A set that is not a subset of some other set in the collection.

CS535 Homework 1

Due: 6pm, Sep. 8, 2022.

General notations: For a digraph $D = (V, A)$, $m := |A|$ and $n := |V|$. For an edge-weighted digraph $D = (V, A; \ell)$, ℓ is the edge-length function.

1. Consider a digraph $D = (V, A)$ with two distinct vertices s and t. Give an algorithm to find an inclusion-wise maximal (not necessarily maximum) edge-disjoint shortest s-t paths in D in $O(m + n)$ time.

2. Consider a digraph $D = (V, A; \ell)$ with arbitrary edge-length function ℓ. Let μ be mean length of a minimum-mean circuit in D. Give an $O(mn)$-time algorithm to find a vertex-price function p on V such that the p-adjusted edge-length function ℓ_p satisfies that
   - for each $a \in A$, $\ell_p(a) \geq \mu$;
   - for each a in a minimum-mean circuit, $\ell_p(a) = \mu$.

3. Let $D = (V, A)$ be a digraph, and s and t be two distinct nodes in V. Show that D has no s-t path if and only if there exists a nonnegative integer-valued labeling p of the nodes satisfying that
   - $p(s) = n$ and $p(t) = 0$;
   - for each $(u, v) \in A$, $p(u) - p(v) \leq 1$.

4. Given a digraph $D = (V, A; \ell)$ in which all but one edge have non-negative lengths, describe an algorithm to test whether D has a negative circuit in $O(m + n \log n)$ time.

5. Suppose $D = (V, A)$ is a digraph and x is a positive vector indexed by A. The lecture presents a reduction from an elementary decomposition of x to an elementary decomposition of circulation in an augmented graph. Show that even if the augmented graph has more than m edges, the elementary decomposition of the circulation described in the lecture still consists of at most m circuits.

6. **[PhD Session only]** Consider a digraph $D = (V, A)$ with two distinct vertices s and t. Give an algorithm to find a shortest s-t even (resp., odd) path in D, if there is any, in $O(m + n)$ time.