Chapter 1 The Foundations:Logic and Proofs

1.4 Predicates and Quantifiers

Question 2

Let $P(\mathrm{x})$ be the statement "The word x contains the letter a." What are these truth values?

1. $P(\text{orange})$
2. $P(\text{lemon})$
3. $P(\text{true})$
4. $P(\text{false})$

Solution

$P(\text{orange})$ and $P(\text{false})$ are truth values.

Question 4

State the value of x after the statement **if** $P(\mathrm{x})$ **then** x := 1 is executed, where $P(\mathrm{x})$ is the statement "x > 1," if the value of x when this statement is reached is

a) x=0.

b) x=1.

c) x=2.

Solution

c)

Question 6

Let N(x) be the statement "x has visited North Dakota," where the domain consists of the students in your school. Express each of these quantifications in English.

1. $\exists x N(x)$
2. $\forall x N(x)$
3. $\neg \exists x N(x)$
4. $\exists x \neg N(x)$
5. $\neg \forall x N(x)$
6. $\forall x \neg N(x)$

Solution

1. There exists a student in the school who has visited North Dakota.
2. All students in the school have visited North Dakota.
3. There does **not** exist a student in the school who has visited North Dakota.
4. There exists a student in the school who has **not** visited North Dakota.
5. **Not** all students in the school have visited North Dakota.
6. All students in the school have **not** visited North Dakota.

Question 8

Translate these statements into English, where R(x) is "x is a rabbit" and H(x) is "x hops" and the domain consists of all animals.

1. $\forall x(R(x) \rightarrow H(x))$
2. $\forall x(R(x) \wedge H(x))$
3. $\exists x(R(x) \rightarrow H(x))$
4. $\exists x(R(x) \wedge H(x))$

Solution

1. All rabbits hop.
2. All animals are rabbits and hop.
3. There exists an animal such that if it is a rabbit, then it hops.
4. There exists an animal that is a rabbit and hops.

Question 10

Let C(x) be the statement "x has a cat," let D(x) be the statement "x has a dog," and let F(x) be the statement "x has a ferret." Express each of these statements in terms of C(x), D(x), F(x), quantifiers, and logical connectives. Let the domain consist of all students in your class.

a) A student in your class has a cat, a dog, and a ferret.

b) All students in your class have a cat, a dog, or a ferret.

c) Some student in your class has a cat and a ferret, but not a dog.

d) No student in your class has a cat, a dog, and a ferret.

e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Solution

a) $\exists x(C(x) \land D(x) \land F(x))$

b) $\forall x(C(x) \land D(x) \land F(x))$

c) $\exists x(C(x) \land \neg D(x) \land F(x))$

d) $\neg\exists x(C(x) \land D(x) \land F(x))$

e) $(\exists x C(x)) \land (\exists x D(x)) \land (\exists F(x))$

Question 16

Determine the truth value of each of these statements if the domain of each variable consists of all real numbers.

a) $\exists x(x^2 = 2)$

b) $\exists x(x^2 = -1)$

c) $\forall x(x^2 + 2 \geq 1)$

d) $\forall x(x^2 \neq x)$

Solution

a) True

b) False

c) True

d) False

Question 18

Suppose that the domain of the propositional function *P*(x) consists of the integers -2, -1, 0, 1, and 2. Write out each of these propositions using disjunctions, conjunctions, and negations.

a) $\exists x P(x)$

b) $\forall x P(x)$

c) $\exists x \neg P(x)$

d) $\forall x \neg P(x)$

e) $\neg \exists x P(x)$

f) $\neg \forall x P(x)$

Solution

a) $P(-1) \vee P(-2) \vee P(0) \vee P(1) \vee P(2)$

b) $P(-1) \wedge P(-2) \wedge P(0) \wedge P(1) \wedge P(2)$

c) $\neg P(-1) \vee \neg P(-2) \vee \neg P(0) \vee \neg P(1) \vee \neg P(2)$

d) $\neg P(-1) \wedge \neg P(-2) \wedge \neg P(0) \wedge \neg P(1) \wedge \neg P(2)$

e) $\neg P(-1) \wedge \neg P(-2) \wedge \neg P(0) \wedge \neg P(1) \wedge \neg P(2)$

f) $\neg P(-1) \vee \neg P(-2) \vee \neg P(0) \vee \neg P(1) \vee \neg P(2)$

Question 24

Translate in two ways each of these statements into logical expressions using predicates, quantifiers, and logical connectives. First, let the domain consist of the students in your class and second, let it consist of all people.

a) Everyone in your class has a cellular phone.

b) Somebody in your class has seen a foreign movie.

c) There is a person in your class who cannot swim.

d) All students in your class can solve quadratic equations.

e) Some student in your class does not want to be rich.

Solution

Let $C(x)$ be the propositional function "x is in your class."

a) Let $P(x)$ be the propositional function "x has a cellular phone." First we have $\forall x P(x)$, and second we have $\forall x(C(x) \to P(x))$

b) Let $F(x)$ be the propositional function "x has seen a foreign movie." First we have $\exists x F(x)$, and second we have $\exists x(C(x) \land F(x))$

c) Let $S(x)$ be the propositional function "x cannot swim." First we have $\exists x S(x)$, and second we have $\exists x(C(x) \land S(x))$

d) Let $Q(x)$ be the propositional function "x can solve quadratic equations." First we have $\forall x Q(x)$, and second we have $\forall x(C(x) \to Q(x))$

e) Let $R(x)$ be the propositional function "x does not want to be rich." First we have $\exists x R(x)$, and second we have $\exists x(C(x) \land R(x))$

Question 32

Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

a) All dogs have fleas.

b) There is a horse that can add.

c) Every koala can climb.

d) No monkey can speak French.

e) There exists a pig that can swim and catch fish.

Solution

Let $C(x)$ be the propositional function "x is in your class."

a) Let $F(x)$ be "x has fleas." and let the domain of discourse be dogs.

- origin statement: $\forall x F(x)$
- negation statement: $\exists x \neg F(x)$
- simple English: There is a dog do not have fleas.

b) Let $F(x)$ be "x can add." and let the domain of discourse be horse.

- origin statement: $\exists x F(x)$
- negation statement: $\forall x \neg F(x)$
- simple English: No horses can add.

c) Let $F(x)$ be "x can climb." and let the domain of discourse be Koala.

- origin statement: $\forall x F(x)$
- negation statement: $\exists x \neg F(x)$
- simple English: There is a Koala can not climb.

d) Let $F(x)$ be "x can speak French." and let the domain of discourse be monky.

- origin statement: $\forall x \neg F(x)$
- negation statement: $\exists x F(x)$
- simple English: There is a monky can speak French.

e) Let $F(x)$ be "x can swim and catch fish." and let the domain of discourse be pig.

- origin statement: $\exists x F(x)$
- negation statement: $\forall x \neg F(x)$
- simple English: Every pig either can not swim or can not catch fish.

Question 38

Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all real numbers.

a) $\forall x(x^2 \neq x)$

b) $\forall x(x^2 \neq 2)$

c) $\forall x(|x| > 0$

Solution

a) Since $1^2 = 1$ and $0^2 = 0$, the statement is false.

b) Since $\sqrt{2}^2 = 2$, the statement is false.

c) Since $|0| \leq 0$, the statement is false.

Question 42

Express each of these system specifications using predicates, quantifiers, and logical connectives.

a) When there is less than 30 megabytes free on the hard disk, a warning message is sent to all users.

b) No directories in the file system can be opened and no files can be closed when system errors have been detected.

c) The file system cannot be backed up if there is a user currently logged on.

d) Video on demand can be delivered when there are at least 8 megabytes of memory available and the connection speed is at least 56 kilobits per second.

Solution

a) Let $F(x)$ be a predicates that means "There is less than x megabytes free on the hard disk", with the domain of discourse being positive numbers, and let $U(x)$ be "User x is sent a warning message". Then we can have$F(30) \to \forall x U(x)$

b) Let S be "System errors have been detected.", and $D(x)$ be "Directorie x in the file system can be opened" $I(x)$ be " File x can be closed.". Then we have $S \to \forall x \neg D(x) \wedge \forall x \neg F(x)$

c) Let I be "The file system can be backed up", and $U(x)$ be "User X currently logged on" Then we have $\exists x U(x) \to \neg I$

d) Let V be "Video on demand can be delivered,", and $M(x)$ be "There are at least x megabytes of memory available" and let $C(x)$ be "The connection speed is at least x kilobits per second" Then we have $(M(8) \wedge C(56)) \to V$

Question 52

Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x(P(x) \vee Q(x))$ are not logically equivalent.

Solution

Let $P(x)$ be true if and only if x is odd, and let $Q(x)$ be true if and only if x is even. Then for $\forall x P(x) \vee \forall x Q(x)$, it is false, while the proposition $\forall x(P(x) \vee Q(x))$ is true. Therefore, the two propositions are not logically equivalent.

1.5 Nested Quantifiers

Question 4

Let $P(x, y)$ be the statement "Student x has taken class y," where the domain for x consists of all students in your class and for y consists of all computer science courses at your school. Express each of these quantifications in English.

1. $\exists x \exists y P(x, y)$
2. $\exists x \forall y P(x, y)$
3. $\forall x \exists y P(x, y)$
4. $\exists y \forall x P(x, y)$
5. $\forall y \exists x P(x, y)$
6. $\forall x \forall y P(x, y)$

Solution

1. There is a student in your class has taken some computer science course.
2. There is a student in your class has taken all computer science courses.
3. Every student in your class has taken some computer science course.
4. There is a computer science course that every students in your class has taken it.
5. Every computer science course has been taken by at least one student in your class.
6. All students in your class have taken all computer science courses.

Question 8

Let $Q(x, y)$ be the statement "Student x has been a contestant on quiz show y." Express each of these sentences in terms of $Q(x, y)$, quantifiers, and logical connectives, where the domain for x consists of all students at your school and for y consists of all quiz shows on television.

a) There is a student at your school who has been a contestant on a television quiz show.

b) No student at your school has ever been a contestant on a television quiz show.

c) There is a student at your school who has been a contestant on *Jeopardy!* and on *Wheel of Fortune*.

d) Every television quiz show has had a student from your school as a contestant.

e) At least two students from your school have been contestants on *Jeopardy!*.

Solution

a) $\exists x \exists y Q(x, y)$

b) $\forall x \forall y \neg Q(x, y)$

c) $\exists x Q(x, Jeopardy!) \wedge \exists x Q(x, WheelofFortune)$

d) $\forall y \exists x Q(x, y)$

e) $\exists x1, x2(Q(x1, Jeopardy!) \wedge Q(x2, Jeopardy!) \wedge x1 \neq x2)$

Question 10

Let $F(x, y)$ be the statement “x can fool y,” where the domain consists of all people in the world. Use quantifiers to express each of these statements.

a) Everybody can fool Fred.

b) Evelyn can fool everybody.

c) Everybody can fool somebody.

d) There is no one who can fool everybody.

e) Everyone can be fooled by somebody.

f) No one can fool both Fred and Jerry.

g) Nancy can fool exactly two people.

h) There is exactly one person whom everybody can fool.

i) No one can fool himself or herself.

j) There is someone who can fool exactly one person besides himself or herself.

Solution

a) $\forall x F(x, Fred)$

b) $\forall x \forall y F(x, y)$

c) $\forall x \exists y F(x, y)$

d) $\forall x \forall y \neg F(x, y)$

e) $\forall y \exists x F(x, y)$

f) $\neg \exists x (F(x, Fred) \land F(x, Jerry))$

g) $\exists y1 \exists y2 (F(Nancy, y1) \land F(Nancy, y2) \land y1 \neq y2 \land \forall y (F(Nancy, y) \rightarrow (y = y1 \lor y = y2)))$

h) $\exists y ((\forall x F(x, y)) \land (\exists z (\forall x F(x, z)) \rightarrow z = y))$

i) $\neg \exists x F(x, x)$

j) $\exists x (\exists y (F(x, y) \land (\exists z F(x, z) \rightarrow z = y) \land y \neq x))$

Question 24

Translate each of these nested quantifications into an English statement that expresses a mathematical fact. The domain in each case consists of all real numbers.

a) $\exists x \wedge y(x + y = y)$

b) $\forall x \forall y(((x \geq 0) \wedge (y \leq 0)) \rightarrow (x - y > 0))$

c) $\exists x \exists y(((x \leq 0) \wedge (y \leq 0)) \wedge (x - y > 0))$

d) $\forall x \forall y((x \neq 0) \wedge (y \neq 0) \leftrightarrow (xy \neq 0))$

Solution

a) There exists a real number x that for all real number y it add x equal to itself.

b) If a real number x is greater than or equal to zero and y is less than or equal to zero, than x minus y is greater than zero.

c) There exists two real number x and y that x and y are both less than or equal to zero and x minus y is greater than zero.

d) For any two real number x and y, the product of x and y is nonzero if and only if both x and y are nonzero.

Question 32

Express the negations of each of these statements so that all negation symbols immediately precede predicates.

a) $\exists z \forall y \forall x T(x, y, z)$

b) $\exists x \exists y P(x, y) \wedge \forall x \forall y Q(x, y)$

c) $\exists x \exists y (Q(x, y) \leftrightarrow Q(y, x))$

d) $\forall y \exists x \exists z (T(x, y, z) \vee Q(x, y))$

Solution

a) $\forall z \exists y \exists x \neg T(x, y, z)$

b) $\forall x \forall y P(x, y) \vee \exists x \exists y Q(x, y)$

c) $\forall x \forall y (Q(x, y) \leftrightarrow \neg Q(y, x))$

d) $\exists y \forall x \forall z (\neg T(x, y, z) \wedge \neg Q(x, y))$

Question 38

Express the negations of these propositions using quantifiers, and in English.

a) Every student in this class likes mathematics.

b) There is a student in this class who has never seen a computer.

c) There is a student in this class who has taken every mathematics course offered at this school.

d) There is a student in this class who has been in at least one room of every building on campus.

Solution

a) In English, the negation is that "Some student in this class does not like mathematics." , and let $M(x)$ be "Student x in this class like mathematics.", then we have $\exists x M(x)$

b) In English, the negation is that "Every student in the class has seen a computer." , and let $C(x)$ be "Student x in this class has seen a computer.", then we have $\forall x C(x)$

c) In English, the negation is that "For every student there is mathematics course that the student has not taken." , and let $F(x, y)$ be "Student x in this class has taken mathematics course y", then we have $\forall x \exists y \neg F(x, y)$

d) In English, the negation is that "For every student there is a building such that for every room in that building, the student has not been in that room.", and let $P(x, y)$ be "Student x in this class has been in room y" and $Q(y, z)$ be "Room y is in building z", then we have $\forall x \exists z \forall y \neg P(x, y) \vee \neg Q(y, z)$

Question 40

Find a counterexample, if possible, to these universally quantified statements, where the domain for all variables consists of all integers.

a) $\forall x \exists y (x = 1/y)$

b) $\forall x \exists y (y^2 - x \leq 100)$

c) $\forall x \forall y (x^2 \neq y^3)$

Solution

a) When x is zero, we can find an integer y such that $x = 1/y$

b) When x is -101, we can find an integer y such that $y^2 - x \leq 100$ since an integers' square is positive.

c) When x is 1 and y is 1 too, then $x^2 = y^3$

1.6 Rules of Inference

Question 6

Use rules of inference to show that the hypotheses “If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on,” “If the sailing race is held, then the trophy will be awarded,” and “The trophy was not awarded” imply the conclusion “It rained.”

Solution

Let proposition r be “It rains”, f be “it is foggy”, s be “the selling race will be held,”, l be “Lifesaving demonstration will go on,” and t be “the trophy will be awarded”. Then we have the premises $\neg r \vee \neg f \rightarrow s \wedge l$, $s \rightarrow t$ and $\neg t$. What we want to conclude is r

1.	$\neg t$	Hypothesis
2.	$s \rightarrow t$	Hypothesis
3.	$\neg s$	Modus tollens using (1) and (2)
4.	$\neg(s \wedge l)$	Domination using (3)
5.	$\neg r \wedge \neg f \rightarrow s \wedge l$	Hypothesis
6.	$\neg(\neg r \vee \neg f)$	Modus tollens using (4)
7.	$r \wedge f$	De Morgan’s law
8.	r	Simplification using (7)

Question 10

For each of these sets of premises, what relevant conclusion or conclusions can be drawn? Explain the rules of inference used to obtain each conclusion from the premises

a) "If I play hockey, then I am sore the next day." "I use the whirlpool if I am sore." "I did not use the whirlpool."

b) "If I work, it is either sunny or partly sunny." "I worked last Monday or I worked last Friday." "It was not sunny on Tuesday." "It was not partly sunny on Friday."

c) "All insects have six legs." "Dragonflies are insects." "Spiders do not have six legs." "Spiders eat dragonflies."

d) "Every student has an Internet account." "Homer does not have an Internet account." "Maggie has an Internet account."

e) "All foods that are healthy to eat do not taste good." "Tofu is healthy to eat." "You only eat what tastes good." "You do not eat tofu." "Cheeseburgers are not healthy to eat."

f) "I am either dreaming or hallucinating." "I am not dreaming." "If I am hallucinating, I see elephants running down the road."

Solution

a) From "I did not use the whirlpool.", we conclude that I am not sore by using modus tollens, and then we conclude that I am not sore by using modus tollens again.

b) We can not conclude anything from these premises.

c) From "All insects have six legs." and "Dragonflies are insects.", we can conclude that dragonflies are insects by using modus ponens, and we can also conclude that Spiders are not insects from "Spiders eat dragonflies." by using modus tollens .

d) From "Every student has an Internet account." and "Homer does not have an Internet account.", we can conclude that Homer is not a student by using modus tollens.

e) We can not conclude anything from these premises.

f) From "I am either dreaming or hallucinating." "I am not dreaming.", we can conclude that I am hallucinating by using disjunctive syllogism. And from "If I am hallucinating, I see elephants running down the road." we can conclude that I see elephants running down the road. by using modus ponens.

Question 12

Show that the argument form with premises $(p \wedge t) \to (r \vee s), q \to (u \wedge t), u \to p$, and $\neg s$ and conclusion $q \to r$ is valid by first using Exercise 11 and then using rules of inference from Table 1.

Solution

By using Exercise 11 we want to conclude $q \to r$ with five premises, $(p \wedge t) \to (r \vee s), q \to (u \wedge t), u \to p$, $\neg s$ and q. From q and $q \to (u \wedge t)$ we conclude that $u \wedge t$ by modus ponens, and we can have u and t by simplification. From $u \to p$ we conclude that p by using modus ponens. From $(p \wedge t) \to (r \vee s)$ we conclude $r \vee s$ by using modus ponens. Finally from $\neg s$ by using disjunctive syllogism we conclude r.

Question 18

What is wrong with this argument? Let $S(x, y)$ be "x is shorter than y." Given the premise $\exists s S(s, Max)$, it follows that $S(Max, Max)$. Then by existential generalization it follows that $\exists x S(x, x)$, so that someone is shorter than himself.

Solution

From the premise $\exists s S(s, Max)$ we can not conclude that Max is one such s. Therefore this first step is invalid.

Question 24

Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \lor Q(x))$ is true then $\forall x P(x) \lor \forall x Q(x)$ is true.

1.	$\forall x(P(x) \lor Q(x))$	Premise
2.	$P(c) \lor Q(c)$	Universal instantiation from (1)
3.	$P(c)$	Simplification from (2)
4.	$\forall x P(x)$	Universal generalization from (3)
5.	$Q(c)$	Simplification from (2)
6.	$\forall x Q(x)$	Universal generalization from (5)
7.	$\forall x(P(x) \lor \forall x Q(x))$	Conjunction from (4) and (6)

Solution

Steps 3 and 5 are incorrect because simplification applies to conjunctions, not disjunctions.

Question 28

Use rules of inference to show that if $\forall x(P(x) \lor Q(x))$ and $\forall x((\neg P(x) \land Q(x)) \to R(x))$ are true, then $\forall x(\neg R(x) \to P(x))$ is also true, where the domains of all quantifiers are the same.

Solution

1.	$\forall x(P(x) \lor Q(x))$	Premise
2.	$P(x) \lor Q(x)$	UI from (1)
3.	$\forall x((\neg P(x) \land Q(x)) \to R(x))$	Premise
4.	$(\neg P(x) \land Q(x)) \to R(x)$	UI from (3)
5.	$\neg R(x) \to (P(x) \lor \neg Q(x))$	Contrapositive
6.	$\neg R(x) \to (P(x) \lor \neg Q(x)) \land (P(x) \lor Q(x)))$	Identity from (2)(5)
7.	$\neg R(x) \to (((P(x) \lor \neg Q(x)) \land P(x)) \lor ((P(x) \lor \neg Q(x)) \land Q(x))))$	Distributive from (6)
8.	$\neg R(x) \to (((P(x) \lor \neg Q(x)) \land P(x)) \lor (P(x) \land Q(x)))$	Distributive from (7)
9.	$\neg R(x) \to (P(x) \lor (P(x) \land Q(x)))$	Absorption from (8)
10.	$\neg R(x) \to P(x)$	Absorption from (9)
11.	$\forall x(\neg R(x) \to P(x))$	UG from (10)

1.7 Introduction to Proofs

Question 8

Prove that if n is a perfect square, then $n + 2$ is not a perfect square.

Solution

If $n + 2$ is not a perfect square, supposed there exists integers $s^2 = n$ and $t^2 = n + 2$, then we have $t^2 - s^2 = (t + s)(t - s) = 2$. Because the factor of 2 is 1 and 2, $s + t = 2$ and $s - t = 1$. We can not find such s and t. Therefore $n + 2$ is not a perfect square.

Question 12

Prove or disprove that the product of a nonzero rational number and an irrational number is irrational.

Solution

This is true. Suppose that a/b is a nonzero rational number and that x is an irrational number. Suppose that xa/b were rational and it equals to c/d. Since $a/b \neq 0$, we know that $a \neq 0$, so b/a is also a rational number. Then we have $(xa/b) * (b/a) = (cb/ad) = x$, so x is rational number, a contradion. Therefore, the product of a nonzero rational number and an irrational number is irrational.

Question 18

Prove that if m and n are integers and mn is even, then m is even or n is even

Solution

Because nm is even, one of factors of nm is 2. This factor is only from n or m because nm is equals to n multyply m. Therefore either n or m is even.

Question 32

Show that these three statements are equivalent, where aand b are real numbers: (i) a is less than b, (ii) the averageof a and b is greater than a, and (iii) the average of a andb is less than b.

Solution

For (i) we can write$a < b$. For (ii) we can write$(a + b)/2 > a$, which equivalent to $a + b > 2a$. For (iii) we can write$(a + b)/2 < b$, which equivalent to $a + b < 2b$. These equations are all equivalent to $a < b$.

1.8 Proof Methods and Strategy

Question 2

Use a proof by cases to show that 10 is not the square of apositive integer. [Hint: Consider two cases: (i) $1 \leq x \leq 3$,(ii) $x \geq 4$.]

Solution

Suppose there exists an integer t such that $t^2 = 10$. Because $10 \geq 9$ and $10 \leq 16$, we have $3 < t < 4$. There is not an integer between 3 and 4. Therefore, 10 is not the square of a positive integer.

Question 6

Use a proof by cases to show that $min(a, min(b, c)) = min(min(a, b), c)$ whenever a, b, and c are real numbers.

Solution

- If a is smallest, then $a < min(b, c)$ and $a < b \wedge a < c$,
  so $min(a, min(b, c)) = min(min(a, b), c) = a$
- If b is smallest, then $b < c \wedge a > b$ and $a > b \wedge b < c$,
  so $min(a, min(b, c)) = min(min(a, b), c) = b$
- If c is smallest, then $b > c \wedge a > c$ and $c < min(a, b)$,
  so $min(a, min(b, c)) = min(min(a, b), c) = c$

Question 12

Prove that either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square. Is your proof constructive or nonconstructive?

Solution

If noth $2 \cdot 10^{500} + 15$ and $2 \cdot 10^{500} + 16$ are perfect squares, then we have two integers s and t such that $s^2 = 2 \cdot 10^{500} + 15$ and $t^2 = 2 \cdot 10^{500} + 16$. And $t^2 - s^2 = (t + s)(t - s) = (2 \cdot 10^{500} + 16) - (2 \cdot 10^{500} + 15) = 1$. We can not find s and t such that $(t + s)(t - s) = 1$. Therefore either $2 \cdot 10^{500} + 15$ or $2 \cdot 10^{500} + 16$ is not a perfect square. This is constructive proof.

Question 16

Prove or disprove that if a and b are rational numbers, then a^b is also rational.

Solution

This is false. One of counterexample is $a = 2$ and $b = 1/2$, and $a^b = \sqrt{2}$ is not rational.

Question 24

Use forward reasoning to show that if x is a nonzeroreal number, then $x^2 + 1/x^2 \geq 2$. [Hint: Start with the inequality $(x - 1/x)^2 \geq 0$, which holds for all nonzero realnumbers x.]

Solution

Because one's square is positive, we have $(x-1/x)^2 = x^2-2+1/x^2 \geq 0$. Therefore, $x^2+1/x^2 \geq 2$.