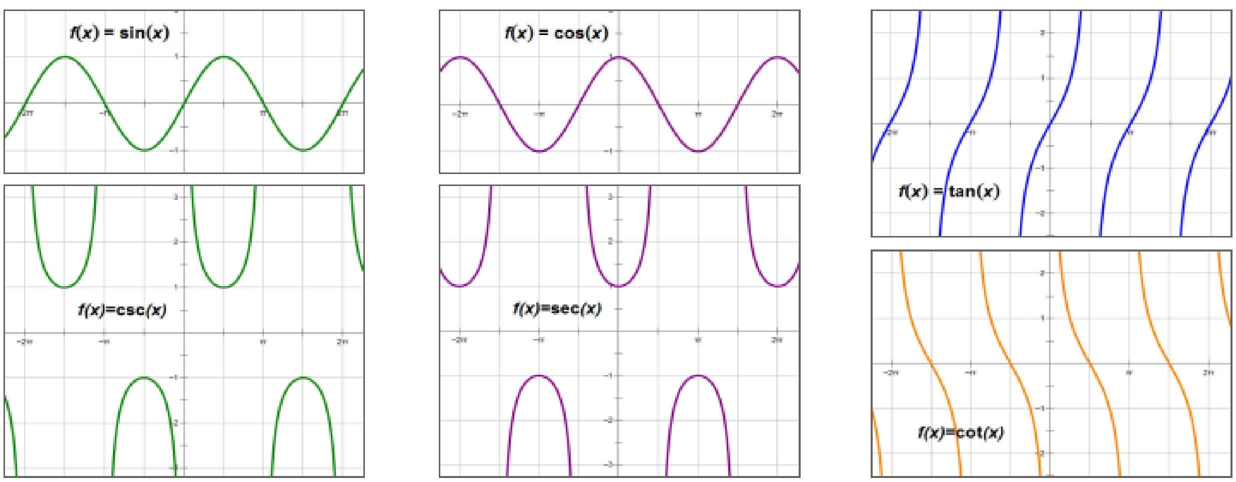
Math Handbook

of Formulas, Processes and Tricks

Trigonometry



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Trigonometry Handbook

This is a work in progress that will eventually result in an extensive handbook on the subject of Trigonometry. In its current form, the handbook covers many of the subjects contained in a Trigonometry course, but is not exhaustive. In the meantime, we are hopeful that this material will be helpful to the student. Revisions to this handbook will be provided on www.mathguy.us as they become available.

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Useful Websites

Mathguy.us – Developed specifically for math students from Middle School to College, based on the author's extensive experience in professional mathematics in a business setting and in math tutoring. Contains free downloadable handbooks, PC Apps, sample tests, and more.

http://www.mathguy.us/

Wolfram Math World – Perhaps the premier site for mathematics on the Web. This site contains definitions, explanations and examples for elementary and advanced math topics.

http://mathworld.wolfram.com/

Khan Academy – Supplies a free online collection of thousands of micro lectures via YouTube on numerous topics. It's math and science libraries are extensive.

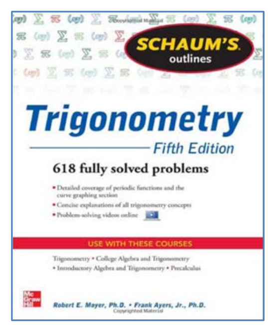
www.khanacademy.org

Analyze Math Trigonometry – Contains free Trigonometry tutorials and problems. Uses Java applets to explore important topics interactively.

http://www.analyzemath.com/Trigonometry.html

Schaum's Outline

An important student resource for any high school or college math student is a Schaum's Outline. Each book in this series provides explanations of the various topics in the course and a substantial number of problems for the student to try. Many of the problems are worked out in the book, so the student can see examples of how they should be solved.

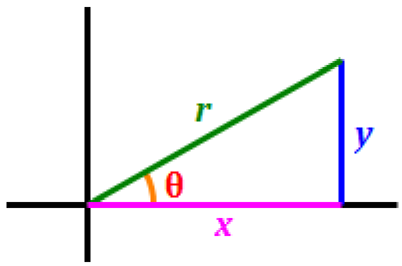


Schaum's Outlines are available at Amazon.com, Barnes & Noble and other booksellers.

Note: This study guide was prepared to be a companion to most books on the subject of High School Trigonometry. Precalculus (4th edition) by Robert Blitzer was used to determine some of the subjects to include in this guide.

Trigonometric Functions

Trigonometric Functions
(x- and y- axes)



$$\sin\theta = \frac{y}{r} \qquad \sin\theta = \frac{1}{\csc\theta}$$

$$\cos\theta = \frac{x}{r} \qquad \cos\theta = \frac{1}{\sec\theta}$$

$$\tan\theta = \frac{y}{x} \qquad \tan\theta = \frac{1}{\cot\theta} \qquad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\cot\theta = \frac{x}{y} \qquad \cot\theta = \frac{1}{\tan\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

$$\sec\theta = \frac{r}{x} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\csc\theta = \frac{r}{y} \qquad \csc\theta = \frac{1}{\sin\theta}$$

Radians ($180° = \pi$ radians)

$$0° = 0 \text{ radians}$$

$$30° = \frac{\pi}{6} \text{ radians}$$

$$45° = \frac{\pi}{4} \text{ radians}$$

$$60° = \frac{\pi}{3} \text{ radians}$$

$$90° = \frac{\pi}{2} \text{ radians}$$

Sine-Cosine Relationship

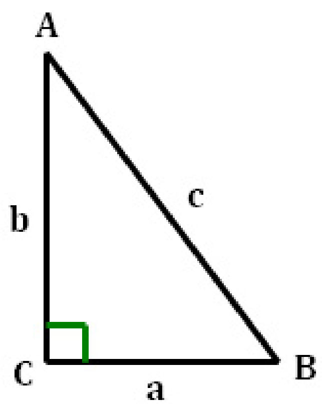
$$\sin\left(\theta + \frac{\pi}{2}\right) = \cos\theta$$

$$\sin\theta = \cos\left(\theta - \frac{\pi}{2}\right)$$

$$\sin^2\theta + \cos^2\theta = 1$$

Trigonometric Functions and Special Angles

Trigonometric Functions (Right Triangle)



SOH-CAH-TOA

$\sin = \frac{opposite}{hypoteneuse}$ $\quad \sin A = \frac{a}{c}$ $\quad \sin B = \frac{b}{c}$

$\cos = \frac{adjacent}{hypoteneuse}$ $\quad \cos A = \frac{b}{c}$ $\quad \cos B = \frac{a}{c}$

$\tan = \frac{opposite}{adjacent}$ $\quad \tan A = \frac{a}{b}$ $\quad \tan B = \frac{b}{a}$

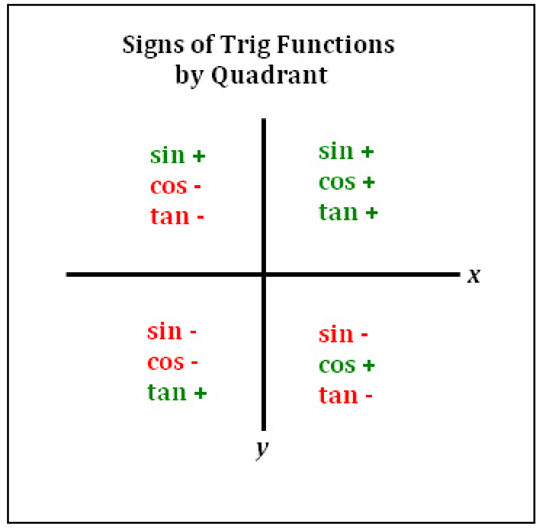
Special Angles

Trig Functions of Special Angles (θ)				
Radians	**Degrees**	$\sin\theta$	$\cos\theta$	$\tan\theta$
0	0^o	$\frac{\sqrt{0}}{2} = 0$	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{\sqrt{4}} = 0$
$\pi/6$	30^o	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
$\pi/4$	45^o	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{\sqrt{2}} = 1$
$\pi/3$	60^o	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{1}}{2} = \frac{1}{2}$	$\frac{\sqrt{3}}{\sqrt{1}} = \sqrt{3}$
$\pi/2$	90^o	$\frac{\sqrt{4}}{2} = 1$	$\frac{\sqrt{0}}{2} = 0$	*undefined*

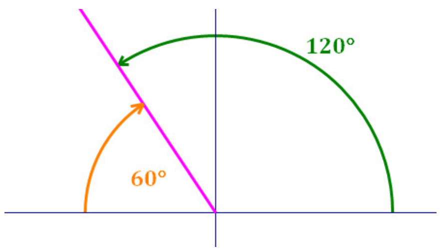
Trigonometric Function Values in Quadrants II, III, and IV

In quadrants other than Quadrant I, trigonometric values for angles are calculated in the following manner:

- Draw the angle $\boldsymbol{\theta}$ on the Cartesian Plane.
- Calculate the measure of the angle from the x-axis to $\boldsymbol{\theta}$.
- Find the value of the trigonometric function of the angle in the previous step.
- Assign a "+" or "–" sign to the trigonometric value based on the function used and the quadrant $\boldsymbol{\theta}$ is in.



Examples:



***Θ* in Quadrant II** – Calculate: $(180^0 \ - \ m\angle\theta)$

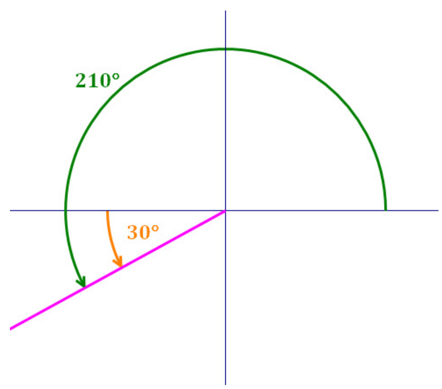
For $\theta\ = 120^0$, base your work on $180° - 120° = 60°$

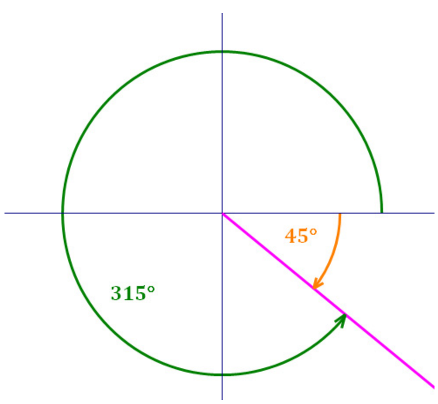
$\sin 60° = \frac{\sqrt{3}}{2}$, so: $\mathbf{\sin 120° = \frac{\sqrt{3}}{2}}$

***Θ* in Quadrant III** – Calculate: $(m\angle\theta - 180^0)$

For $\theta\ = 210^0$, base your work on $210° - 180° = 30°$

$\cos 30° = \frac{\sqrt{3}}{2}$, so: $\mathbf{\cos 210° = -\frac{\sqrt{3}}{2}}$





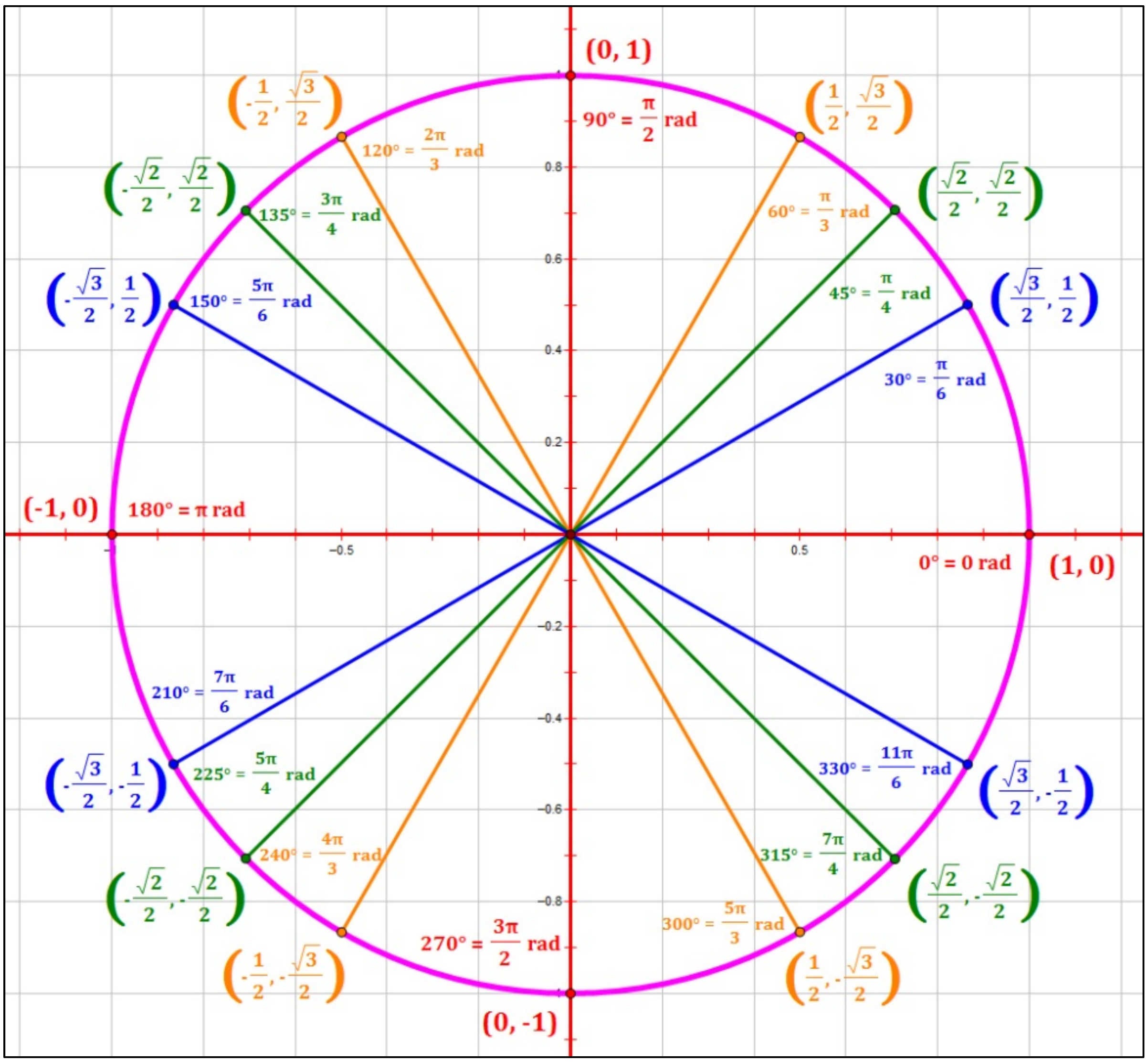
***Θ* in Quadrant IV** – Calculate: $(360^0 \ - \ m\angle\theta)$

For $\theta\ = 315^0$, base your work on $360° - 315° = 45°$

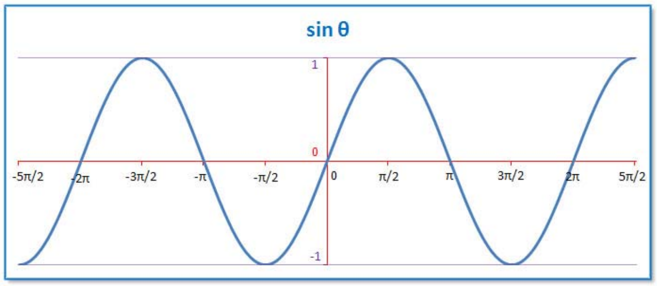
$\tan 45° = 1$, so: $\mathbf{\tan 315° = -1}$

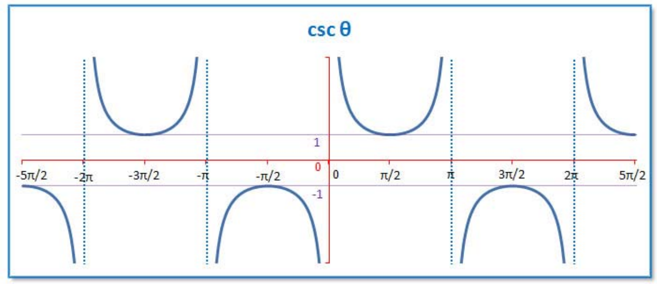
The Unit Circle

The Unit Circle diagram below provides x- and y-values on a circle of radius 1 at key angles. At any point on the unit circle, the x-coordinate is equal to the cosine of the angle and the y-coordinate is equal to the sine of the angle. Using this diagram, it is easy to identify the sines and cosines of angles that recur frequently in the study of Trigonometry.



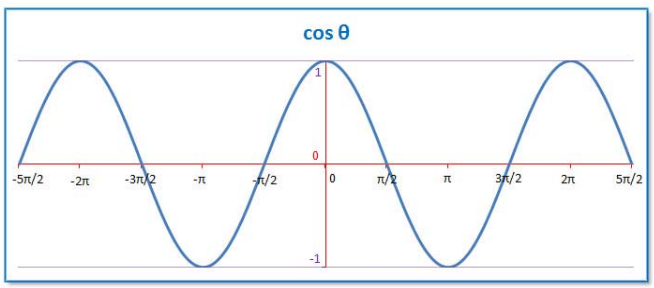
Graphs of Basic (Parent) Trigonometric Functions

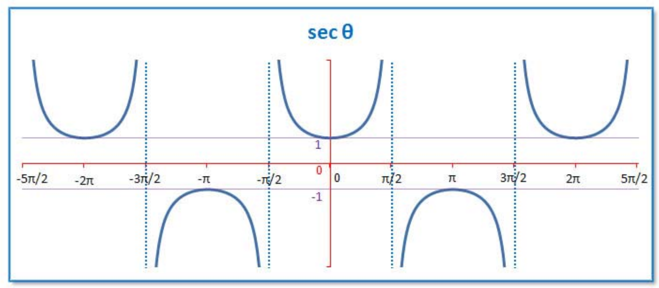




The sine and cosecant functions are reciprocals. So:

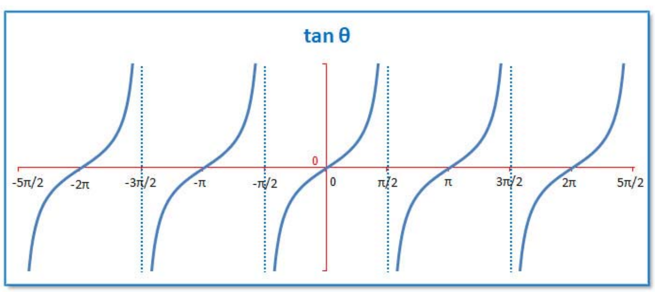
$$\sin\theta = \frac{1}{\csc\theta} \quad \text{and} \quad \csc\theta = \frac{1}{\sin\theta}$$

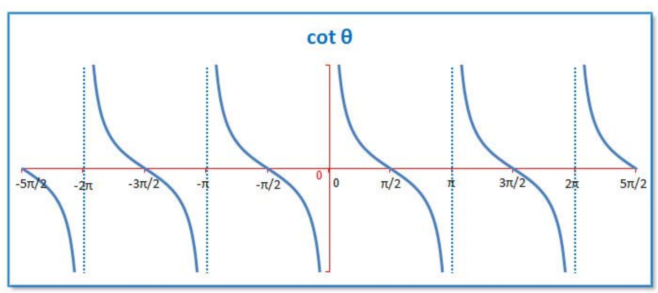




The cosine and secant functions are reciprocals. So:

$$\cos\theta = \frac{1}{\sec\theta} \quad \text{and} \quad \sec\theta = \frac{1}{\cos\theta}$$





The tangent and cotangent functions are reciprocals. So:

$$\tan\theta = \frac{1}{\cot\theta} \quad \text{and} \quad \cot\theta = \frac{1}{\tan\theta}$$

Summary of Characteristics and Key Points – Trigonometric Function Graphs

Function:	Sine	Cosine	Tangent	Cotangent	Secant	Cosecant
Parent Function	$y = \sin(x)$	$y = \cos(x)$	$y = \tan(x)$	$y = \cot(x)$	$y = \sec(x)$	$y = \csc(x)$
Domain	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, \infty)$ except $\frac{n\pi}{2}$, where n is odd	$(-\infty, \infty)$ except $n\pi$, where n is an Integer	$(-\infty, \infty)$ except $\frac{n\pi}{2}$, where n is odd	$(-\infty, \infty)$ except $n\pi$, where n is an Integer
Vertical Asymptotes	none	none	$x = \frac{n\pi}{2}$, where n is odd	$x = n\pi$, where n is an Integer	$x = \frac{n\pi}{2}$, where n is odd	$x = n\pi$, where n is an Integer
Range	$[-1, 1]$	$[-1, 1]$	$(-\infty, \infty)$	$(-\infty, \infty)$	$(-\infty, -1] \cup [1, \infty)$	$(-\infty, -1] \cup [1, \infty)$
Period	2π	2π	π	π	2π	2π
x-intercepts	$n\pi$, where n is an Integer	$\frac{n\pi}{2}$, where n is odd	midway between asymptotes	midway between asymptotes	none	none
Odd or Even Function[1]	Odd Function	Even Function	Odd Function	Odd Function	Even Function	Odd Function
General Form	$y = A\sin(Bx - C) + D$	$y = A\cos(Bx - C) + D$	$y = A\tan(Bx - C) + D$	$y = A\cot(Bx - C) + D$	$y = A\sec(Bx - C) + D$	$y = A\csc(Bx - C) + D$
Amplitude, Period, Phase Shift, Vertical Shift	$\lvert A\rvert, \frac{2\pi}{B}, \frac{C}{B}, D$	$\lvert A\rvert, \frac{2\pi}{B}, \frac{C}{B}, D$	$\lvert A\rvert, \frac{\pi}{B}, \frac{C}{B}, D$	$\lvert A\rvert, \frac{\pi}{B}, \frac{C}{B}, D$	$\lvert A\rvert, \frac{2\pi}{B}, \frac{C}{B}, D$	$\lvert A\rvert, \frac{2\pi}{B}, \frac{C}{B}, D$
$f(x)$ when $x = PS$ [2]	D	$A + D$	D	vertical asymptote	$A + D$	vertical asymptote
$f(x)$ when $x = PS + \frac{1}{4}P$	$A + D$	D	$A + D$	$A + D$	vertical asymptote	$A + D$
$f(x)$ when $x = PS + \frac{1}{2}P$	D	$-A + D$	vertical asymptote	D	$-A + D$	vertical asymptote
$f(x)$ when $x = PS + \frac{3}{4}P$	$-A + D$	D	$-A + D$	$-A + D$	vertical asymptote	$-A + D$
$f(x)$ when $x = PS + P$	D	$A + D$	D	vertical asymptote	$A + D$	vertical asymptote

Notes:

(1) An odd function is symmetric about the origin, i.e. $f(-x) = -f(x)$. An even function is symmetric about the y-axis, i.e., $f(-x) = f(x)$.

(2) All Phase Shifts are defined to occur relative to a starting point of the y-axis (i.e., the vertical line $x = 0$).

Graph of a General Sine Function

General Form

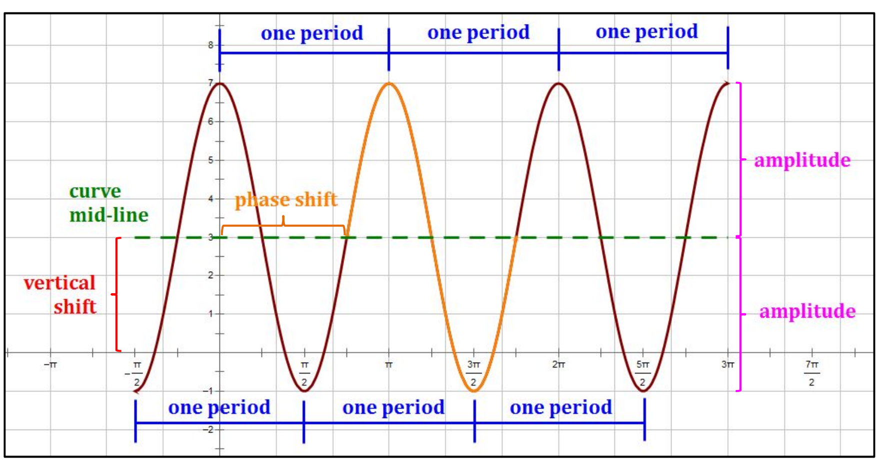
The general form of a sine function is: $\boldsymbol{y = A\ \sin(Bx - C) + D}$.

In this equation, we find several parameters of the function which will help us graph it. In particular:

- Amplitude: $\boldsymbol{Amp = |A|}$. The amplitude is the magnitude of the stretch or compression of the function from its parent function: $y = \sin x$.
- Period: $\boldsymbol{P = \frac{2\pi}{B}}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a sine or cosine function, this is the length of one complete wave; it can be measured from peak to peak or from trough to trough. Note that 2π is the period of $y = \sin x$.
- Phase Shift: $\boldsymbol{PS = \frac{C}{B}}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of $\boldsymbol{C}$ in the general form has a minus sign in front of it, just like $\boldsymbol{h}$ does in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$. So,
  - A minus sign in front of the $\boldsymbol{C}$ implies a translation to the right, and
  - A plus sign in front of the $\boldsymbol{C}$ implies a implies a translation to the left.
- Vertical Shift: $\boldsymbol{VS = D}$. This is the distance of the vertical translation of the function. This is equivalent to $\boldsymbol{k}$ in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$.

Example: $\boldsymbol{y = 4\sin\left(2x - \frac{3}{2}\pi\right) + 3}$

The midline has the equation $\mathrm{y = D}$. In this example, the midline is: $\mathrm{y = 3}$. One wave, shifted to the right, is shown in orange below.



For this example:

$\boldsymbol{A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3}$

Amplitude: $\boldsymbol{Amp = |A| = |4| = 4}$

Period: $\boldsymbol{P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi}$

Phase Shift: $\boldsymbol{PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi}$

Vertical Shift: $\boldsymbol{VS = D = 3}$

Graphing a Sine Function with No Vertical Shift: $y = A\sin(Bx - C)$

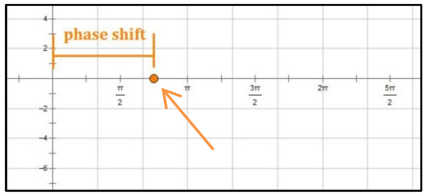
A wave (cycle) of **the sine function** has three zero points (points on the x-axis) – at the beginning of the period, at the end of the period, and halfway in-between.

Example:

$y = 4\sin\left(2x - \frac{3}{2}\pi\right).$

Step 1: Phase Shift: $PS = \frac{C}{B}$.

The first wave begins at the point **PS units to the right of the Origin.**

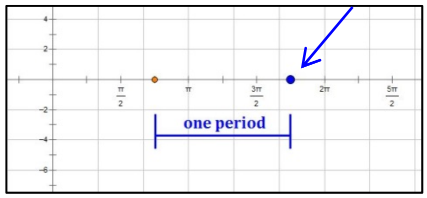


$PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi.$

The point is: $\left(\frac{3}{4}\pi, 0\right)$

Step 2: Period: $P = \frac{2\pi}{B}$.

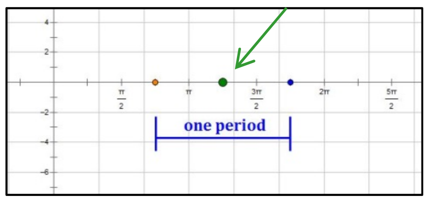
The first wave ends at the point **P units to the right of where the wave begins.**



$P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi.$ The first wave ends at the point:

$\left(\frac{3}{4}\pi + \pi, 0\right) = \left(\frac{7}{4}\pi, 0\right)$

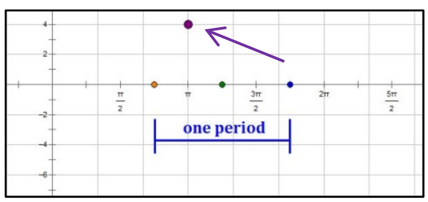
Step 3: The third zero point is located halfway between the first two.



The point is:

$\left(\frac{\frac{3}{4}\pi + \frac{7}{4}\pi}{2}, 0\right) = \left(\frac{5}{4}\pi, 0\right)$

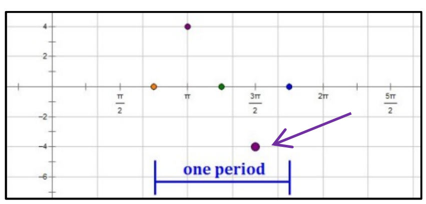
Step 4: The y-value of the point halfway between the left and center zero points is "A".



The point is:

$\left(\frac{\frac{3}{4}\pi + \frac{5}{4}\pi}{2}, 4\right) = (\pi, 4)$

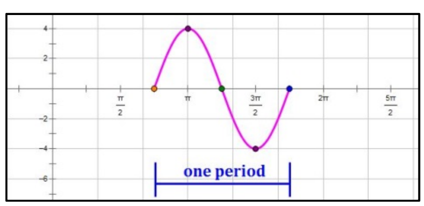
Step 5: The y-value of the point halfway between the center and right zero points is "$-A$".



The point is:

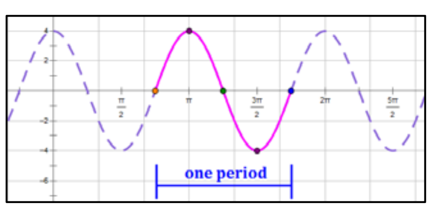
$\left(\frac{\frac{5}{4}\pi + \frac{7}{4}\pi}{2}, -4\right) = \left(\frac{3}{2}\pi, -4\right)$

Step 6: Draw a smooth curve through the five key points.



This will produce the graph of one wave of the function.

Step 7: Duplicate the wave to the left and right as desired.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Graph of a General Cosine Function

General Form

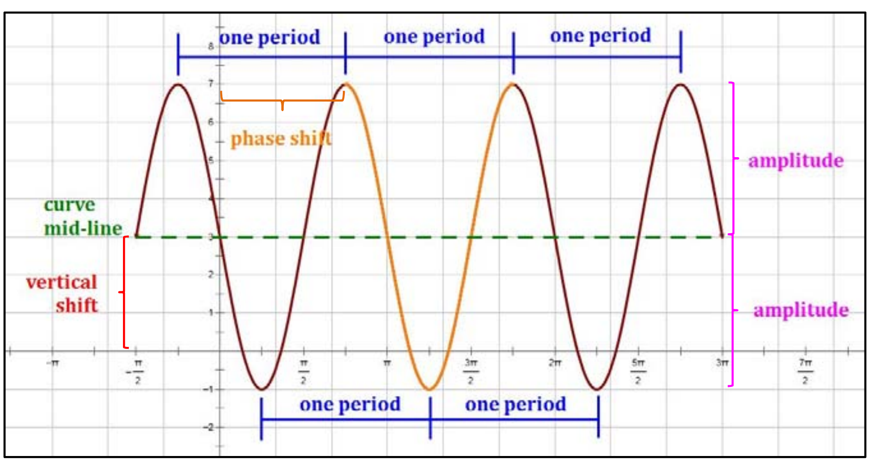
The general form of a cosine function is: $y = A\ \cos(Bx - C) + D$.

In this equation, we find several parameters of the function which will help us graph it. In particular:

- Amplitude: $Amp = |A|$. The amplitude is the magnitude of the stretch or compression of the function from its parent function: $y = \cos x$.
- Period: $P = \frac{2\pi}{B}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a sine or cosine function, this is the length of one complete wave; it can be measured from peak to peak or from trough to trough. Note that 2π is the period of $y = \cos x$.
- Phase Shift: $PS = \frac{C}{B}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of C in the general form has a minus sign in front of it, just like h does in the vertex form of a quadratic equation: $y = (x - h)^2 + k$. So,
  - A minus sign in front of the C implies a translation to the right, and
  - A plus sign in front of the C implies a implies a translation to the left.
- Vertical Shift: $VS = D$. This is the distance of the vertical translation of the function. This is equivalent to k in the vertex form of a quadratic equation: $y = (x - h)^2 + k$.

Example: $y = 4\cos\left(2x - \frac{3}{2}\pi\right) + 3$

The midline has the equation $\mathrm{y} = \mathrm{D}$. In this example, the midline is: $\mathrm{y} = 3$. One wave, shifted to the right, is shown in orange below.



For this example:

$A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3$

Amplitude: $Amp = |A| = |4| = 4$

Period: $P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

Phase Shift: $PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi$

Vertical Shift: $VS = D = 3$

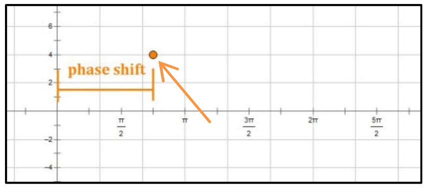
Graphing a Cosine Function with No Vertical Shift: $y = A\cos(Bx - C)$

A wave (cycle) of **the cosine function** has two maxima (or minima if $A < 0$) – one at the beginning of the period and one at the end of the period – and a minimum (or maximum if $A < 0$) halfway in-between.

Example:

$y = 4\cos\left(2x - \frac{3}{2}\pi\right).$

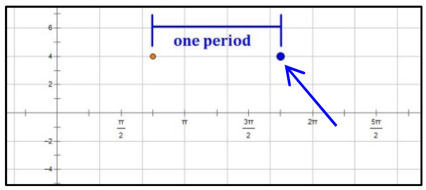
Step 1: Phase Shift: $PS = \frac{C}{B}$. The first wave begins at the point **PS units to the right of the point $(0, A)$.**



$PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi\,,\quad A = 4$

The point is: $\left(\frac{3}{4}\pi, 4\right)$

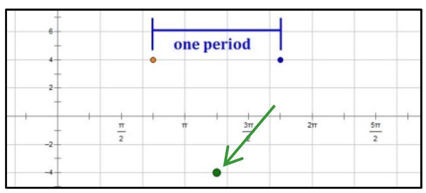
Step 2: Period: $P = \frac{2\pi}{B}$. The first wave ends at the point **P units to the right of where the wave begins.**



$P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$. The first wave ends at the point:

$$\left(\frac{3}{4}\pi + \pi, 4\right) = \left(\frac{7}{4}\pi, 4\right)$$

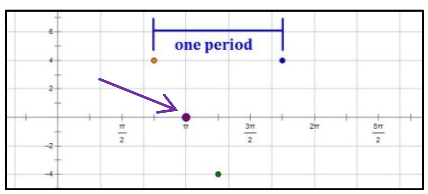
Step 3: The y-value of the point halfway between those in the two steps above is $"-A"$.



The point is:

$$\left(\frac{\frac{3}{4}\pi + \frac{7}{4}\pi}{2}, -4\right) = \left(\frac{5}{4}\pi, -4\right)$$

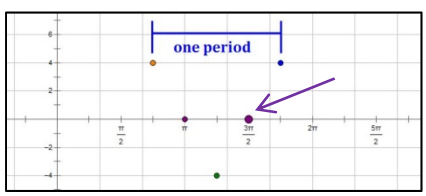
Step 4: The y-value of the point halfway between the left and center extrema is $"0"$.



The point is:

$$\left(\frac{\frac{3}{4}\pi + \frac{5}{4}\pi}{2}, 0\right) = (\pi, 0)$$

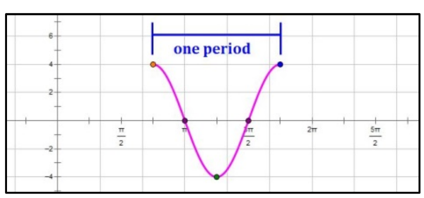
Step 5: The y-value of the point halfway between the center and right extrema is $"0"$.



The point is:

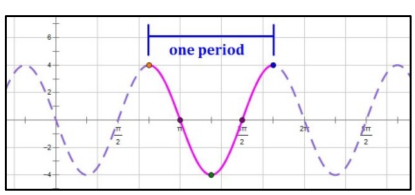
$$\left(\frac{\frac{5}{4}\pi + \frac{7}{4}\pi}{2}, 0\right) = \left(\frac{3}{2}\pi, 0\right)$$

Step 6: Draw a smooth curve through the five key points.



This will produce the graph of one wave of the function.

Step 7: Duplicate the wave to the left and right as desired.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Graph of a General Tangent Function

General Form

The general form of a tangent function is: $\boldsymbol{y = A\ \tan(Bx - C) + D}$.

In this equation, we find several parameters of the function which will help us graph it. In particular:

- Amplitude: $\boldsymbol{Amp = |A|}$. The amplitude is the magnitude of the stretch or compression of the function from its parent function: $y = \tan x$.

- Period: $\boldsymbol{P = \frac{\pi}{B}}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a tangent or cotangent function, this is the horizontal distance between consecutive asymptotes (it is also the distance between x-intercepts). Note that π is the period of $y = \tan x$.

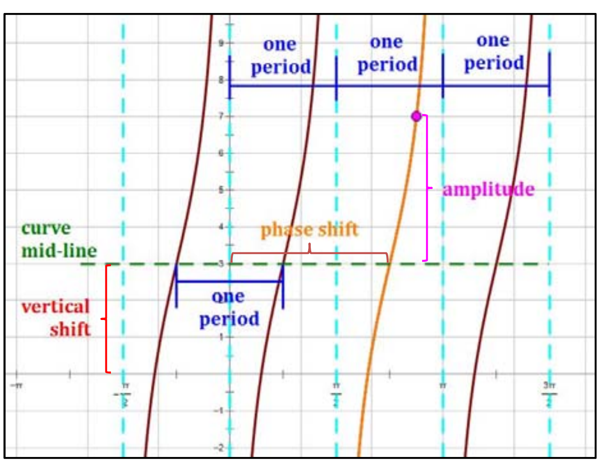
- Phase Shift: $\boldsymbol{PS = \frac{C}{B}}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of $\boldsymbol{C}$ in the general form has a minus sign in front of it, just like $\boldsymbol{h}$ does in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$. So,
  - A minus sign in front of the $\boldsymbol{C}$ implies a translation to the right, and
  - A plus sign in front of the $\boldsymbol{C}$ implies a implies a translation to the left.

- Vertical Shift: $\boldsymbol{VS = D}$. This is the distance of the vertical translation of the function. This is equivalent to $\boldsymbol{k}$ in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$.

Example: $\boldsymbol{y = 4\tan\left(2x - \frac{3}{2}\pi\right) + 3}$

The midline has the equation $\mathrm{y = D}$. In this example, the midline is: $\mathrm{y = 3}$. One cycle, shifted to the right, is shown in orange below.

Note that, for the tangent curve, we typically graph half of the principal cycle at the point of the phase shift, and then fill in the other half of the cycle to the left (see next page).



For this example:

$\boldsymbol{A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3}$

Amplitude: $\boldsymbol{Amp = |A| = |4| = 4}$

Period: $\boldsymbol{P = \frac{\pi}{B} = \frac{\pi}{2} = \frac{1}{2}\pi}$

Phase Shift: $\boldsymbol{PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi}$

Vertical Shift: $\boldsymbol{VS = D = 3}$

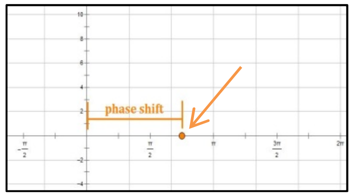
Graphing a Tangent Function with No Vertical Shift: $y = A\tan(Bx - C)$

A cycle of **the tangent function** has two asymptotes and a zero point halfway in-between. It flows upward to the right if $A > 0$ and downward to the right if $A < 0$.

Example:

$y = 4\tan\left(2x - \frac{3}{2}\pi\right).$

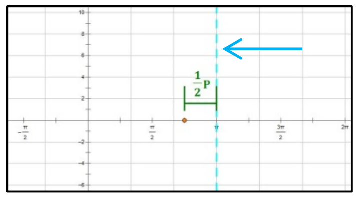
Step 1: Phase Shift: $PS = \frac{C}{B}$.
The first cycle begins at the "zero" point **PS units to the right of the Origin.**



$PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi.$

The point is: $\left(\frac{3}{4}\pi, 0\right)$

Step 2: Period: $P = \frac{\pi}{B}$.
Place a vertical asymptote $\frac{1}{2}P$ units to the right of the beginning of the cycle.

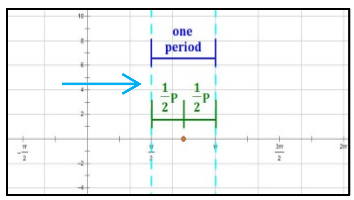


$P = \frac{\pi}{B} = \frac{\pi}{2} = \frac{1}{2}\pi. \quad \frac{1}{2}P = \frac{1}{4}\pi.$

The right asymptote is at:

$x = \frac{3}{4}\pi + \frac{1}{4}\pi = \pi$

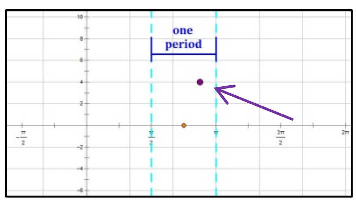
Step 3: Place a vertical asymptote $\frac{1}{2}P$ units to the left of the beginning of the cycle.



The left asymptote is at:

$x = \frac{3}{4}\pi - \frac{1}{4}\pi = \frac{1}{2}\pi$

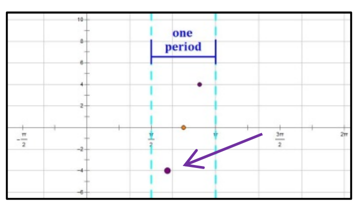
Step 4: The y-value of the point halfway between the zero point and the right asymptote is "A".



The point is:

$\left(\frac{\frac{3}{4}\pi + \pi}{2}, 4\right) = \left(\frac{7}{8}\pi, 4\right)$

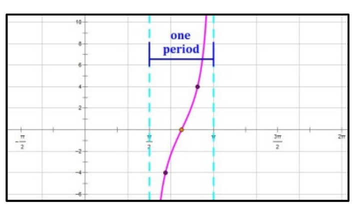
Step 5: The y-value of the point halfway between the left asymptote and the zero point is "$-A$".



The point is:

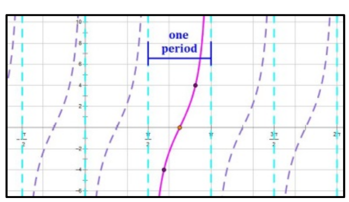
$\left(\frac{\frac{1}{2}\pi + \frac{3}{4}\pi}{2}, -4\right) = \left(\frac{5}{8}\pi, -4\right)$

Step 6: Draw a smooth curve through the three key points, approaching the asymptotes on each side.



This will produce the graph of one wave of the function.

Step 7: Duplicate the wave to the left and right as desired.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Trigonometry

Graph of a General Cotangent Function

General Form

The general form of a cotangent function is: $\boldsymbol{y = A\ \cot(Bx - C) + D}$.

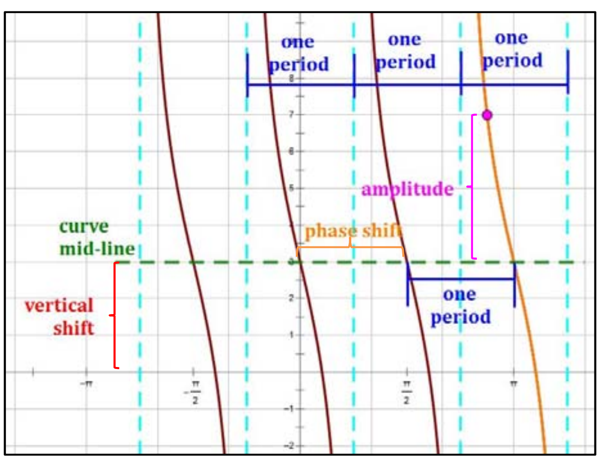
In this equation, we find several parameters of the function which will help us graph it. In particular:

- Amplitude: $\boldsymbol{Amp = |A|}$. The amplitude is the magnitude of the stretch or compression of the function from its parent function: $y = \cot x$.
- Period: $\boldsymbol{P = \frac{\pi}{B}}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a tangent or cotangent function, this is the horizontal distance between consecutive asymptotes (it is also the distance between x-intercepts). Note that π is the period of $y = \cot x$.
- Phase Shift: $\boldsymbol{PS = \frac{C}{B}}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of $\boldsymbol{C}$ in the general form has a minus sign in front of it, just like $\boldsymbol{h}$ does in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$. So,
  - o A minus sign in front of the $\boldsymbol{C}$ implies a translation to the right, and
  - o A plus sign in front of the $\boldsymbol{C}$ implies a implies a translation to the left.
- Vertical Shift: $\boldsymbol{VS = D}$. This is the distance of the vertical translation of the function. This is equivalent to $\boldsymbol{k}$ in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$.

Example: $\boldsymbol{y = 4\cot\left(2x - \frac{3}{2}\pi\right) + 3}$

The midline has the equation $\mathrm{y = D}$. In this example, the midline is: $\mathrm{y = 3}$. One cycle, shifted to the right, is shown in orange below.

Note that, for the cotangent curve, we typically graph the asymptotes first, and then graph the curve between them (see next page).



For this example:

$\boldsymbol{A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3}$

Amplitude: $\boldsymbol{Amp = |A| = |4| = 4}$

Period: $\boldsymbol{P = \frac{\pi}{B} = \frac{\pi}{2} = \frac{1}{2}\pi}$

Phase Shift: $\boldsymbol{PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi}$

Vertical Shift: $\boldsymbol{VS = D = 3}$

Graphing a Cotangent Function with No Vertical Shift: $y = A\cot(Bx - C)$

A cycle of **the cotangent function** has two asymptotes and a zero point halfway in-between. It flows downward to the right if $A > 0$ and upward to the right if $A < 0$.

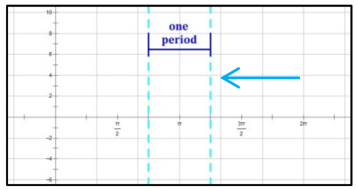
Example:

$y = 4\cot\left(2x - \frac{3}{2}\pi\right)$.

Step 1: Phase Shift: $PS = \frac{C}{B}$. Place a vertical asymptote **PS units to the right of the y-axis.**

$PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi$. The left asymptote is at: $x = \frac{3}{4}\pi$

Step 2: Period: $P = \frac{\pi}{B}$. Place another vertical asymptote P units to the right of the first one.

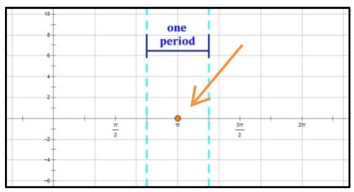


$P = \frac{\pi}{B} = \frac{\pi}{2} = \frac{1}{2}\pi$.

The right asymptote is at:

$x = \frac{3}{4}\pi + \frac{1}{2}\pi = \frac{5}{4}\pi$

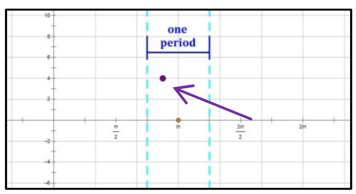
Step 3: A zero point exists halfway between the two asymptotes.



The point is: $\left(\frac{\frac{3}{4}\pi + \frac{5}{4}\pi}{2}, 0\right)$

$= (\pi, 0)$

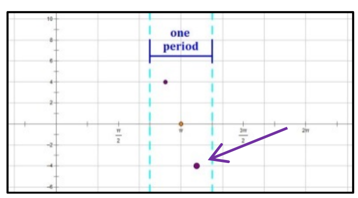
Step 4: The y-value of the point halfway between the left asymptote and the zero point is "A".



The point is:

$\left(\frac{\frac{3}{4}\pi + \pi}{2}, 4\right) = \left(\frac{7}{8}\pi, 4\right)$

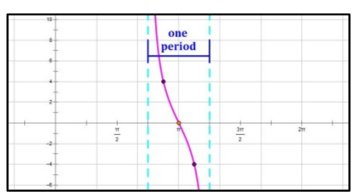
Step 5: The y-value of the point halfway between the zero point and the right asymptote is "$-A$".



The point is:

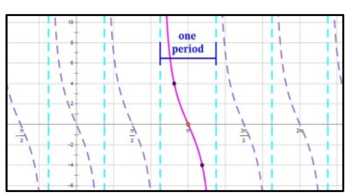
$\left(\frac{\pi + \frac{5}{4}\pi}{2}, -4\right) = \left(\frac{9}{8}\pi, -4\right)$

Step 6: Draw a smooth curve through the three key points, approaching the asymptotes on each side.



This will produce the graph of one wave of the function.

Step 7: Duplicate the wave to the left and right as desired.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Trigonometry

Graph of a General Secant Function

General Form

The general form of a secant function is: $y = A\ \sec(Bx - C) + D$.

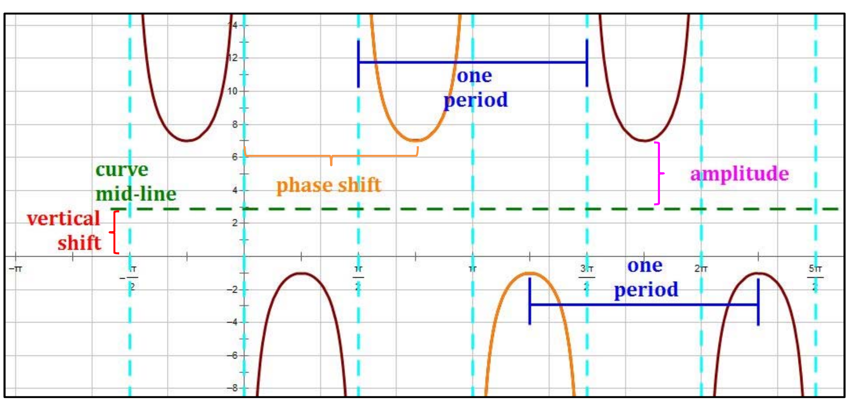
In this equation, we find several parameters of the function which will help us graph it. In particular:

- Amplitude: $Amp = |A|$. The amplitude is the magnitude of the stretch or compression of the function from its parent function: $y = \sec x$.
- Period: $P = \frac{2\pi}{B}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a secant or cosecant function, this is the horizontal distance between consecutive maxima or minima (it is also the distance between every second asymptote). Note that 2π is the period of $y = \sec x$.
- Phase Shift: $PS = \frac{C}{B}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of C in the general form has a minus sign in front of it, just like h does in the vertex form of a quadratic equation: $y = (x - h)^2 + k$. So,
  - A minus sign in front of the C implies a translation to the right, and
  - A plus sign in front of the C implies a implies a translation to the left.
- Vertical Shift: $VS = D$. This is the distance of the vertical translation of the function. This is equivalent to k in the vertex form of a quadratic equation: $y = (x - h)^2 + k$.

Example: $y = 4\sec\left(2x - \frac{3}{2}\pi\right) + 3$

The midline has the equation $y = D$. In this example, the midline is: $y = 3$. One cycle, shifted to the right, is shown in orange below.

One cycle of the secant curve contains two U-shaped curves, one opening up and one opening down.



For this example:

$A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3$

Amplitude: $Amp = |A| = |4| = 4$

Period: $P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi$

Phase Shift: $PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi$

Vertical Shift: $VS = D = 3$

Graphing a Secant Function with No Vertical Shift: $y = A\sec(Bx - C)$

A cycle of **the secant function** can be developed by first plotting a cycle of the corresponding cosine function because $\sec x = \frac{1}{\cos x}$.

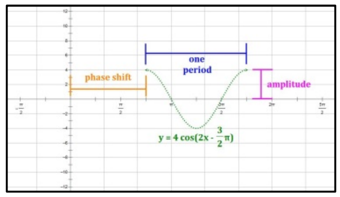
- The cosine function's zero points produce asymptotes for the secant function.
- Maxima for the cosine function produce minima for the secant function.
- Minima for the cosine function produce maxima for the secant function.
- Secant curves are U-shaped, alternately opening up and opening down.

Example:

$y = 4\sec\left(2x - \frac{3}{2}\pi\right)$.

Step 1: Graph one wave of the corresponding cosine function.

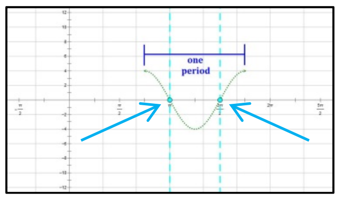
$y = A\cos(Bx - C)$



The equation of the corresponding cosine function for the example is:

$y = 4\cos\left(2x - \frac{3}{2}\pi\right)$

Step 2: Asymptotes for the secant function occur at the zero points of the cosine function.



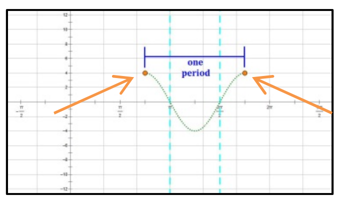
The zero points occur at:

$(\pi, 0)$ and $\left(\frac{3}{2}\pi, 0\right)$

Secant asymptotes are:

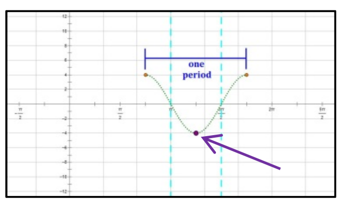
$x = \pi$ and $x = \frac{3}{2}\pi$

Step 3: Each maximum of the cosine function represents a minimum for the secant function.



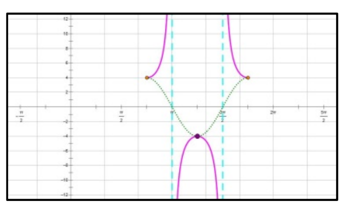
Cosine maxima and, therefore, secant minima are at: $\left(\frac{3}{4}\pi, 4\right)$ and $\left(\frac{7}{4}\pi, 4\right)$

Step 4: Each minimum of the cosine function represents a maximum for the secant function.



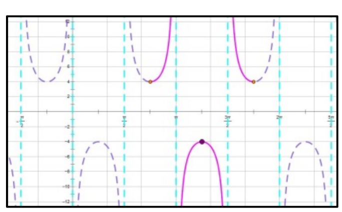
The cosine minimum and, therefore, the secant maximum is at: $\left(\frac{5}{4}\pi, -4\right)$

Step 5: Draw smooth U-shaped curves through each key point, approaching the asymptotes on each side.



This will produce the graph of one wave of the function.

Step 6: Duplicate the wave to the left and right as desired. Erase the cosine function if necessary.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Graph of a General Cosecant Function

General Form

The general form of a cosecant function is: $\boldsymbol{y = A \csc(Bx - C) + D}$.

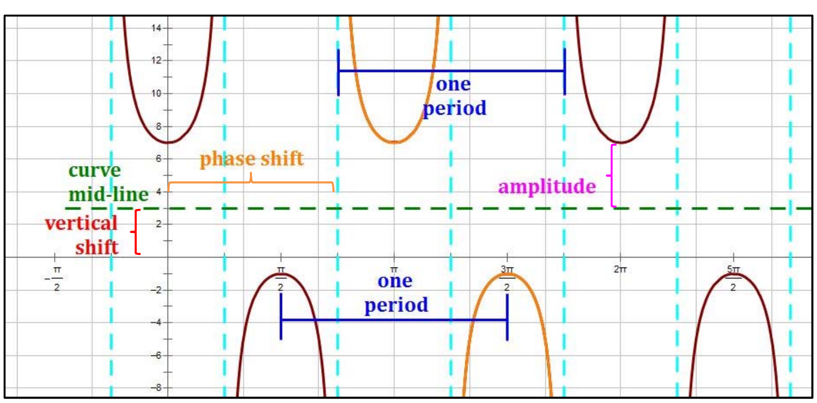
In this equation, we find several parameters of the function which will help us graph it. In particular:

- Amplitude: $\boldsymbol{Amp = |A|}$. The amplitude is the magnitude of the stretch or compression of the function from its parent function: $y = \csc x$.
- Period: $\boldsymbol{P = \frac{2\pi}{B}}$. The period of a trigonometric function is the horizontal distance over which the curve travels before it begins to repeat itself (i.e., begins a new cycle). For a secant or cosecant function, this is the horizontal distance between consecutive maxima or minima (it is also the distance between every second asymptote). Note that 2π is the period of $y = \csc x$.
- Phase Shift: $\boldsymbol{PS = \frac{C}{B}}$. The phase shift is the distance of the horizontal translation of the function. Note that the value of $\boldsymbol{C}$ in the general form has a minus sign in front of it, just like $\boldsymbol{h}$ does in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$. So,
  - A minus sign in front of the $\boldsymbol{C}$ implies a translation to the right, and
  - A plus sign in front of the $\boldsymbol{C}$ implies a implies a translation to the left.
- Vertical Shift: $\boldsymbol{VS = D}$. This is the distance of the vertical translation of the function. This is equivalent to $\boldsymbol{k}$ in the vertex form of a quadratic equation: $\boldsymbol{y = (x - h)^2 + k}$.

Example: $\boldsymbol{y = 4\csc\left(2x - \frac{3}{2}\pi\right) + 3}$

The midline has the equation $y = D$. In this example, the midline is: $y = 3$. One cycle, shifted to the right, is shown in orange below.

One cycle of the cosecant curve contains two U-shaped curves, one opening up and one opening down.



For this example:

$\boldsymbol{A = 4;\ \ B = 2;\ \ C = \frac{3}{2}\pi;\ \ D = 3}$

Amplitude: $\boldsymbol{Amp = |A| = |4| = 4}$

Period: $\boldsymbol{P = \frac{2\pi}{B} = \frac{2\pi}{2} = \pi}$

Phase Shift: $\boldsymbol{PS = \frac{C}{B} = \frac{\frac{3}{2}\pi}{2} = \frac{3}{4}\pi}$

Vertical Shift: $\boldsymbol{VS = D = 3}$

Graphing a Cosecant Function with No Vertical Shift: $y = A\csc(Bx - C)$

A cycle of **the cosecant function** can be developed by first plotting a cycle of the corresponding sine function because $\csc x = \frac{1}{\sin x}$.

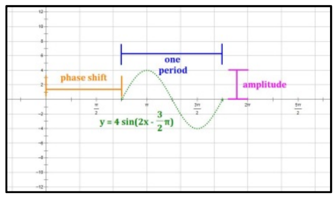
- The sine function's zero points produce asymptotes for the cosecant function.
- Maxima for the sine function produce minima for the cosecant function.
- Minima for the sine function produce maxima for the cosecant function.
- Cosecant curves are U-shaped, alternately opening up and opening down.

Example:

$y = 4\csc\left(2x - \frac{3}{2}\pi\right)$.

Step 1: Graph one wave of the corresponding sine function.

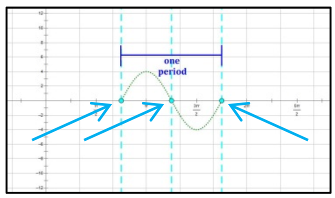
$y = A\sin(Bx - C)$



The equation of the corresponding sine function for the example is:

$y = 4\sin\left(2x - \frac{3}{2}\pi\right)$

Step 2: Asymptotes for the cosecant function occur at the zero points of the sine function.



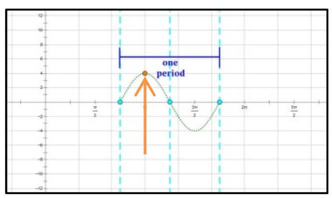
The zero points occur at:

$\left(\frac{3}{4}\pi, 0\right), \left(\frac{5}{4}\pi, 0\right), \left(\frac{7}{4}\pi, 0\right)$

Cosecant asymptotes are:

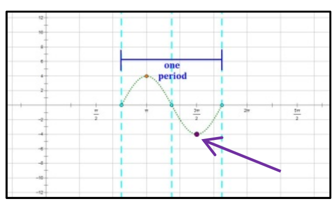
$x = \frac{3}{4}\pi,\ x = \frac{5}{4}\pi,\ x = \frac{7}{4}\pi$

Step 3: Each maximum of the sine function represents a minimum for the cosecant function.



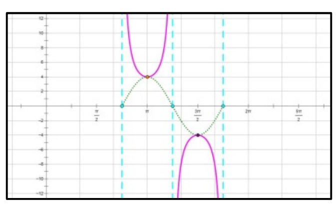
The sine maximum and, therefore, the cosecant minimum is at: $(\pi, 4)$

Step 4: Each minimum of the sine function represents a maximum for the cosecant function.



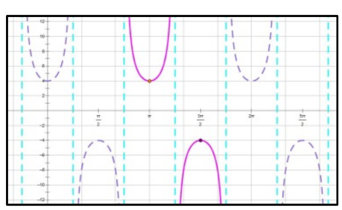
The sine minimum and, therefore, the cosecant maximum is at: $\left(\frac{3}{2}\pi, -4\right)$

Step 5: Draw smooth U-shaped curves through each key point, approaching the asymptotes on each side.



This will produce the graph of one wave of the function.

Step 6: Duplicate the wave to the left and right as desired. Erase the sine function if necessary.



Note: If $D \neq 0$, all points on the curve are shifted vertically by D units.

Inverse Trigonometric Functions

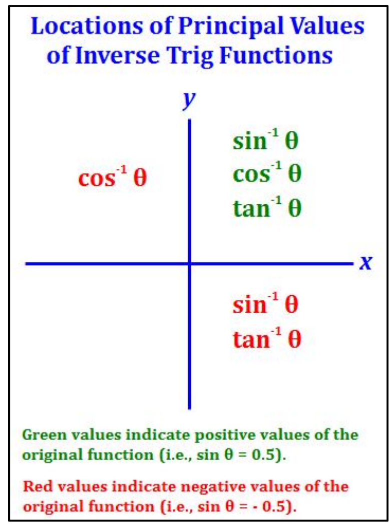
Inverse Trigonometric Functions

Inverse trigonometric functions ask the question: which angle θ has a function value of x? For example:

$\theta = \sin^{-1}(0.5)$ asks which angle has a sine value of 0.5. It is equivalent to: $\sin\theta = 0.5$.

$\theta = \tan^{-1} 1$ asks which angle has a tangent value of 1. It is equivalent to: $\tan\theta = 1$.

Principal Values of Inverse Trigonometric Functions



There are an infinite number of angles that answer these questions. So, mathematicians have defined a principal solution for problems involving inverse trigonometric functions. The angle which is the principal solution (or principal value) is defined to be the solution that lies in the quadrants identified in the figure at right. For example:

The solutions to $\theta = \sin^{-1} 0.5$ are $x \in \left\{\left(\frac{\pi}{6} + 2n\pi\right) \cup \left(\frac{5\pi}{6} + 2n\pi\right)\right\}$. That is, the set of all solutions to this equation contains the two solutions in the interval $[0, 2\pi)$, as well as all angles that are integer multiples of 2π less or greater than those two angles. Given the confusion this can create, mathematicians defined a principal value for the solution to these kinds of equations.

The principal value of θ for which $\theta = \sin^{-1} 0.5$ lies in Q1 because 0.5 is positive, and is $\theta = \frac{\pi}{6}$.

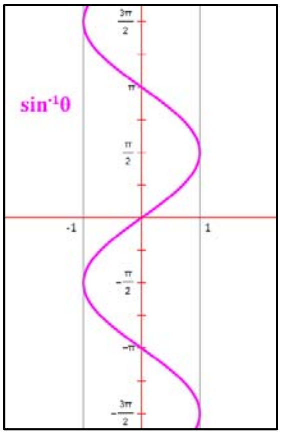
Ranges of Inverse Trigonometric Functions

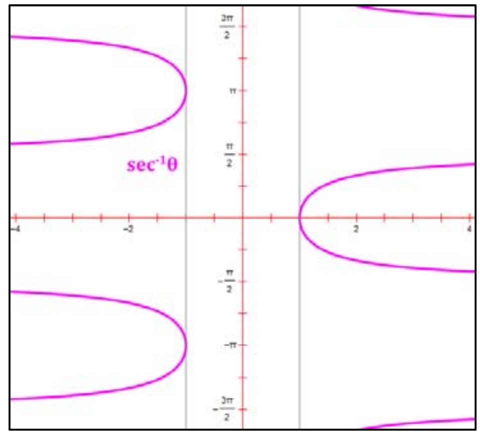
The ranges of the inverse trigonometric functions are the ranges of the principal values of those functions. A table summarizing these is provided in the table at right.

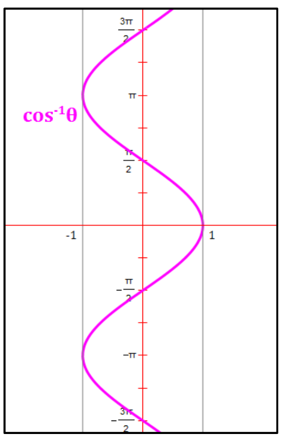
Angles in Q4 are generally expressed as negative angles.

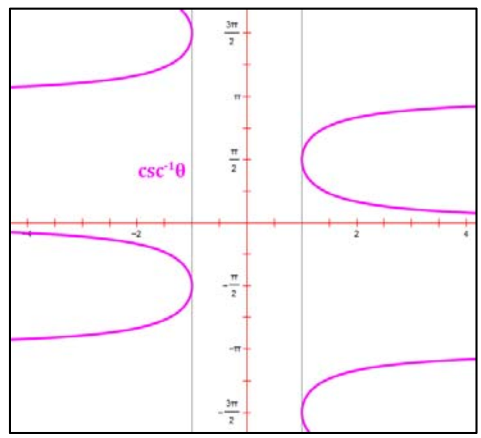
Ranges of Inverse Trigonometric Functions	
Function	**Range**
$\sin^{-1}\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$
$\cos^{-1}\theta$	$0 \le \theta \le \pi$
$\tan^{-1}\theta$	$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

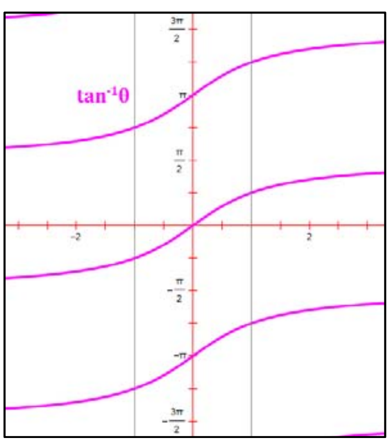
Graphs of Inverse Trigonometric Functions

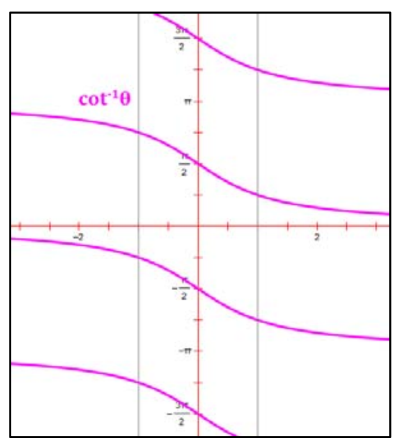












Verifying Identities

A significant portion of any trigonometry course deals with verifying Trigonometric Identities, i.e., statements that are always true (assuming the trigonometric values involved exist). This section deals with how the student may approach verification of identities such as:

$$(1+\tan^2 u)\cdot(1-\sin^2 u)=1$$

In verifying a Trigonometric Identity, the student is asked to work with only one side of the identity and, using the standard rules of mathematical manipulation, derive the other side. The student may work with either side of the identity, so generally it is best to work on the side that is most complex. The steps below present a strategy that may be useful in verifying identities.

Verification Steps

1. **Identify which side you want to work on**. Let's call this **Side A.** Let's call the side you are not working on **Side B.** So, you will be working on Side A to make it look like Side B.
    a. If one side has a multiple of an angle (e.g., $\tan 3x$) and the other side does not (e.g., $\cos x$), work with the side that has the multiple of an angle.
    b. If one side has only sines and cosines and the other does not, work with the one that does not have only sines and cosines.
    c. If you get part way through the exercise and realize you should have started with the other side, start over and work with the other side.
2. If necessary, **investigate Side B** by working on it a little. This is not a violation of the rules as long as, in your verification, you completely manipulate Side A to look like Side B. If you choose to investigate Side B, move your work off a little to the side so it is clear you are "investigating" and not actually "working" side B.
3. **Simplify** as much as possible first, but remember to look at the other side to make sure you are moving in that direction. Do this also at each step along the way, as long as it makes Side A look more like Side B.
    a. Use the Pythagorean Identities to simplify, e.g., if one side contains $(1-\sin^2 x)$ and the other side contains cosines but not sines, replace $(1-\sin^2 x)$ with $\cos^2 x$.
    b. Change any multiples of angles, half angles, etc. to expressions with single angles (e.g., replace $\sin 2x$ with $2\sin x\cos x$).
    c. Look for 1's. Often changing a 1 into $\sin^2\theta+\cos^2\theta$ will be helpful.
4. **Rewrite Side A in terms of sines and cosines.**
5. **Factor** where possible.
6. **Separate or combine fractions** to make Side A look more like Side B.

The following pages illustrate a number of techniques that can be used to verify identities.

Verifying Identities – Techniques

Technique: Investigate Both Sides

Often, when looking at an identity, it is not immediately obvious how to proceed. In many cases, investigating both sides will provide the necessary hints to proceed.

Example:

$$\frac{\dfrac{1}{\sin x} - \dfrac{1}{\cos x}}{\dfrac{1}{\sin x} + \dfrac{1}{\cos x}} \qquad = \frac{\cot x - 1}{\cot x + 1}$$

Yuk! This identity contains a lot of functions that are difficult to deal with. Let's investigate it by converting to sines and cosines on both sides. Note that on the right, I move my new fraction off to the side to indicate I am investigating only. I do this because we must verify an identity by working only one side until we get the other side.

$$\frac{\dfrac{1}{\sin x} - \dfrac{1}{\cos x}}{\dfrac{1}{\sin x} + \dfrac{1}{\cos x}} = \qquad\qquad \frac{\dfrac{\cos x}{\sin x} - \dfrac{\cos x}{\cos x}}{\dfrac{\cos x}{\sin x} + \dfrac{\cos x}{\cos x}}$$

Notice that I changed each 1 in the expression on the right to $\dfrac{\cos x}{\cos x}$ because I want to get something that looks more like the expression on the right.

Looking at what I have now, I notice that the two expressions look a lot alike, except that every place I have a 1 in the expression on the left I have $\cos x$ in the expression on the right.

What is my next step? I need to change all the 1's in the expression on the left to $\cos x$. I can do this my multiplying the expression on the left by $\dfrac{\cos x}{\cos x}$.

$$\frac{\cos x}{\cos x} \cdot \frac{\dfrac{1}{\sin x} - \dfrac{1}{\cos x}}{\dfrac{1}{\sin x} + \dfrac{1}{\cos x}}$$

$$\frac{\dfrac{\cos x}{\sin x} - \dfrac{\cos x}{\cos x}}{\dfrac{\cos x}{\sin x} + \dfrac{\cos x}{\cos x}}$$ Notice that this matches the orange expression above.

$$\frac{\cot x - 1}{\cot x + 1} \qquad = \frac{\cot x - 1}{\cot x + 1}$$

Verifying Identities – Techniques (cont'd)

Technique: Break a Fraction into Pieces

When a fraction contains multiple terms in the numerator, it is sometimes useful to break it into separate terms. This works especially well when the numerator has the same number of terms as exist on the other side of the equal sign.

Example:

$$\frac{\cos(\alpha - \beta)}{\cos\alpha\cos\beta} \qquad = 1 - \tan\alpha\tan\beta$$

First, it's a good idea to replace $\cos(\alpha - \beta)$ with its equivalent:

$$\frac{\cos\alpha\cos\beta - \sin\alpha\sin\beta}{\cos\alpha\cos\beta}$$

Next, break the fraction into two pieces:

$$\frac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \frac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}$$

Finally, simplify the expression:

$$1 - \left(\frac{\sin\alpha}{\cos\alpha}\right) \cdot \left(\frac{\sin\beta}{\cos\beta}\right)$$

$$1 - \tan\alpha\tan\beta \qquad = 1 - \tan\alpha\tan\beta$$

Verifying Identities – Techniques (cont'd)

Technique: Get a Common Denominator on One Side

When a fraction contains multiple terms in the numerator, it is sometimes useful to break it into separate terms. This works especially well when the numerator has the same number of terms as exist on the other side of the equal sign.

Example:

$$\frac{\cos x}{1-\sin x} \qquad = \frac{1+\sin x}{\cos x}$$

If we were to solve this like an equation, we might create a common denominator. Remember, however, that we can only work on one side, so we will use the common denominator only on that side. In this example, the common denominator would be: $\cos x\,(1-\sin x)$.

$$\frac{\cos x}{\cos x}\cdot\frac{\cos x}{1-\sin x}$$

$$\frac{\cos^2 x}{\cos x\,(1-\sin x)}$$

Once we have inserted the denominator from the right side in the expression on the left, the rest of the expression should simplify. To keep the $\cos x$ in the expression, we need to work with the numerator. A common substitution is to work back and forth between $\sin^2 x$ and $\cos^2 x$.

$$\frac{1-\sin^2 x}{\cos x\,(1-\sin x)}$$

Notice that the numerator is a difference of squares. Let's factor it.

$$\frac{(1+\sin x)(1-\sin x)}{\cos x\,(1-\sin x)}$$

Finally, we simplify by eliminating the common factor in the numerator and denominator.

$$\frac{1+\sin x}{\cos x} \qquad = \frac{1+\sin x}{\cos x}$$

Key Angle Formulas

Angle Addition Formulas

$$\sin\,(\alpha+\beta) = \sin\alpha\cos\beta\ +\ \cos\alpha\sin\beta$$
$$\sin\,(\alpha-\beta) = \sin\alpha\cos\beta\ -\ \cos\alpha\sin\beta$$

$$\cos\,(\alpha+\beta) = \cos\alpha\cos\beta\ -\ \sin\alpha\sin\beta$$
$$\cos\,(\alpha-\beta) = \cos\alpha\cos\beta\ +\ \sin\alpha\sin\beta$$

$$\tan\,(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\ \tan\beta}$$

$$\tan\,(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\ \tan\beta}$$

Double Angle Formulas

$$\sin\,2\theta\ =\ 2\sin\theta\cos\theta$$

$$\cos\,2\theta\ =\ \cos^2\theta - \sin^2\theta$$
$$=\ 1 - 2\sin^2\theta$$
$$=\ 2\cos^2\theta - 1$$

$$\tan\,2\theta\ =\ \frac{2\ \tan\theta}{1-\tan^2\theta}$$

Half Angle Formulas

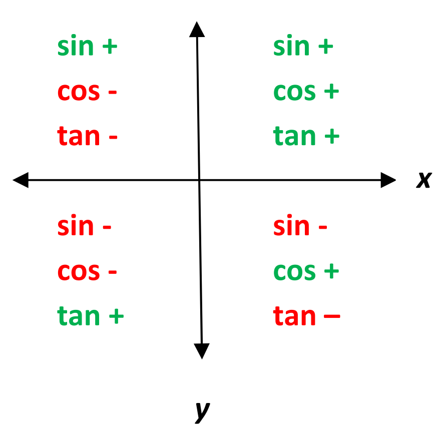
$$\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$$

$$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$$

$$\tan\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$
$$= \frac{1-\cos\theta}{\sin\theta}$$
$$= \frac{\sin\theta}{1+\cos\theta}$$

The use of a "+" or "-" sign in the half angle formulas depends on the quadrant in which the angle $\frac{\theta}{2}$ resides. See chart below.

Signs of Trig Functions
By Quadrant



Key Angle Formulas (cont'd)

Power Reducing Formulas

$$\sin^2\theta = \frac{1-\cos 2\theta}{2} \qquad \cos^2\theta = \frac{1+\cos 2\theta}{2}$$

$$\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$$

Product-to-Sum Formulas

$$\sin\alpha \cdot \sin\beta = \frac{1}{2}\left[\cos(\alpha-\beta) - \cos(\alpha+\beta)\right]$$

$$\cos\alpha \cdot \cos\beta = \frac{1}{2}\left[\cos(\alpha-\beta) + \cos(\alpha+\beta)\right]$$

$$\sin\alpha \cdot \cos\beta = \frac{1}{2}\left[\sin(\alpha+\beta) + \sin(\alpha-\beta)\right]$$

$$\cos\alpha \cdot \sin\beta = \frac{1}{2}\left[\sin(\alpha+\beta) - \sin(\alpha-\beta)\right]$$

Sum-to-Product Formulas

$$\sin\alpha + \sin\beta = 2\cdot\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right)$$

$$\sin\alpha - \sin\beta = 2\cdot\sin\left(\frac{\alpha-\beta}{2}\right)\cdot\cos\left(\frac{\alpha+\beta}{2}\right)$$

$$\cos\alpha + \cos\beta = 2\cdot\cos\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right)$$

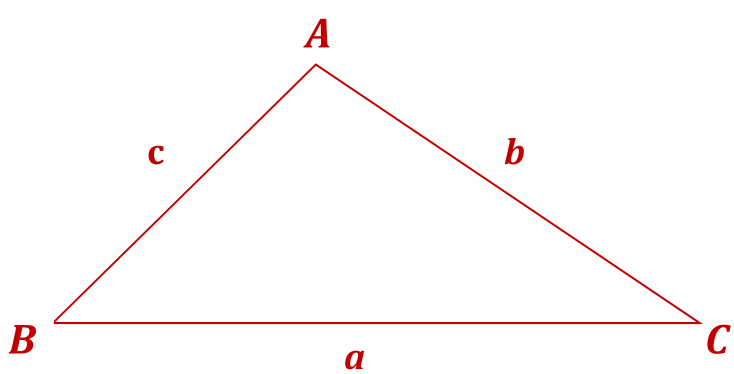
$$\cos\alpha - \cos\beta = -2\cdot\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\sin\left(\frac{\alpha-\beta}{2}\right)$$

Key Angle Formulas (cont'd)

Cofunctions

Each trigonometric function has a cofunction with symmetric properties in Quadrant I. The following identities express the relationships between cofunctions.

$$\sin\theta = \cos(90° - \theta) \qquad \cos\theta = \sin(90° - \theta)$$
$$\tan\theta = \cot(90° - \theta) \qquad \cot\theta = \tan(90° - \theta)$$
$$\sec\theta = \csc(90° - \theta) \qquad \csc\theta = \sec(90° - \theta)$$



Law of Sines (see above illustration)

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Law of Cosines (see above illustration)

$$a^2 = b^2 + c^2 - 2bc\cos A$$
$$b^2 = a^2 + c^2 - 2ac\cos B$$
$$c^2 = a^2 + b^2 - 2ab\cos C$$

Pythagorean Identities (for any angle θ)

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sec^2\theta = 1 + \tan^2\theta$$

$$\csc^2\theta = 1 + \cot^2\theta$$

Solving an Oblique Triangle

Several methods exist to solve an oblique triangle, i.e., a triangle with no right angle. The appropriate method depends on the information available for the triangle. All methods require that the length of at least one side be provided. In addition, one or two angle measures may be provided. Note that if two angle measures are provided, the measure of the third is determined (because the sum of all three angle measures must be 180°). The methods used for each situation are summarized below.

Given Three Sides and no Angles (SSS)

Given three segment lengths and no angle measures, do the following:

- Use the Law of Cosines to determine the measure of one angle.
- Use the Law of Sines to determine the measure of one of the two remaining angles.
- Subtract the sum of the measures of the two known angles from 180° to obtain the measure of the remaining angle.

Given Two Sides and the Angle between Them (SAS)

Given two segment lengths and the measure of the angle that is between them, do the following:

- Use the Law of Cosines to determine the length of the remaining leg.
- Use the Law of Sines to determine the measure of one of the two remaining angles.
- Subtract the sum of the measures of the two known angles from 180° to obtain the measure of the remaining angle.

Given One Side and Two Angles (ASA or AAS)

Given one segment length and the measures of two angles, do the following:

- Subtract the sum of the measures of the two known angles from 180° to obtain the measure of the remaining angle.
- Use the Law of Sines to determine the lengths of the two remaining legs.

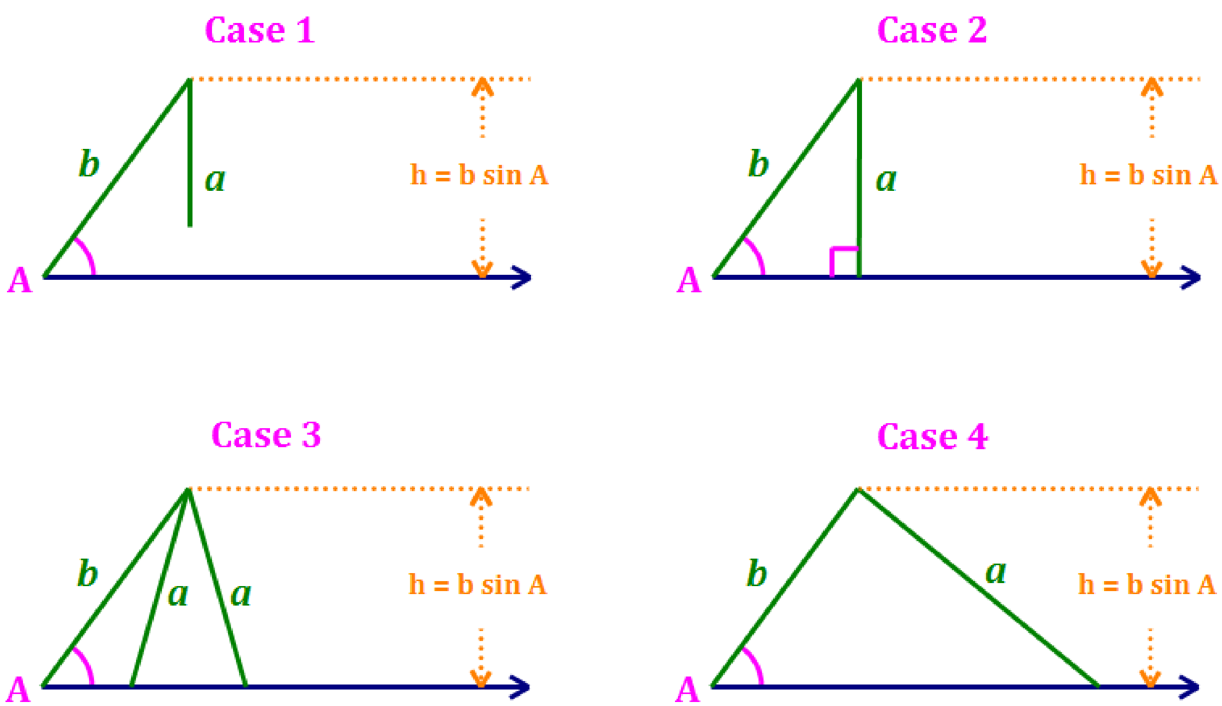
Given Two Sides and an Angle not between Them (SSA)

This is the Ambiguous Case. Several possibilities exist, depending on the lengths of the sides and the measure of the angle. The possibilities are discussed on the next several pages.

Solving an Oblique Triangle (cont'd)

The Ambiguous Case (SSA)

Given two segment lengths and an angle that is not between them, it is not clear whether a triangle is defined. It is possible that the given information will define a single triangle, two triangles, or even no triangle. Because there are multiple possibilities in this situation, it is called the ambiguous case. Here are the possibilities:



There are three cases in which $a < b$.

Case 1: $a < b \sin A$ Produces no triangle because a is not long enough to reach the base.

Case 2: $a = b \sin A$ Produces one (right) triangle because a is exactly long enough to reach the base. a forms a right angle with the base, and is the height of the triangle.

Case 3: $a > b \sin A$ Produces two triangles because a is the right size to reach the base in two places. The angle from which a swings from its apex can take two values.

There is only one case in which $a \geq b$.

Case 4: $a \geq b$ Produces one triangle because a is not long enough to reach the base.

Solving a Triangle (cont'd)

Solving the Ambiguous Case (SSA)

How do you solve the triangle in each of the cases discussed above. Assume the information given is the lengths of sides a and b, and the measure of Angle A. Use the following steps:

Step 1: Calculate the sine of the missing angle (in this development, angle B).

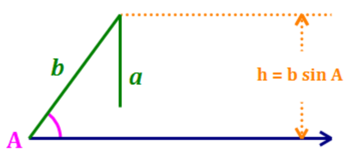
Step 1: Use

$$\frac{a}{\sin\ A} = \frac{b}{\sin\ B}$$

Step 2: Consider the value of $\sin B$:

- If $\sin B > 1$, then we have Case 1 – there is no triangle. Stop here.

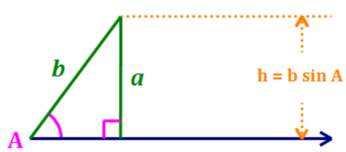
Case 1



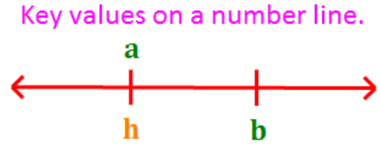
Key values on a number line.

- If $\sin B = 1$, then $B = 90°$, and we have Case 2 – a right triangle. Proceed to Step 4.

Case 2



Key values on a number line.

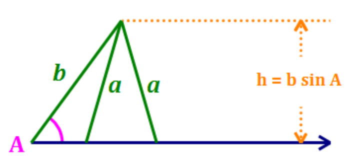


- If $\sin B < 1$, then we have Case 3 or Case 4. Proceed to the next step to determine which.

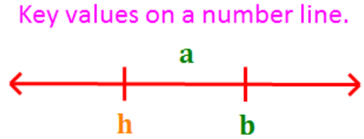
Step 3: Consider whether $a > b$.

- If $a < b$, then we have Case 3 – two triangles. Calculate the values of each angle B, using the Law of Sines. Then, proceed to Step 4 and calculate the remaining values for each triangle.

Case 3

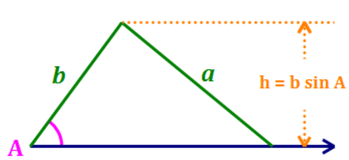


Key values on a number line.

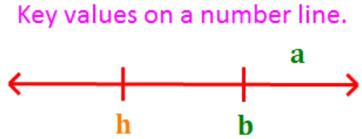


- If $a \geq b$, then we have case 4 – one triangle. Proceed to Step 4.

Case 4



Key values on a number line.



Solving the Ambiguous Case (SSA) – cont'd

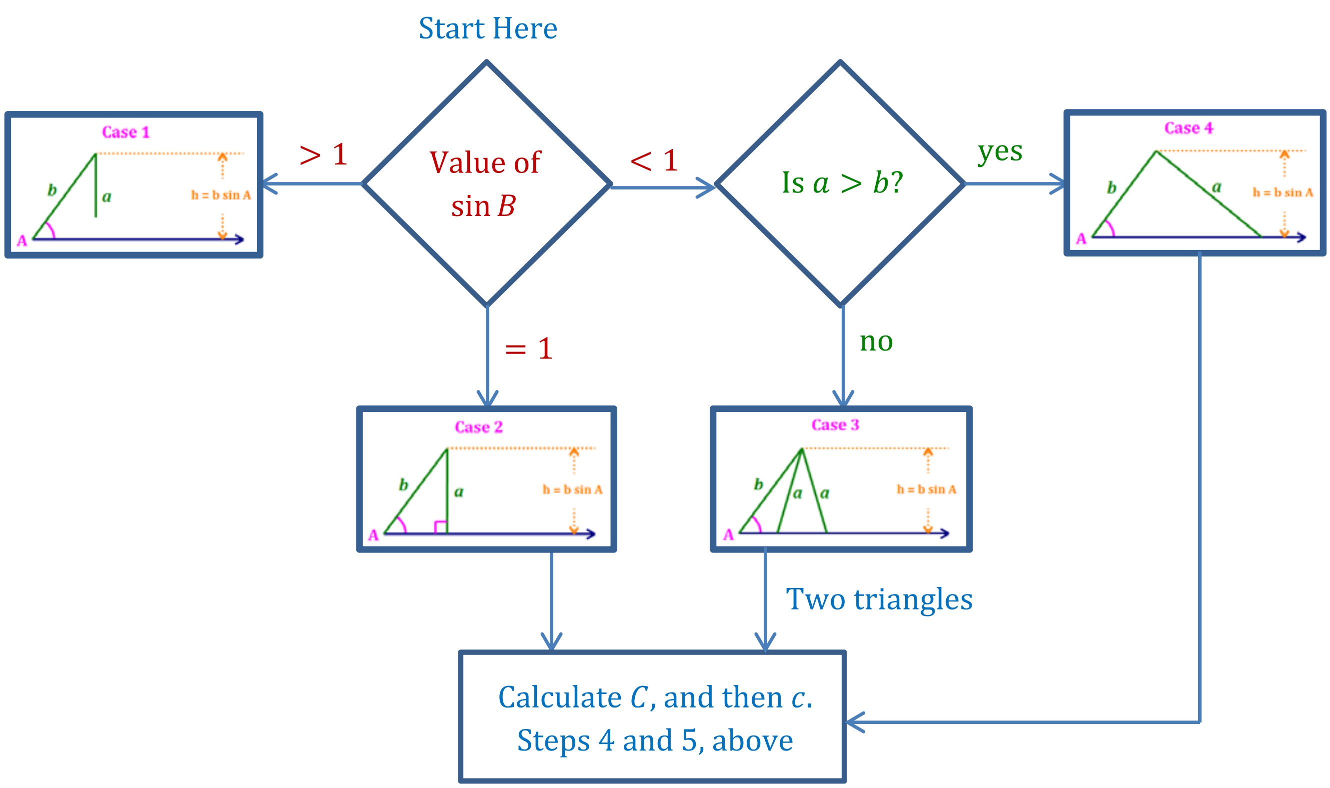
Step 4: Calculate C. At this point, we have the lengths of sides a and b, and the measures of Angles A and B. If we are dealing with Case 3 – two triangles, we must perform Steps 4 and 5 for each angle.

Step 4 is to calculate the measure of Angle C as follows: $c = 180° - A - B$

Step 5: Calculate c. Finally, we calculate the value of c using the Law of Sines. Note that in the case where there are two triangles, there is an Angle B in each. So, the Law of Sines should be used relating Angles B and C.

$$\frac{b}{\sin B} = \frac{c}{\sin C} \quad \Rightarrow \quad c = \frac{b \sin C}{\sin B}$$

Ambiguous Case Flowchart

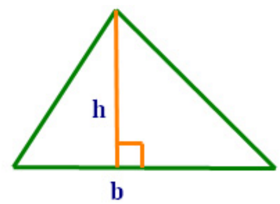


Area of a Triangle

Area of a Triangle

There are two formulas for the area of a triangle, depending on what information about the triangle is available.

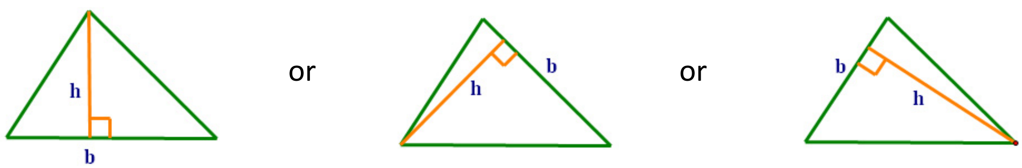
Formula 1: The formula most familiar to the student can be used when the base and height of the triangle are either known or can be determined.



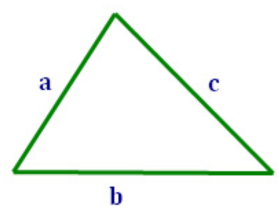
$$A = \frac{1}{2}bh$$

where, b is the length of the base of the triangle.
h is the height of the triangle.

Note: The base can be any side of the triangle. The height is the measure of the altitude of whichever side is selected as the base. So, you can use:



Formula 2: Heron's formula for the area of a triangle can be used when the lengths of all of the sides are known. Sometimes this formula, though less appealing, can be very useful.



$$A = \sqrt{s(s-a)(s-b)(s-c)}$$

where, $s = \frac{1}{2}P = \frac{1}{2}(a+b+c)$. **Note:** s is sometimes called the semi-perimeter of the triangle.

a, b, c are the lengths of the sides of the triangle.

Area of a Triangle (cont'd)

Trigonometric Formulas

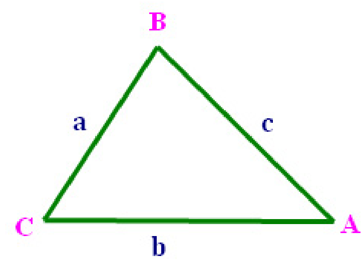
The following formulas for the area of a triangle come from trigonometry. Which one is used depends on the information available:

Two angles and a side:

$$A = \frac{1}{2} \cdot \frac{a^2 \cdot \sin B \cdot \sin C}{\sin A} = \frac{1}{2} \cdot \frac{b^2 \cdot \sin A \cdot \sin C}{\sin B} = \frac{1}{2} \cdot \frac{c^2 \cdot \sin A \cdot \sin B}{\sin C}$$

Two sides and an angle:

$$A = \frac{1}{2}\, ab \, \sin C = \frac{1}{2}\, ac \, \sin B = \frac{1}{2}\, bc \, \sin A$$

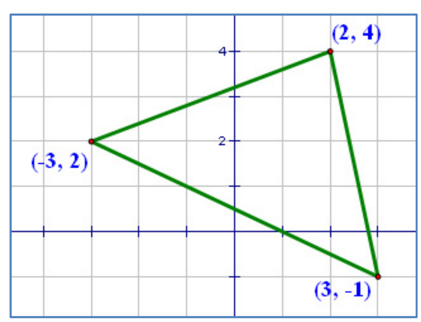


Coordinate Geometry

If the three vertices of a triangle are displayed in a coordinate plane, the formula below, using a determinant, will give the area of a triangle.

Let the three points in the coordinate plane be: (x_1, y_1), (x_2, y_2), (x_3, y_3). Then, the area of the triangle is one half of the absolute value of the determinant below:

$$A = \frac{1}{2} \cdot \left| \begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \right|$$



Example: For the triangle in the figure at right, the area is:

$$A = \frac{1}{2} \cdot \left| \begin{vmatrix} 2 & 4 & 1 \\ -3 & 2 & 1 \\ 3 & -1 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \cdot \left| \left(2 \begin{vmatrix} 2 & 1 \\ -1 & 1 \end{vmatrix} - 4 \begin{vmatrix} -3 & 1 \\ 3 & 1 \end{vmatrix} + \begin{vmatrix} -3 & 2 \\ 3 & -1 \end{vmatrix} \right) \right| = \frac{1}{2} \cdot 27 = \frac{27}{2}$$

Polar Coordinates

Polar coordinates are an alternative method of describing a point in a Cartesian plane based on the distance of the point from the origin and the angle whose terminal side contains the point. First, let's investigate the relationship between a point's rectangular coordinates (x, y) and its polar coordinates (r, θ).

The magnitude, r, is the distance of the point from the origin: $r = \sqrt{x^2 + y^2}$

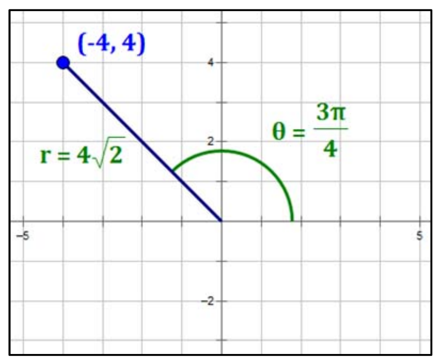
The angle, θ, is the angle whose terminal side contains the point. Generally, this angle is expressed in radians, not degrees.

$$\tan\theta = \frac{y}{x} \quad \text{or} \quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Conversion from polar coordinates to rectangular coordinates is straightforward:

$x = r \cdot \cos\theta$ and $y = r \cdot \sin\theta$

Example 1: Express the rectangular form (-4, 4) in polar coordinates:



Given: $x = -4 \qquad y = 4$

$r = \sqrt{x^2 + y^2} = \sqrt{(-4)^2 + 4^2} = 4\sqrt{2}$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{4}{-4}\right) = \tan^{-1}(-1)$ in Quadrant II,

so $\theta = \frac{3\pi}{4}$

So, the coordinates of the point are as follows:

Rectangular coordinates: $(-4, 4)$ Polar Coordinates: $(4\sqrt{2}, \frac{3\pi}{4})$

Example 2: Express the polar form $(4\sqrt{2}, \frac{3\pi}{4})$ in rectangular coordinates:

Given: $r = 4\sqrt{2} \qquad \theta = \frac{3\pi}{4}$

$x = r \cdot \cos\theta = 4\sqrt{2} \cdot \cos\frac{3\pi}{4} = 4\sqrt{2} \cdot \left(-\frac{\sqrt{2}}{2}\right) = -4$

$y = r \cdot \sin\theta = 4\sqrt{2} \cdot \sin\frac{3\pi}{4} = 4\sqrt{2} \cdot \left(\frac{\sqrt{2}}{2}\right) = 4$

So, the coordinates of the point are as follows:

Polar Coordinates: $(4\sqrt{2}, \frac{3\pi}{4})$ Rectangular coordinates: $(-4, 4)$

Polar Form of Complex Numbers

Expressing Complex Numbers in Polar Form

A complex number can be represented as point in the Cartesian Plane, using the horizontal axis for the real component of the number and the vertical axis for the imaginary component of the number.

If we express a complex number in rectangular coordinates as $z = a + bi$, we can also express it in polar coordinates as $z = r(\cos\theta + i\sin\theta)$, with $\theta \in [0, 2\pi)$. Then, the equivalences between the two forms for z are:

Convert Rectangular to Polar	Convert Polar to Rectangular
Magnitude: $\lvert z \rvert = r = \sqrt{a^2 + b^2}$	x-coordinate: $a = r\cos\theta$
Angle: $\theta = \tan^{-1}\left(\frac{b}{a}\right)$	y-coordinate: $b = r\sin\theta$

Since θ will generally have two values on $[0, 2\pi)$, you need to be careful to select the angle in the quadrant in which $z = a + bi$ resides.

Operations on Complex Numbers in Polar Form

Another expression that may be useful is: $e^{i\theta} = \cos\theta + i\sin\theta$, a complex number can be expressed as an exponential form of e. That is:

$$z = a + bi = r(\cos\theta + i\sin\theta) = r \cdot e^{i\theta}$$

It is this expression that is responsible for the following rules regarding operations on complex numbers. Let: $z_1 = a_1 + b_1 i = r_1(\cos\theta + i\sin\theta)$, $z_2 = a_2 + b_2 i = r_2(\cos\varphi + i\sin\varphi)$. Then,

Multiplication: $z_1 \cdot z_2 = r_1 r_2 [\cos(\theta + \varphi) + i\sin(\theta + \varphi)]$

So, to multiply complex numbers, you multiply their magnitudes and add their angles.

Division: $z_1 \div z_2 = \frac{r_1}{r_2} [\cos(\theta - \varphi) + i\sin(\theta - \varphi)]$

So, to divide complex numbers, you divide their magnitudes and subtract their angles.

Powers: ${z_1}^n = {r_1}^n(\cos n\theta + i\sin n\theta)$

This results directly from the multiplication rule.

Roots: $\sqrt[n]{z_1} = \sqrt[n]{r_1}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$ *also, see "DeMoivre's Theorem" below*

This results directly from the power rule if the exponent is a fraction.

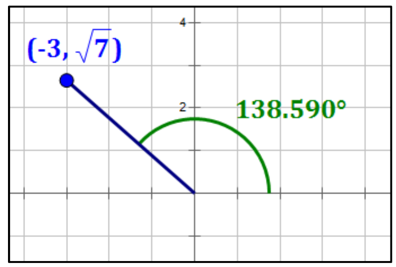
DeMoivre's Theorem

Abraham de Moivre (1667-1754) was a French mathematician who provided us with a very useful Theorem for dealing with operations on complex numbers.

If we let $z = r(\cos\theta + i\sin\theta)$, DeMoivre's Theorem gives us the power rule expressed on the prior page:

$$z^n = r^n(\cos n\theta + i\sin n\theta)$$

Example 1: Find $\left(-3 + i\sqrt{7}\right)^6$



First, since $z = a + bi$, we have $a = -3$ and $b = \sqrt{7}$.

Then, $r = \sqrt{(-3)^2 + (\sqrt{7})^2} = 4;\ r^6 = 4^6 = 4{,}096$

And, $\theta = \tan^{-1}\left(-\frac{\sqrt{7}}{3}\right) = 138.590°$ in Q II

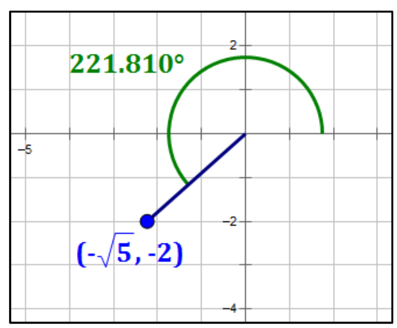
$6\theta = 831.542° \sim 111.542°$

So,

$$\left(-3 + i\sqrt{7}\right)^6 = 4{,}096 \cdot [\cos(111.542°) + i\sin(111.542°)]$$

$$= -1{,}504.0 + 3{,}809.9i$$

Example 2: Find $\left(-\sqrt{5} - 2i\right)^5$



First, since $z = a + bi$, we have $a = -2$ and $b = \sqrt{7}$.

Then, $r = \sqrt{(-\sqrt{5})^2 + (-2)^2} = 3;\ r^5 = 3^5 = 243$

And, $\theta = \tan^{-1}\left(\frac{2}{\sqrt{5}}\right) = 221.810°$ in Q III

$5\theta = 1{,}109.052° \sim 29.052°$

So,

$$\left(-\sqrt{5} - 2i\right)^5 = 243 \cdot [\cos(29.052°) + i\sin(29.052°)]$$

$$= 212.4 + 118.0i$$

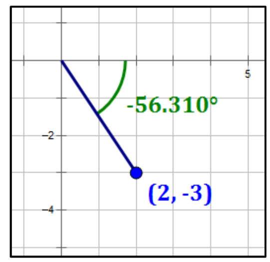
DeMoivre's Theorem for Roots

Let $z = r(\cos\theta + i\sin\theta)$. Then, z has n distinct complex n-th roots that occupy positions equidistant from each other on a circle of radius $\sqrt[n]{r}$. Let's call the roots: $z_1, z_2, \dots, z_n$ Then, these roots can be calculated as follows:

$$z_k = \sqrt[n]{r} \cdot \left[\cos\left(\frac{\theta + k(2\pi)}{n}\right) + i\sin\left(\frac{\theta + k(2\pi)}{n}\right)\right]$$

The formula could also be restated with 2π replaced by $360°$ if this helps in the calculation.

Example: Find the fifth roots of $\mathbf{2 - 3}\boldsymbol{i}$.



First, since $z = a + bi$, we have $a = 2$ and $b = -3$.

Then, $r = \sqrt{2^2 + (-3)^2} = \sqrt{13}$; $\sqrt[5]{r} = \sqrt[10]{13} \sim 1.2924$

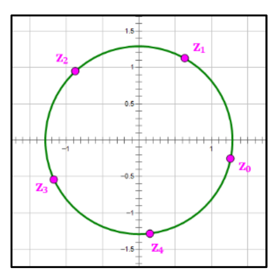
And, $\theta = \tan^{-1}\left(-\frac{3}{2}\right) = -56.310°$; $\frac{\theta}{5} = -11.262°$

The incremental angle for successive roots is: $360° \div 5 \text{ roots} = 72°$.

Then create a chart like this:

Fifth roots of $(\mathbf{2 - 3}\boldsymbol{i})$ $\sqrt[5]{\boldsymbol{r}} = \sqrt[10]{\mathbf{13}} \sim \mathbf{1.2924}$ $\frac{\boldsymbol{\theta}}{\mathbf{5}} = \mathbf{-11.262°}$		
$\boldsymbol{k}$	**Angle** $(\boldsymbol{\theta_k})$	$\boldsymbol{z_k} = \sqrt[n]{\boldsymbol{r}} \cdot \cos\boldsymbol{\theta_k} + \sqrt[n]{\boldsymbol{r}} \cdot \sin\boldsymbol{\theta_k} \cdot \boldsymbol{i}$
0	$-11.262°$	$z_0 = 1.2675 - 0.2524\,i$
1	$-11.262° + 72° = 60.738°$	$z_1 = 0.6317 + 1.1275\,i$
2	$60.738° + 72° = 132.738°$	$z_2 = -0.8771 + 0.9492\,i$
3	$132.738° + 72° = 204.738°$	$z_3 = -1.1738 - 0.5408\,i$
4	$204.738° + 72° = 276.738°$	$z_4 = 0.1516 - 1.2835\,i$

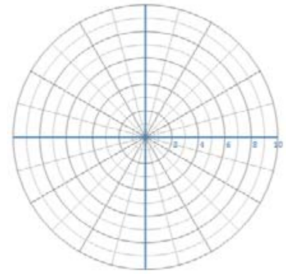
Notice that if we add another $72°$, we get $348.738°$, which is equivalent to our first angle, $-11.262°$ because $348.738° - 360° = -11.262°$. This is a good thing to check. The "next angle" will always be equivalent to the first angle! If it isn't, go back and check your work.



Roots fit on a circle: Notice that, since all of the roots of $\mathbf{2 - 3}\boldsymbol{i}$ have the same magnitude, and their angles that are $72°$ apart from each other, that they occupy equidistant positions on a circle with center $(0, 0)$ and radius $\sqrt[5]{r} = \sqrt[10]{13} \sim 1.2924$.

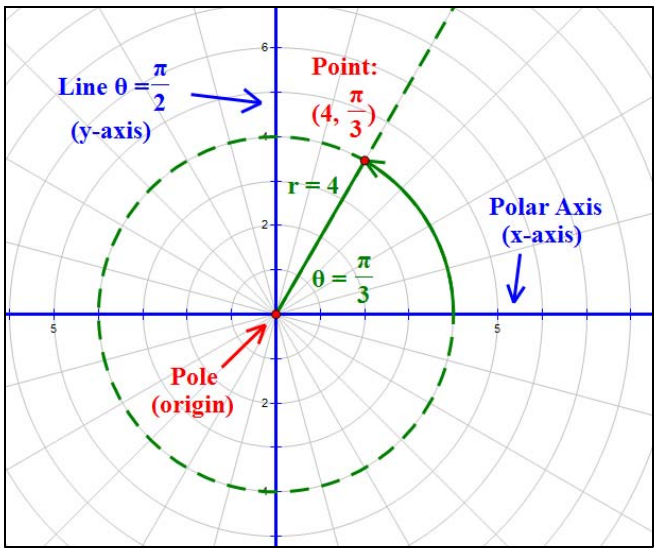
Polar Graphs

Typically, Polar Graphs will be plotted on polar graphs such as the one illustrated at right. On this graph, a point (r, θ) can be considered to be the intersection of the circle of radius r and the terminal side of the angle θ (see the illustration below).



Parts of the Polar Graph

The illustration below shows the key parts of a polar graph, along with a point, $(4, \frac{\pi}{3})$.



The Pole is the point $(0, 0)$ (i.e., the origin).

The Polar Axis is the x-axis.

The Line: $\theta = \frac{\pi}{2}$ is the y-axis.

Many equations that contain the cosine function are symmetric about the Polar Axis.

Many equations that contain the sine function are symmetric about the line $\theta = \frac{\pi}{2}$.

Polar Equations – Symmetry

Following are the three main types of symmetry exhibited in many polar equation graphs:

Symmetry about:	Quadrants Containing Symmetry	Symmetry Test(1)
The Pole	Opposite (I and III or II and IV)	Replace r with $-r$ in the equation
The Polar Axis	Left or right hemispheres (II and III or I and IV)	Replace θ with $-\theta$ in the equation
The Line $\theta = \frac{\pi}{2}$	Upper or lower hemispheres (I and II or III and IV)	Replace (r, θ) with $(-r, -\theta)$ in the equation

(1) If performing the indicated replacement results in an equivalent equation, the equation passes the symmetry test and the indicated symmetry exists. If the equation fails the symmetry test, symmetry may or may not exist.

Graphs of Polar Equations

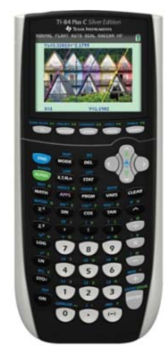
Graphing Methods

Method 1: Point plotting

- Create a two-column chart that calculates values of r for selected values of θ. This is akin to a two-column chart that calculates values of y for selected values of x that can be used to plot a rectangular coordinates equation (e.g., $y = x^2 - 4x + 3$).
- The θ-values you select for purposes of point plotting should vary depending on the equation you are working with (in particular, the coefficient of θ in the equation). However, a safe bet is to start with multiples of ${}^{\pi}/_{6}$ (including $\theta = 0$). Plot each point on the polar graph and see what shape emerges. If you need more or fewer points to see what curve is emerging, adjust as you go.
- If you know anything about the curve (typical shape, symmetry, etc.), use it to facilitate plotting points.
- Connect the points with a smooth curve. Admire the result; many of these curves are aesthetically pleasing.

Method 2: Calculator

Using a TI-84 Plus Calculator or its equivalent, do the following:

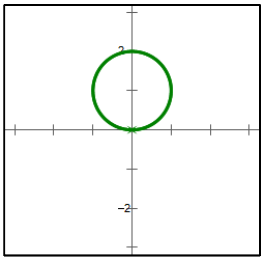


- Make sure your calculator is set to radians and polar functions. Hit the **MODE** key; select **RADIANS** in row 4 and **POLAR** in row 5. After you do this, hitting **CLEAR** will get you back to the main screen.
- Hit **Y=** and enter the equation in the form $r = f(\theta)$. Use the **X,T,θ,n** key to enter θ into the equation. If your equation is of the form $r^2 = f(\theta)$, you may need to enter two functions, $r = \sqrt{f(\theta)}$ and $r = -\sqrt{f(\theta)}$, and plot both.
- Hit **GRAPH** to plot the function or functions you entered in the previous step.
- If necessary, hit **WINDOW** to adjust the parameters of the plot.
  - o If you cannot see the whole function, adjust the **X-** and **Y-** variables (or use **ZOOM**).
  - o If the curve is not smooth, reduce the value of the **θstep** variable. This will plot more points on the screen. Note that smaller values of **θstep** require more time to plot the curve, so choose a value that plots the curve well in a reasonable amount of time.
  - o If the entire curve is not plotted, adjust the values of the **θmin** and **θmax** variables until you see what appears to be the entire plot.

Note: You can view the table of points used to graph the polar function by hitting **2ND – TABLE**.

Graph Types (Polar Equations)

Circle



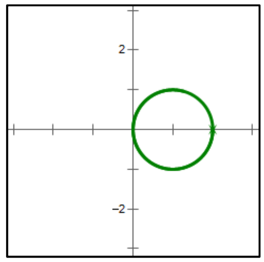
Equation: $r = a\sin\theta$

Location:

above Polar Axis if $a > 0$
below Polar Axis if $a < 0$

Radius: $a/2$

Symmetry: Line $\theta = \pi/2$



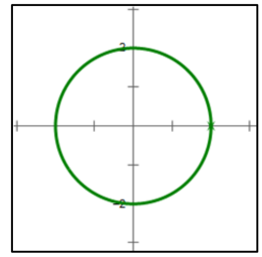
Equation: $r = a\cos\theta$

Location:

right of line $\theta = \pi/2$ if $a > 0$
left of line $\theta = \pi/2$ if $a < 0$

Radius: $a/2$

Symmetry: Polar Axis



Equation: $r = a$

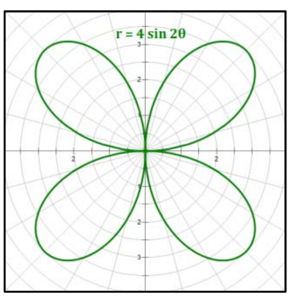
Location:

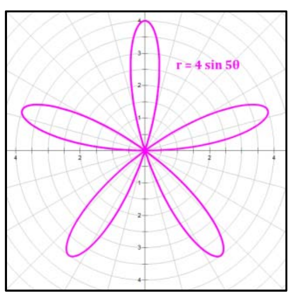
Centered on the Pole

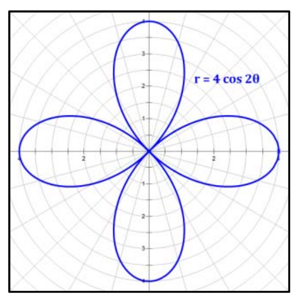
Radius: a

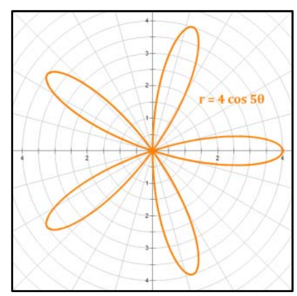
Symmetry: Pole, Polar Axis,
Line $\theta = \pi/2$

Rose







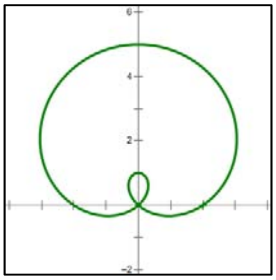


Characteristics of roses:

- Equation: $r = a\sin n\theta$
  - Symmetric about the line $\theta = \pi/2$ (y-axis)
- Equation: $r = a\cos n\theta$
  - Symmetric about the Polar Axis (x-axis)
- Contained within a circle of radius $r = a$
- If n is odd, the rose has n petals.
- If n is even the rose has $2n$ petals.
- Note that a circle is a rose with one petal (i.e, $n = 1$).

Graphs of Polar Equations

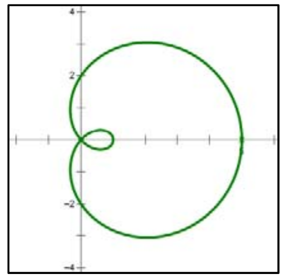
Limaçon of Pascal



Equation: $r = a + b\sin\theta$

Location: bulb above Polar Axis if $b > 0$
bulb below Polar Axis if $b < 0$

Symmetry: Line $\theta = \pi/2$

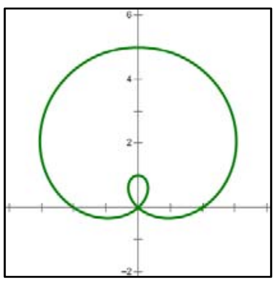


Equation: $r = a + b\cos\theta$

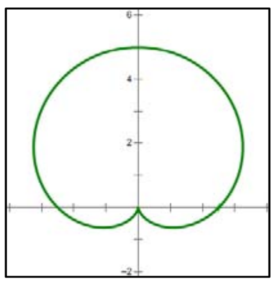
Location: bulb right of Line $\theta = \pi/2$ if $b > 0$
bulb left of Line $\theta = \pi/2$ if $b < 0$

Symmetry: Polar Axis

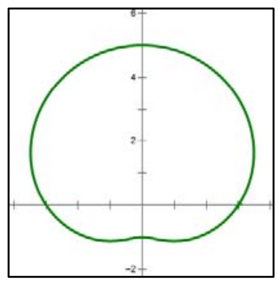
Four Limaçon Shapes



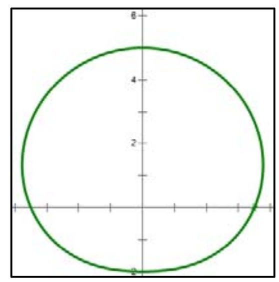
$a < b$
Inner loop



$a = b$
"Cardioid"

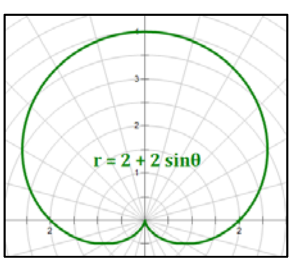


$b < a < 2b$
Dimple

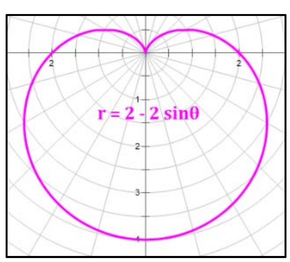


$a \geq 2b$
No dimple

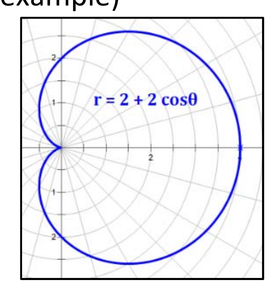
Four Limaçon Orientations (using the Cardioid as an example)



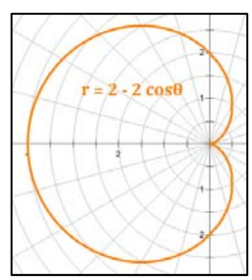
sine function
$b > 0$



sine function
$b < 0$



cosine function
$b > 0$



cosine function
$b < 0$

Graphing Polar Equations – The Rose

Example: $\mathbf{r = 4\sin 2\theta}$

This function is a rose. Consider the forms $r = a\sin b\theta$ and $r = a\cos b\theta$.

The number of petals on the rose depends on the value of b.

- If b is an even integer, the rose will have $2b$ petals.
- If b is an odd integer, it will have b petals.

Let's create a table of values and graph the equation:

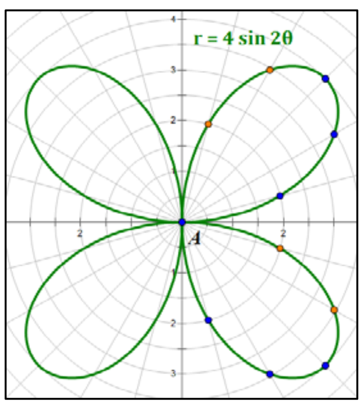
$r = 4\sin 2\theta$			
θ	r	θ	r
0	0		
$\pi/12$	2	$7\pi/12$	-2
$\pi/6$	3.464	$2\pi/3$	-3.464
$\pi/4$	4	$3\pi/4$	-4
$\pi/3$	3.464	$5\pi/6$	-3.464
$5\pi/12$	2	$11\pi/12$	-2
$\pi/2$	0	π	0

Because this function involves an argument of 2θ, we want to start by looking at values of θ in $[0, 2\pi] \div 2 = [0, \pi]$. You could plot more points, but this interval is sufficient to establish the nature of the curve; so you can graph the rest easily.

Once symmetry is established, these values are easily determined.

The values in the table generate the points in the two petals right of the y-axis.

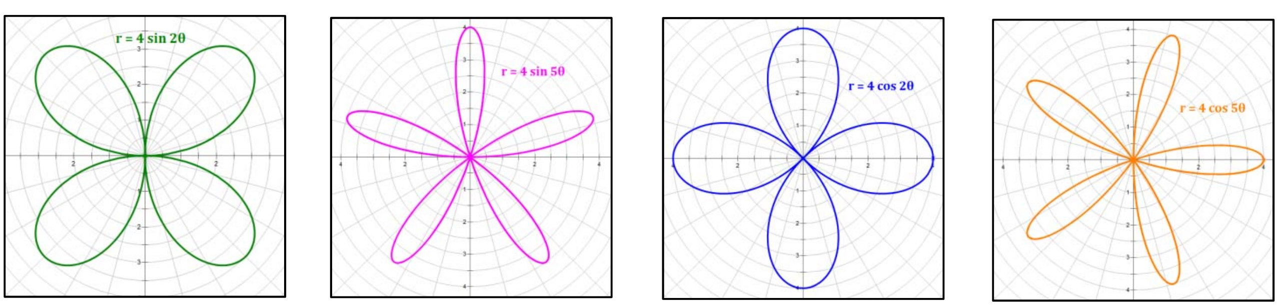
Knowing that the curve is a rose allows us to graph the other two petals without calculating more points.



Blue points on the graph correspond to blue values in the table.

Orange points on the graph correspond to orange values in the table.

The four Rose forms:



Graphing Polar Equations – The Cardioid
Example: $\mathbf{r = 2 + 2\sin\theta}$

This cardioid is also a limaçon of form $r = a + b\sin\theta$ with $a = b$. The use of the sine function indicates that the large loop will be symmetric about the y-axis. The $+$ sign indicates that the large loop will be above the x-axis. Let's create a table of values and graph the equation:

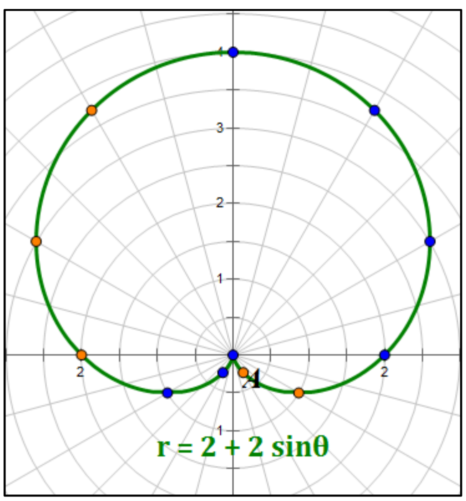
$\mathbf{r = 2 + 2\sin\theta}$			
θ	r	θ	r
0	2		
$\pi/6$	3	$7\pi/6$	1
$\pi/3$	3.732	$4\pi/3$	0.268
$\pi/2$	4	$3\pi/2$	0
$2\pi/3$	3.732	$5\pi/3$	0.268
$5\pi/6$	3	$11\pi/6$	1
π	2	2π	2

Generally, you want to look at values of θ in $[0, 2\pi]$. However, some functions require larger intervals. The size of the interval depends largely on the nature of the function and the coefficient of θ.

Once symmetry is established, these values are easily determined.

The portion of the graph above the x-axis results from θ in Q1 and Q2, where the sine function is positive.

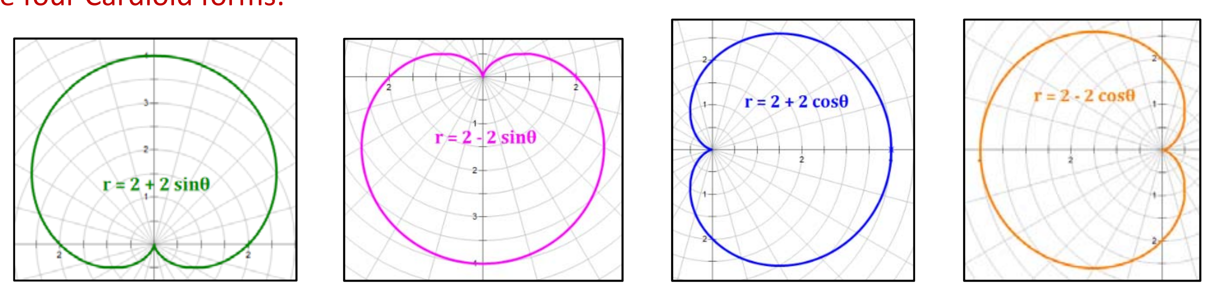
Similarly, the portion of the graph below the x-axis results from θ in Q3 and Q4, where the sine function is negative.



Blue points on the graph correspond to blue values in the table.

Orange points on the graph correspond to orange values in the table.

The four Cardioid forms:



Converting Between Polar and Rectangular Forms of Equations

Rectangular to Polar

To convert an equation from Rectangular Form to Polar Form, use the following equivalences:

$x = r\cos\theta$	Substitute $r\cos\theta$ for x
$y = r\sin\theta$	Substitute $r\sin\theta$ for y
$x^2 + y^2 = r^2$	Substitute r^2 for $x^2 + y^2$

Example: Convert $8x - 3y + 10 = 0$ to a polar equation of the form $r = f(\theta)$.

Starting Equation:	$8x - 3y + 10 = 0$
Substitute $x = r\cos\theta$ and $y = r\sin\theta$:	$8 \cdot r\cos\theta - 3 \cdot r\sin\theta + 10 = 0$
Factor out r:	$r\,(8\cos\theta - 3\sin\theta) = -10$
Divide by $(8\cos\theta - 3\sin\theta)$:	$\boldsymbol{r = \dfrac{-10}{8\cos\theta - 3\sin\theta}}$

Polar to Rectangular

To convert an equation from Polar Form to Rectangular Form, use the following equivalences:

$\cos\theta = \dfrac{x}{r}$	Substitute $\dfrac{x}{r}$ for $\cos\theta$
$\sin\theta = \dfrac{y}{r}$	Substitute $\dfrac{y}{r}$ for $\sin\theta$
$r^2 = x^2 + y^2$	Substitute $x^2 + y^2$ for r^2

Example: Convert r = 8 cos θ + 9 sin θ to a rectangular equation.

Starting Equation:	r = 8 cos θ + 9 sin θ
Substitute $\cos\theta = \frac{x}{r}$, $\sin\theta = \frac{y}{r}$:	$r = 8\left(\frac{x}{r}\right) + 9\left(\frac{y}{r}\right)$
Multiply by r:	$r^2 = 8x + 9y$
Substitute $r^2 = x^2 + y^2$:	$x^2 + y^2 = 8x + 9y$
Subtract $8x + 9y$:	$x^2 - 8x \quad + y^2 - 9y \quad = 0$
Complete the square:	$(x^2 - 8x + 16) + \left(y^2 - 9y + \frac{81}{4}\right) = 16 + \frac{81}{4}$
Simplify to standard form for a circle:	$\boldsymbol{(x-4)^2 + \left(y - \frac{9}{2}\right)^2 = \frac{145}{4}}$

Vectors

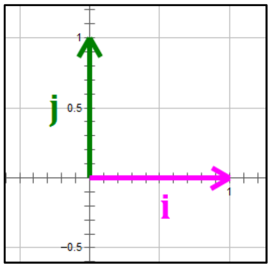
A vector is a quantity that has both magnitude and direction. An example would be wind blowing toward the east at 30 miles per hour. Another example would be the force of 10 kg weight being pulled toward the earth (a force you can feel if you are holding the weight).

Special Unit Vectors

We define unit vectors to be vectors of length 1. Unit vectors having the direction of the positive axes will be quite useful to us. They are described in the chart and graphic below.

Unit Vector	Direction
$\mathbf{i}$	positive x-axis
$\mathbf{j}$	positive y-axis
$\mathbf{k}$	positive z-axis

Graphical representation of unit vectors $\mathbf{i}$ and $\mathbf{j}$ in two dimensions.



Vector Components

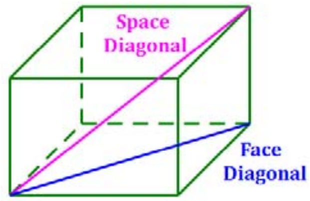
The length of a vector, $\mathbf{v}$, is called its magnitude and is represented by the symbol $\|\mathbf{v}\|$. If a vector's initial point (starting position) is (x_1, y_1, z_1), and its terminal point (ending position) is (x_2, y_2, z_2), then the vector displaces $\boldsymbol{a = x_2 - x_1}$ in the x-direction, $\boldsymbol{b = y_2 - y_1}$ in the y-direction, and $\boldsymbol{c = z_2 - z_1}$ in the z-direction. We can, then, represent the vector as follows:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} + c\mathbf{k}$$

The magnitude of the vector, $\mathbf{v}$, is calculated as:

$$\|\mathbf{v}\| = \sqrt{a^2 + b^2 + c^2}$$

If this looks familiar, it should. The magnitude of a vector in three dimesnsions is determined as the length of the space diagonal of a rectangular prism with sides a, b and c.



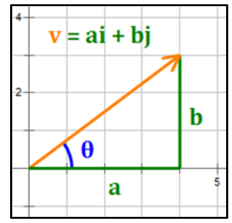
In two dimensions, these concepts contract to the following:

$$\mathbf{v} = a\mathbf{i} + b\mathbf{j} \qquad\qquad \|\mathbf{v}\| = \sqrt{a^2 + b^2}$$

In two dimensions, the magnitude of the vector is the length of the hypotenuse of a right triangle with sides a and b.

Vector Properties

Vectors have a number of nice properties that make working with them both useful and relatively simple. Let m and n be scalars, and let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then,



- If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, then $a = \|\mathbf{v}\| \cos\theta$ and $b = \|\mathbf{v}\| \sin\theta$
- Then, $\mathbf{v} = \|\mathbf{v}\| \cos\theta\ \mathbf{i} + \|\mathbf{v}\| \sin\theta\ \mathbf{j}$ (note: this formula is used in Force calculations)
- If $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j}$, then $\mathbf{u} + \mathbf{v} = (a_1 + a_2)\mathbf{i} + (b_1 + b_2)\mathbf{j}$
- If $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$, then $m\mathbf{v} = (ma)\mathbf{i} + (mb)\mathbf{j}$
- Define $\mathbf{0}$ to be the zero vector (i.e., it has zero length, so that $a = b = 0$). Note: the zero vector is also called the null vector.

Note: $\mathbf{v} = a\mathbf{i} + b\mathbf{j}$ can also be shown with the following notation: $\mathbf{v} = \langle a, b\rangle$. This notation is useful in calculating dot products and performing operations with vectors.

Properties of Vectors

- $\mathbf{0} + \mathbf{v} = \mathbf{v} + \mathbf{0} = \mathbf{v}$ — Additive Identity
- $\mathbf{v} + (-\mathbf{v}) = (-\mathbf{v}) + \mathbf{v} = \mathbf{0}$ — Additive Inverse
- $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ — Commutative Property
- $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ — Associative Property
- $m(n\boldsymbol{u}) = (mn)\boldsymbol{u}$ — Associative Property
- $m(\boldsymbol{u} + \boldsymbol{v}) = m\boldsymbol{u} + m\boldsymbol{v}$ — Distributive Property
- $(m + n)\boldsymbol{u} = m\boldsymbol{u} + n\mathbf{u}$ — Distributive Property
- $1(\mathbf{v}) = \mathbf{v}$ — Multiplicative Identity

Also, note that:

- $\|m\mathbf{v}\| = |m|\ \|\mathbf{v}\|$ — Magnitude Property
- $\dfrac{\mathbf{v}}{\|\mathbf{v}\|}$ — Unit vector in the direction of $\mathbf{v}$

Vector Dot Product

The Dot Product of two vectors, $\mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + c_1\mathbf{k}$ and $\mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + c_2\mathbf{k}$, is defined as follows:

$$\mathbf{u} \cdot \mathbf{v} = (a_1 \cdot a_2) + (b_1 \cdot b_2) + (c_1 \cdot c_2)$$

It is important to note that the dot product is a scalar, not a vector. It describes something about the relationship between two vectors, but is not a vector itself. A useful approach to calculating the dot product of two vectors is illustrated here:

$$\left.\begin{array}{l} \mathbf{u} = a_1\mathbf{i} + b_1\mathbf{j} + \mathbf{c_1}\mathbf{k} = \langle a_1, b_1, c_1 \rangle \\ \mathbf{v} = a_2\mathbf{i} + b_2\mathbf{j} + \mathbf{c_2}\mathbf{k} = \langle a_2, b_2, c_2 \rangle \end{array}\right\} \text{alternative vector notation}$$

In the example at right the vectors are lined up vertically. The numbers in the each column are multiplied and the results are added to get the dot product. In the example, $\langle 4, -3, 2 \rangle \circ \langle 2, -2, 5 \rangle = 8 + 6 + 10 = 24$.

General	Example
$\langle a_1, b_1, c_1 \rangle$	$\langle 4, -3, 2 \rangle$
$\circ \langle a_2, b_2, c_2 \rangle$	$\circ \langle 2, -2, 5 \rangle$
$a_1a_2 + b_1b_2 + c_1c_2$	$8 + 6 + 10$ $= 24$

Properties of the Dot Product

Let m be a scalar, and let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then,

- $\mathbf{0} \circ \mathbf{u} = \mathbf{u} \circ \mathbf{0} = 0$ Zero Property
- $\mathbf{i} \circ \mathbf{j} = \mathbf{j} \circ \mathbf{k} = \mathbf{k} \circ \mathbf{i} = 0$ $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ are orthogonal to each other.
- $\mathbf{u} \circ \mathbf{v} = \mathbf{v} \circ \mathbf{u}$ Commutative Property
- $\mathbf{u} \circ \mathbf{u} = \|\mathbf{u}\|^2$ Magnitude Square Property
- $\mathbf{u} \circ (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \circ \mathbf{v}) + (\mathbf{u} \circ \mathbf{w})$ Distributive Property
- $m(\mathbf{u} \circ \mathbf{v}) = (m\boldsymbol{u}) \circ \mathbf{v} = \boldsymbol{u} \circ (m\mathbf{v})$ Multiplication by a Scalar Property

More properties:

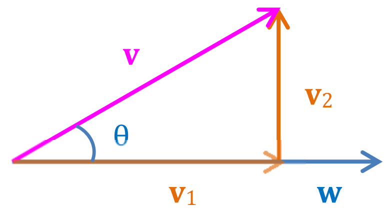
- If $\mathbf{u} \circ \mathbf{v} = 0$ and $\mathbf{u} \neq \mathbf{0}$ and $\mathbf{v} \neq \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (perpendicular).
- If there is a scalar m such that $m\boldsymbol{u} = \mathbf{v}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then $\cos\theta = \dfrac{\mathbf{u} \circ \mathbf{v}}{\|\mathbf{u}\|\ \|\mathbf{v}\|}$

Vector Dot Product (cont'd)

Vector Projection

The projection of a vector, $\mathbf{v}$, onto another vector $\mathbf{w}$, is obtained using the dot product. The formula used to determine the projection vector is:

$$\text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\,\mathbf{w} = \frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\,\mathbf{w}$$



Notice that $\dfrac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}$ is a scalar, and that $\text{proj}_{\mathbf{w}}\mathbf{v}$ is a vector.

In the diagram at right, $\mathbf{v}_1 = \text{proj}_{\mathbf{w}}\mathbf{v}$.

Orthogonal Components of a Vector (Decomposition)

A vector, $\mathbf{v}$, can be expressed as the sum of two orthogonal vectors $\mathbf{v_1}$ and $\mathbf{v_2}$, as shown in the above diagram. The resulting vectors are:

$$\mathbf{v_1} = \text{proj}_{\mathbf{w}}\mathbf{v} = \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{w}\|^2}\,\mathbf{w} \qquad \text{and} \qquad \mathbf{v_2} = \mathbf{v} - \mathbf{v_1}$$

$\mathbf{v_1}$ is parallel to $\mathbf{w}$ $\qquad$ $\mathbf{v_2}$ is orthogonal to $\mathbf{w}$

Work

Work is a scalar quantity in physics that measures the force exerted on an object over a particular distance. It is defined using vectors, as shown below. Let:

- $\mathbf{F}$ be the force vector acting on an object, moving it from point A to point B.
- $\overrightarrow{AB}$ be the vector from A to B.
- θ be the angle between $\mathbf{F}$ and $\overrightarrow{AB}$.

Then, we define work as:

$$W = \mathbf{F}\cdot\overrightarrow{AB}$$

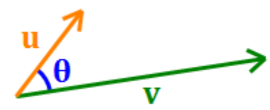
$$W = \underbrace{\|\mathbf{F}\|}_{\text{Magnitude of Force}}\ \underbrace{\left\|\overrightarrow{AB}\right\|}_{\text{Distance Traveled}}\ \underbrace{\cos\theta}_{\text{Angle between Vectors}}$$

Both of these formulas are useful. Which one you use in a particular situation depends on what information is available.

Vector Cross Product

Cross Product

In three dimensions,



Let: $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$

Then, the Cross Product is given by:

$$\mathbf{u} \times \mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2)\,\mathbf{i} + (u_3v_1 - u_1v_3)\,\mathbf{j} + (u_1v_2 - u_2v_1)\,\mathbf{k}$$

$$\mathbf{u} \times \mathbf{v} = \|\mathbf{u}\|\ \|\mathbf{v}\|\ \sin\theta\ \mathbf{n}$$

The cross product of two nonzero vectors in three dimensions produces a third vector that is orthogonal to each of the first two. This resulting vector $\mathbf{u} \times \mathbf{v}$ is, therefore, normal to the plane containing the first two vectors (assuming $\mathbf{u}$ and $\mathbf{v}$ are not parallel). In the second formula above, $\mathbf{n}$ is the unit vector normal to the plane containing the first two vectors. Its orientation (direction) is determined using the right hand rule.

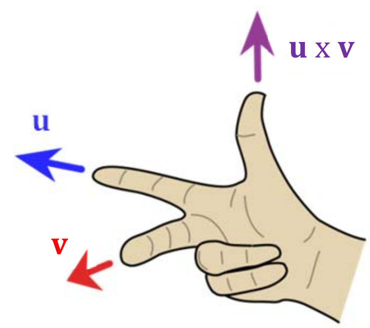
Right Hand Rule

Using your right hand:

- Point your forefinger in the direction of $\mathbf{u}$, and
- Point your middle finger in the direction of $\mathbf{v}$.

Then:

- Your thumb will point in the direction of $\mathbf{u} \times \mathbf{v}$.



In two dimensions,

Let: $\mathbf{u} = u_1\mathbf{i} + u_2\mathbf{j}$ and $\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j}$

Then, $\mathbf{u} \times \mathbf{v} = \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} = (u_1v_2 - u_2v_1)$ which is a scalar (in two dimensions).

The cross product of two nonzero vectors in two dimensions is zero if the vectors are parallel. That is, vectors $\mathbf{u}$ and $\mathbf{v}$ are parallel if $\mathbf{u} \times \mathbf{v} = 0$.

The area of a parallelogram having $\mathbf{u}$ and $\mathbf{v}$ as adjacent sides and angle θ between them:

$$Area = \|\mathbf{u}\|\ \|\mathbf{v}\|\ \sin\theta.$$

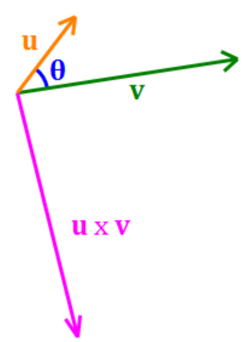
Properties of the Cross Product

Let m be a scalar, and let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be vectors. Then,

$\mathbf{0} \times \mathbf{u} = \mathbf{u} \times \mathbf{0} = \mathbf{0}$	Zero Property
$\mathbf{i} \times \mathbf{j} = \mathbf{k}$, $\mathbf{j} \times \mathbf{k} = \mathbf{i}$, $\mathbf{k} \times \mathbf{i} = \mathbf{j}$	$\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ are orthogonal to each other
$\mathbf{j} \times \mathbf{i} = -\mathbf{k}$, $\mathbf{k} \times \mathbf{j} = -\mathbf{i}$, $\mathbf{i} \times \mathbf{k} = -\mathbf{j}$	Reverse orientation orthogonality
$\mathbf{u} \times \mathbf{u} = \mathbf{0}$	Every non-zero vector is parallel to itself
$\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$	Anti-commutative Property
$\mathbf{u} \times (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) + (\mathbf{u} \times \mathbf{w})$	Distributive Property
$(\mathbf{u} + \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \times \mathbf{w}) + (\mathbf{v} \times \mathbf{w})$	Distributive Property
$(m\mathbf{u}) \times \mathbf{v} = \mathbf{u} \times (m\mathbf{v}) = m(\mathbf{u} \times \mathbf{v})$	Scalar Multiplication

More properties:

- If $\mathbf{u} \times \mathbf{v} = \mathbf{0}$, then $\mathbf{u}$ and $\mathbf{v}$ are parallel.
- If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then
  - $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\| \, \|\mathbf{v}\| \sin\theta$
  - $\sin\theta = \dfrac{\|\mathbf{u} \times \mathbf{v}\|}{\|\mathbf{u}\| \, \|\mathbf{v}\|}$



Vector Triple Products

Scalar Triple Product

Let: $\mathbf{u} = \mathrm{u}_1\mathbf{i} + \mathrm{u}_2\mathbf{j} + \mathrm{u}_3\mathbf{k}.$ Then the triple product $\mathbf{u} \circ (\mathbf{v} \times \mathbf{w})$ gives a scalar representing the volume of a parallelepiped with $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ as edges:

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} \mathrm{u}_1 & \mathrm{u}_2 & \mathrm{u}_3 \\ \mathrm{v}_1 & \mathrm{v}_2 & \mathrm{v}_3 \\ \mathrm{w}_1 & \mathrm{w}_2 & \mathrm{w}_3 \end{vmatrix}$$

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \times \mathbf{v}) \circ \mathbf{w}$$

Other Triple Products

$\mathbf{u} \circ (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \circ (\mathbf{u} \times \mathbf{v}) = \mathbf{0}$ Duplicating a vector results in a product of $\mathbf{0}$

$$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) = (\mathbf{u} \circ \mathbf{w})\,\mathbf{v} - (\mathbf{u} \circ \mathbf{v})\,\mathbf{w}$$

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \circ \mathbf{w})\,\mathbf{v} - (\mathbf{v} \circ \mathbf{w})\,\mathbf{u}$$

$$\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = \mathbf{v} \circ (\mathbf{w} \times \mathbf{x}) = \mathbf{w} \circ (\mathbf{x} \times \mathbf{v})$$

Note: vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are coplanar if and only if $\mathbf{u} \circ (\mathbf{v} \times \mathbf{w}) = 0$.

No Associative Property

The associative property of real numbers does not translate to triple products. In particular,

$(\mathbf{u} \circ \mathbf{v}) \cdot \mathbf{w} \neq \mathbf{u} \cdot (\mathbf{v} \circ \mathbf{w})$ No associative property of dot products/multiplication

$\mathbf{u} \times (\mathbf{v} \times \mathbf{w}) \neq (\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ No associative property of cross products

Appendix A

Summary of Rectangular and Polar Forms

		Rectangular Form	Polar Form
Coordinates	Form	(x, y)	(r, θ)
	Conversion	$x = r\cos\theta$ $y = r\sin\theta$	$r = \sqrt{x^2 + y^2}$ $\theta = \tan^{-1}\left(\frac{y}{x}\right)$
Complex Numbers	Form	$a + bi$	$r(\cos\theta + i\sin\theta)$ or $r\ cis\theta$
	Conversion	$a = r\cos\theta$ $b = r\sin\theta$	$r = \sqrt{a^2 + b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$
Vectors	Form	$a\mathbf{i} + b\mathbf{j}$	$\lVert v \rVert\ \angle\theta$ $\lVert v \rVert$ = magnitude θ = direction angle
	Conversion	$a = \lVert v \rVert\cos\theta$ $b = \lVert v \rVert\sin\theta$	$\lVert v \rVert = \sqrt{a^2 + b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

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