CS535 Homework 4

Due: 6pm, Nov. 10, 2022.

1. Let $D = (V, A)$ be a digraph, and k be a positive integer. Suppose that for three distinct nodes u, v and w, there are k edge-disjoint paths from u to v and k edge-disjoint paths from v to w. Show that there are also k edge-disjoint paths from u to w.

2. A car manufacturer has m manufacturing plants $s_1, \cdots, s_m$ and n retail centers $t_1, \cdots, t_n$. The plant s_i has a supply of a_i cars and the retail center t_j has a demand of b_j cars with $\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j$. The shipment expense per car from plant s_i to center t_j is $\max\{p_i, q_j\}$ with $0 < p_1 \leq \cdots \leq p_m$ and $0 < q_1 \leq \cdots \leq q_m$. The firm seeks a distribution of the products from the plants to the centers minimizing the total shipment cost. Give an $O(m+n)$-time algorithm for this car distribution problem. [Hint: try first the case with $m = n = 2$.]

3. Let $D = (V, A; c, \ell)$ be a flow network with positive *integer* capacity vector c and an arbitrary flow price vector ℓ, and x be a min-cost b-TS in D. Suppose now an edge $a \in A$ has capacity increased by 1. Describe an $O(mn)$-time algorithm for finding a min-cost b-TS in D after the capacity change.

4. Suppose that we have to solve a min-cost TS problem in which the sum of the supplies exceeds the sum of the demands, so some nodes need to retain some amount of the supplies. Show how to transform this problem into an (ordinary) min-cost TS problem.

5. A shipping company wants to phase out a fleet of m (homogeneous) general cargo ships over a period of n years. Its objective is to maximize its cash assets at the end of the n years by considering the possibility of prematurely selling ships and temporary replacing them by charter ships. The company faces a known *decreasing* demands $d_1 \geq d_2 \geq \cdots \geq d_n$ for ships in the n years respectively. Each ship earns a revenue of $r_k > 0$ dollars in year k. At the beginning of year k, the company can sell any ship that it owns, accruing a cash inflow of $s_k > 0$ dollars. If the company does not own sufficiently many ships to meet its demand, it must hire additional charter ships. The (net) cost of hiring a ship for the year k is $h_k > 0$ after the deduction of the revenue r_k. The shipping company wants to meet its commitments and at the same time maximize the cash assets at the end of the year n. Formulate this problem as a min-cost TS problem.

6. **[PhD Session only]** Let $D = (V, A; c, \ell)$ be a flow network with positive *integer* capacity vector c and positive flow price vector ℓ. We wish to maximize the s-t flow subject to the additional constraint that the flow cost does not exceed a given cost budget B.

   (a) Let φ^* be the max-flow value from the source s to the sink t. For each $\varphi \in [0, \varphi^*]$, let $mc(\varphi)$ denote the the min-cost of s-t flows of value φ. Prove that $mc(\varphi)$ strictly increases with $\varphi \in [0, \varphi^*]$.

(b) Give a polynomial-time algorithm for finding a maximum s-t flow subject the cost budget B.