Measuring Yield Curve Risk Using Principal Components Analysis, Value at Risk, and Key Rate Durations

Directing light into the black box.

Bennett W. Golub and Leo M. Tilman

The measurement of interest rate risks embedded in a trade or a portfolio plays a central role in the management of fixed-income portfolios. Over the last decade, researchers and practitioners have developed a number of methodologies that quantify the interest rate exposure of portfolios and securities. Beyond duration, the most popular among them are value at risk (VaR) ("RiskMetrics" [1995]), key rate durations (Ho [1992]), principal components (factor) analysis (Litterman and Scheinkman [1991]), principal components durations (Willner [1996]), and yield curve reshaping durations (Klaffky, Ma, and Nozari [1992]).

Each of these approaches has its pros and cons. Some are more intuitive than others; some are applicable to a broader universe of securities; others have proved to be merely an effective portfolio and risk management tool.

This article attempts to unify these approaches and present them within a comprehensive portfolio management framework. We begin by briefly summarizing the methodologies.

The *variance/covariance* approach to *value at risk* is a cornerstone of J.P. Morgan's RiskMetrics™ methodology. First, for each security, a replicating portfolio of zero-coupon bonds is defined. Then, by using the term structure of historical volatility of spot rates and the correlations between them, RiskMetrics™ constructs a 95% confidence interval for the dollar return, thus determining the interval of "improbable losses." The drawbacks of this approach are that it is not directly

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applicable to securities with embedded options, and, according to Ronn [1996], it underestimates losses on days of dramatic market moves.

Many portfolio and risk managers are still reluctant to use VaR, considering it a "one number from a black box [that] provides little understanding of risk" (Litterman [1996]). Our article eliminates this limitation by decomposing VaR along the curve and providing intuition behind and visual representation of the interest rate risk embedded in a portfolio.

Key rate durations have proven to be an effective and intuitive tool for hedging and yield curve risk management. Their popularity may be attributed to the fact that they provide an intuitively obvious way to think about the curve risk. They may be easily implemented within the option-adjusted spread (OAS) model framework, and visually describe the yield curve exposure of securities and portfolios.[1]

KRD profiles can be potentially misleading, however, since sensitivities to key rates are not equally important. Key rates at the long end of the curve are highly correlated and exhibit similar volatilities.[2] Moreover, KRDs do not take advantage of any knowledge about the covariance of interest rates, which limits their applicability to path-dependent securities.

Finally, the KRD interest rate shocks introduced by Ho [1992] have historically implausible shapes, and produce non-differentiable points on the spot curve; see Exhibit 1. This may impact the valuation of securities with forward rate resets, and could conceivably lead to negative forward rates.

Principal components analysis is usually applied to complex systems (e.g., interest rate dynamics) that depend on a large number of factors (explanatory variables) where one wishes to identify the fewest number of new "composite" variables that explain as much of the variability in the system as possible. Using principal components analysis on changes in the level and shape of the U.S. Treasury spot curve, researchers have discovered that the first three principal components of the spot curve changes explain over 95% of the variation in returns on all fixed-income securities over time (see Litterman and Scheinkman [1991]). These factors — *level, steepness,* and *curvature* — are believed to drive interest rate dynamics and can be formulated in terms of interest rate shocks, which can then be used to compute principal component durations — the sensitivities of a security to these changes in the yield curve (see Willner [1996]). Exhibit 2 is an example.

Several years ago, shortly after principal components analysis was first applied to finance, these durations were fleetingly very popular in estimating the interest rate sensitivity of fixed-income securities.(see Kuberek [1990]). Aside from hedging (Barber and

EXHIBIT 1
EFFECT OF KRD SHOCKS ON SPOT AND FORWARD CURVES

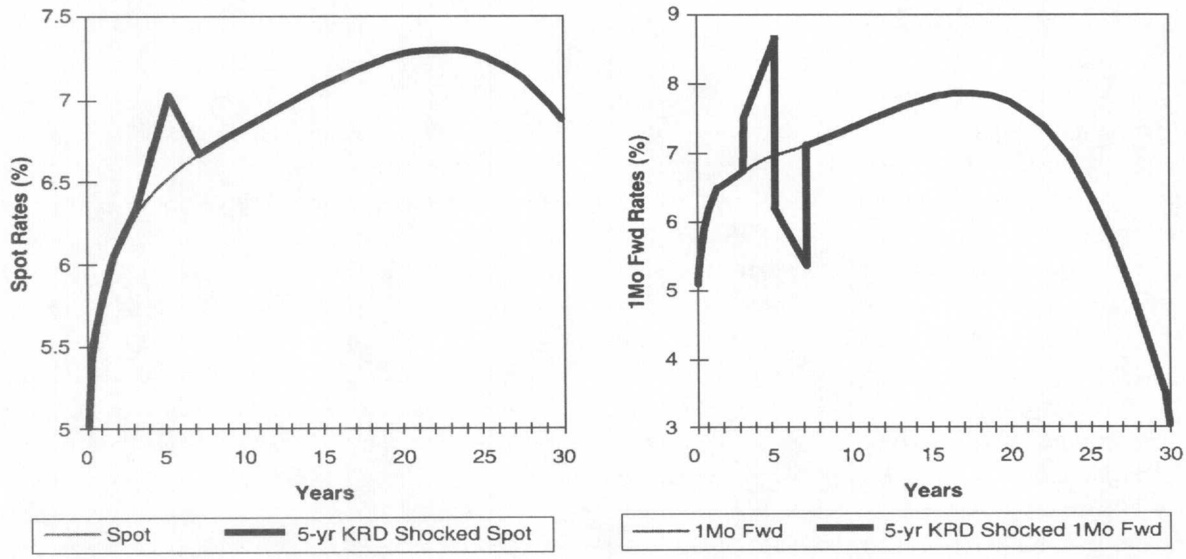
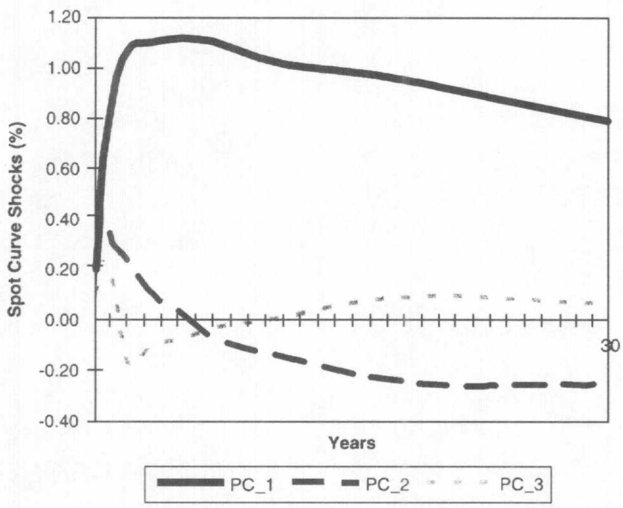


EXHIBIT 2
PRINCIPAL COMPONENT SHOCKS TO SPOT CURVE SMOOTHED VIA CUBIC SPLINES



Data source: J.P. Morgan RiskMetrics™ as of September 30, 1996.

Copper [1996]) and other rare exceptions (Weir [1996]), their actual application today appears to be rather limited, in part, we conjecture, because of lack of intuition and understanding of how to use them when placing yield curve bets.[3]

Yield curve reshaping durations — SEDUR and LEDUR — symbolize a variety of approaches that attempt to estimate the sensitivity of portfolios and securities to the slope of the yield curve. Typically, such durations are computed using predefined *static* interest rate shocks. These measures are very appealing to traders because they are explicitly linked to the commonly traded yield curve spreads. Yet the issues of historical plausibility of these shocks are never considered. Golub and Tilman [1997a] discuss a way to measure historical plausibility of hypothetical interest rate shocks used in immunization, stress tests, and yield curve bets.

This article provides a comprehensive framework for modern yield curve risk and portfolio management. We first use key rate durations to generalize RiskMetrics™ cash flow mappings. This permits us to compute the "variance-covariance" value at risk (VaR) for securities with cash flow uncertainty such as mortgage-backed securities or callable bonds. Next we describe the theory of principal components and its applications to the empirical analysis of yield curve movements. We also provide an alternative method for computing KRDs by establishing relationships among principal components, VaR, and KRDs, and demonstrate that hedging principal component durations is equivalent to hedging value at risk computed via key rate durations. We conclude by discussing practical applications of the theoretical results in everyday portfolio and risk management.

KEY RATE DURATIONS, VALUE AT RISK, AND INTEREST RATE RISK

Recent financial literature discusses the concept of value at risk at length (see "RiskMetrics" [1995], Linsmeier and Pearson [1996], and Jorion [1996]). We briefly summarize this notion and discuss a way to generalize it using key rate durations. We demonstrate that, while for option-free securities VaRs obtained via the KRD methodology and that of J.P. Morgan are quite similar, the former approach is also applicable to securities with embedded interest rate options.

Standard RiskMetrics™ methodology defines value at risk of a portfolio of n assets as follows:

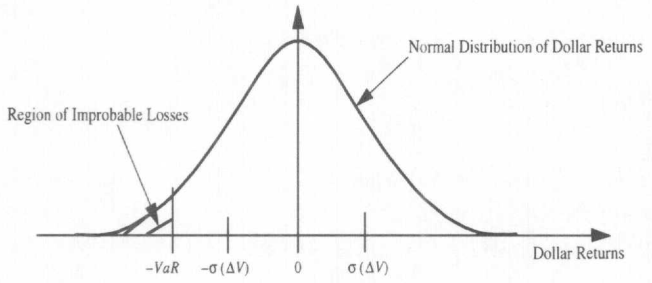
$$\begin{aligned} \text{VaR} &\equiv 1.65\sigma(\Delta V) \\ &= 1.65\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} V_i\sigma_i^P V_j\sigma_j^P \operatorname{corr}(i, j)} \end{aligned} \qquad (1)$$

where V is the total value of the portfolio; V_i is the market value of the i-th asset; σ_i^P is the volatility of percentage returns on the i-th asset; and corr (i, j) is the correlation between the returns on the assets i and j.[4]

There are two competing interpretations of VaR. The first one deals with the "worst case" losses: *Over the specified investment/trading horizon, the position will not lose more than its value at risk with probability 95%.* The other refers to regularly occurring events that one should be comfortable with: *Suppose that our trading horizon is one day; then in 5% of the trading days considered the position will incur losses equal to its daily value at risk or higher.* See Exhibit 3.

Value at risk of an option-free fixed-income security is usually computed via so-called cash flow mappings. They present a security as a collection of zero-coupon bonds by mapping the present values of base case cash flows into key rate maturities. Clearly, this method-

EXHIBIT 3
VARIANCE/COVARIANCE APPROACH TO VALUE AT RISK



ology is not directly applicable to securities with embedded interest rate options, because as interest rates change, cash flows will vary due to option exercise.

Key rate durations provide an alternative method for generating cash flow mappings, presenting a security as a replicating portfolio of zero-coupon bonds. The advantage of such a representation is its applicability to all types of securities including those with embedded options. Thus, if k_i is the i-th KRD of a security, and $MDur_i$ is the modified duration of the zero-coupon bond (ZCB) corresponding to the i-th key rate, the ZCB position v_i delta-equivalent to having a duration of k_i at the i-th key rate is given by the dollar-duration hedging argument:

$$V_i = \frac{Vk_i}{MDur_i} \tag{2}$$

Consider the first-order relationship

$$\sigma^P \approx \sigma^y y MDur \tag{3}$$

among the volatility $\sigma^P \equiv \sigma(\Delta P/P)$ of percentage price changes, the volatility $\sigma^y \equiv \sigma(\Delta y/y)$ of percentage yield changes, the yield y, and the modified duration MDur of an asset (in our case, a zero-coupon bond of maturity i). Then we can use Equations (2) and (3) to rewrite Equation (1) as follows:

$$VaR = 1.65V\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} k_i y_i \sigma_i^y k_j y_j \sigma_j^y \operatorname{corr}(i, j)} \tag{4}$$

In Equation (4), as a direct implication of the approximation in Equation (3), the expression $y_i\sigma_i^y$ is simply the volatility of *yield changes* Δy_i; i.e., $y_i\sigma_i^y = \sigma(\Delta y_i)$ and therefore

$$VaR = 1.65V\sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} k_i k_j \operatorname{cov}(\Delta y_i, \Delta y_j)} \tag{5}$$

Finally, we can rewrite Equation (5) in matrix notation, denoting by $\Im$ the covariance matrix of yield changes and by k the vector of KRDs:

$$VaR = 1.65V\sqrt{k \Im k^T} \tag{6}$$

Exhibit 4 demonstrates the details of the replication of two fixed-income securities — an on-the-run Treasury and a mortgage-backed security — by a portfolio of zero-coupon bonds. Notice that for option-free securities (Treasuries, for example), the replicating portfolios computed via the standard RiskMetrics™ methodology and via KRDs are quite similar, while the VaRs are almost identical.

Typically, the performance of portfolio managers is judged by historical risk-adjusted active return (e.g., "ex post Sharpe ratio"), usually measured as R/TE, where R is the historic average return and TE is the tracking error (see Sharpe [1994]). Tracking error, defined as the standard deviation of the differential return between a portfolio and its benchmark, has been a part of the fixed-income lexicon for over two decades (see Jensen [1969]). It is therefore natural for money managers to measure risk in units of one standard deviation of returns rather than VaR.[5]

There are two types of factors that influence the performance of fixed-income portfolios: basis movements and interest rate movements. While long time periods are required to estimate statistical characteristics of basis movements, much shorter time horizons are needed to measure historical correlations and volatilities of interest rates. Previous studies have failed to create reliable variance/covariance matrices of all such factors (see Dynkin and Hyman [1996]). Therefore a trade-off decision has to be made.

In our judgment, for day-to-day risk and portfolio management, the precision of basis relationships has to be sacrificed in order to achieve better interest rate risk forecasts. As the vast majority of risk in fixed-income securities comes from their exposure to

EXHIBIT 4
SELECTED SECURITIES WITH THEIR VAR AND REPLICATING ZCB PORTFOLIOS

Security Type	VaR	OAD	MV	Cash	3-Mo.	1-Yr.	2-Yr.	3-Yr.	5-Yr.	7-Yr.	10-Yr.	15-Yr.	20-Yr.	30-Yr.
Zero-Coupon Bonds														
Yields					5.35	5.81	6.14	6.31	6.53	6.67	6.84	7.10	7.28	6.86
Prices					98.69	94.43	88.61	83.00	72.52	63.17	51.04	35.12	23.93	13.22
Durations					0.24	0.97	1.94	2.91	4.84	6.77	9.67	14.49	19.30	29.01
30-Yr. OTR TSY														
KRD	18.67	12.42	$100	$0.0	0.01	0.06	0.12	0.27	0.48	0.74	1.36	1.73	1.82	5.82
Equiv. Portfolio (JPM)	18.53	12.42	$100	$0.3	$2.6	$5.8	$6.0	$8.4	$10.0	$10.6	$13.6	$12.3	$11.5	$19.6
Equiv. Portfolio (KRD)	18.67	12.42	$100	–$3.6	$5.9	$6.0	$6.0	$9.2	$9.9	$11.0	$14.1	$12.0	$ 9.5	$20.1
30-Yr. 7% FNMA TBA														
KRD	9.24	5.42	$100	$0.0	0.01	0.05	0.29	0.57	0.87	0.86	1.33	0.88	0.41	0.15
Equiv. Portfolio (KRD)	9.24	5.42	$100	–$1.4	$5.6	$5.3	$15.1	$19.5	$18.0	$12.6	$13.7	$6.1	$2.1	$0.5

OAD = option-adjusted duration, ZCB = zero-coupon bond.

changes in interest rates, we give priority to the most contemporaneous interest rate data rather than to otherwise interesting information about basis relationships.

We define interest rate risk (IntRR) of a security or portfolio to be the a priori component of the total risk arising solely from the variability of default-free spot rates. Formally speaking, IntRR is the standard deviation of percentage changes in value due to changes in interest rates:

$$\mathrm{IntRR} = \sigma(\Delta V/V) \tag{7}$$

Since a portfolio's value V is assumed to be non-stochastic, VaR and IntRR are related linearly:

$$\mathrm{VaR} = 1.65\sigma(\Delta V) = 1.65V\,\mathrm{IntRR} \tag{8}$$

where

$$\mathrm{IntRR} = \sqrt{k\mathfrak{S}k^T} \tag{9}$$

Note that we would obtain the same result if we start with the first-order approximation of V:

$$\frac{\Delta V}{V} = -\sum_{i=1}^{n} k_i \Delta y_i \tag{10}$$

and take the variance of both sides of Equation (10).

The *first-order approximation* of a security's return (and, for that matter, the resulting IntRR and VaR) does not take into account the non-linearity of the price-yield relationship, so its applicability to securities with non-zero convexities may be in doubt. We realize the significance of this limitation. Elsewhere, we discuss a variety of analytical and simulation methods that incorporate non-linearity into VaR measures and focus on the trade-offs among value surface, yield curve dynamics, and computational costs (see Golub and Tilman [1997b]). Other authors have proposed various methods to resolve this issue as well; their approaches include the delta-gamma methodology (see Jorion [1996]), historical simulation, and Monte Carlo simulation (see Linsmeier and Pearson [1996]).

By construction, the KRD shocks as defined by Ho [1992] sum up to a parallel shock. The sum of KRDs is not necessarily equal to the option-adjusted duration of a security, however. The more optionality in a security, the larger the difference.

Therefore, we impose the condition OAD = $\Sigma a_i \mathrm{UKRD}_i$ where UKRDs are the non-normalized KRDs. We define all a_is to be $a_i = \mathrm{OAD}/\Sigma\mathrm{UKRD}_i$ in order to prorate KRDs correctly. Arguably, there may be other methods to normalize KRDs. From now on, unless stated otherwise, KRDs are assumed to be normalized in this manner.

PRINCIPAL COMPONENTS ANALYSIS OF INTEREST RATE DYNAMICS

Some early work on the application of principal components to the analysis of historical movements of the U.S. spot curve is Kuberek [1990] and Litterman and Scheinkman [1991]. Like Barber and Copper [1996], we deal here with principal components of spot curve changes. Although for the time being we focus on theoretical relationships and the notation important for later results, we also touch on the economic interpretation and intuition behind principal components.

Suppose that any spot curve movement can be thought of as an n-vector $\vec{X}$ of key rate changes that are assumed to follow a multivariate normal distribution with zero mean and a given covariance matrix $\mathfrak{I}$. We denote such vector $\vec{X}$ as follows:

$$\vec{X} = (s_1, \ldots, s_n)_{KR} \tag{11}$$

where the random variables s_i are the coordinates of $\vec{X}$ corresponding to the basis KR of *non-orthogonal* key rates.

Principal components analysis finds another set of n random variables called principal components that provide a more parsimonious representation of interest rate movements. Formally, principal components constitute a different basis in the space of spot curve movements and provide a different representation of $\vec{X}$:

$$\vec{X} = (p_1, \ldots, p_n)_{PC} \tag{12}$$

where random variables p_i are the coordinates of $\vec{X}$ corresponding to the basis PC of *orthogonal* unit principal component coefficients. They are the eigenvectors of $\mathfrak{I}$ and are denoted by $(m_{i,1}, \ldots, m_{i,n})$.[6] If $\Omega = (m_{i,j})$ is the matrix whose rows are the principal component coefficients, then the relationships among the different representations of $\vec{X}$ above can be written as follows:[7]

Spot Key Rates → Principal Components:

$$p_i = \sum_{j=1}^{n} m_{i,j} s_j \qquad \text{or } P = \Omega S \tag{13}$$

Principal Components → Spot Key Rates (see the appendix):

$$s_i = \sum_{j=1}^{n} m_{j,i} p_j \qquad \text{or } S = \Omega^T P \tag{14}$$

While Equations (13) and (14) describe the transition from one coordinate system to another, the variance/covariance information about the system is transformed by principal components analysis into the vector $\lambda = (\lambda_i)$ of principal components variances (eigenvalues of $\mathfrak{I}$). Let the diagonal matrix with λ on the diagonal be denoted by Λ.

All principal components (as well as key rates) explain the total variability of historical spot curve movements. If λ_i is the variance of the i-th principal component, important information about historical yield curve movements is embodied in the vector $\hat{\lambda} = (\hat{\lambda}_i)$, where $\hat{\lambda}_i = \lambda_i / \sum_{j=1}^{n} \lambda_j$ is the proportion of total population variance due to the i-th principal component. In effect, the $\hat{\lambda}_i$s rank principal components by their explanatory power. This ranking describes the fundamental dynamics of interest rates. See Exhibit 5.

Principal components analysis of interest rate movements is based on the historical volatilities and correlations of spot key rates, which, by themselves, may lead us to a number of interesting observations. First, notice that the term structure of volatility of spot rates is not flat (Exhibit 5): It sharply increases from three months to two years, and then moderately declines. As for the correlations, they essentially divide key rates into the three distinct highly correlated blocks: short (two- and three-year), intermediate (five-, seven-, and ten-year), and long (ten-, fifteen-, twenty-, and thirty-year). The three-month rate is rather loosely correlated with the rest of the rates, while the one-year rate is moderately correlated.

Usually the first few principal components explain almost all the historical variability. In the monthly RiskMetrics™ data set (Exhibit 5), for instance, nearly 99% of all yield curve movements are explained by the first three principal components.

This approach is different from those discussed in the literature. First, note that the covariance matrix of spot rate changes is the only information needed for the analysis, and no additional regressions or historical fitting are required. Second, if exponential weights or other methods are used while generating the covariance matrix (i.e., are considered appropriate for the types of problems we are dealing with), the principal components will incorporate these techniques.

Third, unlike Willner [1996], we do not define a priori the functional form of the level, steepness, and curvature. Thus, in particular, we do not enforce the

EXHIBIT 5
PRINCIPAL COMPONENTS IMPLIED BY J.P. MORGAN'S RISKMETRICS™ MONTHLY DATA SET (SEPTEMBER 30, 1996)

		3-Mo.	1-Yr.	2-Yr.	3-Yr.	5-Yr.	7-Yr.	10-Yr.	15-Yr.	20-Yr.	30-Yr.
Annualized ZCB Yield Vol. (%)		9.63	16.55	18.33	17.82	17.30	16.62	15.27	14.25	13.26	12.09
One Std. Dev. of ZCB Yields (bp)		52	96	113	112	113	111	104	101	97	83
Correlation Matrix	3-Mo.	1.00	0.80	0.72	0.68	0.65	0.61	0.58	0.54	0.51	0.46
	1-Yr.	0.80	1.00	0.91	0.91	0.89	0.87	0.85	0.81	0.78	0.76
	2-Yr.	0.72	0.91	1.00	0.99	0.97	0.95	0.93	0.89	0.85	0.84
	3-Yr.	0.68	0.91	0.99	1.00	0.99	0.97	0.96	0.92	09.0	0.88
	5-Yr.	0.65	0.89	0.97	0.99	1.00	0.99	0.98	0.96	0.93	0.92
	7-Yr.	0.61	0.87	0.95	0.97	0.99	1.00	0.99	0.98	0.96	0.95
	10-Yr.	0.58	0.85	0.93	0.96	0.98	0.99	1.00	0.99	0.98	0.97
	15-Yr.	0.54	0.81	0.89	0.92	0.96	0.98	0.99	1.00	0.99	0.98
	20-Yr.	0.51	0.78	0.85	0.90	0.93	0.96	0.98	0.99	1.00	0.99
	30-Yr.	0.46	0.76	0.84	0.88	0.92	0.95	0.97	0.98	0.99	1.00

PC No.	Eig. Val.	Vol. PC	Var Expl	CVar Expl	Principal Components 3-Mo.	1-Yr.	2-Yr.	3-Yr.	5-Yr.	7-Yr.	10-Yr.	15-Yr.	20-Yr.	30-Yr.
1	9.24	3.04	92.80	92.80	11.09	28.46	35.69	36.37	36.94	36.30	34.02	32.40	30.33	25.71
2	0.48	0.69	4.80	97.60	43.93	48.66	34.19	20.37	5.23	−9.32	−18.63	−30.09	−37.24	−36.24
3	0.13	0.36	1.27	98.87	42.43	54.96	−44.61	−35.28	−21.02	−8.43	0.31	19.59	27.12	17.76
4	0.06	0.25	0.62	99.49	76.77	−61.47	9.21	−0.18	−0.01	−2.08	−0.65	10.46	11.30	−0.31
5	0.02	0.14	0.20	99.69	12.33	−4.93	−55.03	−3.84	38.06	47.35	33.64	−21.36	−35.74	−14.98
6	0.01	0.10	0.11	99.79	8.94	0.33	18.59	−11.83	−15.02	−2.14	19.64	−44.15	−30.58	77.03
7	0.01	0.09	0.09	99.88	3.02	−0.79	−38.42	49.35	45.01	−48.00	−28.08	−10.93	7.76	27.93
8	0.00	0.07	0.06	99.94	3.26	−1.14	−24.96	66.51	−66.82	17.27	13.02	−0.70	−2.46	−1.38
9	0.00	0.06	0.03	99.97	0.76	−0.46	−1.46	−0.97	0.21	60.38	−72.73	−20.12	19.52	16.59
10	0.00	0.05	0.03	100.00	0.54	0.00	−2.53	1.32	−0.42	5.15	−27.03	67.98	−64.58	21.03

Eig. Val. = $10{,}000\lambda_i$ = eigenvalues × 10,000.
Vol. PC = $100\sqrt{\lambda_i}$ = volatility of principal components × 100.
Var Expl = $\hat{\lambda}_i$ = percentage of variance explained.
CVar Expl = cumulative percentage of variance explained.

first principal component as a parallel shock; this may cause the factors to become correlated. On the contrary, we believe that the "humped" shape of the first principal component is important by itself and can be used as a tool while placing curve bets.

If, nevertheless, one still wants to force the first principal component to be parallel, there exists a way to orthogonalize the new coordinate system properly. Elsewhere we study shapes of principal components in detail and introduce a way to model yield curve movements using orthogonal factors different from principal components (Golub and Tilman [1997a]).

Finally, we would like to note that it is not true that any hypothetical spot curve shift can be explained by the first three principal components. For example, if the spot curve moves exactly one standard deviation of the fourth principal component, the first three will explain 0% of such move.

THE RELATIONSHIPS AMONG IntRR, KRDs, AND PRINCIPAL COMPONENTS

IntRR as a Function of KRDs and Principal Components

Let k be a vector of key rate durations (KRDs) for a given portfolio or security. Express the IntRR element-wise in terms of Equation (9):

$$\mathrm{IntRR}^2 = \sum_{i,j=1}^{n} k_i k_j \operatorname{cov}(s_i, s_j) \tag{15}$$

Note that each element of the covariance matrix $\Im$ is a function of changes in spot rates. Therefore, it can be presented as a function of principal components via Equation (14) as follows:

$$\text{cov}(s_i, s_j) = \text{cov}\left(\sum_{u=1}^{n} m_{u,i}p_u, \sum_{v=1}^{n} m_{v,j}p_v\right) \quad (16)$$

The covariance of a sum is the sum of covariances. Moreover, the principal components are linearly independent, and by definition cov $(p_u, p_u) \equiv \sigma^2(p_u) = \lambda_u$. Hence, Equation (16) reduces to

$$\text{cov}(s_i, s_j) = \sum_{v=1}^{n} m_{v,i}m_{v,j}\lambda_v \quad (17)$$

and the initial expression in Equation (9) for the IntRR transforms into

$$\text{IntRR}^2 = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{v=1}^{n} (k_i m_{v,i}\sqrt{\lambda_v})(k_j m_{v,j}\sqrt{\lambda_v}) \quad (18)$$

Equation (18) identifies the relationship among IntRR, KRDs, and principal components, and has a number of significant implications. First, it pinpoints the exact "sources" of risk, $k_i m_{v,i}\sqrt{\lambda_v}$, which are in units of basis points per year and can be thought of as the partial risk of a security due to the i-th key rate. To see this, notice that $\Delta y_i = m_{v,i}\sqrt{\lambda_v}$ is the annualized change in the i-th key rate implied by v-th principal component. Finally, Equation (18) suggests that there is a direct relationship between KRDs and principal component durations.

Relationship Between Principal Component Durations and Key Rate Durations

The sensitivity of a given portfolio or security to principal component movements in the yield curve can be computed directly (see Willner [1996]). Typically, an OAS model framework would be used to revalue the security after shocking the initial spot curve up and down by each individual principal component.

An effective principal component duration pcdur_i is defined as a percentage change in value V of a security resulting from an annualized one-standard deviation spot curve shock implied by the i-th principal component:

$$\text{pcdur}_i = \frac{-(V_{i,up} - V_{i,down})}{2V} \quad (19)$$

where $V_{i,up}(V_{i,down})$ is the price of a security when the current spot curve is shocked up (down) by the one-standard deviation shock S corresponding to the i-th principal component. Recall that when movements of spot curve are discretized as key rates, S is defined as a vector of key rate changes $(\sqrt{\lambda_i}m_{i,1}, \ldots, \sqrt{\lambda_i}m_{i,n})$. Nevertheless, since inputs to an interest rate model usually require finer time steps than those provided by key rates, interpolation is necessary. We use cubic splines.[8]

Note also that while the unit of ordinary duration is defined to be "the percentage change in price for each unit change in yield," the units of the effective principal component durations are defined as "the percentage change in price for each one-standard deviation change of principal components."

Recall that while deriving expressions for VaR and IntRR, we used the first-order approximation given by Equation (10). We can rewrite it as follows:

$$\frac{\Delta V}{V} = \left.\frac{\Delta V}{V}\right|_{KR;1} + \ldots + \left.\frac{\Delta V}{V}\right|_{KR;n} \quad (20)$$

where $\left.\frac{\Delta V}{V}\right|_{KR;i}$ is the change in price due to a change in the i-th key rate. These components of price change are not independent of each other because the corresponding Δy_i are not independent.

Principal component durations provide a different representation of the IntRR:

$$\frac{\Delta V}{V} = \left.\frac{\Delta V}{V}\right|_{PC;1} + \ldots + \left.\frac{\Delta V}{V}\right|_{PC;n}$$

$$= -\text{pcdur}_1\frac{\Delta PC1}{\sqrt{\lambda_1}} - \ldots - \text{pcdur}_n\frac{\Delta PCn}{\sqrt{\lambda_n}} \quad (21)$$

where $\left.\frac{\Delta V}{V}\right|_{KR;i}$ is the change in price due to a change in the i-th principal component, and are the "standardized" principal components that drive price changes.

The need for "standardized" principal components in Equation (21) is solely because principal component durations measure price sensitivity to the number of *standard deviation* changes in principal components. Finally, since changes in principal components are uncorrelated, taking the variance of both sides of Equation (21) yields:

$$\mathrm{IntRR}^2 \equiv \sigma^2\left(\frac{\Delta P}{P}\right)$$

$$= \mathrm{pcdur}_1^2\,\frac{\sigma^2(\Delta PC1)}{\lambda_1} + \dots +$$

$$\mathrm{pcdur}_n^2\,\frac{\sigma^2(\Delta PCn)}{\lambda_n} \qquad (22)$$

where, by definition, σ^2 $(\Delta PCi) = \lambda_i$. We can rewrite Equation (22) as follows:

$$\mathrm{IntRR} = \sqrt{\sum_{i=1}^{n} \mathrm{pcdur}_i^2} \qquad (23)$$

The result in Equation (23) seems counter-intuitive. In fact, why should pcdur_1 contribute as much to the IntRR as pcdur_2, when the second principal component (PC_2) is one order of magnitude less influential than PC_1? We can reason as follows: Since the variance of PC_1 is much greater than that of PC_2, the interest rate shock implied by PC_1 will be much larger than that implied by PC_2. This will result in a much larger perturbation of the spot curve by PC_1 than PC_2, and therefore the resulting duration pcdur_1 will be much greater than pcdur_2. So, despite the fact that all principal component durations seem to contribute equally to the IntRR in Equation (23), durations corresponding to more influential principal components are much larger by construction, and therefore their contribution to the IntRR is larger.

Finally, we have a simple interpretation of the IntRR in terms of principal components: *Any portfolio is uniquely determined by its "risk vector," whose coordinates are the principal component durations and whose length is the IntRR.*

To establish the relationship between KRDs and principal component durations, denote by $\mathrm{pcdur}/\sqrt{\lambda}$ the element-wise division of the vectors pcdur and $\sqrt{\lambda}$, where the latter is the vector of the volatilities of the principal components. Rewrite Equation (23) as follows:

$$\mathrm{IntRR}^2 = \left(\frac{\mathrm{pcdur}}{\sqrt{\lambda}}\right)\Lambda\left(\frac{\mathrm{pcdur}}{\sqrt{\lambda}}\right)^T \qquad (24)$$

where Λ is defined as a diagonal matrix of principal components' variances. Note that if Ω is the matrix of eigenvectors, then according to the appendix we have $\Lambda = \Omega\Im\Omega^T$. Therefore Equation (24) may be transformed into

$$\mathrm{IntRR}^2 = \left(\frac{\mathrm{pcdur}}{\sqrt{\lambda}}\Omega\right)\Im\left(\frac{\mathrm{pcdur}}{\sqrt{\lambda}}\Omega\right)^T \qquad (25)$$

Finally, through an unambiguous representation of IntRR as a function of the durations and the covariance matrix, the two different representations in Equation (9) and Equation (25) of the IntRR establish a *theoretical* relationship between KRDs and principal component durations:[9]

$$k = \frac{\mathrm{pcdur}}{\sqrt{\lambda}}\Omega \qquad (26)$$

Equation (26) defines a relationship between key rate durations and principal component durations. Suppose a practitioner relies on the variance/covariance matrix and prefers the principal components methodology over key rate durations, while believing that for practical purposes (hedging or yield curve bets) key rate durations are more useful and intuitive than principal component durations. Equation (26) allows the transformation of principal component durations into the key rate durations. Thus the practitioner can enjoy all the benefits of the principal component approach without changing the preferred decision-making paradigm.

Exhibit 6 illustrates these theoretical results. "Method" refers to the way the parameters are computed; on "KRD" lines, for instance, KRDs are computed directly, while the principal component durations are implied via Equation (26) and IntRR is com-

EXHIBIT 6
COMPARISON OF KRD AND PCDUR PROFILES OF TREASURY, CORPORATE, AND MORTGAGE SECURITIES

				Key Rate Durations										PC Durations		
Security	Method	IntRR	OAD	3-Mo.	1-Yr.	2-Yr.	3-Yr.	5-Yr.	7-Yr.	10-Yr.	15-Yr.	20-Yr.	30-Yr.	1	2	3
TSY	KRD	11.32	12.42	0.01	0.06	0.12	0.27	0.48	0.74	1.36	1.73	1.82	5.82	11.04	−2.41	0.56
	PC	11.38	12.42	0.01	0.07	0.07	0.30	0.47	0.62	1.47	1.80	2.36	5.25	11.10	−2.42	0.59
CORP	KRD	8.97	9.30	0.02	0.07	0.13	0.30	0.53	1.15	1.58	1.74	1.71	2.09	8.85	−1.48	0.30
	PC	9.09	9.30	0.00	0.08	0.07	0.35	0.47	1.11	1.17	1.85	2.25	1.43	8.96	−1.49	0.32
MTG	KRD	5.60	5.42	0.01	0.05	0.29	0.57	0.87	0.86	1.33	0.88	0.41	0.15	5.58	−0.35	−0.08
	PC	5.66	5.42	0.00	0.08	0.19	0.65	0.85	0.83	1.44	0.91	0.43	0.09	5.65	−0.37	−0.07

TSY = 30-yr. 6.75% OTR TSY (price: 97.73); Corp = 30-yr. 7.5% A1/A+ Callable Corp Bond (price: 94.49); MTG = 30-yr. 7.0% FNMA TBA (price: 97.54).

puted via Equation (9). On "PC" lines, principal component durations are computed directly, while the KRDs and IntRR are implied.

First, observe that KRDs imply principal component durations quite precisely, and vice versa; this provides the empirical support for Equation (26). Second, note that the IntRRs implied by both methods are almost identical.

Finally, the first principal component durations are extremely close to the IntRR. This is a consequence of Equation (23) and of the fact that in the recent history of the U.S. bond market the first principal component explains the vast majority of the historical variability in the system.

Effective Risk Profile and Other Portfolio Management Applications

Principal component durations take into account the most recent historical covariance of interest rate changes and are therefore most suitable for short-term risk and portfolio management. Therefore, hedges that employ principal component durations directly may be superior to corresponding hedges that use key rate durations, because the latter do not incorporate knowledge about the dynamics of interest rates.

If the number of KRDs is large, an exact KRD hedge would certainly work well. KRD hedges are not parsimonious, though, and may result in an excessive amount of trading.

In effect, one may think of principal components or VaR hedges as immunization strategies (see Barber and Copper [1996]), while the infinite KRD hedges mimic dedication strategies. As mentioned earlier, however, principal component durations lack intuition and are not suitable as a tool for placing curve bets. Equation (26) demonstrates that hedging IntRR via KRDs is equivalent to hedging principal component risk. Thus, market participants who hedge IntRR via KRDs use a very intuitive approach while enjoying all the benefits of principal component duration hedging.

Equation (26) also provides a simple way to approximate the IntRR when the first principal component explains almost all the variability of interest rates:[10]

$$\text{IntRR} = \sqrt{\sum_{i=1}^{n} \text{pcdur}_i^2} \approx \left|\text{pcdur}_1\right| \approx \left|\sum_{i=1}^{n} k_n m_{1,i} \sqrt{\lambda_1}\right| \qquad (27)$$

For a security with KRDs $(k_1, \ldots, k_n)$, the effective risk profile (ERP) is defined as a vector $(k_1 m_{1,1}\sqrt{\lambda_1}, \ldots, k_n m_{1,n}\sqrt{\lambda_1})$.

We intend to demonstrate that ERP is a better measure of yield curve risk than KRDs. First, note that similar to KRDs, ERP provides a visual representation of the sensitivity of a security to different points on the spot curve. Second, via Equation (27), the elements of the effective risk profile add up approximately to the IntRR. Therefore, the ERP is essentially a *decomposition of the IntRR along the curve*, and its introduction eliminates a significant drawback of the VaR methodology, according to Linsmeier and Pearson [1996]. Not only do we now have a "single summary" measure (IntRR or VaR), we also have their decomposition along the curve. We can therefore observe the contribution of each key rate to the overall interest rate risk.

The example in Exhibit 7 uses the effective risk profile to describe the exposure of a portfolio with a "flat" KRD exposure along the curve. Notice that via Equation (27), the "true" yield curve exposure is not exactly flat but should be adjusted for the volatility of the key rates.

Finally, certain types of securities (adjustable-rate mortgages, for example) are extremely sensitive to discontinuities of the forward curve caused by KRD shocks. We therefore expected the KRDs implied by principal component durations to provide substantially better risk statistics, but our experiments to date via return attribution and other methods do not support this conjecture.

APPLICATIONS TO YIELD CURVE PORTFOLIO AND RISK MANAGEMENT

Our theoretical and experimental results define a comprehensive yield curve portfolio and risk management framework. They unify and generalize the approaches currently in use without changing the current intellectual paradigm and while maintaining an intuitive appeal. Assume that the KRD profile of a portfolio is computed either as defined by Ho [1992] or as implied by principal components. The presented framework is then applicable to a variety of problems.

Monitoring/Risk Management

We have three types of information that describe different aspects of a portfolio's interest rate risk exposure — IntRR (or VaR), effective risk profile (ERP), and key rate durations. While each of them measures the first-order interest rate sensitivity of a portfolio, IntRR and ERP do it by considering the historical covariance of interest rates. Both ERP and KRDs visually describe the yield curve exposure of portfolios, but ERP has one more important property: It "decomposes" IntRR along the curve, thus identifying the key rates with the highest risk concentration.

Hedging

As shown earlier, hedging IntRR and KRDs via portfolio optimization is equivalent to hedging principal component durations, and is more intuitive. These approaches are significantly better than traditional duration matching.

Dramatic Market Move Days

Typically, practitioners worry about duration neutrality. Yet recall that on days when the market moves dramatically, it usually moves as the first principal component. If we wish to maintain duration neutrality, it is better to be duration neutral with respect to the first principal component, instead of duration neutral to a parallel shift in the yield curve.

CONCLUSION

We have defined a direct and unambiguous framework to deal with measuring and managing yield curve risk embedded in portfolios and securities. It unifies the three most popular methodologies currently employed by practitioners — value at risk, key rate durations, and principal components. The results present an opportunity to apply alternative conceptual and computational methods to risk and portfolio management without changing the intellectual paradigm currently in use.

APPENDIX
PRINCIPAL COMPONENTS ANALYSIS

Principal components analysis is a statistical technique used to explain the variance/covariance structure of a complex system (in our case, interest rate dynamics). It is especially beneficial when the system depends on a large number of variables, and we attempt to model it with a smaller number of new variables called principal components. The latter are by construction linear combinations of the original variables.

The first principal component is chosen to explain the maximum percentage of the total variability of the system. The second is

EXHIBIT 7
COMPARISON OF KRD AND EFFECTIVE RISK PROFILES OF A "KRD-FLAT" POSITION

	IntRR	Sum	3-Mo.	1-Yr.	2-Yr.	3-Yr.	5-Yr.	7-Yr.	10-Yr.	15-Yr.	20-Yr.	30-Yr.
KRD Profile (yrs.)	185	2.00	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20	0.20
Effective Risk (bp/yr.)		185	7	17	22	22	22	22	20	19	18	15

chosen so that it is independent from the first one, and explains the maximum percentage of the remaining variability in the system, and so on. We are thus left with a number of independent variables that are ranked by their explanatory power. Moreover, a few of them typically explain 95%-99% of the total variability of the system.

It turns out that principal components coefficients are the eigenvectors of the original covariance matrix $\Im$, i.e., the normalized solutions e_i of the equation:

$$\Im e_i = \lambda_i e_i \tag{A-1}$$

where $e^T e = 1$ and the real numbers λ_i are the solutions to the equation:

$$\det(\Im - \lambda E) = 0 \tag{A-2}$$

and E is the identity matrix. The eigenvalues λ_i are the variances of the principal components.

In matrix notation, Equation (A-1) looks like:

$$\begin{bmatrix} \lambda_1 & 0 \\ \dots & \\ 0 & \lambda_n \end{bmatrix} = \begin{bmatrix} m_{1,1} & \dots & m_{1,n} \\ \dots & \dots & \dots \\ m_{n,1} & \dots & m_{n,n} \end{bmatrix} \Im \begin{bmatrix} m_{1,1} & \dots & m_{1,n} \\ \dots & \dots & \dots \\ m_{n,1} & \dots & m_{n,n} \end{bmatrix}^T \quad \text{or}$$

$$\Lambda = \Omega \Im \Omega^T \tag{A-3}$$

Note that the rows of Ω are linearly independent unit vectors by construction, so Ω is an orthogonal matrix: $\Omega^{-1} = \Omega^T$.

ENDNOTES

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[1]For securities requiring Monte Carlo simulation, computing KRDs can be computationally intensive. Assuming 200 simulations per valuation, a full run — OAS, option-adjusted duration (OAD), and KRDs — requires 5,000 simulations.

[2]Key rates are defined as three-month, one-year, two-year, three-year, five-year, seven-year, ten-year, fifteen-year, twenty-year, twenty-five-year, and thirty-year rates. RiskMetrics™ does not track the historical characteristics of the twenty-five-year. We therefore exclude it from the analysis by mapping the twenty-five-year exposure of a security to the portfolio of twenty-year and thirty-year exposures through a variation of J.P. Morgan's cash flow mappings.

[3]We are aware of some proprietary traders who have been using principal components methods to control risk.

[4]We use the terms "volatility" and "standard deviation" interchangeably.

[5]If a portfolio is not managed against a benchmark, the tracking error estimates the absolute level of risk resulting from holding the portfolio. We use the term "tracking error" in this context as well.

[6]Unless stated otherwise, all indexes throughout vary from 1 to n.

[7]This implies that, if all n principal components are considered, any change in spot rates can be *exactly* represented by changes in principal components, and vice versa.

[8]In practice, however, instead of using the annualized one-standard deviation spot curve shocks, one should consider scaled-down shocks in order to reduce the impact of convexity and then scale up the durations accordingly.

[9]It turns out that KRDs that are computed directly are extremely close but not exactly equal to the KRDs implied by principal components. The minor differences can be attributed to convexity, path-dependence, and extreme sensitivity of certain security types to the non-differentiable points on the forward rate curve caused by the KRD shocks as defined by Ho.

[10]This assumption usually holds for the U.S. Treasury market. Thus, between January and September 1996, the percentage of variance explained by the first principal component has been between 86.9% and 95.2%.

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