Lecture 6. MFMC: Applications

Outline

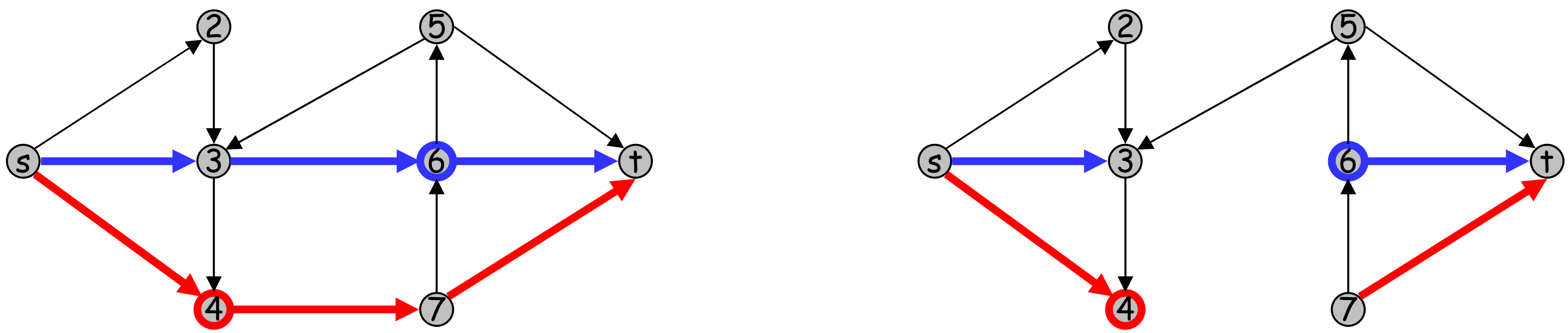
- Disjoint paths
- Min-weighted vertex cover in bipartite graphs
- Max-weighted closed set

1. Disjoint Paths

Edge-disjoint paths

$D = (V, A)$: a digraph with two nodes s and t (assuming $(s, t) \notin A$)

Disjoint path problem. Find the max number of edge-disjoint s-t paths.

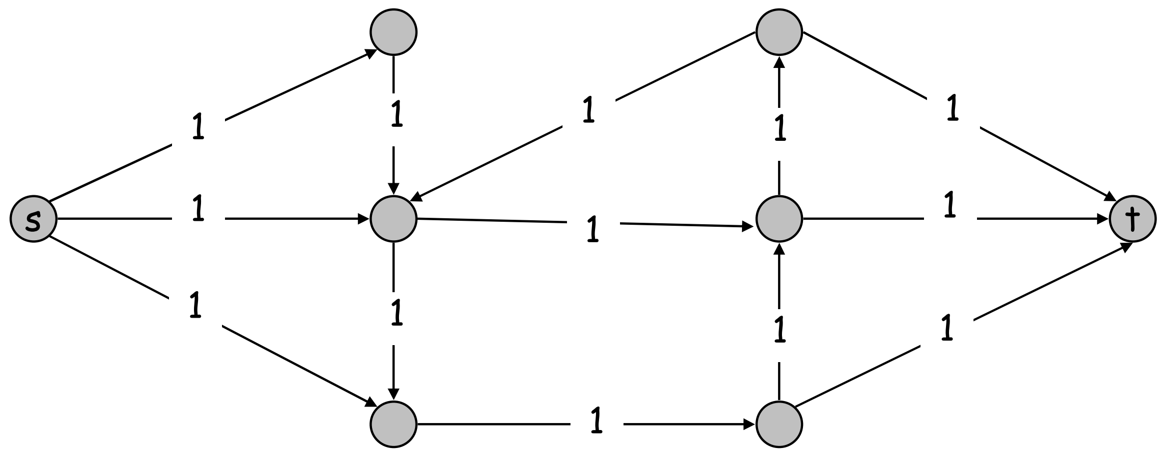


Def. A set of edges $B \subseteq A$ is an s-t edge disconnector if the removal of B disconnects t from s.

Edge connectivity. Find an edge s-t separator of minimum size.

Max-flow min-cut formulation

unit-capacity flow network.



Claim. max number of edge-disjoint s-t paths = max s-t flow value.

Pf.

$\leq$ Given k edge-disjoint paths, sending a unit flow along each path gives a flow of value k.

$\geq$ Given a flow of value k, its decomposition into path/circuit flows gives flows k edge-disjoint paths (and possibly additional circuits). ▪

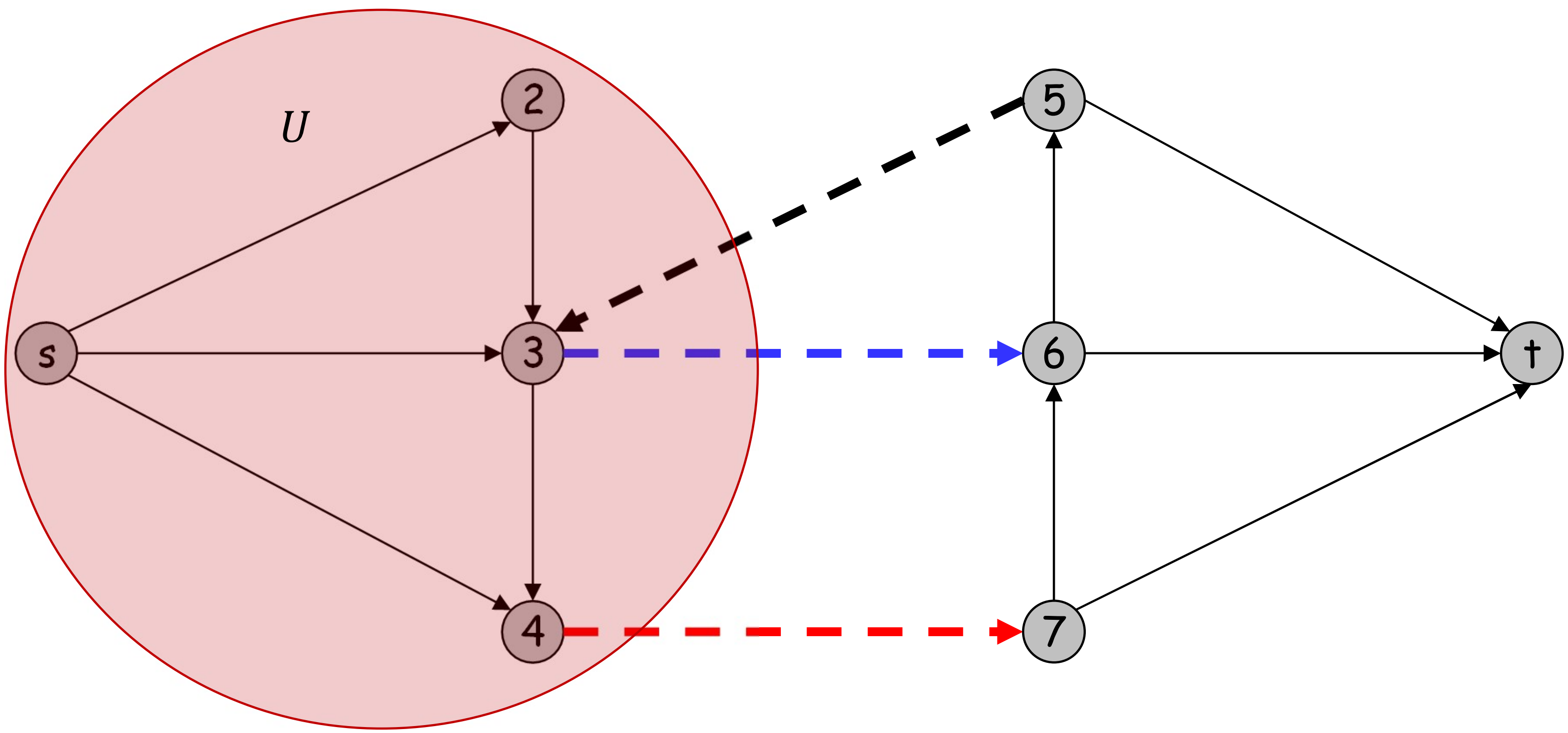
Max-flow min-cut formulation

$B \subseteq A$: an s-t edge disconnector

$U \coloneqq \{$nodes reachable from s via $A \setminus B\}$

Fact. $B \supseteq \delta^{out}(U)$ and $s \in U \subseteq V \setminus \{t\}$

Fact. minimal s-t edge disconnectors $\Leftrightarrow$ s-t cuts



Menger's Theorem: edge version

Theorem. [Menger 1927] max number of edge-disjoint s-t paths = min size of s-t edge-disconnector.

Finding blocking flow via DFS

Each iteration starts at s, and each current vertex v acts as follows:

- Case 1: v has a forward edge. Move along a forward edge to the next node.

- Case 2: v has no forward edge.
  - subcase 2.1: $v = s$. Stop
  - subcase 2.2: $v = t$. Augment and delete ALL arcs on path. If t has no incoming edge, stop; otherwise, move on to the next iteration
  - subcase 2.3: $v \neq s, t$. Delete v (and all its incident edges), and move backward to its predecessor.

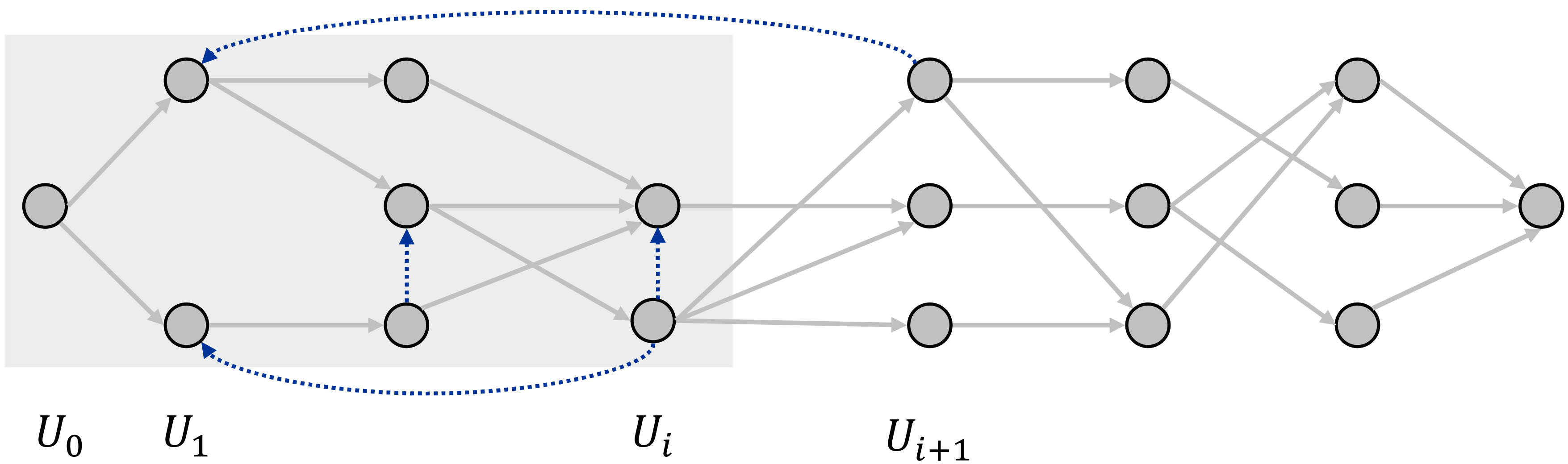
Total running time: $O(m)$

Number of augmentations by blocking flows

Theorem. The total number of augmentations $\leq 2k$, where $k := \lfloor m^{1/2} \rfloor$.
Lemma. After k augmentations, $val(f^*) - val(f) \leq k$.

Pf. Each s-t path in D_f has length $\geq k+1$

- U_i: the set of vertices at distance i from s in D_f
- For some $0 \leq i \leq k$, the number of edges in D_f between U_i and $U_{i+1} \leq m/(k+1) \leq m^{1/2}$, and hence is $\leq k$
- $val(f^*) - val(f) \leq$ the residual cut capacity of $U_0 \cup U_1 \cup \cdots \cup U_i \leq k$



Number of augmentations by blocking flows

Theorem. The total number of augmentations $\leq 2k$ where $k := \lfloor n^{2/3} \rfloor$.

Lemma. After k augmentations, $val(f^*) - val(f) \leq k$.

Pf. Since $\sum_{i=0}^{k}(|U_i| + |U_{i+1}|) \leq 2n$, for some $0 \leq i \leq k$,

$$|U_i| + |U_{i+1}| \leq 2n/(k+1) \ \leq 2n^{1/3}$$

$$\Rightarrow |U_i||U_{i+1}| \leq \left(\frac{|U_i|+|U_{i+1}|}{2}\right)^2 \leq n^{2/3}$$

$$\Rightarrow |U_i||U_{i+1}| \leq k$$

The number of edges in D_f between U_i and $U_{i+1} \leq k$

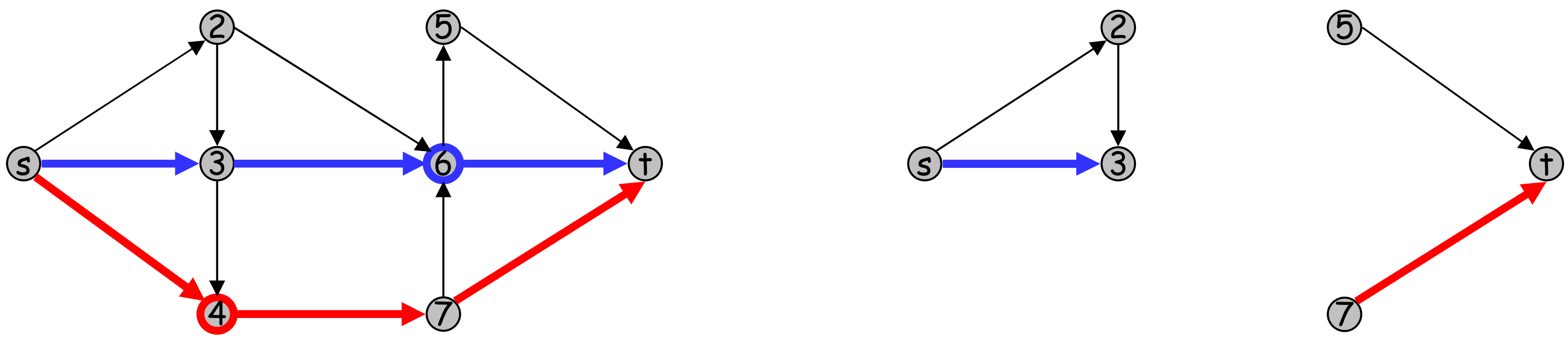
Total running time

Theorem. A maximum number of edge-disjoint s-t paths and a minimum s-t edge disconnector can be computed in $O(m \min\{m^{1/2}, n^{2/3}\})$ time.

Internally node-disjoint paths

$D = (V, A)$: a digraph with two nodes s and t (assuming $(s,t) \notin A$)

Node-disjoint path problem. Find the max number of internally node-disjoint s-t paths.



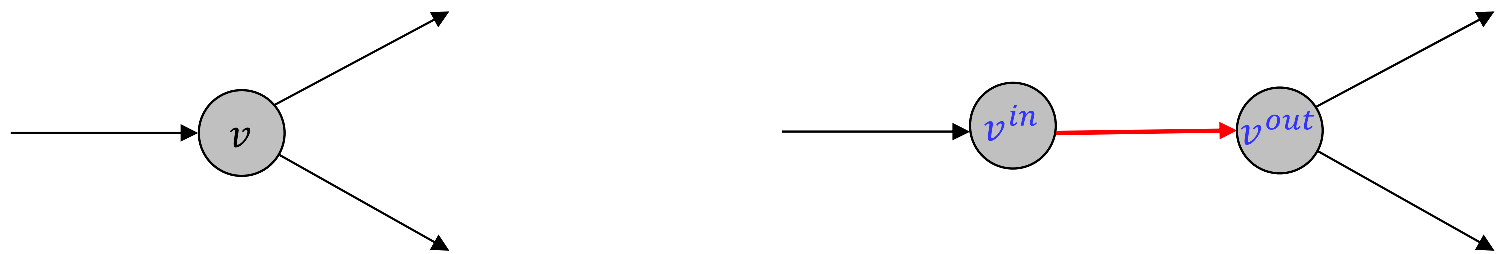
Def. A set of nodes $U \subseteq V \setminus \{s,t\}$ is an s-t node disconnector if the removal of U disconnects t from s

Node connectivity. Find an s-t node disconnector of minimum size.

Reduction to edge-disjoint paths via node-splitting

For each node v other than s and t,

- Replace v by a self edge (v^{in}, v^{out});
- Each edge entering v now enters v^{in};
- Each edge leaving v now leaves v^{out}.



D^+: expanded network

$$s - t \text{ path } P \text{ in } D \leftrightarrow s - t \text{ path } P^+ \text{ in } D^+$$

Claim. Max number of internally node-disjoint $s - t$ paths in D
= max number of edge-disjoint $s - t$ paths in D^+

Reduction to edge-disjoint paths via node-splitting

U: a min set of node $s-t$ disconnectors in D
S: a min $s-t$ cut in D^+

Claim. $|U| = \left|\delta_{D^+}^{out}(S)\right|$

Pf. $\geq$: $\{(v^{in}, v^{out})|v \in U\}$ is an $s-t$ edge disconnector in D^+

$\leq$: Expand S s.t. $\delta_{D^+}^{out}(S)$ consists of only self edges and $\left|\delta_{D^+}^{out}(S)\right|$ is same:

- If a non-self edge $(u^{out}, v^{in}) \in \delta_{D^+}^{out}(S)$, add v^{in} to S.
- $\delta_{D^+}^{out}(S)$ gains 1^- edge (v^{in}, v^{out}) but loses 1^+ edge (u^{out}, v^{in}).

Suppose $\delta_{D^+}^{out}(S) = \{(v^{in}, v^{out})|v \in U'\}$. Then U' is an s-t node disconnector in D ▪

Menger's Theorem: vertex version

Theorem. [Menger 1927] The max number of internal node-disjoint s-t paths = the min size of s-t node disconnectors.

Unit network

- All self edges (v^{in}, v^{out}) have unit capacity
  - each node, except s and t, either has a single unit-capacity incoming edge, or a single unit-capacity outgoing edge
- All other edges have arbitrary positive integer capacity including ∞

Such flow network is called a *unit network*

Finding blocking flow in unit networks via DFS

- For any 0-1 flow f, the f–residual graph and its level graph are also unit networks.
- After an augmenting path is found in the level graph, only unit-flow is sent along it, and ALL internal nodes (and their incident arcs) on path are deleted.

Total running time: $O(m)$

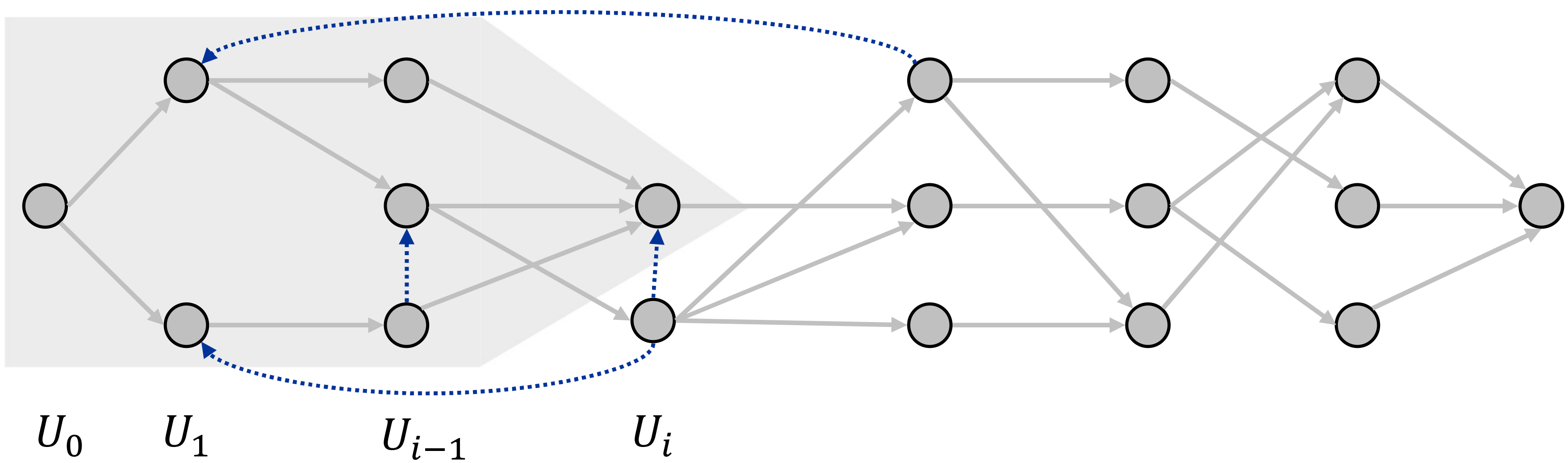
Number of augmentations by blocking flows

Theorem. Total number of augmentations $\leq 2k$, where $k := \lfloor n^{1/2} \rfloor$

Lemma. After k augmentations, $val(f^*) - val(f) \leq k$.

Pf.

- Each s-t path in D_f has length $\geq k + 2$
- U_i: the set of vertices at distance i from s in D_f
- For some $1 \leq i \leq k + 1$, $|U_i| \leq n/(k+1) \leq n^{1/2}$ and hence $|U_i| \leq k$.
- $S := U_0 \cup U_1 \cup \cdots \cup U_{i-1} \cup \{v \in U_i \mid v$ has 1 outgoing residual arc$\}$.
- $val(f^*) - val(f) \leq$ the residue cut capacity of $S \leq |U_i| \leq k$.



Total running time

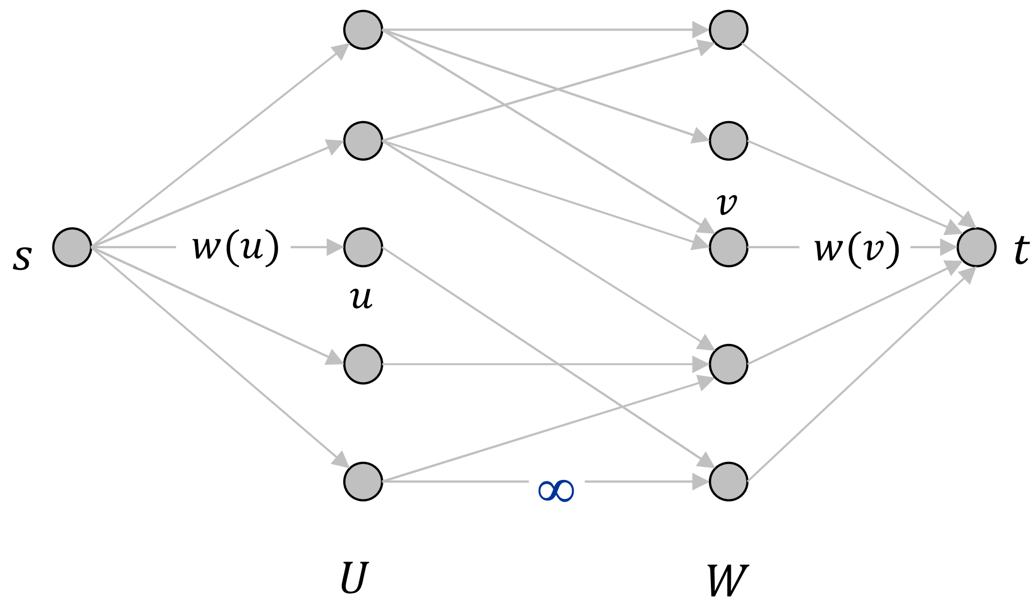
Theorem. A maximum flow and a min-cut in unit networks can be computed in $O(n^{1/2}m)$ time.

Theorem. A maximum number of internally node-disjoint s-t paths and a minimum s-t node separator can be computed in $O(m^{3/2})$ time.

2. WVC in Bipartite Graphs

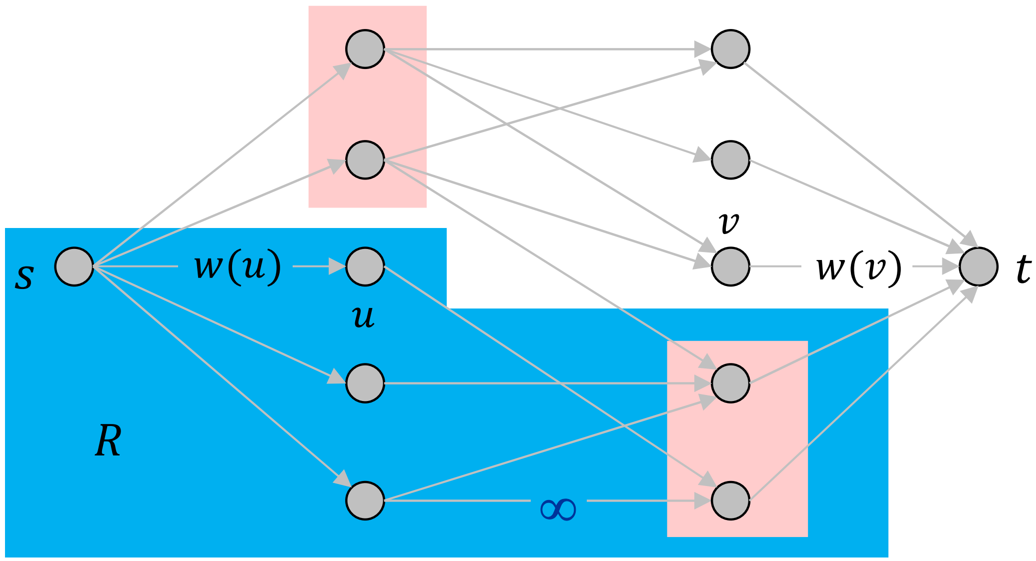
Reduction to Min Cut

Flow network D: turn (positive) weights into capacities



Theorem. min vertex-cover weight of G = min $s - t$ cut capacity of D

Min-VC Weight ≥ Min-Cut Capacity

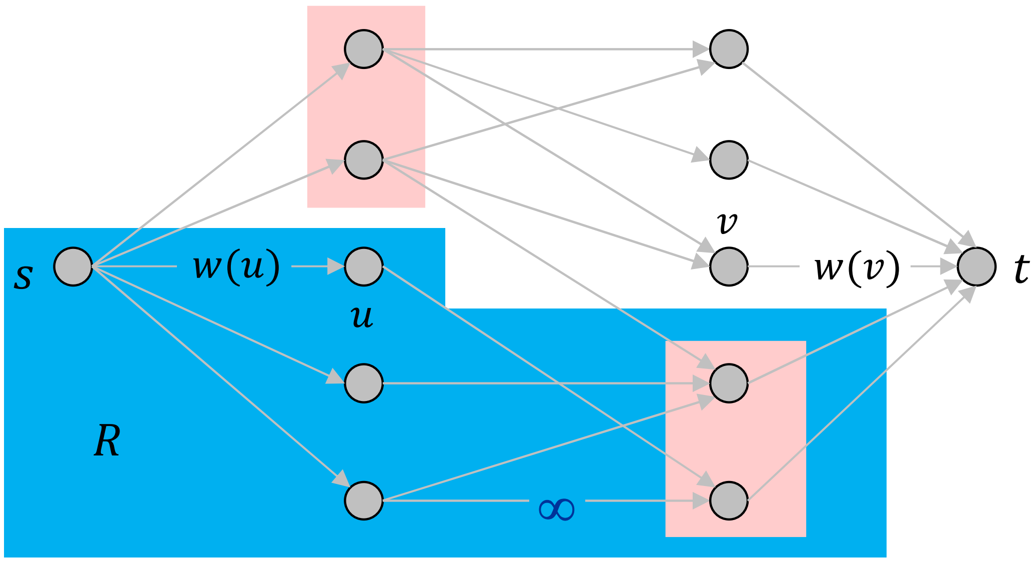


- C: a min-weight vertex cover
- $V \setminus C$ is a stable set
- the cut capacity of $R := \{s\} \cup (U \setminus C) \cup (W \cap C)$ in D is

$$w(U \cap C) + w\,(W \cap C) = w(C)$$

- $N(U \setminus C) = W \cap C,\, N(W \setminus C) = U \cap C$

Min-VC Weight = Min-Cut Capacity



- R: a min s-t cut.
- $\delta^{out}(R)$ can't have ∞ arcs $\Longrightarrow$ no edges between $U \cap R$ and $W \setminus R$
- $I := (U \cap R) \cup (W \setminus R)$ is a stable set
  - $N(U \cap R) = W \cap R$, $N(W \setminus R) = U \setminus R$
- $C := (U \setminus R) \cup (W \cap R)$ is a vertex cover
- $w(C) = w(U \setminus R) + w(W \cap R) =$ capacity of the min-cut R

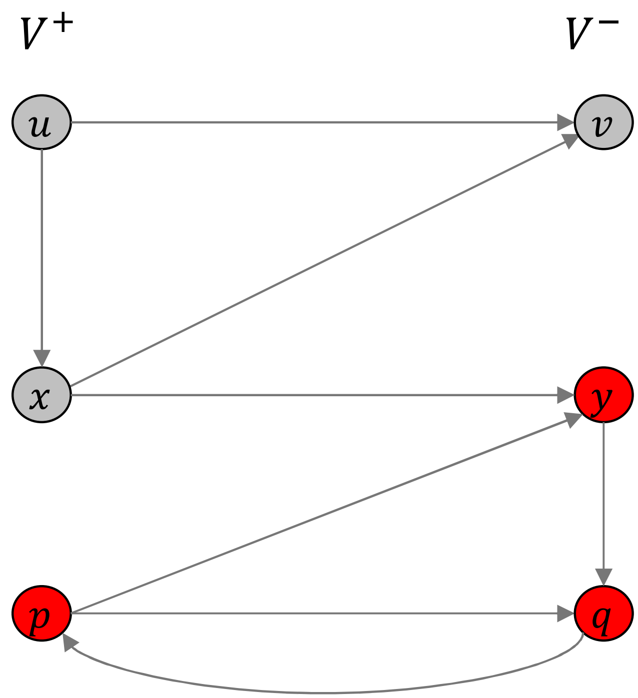
3. Maximum-Weighted Closed Set

Closed subset of vertices

$D = (V, A;\ w)$: a vertex-weighted digraph

can be positive or negative

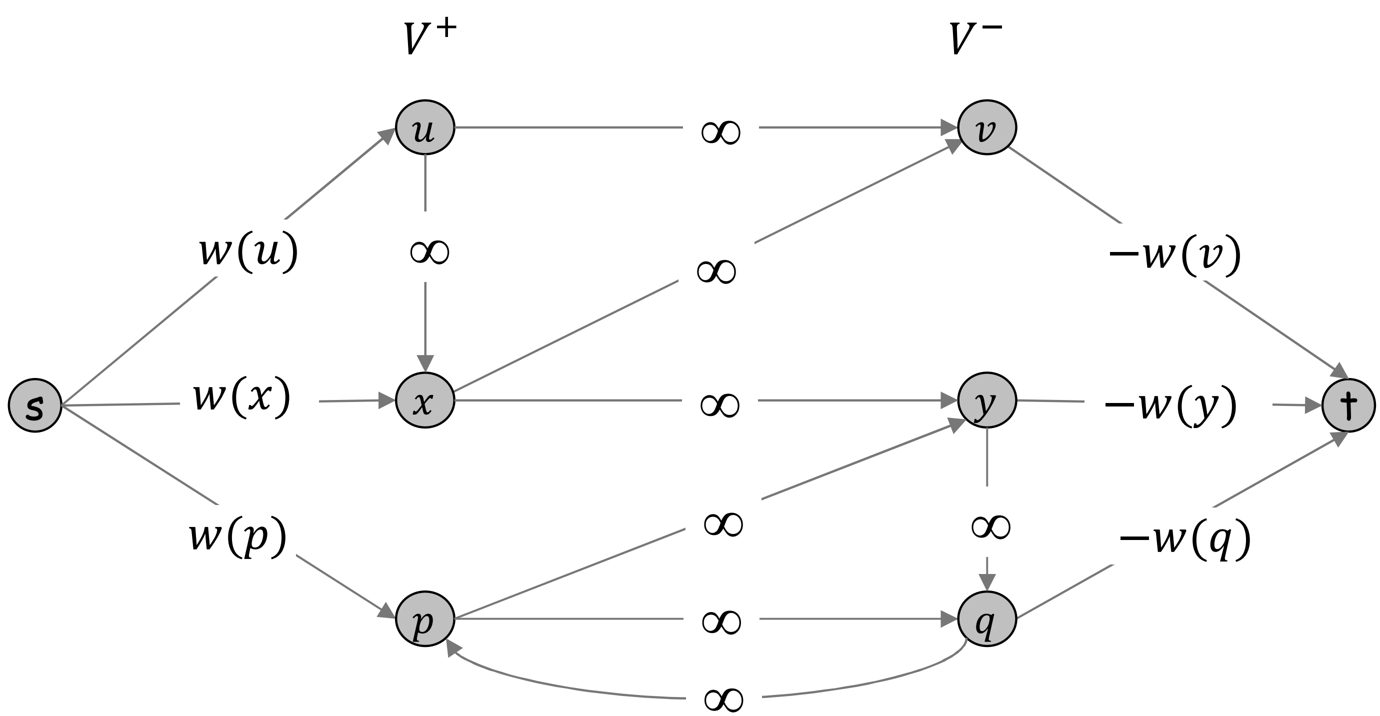
- A subset U of V is closed if for each $u \in U$ its outgoing neighbors are also in U.
- Objective: find a maximum-weighted closed subset of vertices.



p, q, y is closed;
p, q, v is not closed

Min-Cut formulation

- Assign capacity ∞ to all edges in D.
- For each $v \in V^+$, add edge (s, v) with capacity $w(v)$.
- For each $v \in V^-$, add edge (v, t) with capacity $-w(v)$.



Min Cut formulation

- A set U is closed $\Leftrightarrow$ $U \cup \{s\}$ has finite cut capacity
- For any closed U, $U \cup \{s\}$ has cut capacity

$$\begin{aligned} & w(V^+ \setminus U) - \left(-w(V^- \cap U)\right) \\ & = w(V^+) - w(V^+ \cap U) - w(V^- \cap U) \\ & = w(V^+) - w(U) \end{aligned}$$

- U is a max-weighted closed subset $\Leftrightarrow$ $U \cup \{s\}$ has min cut capacity