CS 535 Homework 2

Due: 6pm, Sep. 22, 2022

1. Suppose M and N are two disjoint matchings in a graph. Give an $O(|M| + |N|)$-time algorithm to decompose $M \cup N$ into two disjoint matchings M' and N' such that $|M'| - |N'| \in \{0, \pm 1\}$.

2. Let $G = (U, W, E)$ be a bipartite graph. Given a maximum matching M of G, describe a *linear*-time algorithm to compute a Hall set $T \subseteq U$.

3. Let $G = (U, W, E)$ be a bipartite graph. A vertex in $U \cup W$ is *essential* if it is covered by all maximum matchings. Suppose M is a maximum matching in G. Give a *linear*-time algorithm to find all essential vertices in G.

4. Let $D = (V, A; w)$ be a *directed* graph with positive edge weight function w. Give an $O(|V|(|A| + |V| \log |V|))$-time algorithm to find a collection of vertex-disjoint circuits in D whose total edge weight is maximum.

5. Suppose that there are m machines and n jobs. Each job j has a processing time p_{ij} on machine i. A scheduling for them partitions the n jobs into m *sequences* $J_1, J_2, \cdots, J_m$, and assigns J_i (possibly empty) to the machine i for $1 \leq i \leq m$. If a job j appears as the k-th job in J_i, it would finish at time which is the sum of the processing time of the first k jobs in J_i. Give a polynomial-time algorithm to compute a scheduling which minimizes the total finishing times of all jobs.

6. [**PhD Session only**] Let $G = (U, W, E; \ell)$ be an edge-weighted bipartite graph with $\ell(e) > 0$ for each $e \in E$. Suppose that the total weight of the edges incident to each vertex in $U \cup W$ is exactly one.

   (a) Show that $|U| = |W|$ and G has a perfect matching. (*Hint*: Use the Hall's condition.)

   (b) Let $n := |U|$ and $m := |E|$. Give an $O(m^2)$-time algorithm to decompose ℓ into a linear combination of at most $m - n + 1$ incidence vectors of perfect matchings in G. (*Hint*: Use a warm-start to generate all but the first perfect matchings.)