Chapter 2 Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

2.1 Sets

Question 6

For each of these pairs of sets, determine whether the first is a subset of the second, the second is a subset of the first, or neither is a subset of the other.

a) the set of people who speak English, the set of people who speak English with an Australian accent

b) the set of fruits, the set of citrus fruits

c) the set of students studying discrete mathematics, the set of students studying data structures

Solution

a) $a \nsubseteq b$

b) $a \nsubseteq b$

c) $a \nsubseteq b$

Question 12

Determine whether these statements are true or False.

a) $\varnothing \in \{\varnothing\}$

b) $\varnothing \in \{\varnothing, \{\varnothing\}\}$

c) $\{\varnothing\} \in \{\varnothing\}$

d) $\{\varnothing\} \in \{\{\varnothing\}\}$

e) $\{\varnothing\} \subset \{\varnothing, \{\varnothing\}\}$

f) $\{\{\varnothing\}\} \subset \{\varnothing, \{\varnothing\}\}$

g) $\{\{\varnothing\}\} \subset \{\{\varnothing\}, \{\varnothing\}\}$

Solution

a) True

b) True

c) False

d) True

e) True

f) True

g) False

Question 20

Find two sets A and B such that $A \in B$ and $A \subseteq B$.

Solution

$A = \emptyset$ and $B = \{\emptyset\}$

Question 22

What is the cardinality of each of these sets?

a) $\emptyset$

b) $\{\emptyset\}$

c) $\{\emptyset, \{\emptyset\}\}$

d) $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

Solution

a) 0

b) 1

c) 2

d) 3

Question 26

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

a) $\emptyset$

b) $\{\emptyset, \{a\}\}$

c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Solution

a) Yes

b) Yes

c) No

d) Yes

Question 28

Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$

Solution

For $A \times B$, suppose a pair (a, b) where $a \in A$ and $b \in B$, since A is a subset of C and B is a subset of D, so $a \in C$ and $b \in D$, Therefore $(a, b) \in C \times D$, and $A \times B \subseteq C \times D$.

Question 32

Suppose that $A \times B = \emptyset$, where A and B are sets. What can you conclude?

Solution

$A = \emptyset$ or $b = \emptyset$

Question 38

How many different elements does $A \times B \times C$ have if A has m elements, B has n elements, and C has p elements?

Solution

$A \times B \times C$ has mnp elements.

2.2 Set Operations

Question 4

Let $A = \{a, b, c, d, e\}$ and$B = \{a, b, c, d, e, f, g, h\}$. Find

a) $A \cup B$.

b) $A \cap B$.

c) $A - B$.

d) $B - A$.

Solution

a) $\{a, b, c, d, e, f, g, h\}$

b) $\{a, b, c, d, e\}$

c) $\emptyset$

d) $\{f, g, h\}$

Question 12

Prove the first absorption law from Table 1 by showing that if A and B are sets, then $A \cup (A \cap B) = A$.

Solution

Suppose $x \in A$, then $x \in A \cup (A \cap B)$ since $A \cup (A \cap B)$ is a union of A and another set, so $A \cup (A \cap B) \subseteq A$. Suppose $x \in A \cup (A \cap B)$, then $x \in A$ or $x \in (A \cap B)$ by the defintion of union, if $x \in (A \cap B)$ then $x \in A$ by the defintion of intersection. Therefore, $A \cup (A \cap B) = A$.

Question 16

Let A and B be sets. Show that

a) $(A \cap B) \subseteq A$.

b) $A \subseteq (A \cup B)$.

c) $A - B \subseteq A$.

d) $A \cap (B - A) = \emptyset$.

e) $A \cup (B - A) = A \cup B$.

Solution

a) If x is in $A \cap B$, then perforce it is in A (by definition of intersection).

b) If x is in A, then perforce it is in$A \cup B$ (by definition of union).

c) If x is in $A - B$, then perforce it is in A (by definition of difference).

d) If $x \in A$ then $x \in B - A$. Therefore there can be no elements in $A \cap (B - A)$, so $A \cap (B - A) = \emptyset$.

e) $A \cup (B - A) = A \cup B$. means $x \in A$ or $x \in B\ AND\ x \notin A$, which means $x \in A\ OR\ x \in B$. Therefore $A \cup (B - A) = A \cup B$.

Question 26

Let A, B, and C be sets. Show that $(A - B) - C = (A - C) - (B - C)$.

Solution

Suppose x is in the left side, then x must be in A but in neither B nor C . Thus $x \in A - C$, but $x \notin B - C$, so x is in the right-hand side.
Suppose x is in the right side, then x must be in $A - C$ but not in $B - C$. Thus $x \in A - C$, but $x \notin B$, so x is in the left-hand side.

Question 32

Can you conclude that A = B if A, B, and C are sets such that

a) $A \cup C = B \cup C?$

b) $A \cap C = B \cap C?$

c) $A \cup C = B \cup C and A \cap C = B \cap C?$

Solution

a) No

b) No

c) Yes,By symmetry, it suffices to prove that $A \subseteq B$. Suppose that $x \in A$. There are two cases. If $x \in C$, then $x \in A \cap C = B \cap C$, whichforces $\in B$. On the other hand, if $x \notin C$, then because $x \in A \cup C = B \cup C$, we must have $x \in B$.

Question 56

Find $\bigcup_{i=1}^{\infty} A_i$ and $\bigcap_{i=1}^{\infty} A_i$if for every positive integer i,

a) $A_i = i, i+1, i+2, \ldots$.

b) $A_i = 0, i$.

c) $A_i = (0, i)$, that is, the set of real numbers x with $0 < x < i$.

d) $A_i = (i, \infty)$, that is, the set of real numbers x with $x > i$.

Solution

a) $\bigcup_{i=1}^{\infty} A_i = \{1, 2, 3, 4, ...\}$ and $\bigcap_{i=1}^{\infty} A_i = \emptyset$ since Every positive integer is excluded from at least one of the sets

b) $\bigcup_{i=1}^{\infty} A_i = \{0, 1, 2, 3, 4, ...\}$ and $\bigcap_{i=1}^{\infty} A_i = \{0\}$

c) $\bigcup_{i=1}^{\infty} A_i = \mathbb{R}^+$ and $\bigcap_{i=1}^{\infty} A_i = A_1$

d) $\bigcup_{i=1}^{\infty} A_i = A_1$ and $\bigcap_{i=1}^{\infty} A_i = \emptyset$

Question 58

Suppose that the universal set is $U = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Express each of these sets with bitstrings where the ith bit in the string is 1 if i is in theset and 0 otherwise.

a) $\{3, 4, 5\}$

b) $\{1, 3, 6, 10\}$

c) $\{2, 3, 4, 7, 8, 9\}$

Solution

a) 0011100000

b) 1010010001

c) 0111001110

2.1 Functions

Question 2

Determine whether f is a function from $\mathbb{Z}$ to $\mathbb{R}$ if

a) $f(n) = \pm n$.

b) $f(n) = \sqrt{n^2 + 1}$.

c) $f(n) = 1/(n^2 - 4)$.

Solution

a) No. Because $f(n)$ is not an assignment of exactly one element of $\mathbb{R}$ to each element of $\mathbb{Z}$.

b) Yes.

c) No. Because $f(2)$ and $f(-2)$ are not defined.

Question 4

Find the domain and range of these functions. Note that in each case, to find the domain, determine the set of elements assigned values by the function.

a) the function that assigns to each nonnegative integer its last digit

b) the function that assigns the next largest integer to a positive integer

c) the function that assigns to a bit string the number of one bits in the string

d) the function that assigns to a bit string the number of bits in the string

Solution

a) The domain is the set of nonnegative integers, and the range is the set of digits (0 through 9).

b) The domain is the set of positive integers, and the range is the set of positive integers greater than 1.

c) The domain is the set of strings, and the range is the set of nonnegative integers.

d) The domain is the set of strings, and the range is the set of nonnegative integers.

Question 8

Find these values.

a) $\lfloor 1.1 \rfloor$

b) $\lceil 1.1 \rceil$

c) $\lfloor -0.1 \rfloor$

d) $\lceil -0.1 \rceil$

e) $\lceil 2.99 \rceil$

f) $\lceil -2.99 \rceil$

g) $\lfloor \frac{1}{2} + \lceil \frac{1}{2} \rceil \rfloor$

h) $\lceil \lfloor \frac{1}{2} \rfloor + \lceil \frac{1}{2} \rceil + \frac{1}{2} \rceil$

Solution

a) 1

b) 2

c) -1

d) 0

e) 3

f) -2

g) 1

h) 2

Question 16

Consider these functions from the set of students in a discrete mathematics class. Under what conditions is the function one-to-one if it assigns to a student his or her

a) mobile phone number.

b) student identification number.

c) final grade in the class.

d) home town.

Solution

a) This is normally the one-to-one function.

b) This is one-to-one function.

c) This is normally not the one-to-one function since it is likely that two students have final grade in the class.

d) This is normally not the one-to-one function since it is likely that two students came from same home towm.

Question 30

Let $S = \{-1, 0, 2, 4, 7\}$. Find $f(S)$ if

a) $f(x) = 1$.

b) $f(x) = 2x + 1$.

c) $f(x) = \lceil x/5 \rceil$.

d) $f(x) = \lfloor (x^2 + 1)/3 \rfloor$.

Solution

a) $f(S) = \{1\}$.

b) $f(x) = \{-1, 1, 5, 9, 15\}$.

c) $f(x) = \{0, 1, 2\}$.

d) $f(x) = \{0, 1, 5, 16\}$.

Question 44

Let f be the function from $\mathbb{R}$ to $\mathbb{R}$ defined by $f(x) = x^2$. Find

a) $f^{-1}(1)$.

b) $f^{-1}(x|0 < x < 1)$.

c) $f^{-1}(x|x > 4)$.

Solution

a) $f^{-1}(1) = \{1, -1\}$.

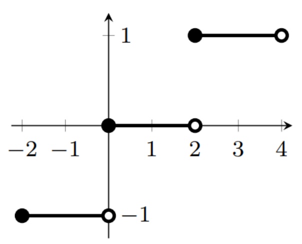
b) $f^{-1}(x|0 < x < 1) = \{x|0 < x < 1 \ \vee \ -1 < x < 0\}$.

c) $f^{-1}(x|x > 4) = \{x|x > 2 \ \vee \ x < -2\}$.

Question 66

Draw the graph of the function $f(x) = \lfloor x/2 \rfloor$ from $\mathbb{R}$ to $\mathbb{R}$.

Solution



Question 78

Let x be a real number. Show that $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

Solution

Suppose $p \leq x < p + 1$, then $\lfloor x \rfloor = p$.

- If $p \leq x < p + \frac{1}{3}$, then $\lfloor 3x \rfloor = 3p$, $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = p + p + p = 3p$
- If $p + \frac{1}{3} \leq x < p + \frac{2}{3}$, then $\lfloor 3x \rfloor = 3p + 1$, $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = p + p + (p + 1) = 3p + 1$
- If $p + \frac{2}{3} \leq x < p + 1$, then $\lfloor 3x \rfloor = 3p + 2$, $\lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor = p + (p + 1) + (p + 1) = 3p + 2$

Therefore, $\lfloor 3x \rfloor = \lfloor x \rfloor + \lfloor x + \frac{1}{3} \rfloor + \lfloor x + \frac{2}{3} \rfloor$.

2.4 Sequences and Summations

Question 4

What are the terms a_0, a_1, a_2, and a_3 of the sequence $\{a_n\}$,where an equals

a) $(-2)^n$?

b) 3?

c) $7 + 4^n$?

d) $2^n + (-2)^n$?

Solution

a) $a_0 = 1, a_1 = -2, a_2 = 4, a_3 = -8$

b) $a_0 = 3, a_1 = 3, a_2 = 3, a_3 = 3$

c) $a_0 = 8, a_1 = 11, a_2 = 23, a_3 = 71$

d) $a_0 = 2, a_1 = 0, a_2 = 8, a_3 = 0$

Question 10

Find the first six terms of the sequence defined by each of these recurrence relations and initial conditions.

a) $a_n = -2a_{n-1}, a_0 = -1$

b) $a_n = a_{n-1} - a_{n-2}, a_0 = 2, a_1 = -1$

c) $a_n = 3a_{n-1}^2, a_0 = 1$

d) $a_n = na_{n-1} + a_{n-2}^2, a_0 = -1, a_1 = 0$

e) $a_n = a_{n-1} - a_{n-2} + a_{n-3}, a_0 = 1, a_1 = 1, a_2 = 2$

Solution

a) $a_0 = -1, a_1 = 2, a_2 = -4, a_3 = 8, a_4 = -16, a_5 = 32$

b) $a_0 = 2, a_1 = -1, a_2 = -3, a_3 = -2, a_4 = 1, a_5 = 3$

c) $a_0 = 1, a_1 = 3, a_2 = 27, a_3 = 2187, a_4 = 14348907, a_5 = 617673396283947$

d) $a_0 = 1, a_1 = 1, a_2 = 2, a_3 = 2, a_4 = 1, a_5 = 1$

Question 20

Assume that the population of the world in 2017 was 7.6 billion and is growing at the rate of 1.12% a year.

a) Set up a recurrence relation for the population of the world n years after 2017.

b) Find an explicit formula for the population of the world n years after 2017.

c) What will the population of the world be in 2050?

Solution

Suppose the population of the world n years after 2017 is a_n (billions).

a) $a_n = (100 + 1.12)\% \cdot a_{n-1}, a_0 = 7.6$

b) $a_n = 7.6 \cdot [(100 + 1.12)\%]^n$

c) $a_{33} \approx 10.98$

Question 24

a) Find a recurrence relation for the balance $B(k)$ owed at the end of k months on a loan at a rate of r if a payment P is made on the loan each month. [*Hint*: Express $B(k)$ in terms of $B(k-1)$ and note that the monthly interest rate is $r/12$.]

b) Determine what the monthly payment P should be so that the loan is paid off after T months.

Solution

a) $B(k) = (1 + r/12)B(k-1) - P$

b) From recurrence relation we can know the explicit formula is $B(k) - 12P/r = (1+r/12)^k(B(0) - 12P/r)$
Then we can solve that $T = \log_{1+r/12}(\frac{-12p}{B(0)r-12p})$

Question 32

Find the value of each of these sums.

a) $\Sigma_{j=0}^{8} = (-1 + (-1)^j)$

b) $\Sigma_{j=0}^{8} = (3^j - 2^j)$

c) $\Sigma_{j=0}^{8} = (2 \cdot 3^j + 3 \cdot 2^j)$

d) $\Sigma_{j=0}^{8} = (2^{j+1} - 2^j)$

Solution

a) $\Sigma_{j=0}^{8} = (-1 + (-1)^j) = 0$

b) $\Sigma_{j=0}^{8} = (3^j - 2^j) = 9330$

c) $\Sigma_{j=0}^{8} = (2 \cdot 3^j + 3 \cdot 2^j) = 21215$

d) $\Sigma_{j=0}^{8} = (2^{j+1} - 2^j) = 511$