Estimating Quantile Production Functions: A Control Function Approach

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Abstract

We propose a new approach to estimating production functions when elasticities are heterogeneous across the conditional distribution of output. We provide a quantile regression framework which controls for unobserved productivity as an extension of the control function approaches of Ackerberg *et al.* (2015) and Levinsohn and Petrin (2003). Production function parameters are estimated in a simple two-stage approach which relies on a location-shift assumption on unobserved productivity. We show that this method allows us to capture heterogeneity in output elasticities which may not be found in the conditional mean estimates. We provide small-sample evidence in a Monte Carlo study to show that this approach is robust compared to other production function estimators. The method is applied to firm and plant-level manufacturing data from the U.S., Chile, and Colombia. The findings confirm that the proposed method captures unobserved heterogeneity in output elasticities.

Keywords: Production functions, productivity, heterogeneous elasticity, quantile regression

JEL Classification: C14, C31, C36, D24

1 Introduction

Production function estimation is an ongoing and historical empirical research topic that links firm's inputs to output decisions. Identification of the output elasticities and consequently the distribution of firm-level productivity is constrained by endogeneity issues. This is because productivity is unobserved by the econometrician, but observed by the firm when making input decisions.

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A popular approach to address this issue is to introduce a proxy variable such as investment, initialized by Olley and Pakes (1996) (henceforth OP) or an intermediate input using Levinsohn and Petrin (2003) (henceforth LP) or Ackerberg *et al.* (2015) (henceforth ACF). These proxies are functions of state variables such as capital and unobserved productivity. This function is assumed to be strictly increasing in its scalar unobserved productivity component. Inverting this function controls for unobserved productivity and the production function parameters can be estimated using a simple two-stage approach.

This approach is commonly applied to production functions with homogeneous coefficients such as the Cobb-Douglas. One implication of this restriction is that firms have constant capital shares. This contradicts recent empirical evidence that find substantial variation in capital shares across firms in similar industries. Allowing for heterogeneous coefficients is one way to reconcile these empirical findings by use of a firm-specific production function. The literature on heterogeneous production functions is small relative to the empirical research using the homogeneous coefficient model, even though many recent empirical studies have found heterogeneity in firm behavior and decisions.[1] This is because estimating the homogeneous coefficient model by itself is very difficult due to the issue of unobserved productivity.

It is well known that there is substantial productivity variation across firms even after correcting for the simultaneity bias of inputs. One could question whether production function estimators that use conditional mean estimates are best suited to capturing this heterogeneity. To address this, we propose a quantile regression framework as an alternative to the traditional mean estimator used in the control function approach. This allows us to estimate a production function that is robust to outliers in the data as well as address the fact that many recent empirical findings of firm heterogeneity are not supported by homogeneous coefficient models. This framework provides new tools to examine features of firm production. For example, in the presence of outliers in the data, a median TFP estimator may be more appropriate for subsequent productivity analysis than a mean estimator. Quantile regression estimates give a richer description of the entire conditional distribution of output. Unobserved productivity remains an important econometric challenge so we propose a quantile estimator that corrects for simultaneity bias in the framework of the control function approach.

The literature on control function approaches for quantile regression models is still a

[1]Some notable examples are Kasahara, Schrimpf and Suzuki (2017), Balat, Brambilla and Sasaki (2019) and Li and Sasaki (2017) to name of few. Also Gandhi *et al.* (2020) who estimate a nonparametric production function and obtain heterogeneous estimates.

developing area. Therefore, it is not straightforward to estimate production functions by allowing for endogenous inputs and their heterogeneous coefficients.[2] This is likely due to the fact that the control function approaches of OP, LP, and ACF rely on conditional moment restrictions on multiple unobservables. For example, the second stage of their approach utilizes the condition $\mathbb{E}[\xi_{it} + \varepsilon_{it}|\mathcal{I}_{it-1}] = 0$, where ξ_{it} are innovations to productivity, ε_{it} are independent and identically distributed (i.i.d.) ex-post shocks to production, and $\mathcal{I}_{it-1}$ is the firm's information set at time $t-1$. This restriction does not extend naturally to conditional quantiles, since unlike expectation operators, quantile operators are non-linear. In contrast to the mean of multidimensional unobservables, there is no consensus about defining the quantile counterpart. As a result, it is a non-trivial task to appropriately define quantile functions in the presence of bivariate unobservables, ξ_{it} and ε_{it}, in the conventional framework of the control function approaches. Similar issues are encountered in the quantile panel data literature where researches have proposed alternatives to individual fixed effects using correlated random effects.[3] However, these approaches are not directly applicable to estimating the conditional quantiles of production functions, since the unobservables in these models are typically assumed to be time-varying.

In our model, we allow for non-neutrality of the unobserved idiosyncratic production shock, while the component of productivity that is anticipated by firms to be Hicks-neutral. We use the control function approach in this framework to control for the part of production unobservables that are correlated with input decisions. We are not aware of any published paper which takes into account the endogeneity issue of production functions in the conventional quantile regression framework. We fill the gap in this paper by proposing an easy-to-implement estimator which is available on the author's Github.

We show the conditional quantiles of firm production are non-parametrically identified using conditional deconvolution arguments under independence of the unobservable technology shocks, productivity, and productivity innovation shocks conditional on inputs and other mild regularity conditions on the conditional characteristic functions. These assumptions are plausible and have been used in identification arguments of other work such as Hu *et al.* (2020). The conditional characteristic functions are identified up to a location normalization. We show that using the conditions in ACF allows us to identify this unknown location. Identification of the conditional characteristic functions implies identification of the conditional distribution functions (CDFs). In turn, identification of the conditional quantiles is guaranteed by the identification of the CDFs, due to the inverse relationship between

[2] See for example Chesher (2003), Ma and Koenker (2006), and Lee (2007).

[3] See for example Harding and Lamarche (2016) and Cai *et al.* (2018) for two separate approaches.

quantiles and the CDF. We propose an estimator motivated by our identification strategy which nests the existing control function approaches as an initial consistent estimator in our two-step approach. Consistency and asymptotic normality of the proposed estimator are provided.

We show through simulation that our estimator performs well, when productivity is estimated using the control function approaches of either Ackerberg *et al.* (2015) or Levinsohn and Petrin (2003). The estimator is successful in capturing both heterogeneous output elasticities and controlling for unobserved productivity. In our empirical application, we consider several popular firm and plant-level manufacturing datasets and compare our estimator to the ACF estimator for a value-added production function. We show that heterogeneity in these estimates implies differences in other features of firm production, such as returns to scale, capital intensity, and TFP.

The rest of the paper is organized as follows. Section 2 reviews the ACF procedure for estimating a homogeneous coefficient production function. Section 3 introduces the econometric model and its economic interpretations. Section 4 presents conditions under which our model is identified. Section 5 proposes a computationally simple estimator and discusses its asymptotic properties. Section 6 presents finite-sample behaviors of the estimator via Monte Carlo experiments and Section 7 applies this estimator to the U.S., Chilean, and Colombian manufacturing datasets. Section 8 concludes with directions for future research.

2 Literature Review

2.1 Production Function Estimation

We briefly review the ACF procedure for estimating a *value-added* production function (in logs):

$$y_{it} = \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + \varepsilon_{it}, \tag{1}$$

where y_{it} denotes value-added output for firm i at time t, l_{it} denotes labor input, k_{it} denotes capital input, ω_{it} is unobserved productivity and ε_{it} denotes an i.i.d. shock to production.[4]

To control for the correlation between ω_{it} and inputs k_{it} and l_{it}, ACF introduce an

[4]We omit the constant β_0 because it is not separately identified from the mean of productivity.

intermediate input demand function defined as

$$m_{it} = m_t(k_{it}, l_{it}, \omega_{it}), \tag{2}$$

where the function m is strictly increasing in ω_{it} for all k_{it} and l_{it}. Productivity can then be expressed as

$$\omega_{it} = m_t^{-1}(k_{it}, l_{it}, m_{it}). \tag{3}$$

Substituting this equation into the production function they obtain

$$y_{it} = \beta_k k_{it} + \beta_l l_{it} + m_t^{-1}(k_{it}, l_{it}, m_{it}) + \varepsilon_{it} = \Phi_t(k_{it}, l_{it}, m_{it}) + \varepsilon_{it}. \tag{4}$$

The function, $\Phi_t(k_{it}, l_{it}, m_{it})$, is identified by the following first stage moment restriction

$$\mathbb{E}[\varepsilon_{it} | \mathcal{I}_{it}] = 0, \tag{5}$$

where $\mathcal{I}_{it}$ denotes the firm's information at time t. The first stage estimate of Φ_t can be obtained by a local linear regression or a polynomial regression in (k_{it}, l_{it}, m_{it}).

For the second stage, productivity can be assumed to follow an AR(1) process given by

$$\omega_{it} = \mathbb{E}[\omega_{it} | \omega_{it-1}] + \xi_{it} = \rho\omega_{it-1} + \xi_{it}, \tag{6}$$

where ξ_{it} denotes an innovation to productivity which satisfies $\mathbb{E}[\xi_{it} | \mathcal{I}_{it-1}] = 0$. Plugging into the production function gives

$$\begin{aligned} y_{it} &= \beta_k k_{it} + \beta_l l_{it} + \rho\omega_{it-1} + \xi_{it} + \varepsilon_{it} \\ &= \beta_k k_{it} + \beta_l l_{it} + \rho(\Phi_{t-1}(k_{it-1}, l_{it-1}, m_{it-1}) - \beta_k k_{it-1} - \beta_l l_{it-1}) + \xi_{it} + \varepsilon_{it}. \end{aligned}$$

The production function parameters β_k, β_l and ρ are identified from the moment restrictions given by

$$\begin{aligned} &\mathbb{E}[\xi_{it} + \varepsilon_{it} | \mathcal{I}_{it-1}] \\ &\quad = \mathbb{E}[y_{it} - \beta_k k_{it} - \beta_l l_{it} - \rho(\Phi_{t-1}(k_{it-1}, l_{it-1}, m_{it-1}) - \beta_k k_{it-1} - \beta_l l_{it-1}) | \mathcal{I}_{it-1}] = 0. \end{aligned} \tag{7}$$

Estimation of the second stage coefficients proceeds by plugging in first stage estimates $\hat{\Phi}_{t-1}$ and forming a Generalized Method of Moments (GMM) criterion function.

3 A Random Coefficient Production Function

We specify a Skorohod representation for a firm's *value-added* production function as

$$y_{it} = \beta_k(\eta_{it})k_{it} + \beta_l(\eta_{it})l_{it} + \omega_{it}, \quad \text{where} \quad \eta_{it} \sim U(0,1), \tag{8}$$

and $Q_\tau(y_{it}|\mathcal{I}_{it})$ is the conditional τ-th quantile of y_{it} given the information set $\mathcal{I}_{it}$ for $\tau \in (0,1)$. The variables in equation (8) have the same definitions as the ones we introduced in the ACF model. The Skorohod representation assumes that the unobserved heterogeneity enters through the rank of a firm on the conditional output distribution. In our model, productivity ω_{it} is additively separable and does not depend on η_{it}. Therefore, ω_{it} is only a location-shift of the conditional output distribution. We assume the constant in the production function does not vary over τ and subsume it into the productivity term. The location-shift specification is crucial in our approach since ω_{it} also appears in the conditional mean counterpart of equation (8). We will argue that if productivity can be consistently estimated using the control function approach of ACF, then the production function parameters $\beta_k(\tau)$ and $\beta_l(\tau)$ can be consistently estimated using quantile regression.

A value-added specification in equation (8) is non-trivial. Value-added production functions are common in the empirical literature. However, the objects recovered from a value-added model such as the output elasticities and TFP can only be mapped to its gross-output counterpart under specific structural production functions such as Leontief value-added. For example, a Leontief value-added production function in our framework could be written as

$$y_{it} = \min\{\beta_k(\eta_{it})k_{it} + \beta_l(\eta_{it})l_{it} + \omega_{it}, \beta_m + m_{it}\},$$

where, thanks to the equivariance properties of quantiles we have

$$Q_\tau(y_{it}|\mathcal{I}_{it}) = \min\{\beta_k(\tau)k_{it} + \beta_l(\tau)l_{it} + \omega_{it}, \beta_m + m_{it}\},$$

so that the conditional quantile of a structural value-added production function can be written as

$$Q_\tau(y_{it}|\mathcal{I}_{it}) = \beta_k(\tau)k_{it} + \beta_l(\tau)l_{it} + \omega_{it},$$

which corresponds directly to the conditional quantile of equation (8). The value-added approach avoids the non-identification results of Gandhi *et al.* (2020). One consequence of this approach is that estimates of the output elasticities, and hence TFP, may be more disperse than gross-output estimates (one which includes intermediate inputs). In the Online

Appendix, we provide results using estimates from a gross-output production function to examine whether our estimator proposed in the next section can still capture heterogeneity in the quantiles.

We make a subtle distinction between the unobserved technology shock η_{it} in our model and the production shock ε_{it} in the standard production function models. The term ε_{it} is usually referred to as an ex-post shock and/or measurement error in output. Here, an ex-post shock simply refers to unanticipated shock to output. However, in the quantile regression literature, we must consider these interpretations carefully because measurement error in the dependent variable will lead to bias in quantile regression estimators. Hausman *et al.* (2021) show that this bias (in a bivariate setting) understates the true dispersion in quantile regression estimates. In the Online Appendix, we discuss how our estimator can be adapted to measurement error in output using their approach. This type of measurement error may be non-negligible in many applications where deflated sales is used as the output measure and the deflator is only industry-specific (common to firms).

A fundamental question is which factors contribute to dispersion in the conditional output distribution. One approach would require describing the relationship between input usage and output variation that is contributed to uncertainty in η_{it}. For example, consider the location-scale model, which is a special case of (8):

$$y_{it} = \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + (\mu_k k_{it} + \mu_l l_{it})\eta_{it}, \tag{9}$$

which implies that the τ-th conditional quantile of y_{it} is given by

$$Q_\tau(y_{it}|\mathcal{I}_{it}) = \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + (\mu_k k_{it} + \mu_l l_{it})F^{-1}(\tau), \tag{10}$$

where $F^{-1}(\tau)$ is the quantile function of production shocks η_{it}.

If input choices can impact firm's production beyond the conditional mean, this implies that inputs have some degree of control over the risk of production. A volume of literature that originated in the late 1970's challenged the standard stochastic specifications of production functions (Just and Pope, 1978, 1979) by considering a specification that allows firm's inputs to either increase or decrease the marginal variability of final output. These models are commonly applied to the agricultural industry where, for example, the production risk over the yield of harvested crops could be caused by unexpected insect infestations. In this case the variability of production could be reduced through pesticide usage. Since manufacturing businesses tend to operate in a more controlled environment, risk is less prevalent in these industries so the conditional variance of η_{it} may be smaller.

Extending this idea to the entire distribution of output requires reformulating how firms form beliefs about uncertainty in profits due to production risk. It is standard to use a rational expectations framework which lead to decision rules for static inputs that are functions of $E[\exp(\eta_{it})]$. In the presence of production risk, firms may have different beliefs regarding the uncertainty of final output. Instead of maximizing expected profit, a firm could maximize the τ-th quantile of profits. Different managers may have different preferences for risk and may choose different optimal expenditures for inputs. A short list of papers have considered quantile utility maximization such as Manski (1988), Rostek (2009), Chambers (2007), and Bhattacharya (2009). Dynamic input choices such as investment are much more difficult to solve using the quantile utility framework and the reader can refer to de Castro and Galvao (2019) for a treatment of dynamic quantile utility models. As far as we know, the quantile utility framework has not been applied to firm decision problems, as it is difficult to incorporate this theory into testable econometric models.[5] We leave this for future research agenda.

The model presented here is also related to the stochastic frontier analysis (SFA) literature. A SFA model of production proposed by Aigner *et al.* (1977) introduces statistical error into a frontier model. Frontier models assume that firms deviate from an optimal frontier of production. The SFA model is typically written as

$$y_i = f(x_i, \beta) + \varepsilon_i, \tag{11}$$

where x_i are inputs to production, and β are the parameters and $\varepsilon_i = \eta_i - u_i$. The error term η_i denotes statistical noise in the model such as measurement error and u_i represents one-sided deviations from the production frontier. Estimates of β are typically obtained using maximum likelihood which requires strong distributional assumptions on the error terms. The production frontier can be characterized by the highest rank in the conditional output distribution. This suggests quantile regression as a natural candidate for frontier analysis. Aragon *et al.* (2005) uses a deterministic frontier which interprets production functions as being from a continuous interval $\tau \in [0, 1]$, where $\tau \to 1$ converges to the efficient frontier. The difficulty with this approach is choosing which quantile to estimate as the frontier and subsequent inference of the extremal quantiles.[6] Predicting differences between the

[5] Although de Castro *et al.* (2019) has made some progress on this front. Their econometric model is based on conditional quantile restrictions from a "quantile" Euler equation for consumption.

[6] See Chernozhukov (2005) on how to extend inference in quantile regression to extreme tails in the conditional distribution.

frontier and a production process for a given firm is also complicated, since the predicted error in a stochastic frontier model is a composite of technical inefficiency and noise. Other econometric issues, such as endogeneity of input choices with respect to inefficiency, are difficult to incorporate in this framework. It may be possible to adapt the model presented here to a stochastic frontier model with statistical noise, but such an application is outside the scope of this paper.

4 Identification

In this section, we show that the model presented in equation (8) is non-parametrically identified. Our results can also be applied to other production functions such as translog, provided that productivity is additively separable. Let $\varepsilon_{it} = k_{it}[\beta_k(\eta_{it}) - \beta_k^\mu] + l_{it}[\beta_l(\eta_{it}) - \beta_l^\mu]$. A conditional mean equation for (8) can be written as

$$y_{it} = \beta_k^\mu k_{it} + \beta_l^\mu l_{it} + \omega_{it} + \varepsilon_{it}, \tag{12}$$

where $\mathbb{E}[\varepsilon_{it}|\mathcal{I}_{it}] = 0$. We show that the production function coefficients $\boldsymbol{\beta}(\tau) = (\beta_k(\tau), \beta_l(\tau))$ are non-parametrically identified with time-period observations $T = 2$ under conditional independence assumptions and other mild regularity conditions. To simplify notation, we let $x_{it} = (k_{it}, l_{it}), x_{it+1} = (k_{it+1}, l_{it+1})$, and $x_i = (x_{it}, x_{it+1})$. Let $Z_{it} = \beta_k(\eta_{it})k_{it} + \beta_l(\eta_{it})l_{it}$. For any random variable X and $\rho \neq 0$, let $\tilde{X} = X/\rho$. We then write two consecutive period of output as $y_{it} = Z_{it} + \omega_{it}$ and $y_{it+1} = Z_{it+1} + \omega_{it+1}$. We assume a linear AR(1) process for productivity, $\omega_{it+1} = \rho\omega_{it} + \xi_{it+1}$. Plugging this into the second period of output equation we can write $y_{it+1} = Z_{it+1} + \rho\omega_{it} + \xi_{it+1}$. Provided that $\rho \neq 0$, we can write $\tilde{y}_{it+1} = y_{it+1}/\rho = \tilde{Z}_{it+1} + \tilde{\xi}_{it+1} + \omega_{it}$. Therefore, we write two repeated measures of productivity as

$$\begin{aligned} y_{it} &= Z_{it} + \omega_{it} \\ \tilde{y}_{it+1} &= \tilde{Z}_{it+1} + \tilde{\xi}_{it+1} + \omega_{it}. \end{aligned} \tag{13}$$

Our goal is identification of the conditional quantile:

$$Q_\tau(Z_{it}|x_i) = x_{it}\boldsymbol{\beta}(\tau),$$

which can be identified if the conditional distribution function:

$$F_{Z_{it}|x_i}(Z_{it}|x_i) = \frac{1}{2} - \lim_{\upsilon\to\infty}\int_{-\upsilon}^{\upsilon} \frac{e^{-\mathrm{i}sZ_{it}}}{2\pi\mathrm{i}s}\phi_{Z_{it}|x_i}(s|x_i)ds, \tag{14}$$

is identified. Since the quantile function is the inverse of the CDF, this implies identification of the conditional quantiles. Therefore, identification relies on the conditional characteristic function $\phi_{Z_{it}|x_i}(s|x_i)$ being identified. We utilize conditional deconvolution arguments to identify conditional characteristic functions up to the coefficient ρ and an unknown location, $\mathbb{E}[\omega_{it}|x_i]$.[7] As we show in the Appendix, identification of ρ and the location relies on identification of $\boldsymbol{\beta}^\mu = (\beta_k^\mu, \beta_l^\mu)$ from equation (12). Similar to ACF, we show that these parameters are identified by the moment restriction $\mathbb{E}[\xi_{it} + \varepsilon_{it}|\mathcal{I}_{it-1}] = 0$ from equation (7). Once this is established, the characteristic functions can be identified using $T = 2$ firm-year observations. The extension to $T > 2$ is straightforward. Let $\mathcal{Y}, \mathcal{X}, \mathcal{Z}, \mathcal{W}$, and Ξ denote the supports of the distributions of the random variables $y_{it}, x_i, z_{it}, \omega_{it}$, and ξ_{it}, respectively. We begin with a set of assumptions.

Assumption 4.1 *Suppose the following set of assumptions*

(a) *Random Sample: The random variables $(y_{it}, Z_{it}, \omega_{it})_{i=1}^N$ are independently and identically distributed on $\mathcal{Y} \times \mathcal{Z} \times \mathcal{W}$ and $t \in \{1, 2\}$.*

(b) *Conditional Independence: (i) $f_{Z_{it}|\omega_{it},x_i}(Z_{it}|\omega_{it}, x_i) = f_{Z_{it}|x_i}(Z_{it}|x_i)$ for all $(Z_{it}, \omega_{it}, x_i) \in \mathcal{Z} \times \mathcal{W} \times \mathcal{X}$, (ii) $f_{Z_{it+1}|Z_{it},\xi_{it+1},\omega_{it},x_i}(Z_{it+1}|Z_{it}, \xi_{it+1}, \omega_{it}, x_i) = f_{Z_{it+1}|x_i}(Z_{it+1}|x_i)$ for all $(Z_{it+1}, Z_{it}, \xi_{it+1}, \omega_{it}, x_i) \in \mathcal{Z} \times \mathcal{Z} \times \Xi \times \mathcal{W} \times \mathcal{X}$, and (iii) $f_{\xi_{it+1}|Z_{it},\omega_{it},x_i}(\xi_{it+1}|Z_{it}, \omega_{it}, x_i) = f_{\xi_{it+1}|x_i}(\xi_{it+1}|x_i)$ for all $(\xi_{it+1}, Z_{it}, \omega_{it}, x_i) \in \Xi \times \mathcal{Z} \times \mathcal{W} \times \mathcal{X}$.*

(c) *Characteristic Functions: The conditional characteristic functions $\phi_{Z_{it}|x_i}(s|x_i)$, $\phi_{\omega_{it}|x_i}(s|x_i)$, and $\phi_{\xi_{it+1}|x_i}(s|x_i)$ do not vanish for all $s \in \mathbb{R}$ and $x_i \in \mathcal{X}$.*

(d) *Quantiles: $\eta_{it} \perp\!\!\!\perp (x_{it}, \omega_{it})$ where $\eta_{it} \sim U[0, 1]$ and Z_{it} is strictly increasing in η_{it}.*

In addition to Assumption 4.1, we restate and augment the assumptions in Ackerberg *et al.* (2015).

Assumption 4.2 *Suppose the following set of assumptions*

(a) *Information Set: The firm's information set at time t is given by $\mathcal{I}_{it}$ and includes current and past productivity shocks, but does not include future productivity shocks.*

(b) *Productivity: The evolution of productivity follows a linear AR(1) process: $\omega_{it} = \rho\omega_{it-1} + \xi_{it}$, where ξ_{it} is a shock to productivity which satisfies $\mathbb{E}[\xi_{it}|\mathcal{I}_{it-1}] = 0$.*

[7] Similar ideas have been used in measurement error models such as Li and Vuong (1998), Schennach (2004), Song *et al.* (2015), and in panel data models such as Neumann (2007).

(c) Timing of Input Choices: Firms accumulate capital according to

$$K_{it} = \kappa(K_{it-1}, I_{it-1}).$$

(d) Scalar Unobservable: Firm's intermediate input demand is given by

$$m_{it} = m_t(k_{it}, l_{it}, \omega_{it}).$$

(e) Strict Monotonicity: $m_t(k_{it}, l_{it}, \omega_{it})$ is strictly increasing in ω_{it}.

(f) Identification: There exists a neighborhood of $(\boldsymbol{\beta}^{\mu}, \rho)$ such that $(\boldsymbol{\beta}^{\mu}, \rho)$ is the unique solution to equation (7).

Assumption 4.1(a) places restrictions on the data generating process. Assumption 4.1(b)(i) implies that η_{it} is independent of ω_{it} conditional on two-period of inputs x_i which is consistent with the timing assumption of η_{it} occurring after productivity realizations and input decisions. Assumption 4.1(b)(ii) implies that η_{it+1} is independent of η_{it}, ξ_{it+1}, and ω_{it} conditional on x_i. The first part of this statement excludes dynamic effects in the production process through lagged output and other feedback effects. The second part states that η_{it+1} and ξ_{it+1} are independent conditional on x_i which is again consistent with the timing of the technology shocks η_{it} which occur after input choices and productivity are realized. The last part states that η_{it+1} is independent of ω_{it} conditional on x_i. Assumption 4.1(b)(iii) implies that ξ_{it+1} is independent of η_{it} and ω_{it} conditional on x_i. A similar assumption is used in Hu *et al.* (2020) who allow the distribution of ξ_{it+1} to depend on inputs x_i.

Assumption 4.1(c) are mild technical conditions on characteristic functions. Most standard families of distributions, such as the family of exponential distributions, satisfy this property. Lastly, 4.1(d) assumes that η_{it} is independent of inputs x_{it} and productivity ω_{it}. This restricts productivity from changing across quantiles. A similar independence assumption is used in Gandhi *et al.* (2020). This assumption is used to identify the relevant conditional quantiles.[8] Assumption 4.2 modifies some of the assumptions in Ackerberg *et al.* (2015) to identify $\boldsymbol{\beta}^{\mu}$ and ρ, which are necessary in our deconvolution strategy. Assumption 4.2(a) defines the information set. Assumption 4.2(b)(i) requires productivity to follow an

[8]This is because:

$$\Pr(y_{it} \le x_{it}\boldsymbol{\beta}(\tau) + \omega_{it} | x_{it}, \omega_{it}) = \Pr(x_{it}\boldsymbol{\beta}(\eta_{it}) \le x_{it}\boldsymbol{\beta}(\tau) | x_{it}, \omega_{it}) = \Pr(\eta_{it} \le \tau | x_{it}, \omega_{it}) = \tau.$$

linear AR(1) process. Although restrictive, this allows us to easily adapt a repeated measure type argument used for deconvolution. Assumption 4.2(f) implies that the location parameters $\boldsymbol{\beta}^\mu$ and ρ are locally identified. With these assumptions we are now ready to state the main identification result.

Theorem 4.1 *Under Assumptions 4.1 and 4.2, the location parameters $\boldsymbol{\beta}^\mu$ and ρ, the function $\boldsymbol{\beta}(\tau)$ for each $\tau \in (0,1)$, and the distribution of productivity are identified.*

Proof: See the Appendix.

5 Econometric Procedure

In this section we introduce a two-step estimator for the conditional quantiles of firm output. We argue that this estimator is consistent and asymptotically normal using conditions previously established in the quantile regression literature.

5.1 A Two-Step Estimator for Quantile Production Functions

Estimating the conditional quantiles can be done in the following steps. First, an estimate of $\mathbb{E}[\omega_{it}|x_i]$ can be obtained by estimating (12) using ACF replacing the expectation with the sample mean. This is then plugged into the sample counterpart of the conditional characteristic function $\phi_{Z_{it}|x_i}(s|x_i)$ from equation (23) in the Appendix. The estimate for the conditional characteristic function of Z_{it} is used to construct estimates of the CDF from equation (14) and therefore an estimate of the conditional quantile is obtained from the inverse relationship between the CDF and the quantile function.

Instead of using sample analogues based on the identification approach in the previous section, we propose a more simple estimation procedure motivated by the fact that identification of productivity guarantees identification of the quantile function through equation (23). Let $\varepsilon_{it} = k_{it}[\beta_k(\eta_{it}) - \beta_k^\mu] + l_{it}[\beta_l(\eta_{it}) - \beta_l^\mu]$ where $(\beta_k^\mu, \beta_l^\mu) = \boldsymbol{\beta}^\mu = \mathbb{E}[\boldsymbol{\beta}(\eta_{it})]$ is the mean of the random coefficients. The conditional mean version of the random coefficient production function can then be written as

$$y_{it} = \beta_k^\mu k_{it} + \beta_l^\mu l_{it} + \omega_{it} + \varepsilon_{it}, \tag{15}$$

where $\mathbb{E}[\varepsilon_{it}|\mathcal{I}_{it}] = 0$. A consistent estimator of ω_{it} is obtained from the ACF approach where productivity is estimated as $\hat{\omega}_{it} = \hat{\Phi}_t(k_{it}, l_{it}, m_{it}) - \hat{\beta}_k^\mu k_{it} - \hat{\beta}_l^\mu l_{it}$. This step is similar to the identification and estimation strategy for the unknown location $\mathbb{E}[\omega_{it}|x_i]$. Although we use ACF to identify ω_{it} in our model, we could also have estimated productivity from the OP or LP approach. We avoid these due to the identification issues surrounding them. It would be interesting to extend this approach using the framework of Gandhi *et al.* (2020), but such an extension is not straightforward and we leave this for future research agenda.

Once estimates of ω_{it} are obtained, estimates of $\beta_k(\tau)$ and $\beta_l(\tau)$ in equation (8) can be obtained from a quantile regression of $\hat{y}_{it} = y_{it} - \hat{\omega}_{it}$ on k_{it} and l_{it}. To summarize these steps:

1. Let $\hat{\beta}_k^\mu$ and $\hat{\beta}_l^\mu$ be consistent estimators of β_k^μ and β_l^μ from the value-added production function. Construct productivity, $\hat{\omega}_{it}$, using these estimates.

2. Let $\boldsymbol{\beta}(\tau) = (\beta_k(\tau), \beta_l(\tau))$ and $\hat{y}_{it} = y_{it} - \hat{\omega}_{it}$. For $\tau \in (0,1)$, define the two-step estimator of $\boldsymbol{\beta}(\tau)$ as:

$$\hat{\boldsymbol{\beta}}(\tau) = \underset{\boldsymbol{\beta} \in \boldsymbol{\mathcal{B}}}{\operatorname{argmin}}\, \mathbb{E}\big[\rho_\tau(\hat{y}_{it} - \beta_k k_{it} - \beta_l l_{it})\big],$$

   where $\boldsymbol{\mathcal{B}}$ is a compact and convex parameter space, $\rho_\tau(u) = u[\tau - \mathbb{I}\{u < 0\}]$, and $\mathbb{I}\{\cdot\}$ denotes the indicator function.

This estimator is computationally simple and efficient and can be found on the author's Github.

5.2 Large Sample Properties

Our two-step estimator in Section 5.1 relies on an initial consistent estimator of productivity. Standard errors from an estimate of the asymptotic covariance matrix include the variance from these estimates. Therefore, our model falls under a class of generated dependent variables in quantile regression with some additional complications. The first stage is semi-parametric due to the non-parametric function, $\Phi_t(k_{it}, l_{it}, m_{it})$. Also, the finite parameters β_k^μ and β_l^μ and the asymptotic covariance matrix for β_k^μ and β_l^μ include the variance from estimating $\Phi_t(k_{it}, l_{it}, m_{it})$.

A consistent estimate for $\Phi_t(k_{it}, l_{it}, m_{it})$ can be obtained from a linear sieve estimator. We let $\Phi_t(k_{it}, l_{it}, m_{it}; \theta) = p^{k_n}(z_{it})\theta_t$ where $z_{it} = (k_{it}, l_{it}, m_{it})$ and $p^{k_n}(z_{it})$ is chosen to be a tensor-product linear sieve basis, for example, B-spline, Fourier, or power series. The term k_n is the smoothing parameter which is required to grow with the sample size. In practice, we use polynomial of degree two which is consistent with the empirical implementations of Levinsohn and Petrin (2003) and Ackerberg *et al.* (2015).

Consistency and asymptotic normality of quantile regression estimators are well established. We argue these results also hold for our estimator, given regularity assumptions in e.g., Chernozhukov and Hansen (2005). We restate these assumptions and theorems in the Online Appendix, where we offer an informal derivation of the influence function of the first stage estimates. In our empirical application, we use nonparametric bootstrap to estimate standard errors and confidence intervals.

6 Monte Carlo Experiments

In this section, we investigate the performance of our estimator in a set of Monte Carlo experiments. We use a location-scale model for the production function given by equation (9) which we provide again in equation (16) below.

$$y_{it} = \beta_k k_{it} + \beta_l l_{it} + \omega_{it} + (\gamma_k k_{it} + \gamma_l l_{it})\eta_{it}, \tag{16}$$

where $\beta_k = 0.4$ and $\beta_l = 0.6$. We replicate Ackerberg *et al.* (2015) simulations by sampling 1000 datasets consisting of 1000 firms. We simulate optimal input choices for 100 time periods, using the last 10 periods for estimation. We consider three different data generating processes (DGPs) for the scale parameters and the distribution of η_{it}. For DGP1, we let $\gamma_k = 0.6$, $\gamma_l = -0.6$, and $\eta_{it} \sim N(0, \sigma_\eta^2)$ where $\sigma_\eta^2 = 0.01$. This follows the original specification for the error term in Ackerberg *et al.* (2015). In DGP2, $\gamma_k = 0.4$, $\gamma_l = -0.4$, and $\eta_{it} \sim (\sqrt{3\sigma_\eta^2/5})t_5$ so that the error variance is the same as that of DGP1. In DGP3, we add considerable skewness to the η_{it} by letting $\gamma_k = 0.5$, $\gamma_l = -0.5$, and $\eta_{it} \sim Lognormal(0.15, \sigma_\eta^2)$.

We show that our estimator is flexible by estimating productivity in both the LP and ACF procedure under the different DGPs. With the LP approach, the labor elasticity may not always be identified in the first stage restriction. This would affect our estimates of productivity which we use in the second stage of our estimator. Therefore, following Ackerberg *et al.* (2015), we consider two DGPs where labor can and cannot be identified.[9] In the DGPs prefixed by "LP" in Table 1, we add optimization error to labor which breaks the functional dependence problem explained by ACF. Therefore, LP and ACF estimates of labor and hence productivity are consistent. We use the LP estimator for this DGP. In the

[9] We refer to Ackerberg *et al.* (2015) for a detailed discussion on how to generate the simulated data.

DGPs prefixed by “ACF”, we add serially correlated wages and allow for labor to be chosen in a sub-period $t - b$, where $b = 0.5$. In this setup, the ACF estimates will be consistent since their estimator conditions on labor in the first stage, whereas LP does not. Therefore, we use the ACF estimator for this DGP. An AR(1) process is specified for productivity $\omega_{it} = \rho\omega_{it-1} + \xi_{it}$ where $\rho = 0.7$. The variance of ξ_{it} and initial value ω_{i0} is set so that the standard deviation of ω_{it} is constant over time and equal to 0.3.

Table 1 provides estimates of the bias and mean-squared error (MSE) for each DGP of our proposed estimator (DS) and naive quantile regression (QR) that does not control for simultaneity bias. Our estimator performs relatively well. The bias of our estimator remains low for each τ under each DGP for the error distribution and each DGP under LP and ACF estimates for productivity. The MSE is also small and reaches its minimum at the median ($\tau = 0.5$). Not surprisingly, the QR estimates do not perform well in any situation. Both the bias and MSE are large compared to our estimates and the sign of the bias is positive for capital estimates and negative for labor estimates.

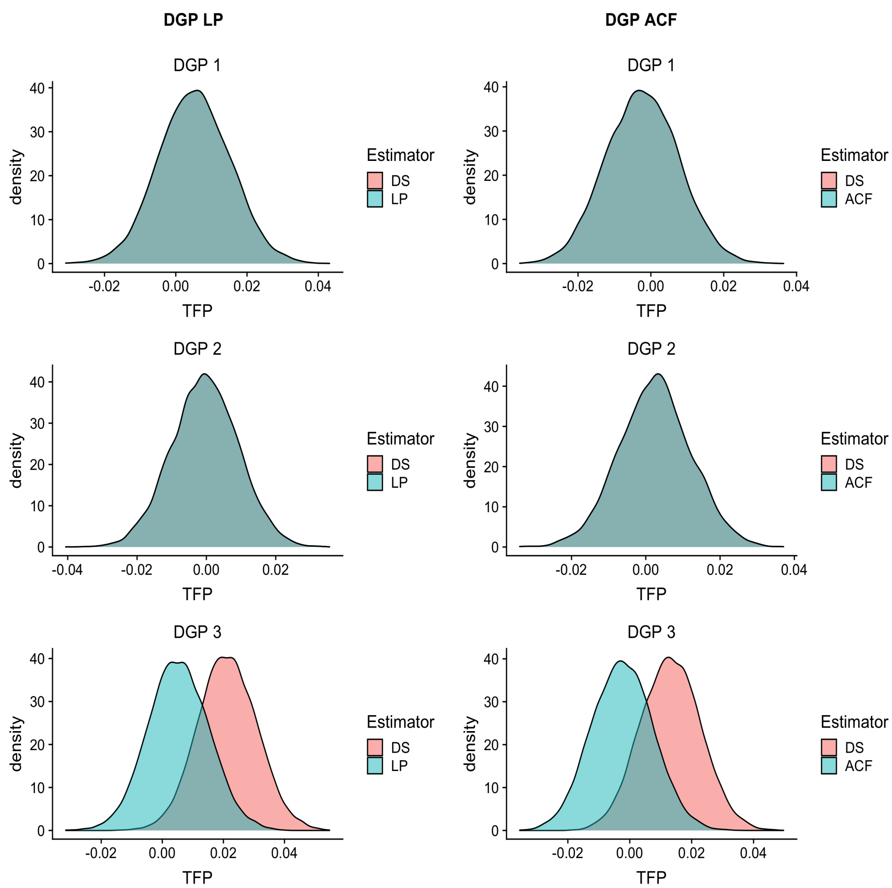
In Figure 1, we plot the estimates of TFP using LP, ACF, and the median DS estimator. The left panel compares the LP estimator and the DS estimator when productivity is estimated using LP. The right panel compares the ACF estimator and the DS estimator when productivity is estimated using ACF. Each row corresponds to the DGP that generates favorable estimates for both approaches. For example, the top left plot corresponds to the DGP with optimization error in labor with $\gamma_k = 0.6$, $\gamma_l = -0.6$, and $\eta_{it} \sim N(0, \sigma_\eta^2)$ where $\sigma_\eta^2 = 0.01$. The top right plot corresponds to the DGP with serially correlated wages with the same specification for the error distribution and scale parameters. For the first two rows, there is no discernible difference between TFP of the two estimators. This is because the error distributions specified in these DGPs are symmetric. However, in the last row which corresponds to the asymmetric error distribution, our median TFP estimates are different from the mean estimates of LP and ACF. This underscores the importance of consistently estimating TFP from quantile regression in the presence of an asymmetric error distribution, by taking into account heterogeneity across the conditional distribution of output.

Table 1: Monte Carlo Results for Capital and Labor Estimates

		DS				QR			
		Capital		Labor		Capital		Labor	
DGP	τ	Bias	MSE	Bias	MSE	Bias	MSE	Bias	MSE
LP 1	0.10	-0.0149	0.0004	0.0199	0.0004	0.2691	0.0830	-0.2771	0.0865
	0.25	-0.0045	0.0002	0.0065	0.0000	0.2905	0.0971	-0.2975	0.1008
	0.50	0.0010	0.0002	0.0000	0.0000	0.3140	0.1143	-0.3210	0.1183
	0.75	0.0065	0.0002	-0.0065	0.0000	0.3375	0.1329	-0.3445	0.1380
	0.90	0.0169	0.0005	-0.0199	0.0004	0.3589	0.1513	-0.3659	0.1560
LP 2	0.10	-0.0097	0.0003	0.0127	0.0002	0.2833	0.0610	-0.2903	0.0635
	0.25	-0.0035	0.0002	0.0035	0.0000	0.2985	0.0870	-0.3065	0.0905
	0.50	0.0000	0.0002	0.0000	0.0000	0.3140	0.1149	-0.3210	0.1190
	0.75	0.0035	0.0002	-0.0035	0.0000	0.3285	0.1460	-0.3365	0.1506
	0.90	0.0097	0.0003	-0.0127	0.0002	0.3447	0.1849	-0.3517	0.1910
LP 3	0.10	-0.0053	0.0002	0.0083	0.0001	0.2857	0.0975	-0.2927	0.1013
	0.25	-0.0015	0.0002	0.0035	0.0000	0.2935	0.0998	-0.3005	0.1036
	0.50	0.0004	0.0002	-0.0004	0.0000	0.3064	0.1056	-0.3144	0.1096
	0.75	0.0049	0.0002	-0.0059	0.0000	0.3269	0.1196	-0.3349	0.1245
	0.90	0.0167	0.0005	-0.0197	0.0004	0.3537	0.1459	-0.3617	0.1513
ACF 1	0.10	-0.0099	0.0003	0.0129	0.0003	0.2881	0.0831	-0.2941	0.0865
	0.25	-0.0035	0.0003	0.0045	0.0001	0.3115	0.0971	-0.3175	0.1008
	0.50	-0.0010	0.0002	0.0000	0.0001	0.3380	0.1143	-0.3440	0.1184
	0.75	0.0025	0.0003	-0.0035	0.0001	0.3645	0.1329	-0.3715	0.1380
	0.90	0.0089	0.0003	-0.0119	0.0002	0.3889	0.1513	-0.3949	0.1560
ACF 2	0.10	-0.0057	0.0002	0.0087	0.0002	0.3053	0.0611	-0.3103	0.0635
	0.25	-0.0015	0.0002	0.0025	0.0001	0.3225	0.0870	-0.3285	0.0906
	0.50	0.0000	0.0002	0.0000	0.0001	0.3390	0.1150	-0.3450	0.1190
	0.75	0.0025	0.0002	-0.0025	0.0001	0.3555	0.1461	-0.3615	0.1506
	0.90	0.0067	0.0003	-0.0077	0.0001	0.3727	0.1850	-0.3787	0.1910
ACF 3	0.10	-0.0043	0.0003	0.0043	0.0001	0.3067	0.0975	-0.3127	0.1013
	0.25	-0.0015	0.0003	0.0015	0.0001	0.3145	0.0998	-0.3205	0.1036
	0.50	-0.0006	0.0003	0.0006	0.0001	0.3294	0.1057	-0.3354	0.1096
	0.75	0.0019	0.0003	-0.0029	0.0001	0.3519	0.1196	-0.3579	0.1245
	0.90	0.0097	0.0004	-0.0137	0.0003	0.3817	0.1459	-0.3877	0.1513

*Bias and MSE from 1000 replications: "DS" is our estimator; "QR" is quantile regression with no correction for simultaneity bias; "LP" uses estimates of productivity from Levinsohn and Petrin (2003) under the DGPs that are favorable to their procedure; "ACF" uses estimates of productivity from Ackerberg *et al.* (2015) under the DGPs that are favorable to their procedure.

Figure 1: Monte Carlo Results for Total Factor Productivity Estimates



*Estimated TFP from LP, ACF, and the median DS estimator for three DGPs: The left panel compares the LP estimator and the DS estimator when productivity is estimated using LP; The right panel compares the ACF estimator and the DS estimator when productivity is estimated using ACF.

7 Application

We apply our estimator to firm and plant-level manufacturing datasets from the U.S., Chile, and Colombia to examine heterogeneity in production. For each country, we examine estimates across different manufacturing industries. We use the DS estimator with productivity estimated using ACF and compare it to the conditional mean estimates from ACF. We also compare our estimates to the quantile regression estimates without controlling for productivity. In the Online Appendix, we estimate a gross-output production function with productivity estimated using LP.

Table 2: Coefficient Estimates and Standard Errors for U.S. Manufacturing Firms

		Capital		Labor		Returns to Scale		Capital Intensity	
NAICS	τ	Coef.	s.e	Coef.	s.e	Coef.	s.e	Coef.	s.e
31	0.10	0.319	0.0323	0.554	0.0383	0.873	0.0161	0.575	0.0787
	0.25	0.345	0.0324	0.500	0.0375	0.845	0.0143	0.689	0.0915
	0.50	0.372	0.0323	0.450	0.0369	0.821	0.0133	0.827	0.1073
	0.90	0.422	0.0327	0.374	0.0390	0.797	0.0204	1.127	0.1420
32	0.10	0.189	0.0280	0.766	0.0359	0.955	0.0132	0.246	0.0478
	0.25	0.217	0.0279	0.692	0.0349	0.909	0.0119	0.313	0.0555
	0.50	0.242	0.0279	0.639	0.0346	0.881	0.0114	0.378	0.0630
	0.90	0.293	0.0278	0.540	0.0354	0.833	0.0127	0.543	0.0835
33	0.10	0.387	0.0179	0.605	0.0226	0.992	0.0070	0.640	0.0283
	0.25	0.415	0.0178	0.522	0.0220	0.937	0.0061	0.796	0.0330
	0.50	0.439	0.0178	0.468	0.0217	0.907	0.0058	0.939	0.0371
	0.90	0.481	0.0179	0.385	0.0216	0.866	0.0057	1.250	0.0459
All	0.10	0.385	0.0136	0.576	0.0172	0.962	0.0055	0.668	0.0262
	0.25	0.416	0.0136	0.495	0.0168	0.910	0.0049	0.841	0.0311
	0.50	0.442	0.0136	0.445	0.0168	0.887	0.0052	0.992	0.0354
	0.90	0.490	0.0137	0.363	0.0171	0.853	0.0064	1.352	0.0458

*Standard errors are obtained using bootstrap with 500 replications. The first stage uses estimates from ACF.

Table 3: ACF Coefficient Estimates and Standard Errors for U.S. Manufacturing Firms

	Capital		Labor		Returns to Scale		Capital Intensity	
NAICS	Coef.	s.e	Coef.	s.e	Coef.	s.e	Coef.	s.e
31	0.370	0.0324	0.463	0.0366	0.833	0.0130	0.800	0.1027
32	0.240	0.0279	0.654	0.0347	0.894	0.0108	0.367	0.0612
33	0.435	0.0178	0.495	0.0218	0.930	0.0057	0.878	0.0352
All	0.438	0.0136	0.467	0.0167	0.906	0.0050	0.938	0.0337

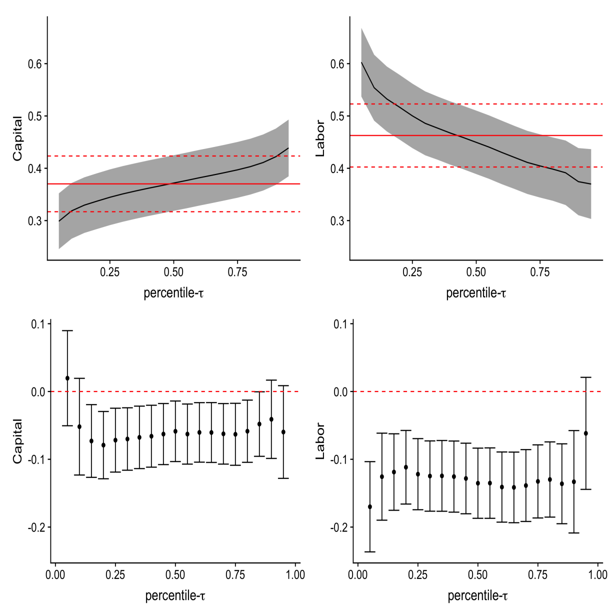
*Standard errors are obtained using bootstrap with 500 replications.

In the ACF estimation procedure, we estimate the non-parametric function Φ_t with a 2nd degree polynomial with interactions in capital, labor, and materials. To estimate the coefficients on capital and labor, we use the ACF criterion function mentioned earlier. Since we only need a consistent estimator of the production function parameters, we do not consider any over-identification conditions in this step. We use bootstrap to estimate standard errors of $\beta_k(\tau)$ and $\beta_l(\tau)$ with the number of iterations set to 500.

7.1 U.S. Compustat

The source for the U.S. manufacturing data is from Compustat which covers publicly traded firms and contains data from their financial statements. We collect a sample between 1961 and 2016 on sales, capital expenditures, number of workers, and other expenses which are deflated to construct measures of output, capital, labor, and material inputs. Data preparation follows Keller and Yeaple (2009) and Dermirer (2020). Manufacturing industries are classified by the first digit NAICS codes 31-33. NAICS 31 includes different types of food and beverage manufacturing as well as manufacturing of textiles and apparels. NAICS 32 includes manufacturing of wood, paper, chemicals, and plastics. NAICS 33 includes steel, mineral, equipment, and electronic manufacturing. We also aggregate the three industries to obtain estimates for all of the manufacturing firms. Summary statistics for these deflated values are provided in the Online Appendix. We present a series of output elasticity estimates in Table 2 which are illustrated graphically in Figures 2, 3, 4, and 5.

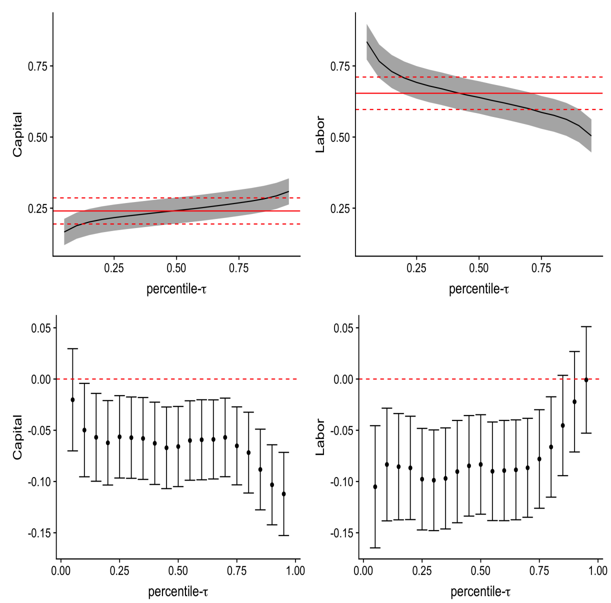
Figure 2: Estimated Coefficients of Capital and Labor for U.S.: NAICS 31



*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

In each of these figures, we plot estimates for $\tau \in \{0.05, 0.1, \ldots, 0.95\}$. In the first row of plots for each figure, the horizontal axis denotes these percentiles and the vertical axis are the estimated output elasticities. The solid black line denotes our estimator for different τ with its corresponding 90% point-wise confidence interval shaded in dark gray. The solid red line denotes the conditional mean estimates of the output elasticities using ACF and the dotted red lines are the corresponding 90% confidence intervals. The bottom row presents the difference between our estimator and the conditional quantile estimator without taking into account the endogeneity of productivity. The horizontal axis denotes the percentiles and the vertical axis denotes the difference between the estimates. The solid black dots are the point estimates of these differences and the solid black lines denote their corresponding 95% confidence intervals.

Figure 3: Estimated Coefficients of Capital and Labor for U.S.: NAICS 32

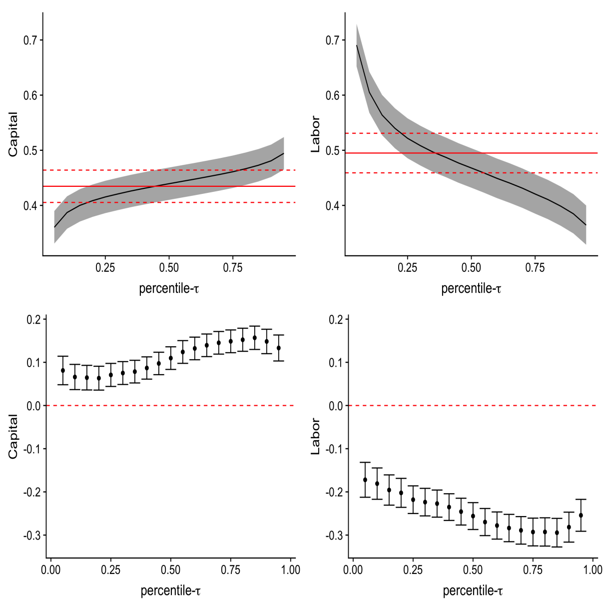


*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

Estimates of the capital elasticity are increasing in τ in every industry as well as the combined sample. The estimates for labor elasticity for each industry and the combined sample are decreasing. In each industry, aside from NAICS 31, there is evidence that our model captures heterogeneity relative to the ACF model. In each industry, there are large differences between estimates at low and high quantiles. In NAICS 31, the capital estimates range from 0.3 to 0.43 and the labor estimates range from 0.6 to 0.37. For NAICS 32, the capital estimates range from 0.17 to 0.31 and the labor estimates range from 0.84 to 0.5. In NAICS 33, capital estimates range from 0.36 to 0.5 and labor estimates range from 0.7 to 0.36. Finally, in the combined sample, capital estimates range from 0.35 to 0.51 and labor estimates range from 0.66 to 0.33.

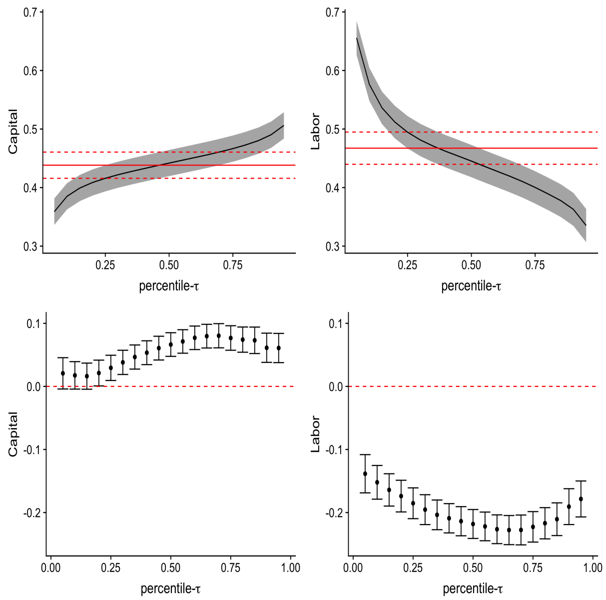
In each industry, we compare the differences between our estimates and QR estimates to examine whether our model corrects for endogeneity from unobserved productivity. Bootstrap is used to construct confidence intervals for the difference between the two estimates. We find significant differences in each industry, with the most pronounced differences in NAICS 33 and the combined sample. We also use the estimates from the output elasticities to construct measures of returns to scale and capital intensity in Table 2. Each industry has estimates of returns to scale that decrease in τ and capital intensity estimates that increase with τ. Table 3 shows the results using the ACF estimator. The estimated coefficients for capital are comparable to those at the median ($\tau = 0.5$) in Table 2, while the labor estimates are different.

Figure 4: Estimated Coefficients of Capital and Labor for U.S.: NAICS 33



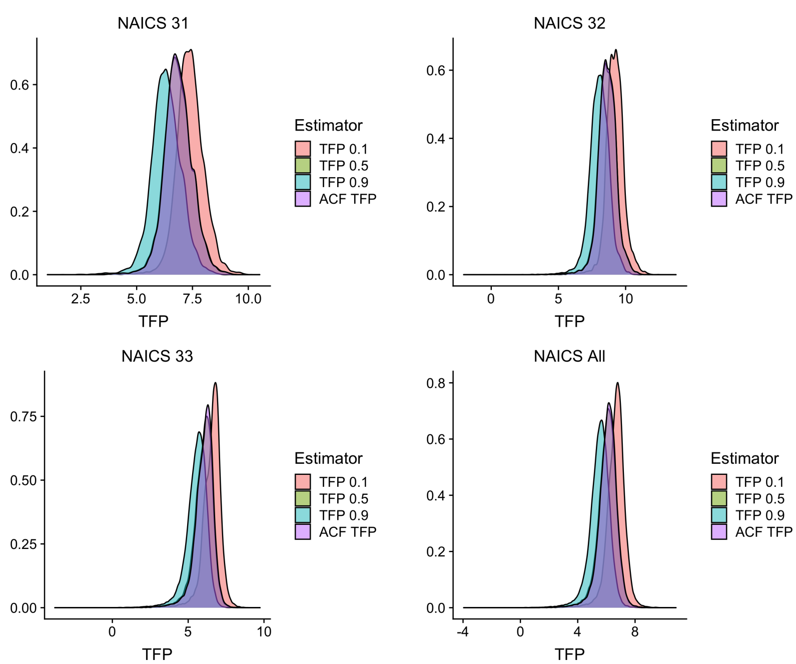
*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

Figure 5: Estimated Coefficients of Capital and Labor for U.S. Manufacturing Firms



*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

Figure 6: DS and ACF Estimates of Log Total Factor Productivity



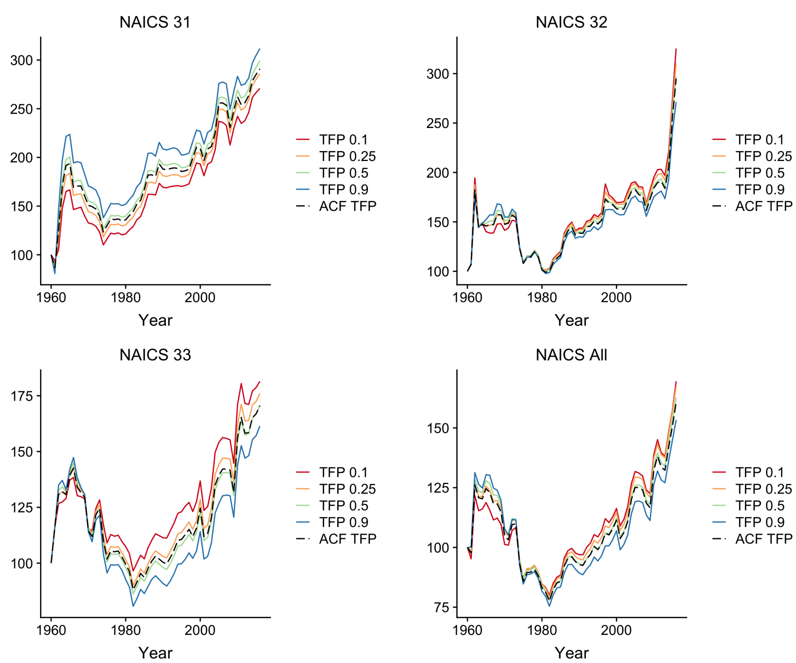
*Estimated distributions of TFP from the DS estimator for $\tau \in \{0.1, 0.5, 0.9\}$ and those from the ACF estimator.

We also use our quantile production function estimates to construct measures of firm level productivity which we define as

$$\widehat{TFP}_{it,\tau} = \exp(y_{it} - \hat{\beta}_k(\tau)k_{it} - \hat{\beta}_l(\tau)l_{it}). \tag{17}$$

We use these estimates to compare the distributions of productivity and productivity growth over time to ACF estimates over the conditional output distribution. Figure 6 plots TFP distributions for $\tau \in \{0.1, 0.5, 0.9\}$ and for the ACF estimator. The plot shows apparent differences between TFP estimates for low and high τ, but not much difference between $\tau = 0.5$ and the mean estimates of ACF.

Figure 7: U.S. Productivity Over Time



*Estimated average productivity (in levels) over time for the U.S. Base year productivity is set to 100.

Figure 7 reports average productivity for each industry with the base year of the sample period set to 100. We observe a period of rapid productivity growth in the beginning of the sample period followed by a gradual decline until the early 1980's. This is followed by productivity growth for the remainder of the sample period. Growth trends for each percentile were similar, although firms who rank lower on the conditional output distribution were more productive than higher ranked firms in all industries except NAICS 31. Lower ranked firms may have higher managerial efficiency and can adapt to market changes faster than higher ranked firms. The ACF estimates are close to the productivity estimates for firms at $\tau = 0.5$. The figures also reveal some degree of productivity dispersion across τ that gradually increase over time. This is consistent with the existing empirical literature which finds increasing productivity dispersion. The highest amount of productivity dispersion is observed in NAICS 31 and NAICS 33. The dispersion in productivity growth across the conditional output distribution is an interesting result of our proposed estimator which cannot be captured by the conditional mean estimator in the control function approach.

Table 4: Productivity Differentials for U.S. Manufacturing Firms using DS

		R&D		Advertisements	
NAICS	τ	Coef.	s.e	Coef.	s.e
31	0.10	0.157	0.0160	0.187	0.0197
	0.25	0.170	0.0143	0.200	0.0178
	0.50	0.181	0.0133	0.211	0.0162
	0.90	0.190	0.0139	0.219	0.0159
32	0.10	0.105	0.0092	0.112	0.0105
	0.25	0.133	0.0093	0.139	0.0103
	0.50	0.148	0.0088	0.154	0.0098
	0.90	0.175	0.0088	0.180	0.0099
33	0.10	0.064	0.0054	0.048	0.0054
	0.25	0.098	0.0047	0.076	0.0047
	0.50	0.115	0.0046	0.091	0.0045
	0.90	0.138	0.0050	0.109	0.0047
All	0.10	0.097	0.0047	0.082	0.0051
	0.25	0.126	0.0042	0.109	0.0045
	0.50	0.138	0.0040	0.120	0.0043
	0.90	0.154	0.0042	0.133	0.0042

*Standard errors are obtained using bootstrap with 500 replications. Log(TFP) is regressed on log(R&D) and log(Advertisements).

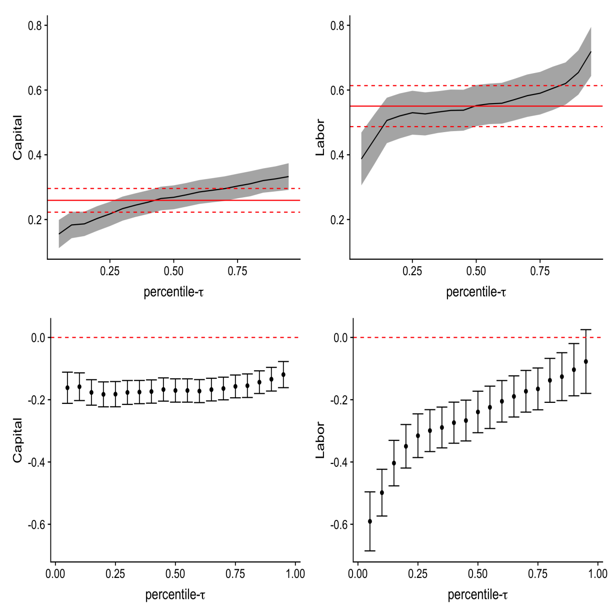
In Table 4 we examine the effect of R&D and advertising intensity on productivity across the conditional output distribution. Specifically, we regress the estimated productivity on R&D and advertising intensity for each τ, where all variables are log-transformed. Our results show that as τ increases, the returns to R&D and advertising also increase. Comparing the median estimates to the ACF estimates in Table 5, we observe that in NAICS 33, the return to R&D for the median firm is 11.5% and the return to advertising is 9.1%, whereas the returns for the average firm are 10% and 7.8%, respectively. Overall, these results suggest that higher ranked firms are better at converting R&D and advertising activities into larger productivity gains.

Table 5: Productivity Differentials for U.S. Manufacturing Firms using ACF

	R&D		Advertisements	
NAICS	Coef.	s.e	Coef.	s.e
31	0.174	0.0132	0.204	0.0161
32	0.140	0.0080	0.146	0.0091
33	0.100	0.0045	0.078	0.0046
All	0.126	0.0040	0.109	0.0043

*Standard errors are obtained using bootstrap with 500 replications. Log(TFP) is regressed on log(R&D) and log(Advertisements).

Figure 8: Estimated Coefficients of Capital and Labor for Chile: ISIC 311

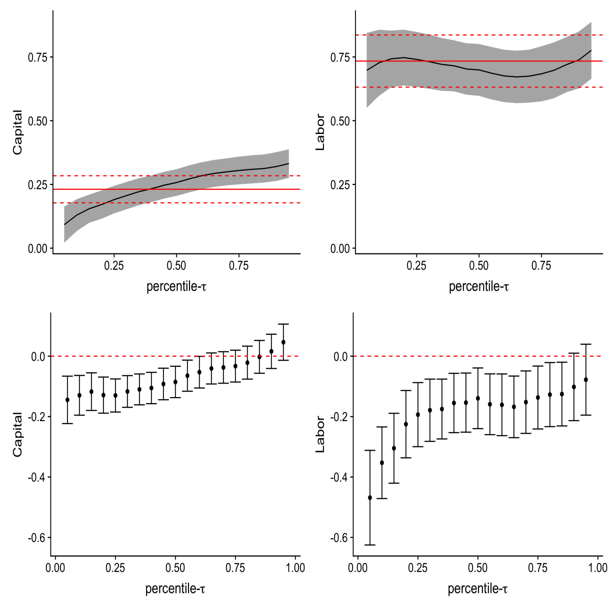


*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

7.2 Chilean Manufacturing

This data comes from the census of Chilean manufacturing plants conducted by the Instituto Nacional de Estadistíca (INE). The sample is collected between 1979 and 1996 for firms with more than 10 employees. We divide our estimates into the three largest manufacturing industries: Food (ISIC 311), Fabricated Metals (ISIC 381), and Textiles (ISIC 321). We also aggregate the three industries with the other smaller industries to obtain estimates for the entire sample. Summary statistics for the data we use are provided in the Online Appendix.

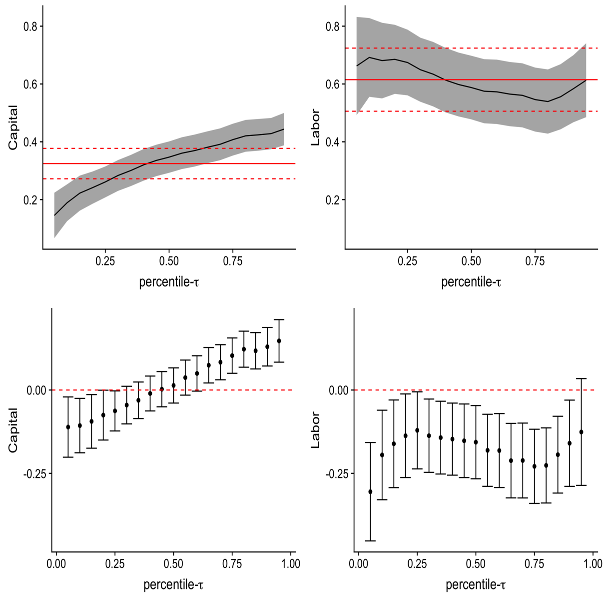
Figure 9: Estimated Coefficients of Capital and Labor for Chile: ISIC 381



*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

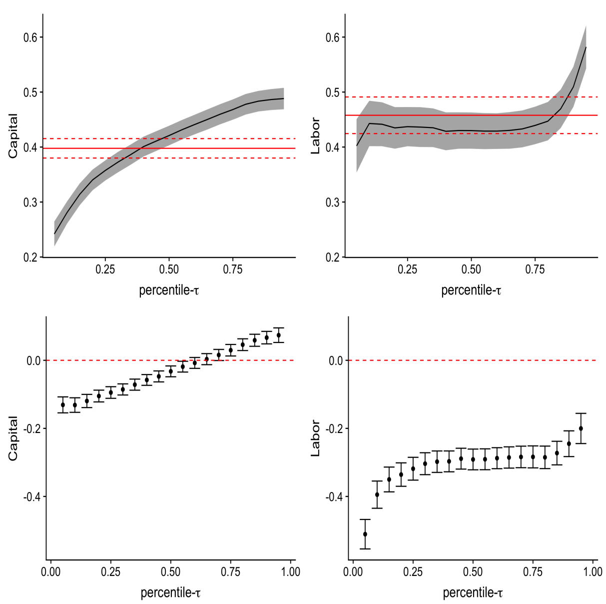
Figures 8, 9, 10, and 11 illustrate the estimates from our model compared to ACF estimates (top row) as well as the differences between our model and QR estimates that does not control for endogeneity (bottom row). In each industry and the combined sample, estimates of capital elasticity are increasing in τ. In ISIC 321 and the combined sample, these differences are most pronounced in the tails of the conditional output distribution, although there is less heterogeneity in the estimates of labor elasticity. In ISIC 311, the estimates of labor elasticity are increasing in τ, but not much different from the mean estimates except at the extreme quantiles. In remaining industries, the estimates do not vary considerably over τ. In ISIC 311, the capital estimates range from 0.18 to 0.36 and the labor estimates range from 0.4 to 0.65. In ISIC 381, capital estimates vary in a similar magnitude. The largest differences occur in ISIC 321 and the combined sample, where the difference between low and high τ in estimates of capital elasticity is about 0.2.

Figure 10: Estimated Coefficients of Capital and Labor for Chile: ISIC 321



*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

Figure 11: Estimated Coefficients of Capital and Labor for all Chilean Manufacturing Plants



*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

Comparing our estimator to QR estimates, we find that there are substantial differences in the estimates for capital and labor for all industries, which supports the importance of controlling for the unobserved productivity in estimating quantile production functions. In the case where our estimates of labor elasticity are not different from the conditional mean estimates, unobserved productivity may explain most of the heterogeneity between firms. In Table 6 we report our estimates for the elasticities as well as the estimates for returns to scale and capital intensity. Opposite of the U.S., we find that returns to scale are increasing in τ for all industries. Capital intensity is increasing in τ for each industry. Firms with higher output use more capital relative to labor in the Chilean manufacturing industry. Table 7 reports the mean estimates using ACF, which are comparable to the median estimates in Table 6. Nevertheless, the estimation results from our estimator provide richer distributional effects of inputs on output in the Chilean production functions, that is not captured by the mean estimator.

Table 6: Coefficient Estimates and Standard Errors for Chilean Manufacturing Plants

		Capital		Labor		Returns to Scale		Capital Intensity	
ISIC	τ	Coef.	s.e	Coef.	s.e	Coef.	s.e	Coef.	s.e
311	0.10	0.183	0.0249	0.447	0.0468	0.630	0.0321	0.410	0.0760
	0.25	0.217	0.0228	0.530	0.0412	0.747	0.0298	0.410	0.0613
	0.50	0.269	0.0224	0.552	0.0387	0.821	0.0279	0.487	0.0601
	0.90	0.326	0.0235	0.654	0.0416	0.980	0.0292	0.498	0.0546
381	0.10	0.129	0.0380	0.728	0.0786	0.857	0.0535	0.178	0.0626
	0.25	0.190	0.0328	0.740	0.0655	0.931	0.0469	0.257	0.0584
	0.50	0.257	0.0318	0.699	0.0617	0.957	0.0447	0.368	0.0674
	0.90	0.320	0.0344	0.738	0.0679	1.059	0.0473	0.434	0.0744
321	0.10	0.189	0.0387	0.692	0.0828	0.881	0.0578	0.273	0.0789
	0.25	0.261	0.0325	0.674	0.0690	0.935	0.0522	0.387	0.0750
	0.50	0.347	0.0330	0.588	0.0669	0.935	0.0502	0.591	0.0989
	0.90	0.428	0.0328	0.584	0.0705	1.012	0.0537	0.734	0.1220
All	0.10	0.281	0.0123	0.443	0.0251	0.724	0.0182	0.634	0.0441
	0.25	0.357	0.0111	0.437	0.0217	0.795	0.0166	0.818	0.0444
	0.50	0.421	0.0108	0.430	0.0200	0.851	0.0155	0.979	0.0468
	0.90	0.486	0.0116	0.510	0.0221	0.996	0.0163	0.954	0.0453

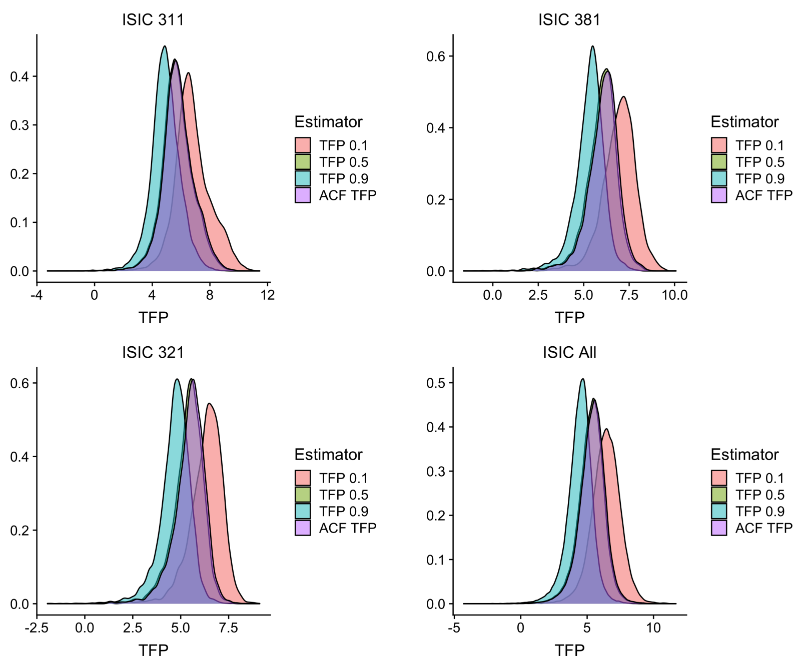
*Standard errors are obtained using bootstrap with 500 replications. The first stage uses estimates from ACF.

Table 7: ACF Coefficient Estimates and Standard Errors for Chilean Manufacturing Plants

	Capital		Labor		Returns to Scale		Capital Intensity	
ISIC	Coef.	s.e	Coef.	s.e	Coef.	s.e	Coef.	s.e
311	0.259	0.0222	0.550	0.0385	0.810	0.0280	0.471	0.0588
381	0.231	0.0323	0.734	0.0622	0.965	0.0447	0.315	0.0615
321	0.325	0.0319	0.615	0.0664	0.940	0.0508	0.528	0.0886
All	0.398	0.0107	0.458	0.0202	0.855	0.0157	0.869	0.0418

*Standard errors are obtained using bootstrap with 500 replications.

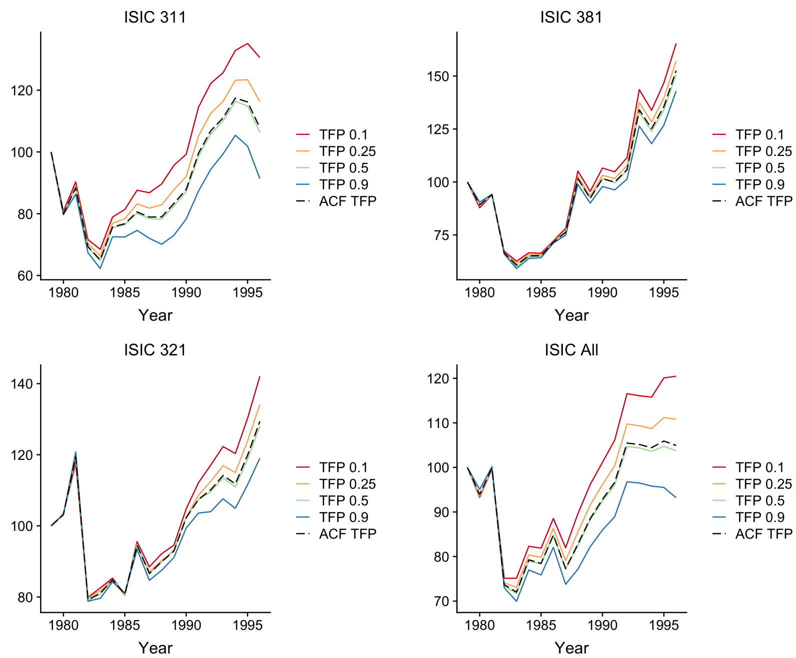
Figure 12: DS and ACF Estimates of Log Total Factor Productivity



*Estimated distributions of TFP from the DS estimator for $\tau \in \{0.1, 0.5, 0.9\}$ and those from the ACF estimator.

Figure 12 plots the distributions of TFP for various quantiles and for ACF estimates. The plot shows significant differences between TFP estimates for low and high τ, but similar distributions of TFP for our estimator at $\tau = 0.5$ and the mean estimate. Thus, the proposed estimator is useful to capture heterogeneous TFP at various quantiles. Figure 13 reports average productivity over the time periods for each industry with base period set to 100. Productivity decreases in the beginning of the 1980s, but then increases for the rest of the sample period. Similar to the U.S. results, These estimates suggest that firms ranked lower on the conditional output distribution had higher productivity levels than higher ranked firms in all industries. The ACF estimates are similar to productivity of firms at $\tau = 0.5$. The figure displays that productivity dispersion across τ has been gradually increasing over time: the highest amount of productivity dispersion is observed in ISIC 311, whereas ISIC 381 shows relatively homogeneous trend across τ. These estimation results confirm that our estimator is useful in capturing heterogeneity in TFP and its evolution over time across the conditional output distribution.

Figure 13: Chile Productivity Over Time



*Estimated average productivity (in levels) over time for Chile. Base year productivity is set to 100.

Table 8 reports the estimated effects of firm characteristics on productivity using our estimator. For each τ, we run a regression of the estimated productivity on the amount of firm exports, the amount of imported raw materials, and advertising expenditure, where all variables are log-transformed. Similarly, Table 9 provides the estimated effects using ACF. Our results show that the return to these activities diminish as τ increases, which suggests significant heterogeneous effects of firm characteristics on productivity across the conditional output distribution. We also observe that there is not much difference between the median estimate $\tau = 0.5$ and the mean estimates in Table 9.

Table 8: Productivity Differentials for Chilean Manufacturing Plants using DS

		Exports		Imports		Advertisements	
ISIC	τ	Coef.	s.e	Coef.	s.e	Coef.	s.e
311	0.10	0.069	0.0377	0.184	0.0397	0.076	0.0279
	0.25	0.057	0.0363	0.144	0.0384	0.054	0.0270
	0.50	0.044	0.0358	0.120	0.0376	0.041	0.0268
	0.90	0.026	0.0369	0.065	0.0374	0.012	0.0270
381	0.10	0.099	0.0299	0.192	0.0348	0.130	0.0377
	0.25	0.068	0.0283	0.149	0.0354	0.105	0.0348
	0.50	0.048	0.0275	0.122	0.0364	0.090	0.0332
	0.90	0.010	0.0277	0.069	0.0388	0.058	0.0338
321	0.10	0.021	0.0278	0.044	0.0374	0.074	0.0327
	0.25	0.005	0.0276	0.018	0.0366	0.056	0.0317
	0.50	0.007	0.0283	0.017	0.0371	0.055	0.0323
	0.90	-0.017	0.0297	-0.020	0.0399	0.030	0.0350
All	0.10	0.101	0.0118	0.192	0.0155	0.143	0.0121
	0.25	0.073	0.0116	0.156	0.0151	0.124	0.0116
	0.50	0.049	0.0116	0.127	0.0150	0.109	0.0114
	0.90	0.005	0.0117	0.067	0.0151	0.073	0.0116

*Standard errors are obtained using bootstrap with 500 replications. Log(TFP) is regressed on log(Exports), log(Imports), and log(Advertisements).

Table 9: Productivity Differentials for Chilean Manufacturing Plants using ACF

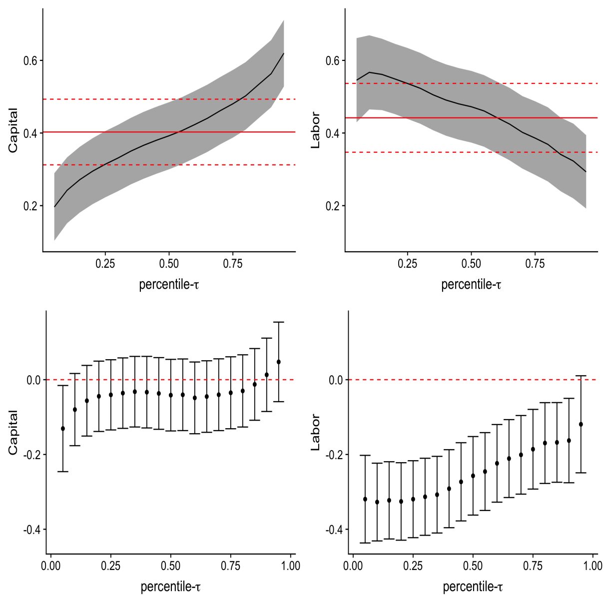
	Exports		Imports		Advertisements	
ISIC	Coef.	s.e	Coef.	s.e	Coef.	s.e
311	0.046	0.0359	0.123	0.0377	0.043	0.0269
381	0.051	0.0278	0.126	0.0362	0.092	0.0338
321	0.005	0.0277	0.015	0.0366	0.054	0.0322
All	0.052	0.0116	0.129	0.0149	0.109	0.0115

*Standard errors are obtained using bootstrap with 500 replications. Log(TFP) is regressed on log(Exports), log(Imports), and log(Advertisements).

7.3 Colombian Manufacturing

This data comes from the Colombian manufacturing census conducted by the Departamento Administrativo Nacional de Estadistica. The sample is collected between 1977 and 1991. We divide our estimates into the three largest manufacturing industries: Food (ISIC 311), Apparel (ISIC 322), and Fabricated Metals (ISIC 381). As we did with the Chilean sample, we also aggregate the three industries with other smaller industries to obtain estimates from the entire sample of manufacturing plants. Summary statistics for this data is provided in the Online Appendix.

Figure 14: Estimated Coefficients of Capital and Labor for Colombia: ISIC 311

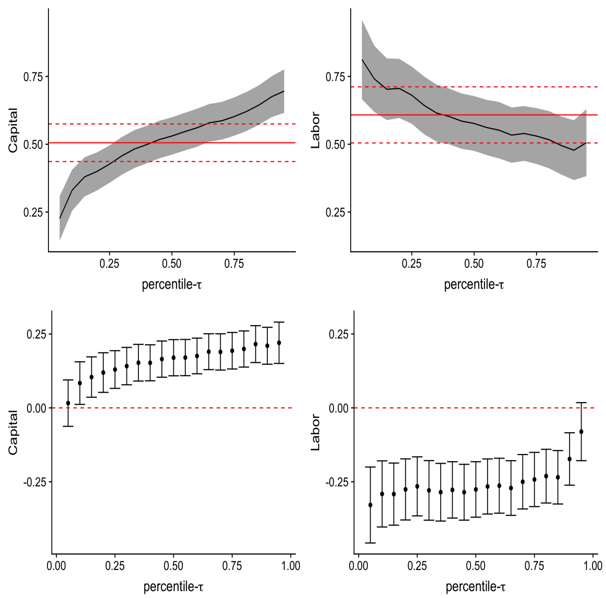


*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

Figures 14, 15, 16, and 17 illustrate the estimates from our model compared to ACF estimates (top row) as well as the differences between our model and QR estimates that does not control for endogeneity (bottom row). The capital estimates in each industry are increasing and the labor estimates are decreasing in τ. Capital estimates are different from ACF for both low and high percentiles in all industries except ISIC 381. In each industry the magnitude of the differences between low an high τ is quite large. In ISIC 311, capital estimates range from 0.24 to 0.56 and labor estimates range from 0.57 to 0.32. In ISIC 322, capital estimates range from 0.25 to 0.7 and labor estimates range from 0.75 to 0.47. In

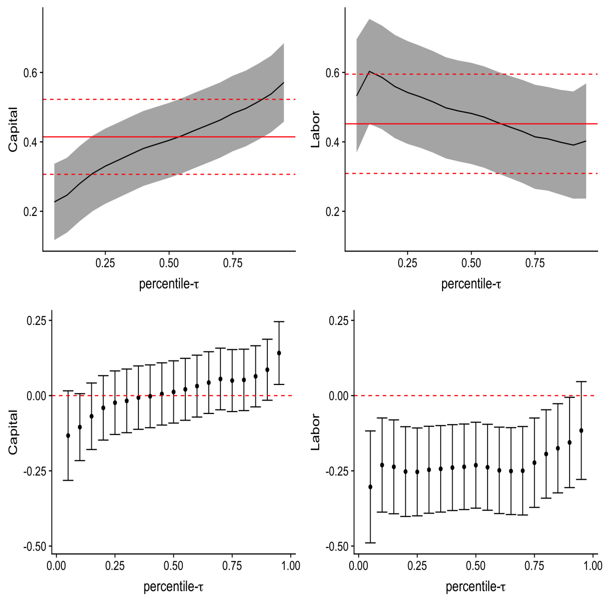
ISIC 381, capital estimates range from 0.25 to 0.54 and labor estimates range from 0.60 to 0.4. Finally, in the combined sample, capital estimates range from 0.26 to 0.58 and labor estimates range from 0.59 to 0.4. Comparing our results to those obtained using QR that does not control for endogeneity, most of the differences appear through the labor estimates which is intuitive as labor is more correlated to current productivity than capital.

Figure 15: Estimated Coefficients of Capital and Labor for Colombia: ISIC 322



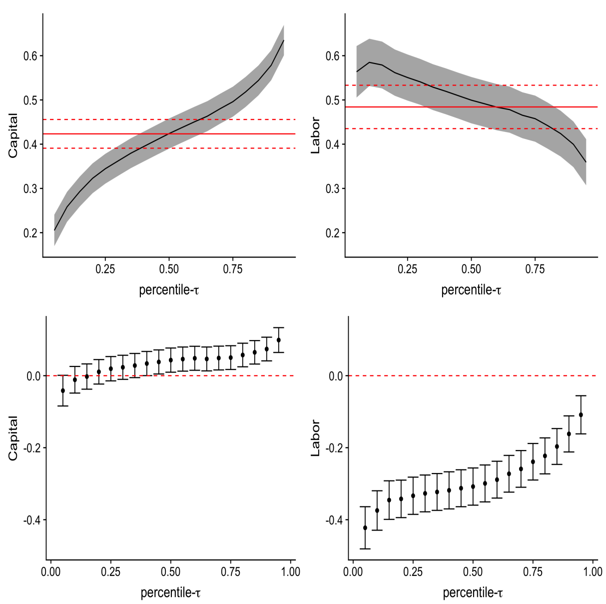
*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

Figure 16: Estimated Coefficients of Capital and Labor for Colombia: ISIC 381



*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

Figure 17: Estimated Coefficients of Capital and Labor for all Colombian Manufacturing Plants



*Top row: Estimated values of production function coefficients and their point-wise 90% confidence interval. Bottom row: Difference between DS and QR estimates that does not control for endogeneity and their 95% confidence intervals.

Using the estimates of capital and labor elasticities, we construct measures of returns to scale and capital intensity for each industry in Table 10. Returns to scale are smallest in ISIC 311 and largest in ISIC 322. Interestingly, both returns to scale and capital intensity are increasing in τ. Table 11 reports the mean estimates from ACF.

Table 10: Coefficient Estimates and Standard Errors for Colombian Manufacturing Plants

		Capital		Labor		Returns to Scale		Capital Intensity	
ISIC	τ	Coef.	s.e	Coef.	s.e	Coef.	s.e	Coef.	s.e
311	0.10	0.242	0.0551	0.567	0.0621	0.809	0.0391	0.427	0.1058
	0.25	0.314	0.0555	0.536	0.0594	0.851	0.0352	0.586	0.1221
	0.50	0.393	0.0562	0.472	0.0600	0.865	0.0344	0.831	0.1575
	0.90	0.563	0.0562	0.323	0.0626	0.886	0.0368	1.746	0.3270
322	0.10	0.331	0.0461	0.741	0.0741	1.071	0.0544	0.446	0.0902
	0.25	0.427	0.0419	0.681	0.0641	1.107	0.0508	0.627	0.1011
	0.50	0.530	0.0419	0.577	0.0614	1.107	0.0489	0.920	0.1409
	0.90	0.675	0.0458	0.478	0.0670	1.153	0.0489	1.413	0.2480
381	0.10	0.247	0.0655	0.603	0.0919	0.850	0.0687	0.409	0.1910
	0.25	0.330	0.0659	0.542	0.0901	0.872	0.0668	0.609	0.2568
	0.50	0.404	0.0656	0.482	0.0886	0.886	0.0660	0.839	0.3616
	0.90	0.538	0.0670	0.391	0.0939	0.929	0.0679	1.377	0.7882
All	0.10	0.258	0.0206	0.585	0.0325	0.843	0.0191	0.442	0.0494
	0.25	0.345	0.0205	0.551	0.0317	0.895	0.0185	0.626	0.0602
	0.50	0.424	0.0207	0.499	0.0318	0.923	0.0183	0.849	0.0780
	0.90	0.578	0.0205	0.400	0.0313	0.978	0.0182	1.447	0.1311

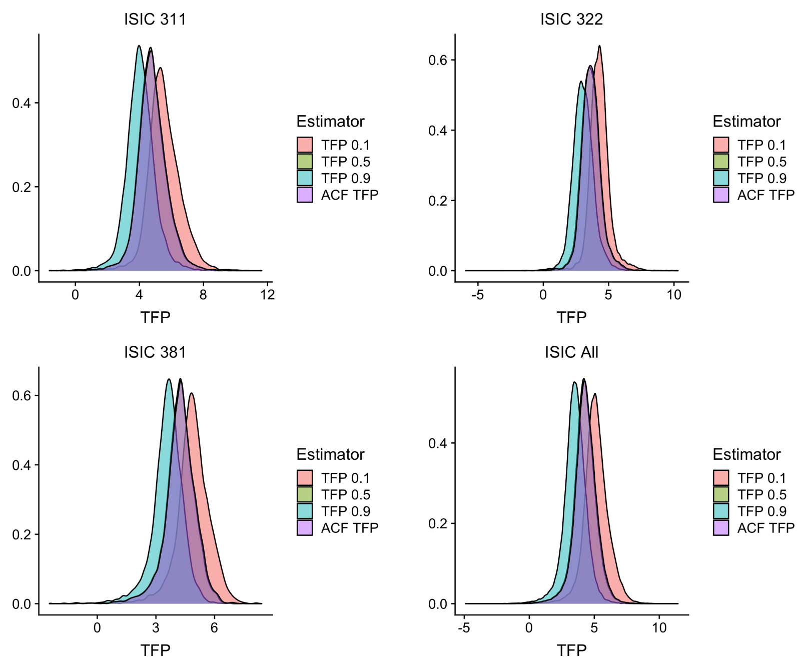
*Standard errors are obtained using bootstrap with 500 replications. The first stage uses estimates from ACF

Table 11: ACF Coefficient Estimates and Standard Errors for Colombian Manufacturing Plants

	Capital		Labor		Returns to Scale		Capital Intensity	
ISIC	Coef.	s.e	Coef.	s.e	Coef.	s.e	Coef.	s.e
311	0.403	0.0550	0.442	0.0576	0.845	0.0346	0.911	0.1660
322	0.506	0.0421	0.608	0.0629	1.114	0.0497	0.831	0.1279
381	0.414	0.0656	0.452	0.0869	0.866	0.0647	0.917	0.4060
All	0.423	0.0197	0.484	0.0298	0.908	0.0177	0.874	0.0765

*Standard errors are obtained using bootstrap with 500 replications.

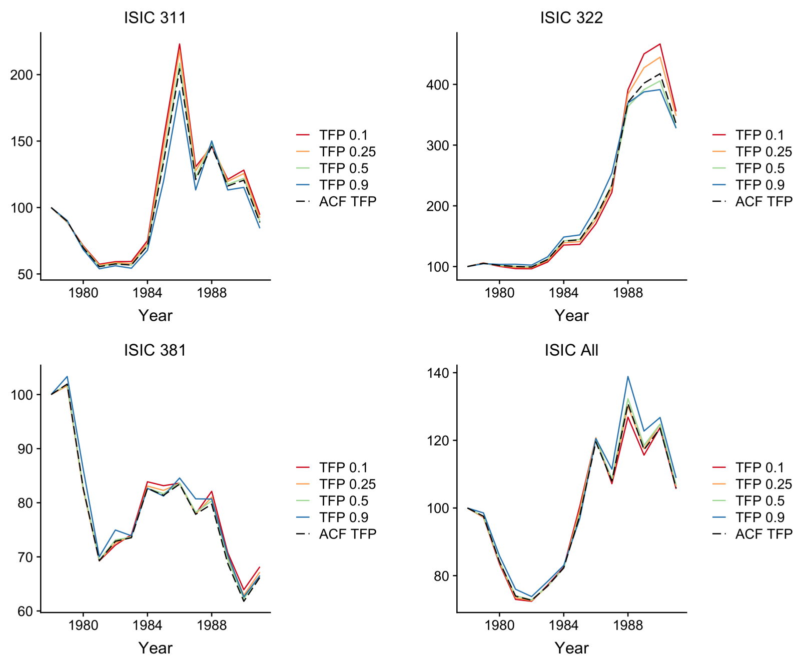
Figure 18: DS and ACF Estimates of Log Total Factor Productivity



*Estimated distributions of TFP from the DS estimator for $\tau \in \{0.1, 0.5, 0.9\}$ and those from the ACF estimator.

Figure 18 plots the densities of TFP for various quantiles and for ACF estimates. Compared to the U.S. and Chile, there is less dispersion between these estimates across τ. Figure 19 reports average productivity over time for each industry with base period set to 100. Productivity decreases in late 1970s and early 1980s for ISIC 311, 381, and the combined sample, but then increases for the rest of the sample period. Productivity in ISIC 322 has increased over time, whereas productivity in ISIC 381 has mostly decreased. The ACF estimates are similar to productivity of firms at $\tau = 0.5$. The figure shows that compared to the U.S. and Chile, productivity dispersion across τ in Colombian manufacturing industries has been relatively constant over time. Thus, it would be interesting to examine whether the pattern in productivity dispersion is similar in more recent years.

Figure 19: Colombia Productivity Over Time



*Estimated average productivity (in levels) over time for Colombia. Base year productivity is set to 100.

Table 12 estimates the effects of firm characteristics on productivity. Some estimates for exports and imports in ISIC 322 are negative, but these are not statistically significant. There is not much difference between the median estimate $\tau = 0.5$ and the mean estimates in Table 13. Similar to the Chilean estimates, our results show that the returns to all three activities diminish as τ increases, which confirms significant heterogeneous effects of firm characteristics across the conditional output distribution. This confirms that the proposed quantile estimator is a useful tool to understand the relationship between productivity and firm activities across the conditional output distribution.

Table 12: Productivity Differentials for Colombian Manufacturing Plants using DS

		Exports		Imports		Advertisements	
ISIC	τ	Coef.	s.e	Coef.	s.e	Coef.	s.e
311	0.10	0.090	0.0323	0.208	0.0420	0.158	0.0350
	0.25	0.074	0.0308	0.194	0.0387	0.151	0.0324
	0.50	0.061	0.0301	0.186	0.0353	0.149	0.0301
	0.90	0.034	0.0300	0.171	0.0305	0.145	0.0271
322	0.10	0.002	0.0243	0.040	0.0294	0.057	0.0254
	0.25	-0.010	0.0230	0.015	0.0291	0.036	0.0253
	0.50	-0.017	0.0225	-0.000	0.0298	0.020	0.0258
	0.90	-0.034	0.0223	-0.035	0.0319	-0.011	0.0274
381	0.10	0.107	0.0384	0.158	0.0424	0.169	0.0372
	0.25	0.091	0.0372	0.141	0.0407	0.151	0.0362
	0.50	0.079	0.0364	0.128	0.0394	0.136	0.0354
	0.90	0.051	0.0347	0.099	0.0369	0.106	0.0341
All	0.10	0.087	0.0118	0.176	0.0117	0.157	0.0096
	0.25	0.068	0.0111	0.148	0.0113	0.136	0.0093
	0.50	0.053	0.0106	0.126	0.0111	0.122	0.0091
	0.90	0.025	0.0099	0.085	0.0110	0.094	0.0091

*Standard errors are obtained using bootstrap with 500 replications. Log(TFP) is regressed on log(Exports), log(Imports), and log(Advertisements).

Table 13: Productivity Differentials for Colombian Manufacturing Plants using ACF

	Exports		Imports		Advertisements	
ISIC	Coef.	s.e	Coef.	s.e	Coef.	s.e
311	0.063	0.0303	0.190	0.0349	0.152	0.0301
322	-0.016	0.0225	0.002	0.0296	0.023	0.0256
381	0.082	0.0363	0.133	0.0391	0.140	0.0352
All	0.056	0.0106	0.130	0.0112	0.125	0.0091

*Standard errors are obtained using bootstrap with 500 replications. Log(TFP) is regressed on log(Exports), log(Imports), and log(Advertisements).

8 Conclusions

We propose a method that extends the control function approach to estimating the conditional quantiles of firm output. The method is computationally attractive, as it resembles many two-stage estimators used in quantile regression models such as Lee (2007) or Chernozhukov and Hansen (2005). As a result, practitioners can easily apply the proposed estimator to production function models where the data may reveal significant heterogeneous elasticities along the conditional output distribution. We show that this estimator works well in finite samples and that it captures heterogeneity under different data generating processes. An application to widely used datasets from the U.S., Chile, and Colombia reveal that in some industries, our estimator captures interesting patterns in unobserved heterogeneity that the conventional mean estimators of the production functions in the literature do not.

Improvements and extensions of this estimator are currently being explored. For example, using a value-added production function may estimate more heterogeneous elasticities than a gross-output production function. Using a gross-output production function with an intermediate input proxy variable suffers from non-identification. However, an adaption to the approach of Gandhi *et al.* (2020) is not as straightforward. This paper also makes a brief connection to the literature on production risk and quantile utility maximization. Currently, quantile utility maximization problems and estimation of these models are being studied by de Castro and Galvao (2019) in the context of dynamic consumption problems. It would be interesting to explore a model for a firm who maximizes quantile utility of profits which could provide a structural interpretation for unobserved heterogeneity that is obtained from quantile regression estimates.

Another extension of this research would be to document the heterogeneity in firms across the *unconditional* quantiles of output as opposed to the conditional distributions we consider here. For example, the estimates from an unconditional model can be interpreted as output elasticities for firms in the τ-th quantile of the marginal distribution of output. Since our model conditions on capital and labor, we can only examine the effect for firms with similar input usage. The unconditional elasticities could then be used to construct estimates of TFP which may have substantial implications for how economists typically measure productivity dispersion. This idea can be explored using the results of Firpo *et al.* (2009) where the dependent variable in their model can be replaced by output net of productivity in our model.

This paper contributes to the growing literature on production functions with unobserved heterogeneity. We show that differences in firm output correspond to the rank of the unobserved technology shock. The location-shift model for productivity we propose here restricts us from examining other dimensions of firm heterogeneity. Therefore, allowing richer distributional effects of productivity would be an interesting extension. This approach also restricts us from examining non-Hicks neutral productivity shocks such as factor-augmenting productivity. An extension of this paper to a non-separable model is being explored to address these last two points, but the estimator we propose here is computationally attractive and easy to implement in empirical research.

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Appendix

Proof of Theorem 4.1:
We first note that the conditional quantile function is given by

$$Q_\tau(Z_{it}|x_i) = Q_\tau(x_{it}\boldsymbol{\beta}(\eta_{it})|x_i) = x_{it}\boldsymbol{\beta}(\tau), \tag{18}$$

where the second equality follows from Assumption (4.1)(d). Since the conditional quantiles are the inverse of the conditional distribution functions, identification of parameters $\boldsymbol{\beta}(\tau)$ comes from identification of the conditional distribution functions.

Using the result of Gil-Pelaez (1951), the conditional distributions of $Z_{it}|x_i$ can be recovered as follows:

$$F_{Z_{it}|x_i}(Z_{it}|x_i) = \frac{1}{2} - \lim_{\upsilon\to\infty}\int_{-\upsilon}^{\upsilon}\frac{e^{-\mathrm{i}sZ_{it}}}{2\pi\mathrm{i}s}\phi_{Z_{it}|x_i}(s|x_i)ds. \tag{19}$$

As a result, identification of the conditional characteristic function $\phi_{Z_{it}|x_i}(s|x_i)$ is sufficient to guarantee identification of the parameters. Recall that for any random variable X and $\rho \neq 0$, we write $\tilde{X} = X/\rho$. The characteristic function of $(y_{it}, \tilde{y}_{it+1})$ conditional on $X_i = x_i$ can then be written as

$$\begin{aligned}
&\Psi_{(y_{it},\tilde{y}_{it+1})|x_i}(s_1, s_2|x_i)\\
=&\mathbb{E}[\exp\{\mathrm{i}[s_1 y_{it} + s_2\tilde{y}_{it+1}]\}|x_i]\\
=&\mathbb{E}[\exp\{\mathrm{i}[s_1 Z_{it} + s_1\omega_{it} + s_2\tilde{Z}_{it+1} + s_2\tilde{\omega}_{it+1}]\}|x_i]\\
=&\mathbb{E}[\exp\{\mathrm{i}[s_1 Z_{it} + s_2\tilde{Z}_{it+1} + s_2\tilde{\xi}_{it+1} + (s_1+s_2)\omega_{it}]\}|x_i]\\
=&\mathbb{E}[\exp\{\mathrm{i}s_2\tilde{Z}_{it+1}\}|x_i]\mathbb{E}[\exp\{\mathrm{i}[s_1 Z_{it} + s_2\tilde{\xi}_{it+1} + (s_1+s_2)\omega_{it}]\}|x_i]\\
=&\mathbb{E}[\exp\{\mathrm{i}s_2\tilde{Z}_{it+1}\}|x_i]\mathbb{E}[\exp\{\mathrm{i}s_2\tilde{\xi}_{it+1}\}|x_i]\mathbb{E}[\exp\{\mathrm{i}[s_1 Z_{it} + (s_1+s_2)\omega_{it}]\}|x_i]\\
=&\mathbb{E}[\exp\{\mathrm{i}s_2\tilde{Z}_{it+1}\}|x_i]\mathbb{E}[\exp\{\mathrm{i}s_2\tilde{\xi}_{it+1}\}|x_i]\mathbb{E}[\exp\{\mathrm{i}s_1 Z_{it}\}|x_i]\mathbb{E}[\exp\{\mathrm{i}(s_1+s_2)\omega_{it}\}|x_i]\\
=&\phi_{\tilde{Z}_{it+1}|x_i}(s_2|x_i)\phi_{\tilde{\xi}_{it+1}|x_i}(s_2|x_i)\phi_{Z_{it}|x_i}(s_1|x_i)\phi_{\omega_{it}|x_i}((s_1+s_2)|x_i),
\end{aligned} \tag{20}$$

where the third equality uses the Markov process for productivity and the fourth line uses Assumption 4.1(b)(ii). The fifth equality uses 4.1(b)(iii) and the sixth equality uses 4.1(b)(i). Taking the derivative of (20) with respect to its first component yields:

$$
\begin{aligned}
&\partial_{s_1}\Psi_{(y_{it},\tilde{y}_{it+1})|x_i}(s_1,s_2|x_i)\\
=&\phi_{\tilde{Z}_{it+1}|x_i}(s_2|x_i)\phi_{\tilde{\xi}_{it+1}|x_i}(s_2|x_i)\partial_{s_1}\phi_{Z_{it}|x_i}(s_1|x_i)\phi_{\omega_{it}|x_i}((s_1+s_2)|x_i)\\
+&\phi_{\tilde{Z}_{it+1}|x_i}(s_2|x_i)\phi_{\tilde{\xi}_{it+1}|x_i}(s_2|x_i)\phi_{Z_{it}|x_i}(s_1|x_i)\partial_{s_1}\phi_{\omega_{it}|x_i}(s_1+s_2|x_i).
\end{aligned}
\tag{21}
$$

Dividing equation (21) by (20) we obtain

$$
\begin{aligned}
\frac{\partial_{s_1}\Psi_{(y_{it},\tilde{y}_{it+1})|x_i}(s,-s|x_i)}{\Psi_{(y_{it},\tilde{y}_{it+1})|x_i}(s,-s|x_i)} =&\frac{\phi'_{Z_{it}|x_i}(s|x_i)}{\phi_{Z_{it}|x_i}(s|x_i)}+\frac{\phi'_{\omega_{it}|x_i}(0|x_i)}{\phi_{\omega_{it}|x_i}(0|x_i)}\\
=&\frac{\phi'_{Z_{it}|x_i}(s|x_i)}{\phi_{Z_{it}|x_i}(s|x_i)}+\mathrm{i}\mathbb{E}[\omega_{it}|x_i],
\end{aligned}
\tag{22}
$$

provided that the conditional characteristic functions are non-vanishing according to Assumption 4.1(c). The second equality follows from the fact that $\phi_{\omega_{it}|x_i}(0|x_i)=1$ and $\phi'_{\omega_{it}|x_i}(0|x_i)=\mathrm{i}\mathbb{E}[\omega_{it}|x_i]$ by properties of characteristic functions. We first discuss identification of $\mathbb{E}[\omega_{it}|x_i]$. Note that,

$$
\begin{aligned}
\mathbb{E}[\omega_{it}|x_i]&=\mathbb{E}[y_{it}-Z_{it}|x_i]=\mathbb{E}[y_{it}-x_{it}\beta(\eta_{it})|x_i]=\mathbb{E}[y_{it}|x_i]-x_{it}\mathbb{E}[\beta(\eta_{it})|x_i]\\
&=\mathbb{E}[y_{it}|x_i]-x_{it}\mathbb{E}[\beta(\eta_{it})]=\mathbb{E}[y_{it}|x_i]-x_{it}\boldsymbol{\beta}^{\mu},
\end{aligned}
$$

where the fourth equality follows from the independence assumption in (4.1)(d). In order to identify $\mathbb{E}[\omega_{it}|x_i]$, we must identify $\boldsymbol{\beta}^{\mu}$. This is satisfied by Assumption 4.2(f). Then the conditional characteristic function $\phi_{Z_{it}|x_i}(s|x_i)$ is identified by rearranging (22):

$$
\phi_{Z_{it}|x_i}(s|x_i)=\exp\left(\int_0^s\frac{\mathrm{i}\mathbb{E}[y_{it}\exp(\mathrm{i}\gamma(y_{it}-\tilde{y}_{it+1}))|x_i]}{\phi_{y_{it}-\tilde{y}_{it+1}|x_i}(\gamma|x_i)}d\gamma-\mathrm{i}s\mathbb{E}[\omega_{it}|x_i]\right),
\tag{23}
$$

given that ρ is also identified from Assumption 4.2(f). To identify the distribution of productivity, it is sufficient to identify its conditional characteristic function. Using Assumption 4.1(b)(i), we can write the characteristic function of y_{it} conditional on $X_i=x_i$ as:

$$
\begin{aligned}
\phi_{y_{it}|x_i}(s|x_i)&=\mathbb{E}[\exp\{\mathrm{i}[s(Z_{it}+\omega_{it}])\}|x_i]\\
&=\mathbb{E}[\exp\{\mathrm{i}sZ_{it}\}|x_i]\mathbb{E}[\exp\{\mathrm{i}s\omega_{it}\}|x_i]\\
&=\phi_{Z_{it}|x_i}(s|x_i)\phi_{\omega_{it}|x_i}(s|x_i),
\end{aligned}
\tag{24}
$$

so that the conditional characteristic function corresponding to productivity is identified by re-writing the last line of equation (24)

$$\phi_{\omega_{it}|x_i}(s|x_i) = \frac{\phi_{y_{it}|x_i}(s|x_i)}{\phi_{Z_{it}|x_i}(s|x_i)}, \tag{25}$$

provided that $\phi_{Z_{it}|x_i}(s|x_i)$ is non-vanishing according to Assumption 4.1(c).