

Mining signed networks: theory and applications

Aristides Gionis,^{1,2} Antonis Matakos,¹ Bruno Ordozgoiti,¹ Han Xiao¹

¹Aalto University, ²KTH Royal Institute of Technology



Slides can be found at
bit.ly/www2020-signed

Contact us



Aris
argioni@kth.se



Antonis
{antonis.matakos, bruno.ordozgoiti, han.xiao}@aalto.fi



Bruno



Han

Outline

Introduction

Theory of signed networks

Problems and applications

Subgraph mining

Graph partitioning

Correlation clustering

Link prediction and link classification

Network dynamics

Graph embedding and representation learning

Conclusions and future directions

Introduction

Signed graphs

Graphs with edge signs,
either *positive* or *negative*

Signed graphs: motivation

Human interaction
friendly or antagonistic



Image source: pxfuel.com

Signed graphs: motivation

Online social media

- ▶ a graph of users (Twitter, Facebook, etc.)
- ▶ Users may express **like** or **dislike** towards others
- ▶ Can be used to study **online polarization**

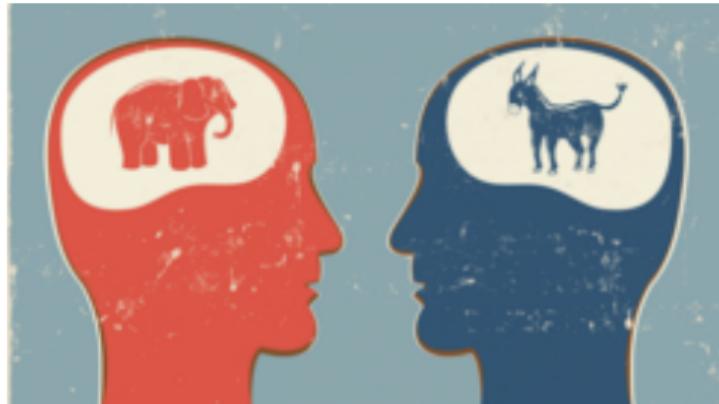
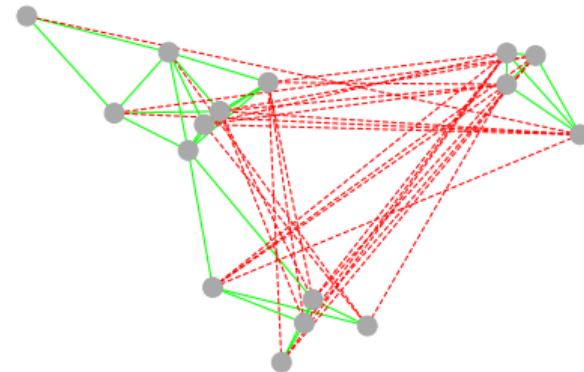


Image source: iStockphoto.com

Signed graphs: motivation

Groups of humans

- ▶ Examples: tribes, political parties, countries, etc.
- ▶ Country relations during war
- ▶ Political polarization



New Guinea Highland Tribes graph Read
(1954)

Signed graphs: motivation

Human language

- ▶ a graph of **words**
- ▶ that captures **synonym** or **antonym** relations

“Happy”

Synonyms for *happy*

cheerful

merry

contented

overjoyed

Antonyms for *happy*

depressed

melancholy

disappointed

miserable

Image source: thesaurus.com

Signed graphs: motivation

Molecular biology

- ▶ a graph of proteins
- ▶ one protein activate or inhibit the functioning of another

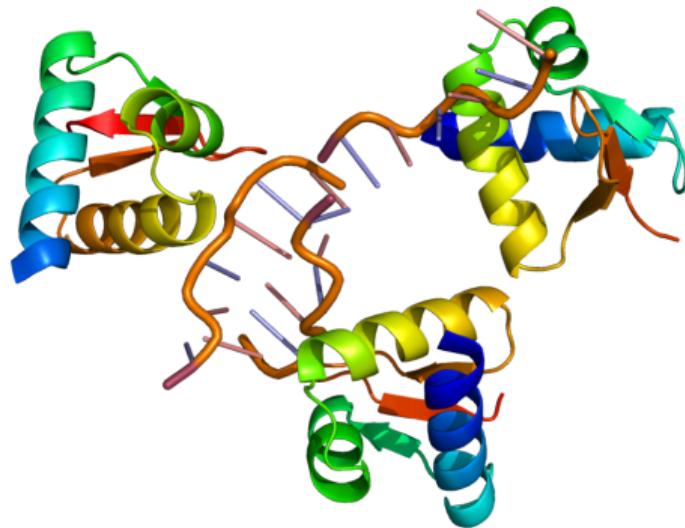


Image source: commons.wikimedia.org

Signed graphs: motivation

Finance

- ▶ a graph of **securities** (tradable assets)
- ▶ a security *correlates positively/negatively* with another.
- ▶ “Correlate” means the joint movement of **price**



Image source: vecteezy.com

Signed graphs: motivation

Linear algebra

- ▶ Speed up linear program solver by finding an embedded network matrix.

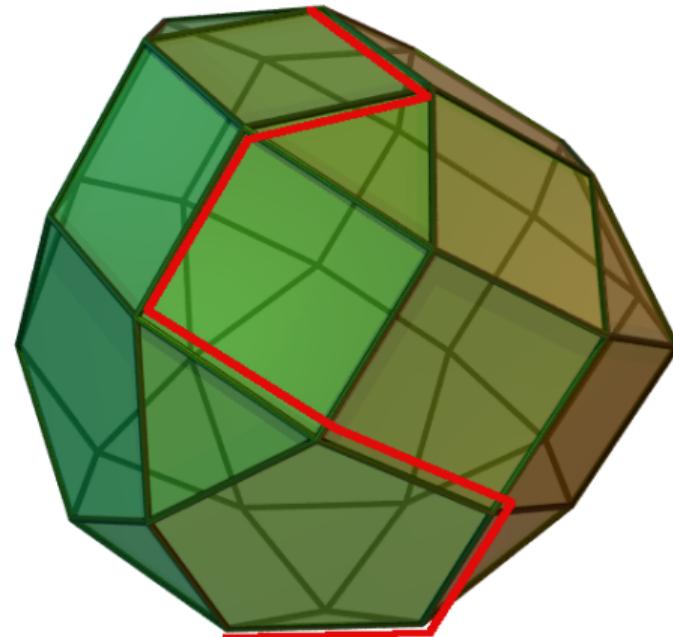


Image source: wiki.ubc.ca

Theory of signed networks

Outline

We will discuss:

Outline

We will discuss:

- ▶ Balance

Outline

We will discuss:

- ▶ Balance
- ▶ Spectrum

Outline

We will discuss:

- ▶ Balance
- ▶ Spectrum
- ▶ Frustration

Signed networks

Signed networks (or graphs): each edge labeled + or -.

Definitions:

- ▶ $G = (V, E^+, E^-)$,
- ▶ $G = (V, E, \sigma), \sigma : E \rightarrow \{-, +\}$.

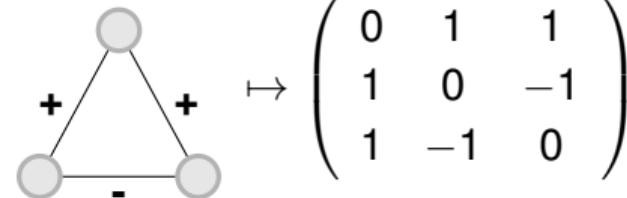
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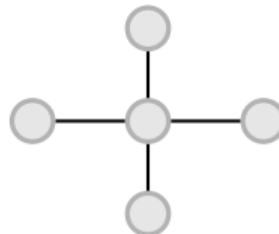
Adjacency matrix: $A = A_{E^+} - A_{E^-}$



Expressiveness of signed graphs

Signed graphs can be quite **expressive**.

Example: Star graph

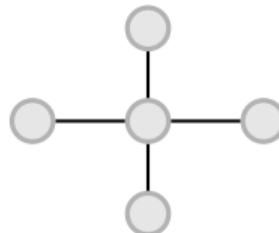


- ▶ Number of possible graphs:
- ▶ Number of non-isomorphic graphs:

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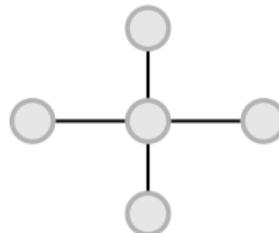


- ▶ Number of possible graphs: $2^{|E|}$
- ▶ Number of non-isomorphic graphs:

Expressiveness of signed graphs

Signed graphs can be quite **expressive**.

Example: Star graph



- ▶ Number of possible graphs: $2^{|E|}$
- ▶ Number of non-isomorphic graphs: $|E|$

Differences in signed graphs

Shortest paths

Signed graphs can be quite **different**... Consider e.g. shortest paths.
How do we even define path length in signed graphs?

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Proposal: distinguish **positive** and **negative** paths (by product of edge signs).

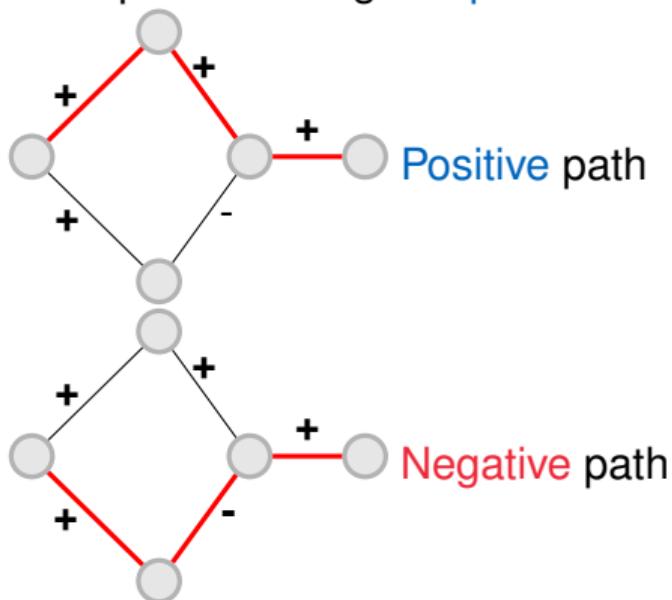
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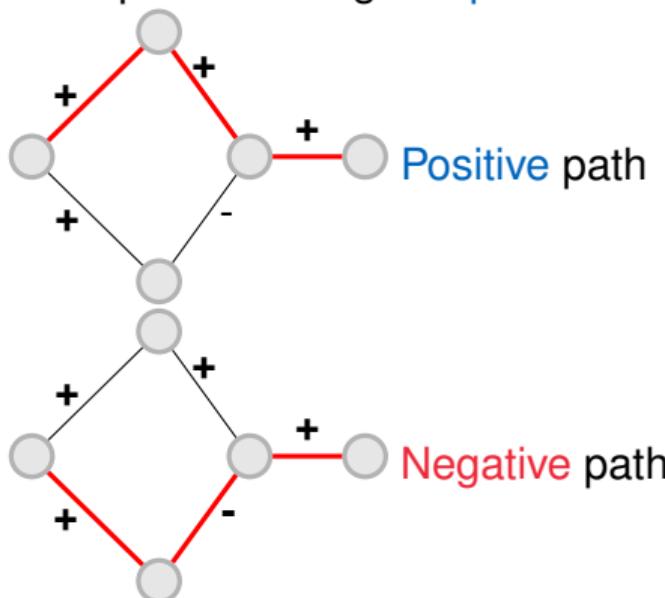
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Finding all shortest simple signed paths is **NP**-complete!

If repetitions are allowed, $\mathcal{O}(|E| \log \log \frac{D}{d})$ algorithm (Hansen, 1984).

Differences in signed graphs

Densest subgraph

Densest subgraph problem in **unsigned** graphs:

$$\max_{x \in \{0,1\}^n} \frac{x^T Ax}{x^T x}$$

Polynomial-time solvable (Goldberg, 1984).

Differences in signed graphs

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Densest subgraph problem in **signed** graphs:

$$\max_{x \in \{-1,0,1\}^n} \frac{x^T Ax}{x^T x}$$

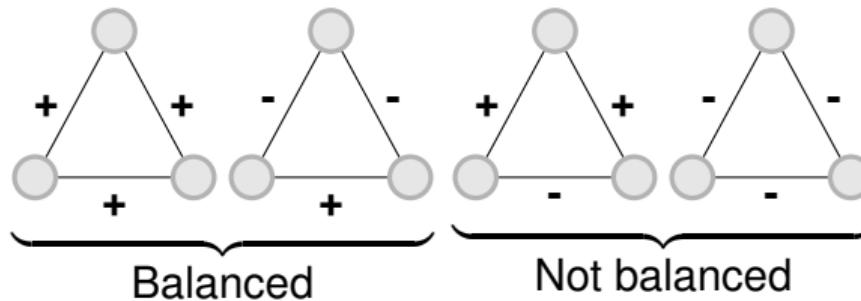
NP-hard ! (Bonchi et al., 2019; Tsourakakis et al., 2019).

Balance

Motivation

Motivation: **balance** in social networks (Harary, 1953).

“*The friend of a friend is a friend*” (or “*the enemy of a friend is an enemy*”).



The four possible non-isomorphic signed triangles.

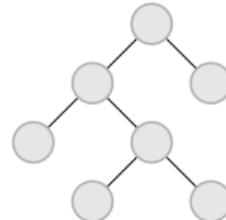
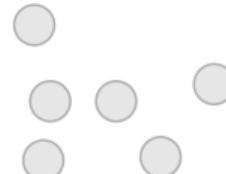
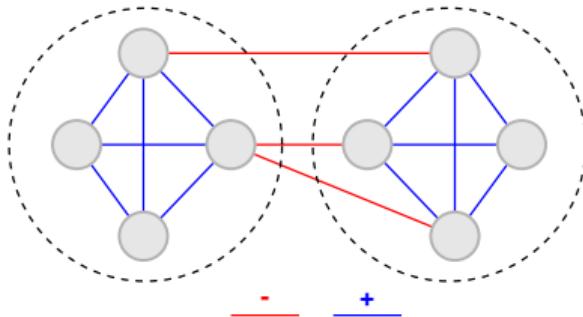
Motivation

Characterizations of balance

G is balanced iff

- ▶ There exists a sign-compliant partition: $V = V_1 \cup V_2$, all + edges within sets, all - edges between sets.

Some balanced graphs



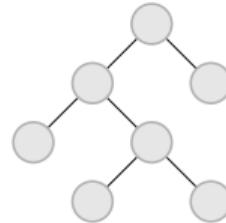
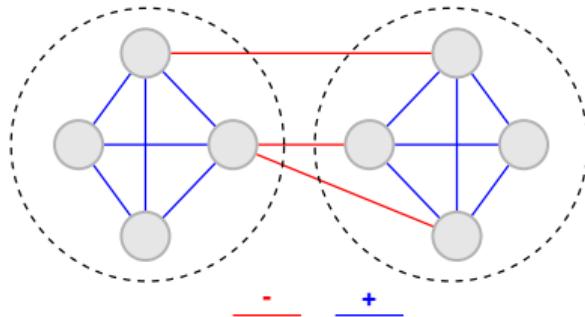
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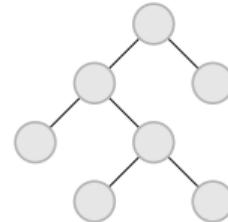
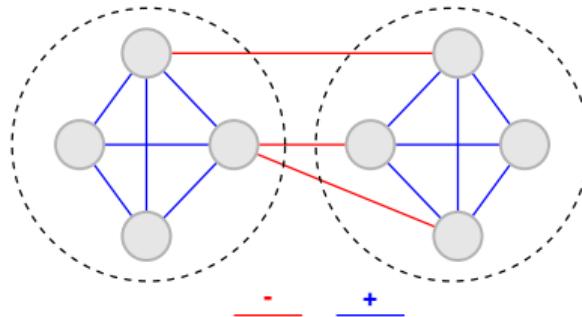
Motivation

Characterizations of balance

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- ▶ There exists a sign-compliant partition: $V = V_1 \cup V_2$, all + edges within sets, all - edges between sets.
- ▶ There are no negative (unbalanced) cycles.
- ▶ All paths between any pair u, v have same sign.

Some balanced graphs



Measures of partial balance

How can we measure **partial** balance?

- ▶ Fraction of balanced cycles
(Cartwright and Harary, 1956; Giscard et al., 2017)
 - ▶ Fraction of balanced triangles
(Terzi and Winkler, 2011) (example in next slide)
- ▶ Spectral methods (discussed later on)

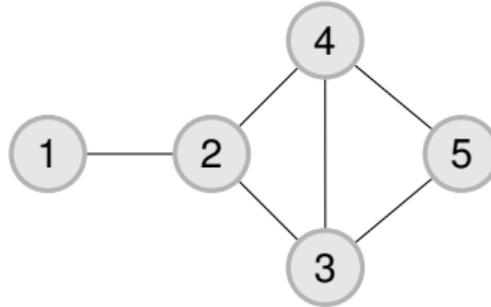
Check Aref and Wilson (2018) for an overview of partial measures of balance.

Measures of partial balance

Example: fraction of balanced triangles

Reminder: counting triangles in **unsigned** graphs:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

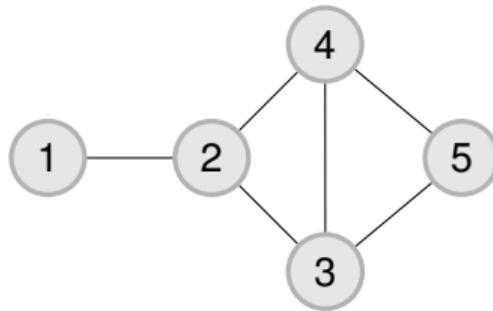


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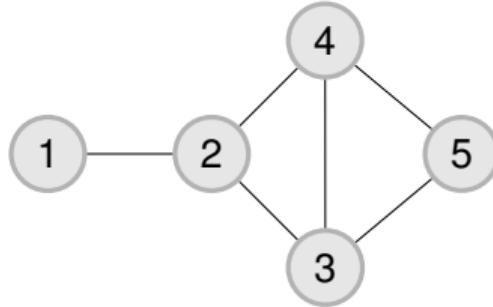


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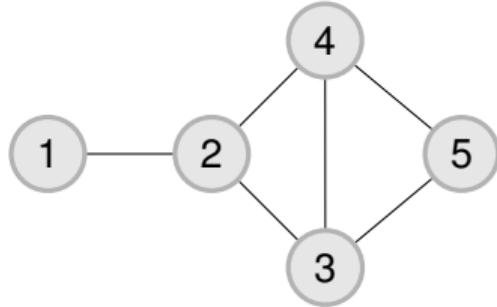


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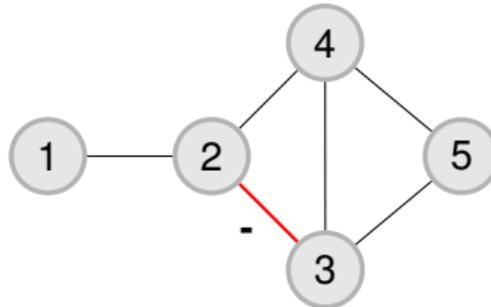
$$A_{ii}^3 = 2 \times \#(\text{3-cycles adjacent to vertex } i).$$

Measures of partial balance

Example: fraction of balanced triangles

Counting triangles in **signed** graphs:

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & -1 & 1 & 0 \\ 0 & -1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}$$

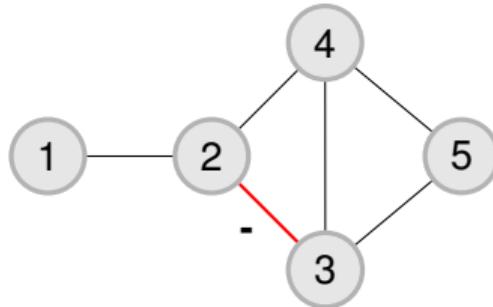


Measures of partial balance

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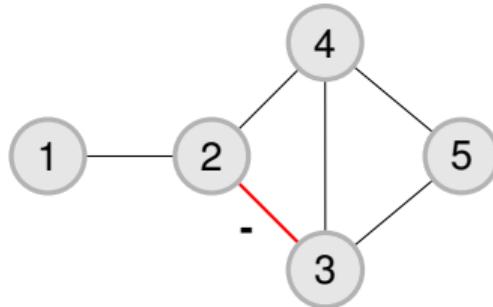


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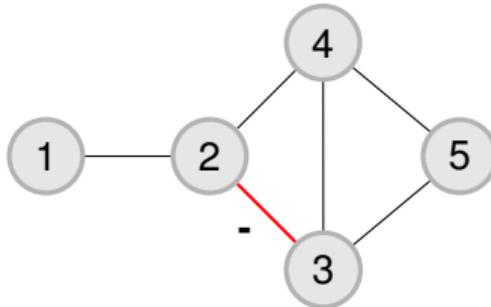
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$$A_{ii}^3 = 2 \times (\# \text{balanced 3-cycles} - \# \text{unbalanced 3-cycles}).$$



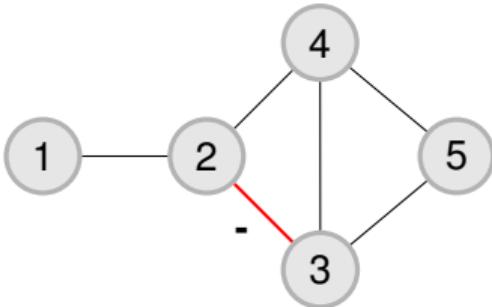
Measures of partial balance

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$$A_{ii}^3 = 2 \times (\# \text{balanced 3-cycles} - \# \text{unbalanced 3-cycles}). \text{ Thus,}$$



$$\frac{\text{Tr}(A^3) + \text{Tr}(|A|^3)}{2\text{Tr}(|A|^3)} = \text{fraction of balanced triangles.}$$

(Terzi and Winkler, 2011)

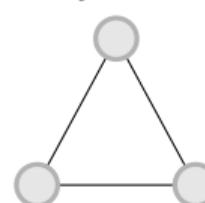
Note: $|A|$ is the adj. matrix of the *underlying* (unsigned) graph.

Spectrum

Spectral theory

Review of **unsigned** spectral theory:

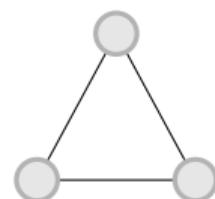
Laplacian: $L = D - A$


$$L = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix}$$

Spectral theory

Review of **unsigned** spectral theory:

Laplacian: $L = D - A$



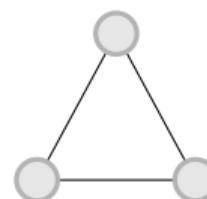
$$L \textcolor{blue}{v}_1 = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

- ▶ $\lambda_{\min}(L) = 0$ (Multiplicity of 0 = n. of connected components)

Spectral theory

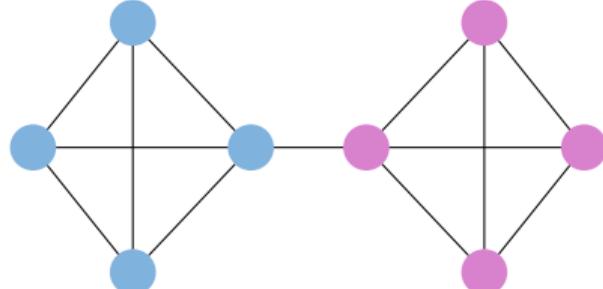
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- ▶ $\lambda_{\min}(L) = 0$ (Multiplicity of 0 = n. of connected components)
- ▶ Eigenvector v_2 gives a “good” partition (Cheeger inequality).



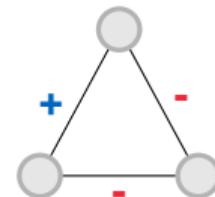
$$v_2 \approx \begin{pmatrix} -0.25 \\ -0.38 \\ -0.38 \\ -0.38 \\ 0.38 \\ 0.38 \\ 0.38 \\ 0.25 \end{pmatrix}, \lambda_2(L) \approx 0.35.$$

Spectral theory

Signed spectral theory:

Laplacian: $L = D - A$

Unsigned	Signed
L is positive semidefinite	
$D_{ii} = \sum_j A_{ij}$	$D_{ii} = \sum_j A_{ij} $
$\lambda_{\min}(L) = 0$	$\lambda_{\min}(L) \geq 0$



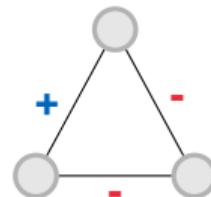
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Spectral theory

Signed spectral theory:

Laplacian: $L = D - A$

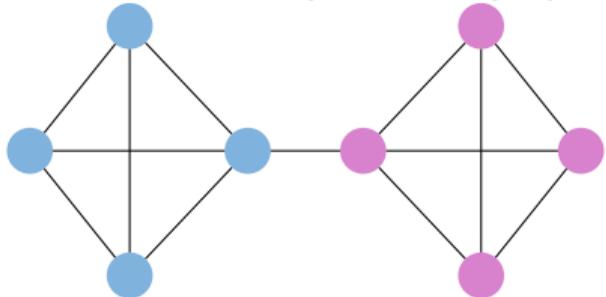
Unsigned	Signed
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$$L \textcolor{violet}{v}_1 = \begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} = 0$$

Spectral theory

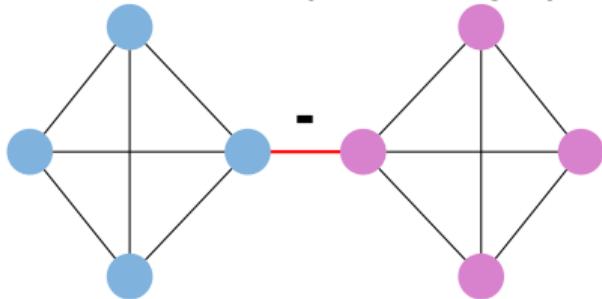
Consider our previous graph.



$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \lambda_{\min}(L) = 0.$$

Spectral theory

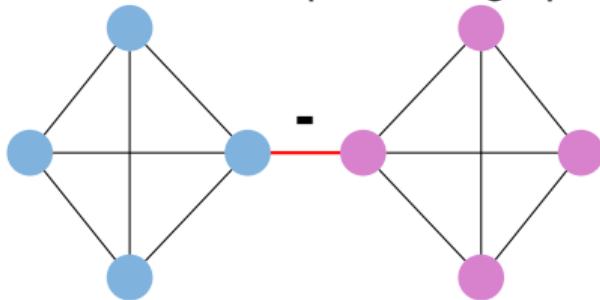
Consider our previous graph. Flip sign of one edge:



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Spectral theory

Consider our previous graph. Flip sign of one edge:



$$v_1 = \begin{pmatrix} -1 \\ -1 \\ -1 \\ -1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \lambda_{\min}(L) = 0.$$

This graph is **balanced**!

Spectral characterizations of balance

1. Connected and $\lambda_{\min} = 0$ (or one zero-eigval per connected component).

Spectral theory

A taste of spectral analysis:

Lemma (Hou et al., 2003)

$\lambda_{max}(L(G)) \leq \lambda_{max}(L(G^-))$, where G^- is the all-negative graph.

Spectral theory

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Easy: $L(G^-)$ is the *signless Laplacian* of the underlying graph $(D_{|G|} + A_{|G|})$. So,

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$$x^T L x = \sum_{(v_i, v_j) \in E} (x_i - \sigma(v_i, v_f)x_j)^2 \leq \sum_{(v_i, v_j) \in E} (|x_i| + |x_j|)^2.$$

Spectral theory

A taste of spectral analysis:

Lemma (Hou et al., 2003)

$\lambda_{\max}(L(G)) \leq \lambda_{\max}(L(G^-))$, where G^- is the all-negative graph.

Easy: $L(G^-)$ is the *signless Laplacian* of the underlying graph $(D_{|G|} + A_{|G|})$. So,

$$x^T L x = \sum_{(v_i, v_j) \in E} (x_i - \sigma(v_i, v_f)x_j)^2 \leq \sum_{(v_i, v_j) \in E} (|x_i| + |x_j|)^2.$$

Lemma (Hou et al., 2003)

$\lambda_{\max}(L(G)) \leq 2(n - 1)$, where n is the number of vertices.

Spectral theory

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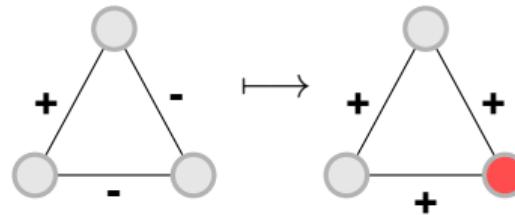
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Easy: $\lambda_{\max}(G) = \lambda_{\max}(D - A) \leq \lambda_{\max}(D_G) + \lambda_{\max}(-A_G) \leq n - 1 + n - 1$

Switching

Switch $S \subset V$: flip edges between S and $V \setminus S$.

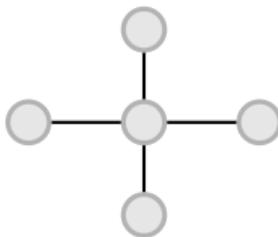


The spectrum is invariant w.r.t. switching.

$$A' = SAS^{-1}, \text{ where } S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Spectral theory

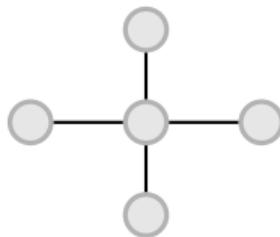
Example: Star graph



- ▶ Number of possible graphs: $2^{|E|}$
- ▶ Number of non-isomorphic graphs: $|E|$
- ▶ Number of distinct spectra: ?

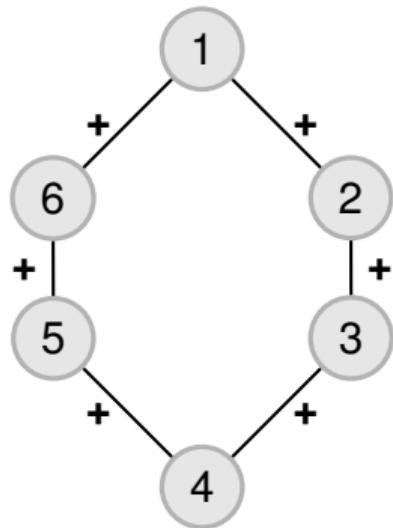
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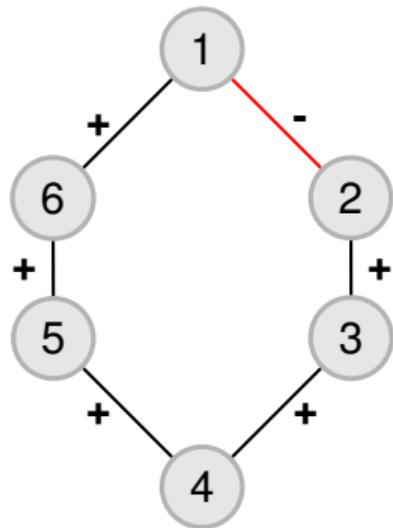
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- ▶ Number of distinct spectra: **Just the one!**

Example: cycle



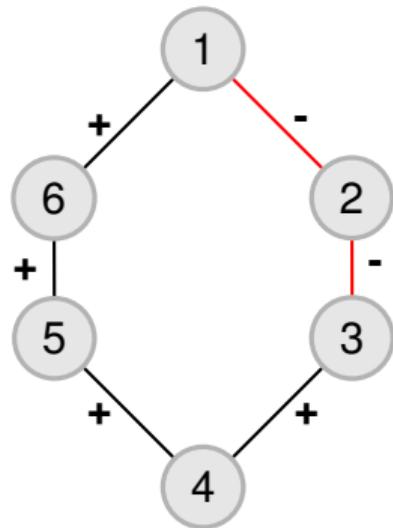
- ▶ Spectrum: $(0, 1, 1, 3, 3, 4)$
- ▶
- ▶

Example: cycle



- ▶
- ▶ Spectrum: $(0.27, 0.27, 2, 2, 3.73, 3.73)$
- ▶

Example: cycle

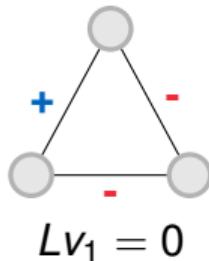


► Spectrum: $(0, 1, 1, 3, 3, 4)$

Switching

Spectral characterizations of balance

1. Connected and $\lambda_{min} = 0$ (or one zero-eigval per connected component).

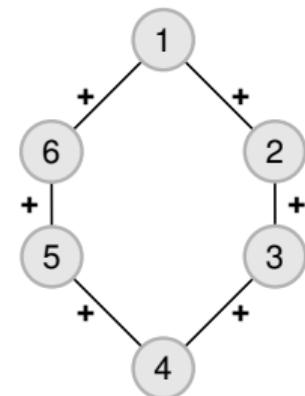
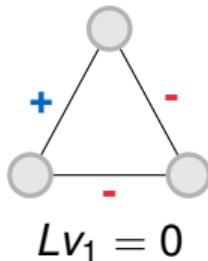


What's more: $\lambda_{min}(G) \leq \lambda_{max}(H)$, where H is the smallest subgraph to remove to make G balanced
Li and Li (2016).

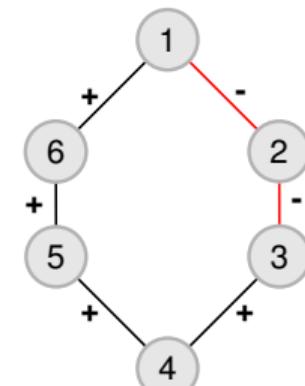
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switch_{v₂}

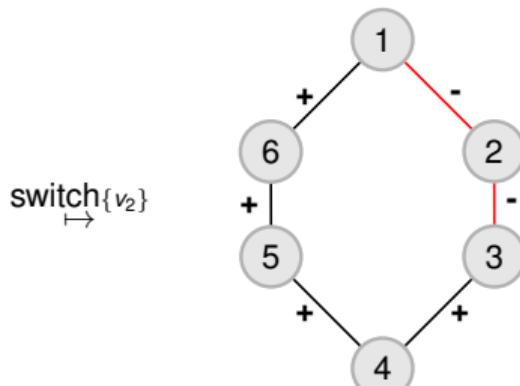
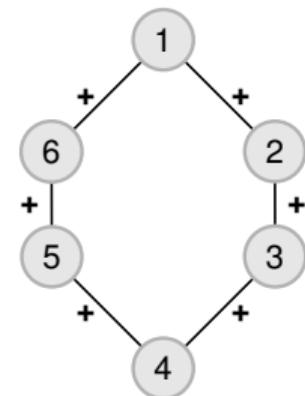
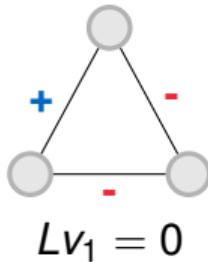


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Switching

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3. Switching equivalent to all-positive.



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Frustration

Frustration

Frustration number (**index**) : N. of **vertices** (**edges**) that need to be removed to achieve balance.

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Proposal: vertex frustration f_v , edge frustration f_e .

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$$\lambda_{\min}(L) \leq f_v \leq f_e.$$

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Spectral frustration inequalities

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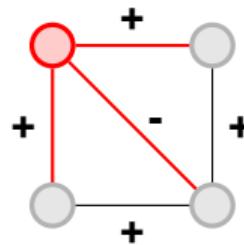
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Second inequality: trivial.



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It is **NP-hard** to find a submatrix with property π , where π is non-trivial, hereditary and holds for permutation matrices.

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π : There exists $S = \text{diag}(\mathbf{1}_\pm)$ such that $SAS = A^+$ (that is, the submatrix A switches to all positive).

In particular, finding f_v is finding submatrix of the adj. matrix satisfying π .

Frustration

How hard is **vertex** frustration?

Dual: find largest balanced subgraph.

Admits **2-approximation** on complete graphs (Bai and Wu, 2012).

Other than that, not that much is known.

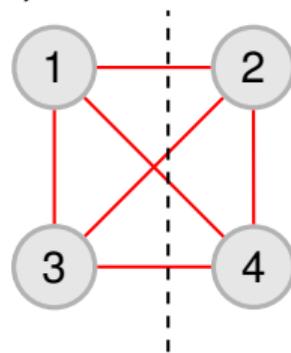
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Consider an **all-negative** signed graph. Finding the minimum f_e is finding the MAXCUT, and thus **NP**-complete.



Corollary: **all-negative** is balanced \Leftrightarrow it is bipartite.

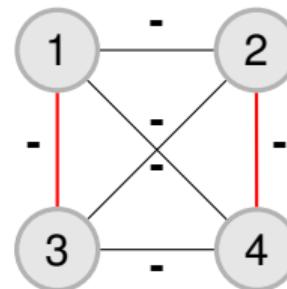
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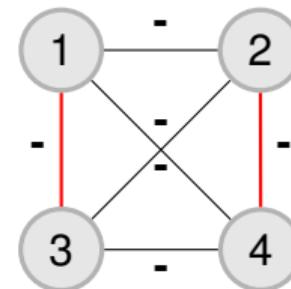
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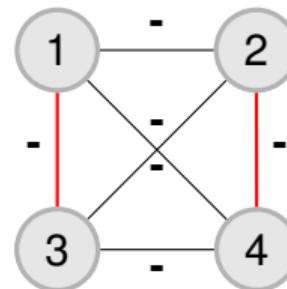


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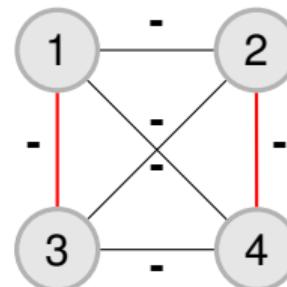
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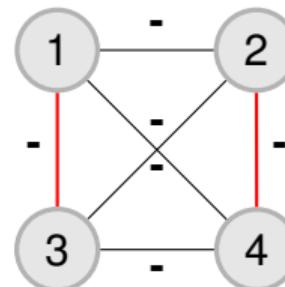
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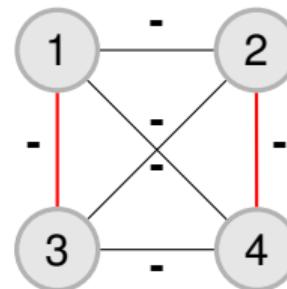
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MINUNCUT problem

Given a graph $G = (V, E)$ find a cut that minimizes the number of uncut edges.

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MINUNCUT problem (alternative formulation)

Consider boolean variables b_1, \dots, b_n and a set of constraints of the form $b_i \oplus b_j = 0$ and $b_i \oplus b_j = 1$. The goal is to minimize the number of unsatisfied constraints.

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Reduction to f_e :

- ▶ $b_i \mapsto v_i$,
- ▶ $b_i \oplus b_j = 1 \mapsto \underline{^-}$ $\mathcal{O}(\sqrt{\log n})$ -approximation (Agarwal et al., 2005).
- ▶ $b_i \oplus b_j = 0 \mapsto \underline{^+}$

Frustration

Reduction example #2

EDGE BIPARTITION problem

Given a graph $G = (V, E)$ delete the minimum number of edges to make it bipartite.

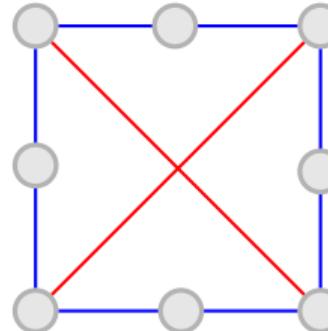
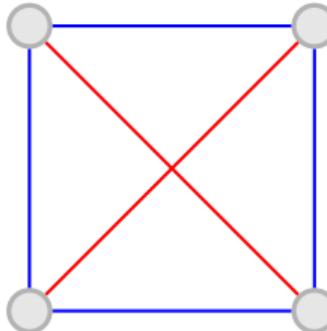
Frustration

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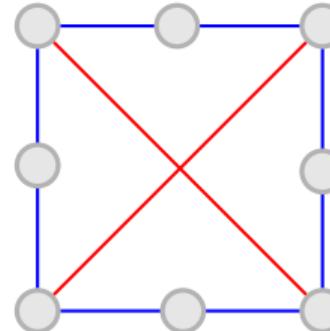
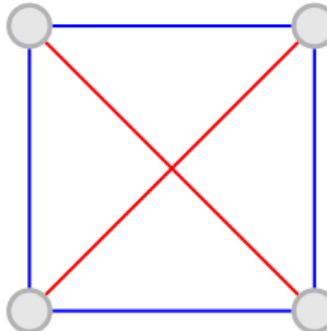
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Other concepts

Signed graph coloring

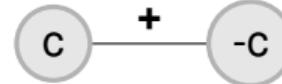
Coloring: $\kappa : V \mapsto \mathbb{N}$ s.t. $\forall (u, v) \in E \quad \kappa(u) \neq \kappa(v)$

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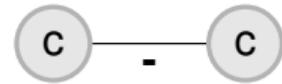
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Motivation: switching-friendly.

If G is balanced, $\chi(G) = \chi(|G|)$.



(Zaslavsky, 1982)

(Máčajová et al., 2014)

Signed graph coloring

Some results.

- ▶ $\chi(G)$: chromatic number of G .
- ▶ $|G|$: underlying graph.

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$$\chi(G) \leq 2\chi(|G|) - 1.$$

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2-colorable signed graphs (Máčajová et al., 2014)

$\chi(G) \leq 2$ iff G is antibalanced (switches to all-negative).

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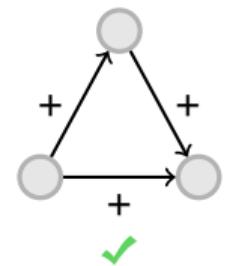
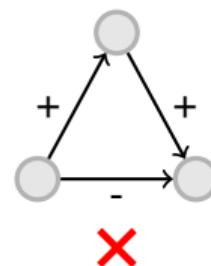
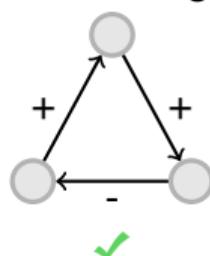
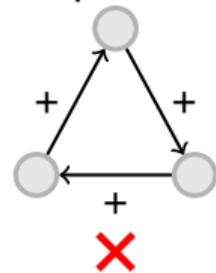
$\chi(G) \leq 2$ iff G is antibalanced (switches to all-negative).

Brooks theorem for signed graphs (Máčajová et al., 2014)

If G is not a balanced complete graph, a balanced odd circuit, or an unbalanced even circuit, then $\chi(G) \leq \Delta(G)$.

Status

Interpretation of signs in directed graphs: **Status**.



If Bob looks up to Alice and Alice looks up to Charlie,
then Charlie probably looks down on Bob.

What else

Some concepts we have overlooked.

- ▶ Signed-graphic matroid
- ▶ Matrix-tree theorem and the chromatic polynomial
- ▶ Two-graphs
- ▶ Full graphs
- ▶ Gain graphs

The bible: Zaslavsky's bibliography (Zaslavsky, 2012)

Problems and applications

Outline

Introduction

Theory of signed networks

Problems and applications

Subgraph mining

Graph partitioning

Correlation clustering

Link prediction and link classification

Network dynamics

Graph embedding and representation learning

Conclusions and future directions

Subgraph mining

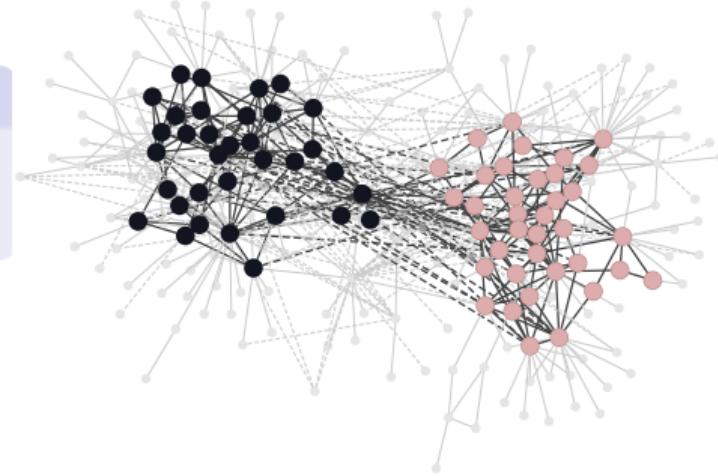
Subgraph mining

Goal

Find **interesting subgraphs** in a signed graph G

Some definitions of “interesting”:

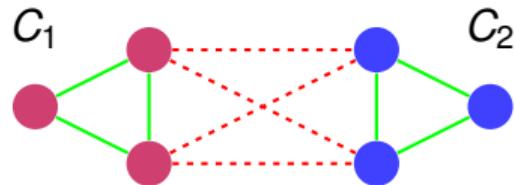
- ▶ maximal balanced subgraph
- ▶ polarized subgraph



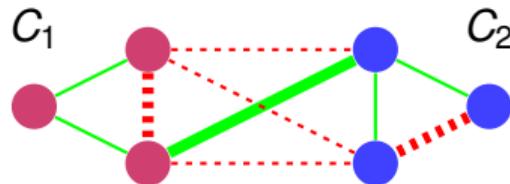
US Congress network (Bonchi et al., 2019)

Subgraph mining: balanced graphs vs. polarized graphs

Balanced graphs

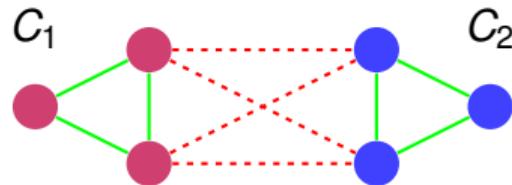


Polarized graphs: “noisy” edges are allowed

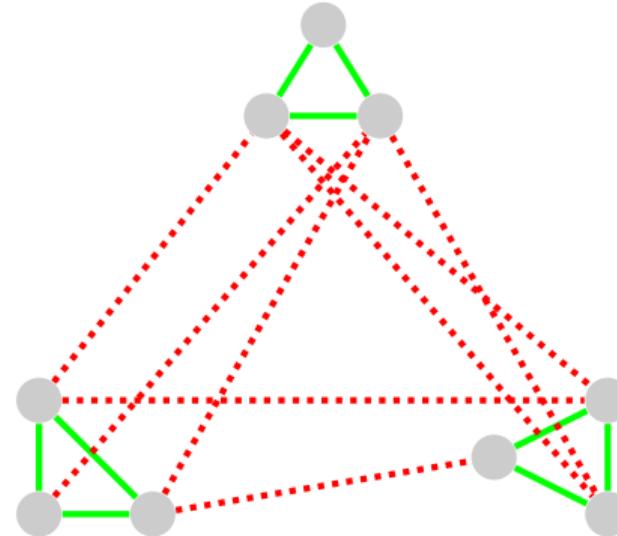


Subgraph mining: balanced graphs vs. polarized graphs

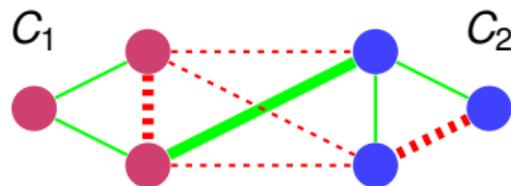
Balanced graphs



Polarized graphs: more than two groups



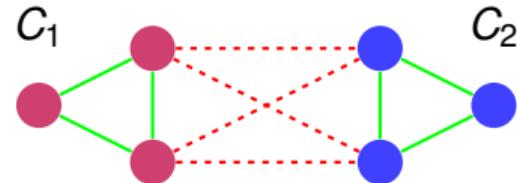
Polarized graphs: “noisy” edges are allowed



Finding maximum balanced subgraph (MBS)

Problem definition

- ▶ **Input:** a signed graph
 $G = (V, E^+, E^-)$
- ▶ **Output:** a **maximum-cardinality** vertex subset $U \subseteq V$ such that $G(U)$ is **balanced**

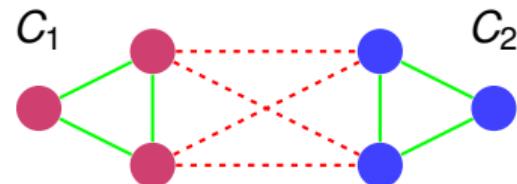


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A balanced graph

- ▶ An equivalent problem: remove the **minimum** number of vertices such that the remaining graph is balanced
- ▶ Solution size of MBS = **frustration index** (defined in theory part)
- ▶ **Edge-version** of MBS: a balanced subgraph with maximum number of edges
- ▶ All these problems are **NP-hard**

Maximum balanced subgraph: applications in linear algebra

- ▶ LP in standard form

$$\begin{array}{ll}\min & c^T x \\ \text{s.t.} & Ax \leq b\end{array}$$

- ▶ Each vertex represents a **row** in A
- ▶ $(i, j) \in E^-$ if $A_{ik} = A_{jk} \neq 0$ for some k
- ▶ $(i, j) \in E^+$ if $A_{ik} = -A_{jk} \neq 0$ for some k
- ▶ A balanced subgraph → **a network sub-matrix**
- ▶ Extracting network sub-matrices **speed up** LP solvers

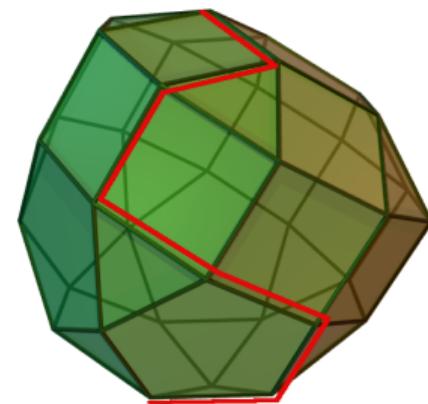


Image source: wiki.ubc.ca

Maximum balanced subgraph: applications in finance

- ▶ Can be used to find a **risk-limiting strategy** for portfolio definition Harary et al. (2002)
- ▶ Each node represents **a security** (a tradable asset)
- ▶ $(i, j) \in E^+$ if the value of two securities move in *same direction*
- ▶ $(i, j) \in E^-$ if the value of two securities move in *opposite direction*
- ▶ A balanced subgraph → a set of assets whose values are **predictable** → low investment risk



Image source: vecteezy.com

Maximum balanced subgraph: applications in political science

- ▶ Setting: find political parties with **opposite political leaning**
- ▶ Each node represents a politician
- ▶ Edges capture supportive/opposive relations
- ▶ Further applications:
 - ▶ **understanding** the nature of polarization
 - ▶ **reducing polarization via applied intervention**

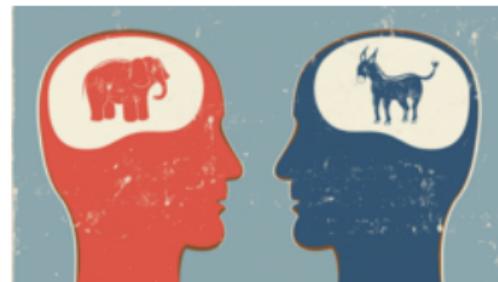


Image source: iStockphoto.com

MBS: Overview of existing results

Paper	graph type	approximation ratio	running time
Gülpınar et al. (2004)	general	—	$\mathcal{O}(m)$
Figueiredo et al. (2011)	general	exact	branch & bound
Bai and Wu (2012)	complete	2	$\mathcal{O}(n^3)$
Bai and Wu (2012)	complete	exact	$\mathcal{O}(1.28^k + n^3)$
Figueiredo and Frota (2014)	general	—	$\mathcal{O}(m)$
Ordozgoiti et al. (2020)	general	—	$\mathcal{O}(n^2)$ per iteration

- ▶ — indicates approximation “not known”, i.e., the algorithm is a heuristic
- ▶ k is the number of nodes to delete

Spanning-tree heuristic for MBS

Notations:

- ▶ Negative graph G^- : induced subgraph on the **negative** edges in G
- ▶ Positive graph G^+ : induced subgraph on the **positive** edges in G
- ▶ $I(G)$: any **maximal independent set** of G

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High-level idea (Gülpinar et al., 2004)

1. Find a **spanning tree** T on G
2. Find a **switch** W such that T^W is **all positive**
3. Switch G by W , yielding G^W
4. Return $I(G^W)^-$

Spanning-tree heuristic: maximal independent set on G^-

Intuition 1

Any maximal independent set on G^- is balanced in G .

G

G^-

$I(G^-)$

Spanning-tree heuristic: maximal independent set on G^-

Intuition 1

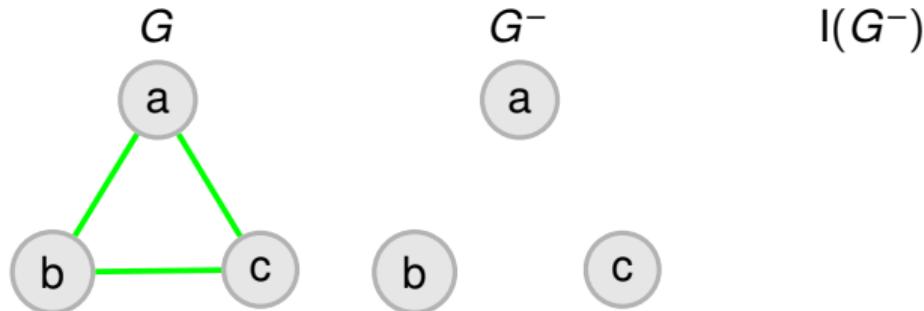
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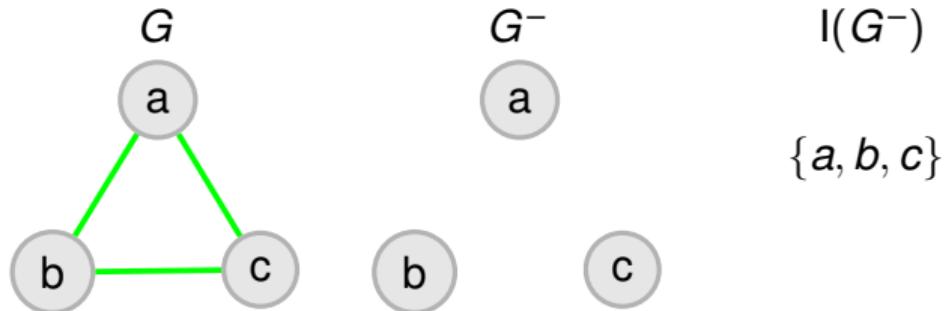
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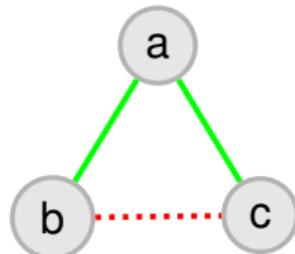
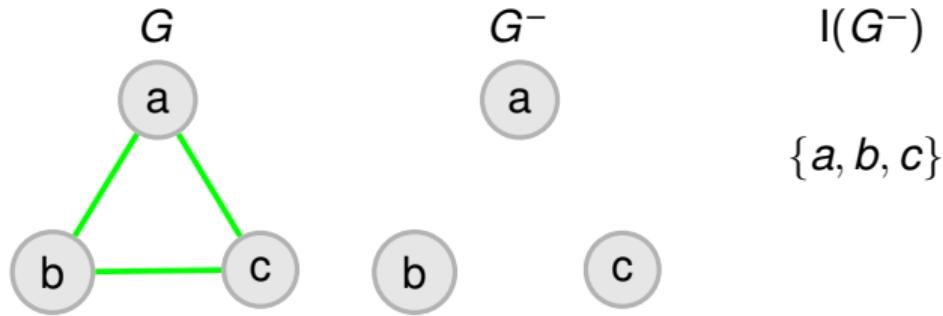
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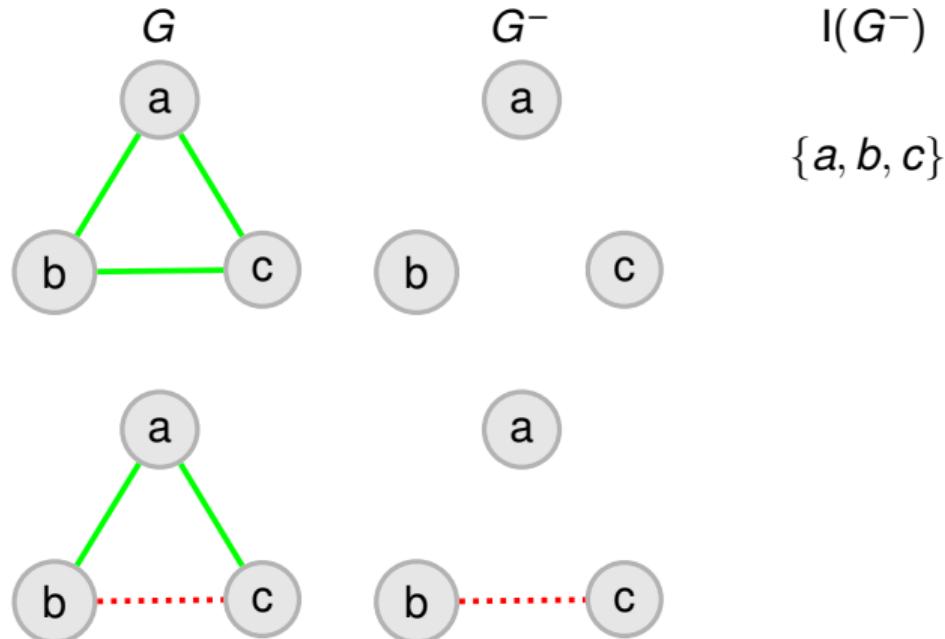
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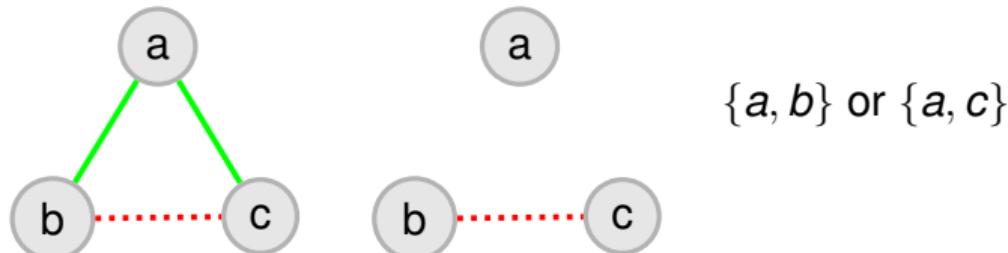
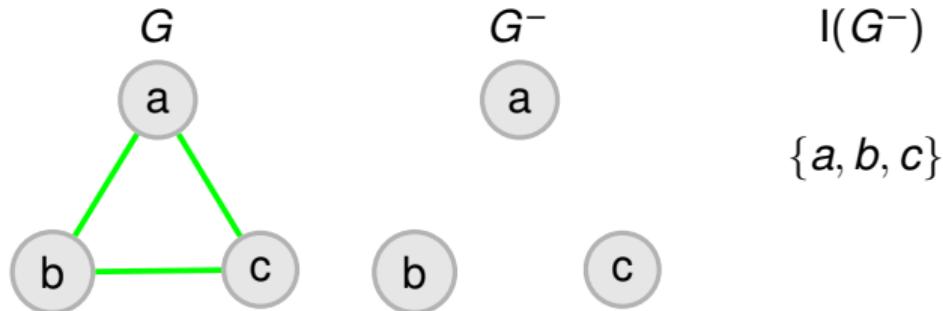
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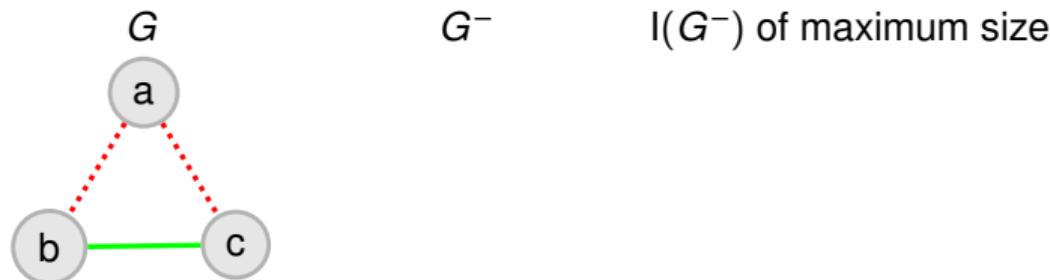
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Quiz: Can we solve MBS optimally by maximizing $|I(G^-)|$?

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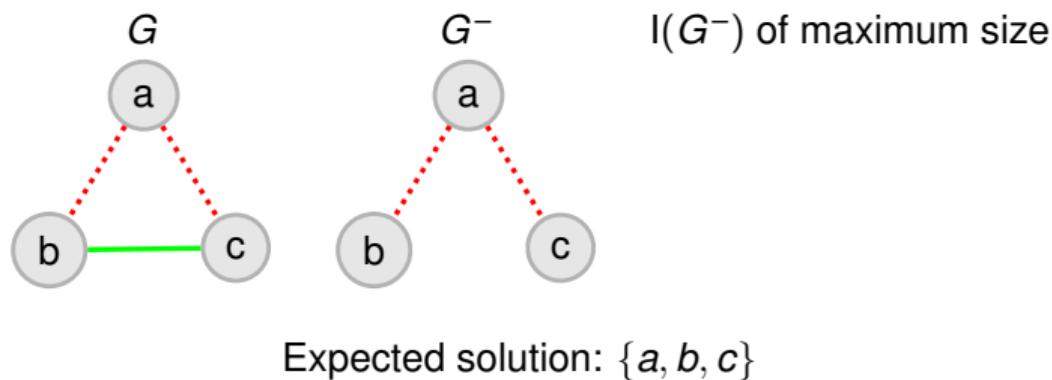


Expected solution: $\{a, b, c\}$

Spanning-tree heuristic: maximal independent set on G^-

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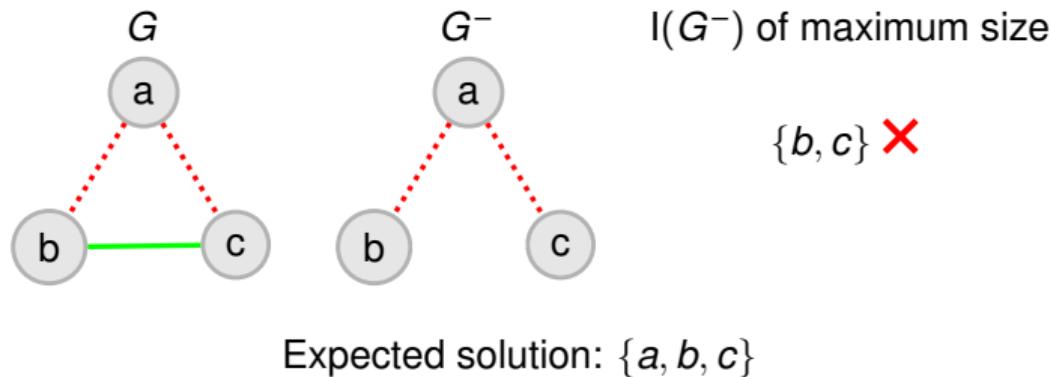
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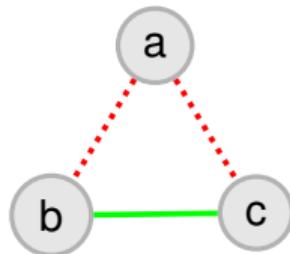
Spanning-tree heuristic: switch

Intuition 2

Switch G to expand size of $I(G^-)$

If we switch on $\{a\}$:

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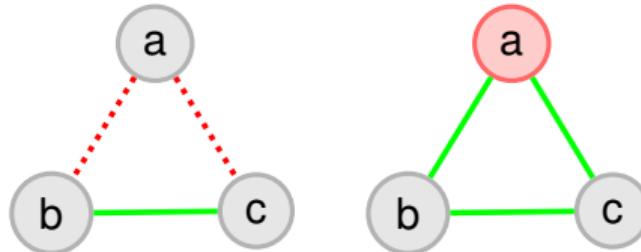
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If we switch on $\{a\}$:

new! ↓

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$G^W, W = \{a\}$



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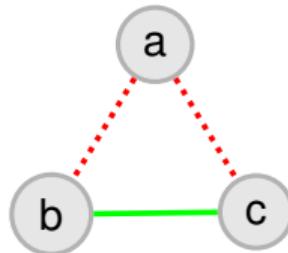
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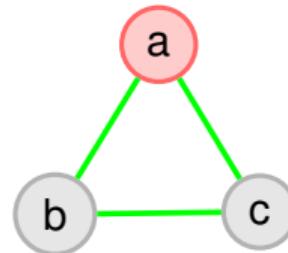
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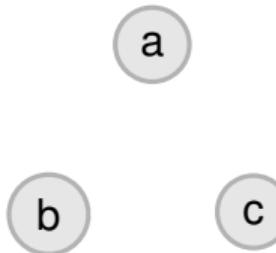
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$(G^W)^-$



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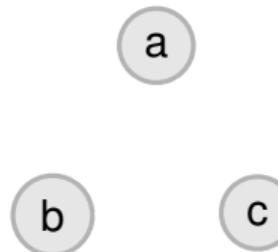
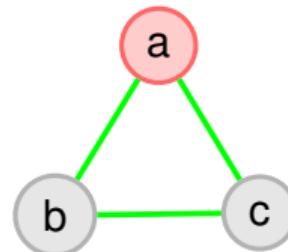
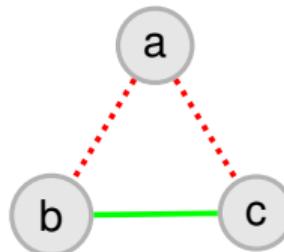
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$G^W, W = \{a\}$

$(G^W)^-$

$I(G^-)$ of maximum size



$\{a, b, c\}$ ✓

Spanning-tree heuristic: combining the previous ideas

An equivalent form of MBS

Find a **switch** W s.t. $|I(G^W)^-|$ is **maximized**

(an **NP-hard** problem)

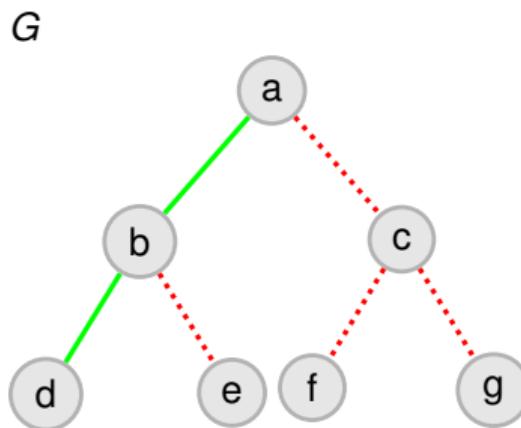
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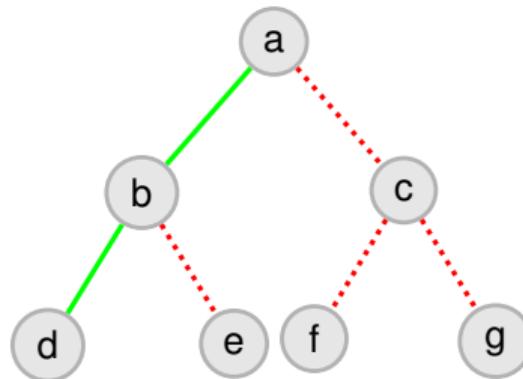
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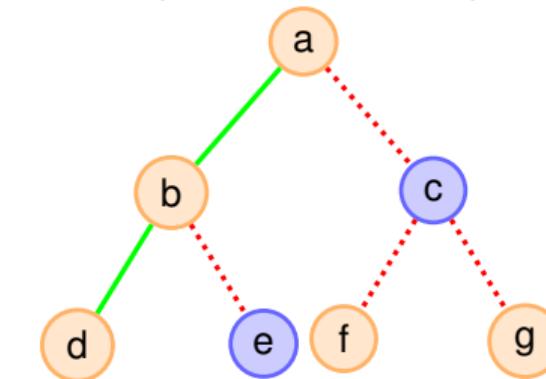
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Switch (either vertex color)



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4. Return **maximal independent set** on $(G^W)^-$ # $I(G^W)^-$ is balanced

A spectral method for MBS

Finding large balanced subgraphs in signed networks (Ordozgoiti et al., 2020)

Intuition

A balanced subgraph has $\lambda_{min}(L) = 0$.

Can we **remove** as few vertices as possible, such that $\lambda_{min}(L)$ of the remaining (connected) graph is zero?

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Perturbation bound for **fast computation**: if \mathbf{v} is the eigenvector for λ_{min} then

$$\lambda_1(L \setminus \mathbf{v}_i) \leq \frac{\lambda_1(L)(1 - 2\mathbf{v}_i^2) - \sum_{j \in N(i)} \mathbf{v}_j^2 + \mathbf{v}_i^2 d(i)}{1 - \mathbf{v}_i^2}.$$

A spectral method for MBS

Finding large balanced subgraphs in signed networks (Ordozgoiti et al., 2020)

Further speed-ups:

- ▶ Remove **multiple** vertices at once (extending perturbation bound)
- ▶ Estimate **eigen-pair** with gradient methods (LOBPCG)
- ▶ **Low-rank** updates
- ▶ **Sampling**: Solve the problem for \sqrt{n} random subgraphs of size \sqrt{n}

A spectral method for MBS

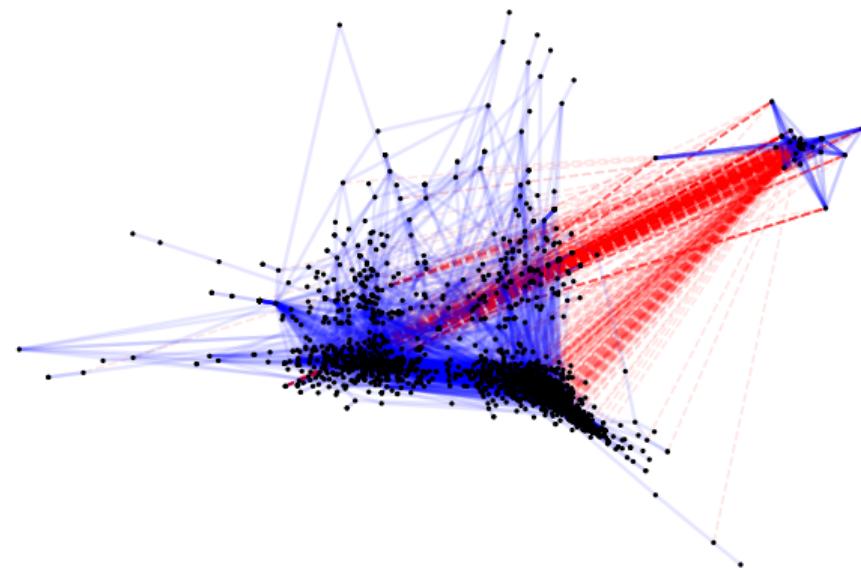
Finding large balanced subgraphs in signed networks (Ordozgoiti et al., 2020)

	HIGHLAND TRIBES		CLOISTER		CONGRESS		BITCOIN		TWITTER REFERENDUM	
method	V	E	V	E	V	E	V	E	V	E
TIMBAL	13	35	10	33	208	452	4 208	10 158	8 944	166 243
GRASP	10	18	6	11	115	145	2 167	3 686	5 425	49 105
GGMZ	10	21	5	7	153	238	1 388	1 683	2 501	2 821
EIGEN	12	37	8	27	11	16	7	17	132	6 140
	WIKI ELECTIONS		SLASHDOT		WIKI CONFLICT		WIKI POLITICS		EPINIONS	
TIMBAL	3 786	18 550	42 205	96 460	48 136	356 204	63 252	218 360	62 010	169 894
GRASP	1 752	4 416	23 289	40 511	18 576	82 726	31 561	81 557	28 189	63 250
GGMZ	713	771	16 389	17 867	6 137	9 145	23 342	37 098	21 009	25 013
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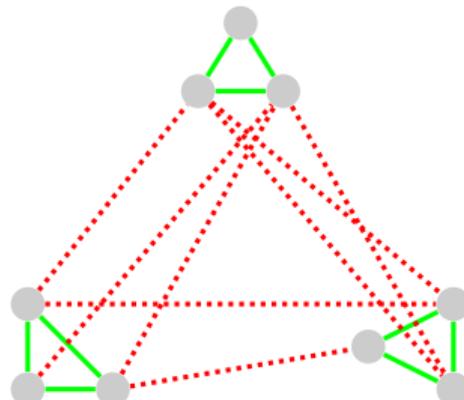
A graph of traders in a
Bitcoin exchange
with trust / distrust relations



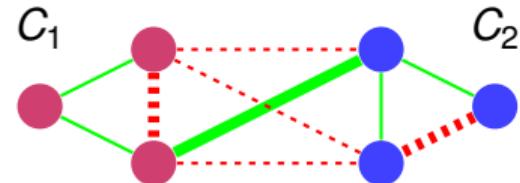
Polarized subgraph detection

Polarized subgraphs as an extension of balanced subgraphs

- ▶ Can have **more than two** components
- ▶ Permits the presence of **noisy** edges:
 - ▶ Positive edges between C_1 and C_2
 - ▶ Negative edges within C_1 or C_2



More than two components



With “noisy” edges (drawn in thick lines)

Polarized subgraph detection: problem dimensions

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Paper	num. groups	num. subgraphs	local / global	approximation guarantee
Chu et al. (2016)	k	≥ 1	global	-
Bonchi et al. (2019)	2	1	global	\sqrt{n}
Xiao et al. (2020)	2	≥ 1	local	$\sqrt{\text{OPT}}$

Polarized subgraph detection: multiple k -way subgraphs

Finding gangs in war from signed networks (Chu et al., 2016)

- ▶ Intuition of the polarization measure:

1. In each group, many **positive** edges
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3. Between two groups, few nodes are **shared**

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$$tr(X^T A^+ X) + \alpha \bar{tr}(X^T A^- X) - \beta \bar{tr}(X^T X)$$

where $X \in \{0, 1\}^{n \times k}$, e.g., each column maps to a group

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- ▶ Main idea of algorithm:

- ▶ Relax X to be continuous
- ▶ The relaxed problem is non-convex → multiple local optimums
- ▶ Each local optimum corresponds to a polarized subgraph
- ▶ Solved by gradient-descent methods

Polarized subgraph detection: single 2-way subgraph

Discovering polarized communities in signed networks (Bonchi et al., 2019)

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- ▶ Spectral algorithms:

- ▶ **Relax** x to be continuous
- ▶ The relaxed problem is solved by finding the **leading** eigenvector
- ▶ Randomized \sqrt{n} -approximation based on **rounding** the leading eigenvector

Polarized subgraph detection: local 2-way subgraph

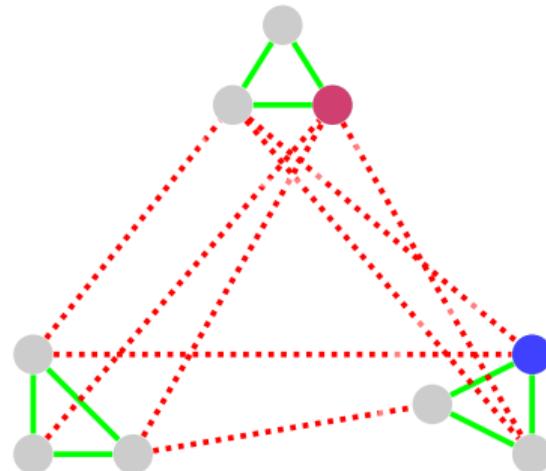
Searching for polarization in signed graphs: a local spectral approach (Xiao et al., 2020)

- ▶ **Input:** a signed graph and **two sets of seed nodes**
- ▶ **Output:** a polarized subgraph that contains the seeds

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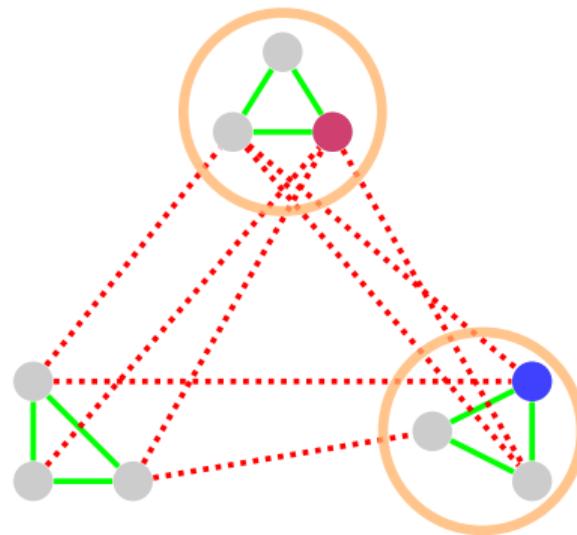


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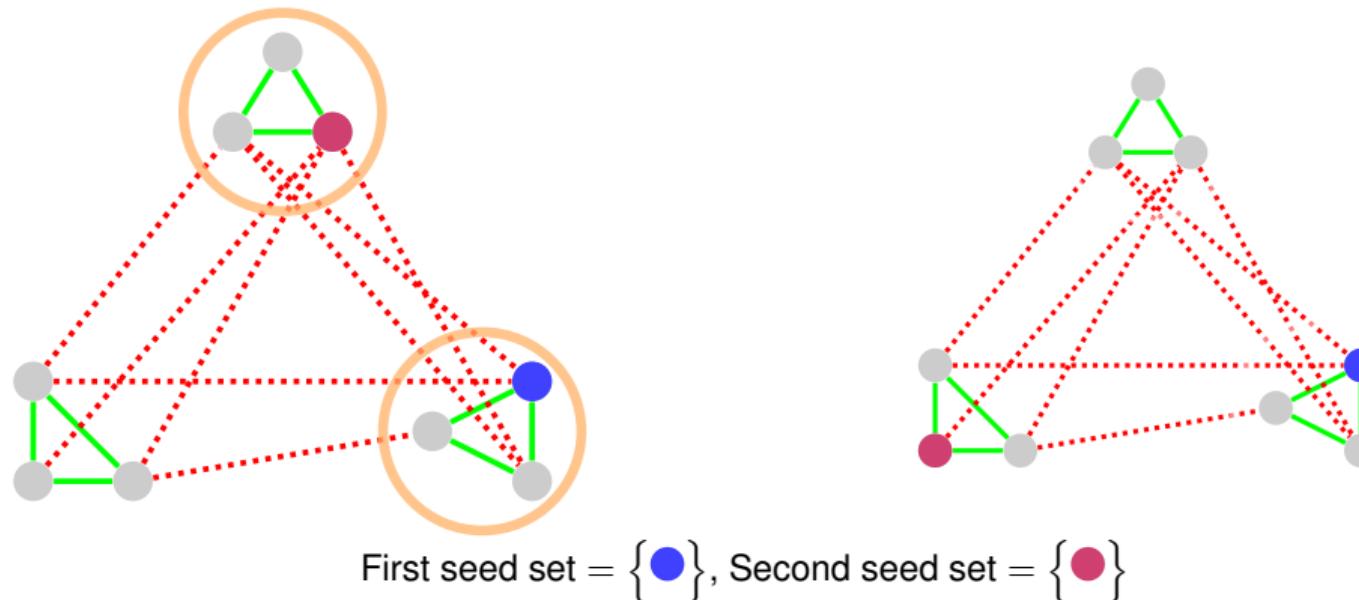


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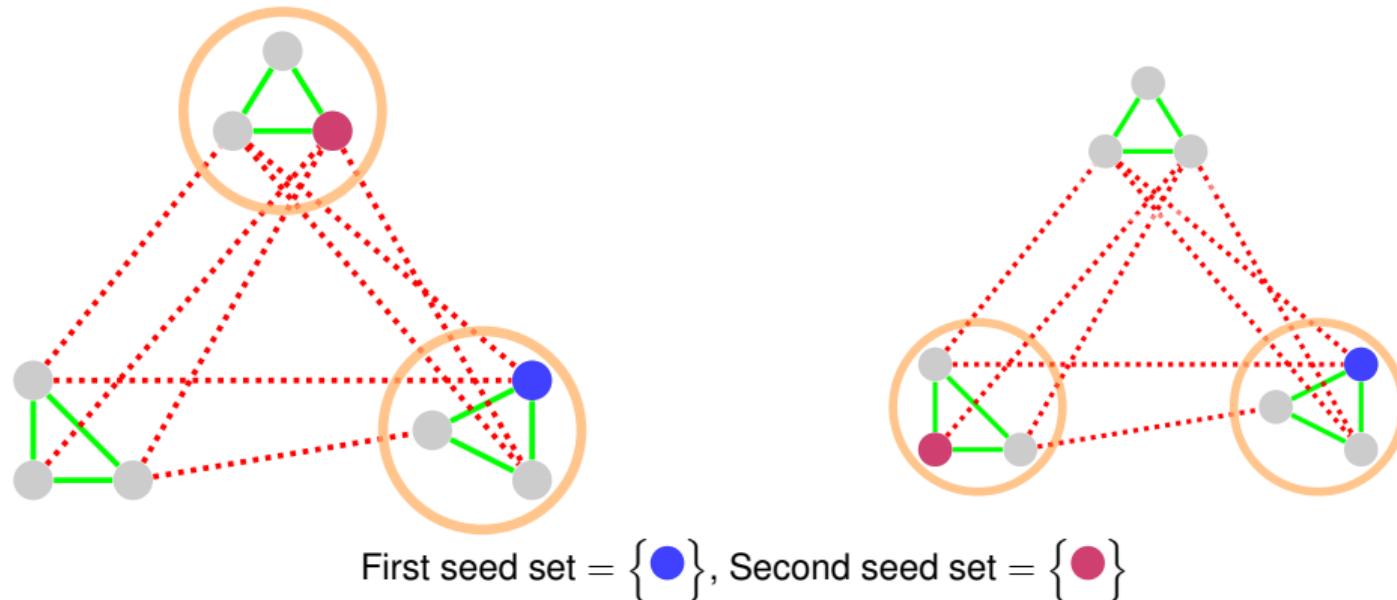
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- ▶ Intuition of the polarization measure:

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Combined: the *fraction* of the above edges in the subgraph is small

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- ▶ Problem in matrix form:

$$\begin{array}{ll} \min_{x \in \{-1, 0, 1\}^n} & \frac{x^T L x}{x^T D x} \\ \text{s.t.} & s^T D x \geq \kappa \end{array}$$

where s encodes the seed information and x correlates with s

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- ▶ A spectral algorithm:

- ▶ Relax x to be continuous
- ▶ The relaxed problem is a locally-biased eigen problem
- ▶ An approximation algorithm based on rounding the locally-based eigenvector.

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Case study: **overlapping communities** in English-word graph

- ▶ **Dataset**: a graph of English words
- ▶ **Word relations**: synonym and antonym

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Two meanings of “fair”

- “without cheating”, e.g., a fair game
- “not excessive”, e.g., fair amount of time

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- ▶ **Word relations**: synonym and antonym

Two meanings of “fair”

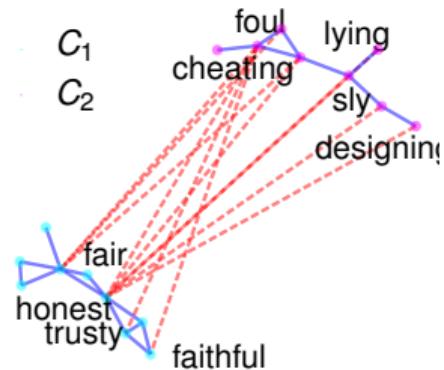
- “without cheating”, e.g., a fair game
- “not excessive”, e.g., fair amount of time

Question

Can we find the synonyms and antonyms w.r.t each meaning of “fair”?

Polarized subgraph detection: local 2-way subgraph

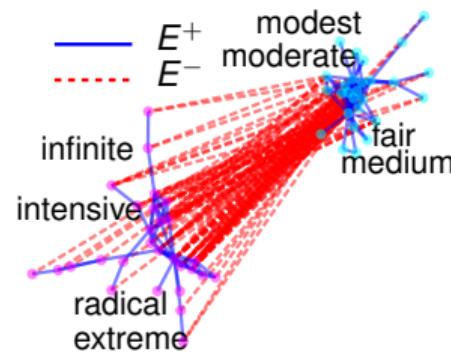
Searching for polarization in signed graphs: a local spectral approach (Xiao et al., 2020)



(a) *fair as without cheating*

$$S_1 = \{fair, honest\}$$

$$S_2 = \{cheating\}$$



(b) *fair as not excessive*

$$S_1 = \{fair, modest\}$$

$$S_2 = \{extreme\}$$

Subgraph mining: summary

- ▶ Types of subgraphs are discussed:

Maximal balanced subgraph:

studied mainly in theoretical computer science

Polarized subgraph:

not necessarily balanced, with different objective functions

- ▶ Applications: from solving linear program solver to analyzing societal debates
- ▶ An emerging and timely topic when polarization is harming our society !

Outline

Introduction

Theory of signed networks

Problems and applications

Subgraph mining

Graph partitioning

Correlation clustering

Link prediction and link classification

Network dynamics

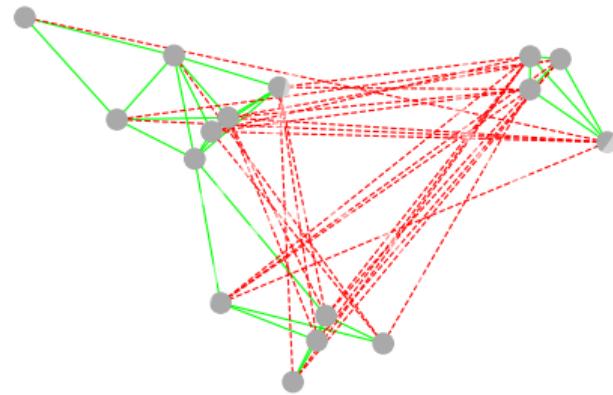
Graph embedding and representation learning

Conclusions and future directions

Graph partitioning

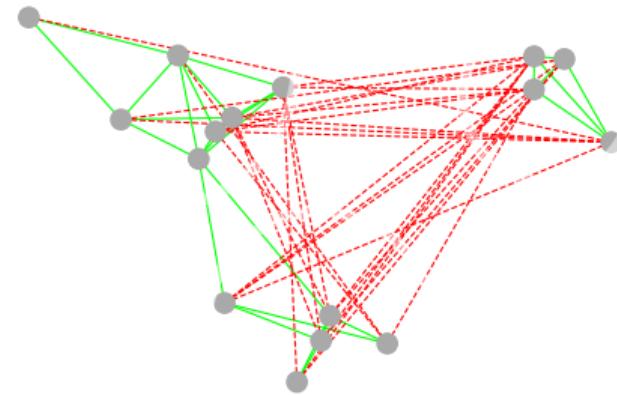
Graph partitioning

- ▶ General goal: partition all vertices into **disjoint** sets



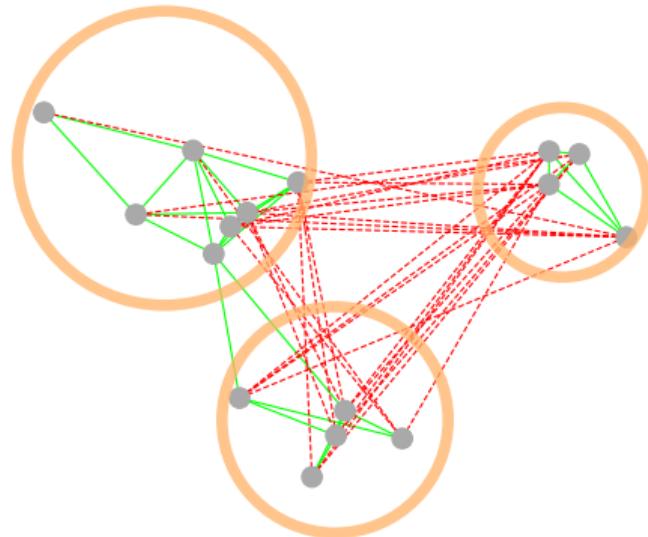
Graph partitioning

- ▶ General goal: partition all vertices into **disjoint** sets
- ▶ such that
 - ▶ edges *inside* each partition are mostly **positive**
 - ▶ edges *between* two partitions are mostly **negative**



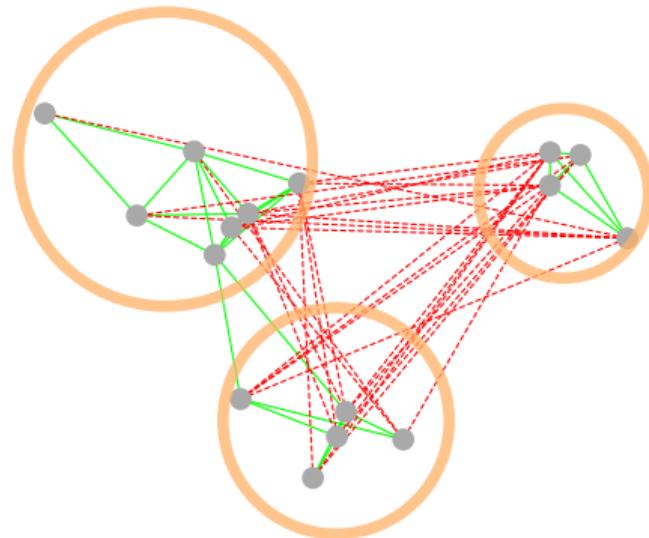
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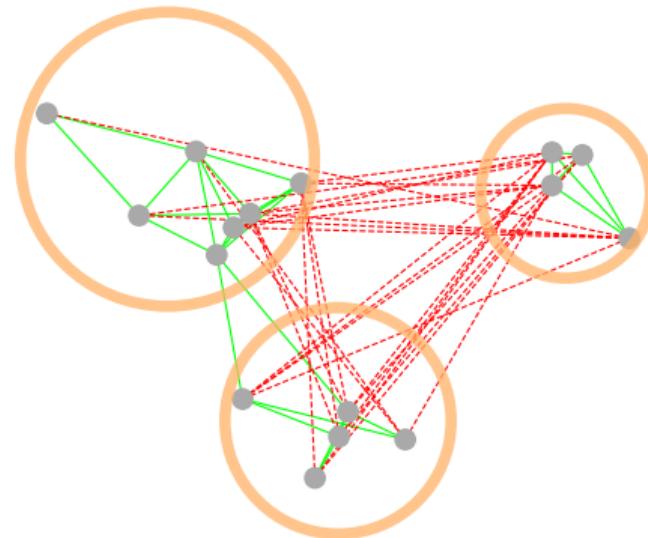
Graph partitioning

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 - ▶ Identifying proteins that have synergetic interaction
 - ▶ Finding cohesive political parties or social-media user groups



Graph partitioning

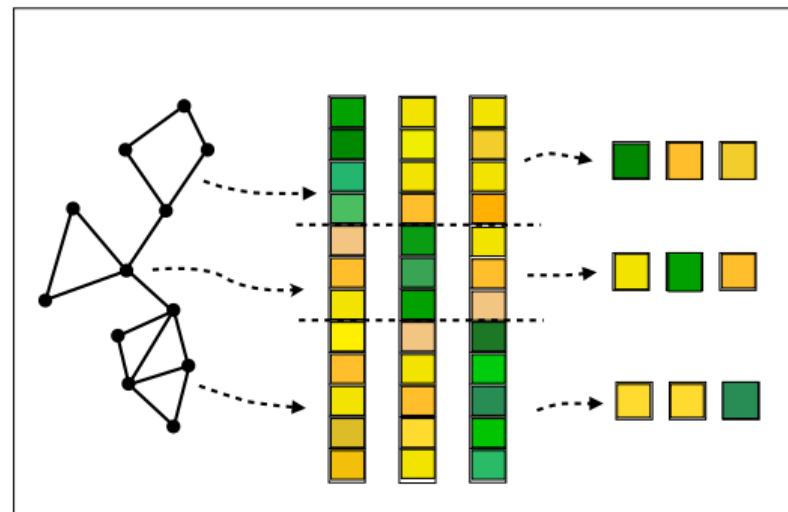
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- ▶ Example applications:
 - ▶ Identifying proteins that have synergetic interaction
 - ▶ Finding cohesive political parties or social-media user groups
- ▶ Our focus: **spectral methods**



Recap: spectral methods on unsigned graphs

Heuristics for k -way partitioning

1. Construct **some** Laplacian matrix of the graph
2. $[\mathbf{v}_1, \dots, \mathbf{v}_k] \leftarrow$ the first k **eigenvectors** of the Laplacian
3. Run a **feature-based** clustering algorithm such as k -means



Spectral clustering methods on signed graphs

Similar heuristics for k -way partitioning

- ▶ Formulate some Laplacian L
- ▶ $[u_1, \dots, u_k] \leftarrow$ the leading k eigenvectors of L , corresponding to smallest eigenvalues)
- ▶ Run a feature-based clustering algorithm on $[u_1, \dots, u_k]$

The Laplacian matrix matters

Spectral clustering methods on signed graphs

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The Laplacian matrix matters

- ▶ Arithmetic-mean Laplacian (Kunegis et al., 2010)
- ▶ Geometric-mean Laplacian (Mercado et al., 2016)
- ▶ Power-mean Laplacian (Mercado et al., 2019)

“Classic” Laplacian

Spectral analysis of signed graphs for clustering, prediction and visualization (Kunegis et al., 2010)

Problem

Find a partition of (X, Y) over all nodes that

$$\text{minimizes } \text{cut}(X, Y) \left(\frac{1}{|X|} + \frac{1}{|Y|} \right)$$

where

$$\text{cut}(X, Y) = (2 \text{ cut}^+(X, Y) + \text{cut}^-(X) + \text{cut}^-(Y))$$

The objective is called **signed ratio cut**

“Classic” Laplacian

Spectral analysis of signed graphs for clustering, prediction and visualization (Kunegis et al., 2010)

Define

$$L = D - A$$

An equivalent problem in **matrix form**

minimize

$$u^T L u$$

s.t.

$$u_i = \begin{cases} \frac{1}{2} \left(\sqrt{|X|/|Y|} + \sqrt{|Y|/|X|} \right) & \text{if } i \in X \\ -\frac{1}{2} \left(\sqrt{|X|/|Y|} + \sqrt{|Y|/|X|} \right) & \text{if } i \in Y \end{cases}$$

“Classic” Laplacian

Spectral analysis of signed graphs for clustering, prediction and visualization (Kunegis et al., 2010)

Define

$$L = D - A$$

An equivalent problem in matrix form

$$\begin{aligned} \text{minimize} \quad & u^T L u \\ \text{s.t.} \quad & u_i = \frac{1}{2} \left(\sqrt{|X|/|Y|} + \sqrt{|Y|/|X|} \right) \quad \text{if } i \in X \\ & u_i = -\frac{1}{2} \left(\sqrt{|X|/|Y|} + \sqrt{|Y|/|X|} \right) \quad \text{if } i \in Y \end{aligned}$$

- ▶ If u is relaxed that $u^T u = 1$,
the problem is equivalent to standard eigen-problem

“Classic” Laplacian

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If normalized Laplacian is used

$$\mathcal{L} = I - D^{-1/2}AD^{-1/2},$$

“Classic” Laplacian

Spectral analysis of signed graphs for clustering, prediction and visualization (Kunegis et al., 2010)

If normalized Laplacian is used

$$\mathcal{L} = I - D^{-1/2}AD^{-1/2},$$

the eigen-problem on \mathcal{L} corresponds to minimizing signed normalized cut:

$$\text{cut}(X, Y) \left(\frac{1}{\text{vol}(X)} + \frac{1}{\text{vol}(Y)} \right)$$

Geometric-mean Laplacian: decomposing classic Laplacian

Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

Classic Laplacian can be re-written:

$$\begin{aligned}L &= D - A \\&= D - (A^+ - A^-) \\&= (D^+ - A^+) + (D^- + A^-) \\&= L^+ + Q^-\end{aligned}$$

Geometric-mean Laplacian: decomposing classic Laplacian

Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

Classic Laplacian can be re-written:

$$\begin{aligned}L &= D - A \\&= D - (A^+ - A^-) \\&= (D^+ - A^+) + (D^- + A^-) \\&= L^+ + Q^-\end{aligned}$$

- ▶ Laplacian on G^+ : $L^+ = D^+ - A^+$
- ▶ Laplacian on G^- : $Q^- = D^- + A^-$
- ▶ L is twice the arithmetic mean of L^+ and Q^- and thus,
is called “*arithmetic-mean Laplacian*”

Geometric-mean Laplacian: decomposing classic Laplacian

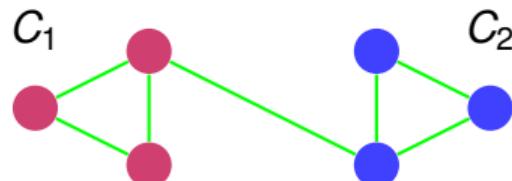
Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

Assortative case

G^+ is well-separated

→ $\lambda_1(L^+)$ is small

→ informative



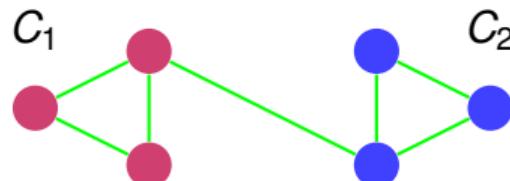
If G^+ is disconnected, $\lambda_2(L^+) = 0$

Geometric-mean Laplacian: decomposing classic Laplacian

Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

Assortative case

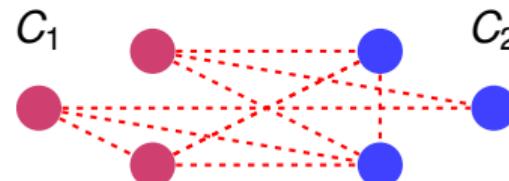
G^+ is well-separated
 $\rightarrow \lambda_1(L^+)$ is small
 \rightarrow informative



If G^+ is disconnected, $\lambda_2(L^+) = 0$

Disassortative case

G^+ is closer to bipartite
 $\rightarrow \lambda_1(Q^-)$ is small
 \rightarrow informative



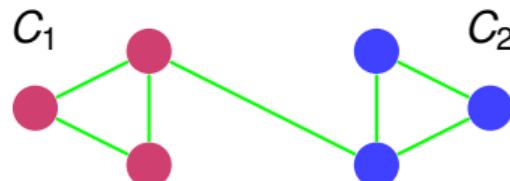
if G^- is bipartite, $\lambda_1(Q^-) = 0$

Geometric-mean Laplacian: decomposing classic Laplacian

Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

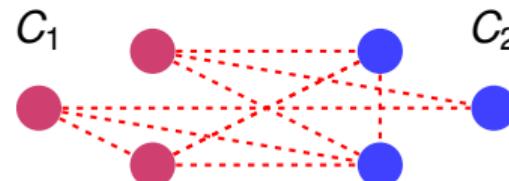
Assortative case

G^+ is well-separated
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 \rightarrow informative



Disassortative case

G^+ is closer to bipartite
 $\rightarrow \lambda_1(Q^-)$ is small
 \rightarrow informative



If G^+ is disconnected, $\lambda_2(L^+) = 0$

if G^- is bipartite, $\lambda_1(Q^-) = 0$

The above intuition also applies to normalized Laplacians:

$$L_{sym}^+ = (D^+)^{-1/2} L^+ (D^+)^{-1/2}$$

$$Q_{sym}^- = (D^-)^{-1/2} Q^- (D^-)^{-1/2}$$

Geometric-mean Laplacian

Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

- ▶ **Definition:** Geometric mean of two matrices A and B

$$A \# B = A^{1/2} (A^{-1/2} B A^{-1/2})^{1/2} A^{1/2}$$

Geometric-mean Laplacian

Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

- ▶ **Definition:** Geometric mean of two matrices A and B

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- ▶ **Geometric-mean Laplacian** (normalized)

$$L_{GM} = L_{sym}^+ \# Q_{sym}^-$$

Geometric-mean Laplacian

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- ▶ **Geometric-mean Laplacian** (normalized)

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- ▶ **Arithmetic-mean Laplacian** (normalized): $L_{AM} = L_{sym}^+ + Q_{sym}^-$

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- ▶ **Geometric-mean Laplacian** (normalized)

$$L_{GM} = L_{sym}^+ \# Q_{sym}^-$$

- ▶ **Arithmetic-mean Laplacian** (normalized): $L_{AM} = L_{sym}^+ + Q_{sym}^-$
- ▶ **Spectral properties:** If x is an eigenvector *shared* by L_{sym}^+ and Q_{sym}^- with eigenvalue λ and μ respectively, then:

$$L_{GM} x = \sqrt{\lambda\mu} x$$

Geometric mean of λ and μ

$$L_{AM} x = (\lambda + \mu) x$$

2 × Arithmetic mean of λ and μ

Geometric-mean Laplacian: motivation

Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

Scalar interpretation

Given two numbers a and b :

- ▶ $a + b$ is small if a and b are small
- ▶ \sqrt{ab} is small if a or b is small

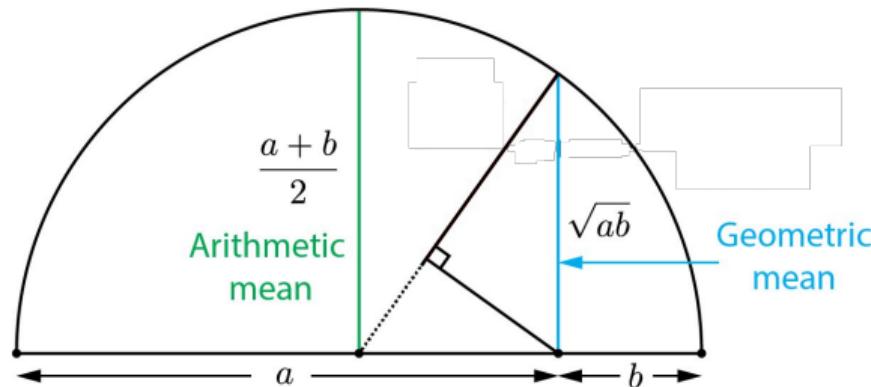


Image source: twitter.com/divbyzero

Geometric-mean Laplacian: motivation

Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

Algorithmic interpretation

- ▶ **Recall:** spectral clustering takes eigenvectors whose corresponding eigenvalues are **small**
- ▶ **Desire:** informative eigenvectors has **small** eigenvalues.

Geometric-mean Laplacian: motivation

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- ▶ **Recall:** spectral clustering takes eigenvectors whose corresponding eigenvalues are **small**
- ▶ **Desire:** informative eigenvectors has **small** eigenvalues.
- ▶ Assume x is an **informative** eigenvector *shared* by L_{AM} and L_{GM}
- ▶ λ and μ are the corresponding eigenvalues of L_{sym}^+ and Q_{sym}^- respectively

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- ▶ **Recall:** spectral clustering takes eigenvectors whose corresponding eigenvalues are **small**
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- ▶ Assume x is an **informative** eigenvector *shared* by L_{AM} and L_{GM}
- ▶ λ and μ are the corresponding eigenvalues of L_{sym}^+ and Q_{sym}^- respectively

To make x among the **top**, L_{AM} and L_{GM} asks different things

- ▶ $L_{AM} x = (\lambda + \mu) x$: **both** λ and μ are small \leftarrow a **stringent** requirement
- ▶ $L_{GM} x = \sqrt{\lambda\mu} x$: **either** λ or μ is small \leftarrow a **loose** requirement

Geometric-mean Laplacian: motivation

Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

Case 1: assortative structures in G^+ \rightarrow small eigenvalue of L_{sym}^+

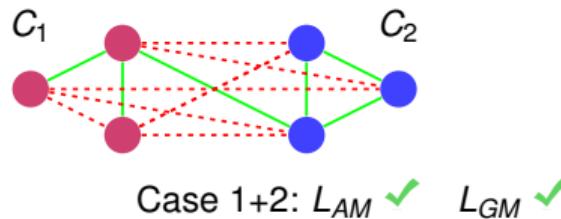
Case 2: disassortative structures in G^- \rightarrow small eigenvalue of Q_{sym}^-

Geometric-mean Laplacian: motivation

Spectral clustering using geometric mean Laplacian (Mercado et al., 2016)

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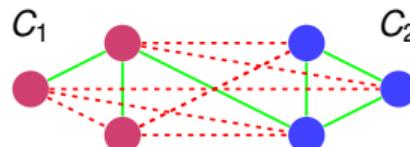


Geometric-mean Laplacian: motivation

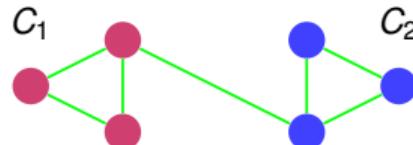
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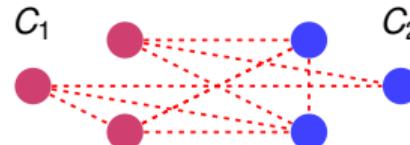
Case 2: disassortative structures in G^- \rightarrow small eigenvalue of Q_{sym}^-



Case 1+2: L_{AM} ✓ L_{GM} ✓



Case 1: L_{AM} ✗ L_{GM} ✓



Case 2: L_{AM} ✗ L_{GM} ✓

Power-mean Laplacian

Spectral clustering of signed graphs via matrix power means (Mercado et al., 2019)

Power mean Laplacian

$$L_p = \left(\frac{L_{sim}^+{}^p + Q_{sim}^-{}^p}{2} \right)^{1/p}$$

where $p \in \mathbb{R}$

p	$m_p(a, b)$	name
$p \rightarrow \infty$	$\max \{a, b\}$	maximum
$p = 1$	$(a + b)/2$	arithmetic mean
$p = 0$	\sqrt{ab}	geometric mean
$p = -1$	$2(\frac{1}{a} + \frac{1}{b})^{-1}$	harmonic mean
$p \rightarrow -\infty$	$\min \{a, b\}$	minimum

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$p \rightarrow -\infty$	$\min \{a, b\}$	minimum

L_p is a **generalization** of L_{AM} and L_{GM}

$$p = 1 \rightarrow L_1 = L_{AM}$$

$$p = 0 \rightarrow L_0 = L_{GM}$$

$p = -1 \rightarrow$ harmonic-mean Laplacian

Power-mean Laplacian: role of the parameter p

Spectral clustering of signed graphs via matrix power means (Mercado et al., 2019)

In one sentence:

- ▶ larger p → **more stringent** clustering recovery condition
- ▶ smaller p → **less stringent** clustering recovery condition

Clustering recovery condition: **informative** eigenvectors have **small** eigenvalues

Power-mean Laplacian: role of the parameter p

Spectral clustering of signed graphs via matrix power means (Mercado et al., 2019)

In one sentence:

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Clustering recovery condition: **informative** eigenvectors have **small** eigenvalues

Extreme cases:

- ▶ $p = \infty$ requires G^+ **AND** G^- are informative
- ▶ $p = -\infty$ requires G^+ **OR** G^- is informative

Power-mean Laplacian: analysis under stochastic block model

Spectral clustering of signed graphs via matrix power means (Mercado et al., 2019)

Definition: Signed Stochastic Block Model (SSBM)

- ▶ k partitions of equal size
- ▶ p_{in}^+ (p_{in}^-): probability of positive (negative) edge **inside** a cluster
- ▶ p_{out}^+ (p_{out}^-): probability of positive (negative) edge **between** clusters

Power-mean Laplacian: analysis under stochastic block model

Spectral clustering of signed graphs via matrix power means (Mercado et al., 2019)

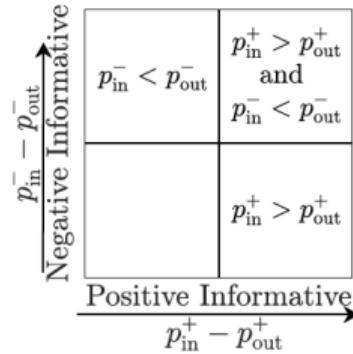
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-
- ▶ G^+ is **informative** if $p_{\text{in}}^+ - p_{\text{out}}^+ > 0$ and is large
assortative case
 - ▶ G^- is **informative** if $p_{\text{in}}^- - p_{\text{out}}^- < 0$ and is small
disassortative case

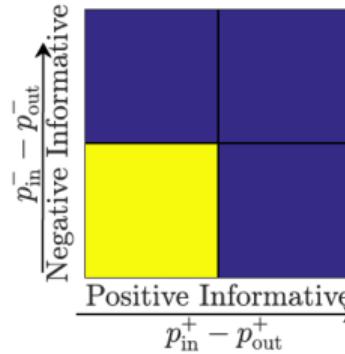
Power-mean Laplacian: recovery probability under SSBM

$p = \infty$ or $-\infty$

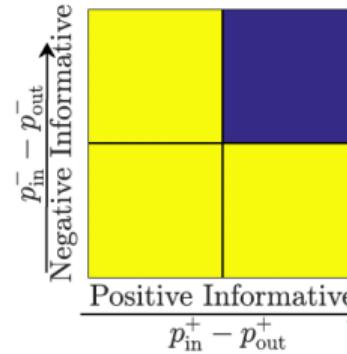
Spectral clustering of signed graphs via matrix power means (Mercado et al., 2019)



(a) SBM Diagram



(b) $L_{-\infty}$ (OR)



(c) L_∞ (AND)

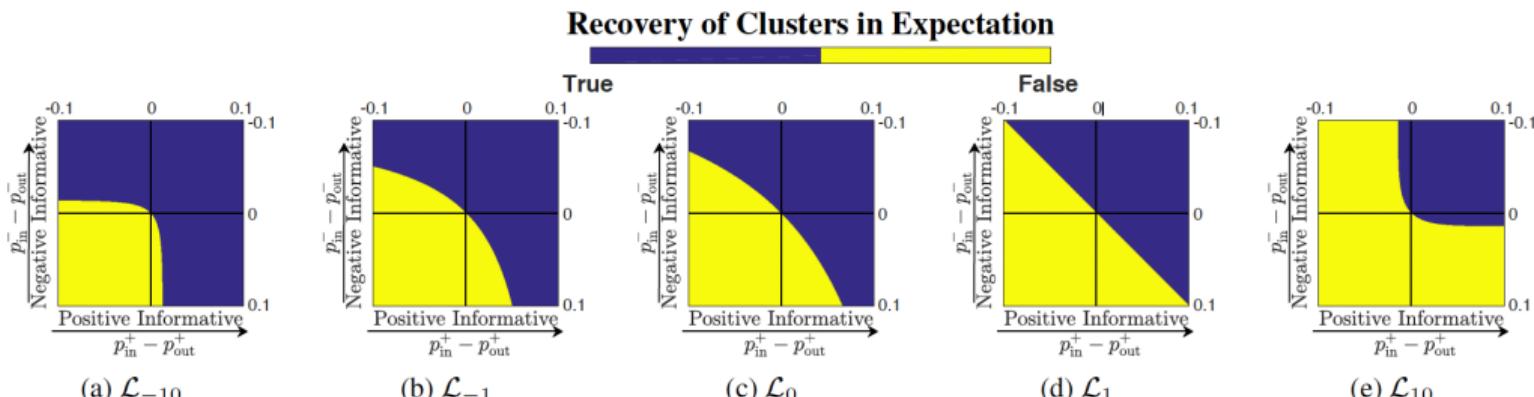
Recovery of Clusters in Expectation



The larger the blue areas, the more likely to recover the ground-truth

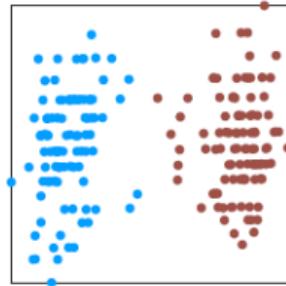
Power-mean Laplacian: recovery probability under SSBM

Spectral clustering of signed graphs via matrix power means (Mercado et al., 2019)

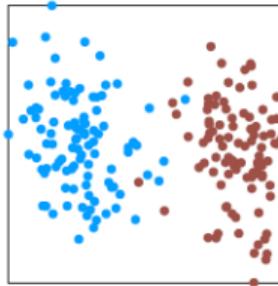


Power-mean Laplacian: node embedding example ($k = 2$)

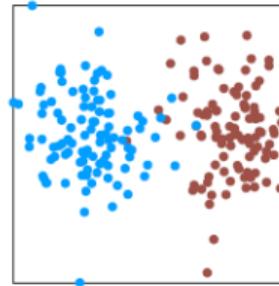
Spectral clustering of signed graphs via matrix power means (Mercado et al., 2019)



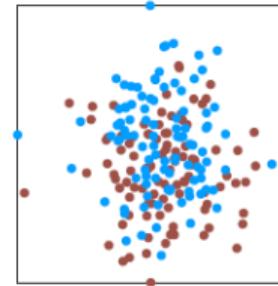
(c) L_{-10}



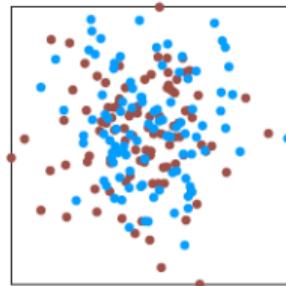
(d) L_{-1}



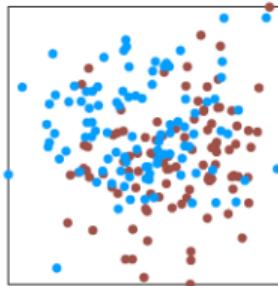
(e) L_0



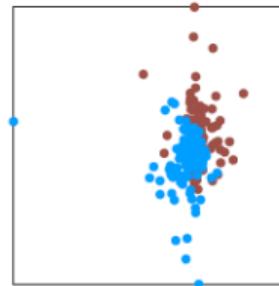
(f) L_1



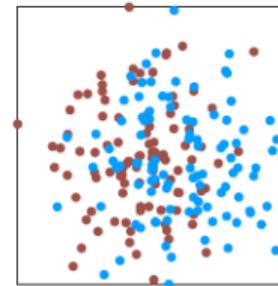
(g) L_{10}



(h) L_{SN}



(i) L_{BN}



(j) H

Power-mean Laplacian: computing eigenvectors

Spectral clustering of signed graphs via matrix power means (Mercado et al., 2019)

- ▶ Materializing the power-mean Laplacian can result in a **dense** matrix not scalable to large graphs
- ▶ Without conducting the above step, the **leading** eigenvectors can be computed **efficiently**
- ▶ The algorithm combines the power method with a Krylov subspace approximation technique (Mercado et al., 2018)

Graph partitioning: summary

- ▶ **Main focus:** spectral methods and the choice of Laplacian matrix
 - ▶ Arithmetic-mean Laplacian
 - ▶ Geometric-mean Laplacian
 - ▶ Power-mean Laplacian (a generalization of the above)
- ▶ **Uncovered topics:**
 - ▶ Probabilistic methods (Jiang, 2015; Yang et al., 2017)
 - ▶ Modularity-based methods (Gómez et al., 2009; Traag and Bruggeman, 2009; Li et al., 2014)
 - ▶ Ad-hoc methods (Chiang et al., 2012; Wu et al., 2016a)

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Correlation clustering

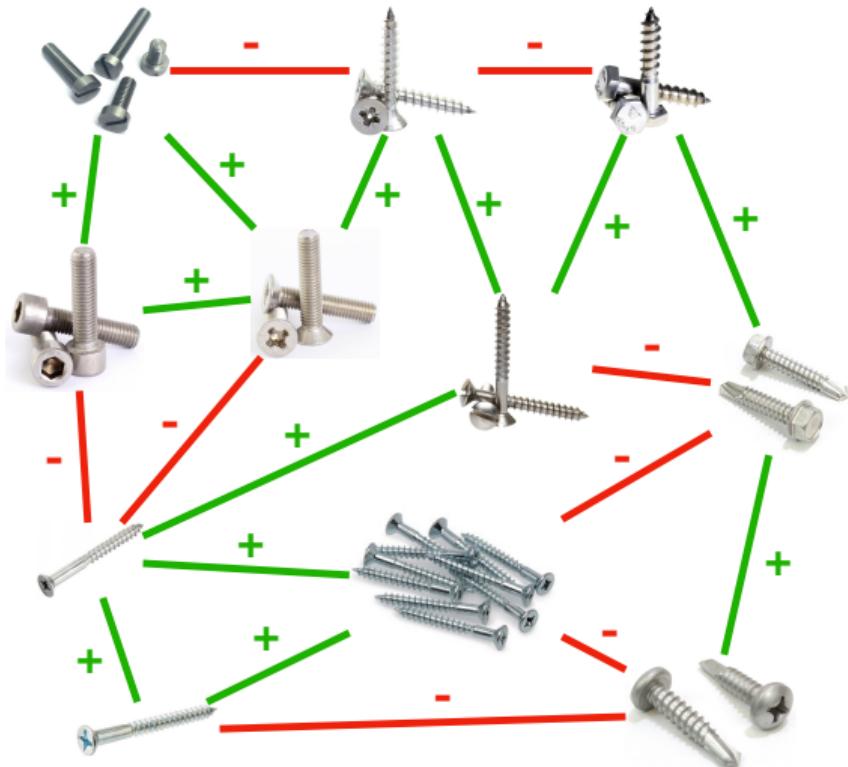
Data clustering — background

- ▶ Data clustering: a fundamental problem in machine learning
- ▶ Intuitively: we want to partition a dataset into clusters so that similar objects are assigned to the same cluster
- ▶ Extensively-studied problem, many different settings, objectives, applications
- ▶ Euclidean setting: data are represented as Euclidean points
 - ▶ minimize an objective function such as **k -means** ($\sum_i \min_j \|x_i - c_j\|_2^2$), **k -median** ($\sum_i \min_j \|x_i - c_j\|_2$) or **k -center** ($\max_i \min_j \|x_i - c_j\|_2$)
- ▶ Graph setting: data are represented as a graph
 - ▶ edges represent affinity, e.g., friends in a social network
 - ▶ often a similarity value is available, e.g., connection strength
 - ▶ optimize an objective function such as **normalized edge cut** across clusters (minimize) or **edge density** within clusters (maximize)

Correlation clustering — motivation

- ▶ In the graph setting described above, edges are positive
 - ▶ presence of an edge suggests that nodes should be clustered together
 - ▶ absence of an edge suggests that nodes should be assigned to different clusters
- ▶ In some cases, we may have a local prediction whether two objects should be assigned to the same cluster or not
 - ▶ positive edge : the two objects should be clustered together
 - ▶ negative edge : the two objects should be assigned to different clusters
 - ▶ no edge : no information
- ▶ We obtain a signed network !

Correlation clustering — motivation

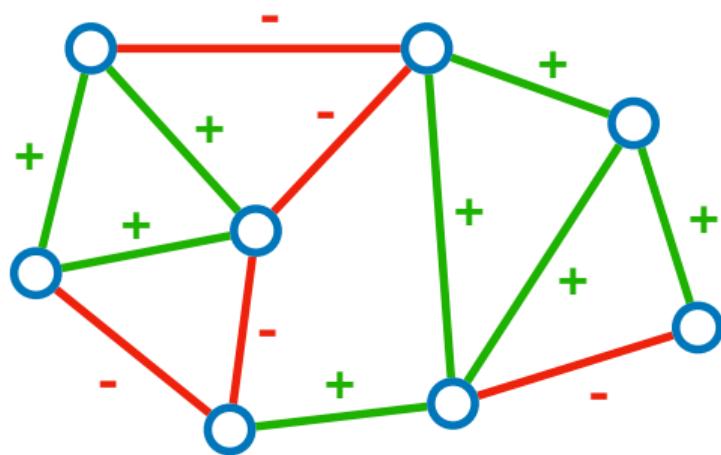


- ▶ **example:** a dataset of images, e.g., screws of different types
- ▶ a machine-learning program, which, given two images, outputs whether the images depict the same type of screws
- ▶ difficult classification task, we expect errors to occur
- ▶ we obtain a signed network
- ▶ we want to cluster the images so that same-type screws are assigned in the same cluster

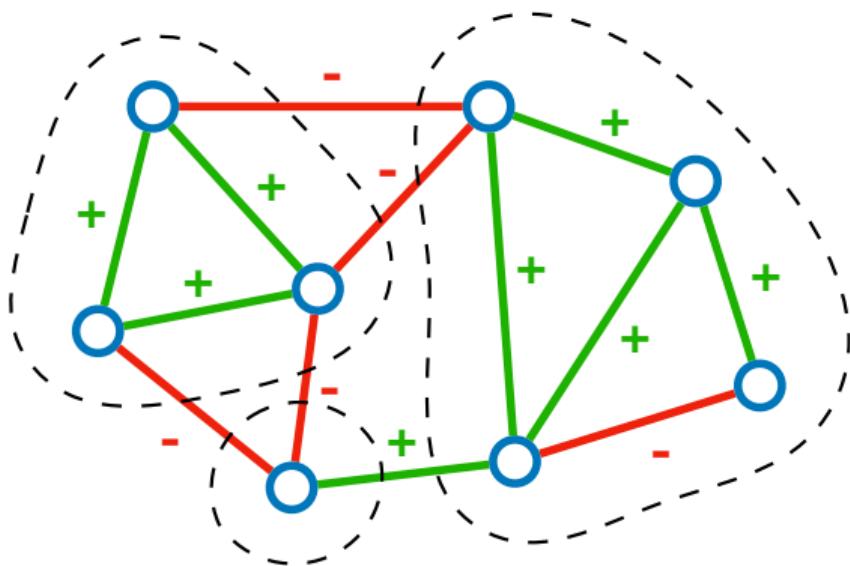
Correlation clustering — motivation

- ▶ Given a signed network, we want to find a clustering of nodes, so that:
 - ▶ all edges **within** clusters are **positive**
 - ▶ all edges **across** clusters are **negative**
- ▶ Due to noise in the data and classification errors in the network construction, we cannot expect to achieve perfect agreement
- ▶ We need an objective function to capture the consistency of the resulting clustering with the input signed network

Correlation clustering — edge agreements and disagreements



Correlation clustering — edge agreements and disagreements



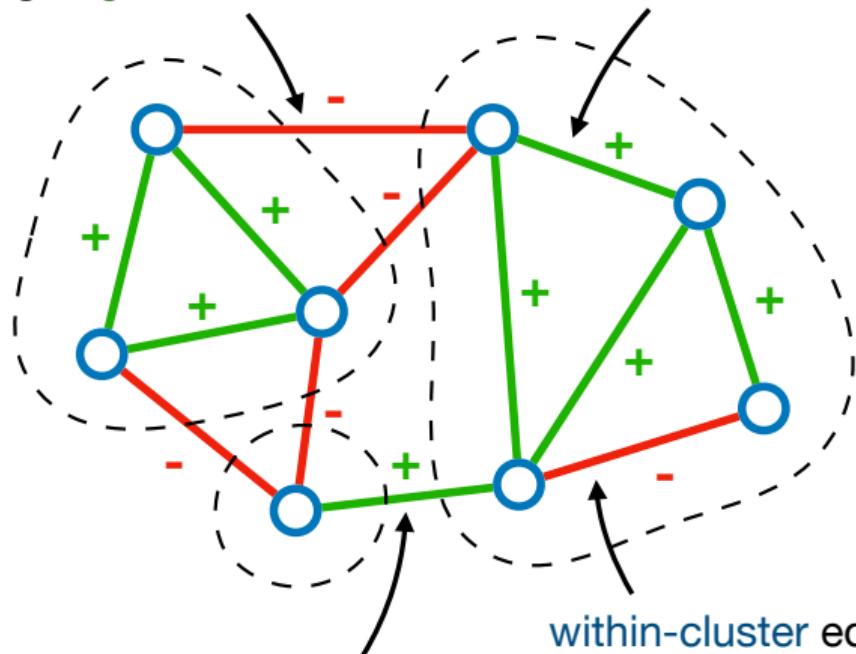
Correlation clustering — edge agreements and disagreements

across-cluster edge agreement

within-cluster edge agreement

within-cluster edge disagreement

across-cluster edge disagreement



Correlation clustering — problem formulation

Given a signed network $G = (V, E^+, E^-)$, find a partitioning $\mathcal{C} = \{C_1, \dots, C_k\}$ of the graph vertices (i.e., $\bigcup_{i=1}^k C_i = V$ and $C_i \cap C_j = \emptyset$, for all $i \neq j$), so as to

Variant 1 : [Maximize agreements]

$$\max \quad a(\mathcal{C}) = \sum_{i,j} \mathbb{I}\{(i,j) \in E^+\} \mathbb{I}\{c(i) = c(j)\} + \sum_{i,j} \mathbb{I}\{(i,j) \in E^-\} \mathbb{I}\{c(i) \neq c(j)\}$$

Variant 2 : [Minimize disagreements]

$$\min \quad d(\mathcal{C}) = \sum_{i,j} \mathbb{I}\{(i,j) \in E^+\} \mathbb{I}\{c(i) \neq c(j)\} + \sum_{i,j} \mathbb{I}\{(i,j) \in E^-\} \mathbb{I}\{c(i) = c(j)\}$$

Correlation clustering — problem formulation

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Variant 2 : [Minimize disagreements]

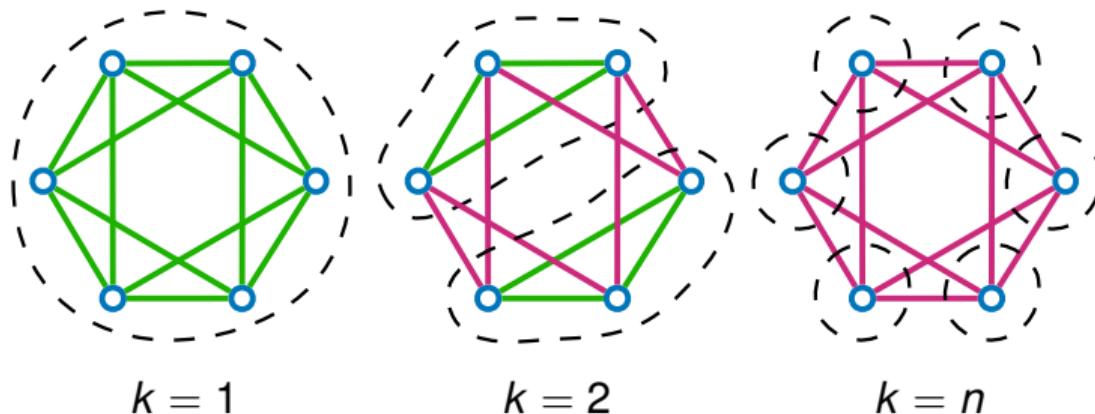
$$\min \quad d(\mathcal{C}) = \sum_{i,j} \mathbb{I}\{(i,j) \in E^+\} \mathbb{I}\{c(i) \neq c(j)\} + \sum_{i,j} \mathbb{I}\{(i,j) \in E^-\} \mathbb{I}\{c(i) = c(j)\}$$

Majority of research focuses on the minimization variant.

Correlation clustering — number of clusters

An important observation

- ▶ the problem formulation **does not (need to)** specify the number of clusters
- ▶ optimal k depends on input network, and does not have trivial minimizers
e.g.,



- ▶ the optimal solution in each of the above cases is the most intuitive one

Correlation clustering — hardness

Both formulations ([max-agree](#) and [min-disagree](#)) are **NP-hard**

The [min-disagree](#) problem is

- ▶ **NP-hard** for complete unweighted graphs
reduction from “partition into triangles” Bansal et al. (2004)
- ▶ **APX-hard** for general (un)weighted graphs
reduction from multiway cut Demaine et al. (2006)

Correlation clustering — approximation algorithms

- ▶ An algorithm ALG is a **c**-approximation algorithm, if for any problem instance I

$$ALG \leq c \cdot OPT$$

c is the **approximation ratio**

- ▶ If ALG is **randomized**, we require that for any problem instance I

$$E[ALG] \leq c \cdot OPT$$

Correlation clustering — existing approximation algorithms

Overview of results for the min-disagree problem

Paper	graph type	approximation ratio	deterministic /randomized	running time
Bansal et al. (2004)	complete	large constant	deterministic	$\mathcal{O}(n^2)$
Demaine et al. (2006)	general	$\mathcal{O}(\log n)$	deterministic	LP
Ailon et al. (2005)	complete	2.5	randomized	LP
Ailon et al. (2005)	complete	3	randomized	$\mathcal{O}(m)$
Chawla et al. (2015)	complete	$2.06 - \epsilon$	deterministic	LP
Giotis and Guruswami (2005) ¹	complete	PTAS	randomized	combinatorial
Coleman et al. (2008) ²	complete ³	2	deterministic	combinatorial

¹ For fixed k ; Recall that the problem is **APX**-hard when k is not fixed

² For $k = 2$ (2-correlation-clustering)

³ Algorithm applicable to general graphs, but analysis for complete graphs

Correlation clustering — the KWIKCLUSTER (or PIVOT) algorithm

KWIKCLUSTER($G = (V, E^+, E^-)$)

If $V = \emptyset$ then return \emptyset
Pick random pivot $i \in V$.
Set $C = \{i\}$, $V' = \emptyset$.

For all $j \in V, j \neq i$:
 If $(i, j) \in E^+$ then
 Add j to C
 Else (If $(i, j) \in E^-$)
 Add j to V'

Let G' be the subgraph induced by V' .

Return $C \cup \text{KWIKCLUSTER}(G')$.

Correlation clustering — the KWIKCLUSTER (or PIVOT) algorithm

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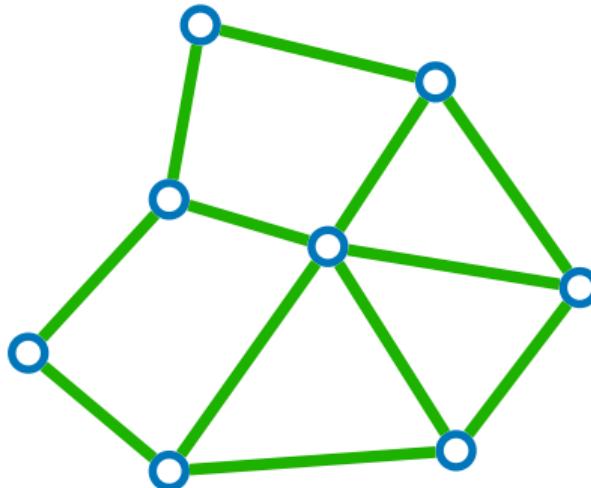
Return $C \cup \text{KWIKCLUSTER}(G')$.

► The PIVOT algorithm

Ailon et al. (2005)

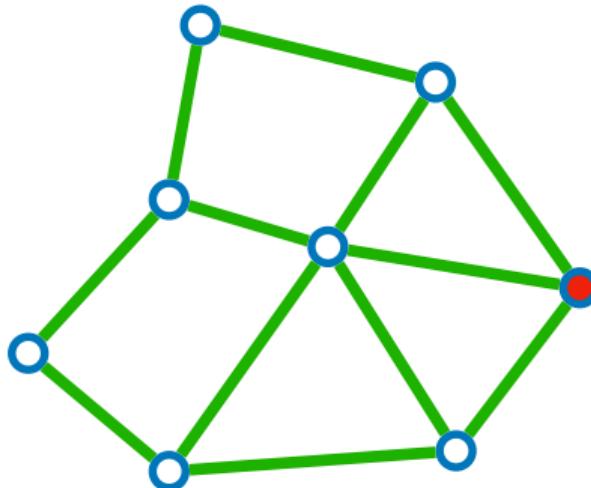
- + An elegant randomized algorithm
- + Approximation ratio 3
- + Running time $\mathcal{O}(m)$
- It assumes a complete graph
- It assumes an unweighted graph

The PIVOT algorithm — example



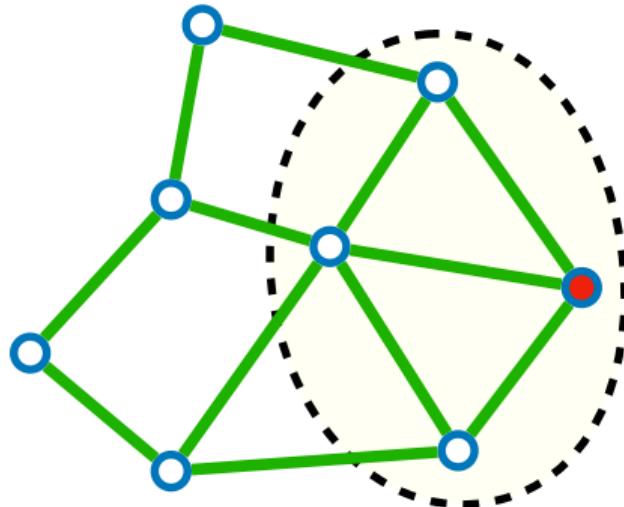
A complete graph: positive edges shown, negative edges not shown

The PIVOT algorithm — example



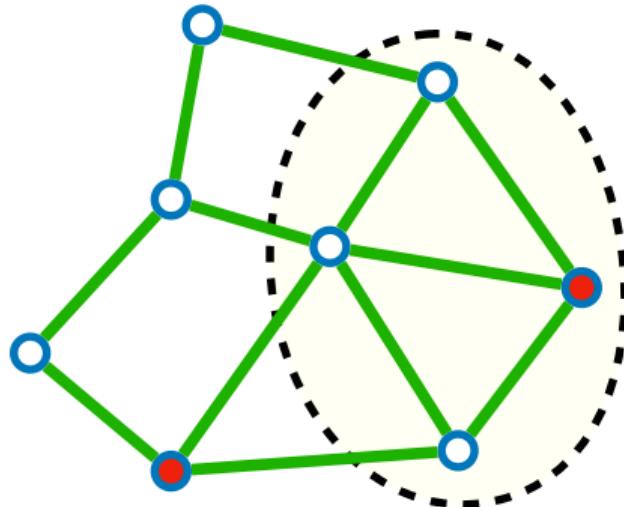
a pivot is selected uniformly at random

The PIVOT algorithm — example



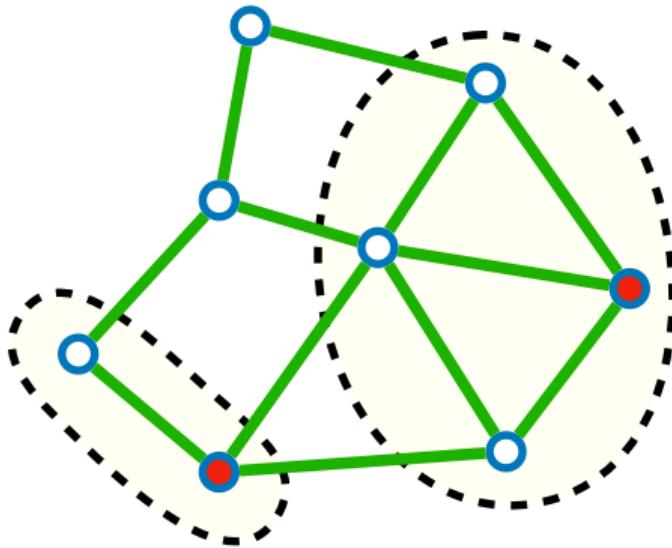
a cluster is formed with the pivot and all its positive neighbors

The PIVOT algorithm — example



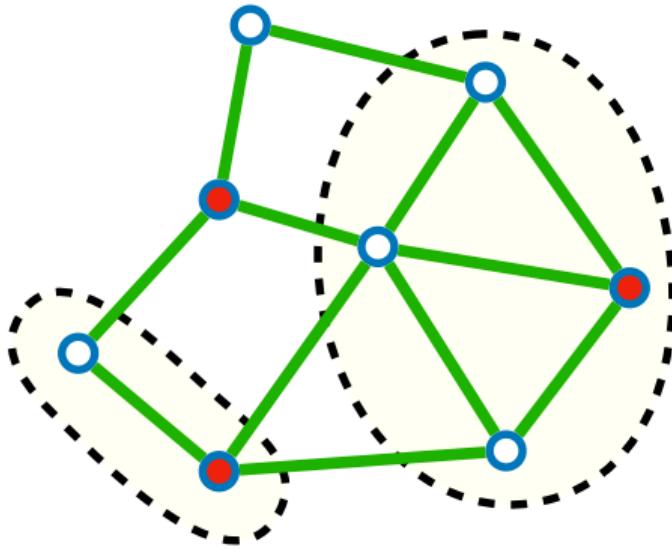
a new pivot is selected from the remaining of the graph vertices

The PIVOT algorithm — example



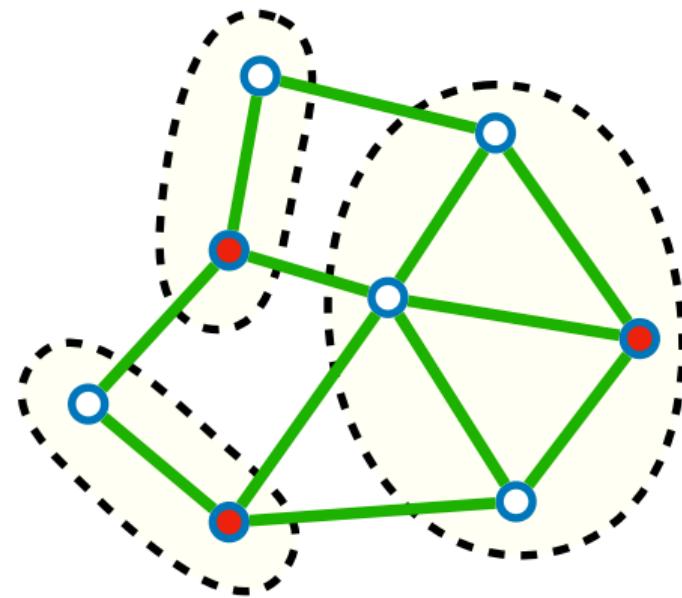
a second cluster is formed with the pivot and all its positive neighbors

The PIVOT algorithm — example



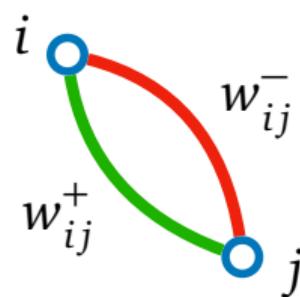
and the process continues ...

The PIVOT algorithm — example



... until the whole graph is consumed.

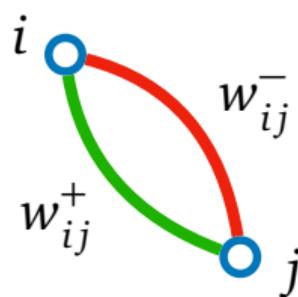
Weighted signed networks



We want to extend the methods to **weighted** signed networks
 $G = (V, w^+, w^-)$

- ▶ w_{ij}^+ : weight of positive edge (i, j)
- ▶ w_{ij}^- : weight of negative edge (i, j)
- ▶ unweighted case : $w_{ij}^+, w_{ij}^- \in \{0, 1\}$
- ▶ **weighted case** : $w_{ij}^+, w_{ij}^- \in \mathbb{R}_{\geq 0}$

Weighted signed networks



We want to extend the methods to **weighted** signed networks
 $G = (V, w^+, w^-)$

- ▶ w_{ij}^+ : weight of positive edge (i, j)
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- ▶ unweighted case : $w_{ij}^+, w_{ij}^- \in \{0, 1\}$
- ▶ **weighted case** : $w_{ij}^+, w_{ij}^- \in \mathbb{R}_{\geq 0}$

Interesting cases :

- ▶ **probability constraints** : $w_{ij}^+ + w_{ij}^- = 1$, for all $i, j \in V$
- ▶ **triangle inequality** : $w_{ik}^- \leq w_{ij}^- + w_{jk}^-$, for all $i, j, k \in V$

The PIVOT algorithm on weighted signed networks

1. Consider a weighted signed networks $G = (V, w^+, w^-)$
2. Assume probability constraints $w_{ij}^+ + w_{ij}^- = 1$, for all $i, j \in V$
3. Form unweighted $G_u = (V, E^+, E^-)$ by taking “majority” on each edge
4. Apply PIVOT on G_u
5. Return solution of PIVOT on G_u , as the solution for G

Theoretical properties of the above algorithm

- ▶ 5 approximation, with probability constraints
- ▶ 2 approximation, with probability constraints and triangle inequality

Using PIVOT for LP rounding

LP relaxation

Ailon et al. (2005)

$$\text{maximize} \quad \sum_{ij} \left(x_{ij}^+ w_{ij}^- + x_{ij}^- w_{ij}^+ \right)$$

$$\text{such that } x_{ik}^- \leq x_{ij}^- + x_{jk}^-, \text{ for all } i, j, k \in V$$

$$x_{ij}^+ + x_{ij}^- = 1, \text{ for all } i, j \in V$$

$$x_{ij}^+, x_{ij}^- \geq 0, \text{ for all } i, j \in V$$

- ▶ Notice that if $x_{ij}^- \in \{0, 1\}$, then x_{ij}^- define an equivalence class (clustering)

Using PIVOT for LP rounding

LP-KWIKCLUSTER(V, x^+, x^-)

A recursive algorithm for rounding the LP for weighted CORRELATION-CLUSTERING. Given an LP solution $x^+ = \{x_{ij}^+\}_{i < j}$, $x^- = \{x_{ij}^-\}_{i < j}$, returns a clustering of the vertices

If $V = \emptyset$ then return \emptyset

Pick random pivot $i \in V$.

Set $C = \{i\}, V' = \emptyset$.

For all $j \in V, j \neq i$:

With probability x_{ij}^+

Add j to C .

Else (With probability $x_{ij}^- = 1 - x_{ij}^+$)

Add j to V' .

Return clustering

$\{C\} \cup \text{LP-KWIKCLUSTER}(V', x^+, x^-)$.

Ailon et al. (2005)

1. Solve the LP relaxation
2. Use the PIVOT for randomized rounding of the LP solution
 - ▶ 2.5-approximation, with probability constraints
 - ▶ 2-approximation, with probability & triangle inequality constraints
 - Expensive; requires solving an LP

Correlation clustering — summary

- ▶ Signed graphs have been studied in theoretical computer science in the context of [correlation clustering](#)
- ▶ A wealth of theoretical results for different problem settings
- ▶ Several applications, e.g., clustering aggregation
- ▶ Many other problem variants not discussed here
 - overlapping, on-line, bipartite, chromatic, local, ...
- ▶ See also KDD tutorial by Bonchi et al. (2014)

Outline

Introduction

Theory of signed networks

Problems and applications

Subgraph mining

Graph partitioning

Correlation clustering

Link prediction and link classification

Network dynamics

Graph embedding and representation learning

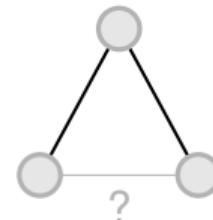
Conclusions and future directions

Link prediction and link classification

Link prediction and link classification

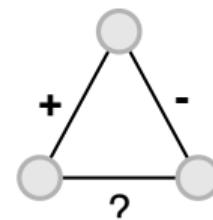
In **unsigned** networks:

- ▶ **Link prediction:**
 - ▶ Is an unobserved edge likely to exist?
 - ▶ Is a non-existing edge likely to materialize?



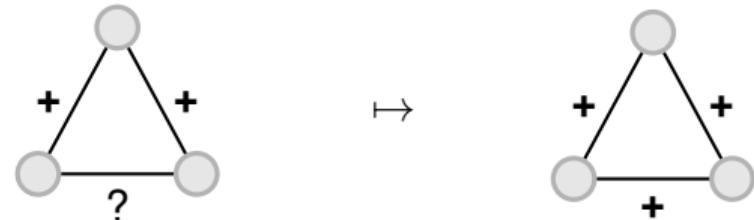
In **signed** networks:

- ▶ **Link prediction:**
 - ▶ Is an unobserved edge of a **particular sign** likely to exist?
- ▶ **Link classification:**
 - ▶ What **sign** an existing edge is likely to have?
Task also known as **sign prediction** or **label prediction**



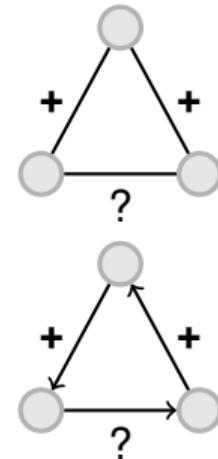
Link classification

Methods based on **balance**.

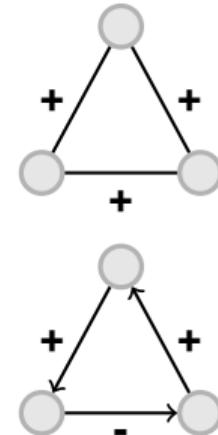


Link classification

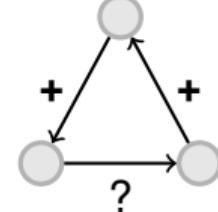
Methods based on **balance**.



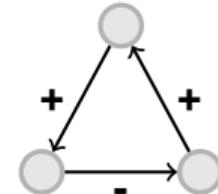
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Methods based on **status**.

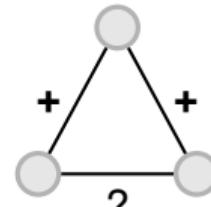


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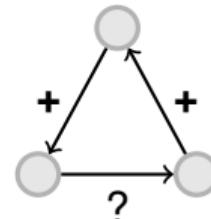


Link classification

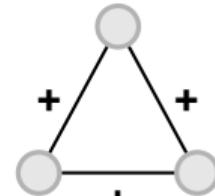
Methods based on **balance**.



→



→



Methods based on **status**.

Methods based on **machine learning**.

$$G \mapsto x \in \mathbb{R}^d, f : \mathbb{R}^d \rightarrow \{-, +\}.$$

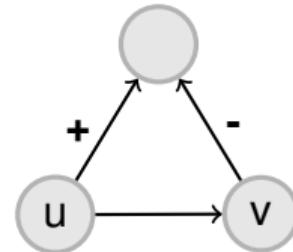
Link classification

Predicting positive and negative links in online social networks (Leskovec et al., 2010)

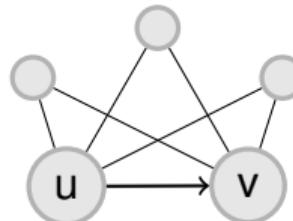
Task: predict sign of (u, v) .

Feature extraction:

- ▶ In & out-degree: $d_{in}^+(u), d_{in}^-(v), \dots$
- ▶ Count vector of directed triads:
Forward/Backward, +/-.
- ▶ Embeddedness (number of common
neighbours of u, v).



FBpn triad



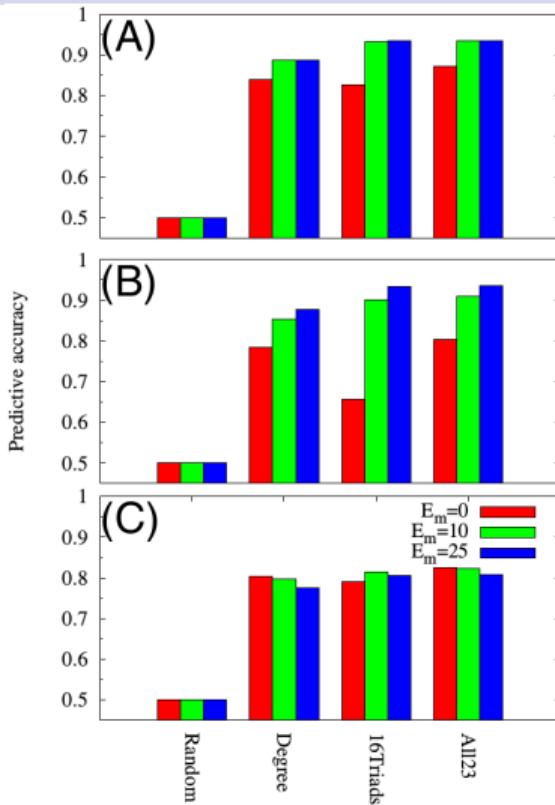
Embeddedness = 3

Link classification

Predicting positive and negative links in online social networks (Leskovec et al., 2010)

	Epinions	Slashdot	Wikipedia
Nodes	119,217	82,144	7,118
Edges	841,200	549,202	103,747
+ edges	85.0%	77.4%	78.7%
- edges	15.0%	22.6%	21.2%

Balance	Epinions	Slashdot	Wikipedia
Netwok	0.8344	0.8105	0.7809
Permuted	0.8562	0.7779	0.7866
Rewired	0.8993	0.8310	0.8316
Status	Epinions	Slashdot	Wikipedia
Network	0.7905	0.8221	0.8538
Permuted	0.7241	0.7568	0.7767
Rewired	0.6377	0.6644	0.6321

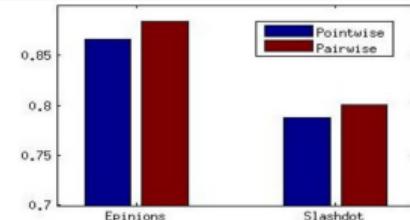


Link classification

Link label prediction in signed social networks (Agrawal et al., 2013)

Given incomplete $n \times n$ adjacency matrix A , learn $X = UV$ to minimize

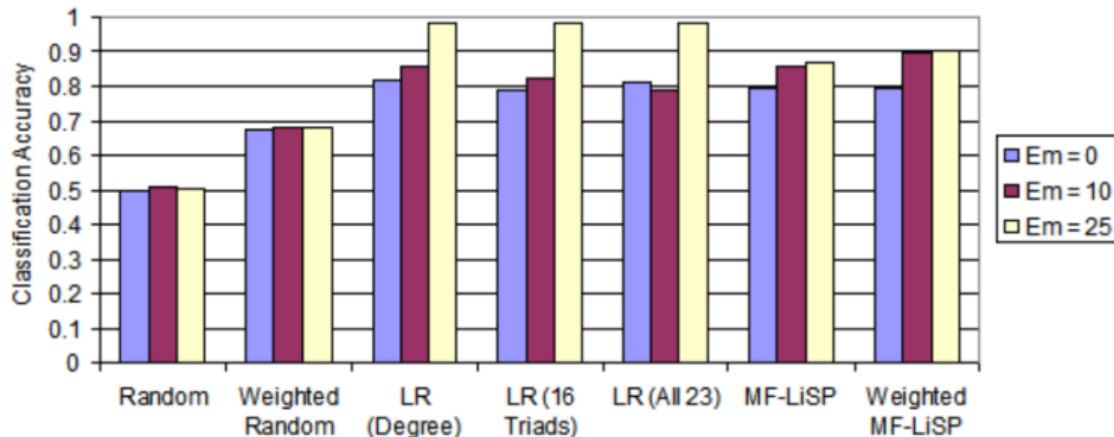
$$\frac{1}{n} \sum_{i=1}^n \frac{1}{|Q_i|} \sum_{(j,k) \in Q_i} ((A_{ij} - A_{ik}) - (X_{ij} - X_{ik}))_+ + \frac{\lambda}{2} \|U\|_F^2 \|V\|_F^2$$



Pairwise vs. pointwise

Wiki-Vote

Better for edges of low embeddedness
(Leskovec et al., 2010).



Link classification

Exploiting longer cycles for link prediction in signed networks (Chiang et al., 2011)

Measure of imbalance: $\mu_k(G) = \sum_{i=3}^k \beta_i \sum_{\sigma \in Cycles_i} \mathbb{I}\{\sigma \text{ is balanced}\}$

Prediction:

$$sign(i, j) = sign(\mu_k(G \cup (i, j, -)) - \mu_k(G \cup (i, j, +)))$$

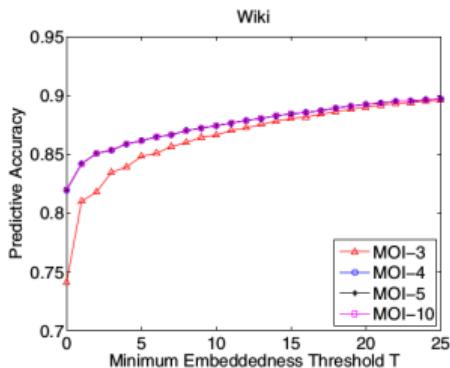
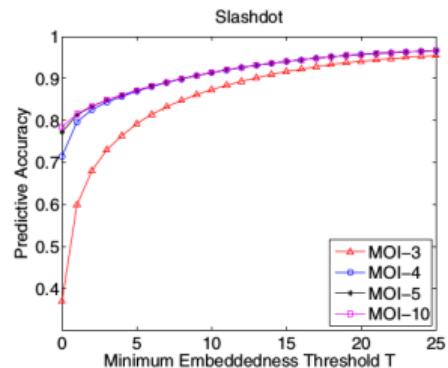
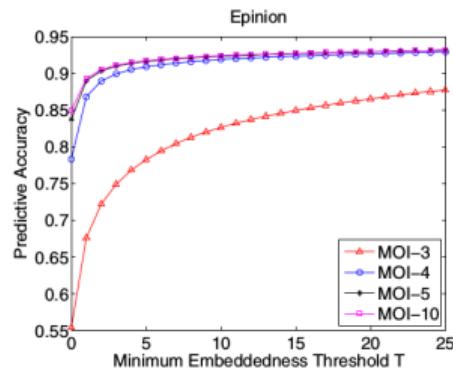
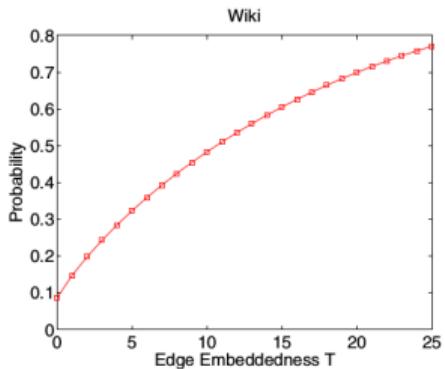
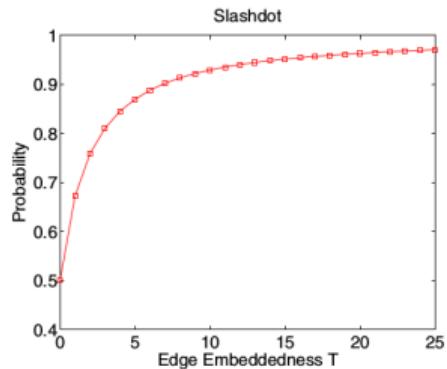
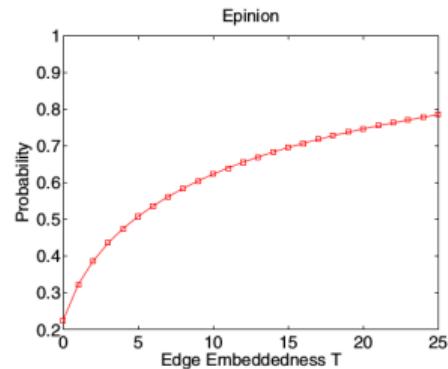
Key insight:

$$sign(\mu_k(G \cup (i, j, -)) - \mu_k(G \cup (i, j, +))) = sign\left(\sum_{t=3}^k \beta_t A_{i,j}^{t-1}\right)$$

So we just need to compute powers of the adjacency matrix!

Two predictors: balance-based and feature based.

Link classification



Learning bounds for link classification

- ▶ Link classification methods discussed in previous slides focus on empirical performance
- ▶ Link classification has also been studied in the context of computational learning theory
- ▶ The goal is to derive theoretical bounds on the number of mistakes a learning algorithm can make
- ▶ Choice of computational learning model is important
- ▶ Several different models have been studied

Link classification in computational learning theory

- ▶ The link-classification task, at high level:
 1. Observe a graph $G = (V, E)$ and the signs of a subset $E_0 \subseteq E$
 2. Build model $f : E \rightarrow \{-1, 1\}$
 3. Use f to predict the signs of the rest of the edges $E \setminus E_0$
- ▶ **Goal:** bound the number of mistakes on $E \setminus E_0$
- ▶ **Questions:** What are the assumptions to ensure a small number of mistakes?
Why observing a subset of edges allows to predict the rest of the edges?
- ▶ **Observation:** Predictions are easy on a perfectly balanced graph
The closer is a graph to be balanced, the easier is to make predictions

Mistake bounds for link classification

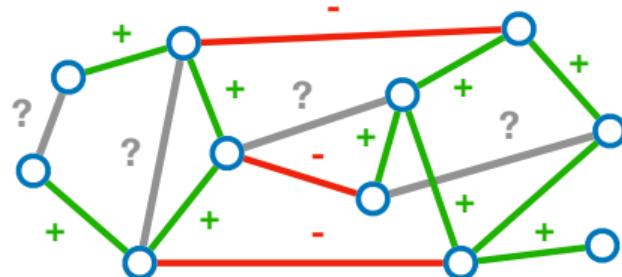
- ▶ Consider: What should a mistake bound depend on?
 $|V|$, $|E|$, $|E_0|$, but also ... a measure of balance of G
- ▶ such a measure is $\Delta(G)$: cost of optimal correlation clustering of G
e.g., $\Delta(G) = 0$ if G is perfectly balanced

Why? What is the intuition?

Mistake bounds for link classification

- ▶ Consider: What should a mistake bound depend on?
 $|V|$, $|E|$, $|E_0|$, but also ... a measure of balance of G
- ▶ such a measure is $\Delta(G)$: cost of optimal correlation clustering of G
e.g., $\Delta(G) = 0$ if G is perfectly balanced

Why? What is the intuition?



- ▶ small $\Delta(G)$ indicates clear community structure
- ▶ a small set of edges E_0 suffices to reveal the community structure
- ▶ rest of edges $E \setminus E_0$ can be predicted, and small $\Delta(G)$ ensures few mistakes

Learning algorithm

1. Observe signs on a subset of edges $E_0 \subseteq E$
2. Solve **correlation-clustering problem** on $G = (V, E_0)$ to obtain solution f
3. Use solution f to predict the rest of the edges

Learning algorithm

1. Observe signs on a subset of edges $E_0 \subseteq E$
2. Solve correlation-clustering problem on $G = (V, E_0)$ to obtain solution f
3. Use solution f to predict the rest of the edges

(Assumptions) If

- ▶ E_0 is a random subset of E
- ▶ correlation clustering on $G = (V, E_0)$ can be approximated within factor α
- ▶ (only for simplicity of exposition, $|E_0| = m/2$)

Learning algorithm

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(Assumptions) If

- ▶ E_0 is a random subset of E
- ▶ correlation clustering on $G = (V, E_0)$ can be approximated within factor α
- ▶ (only for simplicity of exposition, $|E_0| = m/2$)

then, the number of mistakes on $E \setminus E_0$ can be bounded by $\Delta\alpha + \sqrt{|E||V|\log|V|}$

- ▶ An application of a result by El-Yaniv and Pechyony (2009)
- ▶ Recall $\alpha = \mathcal{O}(\log|V|)$ (Demaine et al., 2006)

- ▶ As before, but learner selects subset $E_0 \subseteq E$
- ▶ Adversary returns labeling on E_0
- ▶ Learner makes predictions on $E \setminus E_0$

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- ▶ Learner makes predictions on $E \setminus E_0$

One particular case of interest, *p*-random model :

- ▶ The observed labeling is obtained by a graph that has 2 clusters and $\Delta = 0$, and where each edge is flipped with probability *p*

Active-learning algorithm

1. Select a spanning tree $T = (V_T, E_T)$, and query the labels of E_T
2. Predict label of $(u, v) \in E \setminus E_0$ as the product of signs on $\text{path}_T(u, v)$

Active-learning algorithm

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Number of mistakes :

$$M_T \leq |E_{\text{flip}}| + \sum_{e' \in E \setminus E_T} \sum_{e \in E_T} \mathbb{I}\{e \in \text{path}_T(e')\} \mathbb{I}\{e \in E_{\text{flip}}\}$$

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- ▶ M_T related to average stretch of the spanning tree T ,
which we can approximate within $\mathcal{O}(\log^2 |V| \log \log |V|)$ (Elkin et al., 2008)

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- ▶ M_T related to average stretch of the spanning tree T ,
which we can approximate within $\mathcal{O}(\log^2 |V| \log \log |V|)$ (Elkin et al., 2008)

Thus,

$$\mathbb{E}[M_T] \leq p|E| \mathcal{O}(\log^2 |V| \log \log |V|)$$

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Network dynamics

Network Dynamics in Signed Graphs

Most works in this area model the evolution of social networks to achieve balance

- ▶ A large network is almost surely **imbalanced**
- ▶ Gradual evolution to a more **stable state**
- ▶ Social dynamics → tendency to reduce imbalanced triads

Possible outcomes of dynamics:

- ▶ "Paradise" state: **all positive** signature
- ▶ Bipolar state: two antagonistic factions where all interior edges are **positive** and all exterior **negative**
- ▶ "Jammed" state: system is trapped in local optimum

Analysis often relies on statistical physics. Therefore many works consider complete graphs

T. Antal, P. L. Krapivsky, S. Redner: "Dynamics of social balance on networks" (2005)

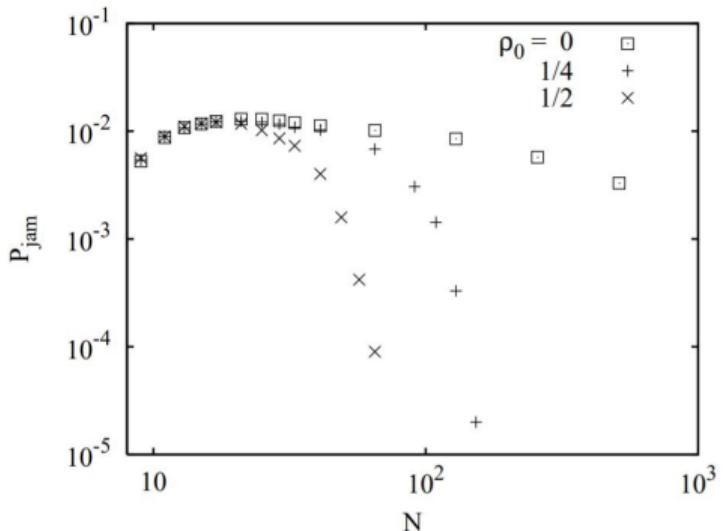
- ▶ Local dynamics: Pick random unbalanced triad and flip sign of one link



- ▶ New imbalances → interaction **cascade**
- ▶ Infinite network:
 1. For $p < 1/2$ **quasistationary** dynamics
 2. For $p \geq 1/2$ **paradise** state
- ▶ Finite network:
 1. For $p < 1/2$ **bipolar** state
 2. For $p \geq 1/2$ **paradise** state

Constrained triad dynamics

- ▶ Select random link
 1. flip sign if # of imbalanced triads **decreases**
 2. flip w.p. 1/2 if # of imbalanced triads **stays the same**
 3. don't flip if # of imbalanced triads **increases**
- ▶ Corresponds to Ising model with a three-spin interaction between links of triad
- ▶ In $\ln N$ time reaches balanced state or gets trapped in jammed state



Probability of reaching a jammed state P_{jam} as a function of N for several values of ρ_0 .

Image

source: Antal et al.

Marvel S., Strogatz S., Kleinberg J.: "Energy Landscape of Social Balance" (2008)

Investigate Jammed states introduced by Antal et al.

Definition (Energy)

The energy of a social network is defined as:

$$U = -\frac{1}{\binom{n}{3}} \sum s_{ij} s_{jk} s_{ik}$$

- ▶ Product of edge signs is **positive** for balanced triangles, **negative** for unbalanced
- ▶ The "correction" of an edge sign to achieve balance **relaxes** the energy of the system

Contributions:

- ▶ Proof that jammed states at most halfway up the energy spectrum
- ▶ Show that jammed states have a natural modular structure

K. Kulakowski, P. Gawronski, P. Gronek: "The Heider balance - a continuous approach"(2005)

Real numbers in $[-R, R]$ instead of ± 1 to describe social distance

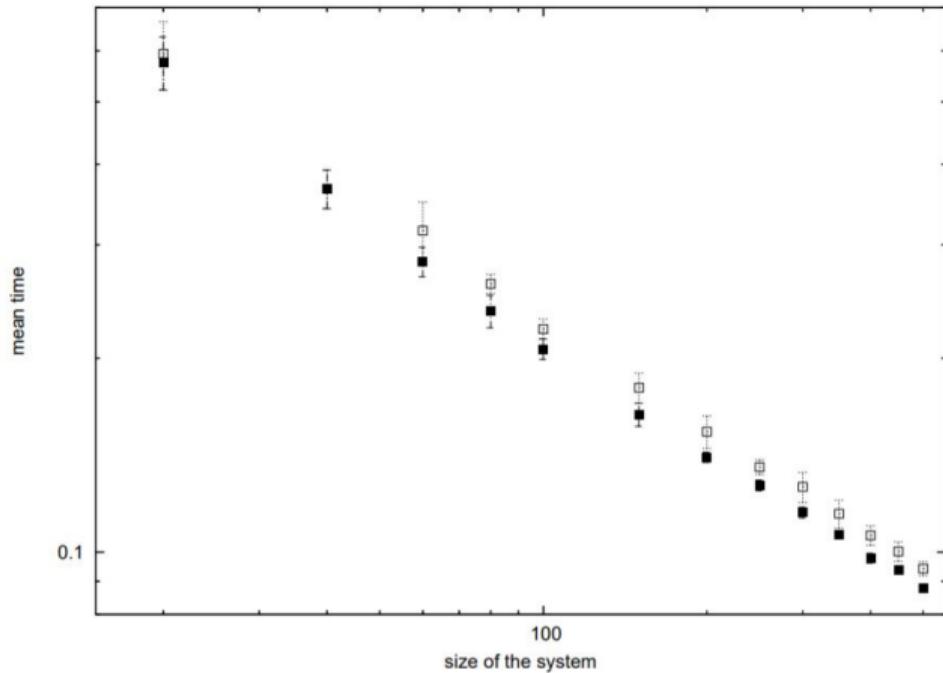
- ▶ Represent a completely connected social network using $n \times n$ matrix X of edge strengths
- ▶ Dynamics expressed in terms of differential equations

Definition (Evolution of social relationships)

$$\frac{dX}{dt} = X^2 \quad \iff \quad \frac{dx_{ij}}{dt} = \sum_k x_{ik}x_{kj}$$

- ▶ x_{ij} pushed in a **positive** or **negative** direction so as to achieve balance in the triangle i, j, k
- ▶ Converges to either **paradise state** or to two **opposing factions**

Reaching Heider balance



Time to reach HB as dependent on the system size N for $R = 5.0$ (open squares) and $R = \infty$ (black squares).

Image source:

S. Marvel, J. Kleinberg, R. Kleinberg, S. Strogatz: "Continuous-time model of structural balance"(2011)

The paper tackles the open questions presented in the paper of Kulakowski et al.

Analytical solution to the model:

- ▶ initial matrix $X(0)$ is real and symmetric so it admits eigendecomposition:
 $QD(0)Q^T$, $D(0) = \text{Diag}(\lambda_1, \dots, \lambda_n)$
- ▶ solution is obtained using matrix recatti equations as
 $D(t) = \text{Diag}(\ell_1(t), \dots, \ell_n(t))$ where $\ell_k(t) = \frac{\ell_k}{1 - \ell_k t}$

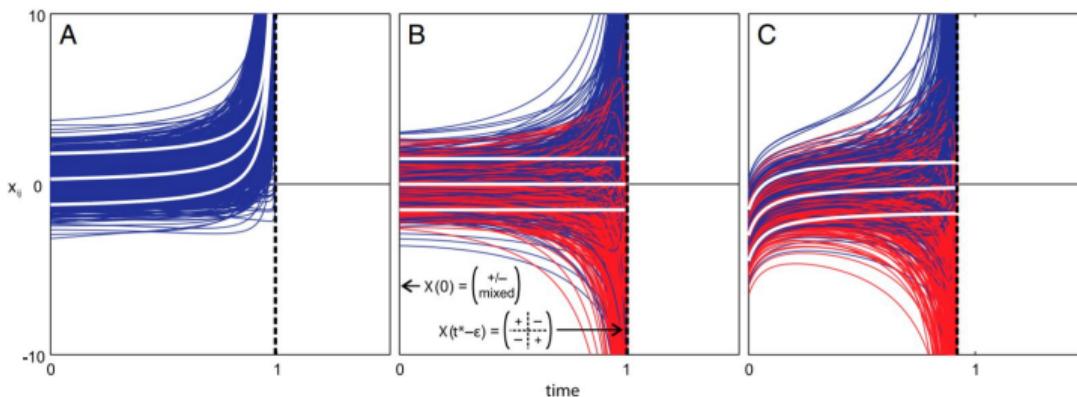
What are the conditions for reaching balance?

Definition (Conditions)

- ▶ $\lambda_1 > 0$ (*holds with $p \rightarrow 1$ as $n \rightarrow \infty$ in finite time*)
- ▶ $\lambda_1 \neq \lambda_2$ (*holds with $p = 1$*)
- ▶ *all components of first eigenvector are nonzero* (*holds with $p = 1$*)

Analysis for classes of random matrices

- ▶ Consider matrices where each entry is drawn from distribution with bounded support, symmetric about μ
- ▶ As μ varies:
 1. $\mu > 0$: the system evolves to all-positive signature
 2. $\mu \leq 0$: the system evolves to two all-positive cliques connected by negative edges



Representative large- n plots of the model for (A) $\mu > 0$ ($\mu = 3/10$ in the plot shown), (B) $\mu = 0$, and (C) $\mu < 0$ ($\mu = -3$ in the plot shown).

Image source: Marvel et al.

C. Altafini: "Dynamics of Opinion Forming in Structurally Balanced Social Networks" (2012)

Main assumption: Outcome of opinion forming process overlaps with bipartition of the network Opinion vector $x \in \mathbb{R}^n$, opinion formation process $f(\cdot)$

Definition (Influence)

$$F_{ij}(x) = \frac{\partial f_i(x)}{\partial x_j}, i, j = 1, \dots, n \rightarrow \text{sgn}(F_{ij}(x)) = \text{sgn}(A_{ij})$$

Definition (Monotonicity(Kamke condition))

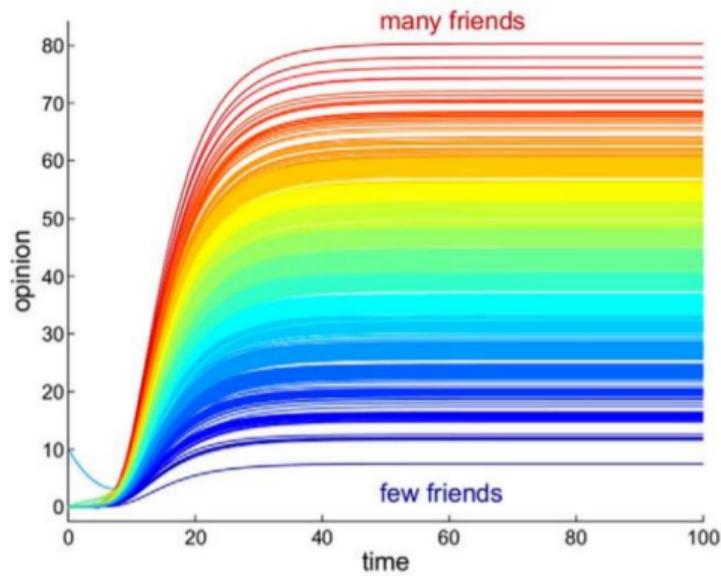
$x = f(x)$ is monotone with respect to order σ iff :

$$\sigma_i \sigma_j F_{ij}(x) \geq 0 \quad x \in \mathbb{R}^n, \quad \forall i, j = 1, \dots, n \quad i \neq j$$

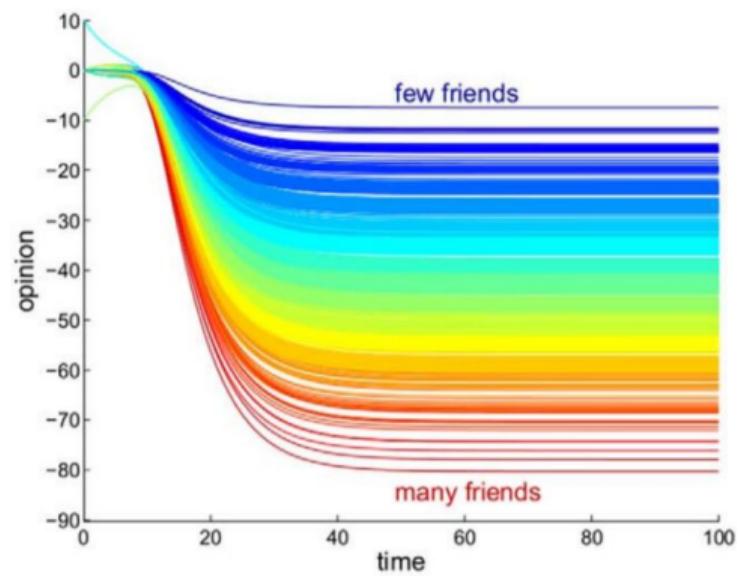
Monotone system \iff corresponding graph is structurally balanced

Results(1)

Cooperative case:



A

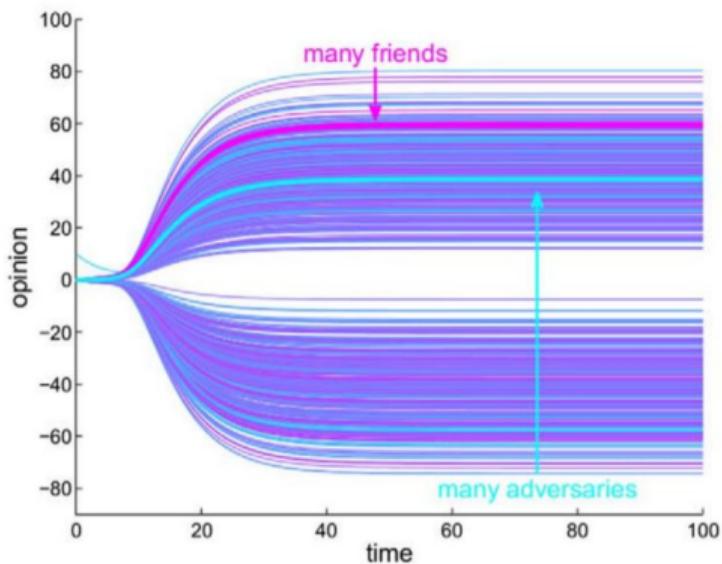


B

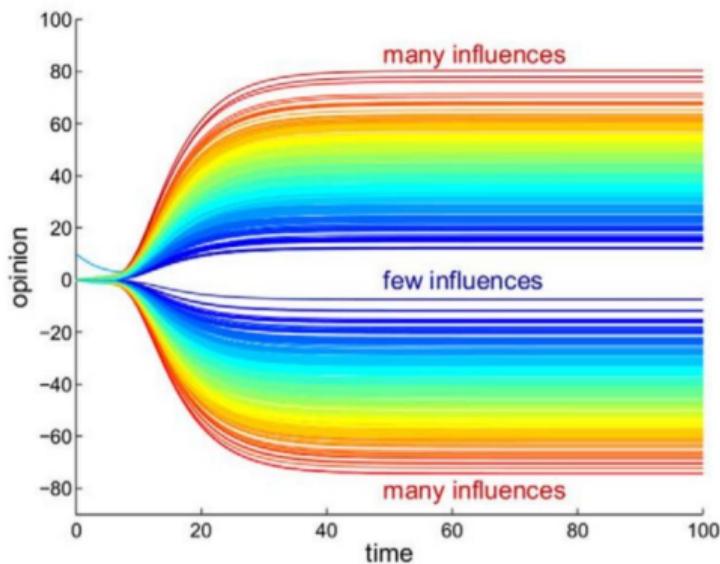
Image

Results(2)

Antagonistic case:



A



B

source: Altafini et al

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Signed graphs in neural networks

Mining signed social networks substantially different from unsigned networks

- ▶ Homophily and social influence applicable in different form
- ▶ Simple extension of unsigned embedding algorithms is not possible

The following papers propose some approaches.

- ▶ We first show a paper that operates within the classical embeddings framework.
- ▶ A more recent paper by Derr et al. extends graph convolutional networks for signed graphs

Attributed Signed Network Embedding, Wang et al. (2017)

Leverages both user attributes and signed links for learning network embeddings

First they perform an analysis of links and attributes

- ▶ Users more likely to have similar attributes with **friends** than with **foes**
- ▶ Users more likely to be similar with **foes** than users without connections

Average user attributes similarities in \mathcal{P} , \mathcal{N} and \mathcal{R}

	Epinions		Slashdot	
	CA	COSINE	CA	COSINE
\mathcal{P}	75.93	0.0650	21.24	0.0332
\mathcal{N}	67.16	0.0540	16.64	0.0289
\mathcal{R}	27.30	0.0168	10.34	0.0162

Image source: Wang et al.

Decomposing the adjacency matrix

The first step is to decompose the adjacency matrix **A**:

$$\min_{\mathbf{U}, \mathbf{H}} \|\mathbf{W} \odot (\mathbf{A} - \mathbf{U}\mathbf{H}\mathbf{U}^T)\|_F^2$$

- ▶ **U** is the latent user feature matrix for **A**
- ▶ $\mathbf{W}_{ij} = 1$ if $\mathbf{A}_{ij} \neq 0$, $\mathbf{W}_{ij} = 0$ otherwise
- ▶ **H** "interaction matrix", captures similarity of user latent features

Incorporating structural balance

Based on Euclidean distance

For three users (u_i, u_j, u_k) :

- ▶ if $\mathbf{A}_{ij} = 1$ and $\mathbf{A}_{ik} = 0$ then it should be $\|u_i - u_j\|_2^2 \leq \|u_i - u_k\|_2^2$
- ▶ if $\mathbf{A}_{ij} = -1$ and $\mathbf{A}_{ik} = 0$ then it should be $\|u_i - u_j\|_2^2 \geq \|u_i - u_k\|_2^2$
- ▶ include all these triplets in \mathcal{H}

$$\min_U \sum_{(u_i, u_j, u_k) \in \mathcal{H}} \max(0, \|u_i - u_j\|_2^2 - \|u_i - u_k\|_2^2)$$

rewrite it as:

$$\min Tr \left(\sum_{(u_i, u_j, u_k) \in \mathcal{H}} I_{ijk} \mathbf{M}^{ijk} \mathbf{U} \mathbf{U}^T \right)$$

where $I_{ijk} = 1$ if $\|u_i - u_j\|_2^2 > \|u_i - u_k\|_2^2$, 0 otherwise and appropriately defined
 $\mathbf{M} \in \{-1, 0, 1\}^{n \times n}$

Learning user attributes

Project \mathbf{U} to **user attribute matrix** \mathbf{X} using projection \mathbf{P}

$$\min_{\mathbf{U}, \mathbf{V}, \mathbf{P}^T \mathbf{P} = \mathbf{I}} \|\mathbf{X} - \mathbf{U}\mathbf{P}\mathbf{V}^T\|_F^2$$

Adding **user similarities** encoded in the laplacian $\mathbf{L} = \mathbf{D} - \mathbf{S}$, where \mathbf{S} is kernel of \mathbf{X} :

$$\min_{\mathbf{U}, \mathbf{P}} \text{Tr}(\mathbf{P}^T \mathbf{U}^T \mathbf{L} \mathbf{U} \mathbf{P})$$

Proposed model

Putting everything together, along with regularization:

$$\begin{aligned} \min_{\mathbf{U}, \mathbf{P}, \mathbf{V}, \mathbf{H}} & \|\mathbf{W} \odot (\mathbf{A} - \mathbf{U}\mathbf{H}\mathbf{U}^T)\|_F^2 + \alpha \operatorname{Tr} \left(\sum_{(u_i, u_j, u_k) \in \mathcal{H}} I_{ijk} \mathbf{M}^{ijk} \mathbf{U} \mathbf{U}^T \right) + \beta \|\mathbf{X} - \mathbf{U}\mathbf{P}\mathbf{V}^T\|_F^2 + \\ & + \gamma \operatorname{Tr}(\mathbf{P}^T \mathbf{U}^T \mathbf{L} \mathbf{U} \mathbf{P}) + \lambda (\|\mathbf{U}\|_F^2 + \|\mathbf{V}\|_F^2 \|\mathbf{H}\|_F^2) \end{aligned}$$

The objective function is optimized using gradient descent

Performance analysis: Link prediction

- ▶ Fextra: learns degree-based and triad-based features
- ▶ MF and CMF: matrix factorization-based, MF: links, CMF: links and attributes
- ▶ SESEN: Spectral analysis for signed graphs
- ▶ ELLR: ranking-based signed network embedding algorithm
- ▶ SiNE: deep learning framework that utilizes structural balance theory, ESiNE extension that aggregates also attributes

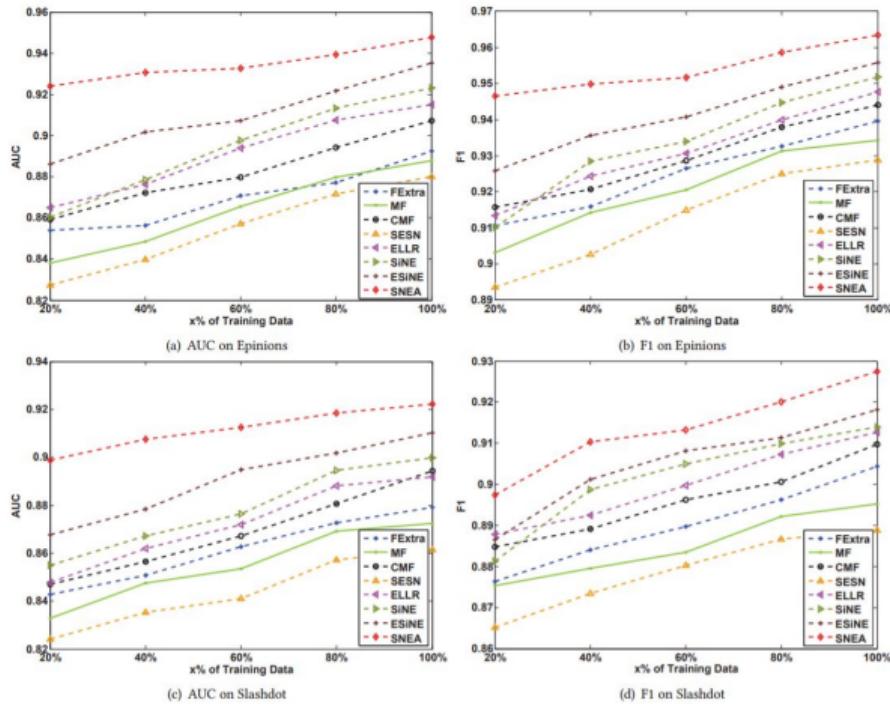


Image source: Wang et al.

Graph convolutional networks

A graph convolutional network(GCN) is a neural network that operates on graphs

- ▶ Learns **node-level** representations
 - ▶ **Convolutional layers** learn different feature representations of nodes
 - ▶ **Aggregation function** on features of each layer to form next layer's features
 - ▶ First layer encodes information about the neighborhood of every node
 - ▶ Second layer encodes 2-hop neighborhood etc...

Useful for node classification, link prediction, community detection, visualization ...

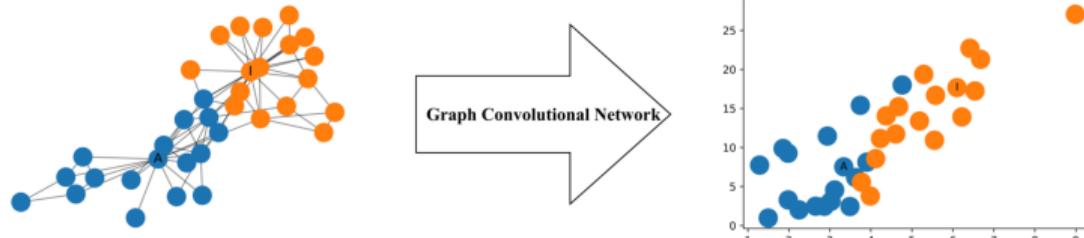


Image source:

Signed graph convolutional networks, Derr et al (2018)

Two different aggregations based on structural balance:

- ▶ Unbalanced paths have an odd number of negative links
- ▶ Balanced paths have an even number of negative links

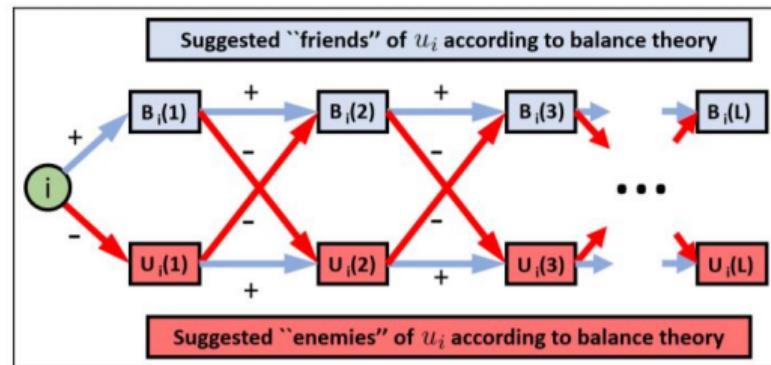


Image source: Derr et al.

Balanced and unbalanced sets $B_i(I + 1)$ and $U_i(I + 1)$ can be obtained from $B_i(I)$ and $U_i(I)$

Extending unsigned GCNs

Rather than maintaining a single representation for each node, keep both friends and enemies.

$$\mathbf{h}_i^{B(I)} = \sigma(\mathbf{W}^{B(I)} [\sum_{j \in \mathcal{N}_i^+} \frac{\mathbf{h}_j^{B(I-1)}}{|\mathcal{N}_i^+|}, \sum_{k \in \mathcal{N}_i^-} \frac{\mathbf{h}_k^{U(I-1)}}{|\mathcal{N}_i^-|}, \mathbf{h}_i^{B(I-1)}])$$

$$\mathbf{h}_i^{U(I)} = \sigma(\mathbf{W}^{U(I)} [\sum_{j \in \mathcal{N}_i^+} \frac{\mathbf{h}_j^{U(I-1)}}{|\mathcal{N}_i^+|}, \sum_{k \in \mathcal{N}_i^-} \frac{\mathbf{h}_k^{B(I-1)}}{|\mathcal{N}_i^-|}, \mathbf{h}_i^{U(I-1)}])$$

- ▶ \mathbf{h} represents node features
- ▶ $\sigma()$ is a non-linear activation function
- ▶ $\mathbf{W}^{B(I)}$ and $\mathbf{W}^{U(I)}$ are linear transformation matrices responsible for "friends" and "enemies"

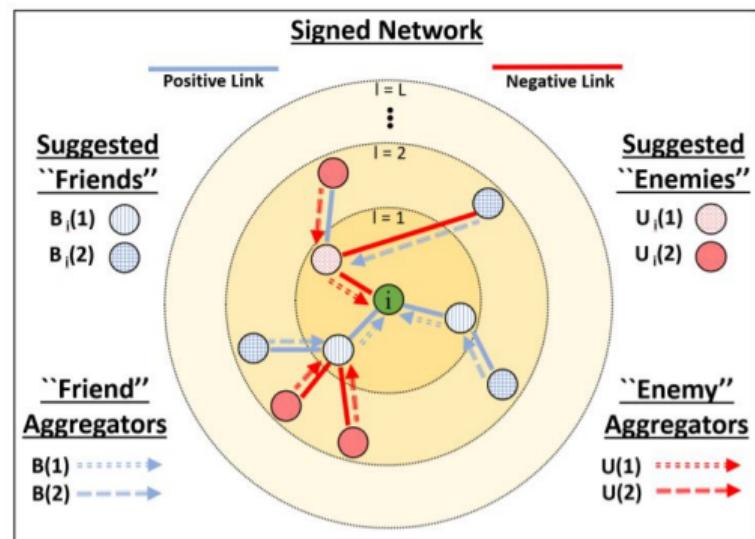


Image source: Derr et al.

Application: Link prediction

- ▶ SSE: spectral clustering algorithm based on signed Laplacian
- ▶ SiNE: deep learning framework that utilizes structural balance theory
- ▶ SIDE: random walk based method that utilizes structural balance theory
- ▶ SGCN-2: Proposed method. SGCN-1 and SGCN-1+ are structural balance-agnostic variants(only consider first aggregation layer)

LINK SIGN PREDICTION RESULTS WITH AUC.

Embedding Method	Bitcoin-Alpha	Bitcoin-OTC	Slashdot	Epinions
SSE	0.764	0.803	0.769	0.822
SiNE	0.778	0.814	0.792	0.849
SIDE	0.630	0.618	0.547	0.571
SGCN-1	0.780	0.818	0.784	0.663
SGCN-1+	0.785	0.817	0.804	0.722
SGCN-2	0.796	0.823	0.804	0.864

LINK SIGN PREDICTION RESULTS WITH F1.

Embedding Method	Bitcoin-Alpha	Bitcoin-OTC	Slashdot	Epinions
SSE	0.898	0.923	0.820	0.901
SiNE	0.888	0.878	0.854	0.914
SIDE	0.738	0.750	0.646	0.711
SGCN-1	0.910	0.918	0.853	0.851
SGCN-1+	0.912	0.923	0.865	0.893
SGCN-2	0.917	0.925	0.864	0.933

Conclusions and future directions

Conclusions

- ▶ Signed networks differ in terms of **basic concepts**, **properties** and present unique **computational challenges**
- ▶ In this tutorial we gave an overview of the literature in mining signed networks
 - ▶ We discussed the most important **theoretical concepts**
 - ▶ We discussed some common **applications**
- ▶ Rapidly growing research area
- ▶ In the remaining time we will mention:
 - ▶ Topics **not covered**
 - ▶ **Open problems**

Topics not covered in this tutorial

Signed networks have been the focus of many research works in the last few years

- ▶ Impossible to cover all topics in one tutorial...

Some important problem areas not covered in this tutorial:

- ▶ Node ranking (Wu et al. (2016b); Jung et al. (2016); Chung et al. (2013))
- ▶ Community detection (Gao et al. (2016); Li et al. (2014); Anchuri and Magdon-Ismail (2012))
- ▶ Dense subgraph discovery (Cadena et al. (2016))
- ▶ Recommendations (Tang et al. (2016); Victor et al. (2011); Ma et al. (2009))
- ▶ Many other papers in these areas not listed here...

Future directions

Fundamental combinatorial optimization problems still open. E.g.

- ▶ Approximability of vertex frustration and MBS.
- ▶ Approximability of polarized communities: $\max_{x \in \{-1,0,1\}} \frac{x^T Ax}{x^T x}$.

For more theoretically-oriented challenges, check open problems in spectral theory listed by Belardo et al. (2019).

Graph construction in social media

- ▶ A lot of information **under-used!**
- ▶ Primary task: **inferring edge signs** (a link classification problem)
- ▶ Can we leverage **multi-media information**, e.g. user posts?
- ▶ Some attempt using crowd-sourcing, e.g. Kumar et al. (2018)
- ▶ **Automatic** way of doing this, e.g. using NLP techniques?



Graph summarization

General idea Koutra et al. (2018)

- ▶ Find a **short representation** of the input graph,
- ▶ often in the form of a **summary** or sparsified graph,
- ▶ which **reveals patterns** in the original data and preserves specific structural or other properties, depending on the application domain.

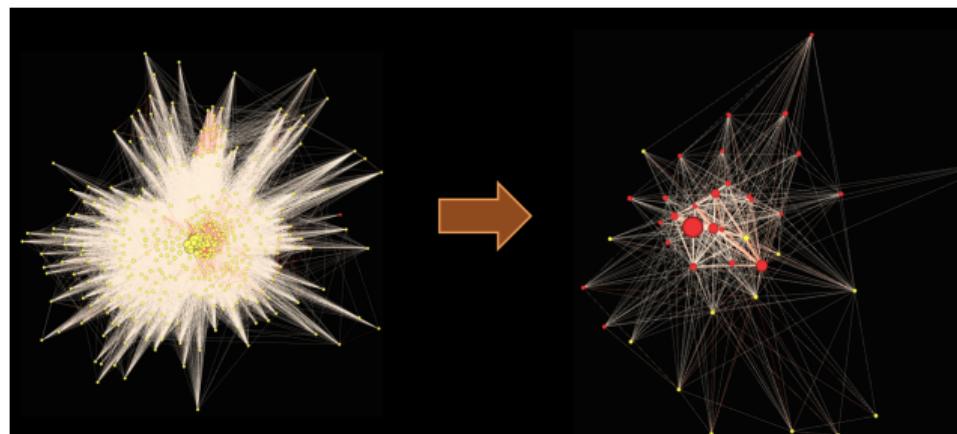
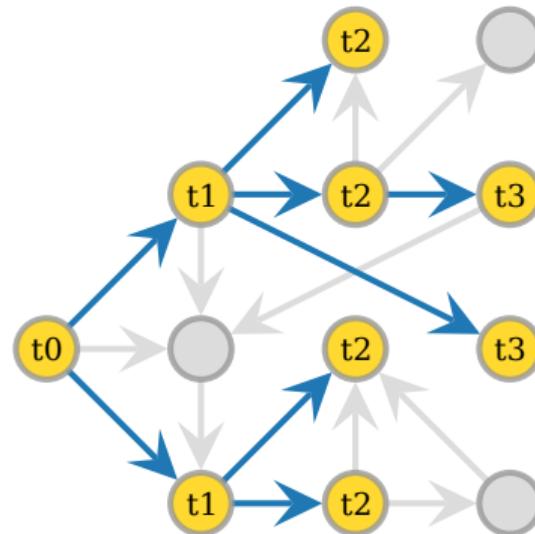


Image source: Koutra et al. (2018)

Propagation process modeling

- ▶ Real-world phenomena
 - ▶ Human disease
 - ▶ Computer virus
 - ▶ piece of information
- ▶ Is the propagation **dynamics** on **signed graphs** different from that on **unsigned graphs**?
- ▶ Inference tasks:
 - ▶ Source identification
 - ▶ Cascade reconstruction
 - ▶ Cascade prediction
(size/popularity, node activation states)
 - ▶ Network structure inference



Local community detection

General setting

- ▶ **Input:** a graph and some **seed** nodes
- ▶ **Output:** a community **relevant** to the seed nodes

A **rarely-touched** topic in the context of signed graphs

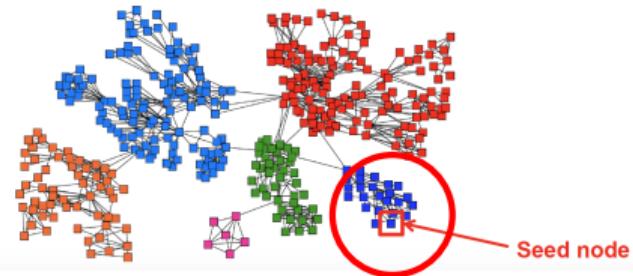


Image source:

<http://web.stanford.edu/class/cs246/>

Thank you and stay healthy !

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