799	813	828	842
301		857	871	885	900	914	929	943	958	972	986
302	48	001	015	029	044	058	073	087	101	116	130
303		144	159	173	187	202	216	230	244	259	273
304		287	302	316	330	344	359	373	387	401	416
305		430	444	458	473	487	501	515	530	544	558
306		572	586	601	615	629	643	657	671	686	700
307		714	728	742	756	770	785	799	813	827	841
308		855	869	883	897	911	926	940	954	968	982
309		996	*010	*024	*038	*052	*066	*080	*094	*108	*122
310	49	136	150	164	178	192	206	220	234	248	262
311		276	290	304	318	332	346	360	374	388	402
312		415	429	443	457	471	485	499	513	527	541
313		554	568	582	596	610	624	638	651	665	679
314		693	707	721	734	748	762	776	790	803	817
315		831	845	859	872	886	900	914	927	941	955
316		969	982	996	*010	*024	*037	*051	*065	*079	*092
317	50	106	120	133	147	161	174	188	202	215	229
318		243	256	270	284	297	311	325	338	352	365
319		379	393	406	420	433	447	461	474	488	501
320		515	529	542	556	569	583	596	610	623	637
321		651	664	678	691	705	718	732	745	759	772
322		786	799	813	826	840	853	866	880	893	907
323		920	934	947	961	974	987	*001	*014	*028	*041
324	51	055	068	081	095	108	121	135	148	162	175
325		188	202	215	228	242	255	268	282	295	308
326		322	335	348	362	375	388	402	415	428	441
327		455	468	481	495	508	521	534	548	561	574
328		587	601	614	627	640	654	667	680	693	706
329		720	733	746	759	772	786	799	812	825	838
330		851	865	878	891	904	917	930	943	957	970
331		983	996	*009	*022	*035	*048	*061	*075	*088	*101
332	52	114	127	140	153	166	179	192	205	218	231
333		244	257	270	284	297	310	323	336	349	362
334		375	388	401	414	427	440	453	466	479	492
335		504	517	530	543	556	569	582	595	608	621
336		634	647	660	673	686	699	711	724	737	750
337		763	776	789	802	815	827	840	853	866	879
338		892	905	917	930	943	956	969	982	994	*007
339	53	020	033	046	058	071	084	097	110	122	135
340		148	161	173	186	199	212	224	237	250	263
341		275	288	301	314	326	339	352	364	377	390
342		403	415	428	441	453	466	479	491	504	517
343		529	542	555	567	580	593	605	618	631	643
344		656	668	681	694	706	719	732	744	757	769
345		782	794	807	820	832	845	857	870	882	895
346		908	920	933	945	958	970	983	995	*008	*020
347	54	033	045	058	070	083	095	108	120	133	145
348		158	170	183	195	208	220	233	245	258	270
349		283	295	307	320	332	345	357	370	382	394

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TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
400	60	206	217	228	239	249	260	271	282	293	304
401	314	325	336	347	358	369	379	390	401	412	
402	423	433	444	455	466	477	487	498	509	520	
403	531	541	552	563	574	584	595	606	617	627	
404	638	649	660	670	681	692	703	713	724	735	
405	746	756	767	778	788	799	810	821	831	842	
406	853	863	874	885	895	906	917	927	938	949	
407	959	970	981	991	*002	*013	*023	*034	*045	*055	
408	61	066	077	087	098	109	119	130	140	151	162
409	172	183	194	204	215	225	236	247	257	268	1 1.1
410	278	289	300	310	321	331	342	352	363	374	2 2.2
411	384	395	405	416	426	437	448	458	469	479	3 3.3
412	490	500	511	521	532	542	553	563	574	584	4 4.4
413	595	606	616	627	637	648	658	669	679	690	5 5.5
414	700	711	721	731	742	752	763	773	784	794	6 6.6
415	805	815	826	836	847	857	868	878	888	899	7 7.7
416	909	920	930	941	951	962	972	982	993	*003	8 8.8
417	62	014	024	034	045	055	066	076	086	097	107
418	118	128	138	149	159	170	180	190	201	211	
419	221	232	242	252	263	273	284	294	304	315	
420	325	335	346	356	366	377	387	397	408	418	
421	428	439	449	459	469	480	490	500	511	521	10 1.0
422	531	542	552	562	572	583	593	603	613	624	2 2.0
423	634	644	655	665	675	685	696	706	716	726	3 3.0
424	737	747	757	767	778	788	798	808	818	829	4 4.0
425	839	849	859	870	880	890	900	910	921	931	5 5.0
426	941	951	961	972	982	992	*002	*012	*022	*033	6 6.0
427	63	043	053	063	073	083	094	104	114	124	134
428	144	155	165	175	185	195	205	215	225	236	8 8.0
429	246	256	266	276	286	296	306	317	327	337	9 9.0
430	347	357	367	377	387	397	407	417	428	438	
431	448	458	468	478	488	498	508	518	528	538	
432	548	558	568	579	589	599	609	619	629	639	
433	649	659	669	679	689	699	709	719	729	739	
434	749	759	769	779	789	799	809	819	829	839	
435	849	859	869	879	889	899	909	919	929	939	
436	949	959	969	979	988	998	*008	*018	*028	*038	9 0.9
437	64	048	058	068	078	088	098	108	118	128	137
438	147	157	167	177	187	197	207	217	227	237	2 1.8
439	246	256	266	276	286	296	306	316	326	335	3 2.7
440	345	355	365	375	385	395	404	414	424	434	4 3.6
441	444	454	464	473	483	493	503	513	523	532	5 4.5
442	542	552	562	572	582	591	601	611	621	631	6 5.4
443	640	650	660	670	680	689	699	709	719	729	6 6.3
444	738	748	758	768	777	787	797	807	816	826	
445	836	846	856	865	875	885	895	904	914	924	
446	933	943	953	963	972	982	992	*002	*011	*021	
447	65	031	040	050	060	070	079	089	099	108	118
448	128	137	147	157	167	176	186	196	205	215	
449	225	234	244	254	263	273	283	292	302	312	

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
450	65	321	331	341	350	360	369	379	389	398	408
451		418	427	437	447	456	466	475	485	495	504
452		514	523	533	543	552	562	571	581	591	600
453		610	619	629	639	648	658	667	677	686	696
454		706	715	725	734	744	753	763	772	782	792
455		801	811	820	830	839	849	858	868	877	887
456		896	906	916	925	935	944	954	963	973	982
457		992	*001	*011	*020	*030	*039	*049	*058	*068	*077
458	66	087	096	106	115	124	134	143	153	162	172
459		181	191	200	210	219	229	238	247	257	266
460		276	285	295	304	314	323	332	342	351	361
461		370	380	389	398	408	417	427	436	445	455
462		464	474	483	492	502	511	521	530	539	549
463		558	567	577	586	596	605	614	624	633	642
464		652	661	671	680	689	699	708	717	727	736
465		745	755	764	773	783	792	801	811	820	829
466		839	848	857	867	876	885	894	904	913	922
467		932	941	950	960	969	978	987	997	*006	*015
468	67	025	034	043	052	062	071	080	089	099	108
469		117	127	136	145	154	164	173	182	191	201
470		210	219	228	237	247	256	265	274	284	293
471		302	311	321	330	339	348	357	367	376	385
472		394	403	413	422	431	440	449	459	468	477
473		486	495	504	514	523	532	541	550	560	569
474		578	587	596	605	614	624	633	642	651	660
475		669	679	688	697	706	715	724	733	742	752
476		761	770	779	788	797	806	815	825	834	843
477		852	861	870	879	888	897	906	916	925	934
478		943	952	961	970	979	988	997	*006	*015	*024
479	68	034	043	052	061	070	079	088	097	106	115
480		124	133	142	151	160	169	178	187	196	205
481		215	224	233	242	251	260	269	278	287	296
482		305	314	323	332	341	350	359	368	377	386
483		395	404	413	422	431	440	449	458	467	476
484		485	494	502	511	520	529	538	547	556	565
485		574	583	592	601	610	619	628	637	646	655
486		664	673	681	690	699	708	717	726	735	744
487		753	762	771	780	789	797	806	815	824	833
488		842	851	860	869	878	886	895	904	913	922
489		931	940	949	958	966	975	984	993	*002	*011
490	69	020	028	037	046	055	064	073	082	090	099
491		108	117	126	135	144	152	161	170	179	188
492		197	205	214	223	232	241	249	258	267	276
493		285	294	302	311	320	329	338	346	355	364
494		373	381	390	399	408	417	425	434	443	452
495		461	469	478	487	496	504	513	522	531	539
496		548	557	566	574	583	592	601	609	618	627
497		636	644	653	662	671	679	688	697	705	714
498		723	732	740	749	758	767	775	784	793	801
499		810	819	827	836	845	854	862	871	880	888

TABLE II — LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
500	69	897	906	914	923	932	940	949	958	966	975
501		984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70	070	079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234	
504	243	252	260	269	278	286	295	303	312	321	
505	329	338	346	355	364	372	381	389	398	406	
506	415	424	432	441	449	458	467	475	484	492	
507	501	509	518	526	535	544	552	561	569	578	
508	586	595	603	612	621	629	638	646	655	663	9
509	672	680	689	697	706	714	723	731	740	749	1 0.9
510	757	766	774	783	791	800	808	817	825	834	2 1.8
511	842	851	859	868	876	885	893	902	910	919	3 2
512	927	935	944	952	961	969	978	986	995	*003	4 3.6
513	71	012	020	029	037	046	054	063	071	079	088
514	096	105	113	122	130	139	147	155	164	172	5 4.5
515	181	189	198	206	214	223	231	240	248	257	6 5.4
516	265	273	282	290	299	307	315	324	332	341	7 6.3
517	349	357	366	374	383	391	399	408	416	425	8 7.2
518	433	441	450	458	466	475	483	492	500	508	9 8.1
519	517	525	533	542	550	559	567	575	584	592	
520	600	609	617	625	634	642	650	659	667	675	
521	684	692	700	709	717	725	734	742	750	759	
522	767	775	784	792	800	809	817	825	834	842	1 0.8
523	850	858	867	875	883	892	900	908	917	925	2 1.6
524	933	941	950	958	966	975	983	991	999	*008	3 4
525	72	016	024	032	041	049	057	066	074	082	090
526	099	107	115	123	132	140	148	156	165	173	5 4.0
527	181	189	198	206	214	222	230	239	247	255	6 8
528	263	272	280	288	296	304	313	321	329	337	7 6
529	346	354	362	370	378	387	395	403	411	419	8 4
530	428	436	444	452	460	469	477	485	493	501	9 2
531	509	518	526	534	542	550	558	567	575	583	
532	591	599	607	616	624	632	640	648	656	665	
533	673	681	689	697	705	713	722	730	738	746	
534	754	762	770	779	787	795	803	811	819	827	
535	835	843	852	860	868	876	884	892	900	908	
536	916	925	933	941	949	957	965	973	981	989	7 0.7
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070	2 1.4
538	73	078	086	094	102	111	119	127	135	143	151
539	159	167	175	183	191	199	207	215	223	231	5 5
540	239	247	255	263	272	280	288	296	304	312	6 2
541	320	328	336	344	352	360	368	376	384	392	7 9
542	400	408	416	424	432	440	448	456	464	472	8 6
543	480	488	496	504	512	520	528	536	544	552	9 3
544	560	568	576	584	592	600	608	616	624	632	
545	640	648	656	664	672	679	687	695	703	711	
546	719	727	735	743	751	759	767	775	783	791	
547	799	807	815	823	830	838	846	854	862	870	
548	878	886	894	902	910	918	926	933	941	949	
549	957	965	973	981	989	997	*005	*013	*020	*028	

TABLE II — LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
550	74	036	044	052	060	068	076	084	092	099	107
551		115	123	131	139	147	155	162	170	178	186
552		194	202	210	218	225	233	241	249	257	265
553		273	280	288	296	304	312	320	327	335	343
554		351	359	367	374	382	390	398	406	414	421
555		429	437	445	453	461	468	476	484	492	500
556		507	515	523	531	539	547	554	562	570	578
557		586	593	601	609	617	624	632	640	648	656
558		663	671	679	687	695	702	710	718	726	733
559		741	749	757	764	772	780	788	796	803	811
560		819	827	834	842	850	858	865	873	881	889
561		896	904	912	920	927	935	943	950	958	966
562		974	981	989	997	*005	*012	*020	*028	*035	*043
563	75	051	059	066	074	082	089	097	105	113	120
564		128	136	143	151	159	166	174	182	189	197
565		205	213	220	228	236	243	251	259	266	274
566		282	289	297	305	312	320	328	335	343	351
567		358	366	374	381	389	397	404	412	420	427
568		435	442	450	458	465	473	481	488	496	504
569		511	519	526	534	542	549	557	565	572	580
570		587	595	603	610	618	626	633	641	648	656
571		664	671	679	686	694	702	709	717	724	732
572		740	747	755	762	770	778	785	793	800	808
573		815	823	831	838	846	853	861	868	876	884
574		891	899	906	914	921	929	937	944	952	959
575		967	974	982	989	997	*005	*012	*020	*027	*035
576	76	042	050	057	065	072	080	087	095	103	110
577		118	125	133	140	148	155	163	170	178	185
578		193	200	208	215	223	230	238	245	253	260
579		268	275	283	290	298	305	313	320	328	335
580		343	350	358	365	373	380	388	395	403	410
581		418	425	433	440	448	455	462	470	477	485
582		492	500	507	515	522	530	537	545	552	559
583		567	574	582	589	597	604	612	619	626	634
584		641	649	656	664	671	678	686	693	701	708
585		716	723	730	738	745	753	760	768	775	782
586		790	797	805	812	819	827	834	842	849	856
587		864	871	879	886	893	901	908	916	923	930
588		938	945	953	960	967	975	982	989	997	*004
589	77	012	019	026	034	041	048	056	063	070	078
590		085	093	100	107	115	122	129	137	144	151
591		159	166	173	181	188	195	203	210	217	225
592		232	240	247	254	262	269	276	283	291	298
593		305	313	320	327	335	342	349	357	364	371
594		379	386	393	401	408	415	422	430	437	444
595		452	459	466	474	481	488	495	503	510	517
596		525	532	539	546	554	561	568	576	583	590
597		597	605	612	619	627	634	641	648	656	663
598		670	677	685	692	699	706	714	721	728	735
599		743	750	757	764	772	779	786	793	801	808

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
500	69	897	906	914	923	932	940	949	958	966	975
501		984	992	*001	*010	*018	*027	*036	*044	*053	*062
502	70	070	079	088	096	105	114	122	131	140	148
503	157	165	174	183	191	200	209	217	226	234	
504	243	252	260	269	278	286	295	303	312	321	
505	329	338	346	355	364	372	381	389	398	406	
506	415	424	432	441	449	458	467	475	484	492	
507	501	509	518	526	535	544	552	561	569	578	
508	586	595	603	612	621	629	638	646	655	663	
509	672	680	689	697	706	714	723	731	740	749	
510	757	766	774	783	791	800	808	817	825	834	
511	842	851	859	868	876	885	893	902	910	919	
512	927	935	944	952	961	969	978	986	995	*003	0
513	71 012	020	029	037	046	054	063	071	079	088	0.9
514	096	105	113	122	130	139	147	155	164	172	1.8
515	181	189	198	206	214	223	231	240	248	257	2.7
516	265	273	282	290	299	307	315	324	332	341	3.6
517	349	357	366	374	383	391	399	408	416	425	4.5
518	433	441	450	458	466	475	483	492	500	508	5.4
519	517	525	533	542	550	559	567	575	584	592	6.3
520	600	609	617	625	634	642	650	659	667	675	7.2
521	684	692	700	709	717	725	734	742	750	759	8.1
522	767	775	784	792	800	809	817	825	834	842	8
523	850	858	867	875	883	892	900	908	917	925	0.8
524	933	941	950	958	966	975	983	991	999	*008	1.6
525	72 016	024	032	041	049	057	066	074	082	090	2.4
526	099	107	115	123	132	140	148	156	165	173	3.2
527	181	189	198	206	214	222	230	239	247	255	4.0
528	263	272	280	288	296	304	313	321	329	337	4.8
529	346	354	362	370	378	387	395	403	411	419	5.6
530	428	436	444	452	460	469	477	485	493	501	6.4
531	509	518	526	534	542	550	558	567	575	583	7.2
532	591	599	607	616	624	632	640	648	656	665	
533	673	681	689	697	705	713	722	730	738	746	
534	754	762	770	779	787	795	803	811	819	827	
535	835	843	852	860	868	876	884	892	900	908	
536	916	925	933	941	949	957	965	973	981	989	0.7
537	997	*006	*014	*022	*030	*038	*046	*054	*062	*070	1.4
538	73 078	086	094	102	111	119	127	135	143	151	2.1
539	159	167	175	183	191	199	207	215	223	231	2.8
540	239	247	255	263	272	280	288	296	304	312	3.5
541	320	328	336	344	352	360	368	376	384	392	4.2
542	400	408	416	424	432	440	448	456	464	472	5.0
543	480	488	496	504	512	520	528	536	544	552	5.6
544	560	568	576	584	592	600	608	616	624	632	6.3
545	640	648	656	664	672	679	687	695	703	711	
546	719	727	735	743	751	759	767	775	783	791	
547	799	807	815	823	830	838	846	854	862	870	
548	878	886	894	902	910	918	926	933	941	949	
549	957	965	973	981	989	997	*005	*013	*020	*028	

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
550	74 036	044	052	060	068	076	084	092	099	107	
551	115	123	131	139	147	155	162	170	178	186	
552	194	202	210	218	225	233	241	249	257	265	
553	273	280	288	296	304	312	320	327	335	343	
554	351	359	367	374	382	390	398	406	414	421	
555	429	437	445	453	461	468	476	484	492	500	
556	507	515	523	531	539	547	554	562	570	578	
557	586	593	601	609	617	624	632	640	648	656	
558	663	671	679	687	695	702	710	718	726	733	
559	741	749	757	764	772	780	788	796	803	811	
560	819	827	834	842	850	858	865	873	881	889	
561	896	904	912	920	927	935	943	950	958	966	
562	974	981	989	997	*005	*012	*020	*028	*035	*043	
563	75 051	059	066	074	082	089	097	105	113	120	8
564	128	136	143	151	159	166	174	182	189	197	1 0.8
565	205	213	220	228	236	243	251	259	266	274	2 1.6
566	282	289	297	305	312	320	328	335	343	351	3 2.4
567	358	366	374	381	389	397	404	412	420	427	4 3.2
568	435	442	450	458	465	473	481	488	496	504	5 4.0
569	511	519	526	534	542	549	557	565	572	580	6 4.8
570	587	595	603	610	618	626	633	641	648	656	7 5.6
571	664	671	679	686	694	702	709	717	724	732	8 6.4
572	740	747	755	762	770	778	785	793	800	808	9 7.2
573	815	823	831	838	846	853	861	868	876	884	
574	891	899	906	914	921	929	937	944	952	959	
575	967	974	982	989	997	*005	*012	*020	*027	*035	
576	76 042	050	057	065	072	080	087	095	103	110	
577	118	125	133	140	148	155	163	170	178	185	
578	193	200	208	215	223	230	238	245	253	260	
579	268	275	283	290	298	305	313	320	328	335	
580	343	350	358	365	373	380	388	395	403	410	
581	418	425	433	440	448	455	462	470	477	485	1 0.7
582	492	500	507	515	522	530	537	545	552	559	2 1.4
583	567	574	582	589	597	604	612	619	626	634	3 2.1
584	641	649	656	664	671	678	686	693	701	708	4 2.8
585	716	723	730	738	745	753	760	768	775	782	5 3.5
586	790	797	805	812	819	827	834	842	849	856	6 4.2
587	864	871	879	886	893	901	908	916	923	930	7 4.9
588	938	945	953	960	967	975	982	989	997	*004	8 5.6
589	77 012	019	026	034	041	048	056	063	070	078	9 6.3
590	085	093	100	107	115	122	129	137	144	151	
591	159	166	173	181	188	195	203	210	217	225	
592	232	240	247	254	262	269	276	283	291	298	
593	305	313	320	327	335	342	349	357	364	371	
594	379	386	393	401	408	415	422	430	437	444	
595	452	459	466	474	481	488	495	503	510	517	
596	525	532	539	546	554	561	568	576	583	590	
597	597	605	612	619	627	634	641	648	656	663	
598	670	677	685	692	699	706	714	721	728	735	
599	743	750	757	764	772	779	786	793	801	808	

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
600	77	815	822	830	837	844	851	859	866	873	880
601		887	895	902	909	916	924	931	938	945	952
602		960	967	974	981	988	996	*003	*010	*017	*025
603	78	032	039	046	053	061	068	075	082	089	097
604		104	111	118	125	132	140	147	154	161	168
605		176	183	190	197	204	211	219	226	233	240
606		247	254	262	269	276	283	290	297	305	312
607		319	326	333	340	347	355	362	369	376	383
608		390	398	405	412	419	426	433	440	447	455
609		462	469	476	483	490	497	504	512	519	526
610		533	540	547	554	561	569	576	583	590	597
611		604	611	618	625	633	640	647	654	661	668
612		675	682	689	696	704	711	718	725	732	739
613		746	753	760	767	774	781	789	796	803	810
614		817	824	831	838	845	852	859	866	873	880
615		888	895	902	909	916	923	930	937	944	951
616		958	965	972	979	986	993	*000	*007	*014	*021
617	79	029	036	043	050	057	064	071	078	085	092
618		099	106	113	120	127	134	141	148	155	162
619		169	176	183	190	197	204	211	218	225	232
620		239	246	253	260	267	274	281	288	295	302
621		309	316	323	330	337	344	351	358	365	372
622		379	386	393	400	407	414	421	428	435	442
623		449	456	463	470	477	484	491	498	505	511
624		518	525	532	539	546	553	560	567	574	581
625		588	595	602	609	616	623	630	637	644	650
626		657	664	671	678	685	692	699	706	713	720
627		727	734	741	748	754	761	768	775	782	789
628		796	803	810	817	824	831	837	844	851	858
629		865	872	879	886	893	900	906	913	920	927
630		934	941	948	955	962	969	975	982	989	996
631	80	003	010	017	024	030	037	044	051	058	065
632		072	079	085	092	099	106	113	120	127	134
633		140	147	154	161	168	175	182	188	195	202
634		209	216	223	229	236	243	250	257	264	271
635		277	284	291	298	305	312	318	325	332	339
636		346	353	359	366	373	380	387	393	400	407
637		414	421	428	434	441	448	455	462	468	475
638		482	489	496	502	509	516	523	530	536	543
639		550	557	564	570	577	584	591	598	604	611
640		618	625	632	638	645	652	659	665	672	679
641		686	693	699	706	713	720	726	733	740	747
642		754	760	767	774	781	787	794	801	808	814
643		821	828	835	841	848	855	862	868	875	882
644		889	895	902	909	916	922	929	936	943	949
645		956	963	969	976	983	990	996	*003	*010	*017
646	81	023	030	037	043	050	057	064	070	077	084
647		090	097	104	111	117	124	131	137	144	151
648		158	164	171	178	184	191	198	204	211	218
649		224	231	238	245	251	258	265	271	278	285

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TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
700	84	510	516	522	528	535	541	547	553	559	566
701		572	578	584	590	597	603	609	615	621	628
702		634	640	646	652	658	665	671	677	683	689
703		696	702	708	714	720	726	733	739	745	751
704		757	763	770	776	782	788	794	800	807	813
705		819	825	831	837	844	850	856	862	868	874
706		880	887	893	899	905	911	917	924	930	936
707		942	948	954	960	967	973	979	985	991	997
708	85	003	009	016	022	028	034	040	046	052	058
709		065	071	077	083	089	095	101	107	114	120
710		126	132	138	144	150	156	163	169	175	181
711		187	193	199	205	211	217	224	230	236	242
712		248	254	260	266	272	278	285	291	297	303
713		309	315	321	327	333	339	345	352	358	364
714		370	376	382	388	394	400	406	412	418	425
715		431	437	443	449	455	461	467	473	479	485
716		491	497	503	509	516	522	528	534	540	546
717		552	558	564	570	576	582	588	594	600	606
718		612	618	625	631	637	643	649	655	661	667
719		673	679	685	691	697	703	709	715	721	727
720		733	739	745	751	757	763	769	775	781	788
721		794	800	806	812	818	824	830	836	842	.848
722		854	860	866	872	878	884	890	896	902	908
723		914	920	926	932	938	944	950	956	962	968
724		974	980	986	992	998	*004	*010	*016	*022	*028
725	86	034	040	046	052	058	064	070	076	082	088
726		094	100	106	112	118	124	130	136	141	147
727		153	159	165	171	177	183	189	195	201	207
728		213	219	225	231	237	243	249	255	261	267
729		273	279	285	291	297	303	308	314	320	326
730		332	338	344	350	356	362	368	374	380	386
731		392	398	404	410	415	421	427	433	439	445
732		451	457	463	469	475	481	487	493	499	504
733		510	516	522	528	534	540	546	552	558	564
734		570	576	581	587	593	599	605	611	617	623
735		629	635	641	646	652	658	664	670	676	682
736		688	694	700	705	711	717	723	729	735	741
737		747	753	759	764	770	776	782	788	794	800
738		806	812	817	823	829	835	841	847	853	859
739		864	870	876	882	888	894	900	906	911	917
740		923	929	935	941	947	953	958	964	970	976
741		982	988	994	999	*005	*011	*017	*023	*029	*035
742	87	040	046	052	058	064	070	075	081	087	093
743		099	105	111	116	122	128	134	140	146	151
744		157	163	169	175	181	186	192	198	204	210
745		216	221	227	233	239	245	251	256	262	268
746		274	280	286	291	297	303	309	315	320	326
747		332	338	344	349	355	361	367	373	379	384
748		390	396	402	408	413	419	425	431	437	442
749		448	454	460	466	471	477	483	489	495	500

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	P.P. Pts.
750	87	506	512	518	523	529	535	541	547	552	558
751		564	570	576	581	587	593	599	604	610	616
752		622	628	633	639	645	651	656	662	668	674
753		679	685	691	697	703	708	714	720	726	731
754		737	743	749	754	760	766	772	777	783	789
755		795	800	806	812	818	823	829	835	841	846
756		852	858	864	869	875	881	887	892	898	904
757		910	915	921	927	933	938	944	950	955	961
758		967	973	978	984	990	996	*001	*007	*013	*018
759	88	024	030	036	041	047	053	058	064	070	076
760		081	087	093	098	104	110	116	121	127	133
761		138	144	150	156	161	167	173	178	184	190
762		195	201	207	213	218	224	230	235	241	247
763		252	258	264	270	275	281	287	292	298	304
764		309	315	321	326	332	338	343	349	355	360
765		366	372	377	383	389	395	400	406	412	417
766		423	429	434	440	446	451	457	463	468	474
767		480	485	491	497	502	508	513	519	525	530
768		536	542	547	553	559	564	570	576	581	587
769		593	598	604	610	615	621	627	632	638	643
770		649	655	660	666	672	677	683	689	694	700
771		705	711	717	722	728	734	739	745	750	756
772		762	767	773	779	784	790	795	801	807	812
773		818	824	829	835	840	846	852	857	863	868
774		874	880	885	891	897	902	908	913	919	925
775		930	936	941	947	953	958	964	969	975	981
776		986	992	997	*003	*009	*014	*020	*025	*031	*037
777	89	042	048	053	059	064	070	076	081	087	092
778		098	104	109	115	120	126	131	137	143	148
779		154	159	165	170	176	182	187	193	198	204
780		209	215	221	226	232	237	243	248	254	260
781		265	271	276	282	287	293	298	304	310	315
782		321	326	332	337	343	348	354	360	365	371
783		376	382	387	393	398	404	409	415	421	426
784		432	437	443	448	454	459	465	470	476	481
785		487	492	498	504	509	515	520	526	531	537
786		542	548	553	559	564	570	575	581	586	592
787		597	603	609	614	620	625	631	636	642	647
788		653	658	664	669	675	680	686	691	697	702
789		708	713	719	724	730	735	741	746	752	757
790		763	768	774	779	785	790	796	801	807	812
791		818	823	829	834	840	845	851	856	862	867
792		873	878	883	889	894	900	905	911	916	922
793		927	933	938	944	949	955	960	966	971	977
794		982	988	993	998	*004	*009	*015	*020	*026	*031
795	90	037	042	048	053	059	064	069	075	080	086
796		091	097	102	108	113	119	124	129	135	140
797		146	151	157	162	168	173	179	184	189	195
798		200	206	211	217	222	227	233	238	244	249
799		255	260	266	271	276	282	287	293	298	304

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
800	90	309	314	320	325	331	336	342	347	352	358
801		363	369	374	380	385	390	396	401	407	412
802		417	423	428	434	439	445	450	455	461	466
803		472	477	482	488	493	499	504	509	515	520
804		526	531	536	542	547	553	558	563	569	574
805		580	585	590	596	601	607	612	617	623	628
806		634	639	644	650	655	660	666	671	677	682
807		687	693	698	703	709	714	720	725	730	736
808		741	747	752	757	763	768	773	779	784	789
809		795	800	806	811	816	822	827	832	838	843
810		849	854	859	865	870	875	881	886	891	897
811		902	907	913	918	924	929	934	940	945	950
812		956	961	966	972	977	982	988	993	998	*004
813	91	009	014	020	025	030	036	041	046	052	057
814		062	068	073	078	084	089	094	100	105	110
815		116	121	126	132	137	142	148	153	158	164
816		169	174	180	185	190	196	201	206	212	217
817		222	228	233	238	243	249	254	259	265	270
818		275	281	286	291	297	302	307	312	318	323
819		328	334	339	344	350	355	360	365	371	376
820		381	387	392	397	403	408	413	418	424	429
821		434	440	445	450	455	461	466	471	477	482
822		487	492	498	503	508	514	519	524	529	535
823		540	545	551	556	561	566	572	577	582	587
824		593	598	603	609	614	619	624	630	635	640
825		645	651	656	661	666	672	677	682	687	693
826		698	703	709	714	719	724	730	735	740	745
827		751	756	761	766	772	777	782	787	793	798
828		803	808	814	819	824	829	834	840	845	850
829		855	861	866	871	876	882	887	892	897	903
830		908	913	918	924	929	934	939	944	950	955
831		960	965	971	976	981	986	991	997	*002	*007
832	92	012	018	023	028	033	038	044	049	054	059
833		065	070	075	080	085	091	096	101	106	111
834		117	122	127	132	137	143	148	153	158	163
835		169	174	179	184	189	195	200	205	210	215
836		221	226	231	236	241	247	252	257	262	267
837		273	278	283	288	293	298	304	309	314	319
838		324	330	335	340	345	350	355	361	366	371
839		376	381	387	392	397	402	407	412	418	423
840		428	433	438	443	449	454	459	464	469	474
841		480	485	490	495	500	505	511	516	521	526
842		531	536	542	547	552	557	562	567	572	578
843		583	588	593	598	603	609	614	619	624	629
844		634	639	645	650	655	660	665	670	675	681
845		686	691	696	701	706	711	716	722	727	732
846		737	742	747	752	758	763	768	773	778	783
847		788	793	799	804	809	814	819	824	829	834
848		840	845	850	855	860	865	870	875	881	886
849		891	896	901	906	911	916	921	927	932	937

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
850	92	942	947	952	957	962	967	973	978	983	988
851	993	998	*003	*008	*013	*018	*024	*029	*034	*039	
852	93	044	049	054	059	064	069	075	080	085	090
853	095	100	105	110	115	120	125	131	136	141	
854	146	151	156	161	166	171	176	181	186	192	
855	197	202	207	212	217	222	227	232	237	242	
856	247	252	258	263	268	273	278	283	288	293	
857	298	303	308	313	318	323	328	334	339	344	
858	349	354	359	364	369	374	379	384	389	394	6
859	399	404	409	414	420	425	430	435	440	445	1 0.6
860	450	455	460	465	470	475	480	485	490	495	2 1.2
861	500	505	510	515	520	526	531	536	541	546	3 1.8
862	551	556	561	566	571	576	581	586	591	596	4 2.4
863	601	606	611	616	621	626	631	636	641	646	5 3.0
864	651	656	661	666	671	676	682	687	692	697	6 3.6
865	702	707	712	717	722	727	732	737	742	747	
866	752	757	762	767	772	777	782	787	792	797	
867	802	807	812	817	822	827	832	837	842	847	
868	852	857	862	867	872	877	882	887	892	897	
869	902	907	912	917	922	927	932	937	942	947	
870	952	957	962	967	972	977	982	987	992	997	
871	94	002	007	012	017	022	027	032	037	042	047
872	052	057	062	067	072	077	082	086	091	096	1 0.5
873	101	106	111	116	121	126	131	136	141	146	2 1.0
874	151	156	161	166	171	176	181	186	191	196	3 1.5
875	201	206	211	216	221	226	231	236	240	245	4 2.0
876	250	255	260	265	270	275	280	285	290	295	5 2.5
877	300	305	310	315	320	325	330	335	340	345	6 3.0
878	349	354	359	364	369	374	379	384	389	394	7 3.5
879	399	404	409	414	419	424	429	433	438	443	8 4.0
880	448	453	458	463	468	473	478	483	488	493	
881	498	503	507	512	517	522	527	532	537	542	
882	547	552	557	562	567	571	576	581	586	591	
883	596	601	606	611	616	621	626	630	635	640	
884	645	650	655	660	665	670	675	680	685	689	
885	694	699	704	709	714	719	724	729	734	738	
886	743	748	753	758	763	768	773	778	783	787	1 0.4
887	792	797	802	807	812	817	822	827	832	836	2 0.8
888	841	846	851	856	861	866	871	876	880	885	3 1.2
889	890	895	900	905	910	915	919	924	929	934	4 1.6
890	939	944	949	954	959	963	968	973	978	983	5 2.0
891	988	993	998	*002	*007	*012	*017	*022	*027	*032	6 2.4
892	95	036	041	046	051	056	061	066	071	075	7 2.8
893	085	090	095	100	105	109	114	119	124	129	8 3.2
894	134	139	143	148	153	158	163	168	173	177	9 3.6
895	182	187	192	197	202	207	211	216	221	226	
896	231	236	240	245	250	255	260	265	270	274	
897	279	284	289	294	299	303	308	313	318	323	
898	328	332	337	342	347	352	357	361	366	371	
899	376	381	386	390	395	400	405	410	415	419	

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
900	95	424	429	434	439	444	448	453	458	463	468
901	472	477	482	487	492	497	501	506	511	516	
902	521	525	530	535	540	545	550	554	559	564	
903	569	574	578	583	588	593	598	602	607	612	
904	617	622	626	631	636	641	646	650	655	660	
905	665	670	674	679	684	689	694	698	703	708	
906	713	718	722	727	732	737	742	746	751	756	
907	761	766	770	775	780	785	789	794	799	804	
908	809	813	818	823	828	832	837	842	847	852	
909	856	861	866	871	875	880	885	890	895	899	
910	904	909	914	918	923	928	933	938	942	947	
911	952	957	961	966	971	976	980	985	990	995	
912	999	*004	*009	*014	*019	*023	*028	*033	*038	*042	
913	96	047	052	057	061	066	071	076	080	085	090
914	095	099	104	109	114	118	123	128	133	137	1
915	142	147	152	156	161	166	171	175	180	185	2
916	190	194	199	204	209	213	218	223	227	232	3
917	237	242	246	251	256	261	265	270	275	280	4
918	284	289	294	298	303	308	313	317	322	327	5
919	332	336	341	346	350	355	360	365	369	374	0.5
920	379	384	388	393	398	402	407	412	417	421	1.0
921	426	431	435	440	445	450	454	459	464	468	1.5
922	473	478	483	487	492	497	501	506	511	515	2.0
923	520	525	530	534	539	544	548	553	558	562	2.5
924	567	572	577	581	586	591	595	600	605	609	3.0
925	614	619	624	628	633	638	642	647	652	656	3.5
926	661	666	670	675	680	685	689	694	699	703	4.0
927	708	713	717	722	727	731	736	741	745	750	4.5
928	755	759	764	769	774	778	783	788	792	797	
929	802	806	811	816	820	825	830	834	839	844	
930	848	853	858	862	867	872	876	881	886	890	
931	895	900	904	909	914	918	923	928	932	937	0.4
932	942	946	951	956	960	965	970	974	979	984	0.8
933	988	993	997	*002	*007	*011	*016	*021	*025	*030	1.2
934	935	035	039	044	049	053	058	063	067	072	1.6
935	081	086	090	095	100	104	109	114	118	123	2.0
936	128	132	137	142	146	151	155	160	165	169	2.4
937	174	179	183	188	192	197	202	206	211	216	2.8
938	220	225	230	234	239	243	248	253	257	262	3.2
939	267	271	276	280	285	290	294	299	304	308	3.6
940	313	317	322	327	331	336	340	345	350	354	
941	359	364	368	373	377	382	387	391	396	400	
942	405	410	414	419	424	428	433	437	442	447	
943	451	456	460	465	470	474	479	483	488	493	
944	497	502	506	511	516	520	525	529	534	539	
945	543	548	552	557	562	566	571	575	580	585	
946	589	594	598	603	607	612	617	621	626	630	
947	635	640	644	649	653	658	663	667	672	676	
948	681	685	690	695	699	704	708	713	717	722	
949	727	731	736	740	745	749	754	759	763	768	

TABLE II.—LOGARITHMS OF NUMBERS.

No.	0	1	2	3	4	5	6	7	8	9	Pp. Pts.
950	97	772	777	782	786	791	795	800	804	809	813
951	818	823	827	832	836	841	845	850	855	859	.
952	864	868	873	877	882	886	891	896	900	905	.
953	909	914	918	923	928	932	937	941	946	950	.
954	955	959	964	968	973	978	982	987	991	996	.
955	98 000	005	009	014	019	023	028	032	037	041	.
956	046	050	055	059	064	068	073	078	082	087	.
957	091	096	100	105	109	114	118	123	127	132	.
958	137	141	146	150	155	159	164	168	173	177	.
959	182	186	191	195	200	204	209	214	218	223	.
960	227	232	236	241	245	250	254	259	263	268	.
961	272	277	281	286	290	295	299	304	308	313	.
962	318	322	327	331	336	340	345	349	354	358	.
963	363	367	372	376	381	385	390	394	399	403	5
964	408	412	417	421	426	430	435	439	444	448	1 0.5
965	453	457	462	466	471	475	480	484	489	493	2 1.0
966	498	502	507	511	516	520	525	529	534	538	3 1.5
967	543	547	552	556	561	565	570	574	579	583	4 2.0
968	588	592	597	601	605	610	614	619	623	628	5 2.5
969	632	637	641	646	650	655	659	664	668	673	6 3.0
970	677	682	686	691	695	700	704	709	713	717	7 3.5
971	722	726	731	735	740	744	749	753	758	762	8 4.0
972	767	771	776	780	784	789	793	798	802	807	9 4.5
973	811	816	820	825	829	834	838	843	847	851	.
974	856	860	865	869	874	878	883	887	892	896	.
975	900	905	909	914	918	923	927	932	936	941	.
976	945	949	954	958	963	967	972	976	981	985	.
977	989	994	998	*003	*007	*012	*016	*021	*025	*029	.
978	99 034	038	043	047	052	056	061	065	069	074	.
979	078	083	087	092	096	100	105	109	114	118	.
980	123	127	131	136	140	145	149	154	158	162	4
981	167	171	176	180	185	189	193	198	202	207	0.4
982	211	216	220	224	229	233	238	242	247	251	0.8
983	255	260	264	269	273	277	282	286	291	295	1.2
984	300	304	308	313	317	322	326	330	335	339	1.6
985	344	348	352	357	361	366	370	374	379	383	2.0
986	388	392	396	401	405	410	414	419	423	427	2.4
987	432	436	441	445	449	454	458	463	467	471	2.8
988	476	480	484	489	493	498	502	506	511	515	3.2
989	520	524	528	533	537	542	546	550	555	559	3.6
990	564	568	572	577	581	585	590	594	599	603	.
991	607	612	616	621	625	629	634	638	642	647	.
992	651	656	660	664	669	673	677	682	686	691	.
993	695	699	704	708	712	717	721	726	730	734	.
994	739	743	747	752	756	760	765	769	774	778	.
995	782	787	791	795	800	804	808	813	817	822	.
996	826	830	835	839	843	848	852	856	861	865	.
997	870	874	878	883	887	891	896	900	904	909	.
998	913	917	922	926	930	935	939	944	948	952	.
999	957	961	965	970	974	978	983	987	991	996	.

TABLE III.—BINOMIAL COEFFICIENTS.

From  $n = 1$  to  $n = 20$ .

$$C_r = \frac{n(n-1)(n-2)\dots(n-r+1)}{1 \times 2 \times 3 \times \dots \times r}. \text{ Also } C_r = C_{n-r} = \frac{n!}{r!(n-r)!}.$$

<b>n</b>	<b>C<sub>0</sub></b>	<b>C<sub>1</sub></b>	<b>C<sub>2</sub></b>	<b>C<sub>3</sub></b>	<b>C<sub>4</sub></b>	<b>C<sub>5</sub></b>	<b>C<sub>6</sub></b>	<b>C<sub>7</sub></b>	<b>C<sub>8</sub></b>	<b>C<sub>9</sub></b>
1	1	1								
2	1	2	1							
3	1	3	3	1						
4	1	4	6	4	1					
5	1	5	10	10	5	1				
6	1	6	15	20	15	6	1			
7	1	7	21	35	35	21	7	1		
8	1	8	28	56	70	56	28	8	1	
9	1	9	36	84	126	126	84	36	9	1
10	1	10	45	120	210	252	210	120	45	10
11	1	11	55	165	330	462	462	330	165	55
12	1	12	66	220	495	792	924	792	495	220
13	1	13	78	286	715	1287	1716	1716	1287	715
14	1	14	91	364	1001	2002	3003	3432	3003	2002
15	1	15	105	455	1365	3003	5005	6435	6435	5005
16	1	16	120	560	1820	4368	8008	11440	12870	11440
17	1	17	136	680	2380	6188	12376	19448	24310	24310
18	1	18	153	816	3060	8568	18564	31824	43758	48620
19	1	19	171	969	3876	11628	27132	50388	75582	92378
20	1	20	190	1140	4945	15504	38760	77520	125970	167960

	<b>C<sub>10</sub></b>	<b>C<sub>11</sub></b>	<b>C<sub>12</sub></b>	<b>C<sub>13</sub></b>	<b>C<sub>14</sub></b>	<b>C<sub>15</sub></b>	<b>C<sub>16</sub></b>	<b>C<sub>17</sub></b>	<b>C<sub>18</sub></b>	<b>C<sub>19</sub></b>	<b>C<sub>20</sub></b>
10	1										
11	11	1									
12	66	12	1								
13	286	78	13	1							
14	1001	364	91	14	1						
15	3003	1365	455	105	15	1					
16	8008	4368	1820	560	120	16	1				
17	19448	12376	6188	2380	680	136	17	1			
18	43758	31824	18564	8568	3060	816	153	18	1		
19	92378	75582	50388	27132	11628	3876	969	171	19	1	
20	184756	167960	125970	77520	38760	15504	4845	1140	190	20	1

**Factorials.**

The continued product of the  $n$  factors  $1 \times 2 \times 3 \times 4 \times \dots \times n$  is called the factorial of  $n$ ; and it is denoted by the symbol  $n!$

Some writers use the symbol  $[n]$ .

0! =	1	11! =	39,916,800
1! =	1	12! =	479,001,600
2! =	2	13! =	6,227,020,800
3! =	6	14! =	87,178,291,200
4! =	24	15! =	1,307,674,368,000
5! =	120	16! =	20,922,789,888,000
6! =	720	17! =	355,687,428,096,000
7! =	5,040	18! =	6,402,373,705,728,000
8! =	40,320	19! =	121,645,100,408,832,000
9! =	362,880	20! =	2,432,902,008,176,640,000
10! =	3,628,800		

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TABLE IV.—NATURAL LOGARITHMS OF NUMBERS.

No.	Nat. Log.								
251	5. 52545	301	5. 70711	351	5. 86079	401	5. 99396	451	6. 11147
252	5. 52943	302	5. 71043	352	5. 86363	402	5. 99645	452	6. 11368
253	5. 53339	303	5. 71373	353	5. 86647	403	5. 99894	453	6. 11589
254	5. 53733	304	5. 71703	354	5. 86930	404	6. 00141	454	6. 11810
255	5. 54126	305	5. 72031	355	5. 87212	405	6. 00389	455	6. 12030
256	5. 54518	306	5. 72359	356	5. 87493	406	6. 00635	456	6. 12249
257	5. 54908	307	5. 72685	357	5. 87774	407	6. 00881	457	6. 12468
258	5. 55296	308	5. 73010	358	5. 88053	408	6. 01127	458	6. 12687
259	5. 55683	309	5. 73334	359	5. 88332	409	6. 01372	459	6. 12905
260	5. 56068	310	5. 73657	360	5. 88610	410	6. 01616	460	6. 13123
261	5. 56452	311	5. 73979	361	5. 88888	411	6. 01859	461	6. 13340
262	5. 56834	312	5. 74300	362	5. 89164	412	6. 02102	462	6. 13556
263	5. 57215	313	5. 74620	363	5. 89440	413	6. 02345	463	6. 13773
264	5. 57595	314	5. 74939	364	5. 89715	414	6. 02587	464	6. 13988
265	5. 57973	315	5. 75257	365	5. 89990	415	6. 02828	465	6. 14204
266	5. 58350	316	5. 75574	366	5. 90263	416	6. 03069	466	6. 14419
267	5. 58725	317	5. 75890	367	5. 90536	417	6. 03300	467	6. 14633
268	5. 59099	318	5. 76205	368	5. 90808	418	6. 03548	468	6. 14847
269	5. 59471	319	5. 76519	369	5. 91080	419	6. 03787	469	6. 15060
270	5. 59842	320	5. 76832	370	5. 91350	420	6. 04025	470	6. 15273
271	5. 60212	321	5. 77144	371	5. 91620	421	6. 04263	471	6. 15486
272	5. 60580	322	5. 77455	372	5. 91889	422	6. 04501	472	6. 15698
273	5. 60947	323	5. 77765	373	5. 92158	423	6. 04737	473	6. 15910
274	5. 61313	324	5. 78074	374	5. 92426	424	6. 04973	474	6. 16121
275	5. 61677	325	5. 78383	375	5. 92693	425	6. 05209	475	6. 16331
276	5. 62040	326	5. 78690	376	5. 92959	426	6. 05444	476	6. 16542
277	5. 62402	327	5. 78996	377	5. 93225	427	6. 05678	477	6. 16752
278	5. 62762	328	5. 79301	378	5. 93489	428	6. 05912	478	6. 16961
279	5. 63121	329	5. 79606	379	5. 93754	429	6. 06146	479	6. 17170
280	5. 63479	330	5. 79909	380	5. 94017	430	6. 06379	480	6. 17379
281	5. 63835	331	5. 80212	381	5. 94280	431	6. 06611	481	6. 17587
282	5. 64191	332	5. 80513	382	5. 94542	432	6. 06843	482	6. 17794
283	5. 64545	333	5. 80814	383	5. 94803	433	6. 07074	483	6. 18002
284	5. 64897	334	5. 81114	384	5. 95064	434	6. 07304	484	6. 18208
285	5. 65249	335	5. 81413	385	5. 95324	435	6. 07535	485	6. 18415
286	5. 65599	336	5. 81711	386	5. 95584	436	6. 07764	486	6. 18621
287	5. 65948	337	5. 82008	387	5. 95842	437	6. 07993	487	6. 18826
288	5. 66296	338	5. 82305	388	5. 96101	438	6. 08222	488	6. 19032
289	5. 66643	339	5. 82600	389	5. 96358	439	6. 08450	489	6. 19236
290	5. 66988	340	5. 82895	390	5. 96615	440	6. 08677	490	6. 19441
291	5. 67332	341	5. 83188	391	5. 96871	441	6. 08904	491	6. 19644
292	5. 67675	342	5. 83481	392	5. 97126	442	6. 09131	492	6. 19848
293	5. 68017	343	5. 83773	393	5. 97381	443	6. 09357	493	6. 20051
294	5. 68358	344	5. 84064	394	5. 97635	444	6. 09582	494	6. 20254
295	5. 68698	345	5. 84354	395	5. 97889	445	6. 09807	495	6. 20456
296	5. 69036	346	5. 84644	396	5. 98141	446	6. 10032	496	6. 20658
297	5. 69373	347	5. 84932	397	5. 98394	447	6. 10256	497	6. 20859
298	5. 69709	348	5. 85220	398	5. 98645	448	6. 10479	498	6. 21060
299	5. 70044	349	5. 85507	399	5. 98896	449	6. 10702	499	6. 21261
300	5. 70378	350	5. 85793	400	5. 99146	450	6. 10925	500	6. 21461

TABLE IV.—NATURAL LOGARITHMS OF NUMBERS.

No.	Nat. Log.								
501	6. 21661	551	6. 31173	601	6. 39859	651	6. 47851	701	6. 55251
502	6. 21860	552	6. 31355	602	6. 40026	652	6. 48004	702	6. 55393
503	6. 22059	553	6. 31536	603	6. 40192	653	6. 48158	703	6. 55536
504	6. 22258	554	6. 31716	604	6. 40357	654	6. 48311	704	6. 55678
505	6. 22456	555	6. 31897	605	6. 40523	655	6. 48464	705	6. 55820
506	6. 22654	556	6. 32077	606	6. 40688	656	6. 48616	706	6. 55962
507	6. 22851	557	6. 32257	607	6. 40853	657	6. 48768	707	6. 56103
508	6. 23048	558	6. 32436	608	6. 41017	658	6. 48920	708	6. 56244
509	6. 23245	559	6. 32615	609	6. 41182	659	6. 49072	709	6. 56386
510	6. 23441	560	6. 32794	610	6. 41346	660	6. 49224	710	6. 56526
511	6. 23637	561	6. 32972	611	6. 41510	661	6. 49375	711	6. 56667
512	6. 23832	562	6. 33150	612	6. 41673	662	6. 49527	712	6. 56808
513	6. 24028	563	6. 33328	613	6. 41836	663	6. 49677	713	6. 56948
514	6. 24222	564	6. 33505	614	6. 41999	664	6. 49828	714	6. 57088
515	6. 24417	565	6. 33683	615	6. 42162	665	6. 49979	715	6. 57228
516	6. 24611	566	6. 33859	616	6. 42325	666	6. 50129	716	6. 57368
517	6. 24804	567	6. 34036	617	6. 42487	667	6. 50279	717	6. 57508
518	6. 24998	568	6. 34212	618	6. 42649	668	6. 50429	718	6. 57647
519	6. 25190	569	6. 34388	619	6. 42811	669	6. 50578	719	6. 57786
520	6. 25383	570	6. 34564	620	6. 42972	670	6. 50728	720	6. 57925
521	6. 25575	571	6. 34739	621	6. 43133	671	6. 50877	721	6. 58064
522	6. 25767	572	6. 34914	622	6. 43294	672	6. 51026	722	6. 58203
523	6. 25958	573	6. 35089	623	6. 43455	673	6. 51175	723	6. 58341
524	6. 26149	574	6. 35263	624	6. 43615	674	6. 51323	724	6. 58479
525	6. 26340	575	6. 35437	625	6. 43775	675	6. 51471	725	6. 58617
526	6. 26530	576	6. 35611	626	6. 43935	676	6. 51619	726	6. 58755
527	6. 26720	577	6. 35784	627	6. 44095	677	6. 51767	727	6. 58893
528	6. 26910	578	6. 35957	628	6. 44254	678	6. 51915	728	6. 59030
529	6. 27099	579	6. 36130	629	6. 44413	679	6. 52062	729	6. 59167
530	6. 27288	580	6. 36303	630	6. 44572	680	6. 52209	730	6. 59304
531	6. 27476	581	6. 36475	631	6. 44731	681	6. 52356	731	6. 59441
532	6. 27664	582	6. 36647	632	6. 44889	682	6. 52503	732	6. 59578
533	6. 27852	583	6. 36819	633	6. 45047	683	6. 52649	733	6. 59715
534	6. 28040	584	6. 36990	634	6. 45205	684	6. 52796	734	6. 59851
535	6. 28227	585	6. 37161	635	6. 45362	685	6. 52942	735	6. 59987
536	6. 28413	586	6. 37332	636	6. 45520	686	6. 53088	736	6. 60123
537	6. 28600	587	6. 37502	637	6. 45677	687	6. 53233	737	6. 60259
538	6. 28786	588	6. 37673	638	6. 45834	688	6. 53379	738	6. 60394
539	6. 28972	589	6. 37843	639	6. 45990	689	6. 53524	739	6. 60530
540	6. 29157	590	6. 38012	640	6. 46147	690	6. 53669	740	6. 60665
541	6. 29342	591	6. 38182	641	6. 46303	691	6. 53814	741	6. 60800
542	6. 29527	592	6. 38351	642	6. 46459	692	6. 53959	742	6. 60935
543	6. 29711	593	6. 38519	643	6. 46614	693	6. 54103	743	6. 61070
544	6. 29895	594	6. 38688	644	6. 46770	694	6. 54247	744	6. 61204
545	6. 30079	595	6. 38856	645	6. 46925	695	6. 54391	745	6. 61338
546	6. 30262	596	6. 39024	646	6. 47080	696	6. 54535	746	6. 61473
547	6. 30445	597	6. 39192	647	6. 47235	697	6. 54679	747	6. 61607
548	6. 30628	598	6. 39359	648	6. 47389	698	6. 54822	748	6. 61740
549	6. 30810	599	6. 39526	649	6. 47543	699	6. 54965	749	6. 61874
550	6. 30992	600	6. 39693	650	6. 47697	700	6. 55108	750	6. 62007

TABLE IV.—NATURAL LOGARITHMS OF NUMBERS.

No.	Nat. Log.	No.	Nat. Log.	No.	Nat. Log.	No.	Nat. Log.	No.	Nat. Log.
751	6.62141	801	6.68586	851	6.74641	901	6.80351	951	6.85751
752	6.62274	802	6.68711	852	6.74759	902	6.80461	952	6.85857
753	6.62407	803	6.68835	853	6.74876	903	6.80572	953	6.85961
754	6.62539	804	6.68960	854	6.74993	904	6.80683	954	6.86066
755	6.62672	805	6.69084	855	6.75110	905	6.80793	955	6.86171
756	6.62804	806	6.69208	856	6.75227	906	6.80904	956	6.86276
757	6.62936	807	6.69332	857	6.75344	907	6.81014	957	6.86380
758	6.63068	808	6.69456	858	6.75460	908	6.81124	958	6.86485
759	6.63200	809	6.69580	859	6.75577	909	6.81235	959	6.86589
760	6.63332	810	6.69703	860	6.75693	910	6.81344	960	6.86693
761	6.63463	811	6.69827	861	6.75809	911	6.81454	961	6.86797
762	6.63595	812	6.69950	862	6.75926	912	6.81564	962	6.86901
763	6.63726	813	6.70073	863	6.76041	913	6.81674	963	6.87005
764	6.63857	814	6.70196	864	6.76157	914	6.81783	964	6.87109
765	6.63988	815	6.70319	865	6.76273	915	6.81892	965	6.87213
766	6.64118	816	6.70441	866	6.76388	916	6.82002	966	6.87316
767	6.64249	817	6.70564	867	6.76504	917	6.82111	967	6.87420
768	6.64379	818	6.70686	868	6.76619	918	6.82220	968	6.87523
769	6.64509	819	6.70808	869	6.76734	919	6.82329	969	6.87626
770	6.64639	820	6.70930	870	6.76849	920	6.82437	970	6.87730
771	6.64769	821	6.71052	871	6.76964	921	6.82546	971	6.87833
772	6.64898	822	6.71174	872	6.77079	922	6.82655	972	6.87936
773	6.65028	823	6.71296	873	6.77194	923	6.82763	973	6.88038
774	6.65157	824	6.71417	874	6.77308	924	6.82871	974	6.88141
775	6.65286	825	6.71538	875	6.77422	925	6.82979	975	6.88244
776	6.65415	826	6.71659	876	6.77537	926	6.83087	976	6.88346
777	6.65544	827	6.71780	877	6.77651	927	6.83195	977	6.88449
778	6.65673	828	6.71901	878	6.77765	928	6.83303	978	6.88551
779	6.65801	829	6.72022	879	6.77878	929	6.83411	979	6.88653
780	6.65929	830	6.72143	880	6.77992	930	6.83518	980	6.88755
781	6.66058	831	6.72263	881	6.78106	931	6.83626	981	6.88857
782	6.66185	832	6.72383	882	6.78219	932	6.83733	982	6.88959
783	6.66313	833	6.72503	883	6.78333	933	6.83841	983	6.89061
784	6.66441	834	6.72623	884	6.78446	934	6.83948	984	6.89163
785	6.66568	835	6.72743	885	6.78559	935	6.84055	985	6.89264
786	6.66696	836	6.72863	886	6.78672	936	6.84162	986	6.89366
787	6.66823	837	6.72982	887	6.78784	937	6.84268	987	6.89467
788	6.66950	838	6.73102	888	6.78897	938	6.84375	988	6.89568
789	6.67077	839	6.73221	889	6.79010	939	6.84482	989	6.89669
790	6.67203	840	6.73340	890	6.79122	940	6.84588	990	6.89770
791	6.67330	841	6.73459	891	6.79234	941	6.84694	991	6.89871
792	6.67456	842	6.73578	892	6.79347	942	6.84801	992	6.89972
793	6.67582	843	6.73697	893	6.79459	943	6.84907	993	6.90073
794	6.67708	844	6.73815	894	6.79571	944	6.85013	994	6.90174
795	6.67834	845	6.73934	895	6.79682	945	6.85118	995	6.90274
796	6.67960	846	6.74052	896	6.79794	946	6.85224	996	6.90375
797	6.68085	847	6.74170	897	6.79906	947	6.85330	997	6.90475
798	6.68211	848	6.74288	898	6.80017	948	6.85435	998	6.90575
799	6.68336	849	6.74406	899	6.80128	949	6.85541	999	6.90675
800	6.68461	850	6.74524	900	6.80239	950	6.85646	1000	6.90776

TABLE V.—TRIGONOMETRIC OR CIRCULAR FUNCTIONS.

Natural Values to Three Places of Decimals.

Angle	Arc	Sin	Csc	Tan	Ctn	Sec	Cos		
		Cos	Sec	Ctn	Tan	Csc	Sin	Arc	Angle
0°	0.000	0.000	infinite	0.000	infinite	1.000	1.000	1.571	90°
1°	0.017	0.017	57.30	0.017	57.29	1.000	1.000	1.553	89
2°	0.035	0.035	28.65	0.035	28.64	1.001	0.999	1.536	88
3°	0.052	0.052	19.11	0.052	19.08	1.001	0.999	1.518	87
4°	0.070	0.070	14.34	0.070	14.30	1.002	0.998	1.501	86
5°	0.087	0.087	11.47	0.087	11.43	1.004	0.996	1.484	85°
6°	0.105	0.105	9.567	0.105	9.514	1.006	0.995	1.466	84
7°	0.122	0.122	8.206	0.123	8.144	1.008	0.993	1.449	83
8°	0.140	0.139	7.185	0.141	7.115	1.010	0.990	1.431	82
9°	0.157	0.156	6.392	0.158	6.314	1.012	0.988	1.414	81
10°	0.175	0.174	5.759	0.176	5.671	1.015	0.985	1.396	80°
11°	0.192	0.191	5.241	0.194	5.145	1.019	0.982	1.379	79
12°	0.209	0.208	4.810	0.213	4.705	1.022	0.978	1.361	78
13°	0.227	0.225	4.445	0.231	4.331	1.026	0.974	1.344	77
14°	0.244	0.242	4.134	0.249	4.011	1.031	0.970	1.326	76
15°	0.262	0.259	3.864	0.268	3.732	1.035	0.966	1.309	75°
16°	0.279	0.276	3.628	0.287	3.487	1.040	0.961	1.292	74
17°	0.297	0.292	3.420	0.306	3.271	1.046	0.956	1.274	73
18°	0.314	0.309	3.236	0.325	3.078	1.051	0.951	1.257	72
19°	0.332	0.326	3.072	0.344	2.904	1.058	0.946	1.239	71
20°	0.349	0.342	2.924	0.364	2.747	1.064	0.940	1.222	70°
21°	0.367	0.358	2.790	0.384	2.605	1.071	0.934	1.204	69
22°	0.384	0.375	2.669	0.404	2.475	1.079	0.927	1.187	68
23°	0.401	0.391	2.559	0.424	2.356	1.086	0.921	1.169	67
24°	0.419	0.407	2.459	0.445	2.246	1.095	0.914	1.152	66
25°	0.436	0.423	2.366	0.466	2.145	1.103	0.906	1.134	65°
26°	0.454	0.438	2.281	0.488	2.050	1.113	0.899	1.117	64
27°	0.471	0.454	2.203	0.510	1.963	1.122	0.891	1.100	63
28°	0.489	0.469	2.130	0.532	1.881	1.133	0.883	1.082	62
29°	0.506	0.485	2.063	0.554	1.804	1.143	0.875	1.065	61
30°	0.524	0.500	2.000	0.577	1.732	1.155	0.866	1.047	60°
31°	0.541	0.515	1.942	0.601	1.664	1.167	0.857	1.030	59
32°	0.559	0.530	1.887	0.625	1.600	1.179	0.848	1.012	58
33°	0.576	0.545	1.836	0.649	1.540	1.192	0.839	0.995	57
34°	0.593	0.559	1.788	0.675	1.483	1.206	0.829	0.977	56
35°	0.611	0.574	1.743	0.700	1.428	1.221	0.819	0.960	55°
36°	0.628	0.588	1.701	0.727	1.376	1.236	0.809	0.942	54
37°	0.646	0.602	1.662	0.754	1.327	1.252	0.799	0.925	53
38°	0.663	0.616	1.624	0.781	1.280	1.269	0.788	0.908	52
39°	0.681	0.629	1.589	0.810	1.235	1.287	0.777	0.890	51
40°	0.698	0.643	1.556	0.839	1.192	1.305	0.766	0.873	50°
41°	0.716	0.656	1.524	0.869	1.150	1.325	0.755	0.855	49
42°	0.733	0.669	1.494	0.900	1.111	1.346	0.743	0.838	48
43°	0.750	0.682	1.466	0.933	1.072	1.367	0.731	0.820	47
44°	0.768	0.695	1.440	0.966	1.036	1.390	0.719	0.803	46
45°	0.785	0.707	1.414	1.000	1.000	1.414	0.707	0.785	45°

TABLE VI.—NATURAL SINES.

Angle	0'	10'	20'	30'	40'	50'	60'	
0°	0.00000	0.00291	0.00582	0.00873	0.01164	0.01454	0.01745	89°
1	0.01745	0.02036	0.02327	0.02618	0.02908	0.03199	0.03490	88
2	0.03490	0.03781	0.04071	0.04362	0.04653	0.04943	0.05234	87
3	0.05234	0.05524	0.05814	0.06105	0.06395	0.06685	0.06976	86
4	0.06976	0.07266	0.07556	0.07846	0.08136	0.08426	0.08716	85
5	0.08716	0.09005	0.09295	0.09585	0.09874	0.10164	0.10453	84
6	0.10453	0.10742	0.11031	0.11320	0.11609	0.11898	0.12187	83
7	0.12187	0.12476	0.12764	0.13053	0.13341	0.13629	0.13917	82
8	0.13917	0.14205	0.14493	0.14781	0.15069	0.15356	0.15643	81
9	0.15643	0.15931	0.16218	0.16505	0.16792	0.17078	0.17365	80°
10°	0.17365	0.17651	0.17937	0.18224	0.18509	0.18795	0.19081	79
11	0.19081	0.19366	0.19652	0.19937	0.20222	0.20507	0.20791	78
12	0.20791	0.21076	0.21360	0.21644	0.21928	0.22212	0.22495	77
13	0.22495	0.22778	0.23062	0.23345	0.23627	0.23910	0.24192	76
14	0.24192	0.24474	0.24756	0.25038	0.25320	0.25601	0.25882	75
15	0.25882	0.26163	0.26443	0.26724	0.27004	0.27284	0.27564	74
16	0.27564	0.27843	0.28123	0.28402	0.28680	0.28959	0.29237	73
17	0.29237	0.29515	0.29793	0.30071	0.30348	0.30625	0.30902	72
18	0.30902	0.31178	0.31454	0.31730	0.32006	0.32282	0.32557	71
19	0.32557	0.32832	0.33106	0.33381	0.33655	0.33929	0.34202	70°
20°	0.34202	0.34475	0.34748	0.35021	0.35293	0.35565	0.35837	69
21	0.35837	0.36108	0.36379	0.36650	0.36921	0.37191	0.37461	68
22	0.37461	0.37730	0.37999	0.38268	0.38537	0.38805	0.39073	67
23	0.39073	0.39341	0.39608	0.39875	0.40141	0.40408	0.40674	66
24	0.40674	0.40939	0.41204	0.41469	0.41734	0.41998	0.42262	65
25	0.42262	0.42525	0.42788	0.43051	0.43313	0.43575	0.43837	64
26	0.43837	0.44098	0.44359	0.44620	0.44880	0.45140	0.45399	63
27	0.45399	0.45658	0.45917	0.46175	0.46433	0.46690	0.46947	62
28	0.46947	0.47204	0.47460	0.47716	0.47971	0.48226	0.48481	61
29	0.48481	0.48735	0.48989	0.49242	0.49495	0.49748	0.50000	60°
30°	0.50000	0.50252	0.50503	0.50754	0.51004	0.51254	0.51504	59
31	0.51504	0.51753	0.52002	0.52250	0.52498	0.52745	0.52992	58
32	0.52992	0.53238	0.53484	0.53730	0.53975	0.54220	0.54464	57
33	0.54464	0.54708	0.54951	0.55194	0.55436	0.55678	0.55919	56
34	0.55919	0.56160	0.56401	0.56641	0.56880	0.57119	0.57358	55
35	0.57358	0.57596	0.57833	0.58070	0.58307	0.58543	0.58779	54
36	0.58779	0.59014	0.59248	0.59482	0.59716	0.59949	0.60182	53
37	0.60182	0.60414	0.60645	0.60876	0.61107	0.61337	0.61566	52
38	0.61566	0.61795	0.62024	0.62251	0.62479	0.62706	0.62932	51
39	0.62932	0.63158	0.63383	0.63608	0.63832	0.64056	0.64279	50°
40°	0.64279	0.64501	0.64723	0.64945	0.65166	0.65386	0.65606	49
41	0.65606	0.65825	0.66044	0.66262	0.66480	0.66697	0.66913	48
42	0.66913	0.67129	0.67344	0.67559	0.67773	0.67987	0.68200	47
43	0.68200	0.68412	0.68624	0.68835	0.69046	0.69256	0.69466	46
44	0.69466	0.69675	0.69883	0.70091	0.70298	0.70505	0.70711	45°

## NATURAL COSINES.

TABLE VI.—NATURAL SINES.

Angle	0'	10'	20'	30'	40'	50'	60'	
45°	o. 70711	o. 70916	o. 71121	o. 71325	o. 71529	o. 71732	o. 71934	44
46	o. 71934	o. 72136	o. 72337	o. 72537	o. 72737	o. 72937	o. 73135	43
47	o. 73135	o. 73333	o. 73531	o. 73728	o. 73924	o. 74120	o. 74314	42
48	o. 74314	o. 74509	o. 74703	o. 74896	o. 75088	o. 75280	o. 75471	41
49	o. 75471	o. 75661	o. 75851	o. 76041	o. 76229	o. 76417	o. 76604	40°
50°	o. 76604	o. 76791	o. 76977	o. 77162	o. 77347	o. 77531	o. 77715	39
51	o. 77715	o. 77897	o. 78079	o. 78261	o. 78442	o. 78622	o. 78801	38
52	o. 78801	o. 78980	o. 79158	o. 79335	o. 79512	o. 79688	o. 79864	37
53	o. 79864	o. 80038	o. 80212	o. 80386	o. 80558	o. 80730	o. 80902	36
54	o. 80902	o. 81072	o. 81242	o. 81412	o. 81580	o. 81748	o. 81915	35
55	o. 81915	o. 82082	o. 82248	o. 82413	o. 82577	o. 82741	o. 82904	34
56	o. 82904	o. 83066	o. 83228	o. 83389	o. 83549	o. 83708	o. 83867	33
57	o. 83867	o. 84025	o. 84182	o. 84339	o. 84495	o. 84650	o. 84805	32
58	o. 84805	o. 84959	o. 85112	o. 85264	o. 85416	o. 85567	o. 85717	31
59	o. 85717	o. 85866	o. 86015	o. 86163	o. 86310	o. 86457	o. 86603	30°
60°	o. 86603	o. 86748	o. 86892	o. 87036	o. 87178	o. 87321	o. 87462	29
61	o. 87462	o. 87603	o. 87743	o. 87882	o. 88020	o. 88158	o. 88295	28
62	o. 88295	o. 88431	o. 88566	o. 88701	o. 88835	o. 88968	o. 89101	27
63	o. 89101	o. 89232	o. 89363	o. 89493	o. 89623	o. 89752	o. 89879	26
64	o. 89879	o. 90007	o. 90133	o. 90259	o. 90383	o. 90507	o. 90631	25
65	o. 90631	o. 90753	o. 90875	o. 90996	o. 91116	o. 91236	o. 91355	24
66	o. 91355	o. 91472	o. 91590	o. 91706	o. 91822	o. 91936	o. 92050	23
67	o. 92050	o. 92164	o. 92276	o. 92388	o. 92499	o. 92609	o. 92718	22
68	o. 92718	o. 92827	o. 92935	o. 93042	o. 93148	o. 93253	o. 93358	21
69	o. 93358	o. 93462	o. 93565	o. 93667	o. 93769	o. 93869	o. 93969	20°
70°	o. 93969	o. 94068	o. 94167	o. 94264	o. 94361	o. 94457	o. 94552	19
71	o. 94552	o. 94646	o. 94740	o. 94832	o. 94924	o. 95015	o. 95106	18
72	o. 95106	o. 95195	o. 95284	o. 95372	o. 95459	o. 95545	o. 95630	17
73	o. 95630	o. 95715	o. 95799	o. 95882	o. 95964	o. 96046	o. 96126	16
74	o. 96126	o. 96206	o. 96285	o. 96363	o. 96440	o. 96517	o. 96593	15
75	o. 96593	o. 96667	o. 96742	o. 96815	o. 96887	o. 96959	o. 97030	14
76	o. 97030	o. 97100	o. 97169	o. 97237	o. 97304	o. 97371	o. 97437	13
77	o. 97437	o. 97502	o. 97566	o. 97630	o. 97692	o. 97754	o. 97815	12
78	o. 97815	o. 97875	o. 97934	o. 97992	o. 98050	o. 98107	o. 98163	11
79	o. 98163	o. 98218	o. 98272	o. 98325	o. 98378	o. 98430	o. 98481	10°
80°	o. 98481	o. 98531	o. 98580	o. 98629	o. 98676	o. 98723	o. 98769	9
81	o. 98769	o. 98814	o. 98858	o. 98902	o. 98944	o. 98986	o. 99027	8
82	o. 99027	o. 99067	o. 99106	o. 99144	o. 99182	o. 99219	o. 99255	7
83	o. 99255	o. 99290	o. 99324	o. 99357	o. 99390	o. 99421	o. 99452	6
84	o. 99452	o. 99482	o. 99511	o. 99540	o. 99567	o. 99594	o. 99619	5
85	o. 99619	o. 99644	o. 99668	o. 99692	o. 99714	o. 99736	o. 99756	4
86	o. 99756	o. 99776	o. 99795	o. 99813	o. 99831	o. 99847	o. 99863	3
87	o. 99863	o. 99878	o. 99892	o. 99905	o. 99917	o. 99929	o. 99939	2
88	o. 99939	o. 99949	o. 99958	o. 99966	o. 99973	o. 99979	o. 99985	1
89°	o. 99985	o. 99989	o. 99993	o. 99996	o. 99998	1. 00000	1. 00000	0°
	60'	50'	40'	30'	20'	10'	0'	Angle

## NATURAL COSINES.

TABLE VII.—NATURAL TANGENTS.

Angle	0'	10'	20'	30'	40'	50'	60'	
0°	0.00000	0.00291	0.00582	0.00873	0.01164	0.01455	0.01746	89°
1	0.01746	0.02036	0.02328	0.02619	0.02910	0.03201	0.03492	88
2	0.03492	0.03783	0.04075	0.04366	0.04658	0.04949	0.05241	87
3	0.05241	0.05533	0.05824	0.06116	0.06408	0.06700	0.06993	86
4	0.06993	0.07285	0.07578	0.07870	0.08163	0.08456	0.08749	85
5	0.08749	0.09042	0.09335	0.09629	0.09923	0.10216	0.10510	84
6	0.10510	0.10805	0.11099	0.11394	0.11688	0.11983	0.12278	83
7	0.12278	0.12574	0.12869	0.13165	0.13461	0.13758	0.14054	82
8	0.14054	0.14351	0.14648	0.14945	0.15243	0.15540	0.15838	81
9	0.15838	0.16137	0.16435	0.16734	0.17033	0.17333	0.17633	80°
10°	0.17633	0.17933	0.18233	0.18534	0.18835	0.19136	0.19438	79
11	0.19438	0.19740	0.20042	0.20345	0.20648	0.20952	0.21256	78
12	0.21256	0.21560	0.21864	0.22169	0.22475	0.22781	0.23087	77
13	0.23087	0.23393	0.23700	0.24008	0.24316	0.24624	0.24933	76
14	0.24933	0.25242	0.25552	0.25862	0.26172	0.26483	0.26795	75
15	0.26795	0.27107	0.27419	0.27732	0.28046	0.28360	0.28675	74
16	0.28675	0.28990	0.29305	0.29621	0.29938	0.30255	0.30573	73
17	0.30573	0.30891	0.31210	0.31530	0.31850	0.32171	0.32492	72
18	0.32492	0.32814	0.33136	0.33460	0.33783	0.34108	0.34433	71
19	0.34433	0.34758	0.35085	0.35421	0.35740	0.36068	0.36397	70°
20°	0.36397	0.36727	0.37057	0.37388	0.37720	0.38053	0.38386	69
21	0.38386	0.38721	0.39055	0.39391	0.39727	0.40065	0.40403	68
22	0.40403	0.40741	0.41081	0.41421	0.41763	0.42105	0.42447	67
23	0.42447	0.42791	0.43136	0.43481	0.43828	0.44175	0.44523	66
24	0.44523	0.44872	0.45222	0.45573	0.45924	0.46277	0.46631	65
25	0.46631	0.46985	0.47341	0.47698	0.48055	0.48414	0.48773	64
26	0.48773	0.49134	0.49495	0.49858	0.50222	0.50587	0.50953	63
27	0.50953	0.51320	0.51688	0.52057	0.52427	0.52798	0.53171	62
28	0.53171	0.53545	0.53920	0.54296	0.54673	0.55051	0.55431	61
29	0.55431	0.55812	0.56194	0.56577	0.56962	0.57348	0.57735	60°
30°	0.57735	0.58124	0.58513	0.58905	0.59297	0.59691	0.60086	59
31	0.60086	0.60483	0.60881	0.61280	0.61681	0.62083	0.62487	58
32	0.62487	0.62892	0.63299	0.63707	0.64117	0.64528	0.64941	57
33	0.64941	0.65355	0.65771	0.66189	0.66608	0.67028	0.67451	56
34	0.67451	0.67875	0.68301	0.68728	0.69157	0.69588	0.70021	55
35	0.70021	0.70455	0.70891	0.71329	0.71769	0.72211	0.72654	54
36	0.72654	0.73100	0.73547	0.73996	0.74447	0.74900	0.75355	53
37	0.75355	0.75812	0.76272	0.76733	0.77196	0.77661	0.78129	52
38	0.78129	0.78598	0.79070	0.79544	0.80020	0.80498	0.80978	51
39	0.80978	0.81461	0.81946	0.82434	0.82923	0.83415	0.83910	50°
40	0.83910	0.84407	0.84906	0.85408	0.85912	0.86419	0.86929	49
41	0.86929	0.87441	0.87955	0.88473	0.88992	0.89515	0.90040	48
42	0.90040	0.90569	0.91099	0.91633	0.92170	0.92709	0.93252	47
43	0.93252	0.93797	0.94345	0.94896	0.95451	0.96008	0.96569	46
44°	0.96569	0.97133	0.97700	0.98270	0.98843	0.99420	1.00000	45°
	60'	50'	40'	30'	20'	10'	0'	Angle

## NATURAL COTANGENTS.

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TABLE VIII.—NATURAL SECANTS.

Angle	0'	10'	20'	30'	40'	50'	60'	
0°	1.00000	1.00001	1.00002	1.00004	1.00007	1.00011	1.00015	89°
1	1.00015	1.00021	1.00027	1.00034	1.00042	1.00051	1.00061	88
2	1.00061	1.00072	1.00083	1.00095	1.00108	1.00122	1.00137	87
3	1.00137	1.00153	1.00169	1.00187	1.00205	1.00224	1.00244	86
4	1.00244	1.00265	1.00287	1.00309	1.00333	1.00357	1.00382	85
5	1.00382	1.00408	1.00435	1.00463	1.00491	1.00521	1.00551	84
6	1.00551	1.00582	1.00614	1.00647	1.00681	1.00715	1.00751	83
7	1.00751	1.00787	1.00825	1.00863	1.00902	1.00942	1.00983	82
8	1.00983	1.01024	1.01067	1.01111	1.01155	1.01200	1.01247	81
9	1.01247	1.01294	1.01342	1.01391	1.01440	1.01491	1.01543	80°
10°	1.01543	1.01595	1.01649	1.01703	1.01758	1.01815	1.01872	79
11	1.01872	1.01930	1.01989	1.02049	1.02110	1.02171	1.02234	78
12	1.02234	1.02298	1.02362	1.02428	1.02494	1.02562	1.02630	77
13	1.02630	1.02700	1.02770	1.02842	1.02914	1.02987	1.03061	76
14	1.03061	1.03137	1.03213	1.03290	1.03368	1.03447	1.03528	75
15	1.03528	1.03609	1.03691	1.03774	1.03858	1.03944	1.04030	74
16	1.04030	1.04117	1.04206	1.04295	1.04385	1.04477	1.04569	73
17	1.04569	1.04663	1.04757	1.04853	1.04950	1.05047	1.05146	72
18	1.05146	1.05246	1.05347	1.05449	1.05552	1.05657	1.05762	71
19	1.05762	1.05869	1.05976	1.06085	1.06195	1.06306	1.06418	70°
20°	1.06418	1.06531	1.06645	1.06761	1.06878	1.06995	1.07115	69
21	1.07115	1.07235	1.07356	1.07479	1.07602	1.07727	1.07853	68
22	1.07853	1.07981	1.08109	1.08239	1.08370	1.08503	1.08636	67
23	1.08636	1.08771	1.08907	1.09044	1.09183	1.09323	1.09464	66
24	1.09464	1.09606	1.09750	1.09895	1.10041	1.10189	1.10338	65
25	1.10338	1.10488	1.10640	1.10793	1.10947	1.11103	1.11260	64
26	1.11260	1.11419	1.11579	1.11740	1.11903	1.12067	1.12233	63
27	1.12233	1.12400	1.12568	1.12738	1.12910	1.13083	1.13257	62
28	1.13257	1.13433	1.13610	1.13789	1.13970	1.14152	1.14335	61
29	1.14335	1.14521	1.14707	1.14896	1.15085	1.15277	1.15470	60°
30°	1.15470	1.15665	1.15861	1.16059	1.16259	1.16460	1.16663	59
31	1.16663	1.16868	1.17075	1.17283	1.17493	1.17704	1.17918	58
32	1.17918	1.18133	1.18350	1.18569	1.18790	1.19012	1.19236	57
33	1.19236	1.19463	1.19691	1.19920	1.20152	1.20386	1.20622	56
34	1.20622	1.20859	1.21099	1.21341	1.21584	1.21830	1.22077	55
35	1.22077	1.22327	1.22579	1.22833	1.23089	1.23347	1.23607	54
36	1.23607	1.23869	1.24134	1.24400	1.24669	1.24940	1.25214	53
37	1.25214	1.25489	1.25767	1.26047	1.26330	1.26615	1.26902	52
38	1.26902	1.27191	1.27483	1.27778	1.28075	1.28374	1.28676	51
39	1.28676	1.28980	1.29287	1.29597	1.29909	1.30223	1.30541	50°
40°	1.30541	1.30861	1.31183	1.31509	1.31837	1.32168	1.32501	49
41	1.32501	1.32838	1.33177	1.33519	1.33864	1.34212	1.34563	48
42	1.34563	1.34917	1.35274	1.35634	1.35997	1.36363	1.36733	47
43	1.36733	1.37105	1.37481	1.37860	1.38242	1.38628	1.39016	46
44°	1.39016	1.39409	1.39804	1.40203	1.40606	1.41012	1.41421	45°
	60'	50'	40'	30'	20'	10'	0'	Angle

NATURAL COSECANTS.

TABLE VIII. — NATURAL SECANTS.

Angle	0'	10'	20'	30'	40'	50'	60'	
45°	1.41421	1.41835	1.42251	1.42672	1.43096	1.43524	1.43956	44°
46	1.43956	1.44391	1.44831	1.45274	1.45721	1.46173	1.46628	43
47	1.46628	1.47087	1.47551	1.48019	1.48491	1.48967	1.49448	42
48	1.49448	1.49933	1.50422	1.50916	1.51415	1.51918	1.52425	41
49	1.52425	1.52938	1.53455	1.53977	1.54504	1.55036	1.55572	40°
50°	1.55572	1.56114	1.56661	1.57213	1.57771	1.58333	1.58902	39
51	1.58902	1.59475	1.60054	1.60639	1.61229	1.61825	1.62427	38
52	1.62427	1.63035	1.63648	1.64268	1.64894	1.65526	1.66164	37
53	1.66164	1.66809	1.67460	1.68117	1.68782	1.69452	1.70130	36
54	1.70130	1.70815	1.71506	1.72205	1.72911	1.73624	1.74345	35
55	1.74345	1.75073	1.75808	1.76552	1.77303	1.78062	1.78829	34
56	1.7829	1.79604	1.80388	1.81180	1.81981	1.82790	1.83608	33
57	1.83608	1.84435	1.85271	1.86116	1.86990	1.87834	1.88708	32
58	1.88708	1.89591	1.90485	1.91388	1.92302	1.93226	1.94160	31
59	1.94160	1.95106	1.96062	1.97029	1.98008	1.98998	2.00000	30°
60°	2.00000	2.01014	2.02039	2.03077	2.04128	2.05191	2.06267	29
61	2.06267	2.07356	2.08458	2.09574	2.10704	2.11847	2.13005	28
62	2.13005	2.14178	2.15366	2.16568	2.17786	2.19019	2.20269	27
63	2.20269	2.21535	2.22817	2.24116	2.25432	2.26766	2.28117	26
64	2.28117	2.29487	2.30875	2.32282	2.33708	2.35154	2.36620	25
65	2.36620	2.38107	2.39614	2.41142	2.42692	2.44264	2.45859	24
66	2.45859	2.47477	2.49119	2.50784	2.52474	2.54190	2.55930	23
67	2.55930	2.57698	2.59491	2.61313	2.63162	2.65040	2.66947	22
68	2.66947	2.68884	2.70851	2.72850	2.74881	2.76945	2.79043	21
69	2.79043	2.81175	2.83342	2.85545	2.87785	2.90063	2.92380	20°
70°	2.92380	2.94737	2.97135	2.99574	3.02057	3.04584	3.07155	19
71	3.07155	3.09774	3.12440	3.15155	3.17920	3.20737	3.23607	18
72	3.23607	3.26531	3.29512	3.32551	3.35649	3.38808	3.42030	17
73	3.42030	3.45317	3.48671	3.52094	3.55587	3.59154	3.62796	16
74	3.62796	3.66515	3.70315	3.74198	3.78166	3.82223	3.86370	15
75	3.86370	3.90613	3.94952	3.99393	4.03938	4.08591	4.13357	14
76	4.13357	4.18238	4.23239	4.28366	4.33622	4.39012	4.44541	13
77	4.44541	4.50216	4.56041	4.62023	4.68167	4.74482	4.80973	12
78	4.80973	4.87649	4.94517	5.01585	5.08863	5.16359	5.24084	11
79	5.24084	5.32049	5.40263	5.48740	5.57493	5.66533	5.75877	10°
80°	5.75877	5.85539	5.95536	6.05886	6.16607	6.27719	6.39245	9
81	6.39245	6.51208	6.63633	6.76547	6.89979	7.03962	7.18530	8
82	7.18530	7.33719	7.49571	7.66130	7.83443	8.01565	8.20551	7
83	8.20551	8.46466	8.61379	8.83367	9.06515	9.30917	9.56677	6
84	9.56677	9.83912	10.12752	10.43343	10.75849	11.10455	11.47371	5
85	11.47371	11.86837	12.29125	12.74550	13.23472	13.76312	14.33559	4
86	14.33559	14.95788	15.63679	16.38041	17.19843	18.10262	19.10732	3
87	19.10732	20.23028	21.49368	22.92559	24.56212	26.45051	28.65371	2
88	28.65371	31.25758	34.38232	38.20155	42.97571	49.11406	57.29869	1
89°	57.29869	68.75736	85.94561	114.59301	171.88831	343.77516	Infinite.	0°
	60'	50'	40'	30'	20'	10'	0'	Angle

## NATURAL COSECANTS.

TABLE IX.—LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

Arc	Angle	Sin	Csc	Tan	Ctn	Sec	Cos		
	0° 00'	-∞	+∞	-∞	+∞	0. 00000	0. 00000	90° 00'	1. 5708
0. 0000	0° 00'	-∞	+∞	-∞	+∞	0. 00000	0. 00000	90° 00'	1. 5708
0. 0020	10°	7. 46373	2. 53627	7. 46373	2. 53627	00000	00000	50°	1. 5679
0. 0058	20°	76475	23525	76476	23524	00001	9. 99999	40°	1. 5650
0. 0087	30°	94084	05916	94086	05914	00002	99998	30°	1. 5621
0. 0116	40°	8. 06578	1. 93422	8. 06581	1. 93419	00003	99997	20°	1. 5592
0. 0145	50°	16268	83732	16273	83727	00005	99995	10°	1. 5563
0. 0175	1° 00'	8. 24186	1. 75814	8. 24192	1. 75808	0. 00007	9. 99993	89° 00'	1. 5533
0. 0204	10°	30879	69121	30888	69112	00009	99991	50°	1. 5504
0. 0233	20°	36678	63322	36689	63311	00012	99988	40°	1. 5475
0. 0262	30°	41792	58208	41807	58193	00015	99985	30°	1. 5446
0. 0291	40°	46366	53634	46385	53615	00018	99982	20°	1. 5417
0. 0320	50°	50504	49496	50527	49473	00022	99978	10°	1. 5388
0. 0349	2° 00'	8. 54282	1. 45718	8. 54308	1. 45692	0. 00026	9. 99974	88° 00'	1. 5359
0. 0378	10°	57757	42243	57788	42212	00031	99969	50°	1. 5330
0. 0407	20°	60973	39027	61009	38991	00036	99964	40°	1. 5301
0. 0436	30°	63968	36032	64009	35991	00041	99959	30°	1. 5272
0. 0465	40°	66769	33231	66816	33184	00047	99953	20°	1. 5243
0. 0495	50°	69400	30600	69453	30547	00053	99947	10°	1. 5213
0. 0524	3° 00'	8. 71880	1. 28120	8. 71940	1. 28060	0. 00060	9. 99940	87° 00'	1. 5184
0. 0553	10°	74226	25774	74292	25708	00066	99934	50°	1. 5155
0. 0582	20°	76451	23549	76525	23475	00074	99926	40°	1. 5126
0. 0611	30°	78568	21432	78649	21351	00081	99919	30°	1. 5097
0. 0640	40°	80585	19415	80674	19326	00089	99911	20°	1. 5068
0. 0669	50°	82513	17487	82610	17390	00097	99903	10°	1. 5039
0. 0698	4° 00'	8. 84358	1. 15642	8. 84464	1. 15536	0. 00106	9. 99894	86° 00'	1. 5010
0. 0727	10°	86128	13872	86243	13757	00115	99885	50°	1. 4981
0. 0756	20°	87829	12171	87953	12047	00124	99876	40°	1. 4952
0. 0785	30°	89464	10536	89598	10402	00134	99866	30°	1. 4923
0. 0814	40°	91040	08960	91185	08815	00144	99856	20°	1. 4893
0. 0844	50°	92561	07439	92716	07284	00155	99845	10°	1. 4864
0. 0873	5° 00'	8. 94030	1. 05970	8. 94195	1. 05805	0. 00166	9. 99834	85° 00'	1. 4835
0. 0902	10°	95450	04550	95627	04373	00177	99823	50°	1. 4806
0. 0931	20°	96825	03175	97013	02987	00188	99812	40°	1. 4777
0. 0960	30°	98157	01843	98358	01642	00200	99800	30°	1. 4748
0. 0989	40°	99450	00550	99662	00338	00213	99787	20°	1. 4719
0. 1018	50°	9. 00704	0. 99296	9. 00930	0. 99070	00225	99775	10°	1. 4690
0. 1047	6° 00'	9. 01923	0. 98077	9. 02162	0. 97838	0. 00239	9. 99761	84° 00'	1. 4661
0. 1076	10°	03109	96891	03361	96639	00252	99748	50°	1. 4632
0. 1105	20°	04262	95738	04528	95472	00266	99734	40°	1. 4603
0. 1134	30°	05386	94614	05666	94334	00280	99720	30°	1. 4574
0. 1164	40°	06481	93519	06775	93225	00295	99705	20°	1. 4544
0. 1193	50°	07548	92452	07858	92142	00310	99690	10°	1. 4515
0. 1222	7° 00'	9. 08589	0. 91411	9. 08914	0. 91086	0. 00325	9. 99675	83° 00'	1. 4486
0. 1251	10°	09606	90394	09947	90053	00341	99659	50°	1. 4457
0. 1280	20°	10599	89401	10956	89044	00357	99643	40°	1. 4428
0. 1309	30°	11570	88430	11943	88057	00373	99627	30°	1. 4399
0. 1338	40°	12519	87481	12909	87091	00390	99610	20°	1. 4370
0. 1367	50°	13447	86553	13854	86146	00407	99593	10°	1. 4341
0. 1396	8° 00'	9. 14356	0. 85644	9. 14780	0. 85220	0. 00425	9. 99575	82° 00'	1. 4312
0. 1425	10°	15245	84755	15688	84312	00443	99557	50°	1. 4283
0. 1454	20°	16116	83884	16577	83423	00461	99539	40°	1. 4254
0. 1484	30°	16970	83030	17450	82550	00480	99520	30°	1. 4224
0. 1513	40°	17807	82193	18306	81694	00499	99501	20°	1. 4195
0. 1542	50°	18628	81372	19146	80854	00518	99482	10°	1. 4166
0. 1571	9° 00'	9. 19433	0. 80567	9. 19971	0. 80029	0. 00538	9. 99462	81° 00'	1. 4137
		Cos	Sec	Ctn	Tan	Csc	Sin	Angle	Arc

TABLE IX.—LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

Arc	Angle	Sin	Csc	Tan	Ctn	Sec	Cos		
0.1571	9° 00'	0.19433	0.80567	0.19971	0.80029	0.00538	0.99462	81° 00'	1.4137
0.1600	10	20223	79777	20782	79218	00558	99442	50	1.4108
0.1629	20	20999	79001	21578	78422	00579	99421	40	1.4079
0.1658	30	21761	78239	22361	77639	00600	99400	30	1.4050
0.1687	40	22509	77491	23130	76870	00621	99379	20	1.4021
0.1716	50	23244	76756	23887	76113	00643	99357	10	1.3992
0.1745	10° 00'	0.23967	0.76033	0.24632	0.75368	0.00665	0.99335	80° 00'	1.3963
0.1774	10	24677	75323	25365	74635	00687	99313	50	1.3934
0.1804	20	25376	74624	26086	73914	00710	99290	40	1.3904
0.1833	30	26063	73937	26797	73203	00733	99267	30	1.3875
0.1862	40	26739	73261	27496	72504	00757	99243	20	1.3846
0.1891	50	27405	72595	28186	71814	00781	99219	10	1.3817
0.1920	11° 00'	0.28060	0.71940	0.28865	0.71135	0.00805	0.99195	79° 00'	1.3788
0.1949	10	28705	71295	29535	70465	00830	99170	50	1.3759
0.1978	20	29340	70660	30195	69805	00855	99145	40	1.3730
0.2007	30	29966	70034	30846	69154	00881	99119	30	1.3701
0.2036	40	30582	69418	31489	68511	00907	99093	20	1.3672
0.2065	50	31189	68811	32122	67878	00933	99067	10	1.3643
0.2094	12° 00'	0.31788	0.68212	0.32747	0.67253	0.00960	0.99040	78° 00'	1.3614
0.2123	10	32378	67622	33365	66635	00987	99013	50	1.3584
0.2153	20	32960	67040	33974	66026	01014	98986	40	1.3555
0.2182	30	33534	66466	34576	65424	01042	98958	30	1.3526
0.2211	40	34100	65900	35170	64830	01070	98930	20	1.3497
0.2240	50	34658	65342	35757	64243	01099	98901	10	1.3468
0.2269	13° 00'	0.35209	0.64791	0.36336	0.63664	0.01128	0.98872	77° 00'	1.3439
0.2298	10	35752	64248	36909	63091	01157	98843	50	1.3410
0.2327	20	36289	63711	37476	62524	01187	98813	40	1.3381
0.2356	30	36819	63181	38035	61965	01217	98783	30	1.3352
0.2385	40	37341	62659	38589	61411	01247	98753	20	1.3323
0.2414	50	37858	62142	39136	60864	01278	98722	10	1.3294
0.2443	14° 00'	0.38368	0.61632	0.39677	0.60323	0.01310	0.98690	76° 00'	1.3265
0.2473	10	38871	61129	40212	59788	01341	98659	50	1.3235
0.2502	20	39369	60631	40742	59258	01373	98627	40	1.3206
0.2531	30	39860	60140	41266	58734	01406	98594	30	1.3177
0.2560	40	40346	59654	41784	58216	01439	98561	20	1.3148
0.2589	50	40825	59175	42297	57703	01472	98528	10	1.3119
0.2618	15° 00'	0.41300	0.58700	0.42805	0.57195	0.01506	0.98494	75° 00'	1.3090
0.2647	10	41768	58232	43308	56692	01540	98460	50	1.3061
0.2676	20	42232	57768	43806	56194	01574	98426	40	1.3032
0.2705	30	42690	57310	44299	55701	01609	98391	30	1.3003
0.2734	40	43143	56857	44787	55213	01644	98356	20	1.2974
0.2763	50	43591	56409	45271	54729	01680	98320	10	1.2945
0.2793	16° 00'	0.44034	0.55966	0.45750	0.54250	0.01716	0.98284	74° 00'	1.2915
0.2822	10	44472	55528	46224	53776	01752	98248	50	1.2886
0.2851	20	44905	55095	46694	53306	01789	98211	40	1.2857
0.2880	30	45334	54666	47160	52840	01826	98174	30	1.2828
0.2909	40	45758	54242	47622	52378	01864	98136	20	1.2799
0.2938	50	46178	53822	48080	51920	01902	98098	10	1.2770
0.2967	17° 00'	0.46594	0.53406	0.48534	0.51466	0.01940	0.98060	73° 00'	1.2741
0.2996	10	47005	52995	48984	51016	01979	98021	50	1.2712
0.3025	20	47411	52589	49430	50570	02018	97982	40	1.2683
0.3054	30	47814	52186	49872	50128	02058	97942	30	1.2654
0.3083	40	48213	51787	50311	49689	02098	97902	20	1.2625
0.3113	50	48607	51393	50746	49254	02139	97861	10	1.2595
0.3142	18° 00'	0.48998	0.51002	0.51178	0.48822	0.02179	0.97821	72° 00'	1.2566
		Cos	Sec	Ctn	Tan	Csc	Sin	Angle	Arc

TABLE IX.—LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

Arc	Angle	Sin	Csc	Tan	Ctn	Sec	Cos		
		Cos	Sec	Ctn	Tan	Csc	Sin	Angle	Arc
o. 3142	18° 00'	9. 48998	o. 51002	9. 51178	o. 48822	o. 02179	9. 97821	72° 00'	1. 2566
o. 3171	10	49385	50615	51606	48394	o. 02221	97779	50	1. 2537
o. 3200	20	49768	50232	52031	47969	o. 02262	97738	40	1. 2508
o. 3229	30	50148	49852	52452	47548	o. 02304	97696	30	1. 2479
o. 3258	40	50523	49477	52870	47130	o. 02347	97653	20	1. 2450
o. 3287	50	50896	49104	53285	46715	o. 02390	97610	10	1. 2421
o. 3316	19° 00'	9. 51264	o. 48736	9. 53697	o. 46303	o. 02433	9. 97567	71° 00'	1. 2392
o. 3345	10	51629	48371	54106	45804	o. 02477	97523	50	1. 2363
o. 3374	20	51991	48009	54512	45488	o. 02521	97479	40	1. 2334
o. 3403	30	52350	47650	54915	45085	o. 02565	97435	30	1. 2305
o. 3432	40	52705	47295	55315	44685	o. 02610	97390	20	1. 2275
o. 3462	50	53056	46944	55712	44288	o. 02656	97344	10	1. 2246
o. 3491	20° 00'	9. 53405	o. 46595	9. 56107	o. 43893	o. 02701	9. 97299	70° 00'	1. 2217
o. 3520	10	53751	46249	56498	43502	o. 02748	97252	50	1. 2188
o. 3549	20	54093	45907	56887	43113	o. 02794	97206	40	1. 2159
o. 3578	30	54433	45567	57274	42726	o. 02841	97159	30	1. 2130
o. 3607	40	54769	45231	57658	42342	o. 02889	97111	20	1. 2101
o. 3636	50	55102	44898	58039	41961	o. 02937	97063	10	1. 2072
o. 3665	21° 00'	9. 55433	o. 44567	9. 58418	o. 41582	o. 02985	o. 97015	69° 00'	1. 2043
o. 3694	10	55761	44239	58794	41206	o. 03034	96966	50	1. 2014
o. 3723	20	56085	43915	59168	40832	o. 03083	96917	40	1. 1985
o. 3752	30	56408	43592	59540	40460	o. 03132	96868	30	1. 1956
o. 3782	40	56727	43273	59909	40091	o. 03182	96818	20	1. 1926
o. 3811	50	57044	42956	60276	39724	o. 03233	96767	10	1. 1897
o. 3840	22° 00'	9. 57358	o. 42642	9. 60641	o. 39359	o. 03283	9. 96717	68° 00'	1. 1868
o. 3869	10	57669	42331	61004	38996	o. 03335	96665	50	1. 1839
o. 3898	20	57978	42022	61364	38636	o. 03386	96614	40	1. 1810
o. 3927	30	58284	41716	61722	38278	o. 03438	96562	30	1. 1781
o. 3956	40	58588	41412	62079	37921	o. 03491	96509	20	1. 1752
o. 3985	50	58889	41111	62433	37567	o. 03544	96456	10	1. 1723
o. 4014	23° 00'	9. 59188	o. 40812	9. 62785	o. 37215	o. 03597	9. 96403	67° 00'	1. 1694
o. 4043	10	59484	40516	63135	36865	o. 03651	96349	50	1. 1665
o. 4072	20	59778	40222	63484	36516	o. 03706	96294	40	1. 1636
o. 4102	30	60070	39930	63830	36170	o. 03760	96240	30	1. 1606
o. 4131	40	60359	39641	64175	35825	o. 03815	96185	20	1. 1577
o. 4160	50	60646	39354	64517	35483	o. 03871	96129	10	1. 1548
o. 4189	24° 00'	9. 60931	o. 39069	9. 64858	o. 35142	o. 03927	9. 96073	66° 00'	1. 1519
o. 4218	10	61214	38786	65197	34803	o. 03983	96017	50	1. 1490
o. 4247	20	61494	38506	65535	34465	o. 04040	95960	40	1. 1461
o. 4276	30	61773	38227	65870	34130	o. 04098	95902	30	1. 1432
o. 4305	40	62049	37951	66204	33796	o. 04156	95844	20	1. 1403
o. 4334	50	62323	37677	66537	33463	o. 04214	95786	10	1. 1374
o. 4363	25° 00'	9. 62595	o. 37405	9. 66867	o. 33133	o. 04272	9. 95728	65° 00'	1. 1345
o. 4392	10	62865	37135	67196	32804	o. 04332	95668	50	1. 1316
o. 4422	20	63133	36867	67524	32476	o. 04391	95609	40	1. 1286
o. 4451	30	63398	36602	67850	32150	o. 04451	95549	30	1. 1257
o. 4480	40	63662	36338	68174	31826	o. 04512	95488	20	1. 1228
o. 4509	50	63924	36076	68497	31503	o. 04573	95427	10	1. 1199
o. 4538	26° 00'	9. 64184	o. 35816	9. 68818	o. 31182	o. 04634	9. 95366	64° 00'	1. 1170
o. 4567	10	64442	35558	69138	30862	o. 04696	95304	50	1. 1141
o. 4596	20	64698	35302	69457	30543	o. 04758	95242	40	1. 1112
o. 4625	30	64953	35047	69774	30226	o. 04821	95179	30	1. 1083
o. 4654	40	65205	34795	70089	29911	o. 04884	95116	20	1. 1054
o. 4683	50	65456	34544	70404	29596	o. 04948	95052	10	1. 1025
o. 4712	27° 00'	9. 65705	o. 34295	9. 70717	o. 29283	o. 05012	9. 94988	63° 00'	1. 0996

TABLE IX.—LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

Arc	Angle	Sin	Csc	Tan	Ctn	Sec	Cos		
o. 4712	27° 00'	9. 65705	o. 34295	9. 70717	o. 29283	o. 05012	9. 94988	63° 00'	1. 0096
o. 4741	10	65952	34048	71028	28972	05077	94923	50	1. 0966
o. 4771	20	66197	33803	71339	28661	05142	94858	40	1. 0937
o. 4800	30	66441	33559	71648	28352	05207	94793	30	1. 0908
o. 4829	40	66682	33318	71955	28045	05273	94727	20	1. 0879
o. 4858	50	66922	33078	72262	27738	05340	94660	10	1. 0850
o. 4887	28° 00'	9. 67161	o. 32839	9. 72567	o. 27433	o. 05407	9. 94593	62° 00'	1. 0821
o. 4916	10	67398	32602	72872	27128	05474	94526	50	1. 0792
o. 4945	20	67633	32367	73175	26825	05542	94458	40	1. 0763
o. 4974	30	67866	32134	73476	26524	05610	94390	30	1. 0734
o. 5003	40	68098	31902	73777	26223	05679	94321	20	1. 0705
o. 5032	50	68328	31672	74077	25923	05748	94252	10	1. 0676
o. 5061	29° 00'	9. 68557	o. 31443	9. 74375	o. 25625	o. 05818	9. 94182	61° 00'	1. 0647
o. 5091	10	68784	31216	74673	25327	05888	94112	50	1. 0617
o. 5120	20	69010	30990	74969	25031	05959	94041	40	1. 0588
o. 5149	30	69234	30766	75264	24736	06030	93970	30	1. 0559
o. 5178	40	69456	30544	75558	24442	06102	93898	20	1. 0530
o. 5207	50	69677	30323	75852	24148	06174	93826	10	1. 0501
o. 5236	30° 00'	9. 69897	o. 30103	9. 76144	o. 23856	o. 06247	9. 93753	60° 00'	1. 0472
o. 5265	10	70115	29885	76435	23565	06320	93680	50	1. 0443
o. 5294	20	70332	29668	76725	23275	06394	93606	40	1. 0414
o. 5323	30	70547	29453	77015	22985	06468	93532	30	1. 0385
o. 5352	40	70761	29239	77303	22697	06543	93457	20	1. 0356
o. 5381	50	70973	29027	77591	22409	06618	93382	10	1. 0327
o. 5411	31° 00'	9. 71184	o. 28816	9. 77877	o. 22123	o. 06693	9. 93307	59° 00'	1. 0297
o. 5440	10	71393	28607	78163	21837	06770	93230	50	1. 0268
o. 5469	20	71602	28398	78448	21552	06846	93154	40	1. 0239
o. 5498	30	71809	28191	78732	21268	06923	93077	30	1. 0210
o. 5527	40	72014	27986	79015	20985	07001	92999	20	1. 0181
o. 5556	50	72218	27782	79297	20703	07079	92921	10	1. 0152
o. 5585	32° 00'	9. 72421	o. 27579	9. 79579	o. 20421	o. 07158	9. 92842	58° 00'	1. 0123
o. 5614	10	72622	27378	79860	20140	07237	92763	50	1. 0094
o. 5643	20	72823	27177	80140	19860	07317	92683	40	1. 0065
o. 5672	30	73022	26978	80419	19581	07397	92603	30	1. 0036
o. 5701	40	73219	26781	80697	19303	07478	92522	20	1. 0007
o. 5730	50	73416	26584	80975	19025	07559	92441	10	0. 9977
o. 5760	33° 00'	9. 73611	o. 26389	9. 81252	o. 18748	o. 07641	9. 92359	57° 00'	0. 9948
o. 5789	10	73805	26195	81528	18472	07723	92277	50	0. 9919
o. 5818	20	73997	26003	81803	18197	07806	92194	40	0. 9890
o. 5847	30	74189	25811	82078	17922	07889	92111	30	0. 9861
o. 5876	40	74379	25621	82352	17648	07973	92027	20	0. 9832
o. 5905	50	74568	25432	82626	17374	08058	91942	10	0. 9803
o. 5934	34° 00'	9. 74756	o. 25244	9. 82899	o. 17101	o. 08143	9. 91857	56° 00'	0. 9774
o. 5963	10	74943	25057	83171	16829	08228	91772	50	0. 9745
o. 5992	20	75128	24872	83442	16558	08314	91686	40	0. 9716
o. 6021	30	75313	24687	83713	16287	08401	91599	30	0. 9687
o. 6050	40	75496	24504	83984	16016	08488	91512	20	0. 9657
o. 6080	50	75678	24322	84254	15746	08575	91425	10	0. 9628
o. 6109	35° 00'	9. 75859	o. 24141	9. 84523	o. 15477	o. 08664	9. 91336	55° 00'	0. 9599
o. 6138	10	76039	23961	84791	15209	08752	91248	50	0. 9570
o. 6167	20	76218	23782	85059	14941	08842	91158	40	0. 9541
o. 6196	30	76395	23605	85327	14673	08931	91069	30	0. 9512
o. 6225	40	76572	23428	85594	14406	09022	90978	20	0. 9483
o. 6254	50	76747	23253	85860	14140	09113	90887	10	0. 9454
o. 6283	36° 00'	9. 76922	o. 23078	9. 86126	o. 13874	o. 09204	9. 90796	54° 00'	0. 9425
		Cos	Sec	Ctn	Tan	Csc	Sin	Angle	Arc

TABLE IX.—LOGARITHMS OF TRIGONOMETRIC FUNCTIONS.

Arc	Angle	Sin	Csc	Tan	Ctn	Sec	Cos		
o.6283	36°00'	9.76922	o.23078	9.86126	o.13874	o.09204	9.90796	54°00'	o.9425
o.6312	10	77095	22905	86392	13608	09296	90704	50	o.9396
o.6341	20	77268	22732	86656	13344	09389	90611	40	o.9367
o.6370	30	77439	22561	86921	13079	09482	90518	30	o.9338
o.6400	40	77609	22391	87185	12815	09576	90424	20	o.9308
o.6429	50	77778	22222	87448	12552	09670	90330	10	o.9279
o.6458	37°00'	9.77946	o.22054	9.87711	o.12289	o.09765	9.90235	53°00'	o.9250
o.6487	10	78113	21887	87974	12026	09861	90139	50	o.9221
o.6516	20	78280	21720	88236	11764	09957	90043	40	o.9192
o.6545	30	78445	21555	88498	11502	10053	89947	30	o.9163
o.6574	40	78609	21391	88759	11241	10151	89849	20	o.9134
o.6603	50	78772	21228	89020	10980	10248	89752	10	o.9105
o.6632	38°00'	9.78934	o.21066	9.89281	o.10719	o.10347	9.89653	52°00'	o.9076
o.6661	10	79095	20905	89541	10459	10446	89554	50	o.9047
o.6690	20	79256	20744	89801	10199	10545	89455	40	o.9018
o.6720	30	79415	20585	90061	09939	10646	89354	30	o.8988
o.6749	40	79573	20427	90320	09680	10746	89254	20	o.8959
o.6778	50	79731	20269	90578	09422	10848	89152	10	o.8930
o.6807	39°00'	9.79887	o.20113	9.90837	o.09163	o.10950	9.89050	51°00'	o.8901
o.6836	10	80043	19957	91095	08905	11052	88948	50	o.8872
o.6865	20	80197	19803	91353	08647	11156	88844	40	o.8843
o.6894	30	80351	19649	91610	08390	11259	88741	30	o.8814
o.6923	40	80504	19496	91868	08132	11364	88636	20	o.8785
o.6952	50	80656	19344	92125	07875	11469	88531	10	o.8756
o.6981	40°00'	9.80807	o.19193	9.92381	o.07619	o.11575	9.88425	50°00'	o.8727
o.7010	10	80957	19043	92638	07362	11681	88319	50	o.8698
o.7039	20	81100	18894	92894	07106	11788	88212	40	o.8668
o.7069	30	81254	18746	93150	06850	11895	88105	30	o.8639
o.7098	40	81402	18598	93406	06594	12004	87996	20	o.8610
o.7127	50	81549	18451	93661	06339	12113	87887	10	o.8581
o.7156	41°00'	9.81694	o.18306	9.93916	o.06084	o.12222	9.87778	49°00'	o.8552
o.7185	10	81839	18161	94171	05829	12332	87668	50	o.8523
o.7214	20	81983	18017	94426	05574	12443	87557	40	o.8494
o.7243	30	82126	17874	94681	05319	12554	87446	30	o.8465
o.7272	40	82269	17731	94935	05065	12666	87334	20	o.8436
o.7301	50	82410	17590	95190	04810	12779	87221	10	o.8407
o.7330	42°00'	9.82551	o.17449	9.95444	o.04556	o.12893	9.87107	48°00'	o.8378
o.7359	10	82691	17309	95698	04302	13007	86993	50	o.8348
o.7389	20	82830	17170	95952	04048	13121	86879	40	o.8319
o.7418	30	82968	17032	96205	03795	13237	86763	30	o.8290
o.7447	40	83106	16894	96459	03541	13353	86647	20	o.8261
o.7476	50	83242	16758	96712	03288	13470	86530	10	o.8232
o.7505	43°00'	9.83378	o.16622	9.96966	o.03034	o.13587	9.86413	47°00'	o.8203
o.7534	10	83513	16487	97219	02781	13705	86295	50	o.8174
o.7563	20	83648	16352	97472	02528	13824	86176	40	o.8145
o.7592	30	83781	16219	97725	02275	13944	86056	30	o.8116
o.7621	40	83914	16086	97978	02022	14064	85936	20	o.8087
o.7650	50	84046	15954	98231	01769	14185	85815	10	o.8058
o.7679	44°00'	9.84177	o.15823	9.98484	o.01516	o.14307	9.85693	46°00'	o.8029
o.7709	10	84308	15692	98737	01263	14429	85571	50	o.7999
o.7738	20	84437	15563	98989	01011	14552	85448	40	o.7970
o.7767	30	84566	15434	99242	00758	14676	85324	30	o.7941
o.7796	40	84694	15306	99495	00505	14800	85200	20	o.7912
o.7825	50	84822	15178	99747	00253	14926	85074	10	o.7883
o.7854	45°00'	9.84949	o.15051	o.00000	o.00000	o.15051	9.84949	45°00'	o.7854
		Cos	Sec	Ctn'	Tan	Csc	Sin	Angle	Arc

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TABLE XI.

Dia-meter <i>d</i>	Circum-ference of the Circle. $\pi d$	Area of the Circle $\frac{1}{4} \pi d^2$	Volume of the Sphere $\frac{4}{3} \pi d^3$	Di-a-meter <i>d</i>	Circum-ference of the Circle $\pi d$	Area of the Circle $\frac{1}{4} \pi d^2$	Volume of the Sphere $\frac{4}{3} \pi d^3$
1	3. 1416	0. 7854	0. 523599	51	160. 22	2042. 82	69455. 91
2	6. 2832	3. 1416	4. 188790	52	163. 36	2123. 72	73622. 18
3	9. 4248	7. 0686	14. 13717	53	166. 50	2206. 18	77951. 81
4	12. 5664	12. 5664	33. 51032	54	169. 65	2290. 22	82447. 92
5	15. 7080	19. 6350	65. 44985	55	172. 79	2375. 83	87113. 75
6	18. 8496	28. 2743	113. 0973	56	175. 93	2463. 01	91952. 32
7	21. 9911	38. 4845	179. 5944	57	179. 07	2551. 76	96966. 83
8	25. 1327	50. 2655	268. 0826	58	182. 21	2642. 08	102160. 4
9	28. 2743	63. 6173	381. 7035	59	185. 35	2733. 97	107536. 2
10	31. 4159	78. 5398	523. 5988	60	188. 50	2827. 43	113097. 3
11	34. 5575	95. 0332	696. 9100	61	191. 64	2922. 47	118847. 0
12	37. 6991	113. 097	904. 7787	62	194. 78	3019. 07	124788. 2
13	40. 8407	132. 732	1150. 347	63	197. 92	3117. 25	130924. 3
14	43. 9823	153. 938	1436. 755	64	201. 06	3216. 99	137258. 2
15	47. 1239	176. 715	1767. 146	65	204. 20	3318. 31	143793. 3
16	50. 2655	201. 062	2144. 660	66	207. 35	3421. 19	150532. 6
17	53. 4071	226. 980	2572. 441	67	210. 49	3525. 65	157479. 1
18	56. 5487	254. 469	3053. 628	68	213. 63	3631. 68	164636. 2
19	59. 6903	283. 529	3591. 364	69	216. 77	3739. 28	172006. 9
20	62. 8319	314. 159	4188. 790	70	219. 91	3848. 45	179594. 4
21	65. 9734	346. 361	4849. 048	71	223. 05	3959. 19	187401. 8
22	69. 1150	380. 133	5575. 280	72	226. 19	4071. 50	195432. 2
23	72. 2566	415. 476	6370. 626	73	229. 34	4185. 39	203688. 8
24	75. 3982	452. 389	7238. 229	74	232. 48	4300. 84	212174. 8
25	78. 5398	490. 874	8181. 231	75	235. 62	4417. 86	220893. 2
26	81. 6814	530. 929	9202. 772	76	238. 76	4536. 46	229847. 3
27	84. 8230	572. 555	10305. 99	77	241. 90	4656. 63	239040. 1
28	87. 9646	615. 752	11494. 04	78	245. 04	4778. 36	248474. 9
29	91. 1062	660. 520	12770. 05	79	248. 19	4901. 67	258154. 6
30	94. 2478	706. 858	14137. 17	80	251. 33	5026. 55	268082. 6
31	97. 389	754. 768	15598. 53	81	254. 47	5153. 00	278261. 8
32	100. 531	804. 248	17157. 28	82	257. 61	5281. 02	288695. 6
33	103. 673	855. 299	18816. 57	83	260. 75	5410. 61	299387. 0
34	106. 814	907. 920	20579. 53	84	263. 89	5541. 77	310339. 1
35	109. 956	962. 113	22449. 30	85	267. 04	5674. 50	321555. 1
36	113. 097	1017. 88	24429. 02	86	270. 18	5808. 80	333038. 2
37	116. 239	1075. 21	26521. 85	87	273. 32	5944. 68	344791. 4
38	119. 381	1134. 11	28730. 91	88	276. 46	6082. 12	356817. 9
39	122. 522	1194. 59	31059. 36	89	279. 60	6221. 14	369120. 9
40	125. 664	1256. 64	33510. 32	90	282. 74	6361. 73	381703. 5
41	128. 81	1320. 25	36086. 95	91	285. 88	6503. 88	394568. 9
42	131. 95	1385. 44	38792. 39	92	289. 03	6647. 61	407720. 1
43	135. 09	1452. 20	41629. 77	93	292. 17	6792. 91	421160. 3
44	138. 23	1520. 53	44602. 24	94	295. 31	6939. 78	434892. 8
45	141. 37	1590. 43	47712. 94	95	298. 45	7088. 22	448920. 5
46	144. 51	1661. 90	50965. 01	96	301. 59	7238. 23	463246. 7
47	147. 65	1734. 94	54361. 60	97	304. 73	7389. 81	477874. 5
48	150. 80	1809. 56	57905. 84	98	307. 88	7542. 96	492807. 0
49	153. 94	1885. 74	61600. 87	99	311. 02	7697. 69	508047. 4
50	157. 08	1963. 50	65449. 85	100	314. 16	7853. 98	523598. 8

TABLE XI.

Dia-meter <i>d</i>	Circum-ference of the Circle $\pi d$	Area of the Circle $\frac{1}{4} \pi d^2$	Volume of the Sphere $\frac{4}{3} \pi d^3$	Dia-meter <i>d</i>	Circum-ference of the Circle $\pi d$	Area of the Circle $\frac{1}{4} \pi d^2$	Volume of the Sphere $\frac{4}{3} \pi d^3$
101	317.30	8011.85	539464.3	151	474.38	17907.9	1802725
102	320.44	8171.28	555647.2	152	477.52	18145.8	1838778
103	323.58	8332.29	572150.5	153	480.66	18385.4	1875309
104	326.73	8494.87	588977.4	154	483.81	18626.5	1912321
105	329.87	8659.01	606131.0	155	486.95	18869.2	1949816
106	333.01	8824.73	623614.5	156	490.09	19113.4	1987799
107	336.15	8992.02	641431.0	157	493.23	19359.3	2026271
108	339.29	9160.88	659583.7	158	496.37	19606.7	2065237
109	342.43	9331.32	678075.6	159	499.51	19855.7	2104699
110	345.58	9503.32	696910.0	160	502.65	20106.2	2144660
111	348.72	9676.89	716090.0	161	505.80	20358.3	2185125
112	351.86	9852.03	735618.6	162	508.94	20612.0	2226094
113	355.00	10028.7	755499.1	163	512.08	20867.2	2267574
114	358.14	10207.0	775734.6	164	515.22	21124.1	2309565
115	361.28	10386.9	796328.3	165	518.36	21382.5	2352071
116	364.42	10568.3	817283.2	166	521.50	21642.4	2395096
117	367.57	10751.3	838602.7	167	524.65	21904.0	2438642
118	370.71	10935.9	860289.5	168	527.79	22167.1	2482713
119	373.85	11122.0	882347.3	169	530.93	22431.8	2527311
120	376.99	11309.7	904778.7	170	534.07	22698.0	2572441
121	380.13	11499.0	927587.2	171	537.21	22965.8	2618104
122	383.27	11689.9	950775.8	172	540.35	23235.2	2664305
123	386.42	11882.3	974347.7	173	543.50	23506.2	2711046
124	389.56	12076.3	998305.9	174	546.64	23778.7	2758331
125	392.70	12271.8	1022654	175	549.78	24052.8	2806162
126	395.84	12469.0	1047394	176	552.92	24328.5	2854543
127	398.98	12667.7	1072531	177	556.06	24605.7	2903477
128	402.12	12868.0	1098066	178	559.20	24884.6	2952967
129	405.27	13069.8	1124004	179	562.35	25164.9	3003006
130	408.41	13273.2	1150347	180	565.49	25446.9	3053628
131	411.55	13478.2	1177098	181	568.63	25730.4	3104805
132	414.69	13684.8	1204260	182	571.77	26015.5	3156551
133	417.83	13892.9	1231838	183	574.91	26302.2	3208869
134	420.97	14102.6	1259833	184	578.05	26590.4	3261761
135	424.12	14313.9	1288249	185	581.19	26880.3	3315231
136	427.26	14526.7	1317090	186	584.34	27171.6	3369282
137	430.40	14741.1	1346357	187	587.48	27464.6	3423919
138	433.54	14957.1	1376055	188	590.62	27759.1	3479142
139	436.68	15174.7	1406187	189	593.76	28055.2	3534956
140	439.82	15393.8	1436755	190	596.90	28352.9	3591364
141	442.96	15614.5	1467763	191	600.04	28652.1	3648369
142	446.11	15836.8	1499214	192	603.19	28952.9	3705973
143	449.25	16060.6	1531112	193	606.33	29255.3	3764181
144	452.39	16286.0	1563457	194	609.47	29559.2	3822996
145	455.53	16513.0	1596256	195	612.61	29864.8	3882419
146	458.67	16741.5	1629511	196	615.75	30171.9	3942456
147	461.81	16971.7	1663224	197	618.89	30480.5	4003108
148	464.96	17203.4	1697398	198	622.04	30790.7	4064379
149	468.10	17436.6	1732038	199	625.18	31102.6	4126272
150	471.24	17671.5	1767146	200	628.32	31415.9	4188790

TABLE XII.—SEGMENTS OF THE CIRCLE OF WHICH  
THE RADIUS IS 1.

Central Angle	Length of Arc	Rise of Arc	Length of Chord	Area of Segment	Central Angle	Length of Arc	Rise of Arc	Length of Chord	Area of Segment
1°	0.0175	0.0000	0.0175	0.00000	46°	0.8029	0.0795	0.7815	0.04176
2	0.0349	0.0002	0.0349	0.00000	47	0.8203	0.0829	0.7975	0.04448
3	0.0524	0.0003	0.0524	0.00001	48	0.8378	0.0865	0.8135	0.04731
4	0.0698	0.0006	0.0698	0.00003	49	0.8552	0.0900	0.8294	0.05025
5	0.0873	0.0010	0.0872	0.00006	50	0.8727	0.0937	0.8452	0.05331
6	0.1047	0.0014	0.1047	0.00010	51	0.8901	0.0974	0.8610	0.05649
7	0.1222	0.0019	0.1221	0.00015	52	0.9076	0.1012	0.8767	0.05978
8	0.1396	0.0024	0.1395	0.00023	53	0.9250	0.1051	0.8924	0.06319
9	0.1571	0.0031	0.1569	0.00032	54	0.9425	0.1090	0.9080	0.06673
10	0.1745	0.0038	0.1743	0.00044	55	0.9599	0.1130	0.9235	0.07039
11	0.1920	0.0046	0.1917	0.00059	56	0.9774	0.1171	0.9389	0.07417
12	0.2094	0.0055	0.2091	0.00076	57	0.9948	0.1212	0.9543	0.07808
13	0.2269	0.0064	0.2264	0.00097	58	1.0123	0.1254	0.9696	0.08212
14	0.2443	0.0075	0.2437	0.00121	59	1.0297	0.1296	0.9848	0.08629
15	0.2618	0.0086	0.2611	0.00149	60	1.0472	0.1340	1.0000	0.09059
16	0.2793	0.0097	0.2783	0.00181	61	1.0647	0.1384	1.0151	0.09502
17	0.2967	0.0110	0.2956	0.00217	62	1.0821	0.1428	1.0301	0.09958
18	0.3142	0.0123	0.3129	0.00257	63	1.0996	0.1474	1.0450	0.10428
19	0.3316	0.0137	0.3301	0.00302	64	1.1170	0.1520	1.0598	0.10911
20	0.3491	0.0152	0.3473	0.00352	65	1.1345	0.1566	1.0746	0.11408
21	0.3665	0.0167	0.3645	0.00408	66	1.1519	0.1613	1.0893	0.11919
22	0.3840	0.0184	0.3816	0.00468	67	1.1694	0.1661	1.1039	0.12443
23	0.4014	0.0201	0.3987	0.00535	68	1.1868	0.1710	1.1184	0.12982
24	0.4189	0.0219	0.4158	0.00607	69	1.2043	0.1759	1.1328	0.13535
25	0.4363	0.0237	0.4329	0.00686	70	1.2217	0.1808	1.1472	0.14102
26	0.4538	0.0256	0.4499	0.00771	71	1.2392	0.1859	1.1614	0.14683
27	0.4712	0.0276	0.4669	0.00862	72	1.2566	0.1910	1.1756	0.15279
28	0.4887	0.0297	0.4838	0.00961	73	1.2741	0.1961	1.1896	0.15889
29	0.5061	0.0319	0.5008	0.01067	74	1.2915	0.2014	1.2036	0.16514
30	0.5236	0.0341	0.5176	0.01180	75	1.3090	0.2066	1.2175	0.17154
31	0.5411	0.0364	0.5345	0.01301	76	1.3265	0.2120	1.2313	0.17808
32	0.5585	0.0387	0.5513	0.01429	77	1.3439	0.2174	1.2450	0.18477
33	0.5760	0.0412	0.5680	0.01566	78	1.3614	0.2229	1.2586	0.19160
34	0.5934	0.0437	0.5847	0.01711	79	1.3788	0.2284	1.2722	0.19859
35	0.6109	0.0463	0.6014	0.01864	80	1.3963	0.2340	1.2856	0.20573
36	0.6283	0.0489	0.6180	0.02027	81	1.4137	0.2396	1.2989	0.21301
37	0.6458	0.0517	0.6346	0.02198	82	1.4312	0.2453	1.3121	0.22045
38	0.6632	0.0545	0.6511	0.02378	83	1.4486	0.2510	1.3252	0.22804
39	0.6807	0.0574	0.6676	0.02568	84	1.4661	0.2569	1.3383	0.23578
40	0.6981	0.0603	0.6840	0.02767	85	1.4835	0.2627	1.3512	0.24367
41	0.7156	0.0633	0.7004	0.02976	86	1.5010	0.2686	1.3640	0.25171
42	0.7330	0.0664	0.7167	0.03195	87	1.5184	0.2746	1.3767	0.25990
43	0.7505	0.0696	0.7330	0.03425	88	1.5359	0.2807	1.3893	0.26825
44	0.7679	0.0728	0.7492	0.03664	89	1.5533	0.2867	1.4018	0.27675
45°	0.7854	0.0761	0.7654	0.03915	90°	1.5708	0.2929	1.4142	0.28540

If in any circle  $r$  is the length of the radius and  $\phi$  is the number of degrees in the central angle, then

$$(1) \text{ Length of the arc, } a = \frac{\phi}{180^\circ} \pi r. \quad (2) \text{ Rise of the arc, } h = r \left( 1 - \cos \frac{\phi}{2} \right) = 2r \sin^2 \frac{\phi}{4}.$$

TABLE XII.—SEGMENTS OF THE CIRCLE OF WHICH  
THE RADIUS IS 1.

Central Angle	Length of Arc	Rise of Arc	Length of Chord	Area of Segment	Central Angle	Length of Arc	Rise of Arc	Length of Chord	Area of Segment
91°	1.5882	0.2991	1.4265	0.29420	136°	2.3736	0.6254	1.8544	0.83949
92	1.6057	0.3053	1.4387	0.30316	137	2.3911	0.6335	1.8608	0.85455
93	1.6232	0.3116	1.4507	0.31226	138	2.4086	0.6416	1.8672	0.86971
94	1.6406	0.3180	1.4627	0.32152	139	2.4260	0.6498	1.8733	0.88497
95	1.6580	0.3244	1.4746	0.33093	140	2.4435	0.6580	1.8794	0.90034
96	1.6755	0.3309	1.4863	0.34050	141	2.4609	0.6662	1.8853	0.91580
97	1.6930	0.3374	1.4979	0.35021	142	2.4784	0.6744	1.8910	0.93135
98	1.7104	0.3439	1.5094	0.36008	143	2.4958	0.6827	1.8966	0.94700
99	1.7279	0.3506	1.5208	0.37009	144	2.5133	0.6910	1.9021	0.96274
100	1.7453	0.3572	1.5321	0.38026	145	2.5307	0.6993	1.9074	0.97858
101	1.7628	0.3639	1.5432	0.39058	146	2.5482	0.7076	1.9126	0.99449
102	1.7802	0.3707	1.5543	0.40104	147	2.5656	0.7160	1.9176	1.01050
103	1.7977	0.3775	1.5652	0.41166	148	2.5831	0.7244	1.9225	1.02658
104	1.8151	0.3843	1.5760	0.42242	149	2.6005	0.7328	1.9273	1.04275
105	1.8326	0.3912	1.5867	0.43333	150	2.6180	0.7412	1.9319	1.05900
106	1.8500	0.3982	1.5973	0.44439	151	2.6354	0.7496	1.9363	1.07532
107	1.8675	0.4052	1.6077	0.45560	152	2.6529	0.7581	1.9406	1.09171
108	1.8850	0.4122	1.6180	0.46695	153	2.6704	0.7666	1.9447	1.10818
109	1.9024	0.4193	1.6282	0.47844	154	2.6878	0.7750	1.9487	1.12472
110	1.9199	0.4264	1.6383	0.49008	155	2.7053	0.7836	1.9526	1.14132
111	1.9373	0.4336	1.6483	0.50187	156	2.7227	0.7921	1.9563	1.15799
112	1.9548	0.4408	1.6581	0.51379	157	2.7402	0.8006	1.9598	1.17472
113	1.9722	0.4481	1.6678	0.52586	158	2.7576	0.8092	1.9633	1.19151
114	1.9897	0.4554	1.6773	0.53806	159	2.7751	0.8178	1.9665	1.20835
115	2.0071	0.4627	1.6868	0.55041	160	2.7925	0.8264	1.9696	1.22525
116	2.0246	0.4701	1.6961	0.56289	161	2.8100	0.8350	1.9726	1.24221
117	2.0420	0.4775	1.7053	0.57551	162	2.8274	0.8436	1.9754	1.25921
118	2.0595	0.4850	1.7143	0.58827	163	2.8449	0.8522	1.9780	1.27626
119	2.0769	0.4925	1.7233	0.60116	164	2.8623	0.8608	1.9805	1.29335
120	2.0944	0.5000	1.7321	0.61418	165	2.8798	0.8695	1.9829	1.31049
121	2.1118	0.5076	1.7407	0.62734	166	2.8972	0.8781	1.9851	1.32766
122	2.1293	0.5152	1.7492	0.64063	167	2.9147	0.8868	1.9871	1.34487
123	2.1468	0.5228	1.7576	0.65404	168	2.9322	0.8955	1.9890	1.36212
124	2.1642	0.5305	1.7659	0.66759	169	2.9496	0.9042	1.9908	1.37940
125	2.1817	0.5383	1.7740	0.68125	170	2.9671	0.9128	1.9924	1.39671
126	2.1991	0.5460	1.7820	0.69505	171	2.9845	0.9215	1.9938	1.41404
127	2.2166	0.5538	1.7899	0.70897	172	3.0020	0.9302	1.9951	1.43140
128	2.2340	0.5616	1.7976	0.72301	173	3.0194	0.9390	1.9963	1.44878
129	2.2515	0.5695	1.8052	0.73716	174	3.0369	0.9477	1.9973	1.46617
130	2.2689	0.5774	1.8126	0.75144	175	3.0543	0.9564	1.9981	1.48359
131	2.2864	0.5853	1.8199	0.76584	176	3.0718	0.9651	1.9988	1.50101
132	2.3038	0.5933	1.8271	0.78034	177	3.0892	0.9738	1.9993	1.51845
133	2.3213	0.6013	1.8341	0.79497	178	3.1067	0.9825	1.9997	1.53589
134	2.3387	0.6093	1.8410	0.80970	179	3.1241	0.9913	1.9999	1.55334
135°	2.3562	0.6173	1.8478	0.82454	180°	3.1416	1.0000	2.0000	1.57080

$$(3) \text{Chord, } c = 2r \sin \frac{\phi}{2}.$$

$$(4) \text{Area of the segment} = \frac{r^2}{2} \left( \frac{\pi\phi}{180^\circ} - \sin \phi \right).$$

$$(5) \text{Area of the sector} = \frac{\pi\phi}{360^\circ} r^2.$$

TABLE XIIIa.—NATURAL VALUES OF THE HYPERBOLIC  
FUNCTION

$\text{Sinh } u = \frac{1}{2} (e^u - e^{-u})$  from  $u = 0$  to  $u = 5.09$

$u$	0	1	2	3	4	5	6	7	8	9
0.0	0000	0100	0.0200	0.0300	0400	0500	0600	0701	0801	0901
0.1	1002	1102	0.1203	0.1304	1405	1506	1607	1708	1810	1911
0.2	2013	2115	0.2218	0.2320	2423	2526	2629	2733	2837	2941
0.3	3045	3150	0.3255	0.3360	3466	3572	3678	3785	3892	4000
0.4	4108	4216	0.4325	0.4434	4543	4653	4764	4875	4986	5098
0.5	5211	5324	0.5438	0.5552	5666	5782	5897	6014	6131	6248
0.6	6367	6485	0.6605	0.6725	6846	6967	7090	7213	7336	7461
0.7	7586	7712	0.7838	0.7966	8094	8223	8353	8484	8615	8748
0.8	8881	9015	0.9150	0.9286	9423	9561	9700	9840	9981	0122
0.9	0265	0409	1.0554	1.0700	0847	0995	1144	1.1294	1446	1598
1.0	1752	1907	1.2063	.2220	2379	2539	2700	1.2862	3025	3190
1.1	3356	3524	1.3093	3863	4035	4208	4382	1.4558	4735	4914
1.2	5095	5276	1.5460	5645	5831	6019	6209	1.6400	6593	6788
1.3	6984	7182	1.7381	7583	7786	7991	8198	1.8406	8617	8829
1.4	9043	9259	1.9477	9697	9919	0143	0369	2.0597	0827	1059
1.5	1293	1529	2.1768	2008	2251	2496	2743	2.2993	3245	3499
1.6	3756	4015	2.4276	4540	4806	5075	5346	2.5620	5896	6175
1.7	6456	6740	2.7027	7317	7609	7904	8202	2.8503	8806	9112
1.8	9422	9734	3.0049	0367	0689	1013	1340	3.1671	2005	2341
1.9	2682	3025	3.3372	3722	4075	4432	4792	3.5156	5523	5894
2.0	6269	6647	3.7028	7414	7803	8196	8593	3.8993	9398	9806
2.1	0219	0635	4.1056	1480	1909	2342	2779	4.3221	3666	4117
2.2	4571	5030	4.5494	5962	6434	6912	7394	4.7880	8372	8868
2.3	9370	9876	5.0387	0903	1425	1951	2483	5.3020	3562	4109
2.4	4662	5221	5.5785	6354	6929	7510	8097	5.8689	9288	9892
2.5	0502	1118	6.1741	2369	3004	3645	4293	6.4946	5607	6274
2.6	6947	7628	6.8315	9009	9709	0417	1132	7.1854	2583	3319
2.7	4063	4814	7.5572	6338	7112	7894	8683	7.9480	0285	1098
2.8	1919	2749	8.3586	4432	5287	6150	7021	8.7902	8791	9689
2.9	0596	1512	9.2437	3371	4315	5268	6231	9.7203	8185	9177
3.0	10.018	10.110	10.221	10.324	10.429	10.534	10.640	10.748	10.856	10.966
3.1	11.076	11.188	11.301	11.415	11.530	11.647	11.764	11.883	12.003	12.124
3.2	12.246	12.369	12.494	12.620	12.747	12.876	13.006	13.137	13.269	13.403
3.3	13.538	13.674	13.812	13.951	14.092	14.234	14.377	14.522	14.668	14.816
3.4	14.965	15.116	15.268	422	15.577	15.734	15.893	16.053	16.214	16.378
3.5	16.543	16.709	16.877	17.047	219	17.392	17.567	744	17.923	18.103
3.6	18.285	18.470	18.655	18.843	19.033	19.224	19.418	19.613	19.811	20.010
3.7	20.211	20.415	20.620	20.828	21.037	21.249	21.463	21.679	21.897	22.117
3.8	22.339	22.564	22.791	23.020	23.252	23.486	23.722	23.961	24.202	24.445
3.9	24.691	24.939	25.190	25.444	25.700	25.958	26.219	26.483	26.749	27.018
4.0	27.290	27.564	27.842	28.122	28.404	28.690	28.979	29.270	29.564	29.862
4.1	30.162	30.465	30.772	31.081	31.393	31.709	32.028	32.350	32.675	33.004
4.2	33.336	33.671	34.009	34.351	34.697	35.046	35.398	35.754	36.113	36.476
4.3	36.843	37.214	37.588	37.966	38.347	38.733	39.122	39.515	39.913	40.314
4.4	40.719	41.129	41.542	41.960	42.382	42.808	43.238	43.673	44.112	44.555
4.5	45.003	45.455	45.912	46.374	46.840	47.311	47.787	48.267	48.752	49.242
4.6	49.737	50.237	50.742	51.252	567	52.288	52.813	53.344	53.880	54.422
4.7	54.909	55.522	56.080	56.643	57.213	57.788	58.369	58.955	59.548	60.147
4.8	60.751	61.362	61.979	62.601	63.231	63.866	64.508	65.157	65.812	66.473
4.9	67.141	67.816	68.498	69.186	69.882	70.584	71.293	72.010	72.734	73.465
5.0	74.203	74.949	75.702	76.463	77.232	78.008	78.792	79.584	80.384	81.192

TABLE XIIIb.—COMMON LOGARITHMS OF THE HYPERBOLIC FUNCTION

Sinh  $u$ , from  $u = 0$  to  $u = 5.09$

$u$	0	1	2	3	4	5	6	7	8	9
0.0	—∞	8.0000	3011	4772	6022	6992	7784	8455	9036	9548
0.1	9.0007	0423	0802	1152	1475	1777	2060	2325	2576	2814
0.2	9.3039	3254	3459	3656	3844	4025	4199	4366	4528	4685
0.3	9.4836	4983	5125	5264	5398	5529	5656	5781	5902	6020
0.4	9.6136	6249	6359	6468	6574	6678	6780	6880	6978	7074
0.5	9.7169	7262	7354	7444	7533	7620	7707	7791	7875	7958
0.6	9.8039	8119	8199	8277	8354	8431	8506	8581	8655	8728
0.7	9.8800	8872	8942	9012	9082	9150	9218	9286	9353	9419
0.8	9.9485	9550	9614	9678	9742	9805	9868	9930	9992	*0053
0.9	0.0114	0174	0234	0294	0353	0412	0470	0529	0586	0644
1.0	0.0701	0758	0815	0871	0927	0982	1038	1093	1148	1203
1.1	0.1257	1311	1365	1419	1472	1525	1578	1631	1684	1736
1.2	0.1788	1840	1892	1944	1995	2046	2098	2148	2199	2250
1.3	0.2300	2351	2401	2451	2501	2551	2600	2650	2699	2748
1.4	0.2797	2846	2895	2944	2993	3041	3090	3138	3186	3234
1.5	0.3282	3330	3378	3426	3474	3521	3569	3616	3663	3711
1.6	0.3758	3805	3852	3899	3946	3992	4039	4086	4132	4179
1.7	0.4225	4272	4318	4364	4411	4457	4503	4549	4595	4641
1.8	0.4687	4733	4778	4824	4870	4915	4961	5007	5052	5098
1.9	0.5143	5188	5234	5279	5324	5370	5415	5460	5505	5550
2.0	0.5595	5640	5685	5730	5775	5820	5865	5910	5955	6000
2.1	0.6044	6089	6134	6178	6223	6268	6312	6357	6401	6446
2.2	0.6491	6535	6580	6624	6668	6713	6757	6802	6846	6890
2.3	0.6935	6979	7023	7067	7112	7156	7200	7244	7289	7333
2.4	0.7377	7421	7465	7509	7553	7597	7642	7686	7730	7774
2.5	0.7818	7862	7906	7950	7994	8038	8082	8126	8169	8213
2.6	0.8257	8301	8345	8389	8433	8477	8521	8564	8608	8652
2.7	0.8696	8740	8784	8827	8871	8915	8959	9003	9046	9090
2.8	0.9134	9178	9221	9265	9309	9353	9396	9440	9484	9527
2.9	0.9571	9615	9658	9702	9746	9789	9833	9877	9920	9964
3.0	1.0008	0051	0095	0139	0182	0226	0270	0313	0357	0400
3.1	1.0444	0488	0531	0575	0618	0662	0706	0749	0793	0836
3.2	1.0880	0923	0967	1011	1054	1098	1141	1185	1228	1272
3.3	1.1316	1359	1403	1446	1490	1533	1577	1620	1664	1707
3.4	1.1751	1794	1838	1881	1925	1968	2012	2056	2099	2143
3.5	1.2186	2230	2273	2317	2360	2404	2447	2491	2534	2578
3.6	1.2621	2665	2708	2752	2795	2839	2882	2925	2969	3012
3.7	1.3056	3099	3143	3186	3230	3273	3317	3360	3404	3447
3.8	1.3491	3534	3578	3621	3665	3708	3752	3795	3838	3882
3.9	1.3925	3969	4012	4056	4099	4143	4186	4230	4273	4317
4.0	1.4360	4403	4447	4490	4534	4577	4621	4664	4708	4751
4.1	1.4795	4838	4881	4925	4968	5012	5055	5099	5142	5186
4.2	1.5229	5273	5316	5359	5403	5446	5490	5533	5577	5620
4.3	1.5664	5707	5750	5794	5837	5881	5924	5968	6011	6055
4.4	1.6098	6141	6185	6228	6272	6315	6359	6402	6446	6489
4.5	1.6532	6576	6619	6663	6706	6750	6793	6836	6880	6923
4.6	1.6967	7010	7054	7097	7141	7184	7227	7271	7314	7358
4.7	1.7401	7445	7488	7531	7575	7618	7662	7705	7749	7792
4.8	1.7836	7879	7922	7966	8009	8053	8096	8140	8183	8226
4.9	1.8270	8313	8357	8400	8444	8487	8530	8574	8617	8661
5.0	1.8704	8748	8791	8835	8878	8921	8965	9008	9052	9095

TABLE XIVa.—NATURAL VALUES OF THE HYPERBOLIC  
FUNCTION

$\text{Cosh } u = \frac{1}{2}(e^u + e^{-u})$  from  $u = 0$  to  $u = 5.09$

<b>u</b>	<b>0</b>	<b>1</b>	<b>2</b>	<b>3</b>	<b>4</b>	<b>5</b>	<b>6</b>	<b>7</b>	<b>8</b>	<b>9</b>
0.0	1.0000	1.0001	1.0002	1.0005	1.0008	1.0013	1.0018	1.0025	1.0032	1.0041
0.1	1.0050	1.0061	1.0072	1.0085	1.0098	1.0113	1.0128	1.0145	1.0162	1.0181
0.2	1.0201	1.0221	1.0243	1.0266	1.0289	1.0314	1.0340	1.0367	1.0395	1.0423
0.3	1.0453	1.0484	1.0516	1.0549	1.0584	1.0619	1.0655	1.0692	1.0731	1.0770
0.4	1.0811	1.0852	1.0895	1.0939	1.0984	1.1030	1.1077	1.1125	1.1174	1.1225
0.5	1.1276	1.1329	1.1383	1.1438	1.1494	1.1551	1.1609	1.1669	1.1730	1.1792
0.6	1.1855	1.1919	1.1984	1.2051	1.2119	1.2188	1.2258	1.2330	1.2402	1.2476
0.7	1.2552	1.2628	1.2706	1.2785	1.2865	1.2947	1.3030	1.3114	1.3199	1.3286
0.8	1.3374	1.3464	1.3555	1.3647	1.3740	1.3835	1.3932	1.4029	1.4128	1.4229
0.9	1.4331	1.4434	1.4539	1.4645	1.4753	1.4862	1.4973	1.5085	1.5199	1.5314
1.0	1.5431	1.5549	1.5669	1.5790	1.5913	1.6038	1.6164	1.6292	1.6421	1.6552
1.1	1.6685	1.6820	1.6956	1.7093	1.7233	1.7374	1.7517	1.7662	1.7808	1.7956
1.2	1.8107	1.8258	1.8412	1.8568	1.8725	1.8884	1.9045	1.9208	1.9373	1.9540
1.3	1.9709	1.9880	2.0053	2.0228	2.0404	2.0583	2.0764	2.0947	2.1132	2.1320
1.4	2.1509	2.1700	2.1894	2.2090	2.2288	2.2488	2.2691	2.2896	2.3103	2.3312
1.5	2.3524	2.3738	2.3955	2.4174	2.4395	2.4619	2.4845	2.5073	2.5305	2.5538
1.6	2.5775	2.6013	2.6255	2.6499	2.6746	2.6995	2.7247	2.7502	2.7760	2.8020
1.7	2.8283	2.8549	2.8818	2.9090	2.9364	2.9642	2.9922	3.0206	3.0492	3.0782
1.8	3.1075	3.1371	3.1669	3.1972	3.2277	3.2585	3.2897	3.3212	3.3530	3.3852
1.9	3.4177	3.4506	3.4838	3.5173	3.5512	3.5855	3.6201	3.6551	3.6904	3.7261
2.0	3.7622	3.7987	3.8355	3.8727	3.9103	3.9483	3.9867	4.0255	4.0647	4.1043
2.1	4.1443	4.1847	4.2256	4.2668	4.3085	4.3507	4.3932	4.4362	4.4797	4.5236
2.2	4.5679	4.6127	4.6580	4.7037	4.7499	4.7966	4.8437	4.8914	4.9395	4.9881
2.3	5.0372	5.0868	5.1370	5.1876	5.2388	5.2905	5.3427	5.3954	5.4487	5.5026
2.4	5.5569	5.6119	5.6674	5.7235	5.7801	5.8373	5.8951	5.9535	6.0125	6.0721
2.5	6.1323	6.1931	6.2545	6.3166	6.3793	6.4426	6.5066	6.5712	6.6365	6.7024
2.6	6.7690	6.8363	6.9043	6.9729	7.0423	7.1123	7.1831	7.2546	7.3268	7.3998
2.7	7.4735	7.5479	7.6231	7.6990	7.7758	7.8533	7.9136	8.0106	8.0905	8.1712
2.8	8.2527	8.3351	8.4182	8.5022	8.5871	8.6728	8.7594	8.8469	8.9352	9.0244
2.9	9.1146	9.2056	9.2976	9.3905	9.4844	9.5791	9.6749	9.7716	9.8693	9.9680
3.0	10.068	10.168	10.270	10.373	10.477	10.581	10.687	10.794	10.902	11.011
3.1	11.122	11.233	11.345	11.459	11.574	11.689	11.807	11.925	12.044	12.165
3.2	12.287	12.410	12.534	12.660	12.786	12.915	13.044	13.175	13.307	13.440
3.3	13.575	13.711	13.848	13.987	14.127	14.269	14.412	14.556	14.702	14.850
3.4	14.999	15.149	15.301	15.455	15.610	15.766	15.924	16.084	16.245	16.408
3.5	16.573	16.739	16.907	17.077	17.248	17.421	17.596	17.772	17.951	18.131
3.6	18.313	18.497	18.682	18.870	19.059	19.250	19.444	19.639	19.836	20.035
3.7	20.236	20.430	20.644	20.852	21.061	21.272	21.486	21.702	21.919	22.130
3.8	22.362	22.586	22.813	23.042	23.273	23.507	23.743	23.982	24.222	24.466
3.9	24.711	24.959	25.210	25.463	25.719	25.977	26.238	26.502	26.768	27.037
4.0	27.308	27.582	27.860	28.139	28.422	28.707	28.996	29.287	29.581	29.878
4.1	30.178	30.482	30.788	31.097	31.409	31.725	32.044	32.365	32.691	33.019
4.2	33.351	33.686	34.024	34.366	34.711	35.060	35.412	35.768	36.127	36.490
4.3	36.857	37.227	37.601	37.979	38.360	38.746	39.135	39.528	39.925	40.326
4.4	40.732	41.141	41.554	41.972	42.393	42.819	43.250	43.684	44.123	44.566
4.5	45.014	45.466	45.923	46.385	46.851	47.321	47.797	48.277	48.762	49.252
4.6	49.747	50.247	50.752	51.262	51.777	52.297	52.823	53.354	53.890	54.431
4.7	54.978	55.531	56.089	56.652	57.221	57.796	58.377	58.964	59.556	60.155
4.8	60.759	61.370	61.987	62.609	63.239	63.874	64.516	65.164	65.819	66.481
4.9	67.149	67.823	68.505	69.193	69.889	70.591	71.300	72.017	72.741	73.472
5.0	74.210	74.956	75.709	76.470	77.238	78.014	78.798	79.590	80.390	81.198

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TABLE XVa.—NATURAL VALUES OF THE HYPERBOLIC  
FUNCTION  $\tanh u$  from  $u = 0$  to  $u = 2.39$ .

$u$	0	1	2	3	4	5	6	7	8	9
0.0	0.0000	0100	0200	0300	0400	0500	0599	0609	0798	0898
0.1	0.0997	1096	1194	1293	1391	1489	1587	1684	1781	1878
0.2	0.1974	2070	2165	2260	2355	2449	2543	2636	2729	2821
0.3	0.2913	3004	3095	3185	3275	3364	3452	3540	3627	3714
0.4	0.3800	3885	3969	4053	4137	4219	4301	4382	4462	4542
0.5	0.4621	4700	4777	4854	4930	5005	5080	5154	5227	5299
0.6	0.5370	5441	5511	5581	5649	5717	5784	5850	5915	5980
0.7	0.6044	6107	6169	6231	6291	6352	6411	6469	6527	6584
0.8	0.6640	6696	6751	6805	6858	6911	6963	7014	7064	7114
0.9	0.7163	7211	7259	7306	7352	7398	7443	7487	7531	7574
1.0	0.7616	7658	7699	7739	7779	7818	7857	7895	7932	7969
1.1	0.8005	8041	8076	8110	8144	8178	8210	8243	8275	8306
1.2	0.8337	8367	8397	8426	8455	8483	8511	8538	8565	8591
1.3	0.8617	8643	8668	8693	8717	8741	8764	8787	8810	8832
1.4	0.8854	8875	8896	8917	8937	8957	8977	8996	9015	9033
1.5	0.9052	9069	9087	9104	9121	9138	9154	9170	9186	9202
1.6	0.9217	9232	9246	9261	9275	9289	9302	9316	9329	9342
1.7	0.9354	9367	9379	9391	9402	9414	9425	9436	9447	9458
1.8	0.9468	9478	9488	9498	9508	9518	9527	9536	9545	9554
1.9	0.9562	9571	9579	9587	9595	9603	9611	9619	9626	9633
2.0	0.9640	9647	9654	9661	9668	9674	9680	9687	9693	9699
2.1	0.9705	9710	9716	9722	9727	9732	9738	9743	9748	9753
2.2	0.9757	9762	9767	9771	9776	9780	9785	9789	9793	9797
2.3	0.9801	9805	9809	9812	9816	9820	9823	9827	9830	9834

TABLE XVb.—COMMON LOGARITHMS OF THE HYPERBOLIC  
FUNCTION  $\tanh u$  from  $u = 0$  to  $u = 2.39$ .

$u$	0	1	2	3	4	5	6	7	8	9
0.0	—∞	8.0000	3010	4770	6018	6986	7776	8444	9022	9531
0.1	8.9986	*0396	*0771	*1115	*1433	*1729	*2004	*2263	*2506	*2736
0.2	9.2953	3159	3355	3542	3720	3890	4053	4210	4360	4505
0.3	9.4644	4778	4907	5031	5152	5268	5381	5490	5596	5698
0.4	9.5797	5894	5987	6078	6166	6252	6336	6417	6496	6573
0.5	9.6648	6720	6792	6861	6928	6994	7058	7121	7182	7242
0.6	9.7300	7357	7413	7467	7520	7571	7622	7671	7720	7767
0.7	9.7813	7858	7902	7945	7988	8029	8069	8109	8147	8185
0.8	9.8222	8258	8293	8328	8362	8395	8428	8459	8491	8521
0.9	9.8551	8580	8609	8637	8664	8691	8717	8743	8768	8793
1.0	9.8817	8841	8864	8887	8909	8931	8952	8973	8994	9014
1.1	9.9034	9053	9072	9090	9108	9126	9144	9161	9177	9194
1.2	9.9210	9226	9241	9256	9271	9285	9300	9314	9327	9341
1.3	9.9354	9367	9379	9391	9404	9415	9427	9438	9450	9460
1.4	9.9471	9482	9492	9502	9512	9522	9531	9540	9550	9558
1.5	9.9567	9576	9584	9592	9601	9608	9616	9624	9631	9639
1.6	9.9646	9653	9660	9666	9673	9679	9686	9692	9698	9704
1.7	9.9710	9716	9721	9727	9732	9738	9743	9748	9753	9758
1.8	9.9763	9767	9772	9776	9781	9785	9789	9794	9798	9802
1.9	9.9806	9810	9813	9817	9821	9824	9828	9831	9834	9838
2.0	9.9841	9844	9847	9850	9853	9856	9859	9862	9864	9867
2.1	9.9870	9872	9875	9877	9880	9882	9884	9887	9889	9891
2.2	9.9893	9895	9898	9900	9902	9904	9905	9907	9909	9911
2.3	9.9913	9914	9916	9918	9919	9921	9923	9924	9926	9927

## CONSTANTS.

	NUMERICAL VALUE	COMMON LOGARITHM
Ratio of circumference to diameter of a circle, Area of circle of which the radius is 1, Surface of sphere of which the diameter is 1,	$\pi = 3.14159\ 26535\ 89793$	0.4971499
	$\pi^2 = 9.86960\ 44010\ 89359$	0.9942997
	$\pi^3 = 31.00627\ 66802\ 93493$	1.4914496
	$\sqrt{\pi} = 1.77245\ 38509\ 05516$	0.2485749
	$\sqrt[3]{\pi} = 1.46459\ 18814\ 91298$	0.1657166
	$\pi^{-1} = 0.31830\ 98861\ 83791$	1.5028501
The Exponential Base ) = the base of Natural Logarithms.	$e = 2.71828\ 18284\ 59045$	0.4342945
Modulus of Common Logarithms	$\log_{10} e = M = 0.43429\ 44819\ 03252$ $\log_{10} 10 = 2.30258\ 50929\ 94046$ $\log_{10} 2 = 0.30102\ 99956\ 63981$ $\log_{10} 2 = 0.69314\ 71805\ 59945$ $\log_{10} 3 = 0.47712\ 12547\ 19662$ $\log_{10} 3 = 1.09861\ 22886\ 68110$ $\log_e \pi = 1.14472\ 98858\ 49400$	
Radius expressed in degrees of arc = in minutes = in seconds =	$57^\circ. 29577\ 95130$ $3437'. 74677\ 07849$ $206264''. 80624\ 70964$	1.7581226 3.5362739 5.3144251
Arc $1^\circ = \pi : 180 = 0.01745\ 32925\ 19943$ Arc $1' = \pi : 10800 = 0.00029\ 08882\ 08666$ Arc $1'' = \pi : 648000 = 0.00000\ 48481\ 36811\ 09536$		• 2.2418774 4.4637261 6.6855749
$\sin 1'' = 0.00000\ 48481\ 36811\ 07637$ $\tan 1'' = 0.00000\ 48481\ 36811\ 15234$		

## WEIGHTS AND MEASURES.

## I. Time.

60 seconds (s.) = 1 minute (m.),

60 minutes = 1 hour (h.),

24 hours = 1 mean solar day (d.).

	SECONDS.	MINUTES.	HOURS.	DAY.
Second . . . . .	1			
Minute . . . . .	60	1		
Hour . . . . .	3600	60	1	
Day . . . . .	86400	1440	24	1

## II. Arc or Angle.

60 seconds ("') 1 minute ('),

60 minutes = 1 degree (°),

90 degrees = 1 quadrant,

4 quadrants = 1 circumference =  $360^\circ = 21600'$

= 1296000".

## Correspondence of Time and Arc.

24 h. =  $360^\circ$                    $1^\circ = 4$  m.

6 h. =  $90^\circ$                    $1' = 4$  s.

1 h. =  $15^\circ$                    $1'' = \frac{1}{60}$  of one second of time.

1 m. =  $15'$

1 s. =  $15''$

## III. Long Measure.

12 inches (in.) = 1 foot (ft.),  
 3 feet = 1 yard (yd.)  
 5½ yards or } = 1 rod (rd.),  
 16½ feet }  
 40 rods = 1 furlong (f.)  
 8 furlongs = 1 mile (m.).

	INCHES.	FEET.	YARDS.	RODS.	FURLONGS.	MILE.
Inch . . . .	1					
Foot . . . .	12	1				
Yard . . . .	36	3	1			
Rod . . . .	198	10.5	5.5	1		
Furlong . . .	7920	660	220	40	1	
Mile . . . .	63360	5280	1760	320	8	1

A Gunter's Chain, used by surveyors, is 66 feet or 4 rods long, and is divided into 100 links. 80 chains = 1 mile. A fathom = 6 feet.

The standard yard of the United States is a copy of the old English standard yard, and is slightly longer than the Imperial standard yard now used in England.

United States standard yard = 1.000024 Imperial Yards.

A geographical, nautical, or sea mile (also called a knot) is the length of one minute of longitude on the Earth's equator. It is equal to 6086.07 feet = 1.152664 statute or land miles.

1 League = 3 nautical miles.

## IV. Square Measure.

144 square inches (sq. in.) = 1 square foot (sq. ft.),  
 9 square feet = 1 square yard (sq. yd.),  
 or 3 ¼ square yards } = 1 square rod (sq. rd.),  
 272 ¼ square feet }  
 160 square rods = 1 acre (A.),  
 640 acres = 1 square mile (sq. m.).

	SQUARE INCHES.	SQUARE FEET.	SQUARE YARDS.	SQUARE RODS.	ACRES.	SQUARE MILE.
Sq. in . . . .	1					
Sq. ft. . . .	144	1				
Sq. yd. . . .	1296	9	1			
Sq. rd. . . .	· · ·	272 ¼	30 ¼	1		
Acre . . . .	· · · ·	43560	4840	160	1	
Sq. mile . . .	· · · ·	· · · ·	· · · ·	· · · ·	640	1

1 acre = 10 square chains.

1 square rod = 625 square links.

**V. Cubic Measure.**

$1728$  cubic inches (cu. in.) =  $1$  cubic foot (cu. ft.),  
 $27$  cubic feet =  $1$  cubic yard (cu. yd.).

	CUBIC INCHES.	CUBIC FEET.	CUBIC YARDS.
Cubic inch . . . . .	$1$		
Cubic foot . . . . .	$1728$	$1$	
Cubic yard . . . . .	$46656$	$27$	$1$

$1$  Cord =  $4' \times 4' \times 8' = 128$  cubic feet.

$1$  Cord-foot =  $4' \times 4' \times 1' = 16$  cubic feet.

$1$  Perch of masonry =  $16\frac{1}{2}' \times 1\frac{1}{2}' \times 1' = 24.75$  cubic feet,  
but usually assumed =  $25$  cubic feet.

**VI. Liquid Measure.**

$4$  gills =  $1$  pint (pt.) =  $28.875$  cubic inches,  
 $2$  pints =  $1$  quart (qt.) =  $57.75$  cubic inches,  
 $4$  quarts =  $1$  gallon (gal.) =  $231$  cubic inches.

The United States Gallon is the old wine gallon of England, and contains  $231$  cubic inches.

The Imperial Gallon, the present standard in England, contains  $277.274$  cubic inches. An imperial gallon of distilled water weighed in air at temperature  $62^{\circ}$  Fah., barometer at  $30$  inches, weighs  $10$  pounds avoirdupois =  $70,000$  grains.

A United States Gallon of distilled water, weighed under the same conditions, weighs  $8.33$  pounds avoirdupois, or (more exactly)  $58317.798$  grains.

A hogshead contains  $63$  United States gallons, or  $52\frac{1}{2}$  Imperial gallons.

The ratio of United States liquid measures to the Imperial liquid measures of the same name is very nearly as  $5$  to  $6$ .

**VII. Dry Measure.**

$2$  pints =  $1$  quart (qt.) =  $67.2$  cubic inches,  
 $8$  quarts =  $1$  peck (pk.) =  $537.6$  cubic inches,  
 $4$  pecks =  $1$  bushel (bu.) =  $2150.42$  cubic inches.

The United States Bushel is the Winchester Bushel, formerly the standard in England; and it contains  $2150.42$  cubic inches.

The Imperial Bushel, the present standard in England, contains  $8$  Imperial gallons =  $2218.192$  cubic inches.

### Comparison of English and United States Units of Capacity.

- 1 Imperial Gallon = 277.274 cubic inches = 1.2003 U. S. Gallons.  
 1 U. S. Gallon = 231.000 cubic inches = 0.8331 Imperial Gallon.  
 1 Imperial Bushel = 2218.192 cubic inches = 1.0315 U. S. Bushels.  
 1 U. S. Bushel = 2150.420 cubic inches = 0.9695 Imperial Bushels.

### VIII. Avoirdupois Weights.

	GRAINS.
16 drachms (dr.)	= 1 ounce (oz.)
16 ounces	= 1 pound (lb.)
25 pounds	= 1 quarter (qr.),
4 quarters	= 1 hundred weight (cwt.),
20 hundred weight	= 1 ton.

The Long Ton or Gross Ton contains 2240 pounds, making the hundred-weight = 112 pounds, and the quarter = 28 pounds.

The principal unit is the pound, which contains 7000 grains. One cubic inch of distilled water weighed in air at temperature 62° Fah., barometer at 30 inches, weighs 252.458 grains. Therefore one avoirdupois pound of matter weighs the same as 27.7274 cubic inches of distilled water weighed under the conditions above stated.

$$\begin{array}{lcl} \text{Weight of 1 cubic foot of distilled water, temperature } 62^\circ, \\ \text{barometer 30 in.} & = & \begin{cases} \text{in grains, } 436247.424 \\ \text{in oz. avoir., } 997.137 \\ \text{in lb. avoir., } 62.321 \end{cases} \end{array}$$

Roughly, a cubic foot of water weighs 1000 ounces, or 62½ pounds.

### IX. Troy Weights.

24 grains (gr.)	= 1 pennyweight (dwt.),
20 pennyweights	= 1 ounce (oz.),
12 ounces	= 1 pound (lb.).

The Troy Pound = 5760 grains.

### X. Apothecaries' Weights.

20 grains (gr.)	= 1 scruple (ʒ),
3 scruples	= 1 drachm (ʒ),
8 drachms	= 1 ounce (ʒ).

The Apothecaries' ounce is identical with the Troy ounce, each contains 480 grains.

The actual standard of weight in the United States is a Troy pound of brass deposited in the United States Mint. It was intended to be an exact copy of the English standard, but has been found to be slightly less heavy.

United States Standard Troy Pound – 0.9999986 Imperial Troy Pound.

**METRIC SYSTEM.****XI. Long Measure.**Principal unit, the Meter, *m.*

1 Myriameter	= 10,000 <sup>m</sup>
1 Kilometer (km.)	= 1,000 <sup>m</sup>
1 Hectometer	= 100 <sup>m</sup>
1 Dekameter	= 10 <sup>m</sup>
1 METER (m.)	= 1 <sup>m</sup>
1 Decimeter (dm.)	= 0 <sup>m</sup> . 1
1 Centimeter (cm.)	= 0 <sup>m</sup> . 01
1 Millimeter (mm.)	= 0 <sup>m</sup> . 001

**XII. Square Measure.**Principal unit, the Are, *a.*

1 Square Kilometer (km. <sup>2</sup> )	= 1,000,000 <sup>m</sup> <sup>2</sup>
1 Hectare (ha.)	= 100 Are = 10,000 <sup>m</sup> <sup>2</sup>
1 ARE (a.)	= 100 <sup>m</sup> <sup>2</sup>
1 Centare = 1 Square Meter	= 1 <sup>m</sup> <sup>2</sup>
1 Square decimeter (dm. <sup>2</sup> )	= 0 <sup>m</sup> <sup>2</sup> . 01
1 Square centimeter (cm. <sup>2</sup> )	= 0 <sup>m</sup> <sup>2</sup> . 0001
1 Square millimeter (mm. <sup>2</sup> )	= 0 <sup>m</sup> <sup>2</sup> . 000001

**XIII. Cubic Measure.**

Including all measures of volume or capacity.

Principal unit, the Liter, *l.*

1 Kiloliter = 1 Stere (s.) = 1 Cubic Meter	= 1000 l
1 Hectoliter (hl.)	= 100 l
1 Dekaliter (dal.)	= 10 l
1 LITER (l.) = 1 Cubic Decimeter	= 1 l
1 Deciliter (dl.)	= 0.1 l
1 Centiliter (cl.)	= 0.01 l
1 Milliliter = 1 Microliter (λ) = 1 Cubic Centimeter	= 0.001 l
1 Cubic Millimeter (mm. <sup>3</sup> )	= 0.000001 l

**XIV. Weights.**Principal units, the Gram, *g* and the Kilogram, *kg.*

1 Miller = 1 Traction = 1 Metric Ton = 1000 kg	
1 Quintal (q.)	= 100 kg
1 Metagram	= 10 <sup>3</sup> kg
1 Kilogram = 1 Kg	= 10 <sup>-3</sup> = 1000 g
1 Hectogram	= 10 <sup>-4</sup>
1 Dekagram	= 10 <sup>-5</sup>
1 Gram	= 10 <sup>-6</sup>
1 Decigram (dg)	= 10 <sup>-7</sup>
1 Centigram (cg)	= 10 <sup>-8</sup>
1 Milligram (mg) = 1 Microgram (μ)	= 10 <sup>-9</sup>

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### Observed Values of Gravity.

Observed values of  $g$  for some important places in the United States taken from the Report of the United States Coast and Geodetic Survey for 1894 and changed into feet, are given in the following table.

PLACE.	LATITUDE.	HEIGHT ABOVE SEA LEVEL.	OBSERVED VALUE OF $g$ IN FEET.
Boston . . . . .	42° 21' 33"	72 ft.	32.16470
Philadelphia . . . . .	39 57 06	52	32.15814
Washington . . . . .	38 53 13	46	32.15539
Cleveland . . . . .	41 30 22	689	32.15962
Cincinnati . . . . .	39 08 20	804	32.15184
Chicago . . . . .	41 47 25	597	32.16084
St. Louis. . . . .	38 38 03	505	32.15174

At Greenwich, England, latitude  $51^{\circ} 28' 38''$ ,  $g = 32.1908$

At Paris, France,      latitude  $48^{\circ} 50' 11''$ ,  $g = 32.1843$

At Berlin, Germany,    latitude  $52^{\circ} 30' 17''$ ,  $g = 32.1937$

The value of  $g$  which has been taken as the basis of the Table of Velocities, printed on the next page is  $g = 32.16$ . If it be desirable to use any other value of  $g$  as the basis, the alterations in the values of  $v$  can easily be made by means of the numbers given in the column  $\Delta_g v$ . These numbers are the differences in the value of  $v$  due to a change of 0.01 foot in the value of  $g$ . They are decimals of the same order as the fractional part of the corresponding values of  $v$ .

### The Seconds Pendulum.

The relation of gravity to the length of the seconds pendulum is expressed by the equation

$$p\pi^2 = g,$$

wherein

$p$  = length of pendulum beating seconds,  
 $\pi^2 = 9.8696044$ ,  $\log_{10} \pi^2 = 0.9942997$ .

**VELOCITY,  $v$ , IN FEET PER SECOND, ACQUIRED BY A  
BODY FALLING THROUGH  $h$  FEET.**

$$v = \sqrt{2gh}, \text{ wherein } g = 32.16.$$

$h$	$v$	$\Delta_g v$	$h$	$v$	$\Delta_g v$	$h$	$v$	$\Delta_g v$	$h$	$v$	$\Delta_g v$
1	8.0200	12	51	57.2741	89	105	82.180	13	425	165.336	26
2	11.3420	18	52	57.8329	90	110	84.114	13	450	170.129	26
3	13.8910	22	53	58.3863	91	115	86.005	13	475	174.791	27
4	16.0399	25	54	58.9345	92	120	87.854	14	500	179.332	28
5	17.9332	28	55	59.4777	93	125	89.666	14	525	183.761	29
6	19.6448	31	56	60.0160	93	130	91.442	14	550	188.085	29
7	21.2189	33	57	60.5495	94	135	93.184	14	575	192.312	30
8	22.6839	35	58	61.0783	95	140	94.894	15	600	196.448	31
9	24.0599	37	59	61.6026	96	145	96.573	15	625	200.499	31
10	25.3614	39	60	62.1225	97	150	98.224	15	650	204.470	32
11	26.5992	41	61	62.6380	97	155	99.848	16	675	208.365	32
12	27.7820	43	62	63.1493	98	160	101.446	16	700	212.189	33
13	28.9164	45	63	63.6566	99	165	103.018	16	725	215.944	34
14	30.0080	47	64	64.1598	100	170	104.568	16	750	219.636	34
15	31.0612	48	65	64.6591	100	175	106.094	16	775	223.267	35
16	32.0799	50	66	65.1546	101	180	107.599	17	800	226.839	35
17	33.0672	51	67	65.6463	102	185	109.083	17	825	230.356	36
18	34.0259	53	68	66.1344	103	190	110.548	17	850	233.820	36
19	34.9583	54	69	66.6189	104	195	111.993	17	875	237.234	37
20	35.8664	56	70	67.0999	104	200	113.420	18	900	240.599	37
21	36.7521	57	71	67.5775	105	205	114.829	18	925	243.918	38
22	37.6170	58	72	68.0517	106	210	116.220	18	950	247.192	38
23	38.4624	60	73	68.5227	107	215	117.596	18	975	250.424	39
24	39.2897	61	74	68.9904	108	220	118.955	18	1000	253.614	39
25	40.0999	62	75	69.4550	108	225	120.300	19	1025	256.764	40
26	40.8940	64	76	69.9165	109	230	121.629	19	1050	259.877	40
27	41.6730	65	77	70.3750	109	235	122.944	19	1075	262.952	41
28	42.4377	66	78	70.8305	110	240	124.245	19	1100	265.992	41
29	43.1889	67	79	71.2831	111	245	125.532	20	1125	268.998	42
30	43.9272	68	80	71.7328	112	250	126.807	20	1150	271.971	42
31	44.6533	69	81	72.1798	112	255	128.069	20	1175	274.911	43
32	45.3678	71	82	72.6240	113	260	129.318	20	1200	277.820	43
33	46.0712	72	83	73.0654	114	265	130.556	20	1225	280.699	44
34	46.7641	73	84	73.5043	114	270	131.782	20	1250	283.549	44
35	47.4468	74	85	73.9405	115	275	132.996	21	1275	286.370	44
36	48.1199	75	86	74.3742	116	280	134.200	21	1300	289.164	45
37	48.7836	76	87	74.8054	116	285	135.393	21	1325	291.932	45
38	49.4384	77	88	75.2340	117	290	136.575	21	1350	294.673	46
39	50.0847	78	89	75.6603	118	295	137.748	21	1375	297.389	46
40	50.7228	79	90	76.0842	118	300	138.919	22	1400	300.080	47
41	51.3529	80	91	76.5057	119	310	141.206	22	1500	310.612	48
42	51.9754	81	92	76.9249	120	320	143.466	22	1600	320.799	50
43	52.5905	82	93	77.3418	120	330	145.690	23	1700	330.672	51
44	53.1985	83	94	77.7565	121	340	147.881	23	1800	340.259	53
45	53.7996	84	95	78.1690	122	350	150.040	23	1900	349.583	54
46	54.3941	85	96	78.5794	122	360	152.168	24	2000	358.664	56
47	54.9822	85	97	78.9875	123	370	154.267	24	3000	439.272	68
48	55.5640	86	98	79.3937	123	380	156.338	24	4000	507.228	79
49	56.1398	87	99	79.7977	124	390	158.382	25	5000	567.098	88
50	56.7098	88	100	80.1997	125	400	160.399	25	6000	621.225	97

