Back-propagation through convolution

Dr. Thanh-Sach LE
LTSACH@hcmut.edu.vn

GVLab:
Graphics and Vision Laboratory

Faculty of Computer Science and Engineering,
HCMUT

Contents

- Introduction
- Notation
- Derivation of ΔW
- Derivation of ΔX
- Summary

Back-propagation through convolution

Introduction

Dr. Thanh-Sach LE
LTSACH@hcmut.edu.vn

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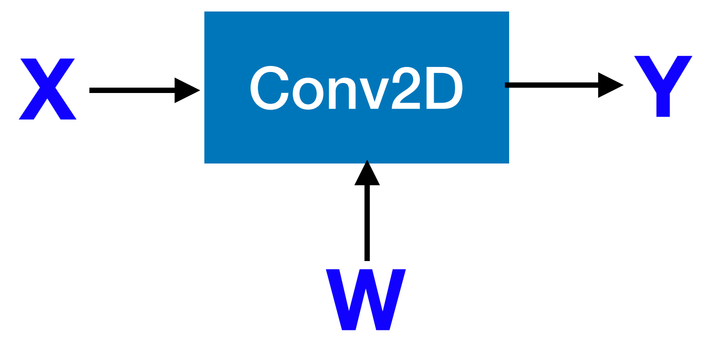
Introduction

Convolution:

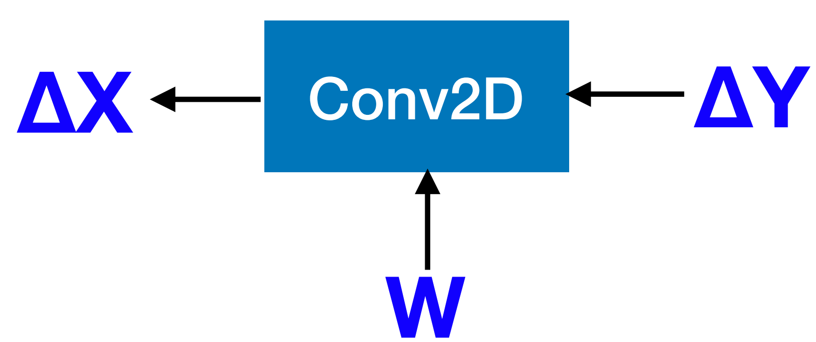


- **Foward-pass:**
  - ∂b **Input:**
    (a) X : feature map from previous layer
    (b) W: Learnable parameter of the convolution
  - **Goal:**
    (a) Compute Y, and it is cached for using in backward-pass

Introduction

- **Backward-pass:**
  - **Input:**
    (a) ΔY: partial derivatives of error function with respect to each value in Y
    (b) X: cached from the forward-pass
    (c) W: current parameters of the convolution

  - **Goal:**
    (a) Compute ΔX: partial derivatives of error function with respect to each value in X



Introduction

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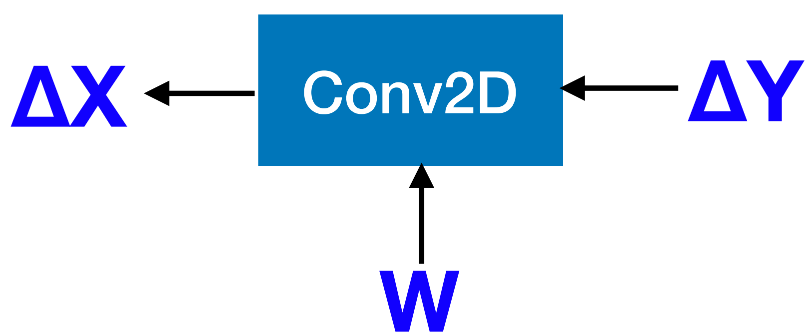
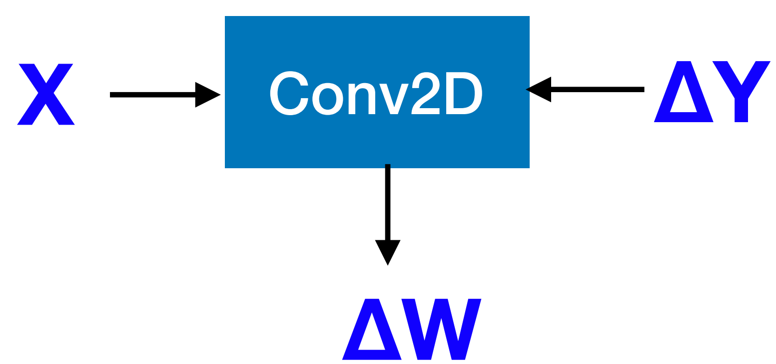
- **Backward-pass:**
  - **Input:**
    - (a) ΔY: partial derivatives of error function with respect to each value in Y
    - (b) X: cached from the forward-pass
    - (c) W: current parameters of the convolution
  - **Goal:**
    - (a) Compute ΔX: partial derivatives of error function with respect to each value in X



    - (b) Compute ΔW: partial derivatives of error function with respect to each value in W



Introduction

- **How to learn parameters**
  - **After backward-pass done, update weights according to the equation below**

$$W = W - \alpha \times \Delta W$$

∂b

α **: learning rate, a hyper-parameter**

Introduction

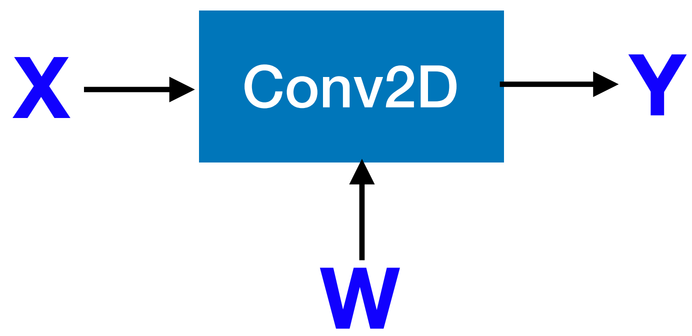
Question:

(1) How to perform backward-pass

∂b

Notation

- **Convolution:**



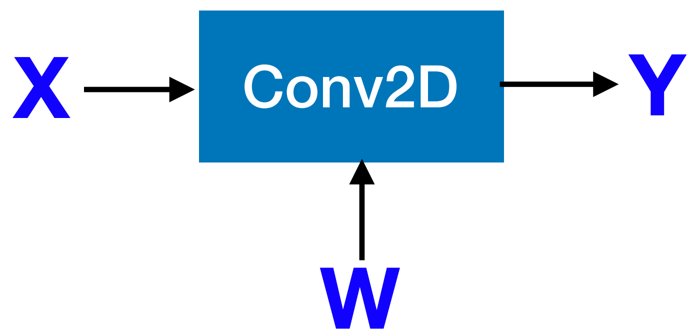
∂b

$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

- X is a the feature map to the convolution, in forward-pass
- X is the output of previous layer in forward-pass, it’s cached for using in backward-pass

Notation

- **Convolution:**



∂b

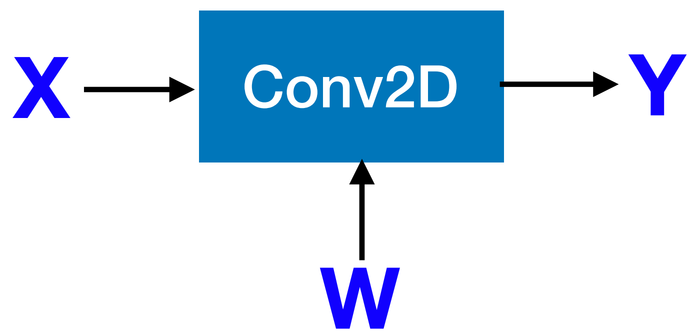
$$\mathbf{X} = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} \qquad \mathbf{\Delta X} = \begin{bmatrix} \delta x_{11} & \delta x_{12} & \delta x_{13} \\ \delta x_{21} & \delta x_{22} & \delta x_{23} \\ \delta x_{31} & \delta x_{31} & \delta x_{33} \end{bmatrix}$$

$\delta x_{ij} = \dfrac{\partial J}{\partial x_{ij}}$: partial derivative of error function with respect to δx_{ij}

- We have to compute **ΔX** because it's the input to previous layer in backward-pass

Notation

- **Convolution:**

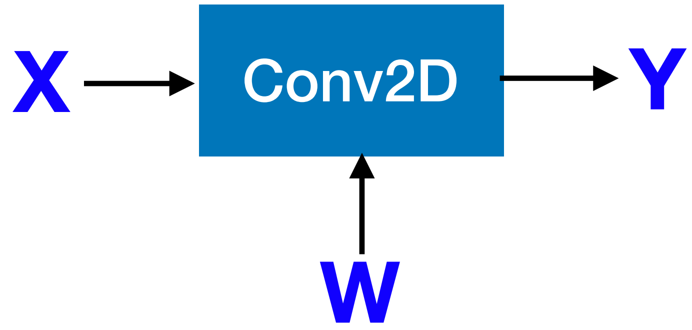


$$\mathbf{W} = \begin{array}{|c|c|} \hline w_{11} & w_{12} \\ \hline w_{21} & w_{22} \\ \hline \end{array}$$

- Current parameters of the convolution

Notation

- **Convolution:**

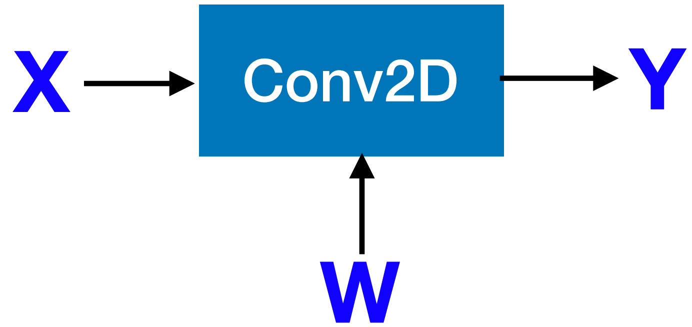


$$\mathbf{W} = \begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix} \qquad \mathbf{\Delta W} = \begin{bmatrix} \delta w_{11} & \delta w_{12} \\ \delta w_{21} & \delta w_{22} \end{bmatrix}$$

$\delta w_{ij} = \dfrac{\partial J}{\partial w_{ij}}$: partial derivative of error function with respect to w_{ij}

Notation

- **Convolution:**

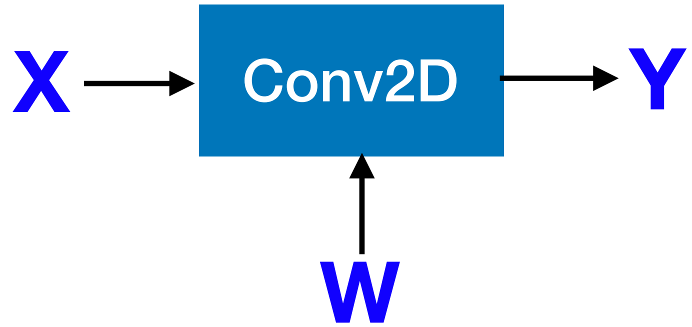


$$\mathbf{Y}_{\partial b} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}$$

- Output of the convolution

Notation

- **Convolution:**



$$\mathbf{Y} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \qquad \Delta\mathbf{Y} = \begin{bmatrix} \delta y_{11} & \delta y_{12} \\ \delta y_{21} & \delta y_{22} \end{bmatrix}$$

$\delta y_{ij} = \dfrac{\partial J}{\partial y_{ij}}$: partial derivative of error function with respect to y_{ij}

Back-propagation through convolution layer

Derivation of ΔW

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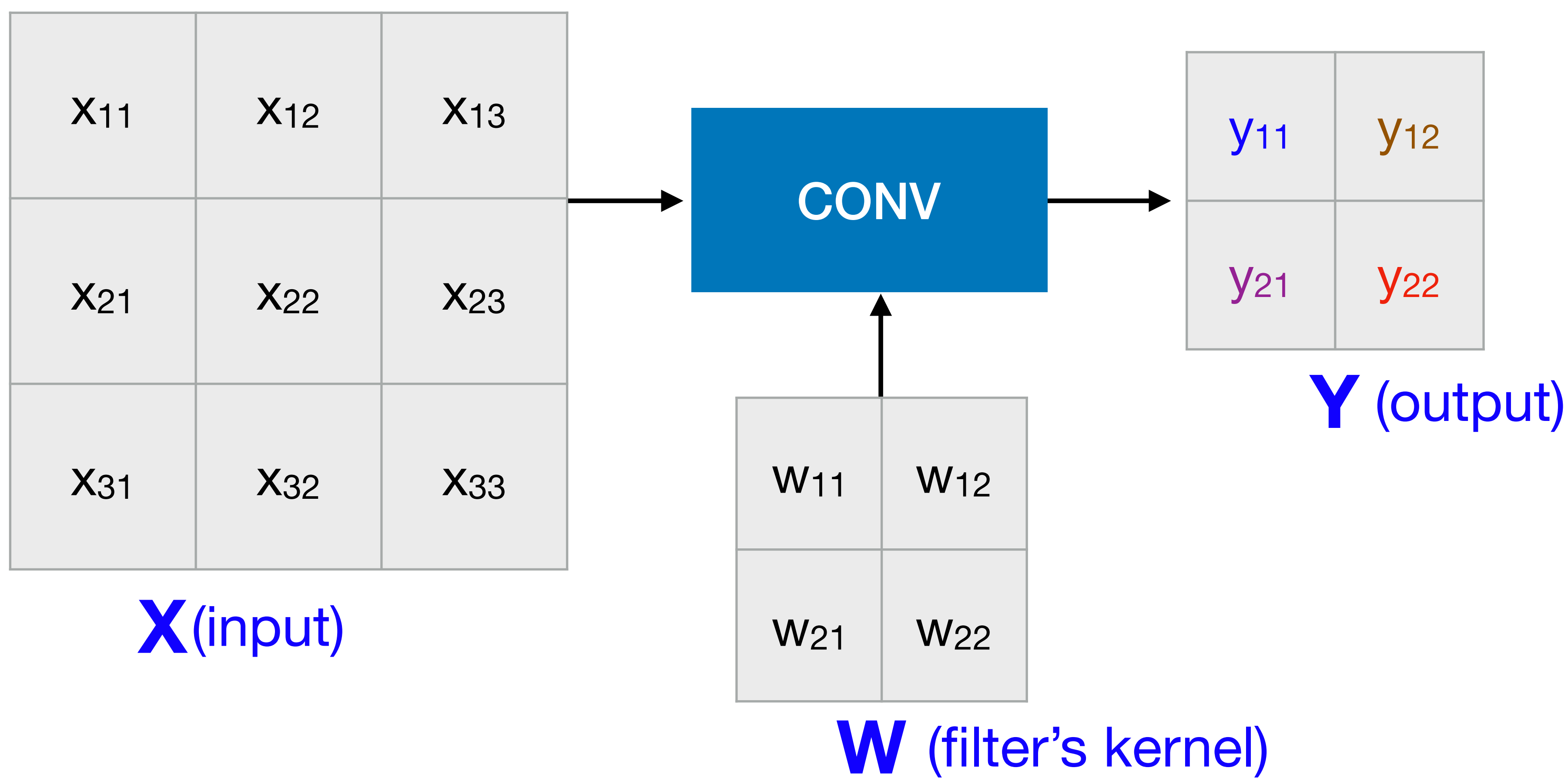
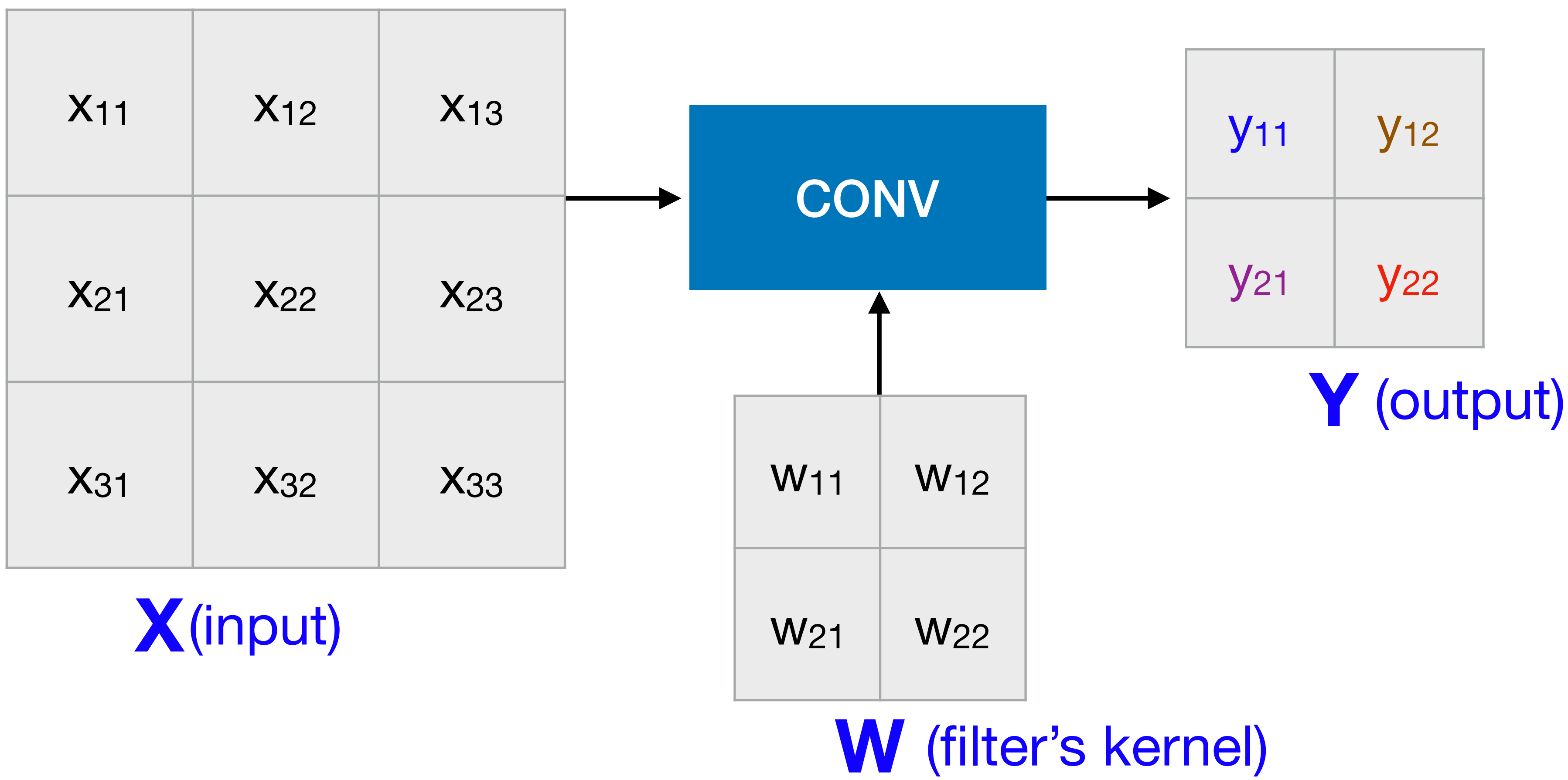
Derivation of ΔW

- **Input-output relationship**



Derivation of ΔW

- **Input-output relationship**



$$
\mathbf{Y} = \begin{bmatrix} y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22} & y_{12} = w_{22}x_{12} + w_{21}x_{13} + w_{12}x_{22} + w_{11}x_{23} \\ y_{21} = w_{22}x_{21} + w_{21}x_{22} + w_{12}x_{31} + w_{11}x_{32} & y_{22} = w_{22}x_{22} + w_{21}x_{23} + w_{12}x_{32} + w_{11}x_{33} \end{bmatrix}
$$

HOW ?

Derivation of ΔW

w_{11}	w_{12}
w_{21}	w_{22}

W

Rotate W 180^0 $\longrightarrow$

w_{22}	w_{21}
w_{12}	w_{11}

Rot180^0(W)

∂b

Derivation of ΔW

$w_{22}x_{11}$	$w_{21}x_{12}$	x_{13}
$w_{12}x_{21}$	$w_{11}x_{22}$	x_{23}
x_{31}	x_{32}	x_{33}

X

$y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22}$	

Y

w_{22}	w_{21}
w_{12}	w_{11}

Rot180^0(W)

Derivation of ΔW

x_{11}	$w_{22}x_{12}$	$w_{21}x_{13}$
x_{21}	$w_{12}x_{22}$	$w_{11}x_{23}$
x_{31}	x_{32}	x_{33}

X

$y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22}$	$y_{12} = w_{22}x_{12} + w_{21}x_{13} + w_{12}x_{22} + w_{11}x_{23}$

Y

w_{22}	w_{21}
w_{12}	w_{11}

Rot180^0(W)

Derivation of ΔW

Similarly,

$y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22}$	$y_{12} = w_{22}x_{12} + w_{21}x_{13} + w_{12}x_{22} + w_{11}x_{23}$
$y_{21} = w_{22}x_{21} + w_{21}x_{22} + w_{12}x_{31} + w_{11}x_{32}$	$y_{22} = w_{22}x_{22} + w_{21}x_{23} + w_{12}x_{32} + w_{11}x_{33}$

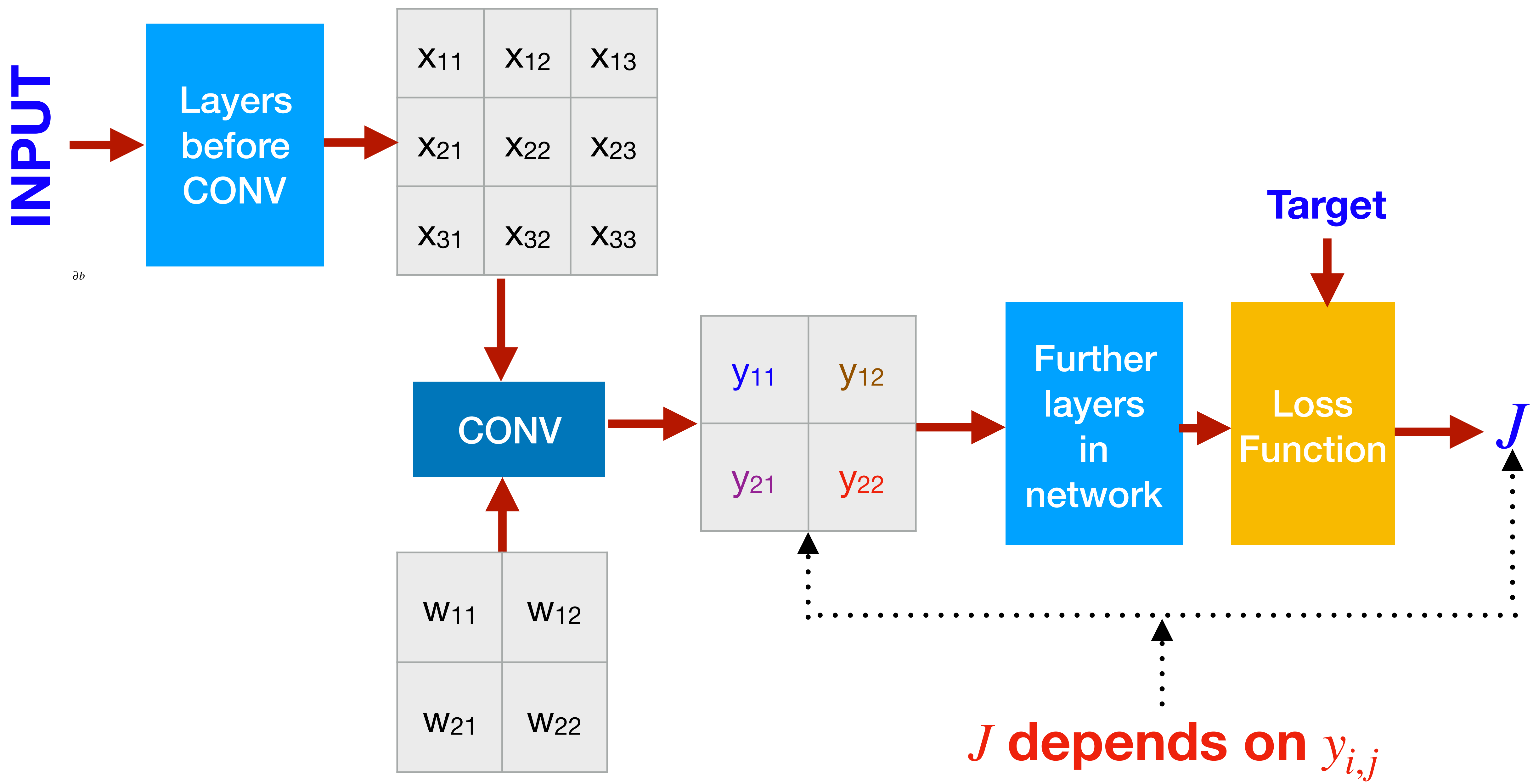
Y

∂b

$y_{i,j}$ **depends on** $w_{m,n}$ **and** $x_{u,v}$

Derivation of ΔW

Convolution layer in the whole network:



Derivation of ΔW

Because J depends on $y_{i,j}$ and $y_{i,j}$ depends on $w_{m,n}$:

∂b

Chain rule:

$$\frac{\partial J}{\partial w_{11}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{11}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{11}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{11}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{11}}$$

$$\frac{\partial J}{\partial w_{12}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{12}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{12}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{12}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{12}}$$

$$\frac{\partial J}{\partial w_{21}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{21}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{21}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{21}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{21}}$$

$$\frac{\partial J}{\partial w_{22}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{22}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{22}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{22}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{22}}$$

Derivation of ΔW

$$y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22}$$

$$y_{12} = w_{22}x_{12} + w_{21}x_{13} + w_{12}x_{22} + w_{11}x_{23}$$

$$y_{21} = w_{22}x_{21} + w_{21}x_{22} + w_{12}x_{31} + w_{11}x_{32}$$

$$y_{22} = w_{22}x_{22} + w_{21}x_{23} + w_{12}x_{32} + w_{11}x_{33}$$

$$\begin{aligned}\frac{\partial J}{\partial w_{11}} &= \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{11}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{11}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{11}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{11}} \\ &= \delta y_{11}x_{22} + \delta y_{12}x_{23} + \delta y_{21}x_{32} + \delta y_{22}x_{33}\end{aligned}$$

∂b

Notation: $$\frac{\partial J}{\partial w_{11}} \triangleq \delta w_{11}$$

$$\frac{\partial J}{\partial y_{ij}} \triangleq \delta y_{ij}$$

Derivation of ΔW

$$y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22}$$

$$y_{12} = w_{22}x_{12} + w_{21}x_{13} + w_{12}x_{22} + w_{11}x_{23}$$

$$y_{21} = w_{22}x_{21} + w_{21}x_{22} + w_{12}x_{31} + w_{11}x_{32}$$

$$y_{22} = w_{22}x_{22} + w_{21}x_{23} + w_{12}x_{32} + w_{11}x_{33}$$

$$\frac{\partial J}{\partial w_{11}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{11}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{11}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{11}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{11}}$$
$$= \delta y_{11}x_{22} + \delta y_{12}x_{23} + \delta y_{21}x_{32} + \delta y_{22}x_{33}$$

∂b

$$\frac{\partial J}{\partial w_{12}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{12}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{12}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{12}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{12}}$$
$$= \delta y_{11}x_{21} + \delta y_{12}x_{22} + \delta y_{21}x_{31} + \delta y_{22}x_{32}$$

Derivation of ΔW

$$y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22}$$

$$y_{12} = w_{22}x_{12} + w_{21}x_{13} + w_{12}x_{22} + w_{11}x_{23}$$

$$y_{21} = w_{22}x_{21} + w_{21}x_{22} + w_{12}x_{31} + w_{11}x_{32}$$

$$y_{22} = w_{22}x_{22} + w_{21}x_{23} + w_{12}x_{32} + w_{11}x_{33}$$

$$\frac{\partial J}{\partial w_{11}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{11}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{11}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{11}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{11}} = \delta y_{11}x_{22} + \delta y_{12}x_{23} + \delta y_{21}x_{32} + \delta y_{22}x_{33}$$

∂b

$$\frac{\partial J}{\partial w_{12}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{12}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{12}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{12}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{12}} = \delta y_{11}x_{21} + \delta y_{12}x_{22} + \delta y_{21}x_{31} + \delta y_{22}x_{32}$$

$$\frac{\partial J}{\partial w_{21}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{21}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{21}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{21}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{21}} = \delta y_{11}x_{12} + \delta y_{12}x_{13} + \delta y_{21}x_{22} + \delta y_{22}x_{23}$$

Derivation of ΔW

$$y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22}$$

$$y_{12} = w_{22}x_{12} + w_{21}x_{13} + w_{12}x_{22} + w_{11}x_{23}$$

$$y_{21} = w_{22}x_{21} + w_{21}x_{22} + w_{12}x_{31} + w_{11}x_{32}$$

$$y_{22} = w_{22}x_{22} + w_{21}x_{23} + w_{12}x_{32} + w_{11}x_{33}$$

$$\frac{\partial J}{\partial w_{11}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{11}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{11}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{11}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{11}}$$
$$= \delta y_{11}x_{22} + \delta y_{12}x_{23} + \delta y_{21}x_{32} + \delta y_{22}x_{33}$$

∂b

$$\frac{\partial J}{\partial w_{12}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{12}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{12}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{12}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{12}}$$
$$= \delta y_{11}x_{21} + \delta y_{12}x_{22} + \delta y_{21}x_{31} + \delta y_{22}x_{32}$$

$$\frac{\partial J}{\partial w_{21}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{21}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{21}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{21}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{21}}$$
$$= \delta y_{11}x_{12} + \delta y_{12}x_{13} + \delta y_{21}x_{22} + \delta y_{22}x_{23}$$

$$\frac{\partial J}{\partial w_{22}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial w_{22}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial w_{22}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial w_{22}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial w_{22}}$$
$$= \delta y_{11}x_{11} + \delta y_{12}x_{12} + \delta y_{21}x_{21} + \delta y_{22}x_{22}$$

Derivation of ΔW

$$\delta w_{11} = \delta y_{11}x_{22} + \delta y_{12}x_{23} + \delta y_{21}x_{32} + \delta y_{22}x_{33}$$

$$\delta w_{12} = \delta y_{11}x_{21} + \delta y_{12}x_{22} + \delta y_{21}x_{31} + \delta y_{22}x_{32}$$

$$\delta w_{21} = \delta y_{11}x_{12} + \delta y_{12}x_{13} + \delta y_{21}x_{22} + \delta y_{22}x_{23}$$

$$\delta w_{22} = \delta y_{11}x_{11} + \delta y_{12}x_{12} + \delta y_{21}x_{21} + \delta y_{22}x_{22}$$

∂b

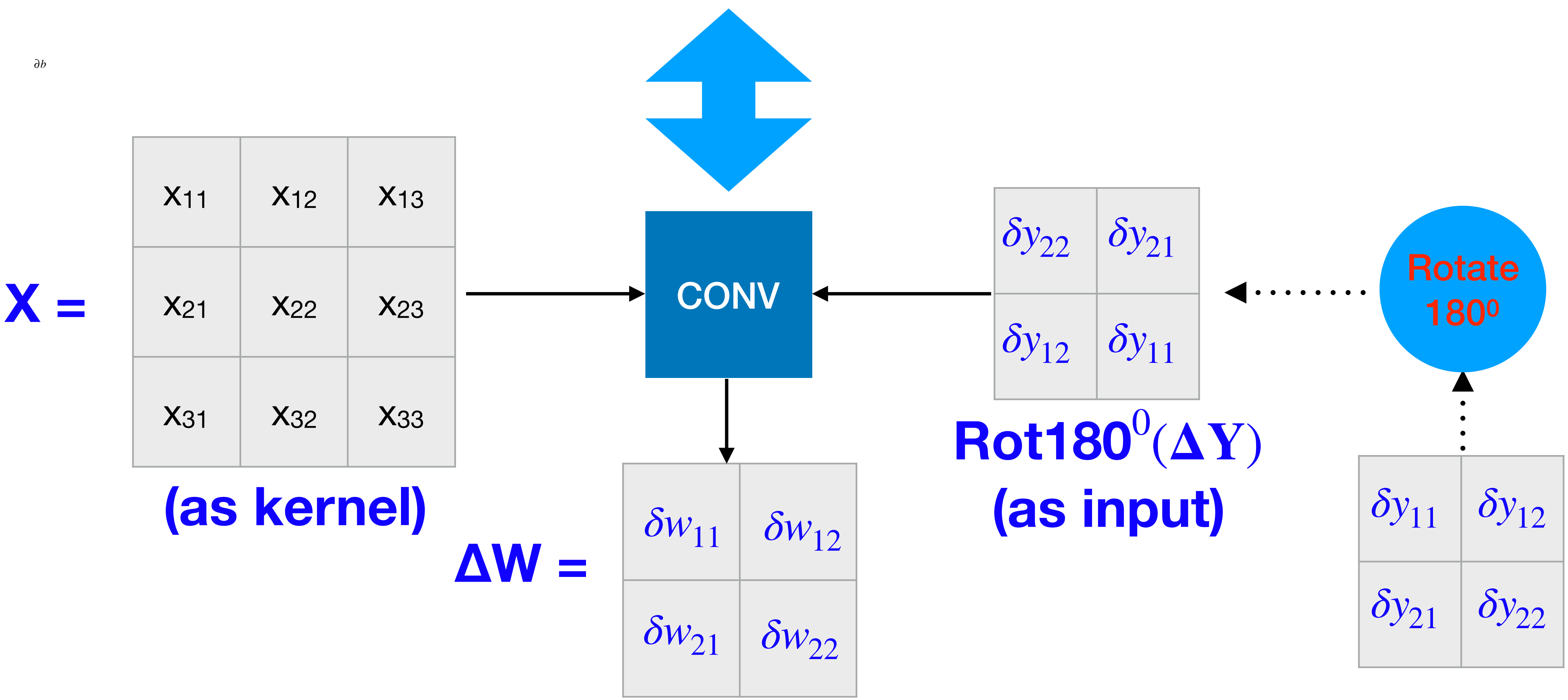
Derivation of ΔW

$$\delta w_{11} = \delta y_{11}x_{22} + \delta y_{12}x_{23} + \delta y_{21}x_{32} + \delta y_{22}x_{33}$$

$$\delta w_{12} = \delta y_{11}x_{21} + \delta y_{12}x_{22} + \delta y_{21}x_{31} + \delta y_{22}x_{32}$$

$$\delta w_{21} = \delta y_{11}x_{12} + \delta y_{12}x_{13} + \delta y_{21}x_{22} + \delta y_{22}x_{23}$$

$$\delta w_{22} = \delta y_{11}x_{11} + \delta y_{12}x_{12} + \delta y_{21}x_{21} + \delta y_{22}x_{22}$$



Derivation of ΔW

Formal result:

- **Input from the consecutive layer in backward mode:**

$$\Delta \mathbf{Y} = \begin{bmatrix} \delta y_{11} & \delta y_{12} \\ \delta y_{21} & \delta y_{22} \end{bmatrix}$$

- **ΔW is computed as follows:**

  (1) Step 1: Rotate **ΔY** to obtain **Rot180⁰(ΔY)**

  (2) Step 2: Compute convolution between **Rot180⁰(ΔY)** (as input) and **X** (as kernel)

$$\boxed{\Delta \mathbf{W} = \mathbf{Rot180}^{0}(\Delta \mathbf{Y}) * \mathbf{X}}$$

Derivation of ΔW

Verification:

x_{11}	x_{12}	x_{13}
x_{21}	x_{22}	x_{23}
x_{31}	x_{32}	x_{33}

X (as kernel)

Rotate 180^0

x_{33}	x_{32}	x_{31}
x_{23}	x_{22}	x_{21}
x_{13}	x_{12}	x_{11}

Derivation of ΔW

Verification:

$\delta y_{22}x_{33}$	$\delta y_{21}x_{32}$	x_{31}
$\delta y_{12}x_{23}$	$\delta y_{11}x_{22}$	x_{21}
x_{13}	x_{12}	x_{11}

(Rotated kernel)

$$\delta w_{11} = \delta y_{11}x_{22} + \delta y_{12}x_{23} + \delta y_{21}x_{32} + \delta y_{22}x_{33}$$

δy_{22}	δy_{21}
δy_{12}	δy_{11}

Rot180^{0}($\boldsymbol{\Delta Y}$)

(as input)

Derivation of ΔW

Verification:

x_{33}	$\delta y_{22} x_{32}$	$\delta y_{21} x_{31}$
x_{23}	$\delta y_{12} x_{22}$	$\delta y_{11} x_{21}$
x_{13}	x_{12}	x_{11}

(Rotated kernel)

$$\delta w_{11} = \delta y_{11} x_{22} + \delta y_{12} x_{23} + \delta y_{21} x_{32} + \delta y_{22} x_{33}$$

$$\delta w_{12} = \delta y_{11} x_{21} + \delta y_{12} x_{22} + \delta y_{21} x_{31} + \delta y_{22} x_{32}$$

δy_{22}	δy_{21}
δy_{12}	δy_{11}

Rot180$^{0}(\boldsymbol{\Delta Y})$

(as input)

Derivation of ΔW

Verification:

x_{33}	$\delta y_{22} x_{32}$	$\delta y_{21} x_{31}$
x_{23}	$\delta y_{12} x_{22}$	$\delta y_{11} x_{21}$
x_{13}	x_{12}	x_{11}

(Rotated kernel)

δy_{22}	δy_{21}
δy_{12}	δy_{11}

Rot180$^{0}(\boldsymbol{\Delta Y})$
(as input)

$$\delta w_{11} = \delta y_{11} x_{22} + \delta y_{12} x_{23} + \delta y_{21} x_{32} + \delta y_{22} x_{33}$$

$$\delta w_{12} = \delta y_{11} x_{21} + \delta y_{12} x_{22} + \delta y_{21} x_{31} + \delta y_{22} x_{32}$$

Similarly,

$$\delta w_{21} = \delta y_{11} x_{12} + \delta y_{12} x_{13} + \delta y_{21} x_{22} + \delta y_{22} x_{23}$$

$$\delta w_{22} = \delta y_{11} x_{11} + \delta y_{12} x_{12} + \delta y_{21} x_{21} + \delta y_{22} x_{22}$$

Back-propagation through convolution layer

Derivation of ΔX

Dr. Thanh-Sach LE
LTSACH@hcmut.edu.vn

GVLab:
Graphics and Vision Laboratory

Faculty of Computer Science and Engineering,
HCMUT

Derivation of ΔX

Because J depends on $y_{i,j}$ and $y_{i,j}$ depends on $x_{u,v}$:

∂b

Chain rule:

$$\frac{\partial J}{\partial x_{11}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{11}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{11}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{11}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{11}}$$

$$\frac{\partial J}{\partial x_{12}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{12}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{12}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{12}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{12}}$$

$$\frac{\partial J}{\partial x_{13}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{13}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{13}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{13}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{13}}$$

$$\frac{\partial J}{\partial x_{21}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{21}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{21}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{21}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{21}}$$

$$\frac{\partial J}{\partial x_{22}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{22}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{22}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{22}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{22}}$$

$$\frac{\partial J}{\partial x_{23}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{23}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{23}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{23}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{23}}$$

Derivation of ΔX

Because J depends on $y_{i,j}$ and $y_{i,j}$ depends on $x_{u,v}$:

∂b

Chain rule:

$$\frac{\partial J}{\partial x_{31}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{31}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{31}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{31}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{31}}$$

$$\frac{\partial J}{\partial x_{32}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{32}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{32}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{32}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{32}}$$

$$\frac{\partial J}{\partial x_{33}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{33}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{33}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{33}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{33}}$$

Derivation of ΔX

We have:

$$\begin{cases} y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22} & y_{12} = w_{22}x_{12} + w_{21}x_{13} + w_{12}x_{22} + w_{11}x_{23} \\ y_{21} = w_{22}x_{21} + w_{21}x_{22} + w_{12}x_{31} + w_{11}x_{32} & y_{22} = w_{22}x_{22} + w_{21}x_{23} + w_{12}x_{32} + w_{11}x_{33} \end{cases}$$

$$\frac{\partial J}{\partial x_{11}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{11}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{11}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{11}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{11}} = \quad \delta y_{11} w_{22}$$

$$\frac{\partial J}{\partial x_{12}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{12}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{12}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{12}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{12}} = \quad \delta y_{11} w_{21} + \delta y_{12} w_{22}$$

$$\frac{\partial J}{\partial x_{13}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{13}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{13}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{13}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{13}} = \quad \delta y_{12} w_{21}$$

Derivation of ΔX

We have:

$$\begin{cases} y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22} & y_{12} = w_{22}x_{12} + w_{21}x_{13} + w_{12}x_{22} + w_{11}x_{23} \\ y_{21} = w_{22}x_{21} + w_{21}x_{22} + w_{12}x_{31} + w_{11}x_{32} & y_{22} = w_{22}x_{22} + w_{21}x_{23} + w_{12}x_{32} + w_{11}x_{33} \end{cases}$$

$$\frac{\partial J}{\partial x_{21}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{21}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{21}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{21}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{21}} = \delta y_{11}w_{12} + \delta y_{21}w_{22}$$

$$\frac{\partial J}{\partial x_{22}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{22}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{22}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{22}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{22}} = \delta y_{11}w_{11} + \delta y_{12}w_{12} + \delta y_{21}w_{21} + \delta y_{11}w_{22}$$

$$\frac{\partial J}{\partial x_{23}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{23}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{23}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{23}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{23}} = \delta y_{12}w_{11} + \delta y_{22}w_{21}$$

Derivation of ΔX

We have:

$$\begin{cases} y_{11} = w_{22}x_{11} + w_{21}x_{12} + w_{12}x_{21} + w_{11}x_{22} & y_{12} = w_{22}x_{12} + w_{21}x_{13} + w_{12}x_{22} + w_{11}x_{23} \\ y_{21} = w_{22}x_{21} + w_{21}x_{22} + w_{12}x_{31} + w_{11}x_{32} & y_{22} = w_{22}x_{22} + w_{21}x_{23} + w_{12}x_{32} + w_{11}x_{33} \end{cases}$$

$$\frac{\partial J}{\partial x_{31}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{31}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{31}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{31}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{31}} = \delta y_{21}w_{12}$$

$$\frac{\partial J}{\partial x_{32}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{32}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{32}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{32}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{32}} = \delta y_{21}w_{11} + \delta y_{22}w_{12}$$

$$\frac{\partial J}{\partial x_{33}} = \frac{\partial J}{\partial y_{11}}\frac{\partial y_{11}}{\partial x_{33}} + \frac{\partial J}{\partial y_{12}}\frac{\partial y_{12}}{\partial x_{33}} + \frac{\partial J}{\partial y_{21}}\frac{\partial y_{21}}{\partial x_{33}} + \frac{\partial J}{\partial y_{22}}\frac{\partial y_{22}}{\partial x_{33}} = \delta y_{22}w_{11}$$

Derivation of ΔX

$$\delta x_{11} = \delta y_{11} w_{22}$$

$$\delta x_{12} = \delta y_{11} w_{21} + \delta y_{12} w_{22}$$

$$\delta x_{13} = \delta y_{12} w_{21}$$

$$\delta x_{21} = \delta y_{11} w_{12} + \delta y_{21} w_{22}$$

$$\delta x_{22} = \delta y_{11} w_{11} + \delta y_{12} w_{12} + \delta y_{21} w_{21} + \delta y_{11} w_{22}$$

$$\delta x_{23} = \delta y_{12} w_{11} + \delta y_{22} w_{21}$$

$$\delta x_{31} = \delta y_{21} w_{12}$$

$$\delta x_{32} = \delta y_{21} w_{11} + \delta y_{22} w_{12}$$

$$\delta x_{33} = \delta y_{22} w_{11}$$

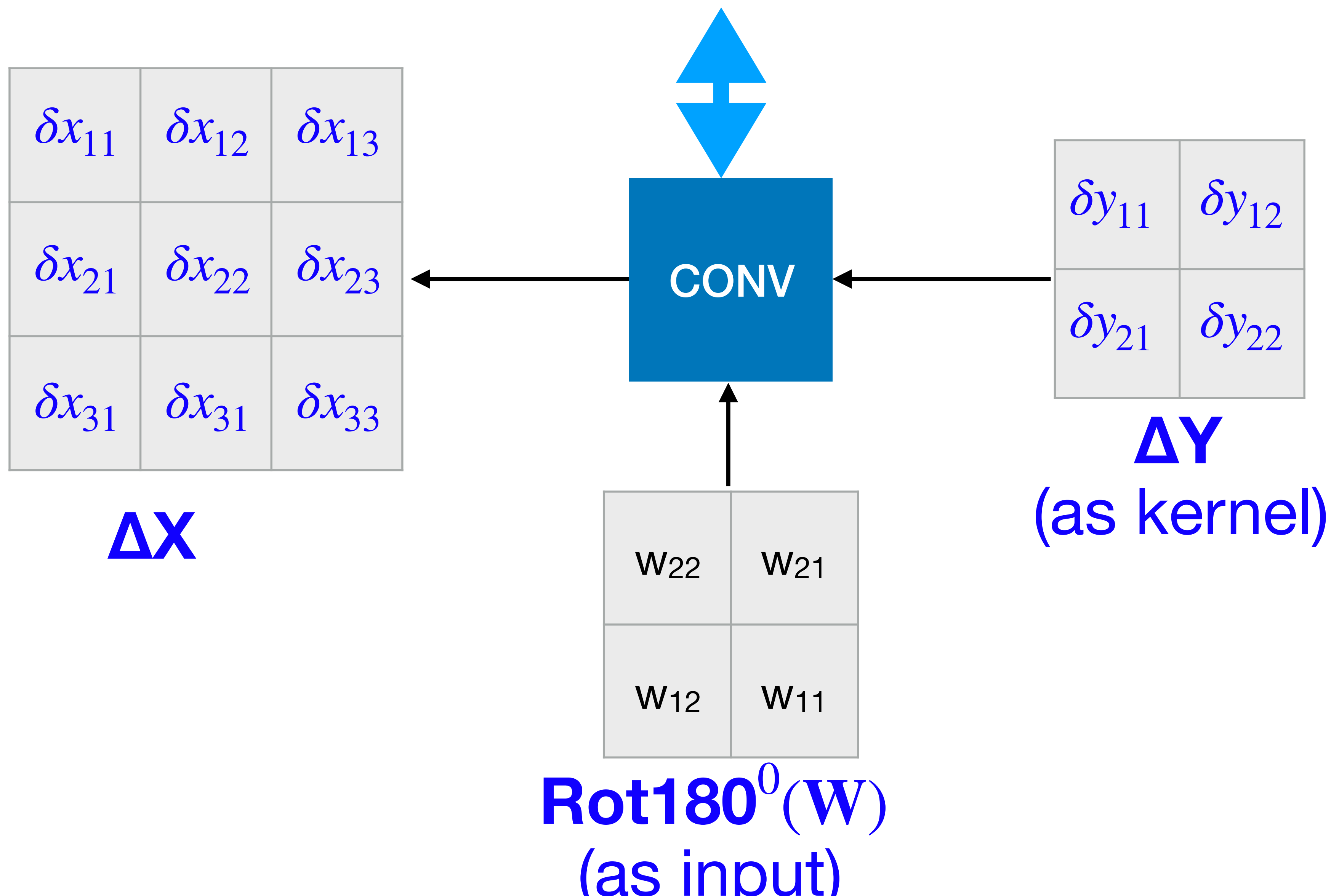
Derivation of ΔX

$$\delta x_{11} = \delta y_{11} w_{22} \qquad \delta x_{12} = \delta y_{11} w_{21} + \delta y_{12} w_{22} \qquad \delta x_{13} = \delta y_{12} w_{21}$$

$$\delta x_{21} = \delta y_{11} w_{12} + \delta y_{21} w_{22} \qquad \delta x_{22} = \delta y_{11} w_{11} + \delta y_{12} w_{12} + \delta y_{21} w_{21} + \delta y_{11} w_{22} \qquad \delta x_{23} = \delta y_{12} w_{11} + \delta y_{22} w_{21}$$

$$\delta x_{31} = \delta y_{21} w_{12} \qquad \delta x_{32} = \delta y_{21} w_{11} + \delta y_{22} w_{12} \qquad \delta x_{33} = \delta y_{22} w_{11}$$

∂b



Derivation of ΔX

Formal result:

- Input from the consecutive layer in backward mode:

$$\Delta \mathbf{Y} = \begin{array}{|c|c|} \hline \delta y_{11} & \delta y_{12} \\ \hline \delta y_{21} & \delta y_{22} \\ \hline \end{array}$$

∂b

- **ΔX is computed as follows:**

(1) Step 1: Rotate **W** to obtain **Rot180⁰(W)**

(2) Step 2: Compute convolution between **Rot180⁰(W)** (as input) and **ΔY** (as kernel); using full padding convolution.

$$\boxed{\Delta \mathbf{X} = \mathbf{Rot180}^0(\mathbf{W}) * \Delta \mathbf{Y}}$$

Derivation of ΔX

Verification:

w_{22}	w_{21}
w_{12}	w_{11}

$\mathbf{W}$

$\xrightarrow{\text{Rotate } 180^0}$

w_{22}	w_{21}
w_{12}	w_{11}

$\mathbf{Rot180^0(W)}$

(use this backward mode, as input)

Derivation of ΔX

Verification:

$$\underbrace{\begin{bmatrix} \delta y_{11} & \delta y_{12} \\ \delta y_{21} & \delta y_{22} \end{bmatrix}}_{\Delta \mathbf{Y}} \xrightarrow[\textbf{Rotate } \mathbf{180^0}]{\text{using as kernel, therefore need rotation}} \underbrace{\begin{bmatrix} \delta y_{22} & \delta y_{21} \\ \delta y_{12} & \delta y_{11} \end{bmatrix}}_{\textbf{Rot180}^0(\Delta \mathbf{Y})}$$

∂b

Derivation of ΔX

Verification:

w_{22}	w_{21}
w_{12}	w_{11}

∂b

Rot180^0($\mathbf{W}$)

(use this backward mode, as input)

δy_{22}	δy_{21}
δy_{12}	δy_{11}

Rot180^0($\mathbf{\Delta Y}$)

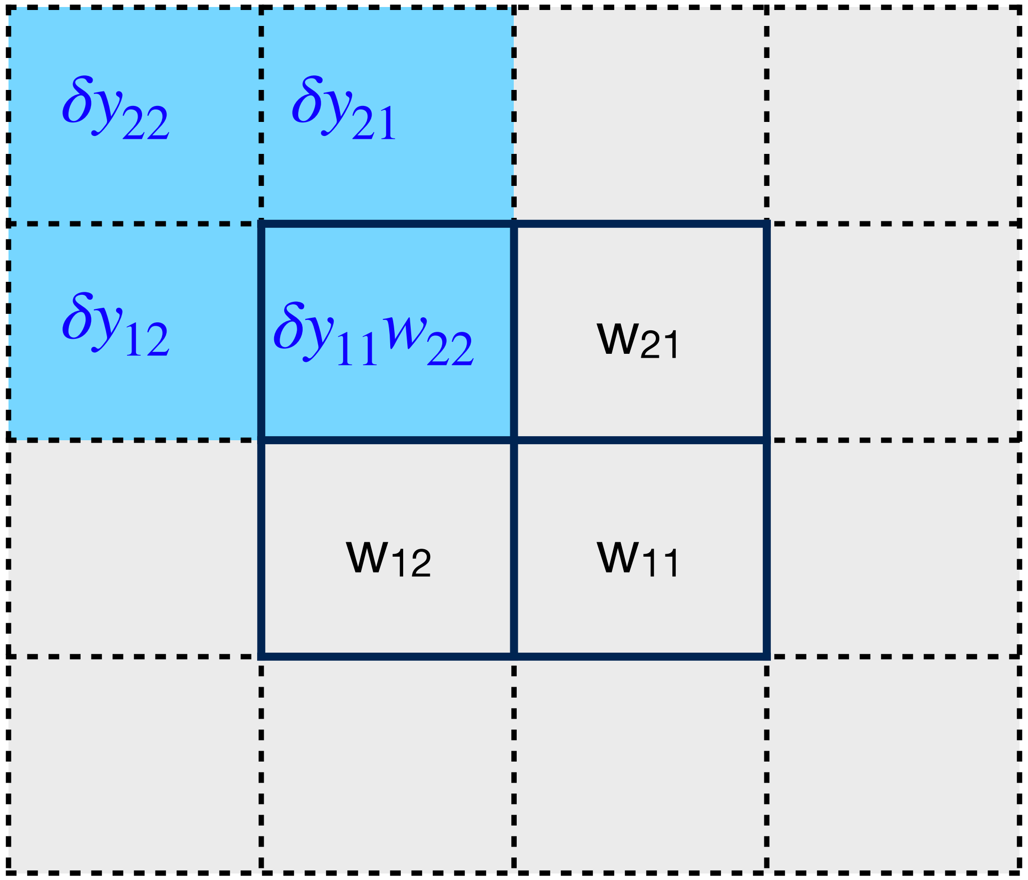
(rotated kernel)

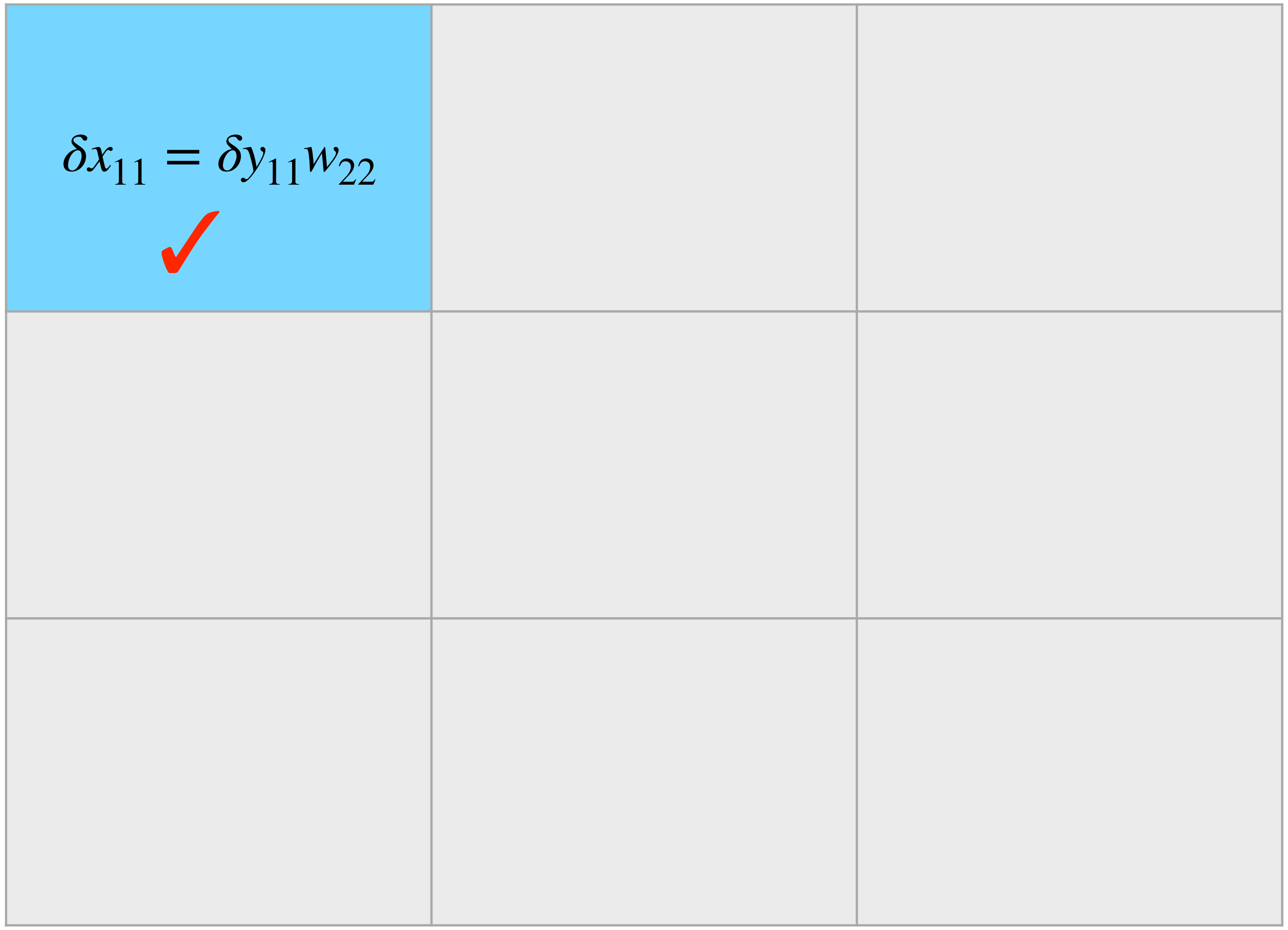
Now, compute cross-correlation

Output

Derivation of ΔX

Verification:





Output

✓ :The result is identical to the expectation

Derivation of ΔX

Verification:

	δy_{22}	δy_{21}	
	$\delta y_{12} w_{22}$	$\delta y_{11} w_{21}$	
	w_{12}	w_{11}	

$\delta x_{11} = \delta y_{11} w_{22}$ ✓	$\delta x_{12} = \delta y_{11} w_{21} + \delta y_{12} w_{22}$ ✓	

Output

✓ :The result is identical to the expectation

Derivation of ΔX

Verification:

		δy_{22}	δy_{21}
	w_{22}	$\delta y_{12} w_{21}$	δy_{11}
	w_{12}	w_{11}	

$\delta x_{11} = \delta y_{11} w_{22}$ ✓	$\delta x_{12} = \delta y_{11} w_{21} + \delta y_{12} w_{22}$ ✓	$\delta x_{13} = \delta y_{12} w_{21}$ ✓

Output

✓ :The result is identical to the expectation

Derivation of ΔX

Verification:

$\delta x_{11} = \delta y_{11} w_{22}$ ✓	$\delta x_{12} = \delta y_{11} w_{21} + \delta y_{12} w_{22}$ ✓	$\delta x_{13} = \delta y_{12} w_{21}$ ✓
$\delta x_{21} = \delta y_{11} w_{12} + \delta y_{21} w_{22}$ ✓		

Output

✓ :The result is identical to the expectation

Derivation of ΔX

Verification:

	$\delta y_{22} w_{22}$	$\delta y_{21} w_{21}$	
	$\delta y_{12} w_{12}$	$\delta y_{11} w_{11}$	

$\delta x_{11} = \delta y_{11} w_{22}$ ✓	$\delta x_{12} = \delta y_{11} w_{21} + \delta y_{12} w_{22}$ ✓	$\delta x_{13} = \delta y_{12} w_{21}$ ✓
$\delta x_{21} = \delta y_{11} w_{12} + \delta y_{21} w_{22}$ ✓	$\delta x_{22} = \delta y_{11} w_{11} + \delta y_{12} w_{12} + \delta y_{21} w_{21} + \delta y_{11} w_{22}$ ✓	

Output

✓ :The result is identical to the expectation

Derivation of ΔX

Verification:

	w_{22}	$\delta y_{22} w_{21}$	δy_{21}
	w_{12}	$\delta y_{12} w_{11}$	δy_{11}

$\delta x_{11} = \delta y_{11} w_{22}$ ✓	$\delta x_{12} =$ $\delta y_{11} w_{21} + \delta y_{12} w_{22}$ ✓	$\delta x_{13} = \delta y_{12} w_{21}$ ✓
$\delta x_{21} =$ $\delta y_{11} w_{12} + \delta y_{21} w_{22}$ ✓	$\delta x_{22} =$ $\delta y_{11} w_{11} + \delta y_{12} w_{12} +$ $\delta y_{21} w_{21} + \delta y_{11} w_{22}$ ✓	$\delta x_{23} =$ $\delta y_{12} w_{11} + \delta y_{22} w_{21}$ ✓

Output

✓ :The result is identical to the expectation

Derivation of ΔX

Verification:

	w_{22}	w_{21}	
δy_{22}	$\delta y_{21} w_{12}$	w_{11}	
δy_{12}	δy_{11}		

$\delta x_{11} = \delta y_{11} w_{22}$ ✓	$\delta x_{12} = \delta y_{11} w_{21} + \delta y_{12} w_{22}$ ✓	$\delta x_{13} = \delta y_{12} w_{21}$ ✓
$\delta x_{21} = \delta y_{11} w_{12} + \delta y_{21} w_{22}$ ✓	$\delta x_{22} = \delta y_{11} w_{11} + \delta y_{12} w_{12} + \delta y_{21} w_{21} + \delta y_{11} w_{22}$ ✓	$\delta x_{23} = \delta y_{12} w_{11} + \delta y_{22} w_{21}$ ✓
$\delta x_{31} = \delta y_{21} w_{12}$ ✓		

Output

✓ :The result is identical to the expectation

Derivation of ΔX

Verification:

$\delta x_{11} = \delta y_{11} w_{22}$ ✓	$\delta x_{12} = \delta y_{11} w_{21} + \delta y_{12} w_{22}$ ✓	$\delta x_{13} = \delta y_{12} w_{21}$ ✓
$\delta x_{21} = \delta y_{11} w_{12} + \delta y_{21} w_{22}$ ✓	$\delta x_{22} = \delta y_{11} w_{11} + \delta y_{12} w_{12} + \delta y_{21} w_{21} + \delta y_{11} w_{22}$ ✓	$\delta x_{23} = \delta y_{12} w_{11} + \delta y_{22} w_{21}$ ✓
$\delta x_{31} = \delta y_{21} w_{12}$ ✓	$\delta x_{32} = \delta y_{21} w_{11} + \delta y_{22} w_{12}$ ✓	

Output

✓ :The result is identical to the expectation

Derivation of ΔX

Verification:

	w_{22}	w_{21}	
	w_{12}	$\delta y_{22} w_{11}$	δy_{21}
		δy_{12}	δy_{11}

$\delta x_{11} = \delta y_{11} w_{22}$ ✓	$\delta x_{12} = \delta y_{11} w_{21} + \delta y_{12} w_{22}$ ✓	$\delta x_{13} = \delta y_{12} w_{21}$ ✓
$\delta x_{21} = \delta y_{11} w_{12} + \delta y_{21} w_{22}$ ✓	$\delta x_{22} = \delta y_{11} w_{11} + \delta y_{12} w_{12} + \delta y_{21} w_{21} + \delta y_{11} w_{22}$ ✓	$\delta x_{23} = \delta y_{12} w_{11} + \delta y_{22} w_{21}$ ✓
$\delta x_{31} = \delta y_{21} w_{12}$ ✓	$\delta x_{32} = \delta y_{21} w_{11} + \delta y_{22} w_{12}$ ✓	$\delta x_{33} = \delta y_{22} w_{11}$ ✓

Output

✓ :The result is identical to the expectation

Back-propagation through convolution layer

Summary

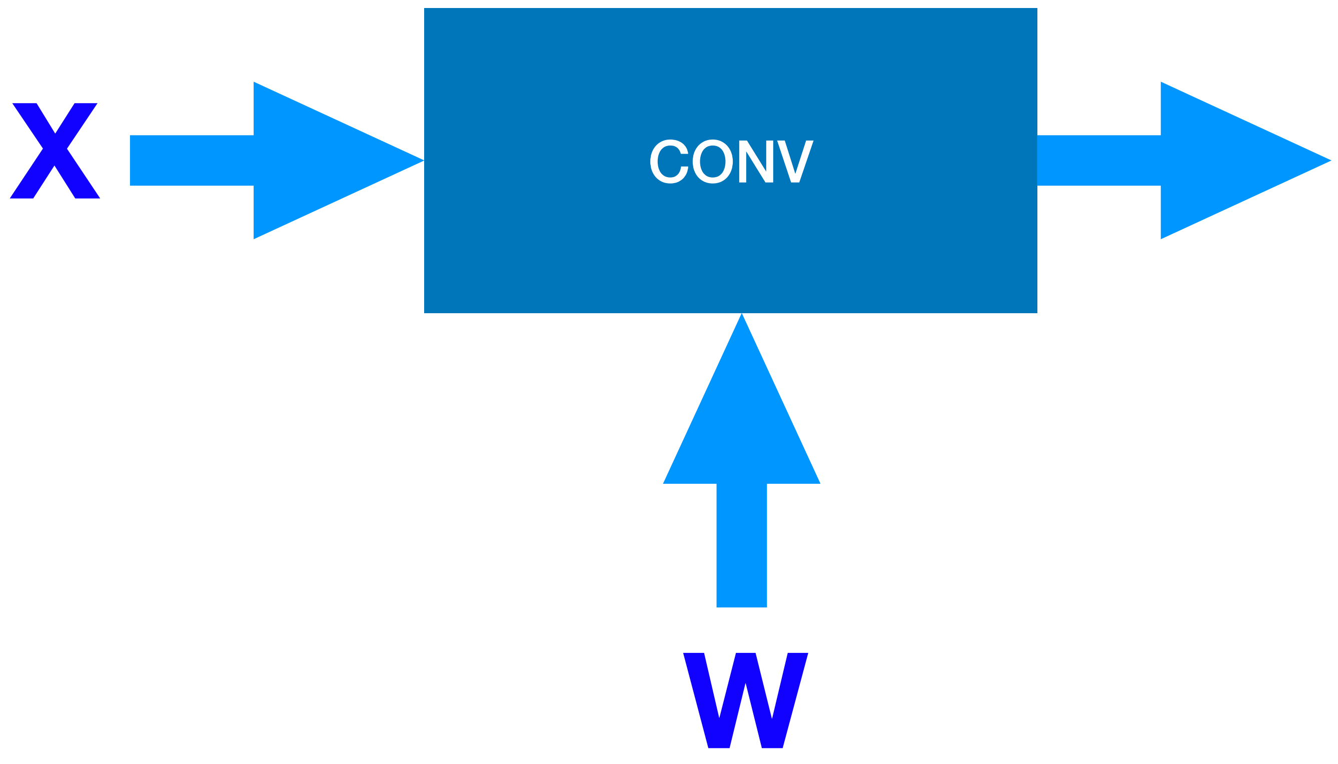
Dr. Thanh-Sach LE
LTSACH@hcmut.edu.vn

GVLab:
Graphics and Vision Laboratory

Faculty of Computer Science and Engineering, HCMUT

Backprop with convolution layer

Convolution:



∂b

Backprop through convolution layer

Forward-pass:

$$\boxed{\mathbf{Y} = \mathbf{X} * \mathbf{W}}$$

∂b

✳ Cache Y for using in backward mode

Backprop through convolution layer

Backward-pass:

✳ (1) Compute **ΔX**, Eq. (1), and use it as the input to previous layer in backward mode.

✦ **Rot180⁰(W)** as input, full padding

✦ **ΔY** as kernel

∂b

$$\boxed{\Delta \mathbf{X} = \mathbf{Rot180}^0(\mathbf{W}) * \Delta \mathbf{Y}} \quad \text{Eq. (1)}$$

Backprop through convolution layer

Backward-pass:

- (2) Compute **ΔW**, Eq. (2), and use it to update convolution's kernel.
  - **Rot180⁰**(**ΔY**) as input, no padding
  - **X** as kernel

∂b

$$\boxed{\Delta \mathbf{W} = \mathbf{Rot180}^{0}(\Delta \mathbf{Y}) * \mathbf{X}} \quad \text{Eq. (2)}$$