Birth-and-Dead Processes

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What is it about?

We will obtain the dynamics governing a very general process. This process explains the Poisson process.

Useful results

Consider the positive random variable $X : \Omega \to \mathbb{R}_+$. Its expected value is

$$\mathrm{E}(X) = \int_{\mathbb{R}_+} x f_X(x) dx = \int_{\mathbb{R}_+} \big[1 - F_X(x)\big] dx,$$

where f_X and F_X are, respectively, the density and cumulative distribution function that characterize the behaviour of X.

As a consequence, if $X : \Omega \to \mathbb{N}_0$, then

$$\mathrm{E}(X) = \sum_{\ell=0}^{\infty} \Pr(X > \ell) = \sum_{\ell=1}^{\infty} \Pr(X \geq \ell).$$

Law of Total Probability

Consider the sample space Ω and a partition $C_1, C_2, \ldots, C_m$ of subsets of Ω. These subsets form a partition if they are mutually disjoint ($C_i \cap C_j = \emptyset$ for every $i \neq j$) and exhaustive ($C_1 \cup C_2 \cup \cdots \cup C_m = \Omega$).

We can compute the probability of any event $A \subset \Omega$ as a sum of weighted conditional probabilities:

$$\Pr(A) = \sum_{\ell=1}^{m} \Pr(A \mid C_\ell) \Pr(C_\ell).$$

See examples in Dekking et al. (2005, Section 3.3).

Recap about the Poisson Process

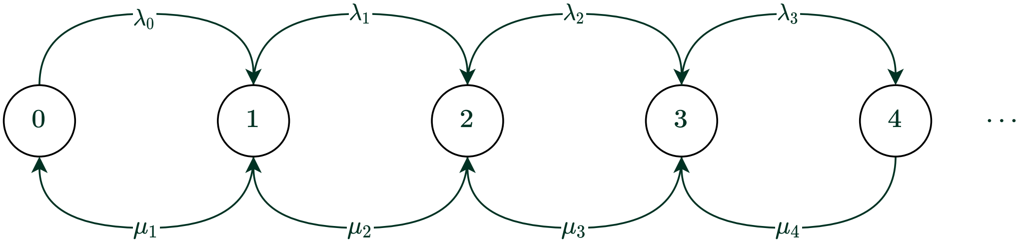
H1: The number of arrivals on non-overlapping time intervals are independent.

H2: For a small time interval Δt holds that

- $\Pr\big(\text{exactly one arrival in } (t, t+\Delta t)\big) = \theta\Delta t + o(\Delta t)$;
- $\Pr\big(\text{no arrivals in } (t, t+\Delta t)\big) = 1 - \theta\Delta t + o(\Delta t)$;
- $\Pr\big(\text{more than one arrival in } (t, t+\Delta t)\big) = o(\Delta t)$,

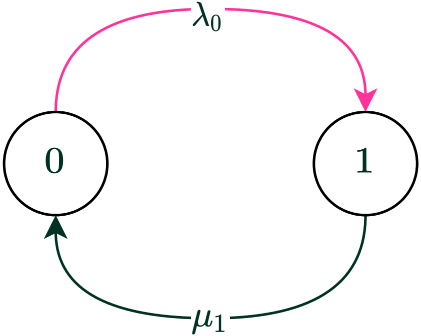
and $\lim_{\Delta t \to 0} o(\Delta t)/\Delta t = 0$.

Making it more general...



- $\Pr\left(\text{birth during } (t, t+\Delta t) \mid S_i\right) = \lambda_i \Delta t + o(\Delta t)$.
- $\Pr\left(\text{death during } (t, t+\Delta t) \mid S_i\right) = \mu_i \Delta t + o(\Delta t)$, and $\mu_0 = 0$.

Reaching the state of no customer in the system



$$
\begin{aligned}
P_0(t+\Delta t) &= \Pr(\text{no birth} \mid S_0)\Pr(S_0) + \Pr(\text{death} \mid S_1)\Pr(S_1) \\
&= (1-\lambda_0\Delta t)P_0(t) + \mu_1 \Delta t P_1(t)
\end{aligned}
$$

Reaching the state of no customer in the system

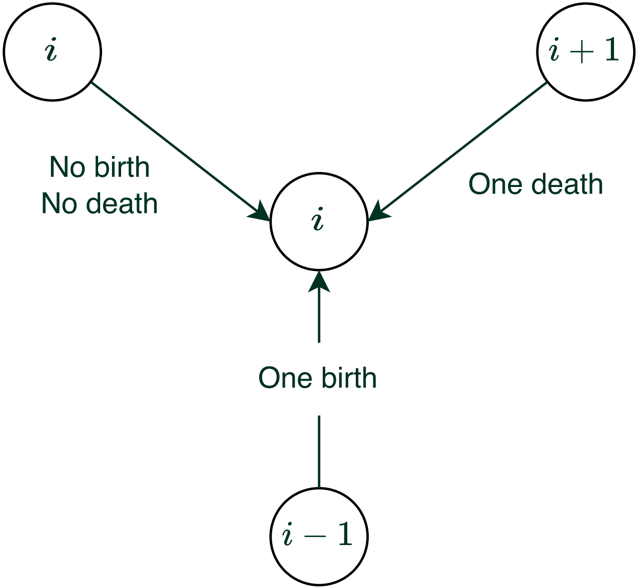
We want to obtain $P_0(t)$ by first finding $P_0'(t)$:

$$
\begin{aligned}
P_0(t+\Delta t) &= P_0(t)(1-\lambda_0\Delta t) + P_1(t)\mu_1\Delta t \\
\frac{P_0(t+\Delta t) - P_0(t)}{\Delta t} &= \frac{P_0(t)(1-\lambda_0\Delta t) + P_1(t)\mu_1\Delta t - P_0(t)}{\Delta t} \\
&= \frac{P_0(t) - P_0(t)\lambda_0\Delta t + P_1(t)\mu_1\Delta t - P_0(t)}{\Delta t} \\
&= \frac{-P_0(t)\lambda_0\Delta t + P_1(t)\mu_1\Delta t}{\Delta t} = -P_0(t)\lambda_0 + P_1(t)\mu_1
\end{aligned}
$$

Taking the limit $\Delta t \to 0$

$$
P_0'(t) = -P_0(t)\lambda_0 + P_1(t)\mu_1.
$$

Reaching i customers in the system



Reaching i customers in the system

$$
\begin{aligned}
P_i(t+\Delta t) &= \Pr(\text{No birth, no death} \mid S_i)P_i(t) + \\
&\quad \Pr(\text{One death} \mid S_{i+1})P_{i+1}(t) + \\
&\quad \Pr(\text{One birth} \mid S_{i-1})P_{i-1}(t) \\
&= P_i(t)(1-\lambda_i\Delta t)(1-\mu_i\Delta t) + P_{i+1}(t)\mu_{i+1}\Delta t + P_{i-1}(t)\lambda_{i-1}\Delta t \\
\frac{P_i(t+\Delta t)-P_i(t)}{\Delta t} &= \frac{P_i(t)(1-\lambda_i\Delta t)(1-\mu_i\Delta t) + P_{i+1}(t)\mu_{i+1}\Delta t + P_{i-1}(t)\lambda_{i-1}\Delta t - P_i(t)}{\Delta t}
\end{aligned}
$$

Go through the details, and take the limit to obtain

$$
P_i'(t) = -(\lambda_i+\mu_i)P_i(t) + \lambda_{i-1}P_{i-1}(t) + \mu_{i+1}P_{i+1}(t),
$$

which holds for $i = 1, 2, \ldots$.

Our full description

$$
\begin{aligned}
P_0'(t) &= -P_0(t)\lambda_0 + P_1(t)\mu_1, \\
P_i'(t) &= -(\lambda_i + \mu_i)P_i(t) + \lambda_{i-1}P_{i-1}(t) + \mu_{i+1}P_{i+1}(t).
\end{aligned}
$$

This is a system of Ordinary Differential Equations (ODE).

Nice! But we want to get rid of the variability on t.

If the time is not so small, we should check all the ways in which the system is able to reach state i.

After a very long time, we have reached the steady state or equilibrium. The probabilities no longer depend on the initial state.

The system is characterized by a vector of probabilities

$$\boldsymbol{\pi} = \left(\pi_0, \pi_1, \dots\right)$$

which obeys

$$-\lambda_0 \pi_0 + \mu_1 \pi_1 = 0 \tag{1}$$

$$-(\lambda_i + \mu_i)\pi_i + \lambda_{i-1}\pi_{i-1} + \mu_{i+1}\pi_{i+1} = 0 \tag{2}$$

Long-term behaviour ii

This is a system of linear equations for $i = 1, 2, \ldots$.

Since $\boldsymbol{\pi}$ is a vector of probabilities, we have that

$$\sum_{i=0}^{\infty} \pi_i = 1. \tag{3}$$

Let's solve the system.

$$\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$$
$$\pi_2 = \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi_0$$
$$\pi_3 = \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}\pi_0$$
$$\vdots$$
$$\pi_{k+1} = \pi_0 \prod_{i=0}^{k} \frac{\lambda_i}{\mu_{i+1}}$$

Long-term behaviour iv

Using these results on (3),

$$1 = \pi_0 + \pi_1 + \pi_2 + \pi_3 + \cdots = \pi_0 + \frac{\lambda_0}{\mu_1}\pi_0 + \frac{\lambda_0\lambda_1}{\mu_1\mu_2}\pi_0 + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3}\pi_0 + \cdots \tag{4}$$

$$= \pi_0\left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_0\lambda_1}{\mu_1\mu_2} + \frac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} + \cdots\right) = \pi_0\left(1 + \sum_{k=0}^{\infty}\prod_{i=0}^{k}\frac{\lambda_i}{\mu_{i+1}}\right) \tag{5}$$

therefore,

$$\pi_0 = \frac{1}{1 + \sum_{k=0}^{\infty}\prod_{i=0}^{k}\frac{\lambda_i}{\mu_{i+1}}}. \tag{6}$$

The steady state distribution is characterized by the birth rates $\lambda_0, \lambda_1, \ldots$ and death rates $\mu_1, \mu_2, \ldots$.

Long-term behaviour v

Knowing $\boldsymbol{\pi}$, the steady state distribution of the system, allows us to compute many quantities of interest.

Example 1

Customers arrive at rate λ to a single-queue single-server system. They are served at rate μ. Assuming that $\lambda < \mu$, what is the proportion of time the system is idle?

The system is idle when there is no customer. This happens with probability

$$\pi_0 = \frac{1}{1 + \sum_{k=0}^{\infty} \prod_{i=0}^{k} \frac{\lambda_i}{\mu_{i+1}}}$$

using that $\lambda_0 = \lambda_1 = \cdots = \lambda$ and that $\mu_1 = \mu_2 = \cdots = \mu$, and writing $\eta = \lambda/\mu$:

$$\pi_0 = \frac{1}{1 + \sum_{k=0}^{\infty} \prod_{i=0}^{k} \frac{\lambda}{\mu}} = \frac{1}{1 + \sum_{k=0}^{\infty} \eta^{k+1}} = \frac{1}{\sum_{k=0}^{\infty} \eta^{k}} = \frac{1}{\frac{1}{1-\eta}} = 1 - \eta.$$

Moreover, the probability of having one customer being served is

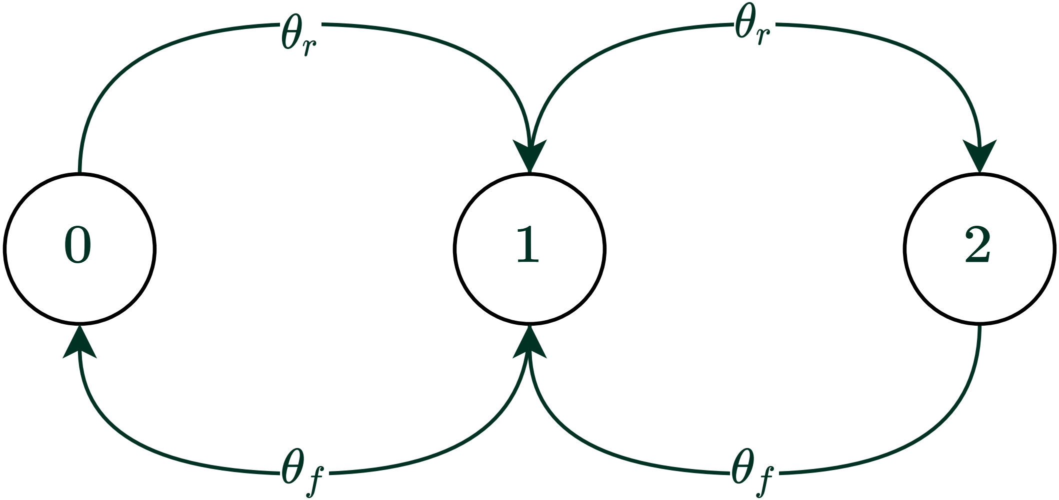
$$\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0 = \frac{\lambda}{\mu}\pi_0 = \eta(1 - \eta).$$

Compute the probability of having one customer waiting to be served: π_2.

A ticket reservation system has two computers, one on-line and one standby. The operating computer fails after an exponentially distributed time, having mean T_f and then it is replaced by the standby computer, i.e., at any time only one or zero computers are operating. There is one repair facility, i.e., at any time only one or zero computers can be repaired. The repair times are exponentially distributed with mean T_r. What fraction of the time will the system be down?

Denote $\theta_f = 1/T_f$ and $\theta_r = 1/T_r$, the rates of failure and repair, respectively.

The birth-and-death graphical representation of the number of working tickets machines is as follows.



θ_r
θ_r
0
1
2
θ_f
θ_f

Example 2 iii

We now compute the probability of zero working machine: π_0.

As seen before, $-\theta_r\pi_0 + \theta_f\pi_1 = 0$, so Also,

$$\pi_1 = \pi_0 \frac{\theta_r}{\theta_f};$$

moreover,

$$\pi_2 = \pi_0 \frac{\theta_r}{\theta_f} \frac{\theta_r}{\theta_f} = \pi_0 \left(\frac{\theta_r}{\theta_f}\right)^2$$

and

$$1 = \pi_0 + \pi_1 + \pi_2,$$

Example 2 iv

so

$$\pi_0 = 1 - \pi_1 - \pi_2 = 1 - \pi_0 \frac{\theta_r}{\theta_f} - \pi_0 \left(\frac{\theta_r}{\theta_f}\right)^2.$$

By solving on π_0:

$$\pi_0 = \frac{1}{1 + \frac{\theta_r}{\theta_f} + \left(\frac{\theta_r}{\theta_f}\right)^2}.$$

If the expected repair time is one tenth of the expected working time (discuss if this makes sense), then $\theta_r/\theta_f = 10$, and $\pi_0 = \frac{1}{1+10+10^2} = \frac{1}{111}$. In other words, the system is out-of-order approximately $1\,\%$.

If the machines are prone to fail and the expected repair time is half the expected working time, then $\theta_r/\theta_f = 2$, and $\frac{1}{1+2+2^2} = \frac{1}{7}$. In other words, the system is out-of-order approximately $14\,\%$ of the time.

If the expected repair time is the same as the expected working time, then $\theta_r/\theta_f = 1$, and $\pi_0 = \frac{1}{1+1+1^2} = \frac{1}{3}$. The system is out-of-order approximately $33\,\%$ of the time.

References

Dekking, F. M., Kraaikamp, C., Lopuhaä, H. P. & Meester, L. E. (2005), *A Modern Introduction to Probability and Statistics: Understanding Why and How*, Springer.