COMP312/DATA304/DATA474
Simulation & Stochastic Models

The Poisson Process

Alejandro C. Frery

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VICTORIA UNIVERSITY OF WELLINGTON
TE HERENGA WAKA

School of Mathematics and Statistics
New Zealand

What is it about?

We will see models which may be useful and interesting for describing the times between arrivals and between servings, along with their properties and limitations.

Remember:

A quote by Box et al. (2005)
The model is a hypothetical conjecture that might or might not summarize and/or explain important features of the data.

All models are wrong; some models are useful.

Basic hypotheses about arrival and serving times

- At most one arrival occurs at any instant. Denote t_i the time at which the ith arrival occurred.
- Denote $T_i = t_i - t_{i-1}$ the time between arrivals i and $i-1$. We will assume that $T_1, T_2, \ldots$ are independent and identically distributed random variables (strongly stationary process).
- We will characterize the distribution of the inter-arrival times by the probability density function f_T. The mean inter-arrival time is $\mathrm{E}(T) = \int_{\mathbb{R}_+} t f_T(t) dt$, and the arrival rate is $1/\mathrm{E}(T)$.

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Further assumptions

H1: The number of arrivals on non-overlapping time intervals are independent.

H2: For a small time interval h holds that

- $\Pr\left(\text{exactly one arrival in } (t, t+h)\right) = \theta h + o(h)$;
- $\Pr\left(\text{no arrivals in } (t, t+h)\right) = 1 - \theta h + o(h)$;
- $\Pr\left(\text{more than one arrival in } (t, t+h)\right) = o(h)$,

and $\lim_{h \to 0} o(h)/h = 0$.

Consequences

Theorem
If assumptions H1 and H2 hold, then the number of arrivals on intervals of length t has Poisson distribution with parameter θt, and the inter-arrival times are exponentially distributed with parameter θ.

Theorem
Inter-arrival times are independent and exponentially distributed with rate $\theta > 0$ if and only if the number of arrivals that occur in an interval of length t follows a Poisson distribution with parameter θt.

Voilà

Finally: A process in which

- inter-arrival times are independent identically distributed random variables following an Exponential law with rate parameter $\theta > 0$ or, equivalently,
- in which the number of arrivals at intervals of length t follows a Poisson distribution with parameter θt and the number of arrivals in disjoint intervals are independent,

is called a Poisson Process with arrival rate $\theta > 0$, or PP(θ) for short.

References

Box, G. E., Hunter, J. S. & Hunter, W. G. (2005), *Statistics for Experimenters: design, discovery and innovation*, 2 edn, Wiley, Hoboken, NJ.