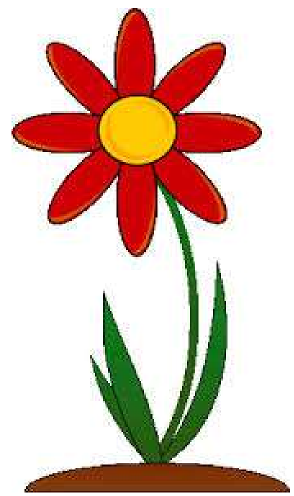
# Lec 8: Maximum Non-Bipartite Matching



# Outline

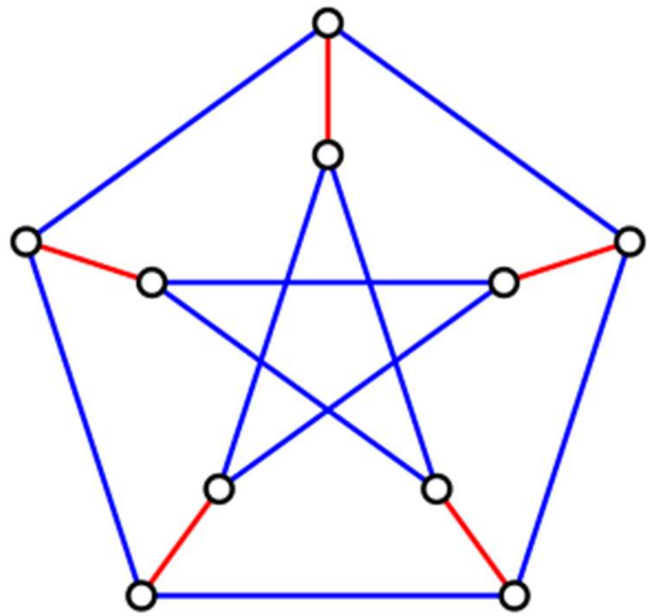
- Curse of odd circuits
- Tutte-Berge Set
- Alternating Forest
- Blossom Shrinking
- Blossom Algorithm for Augmenting-Path
- Edmonds-Gallai Decomposition

# 0. Curse of Odd Circuits

# Recap: Maximum matching

- Input: undirected graph $G = (V, E)$.
- $M \subseteq E$ is a matching if each node appears in at most edge in $M$.
- Max matching: find a max cardinality matching.

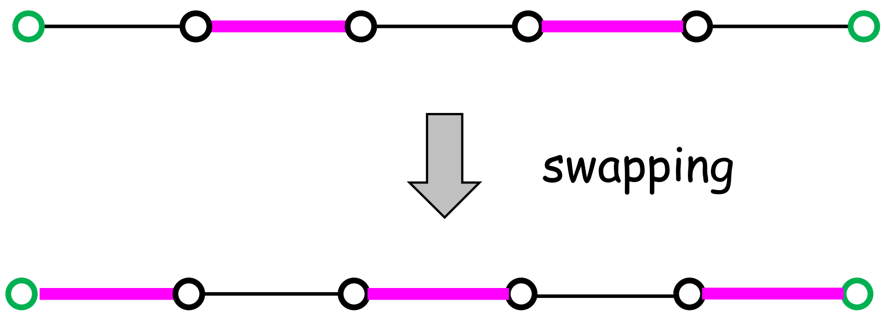
Matching number $\alpha'(G) :=$ the maximum size of all matchings in $G$



Def. A matching is perfect if it covers all vertices.

# Recap: Grow a matching via augmenting paths

For each $M$-augmenting path $P$, $M \oplus P$ is a matching of size $|M| + 1$



# Recap: Augmenting-path method

Theorem: [Petersen 1891] [Kőnig 1931] [Berge 1957]

$M$ is a maximum matching $\Leftrightarrow$ there is no $M$-augmenting path.

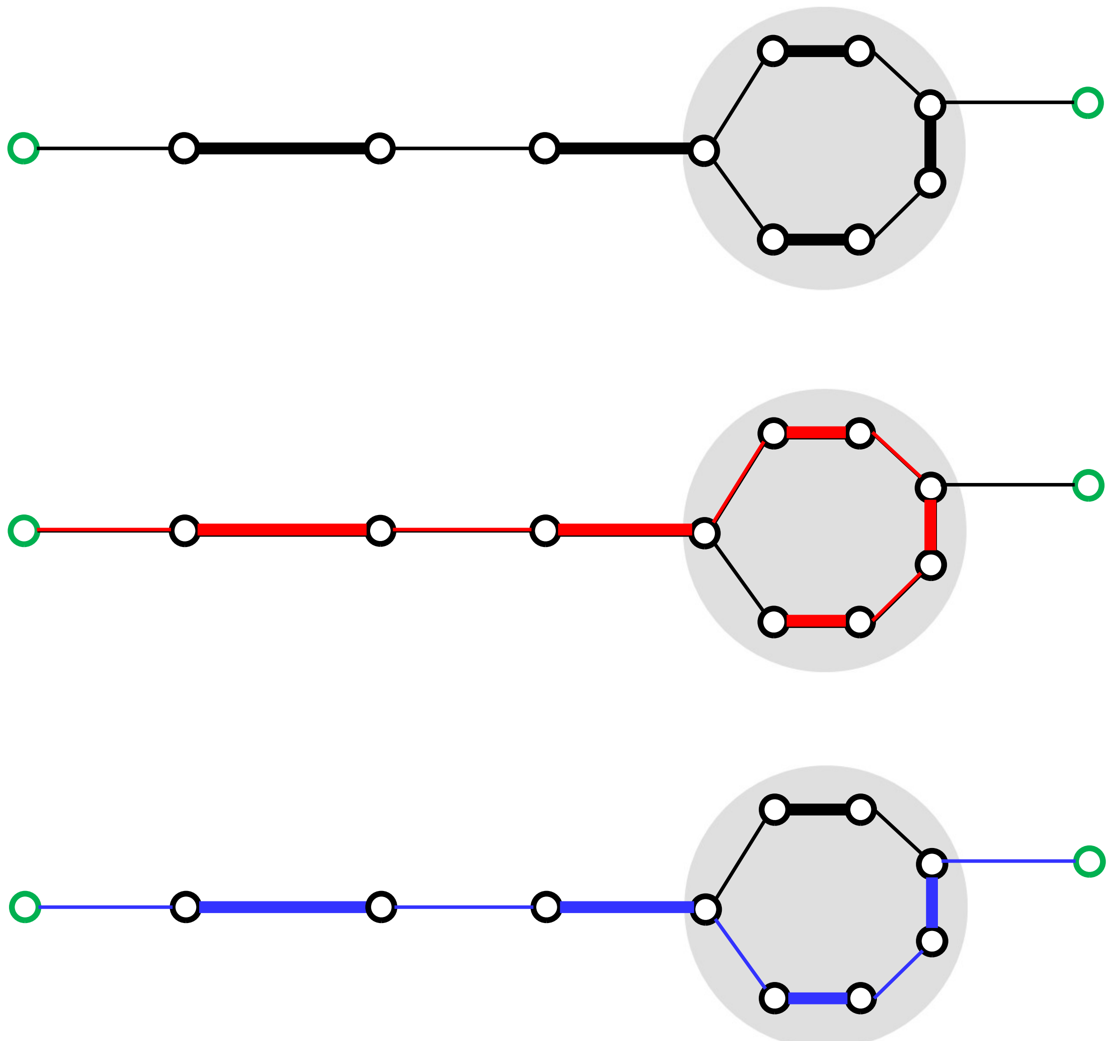
**Augmenting-path method:**

- Start with some initial matching $M$, possibly the empty one.
- While there is an $M$-augmenting path $P$, augment $M$ using $P$

Challenge: How to find augmenting path?

# Curse of odd circuits

A graph $G$ is bipartite $\Leftrightarrow$ it contains no odd circuit



# Essential nodes

Def. A node is essential if it is covered by all maximum matchings, and inessential otherwise.

Recap. Let $M$ be a maximum matching, and $v$ be a node matched by $M$. Then $v$ is inessential $\Leftrightarrow$ $v$ is reachable from an unmatched node along an even $M$-alternating path.

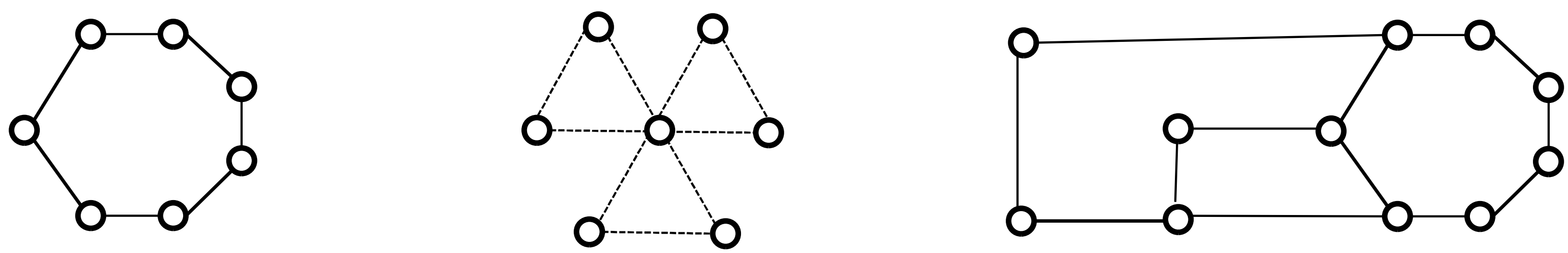
Fact. A bipartite graph with at least one edge has at least one essential vertex.

Extreme cases:

- All nodes are essential: perfectly matchable graph
- All nodes are inessential: ??

# Factor-critical graphs

Def. $G$ is factor-critical if $G - v$ has a perfect matching for every $v \in V$



Fact. odd, non-bipartite, bridgeless

Exercise: Every edge lies in an odd circuit

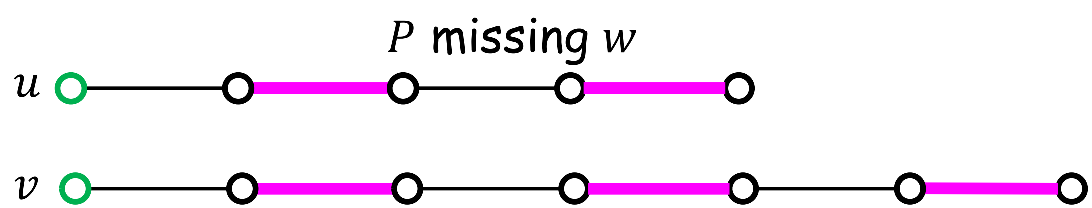
# Factor-critical graphs

**Lem.** $G$ is factor-critical $\Leftrightarrow$ $G$ is connected and all nodes are inessential.

**Pf.** ($\Rightarrow$) simple; ($\Leftarrow$) by contradiction:

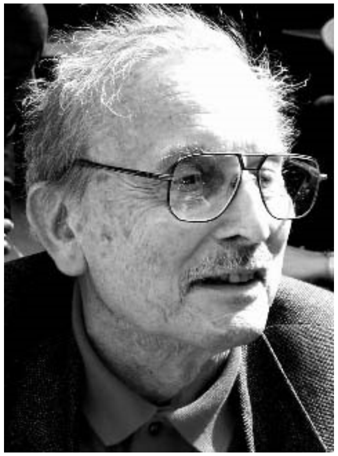
- $\{u, v\}$: a closest pair missed by some max matching $M$ //non-adjacent
- $w$: an internal node in a shortest $u - v$ path //covered by $M$


- $N$: a max. matching missing $w$ //covers $u, v$
- components of $M \oplus N$: even and alternating




- $N \oplus P$ is a max matching missing $\{u, w\}$, contradiction

# 1. Tutte-Berge Set

# Number of nodes missed by a matching

$M$: a matching
$|V| - 2|M|$: # of nodes missed by $M$, has the same parity as $|V|$

Observation. $M$ misses at least one node from each odd component.

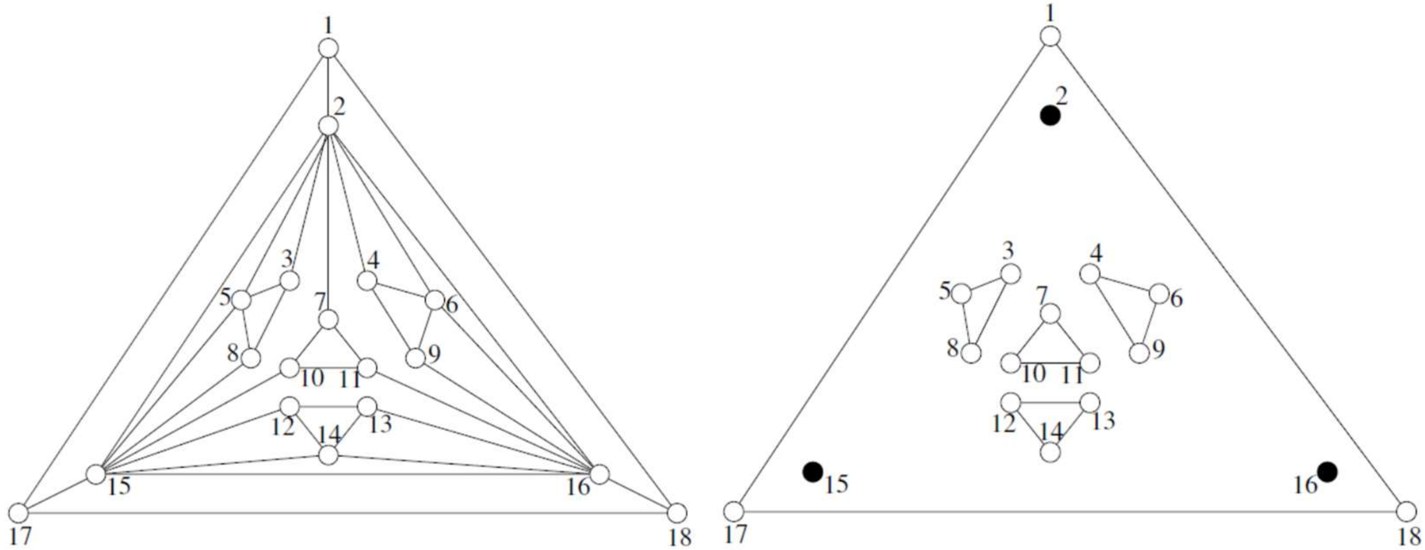
$o(G)$: # of odd components of $G$, has the same parity as $|V|$

Fact. $|V| - 2|M| \geq o(G)$ hence $|V| - 2\alpha'(G) \geq o(G)$

$|V| - 2\alpha'(G) = \min\{|V| - 2|M|: M$ is a matching$\}$:
# of nodes missed by any max matching

# Deficiency of a subset of nodes

Claim. For any $U \subseteq V$, $M$ misses at least $o(G-U) - |U|$ nodes.




Pf. Trivial if $o(G-U) \leq |U|$. Suppose $o(G-U) > |U|$.
Fully covering an odd component of $G-U$ by $M$ requires matching one of its nodes with a node of $U$.
So $\geq o(G-U) - |U|$ odd components in $G-U$ are not fully covered.

Def. $o(G-U) - |U|$ is called the deficiency of $U$.

# Tutte-Berge set

Def. A subset with maximum deficiency is called a Tutte-Berge set.

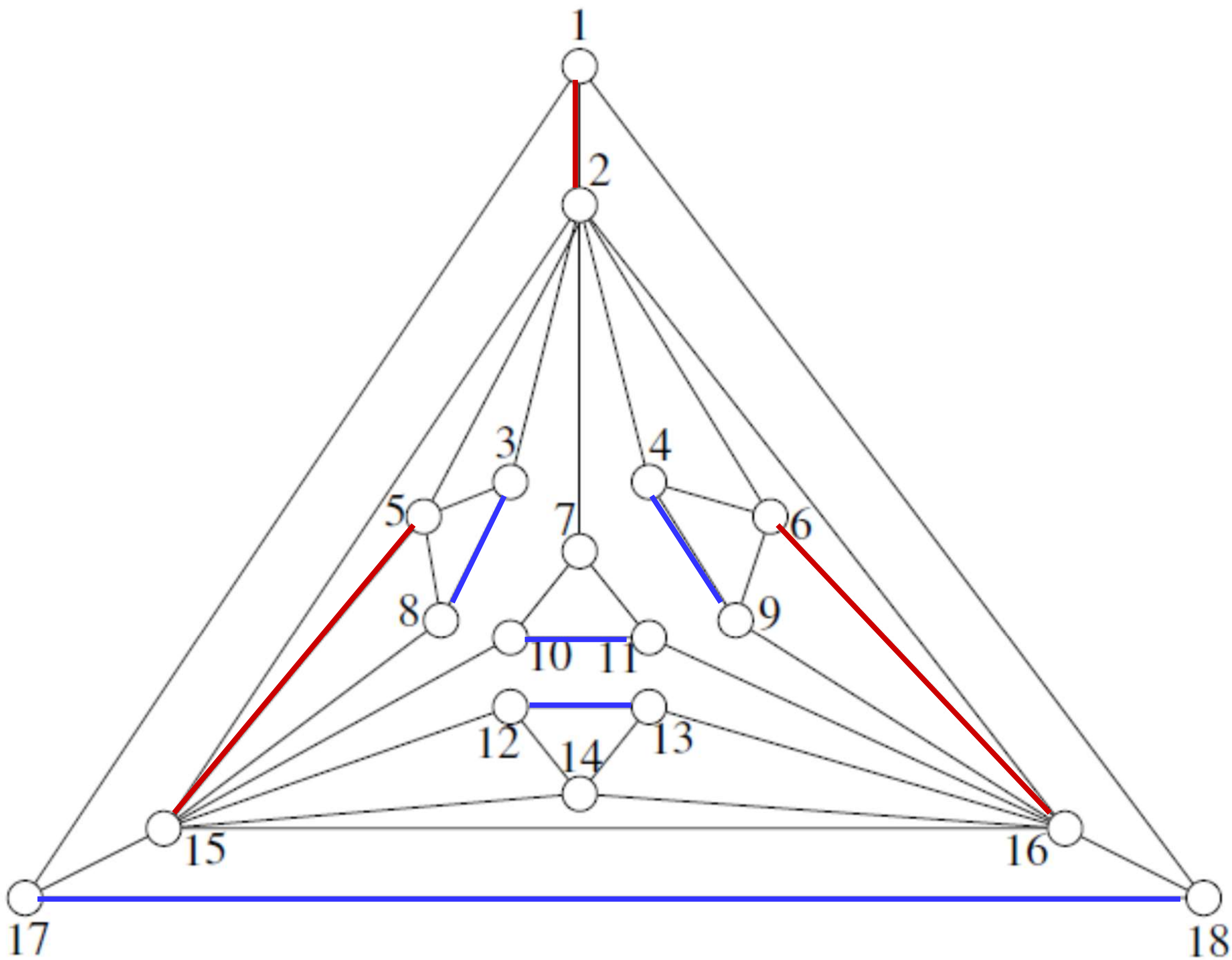
Weak duality. $|V| - 2|M| \geq o(G - U) - |U|$

If the equality holds, $M$ is a max matching and $U$ is a Tutte-Berge set.

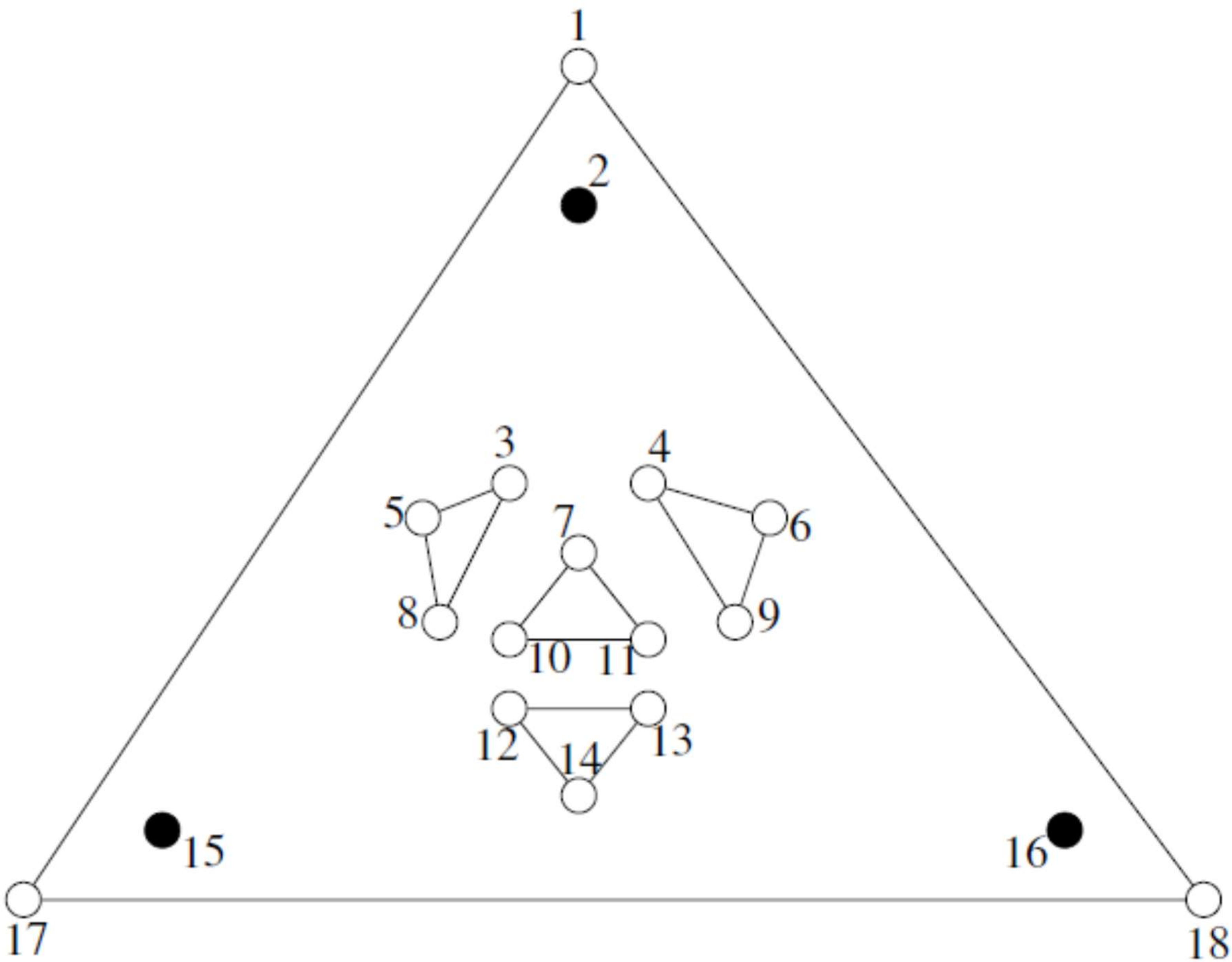
$$|V| - 2\alpha'(G) \geq \text{deficiency of a Tutte-Berge set}$$

Applications: linear-time test, perfectly as an optimality certificate.

# Example

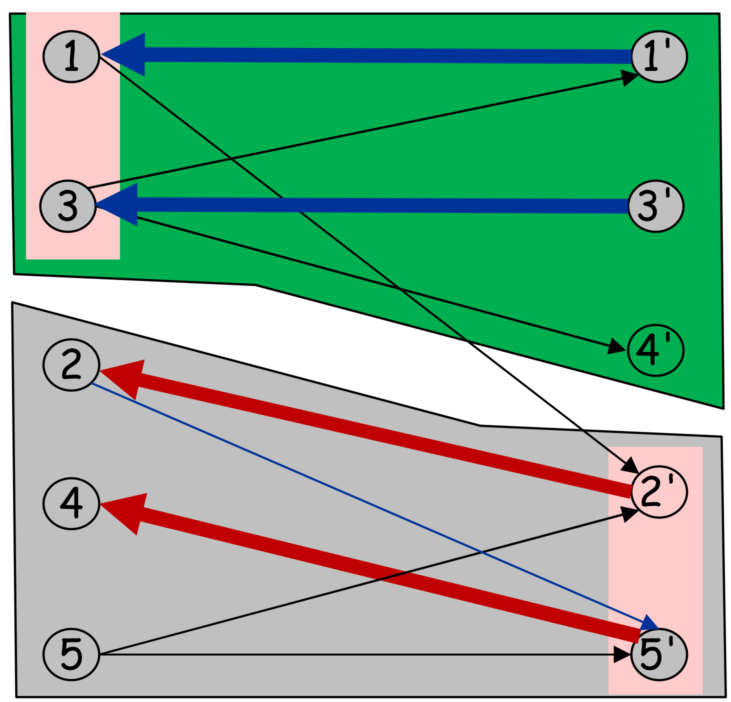


# Example



# Tutte-Berge set in bipartite graph

Fact. Any MVC in a biparite graph is Tutte-Berge set, and strong duality holds.



# Stong duality in general (Tutte-Berge Formular)

[Tutte 1947, Berge 1958].

Thm. $|V| - 2\alpha'(G) =$ deficiency of a Tutte-Berge set

Pf. By induction on $|V|$. Trivial when $|V| = 0,1$. Induction with $|V| > 1$:

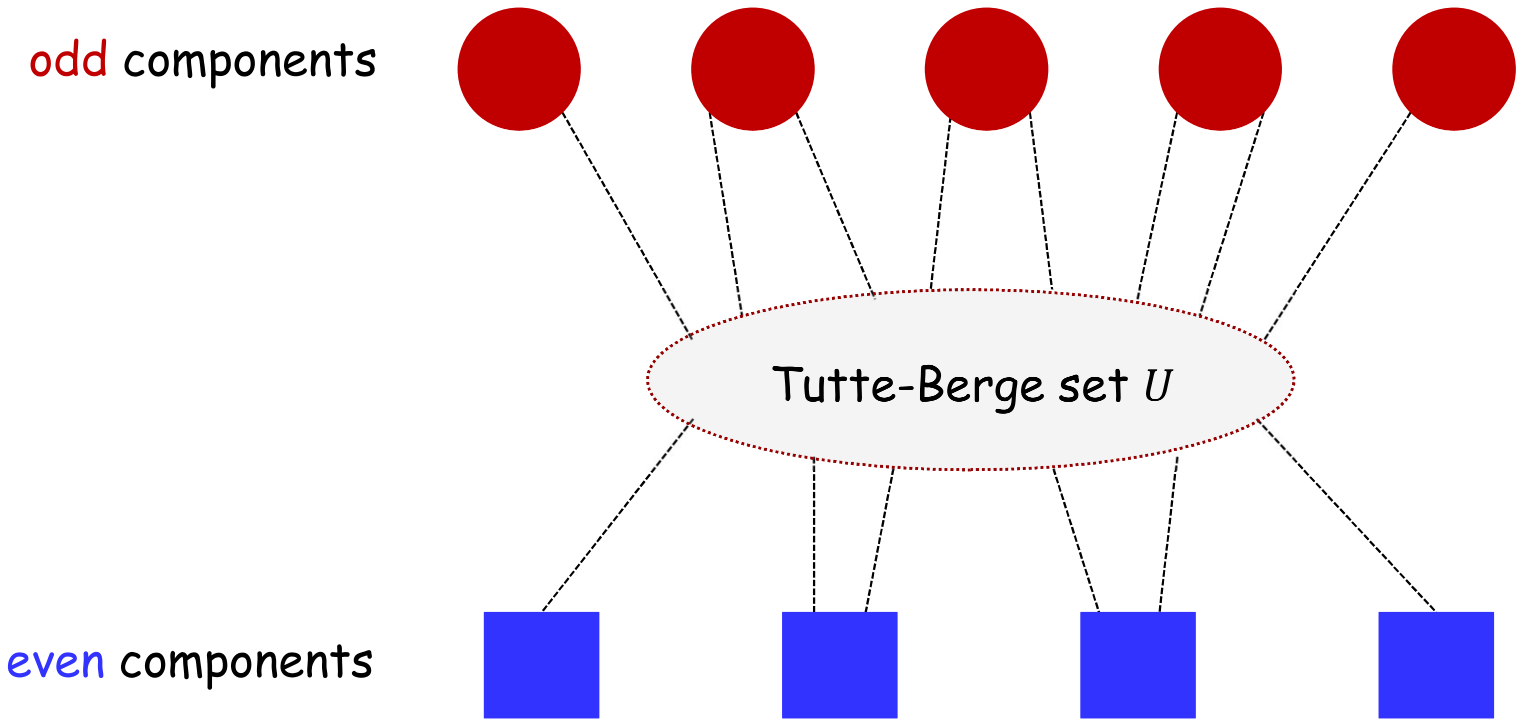
- If $G$ is not connected, apply induction to the components of $G$
- If $G$ is factor-critical, $|V| - 2\alpha'(G) = 1$ and $\emptyset$ is a Tutte-Berge set.

- Otherwise, $G$ has an essential $v$. Apply induction to $G - v$ with a max matching $M$ and a Tutte-Berge set $U'$.

deficiency of $U' \cup \{v\}$ in $G = -1 +$ deficiency of $U'$ in $G - v$

$= -1 + (|V| - 1) - 2|M| = |V| - 2(|M| + 1) = |V| - 2\alpha'(G)$

# Tutte-Berge Decomposition




Every maximum matching $M$

- contains a perfect matching in each even component of $G - U$
- contains a near-perfect matching in each odd component of $G - U$
- matches $U$ with nodes in (distinct, odd) component of $G - U$

All nodes in $U$ and even components of $G - U$ are essential

# Implications

$$\alpha'(G) = \frac{1}{2}\min_{U\subseteq V}\big(|V| + |U| - o(G-U)\big).$$

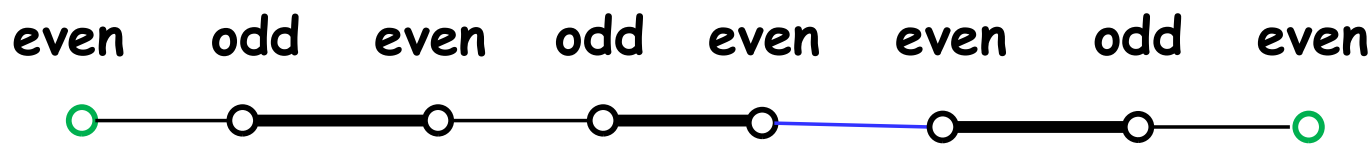
Tutte's perfect matching theorem: $G$ has a perfect matching $\Leftrightarrow$ $o(G-U) \le |U|$ for all $U \subseteq V$.

Edge-cover theorem: For any graph $G = (V,E)$ without isolated vertices,

$$\tau'(G) = \frac{1}{2}\max_{U\subseteq V}\big(|U| + o(G[U])\big).$$

# 2. Alternating Forest

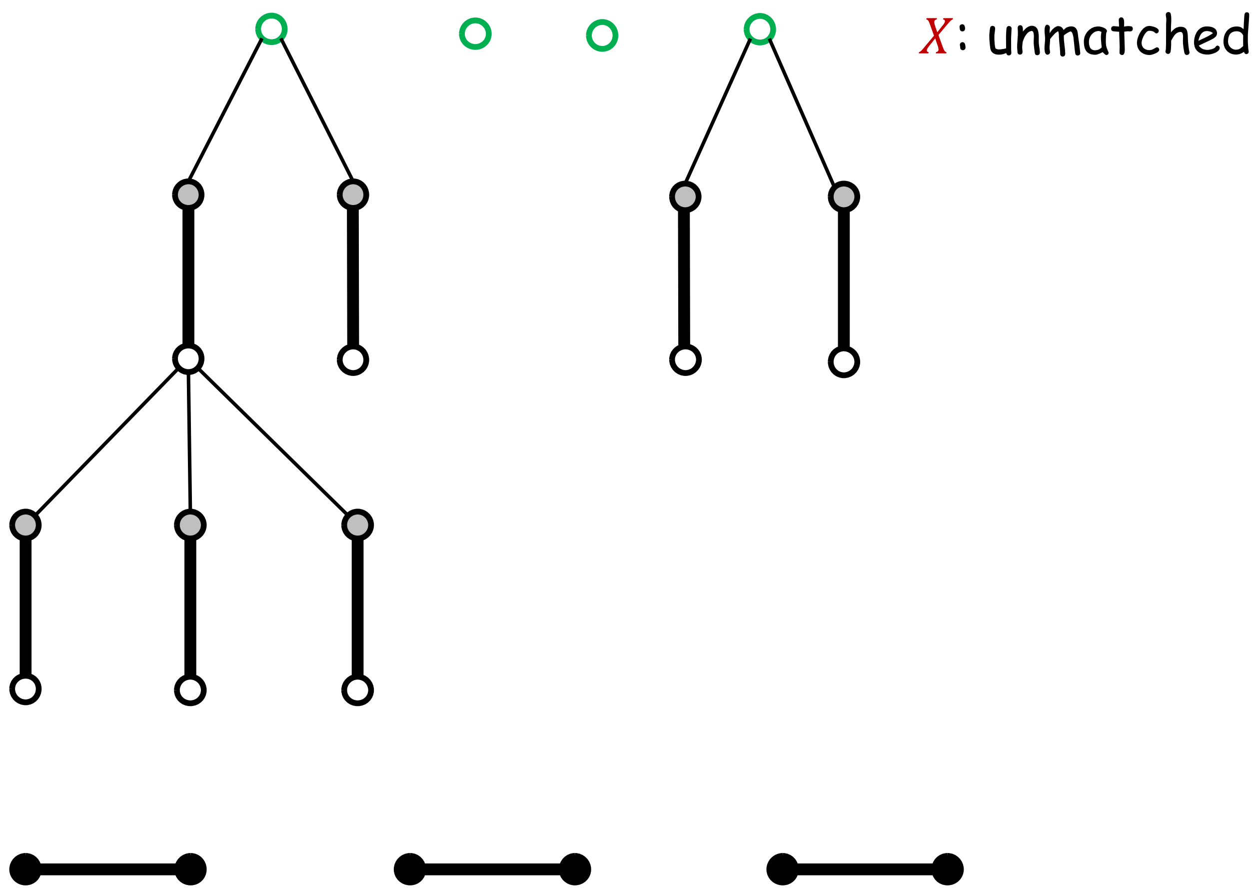
# Search from both (multiple) unmatched nodes




- disjoint even alternating paths fanning out from unmatched nodes
- joined by an unmatched edge (successfully!)
- **parity**:
  - successful joining edge is between two **even** nodes

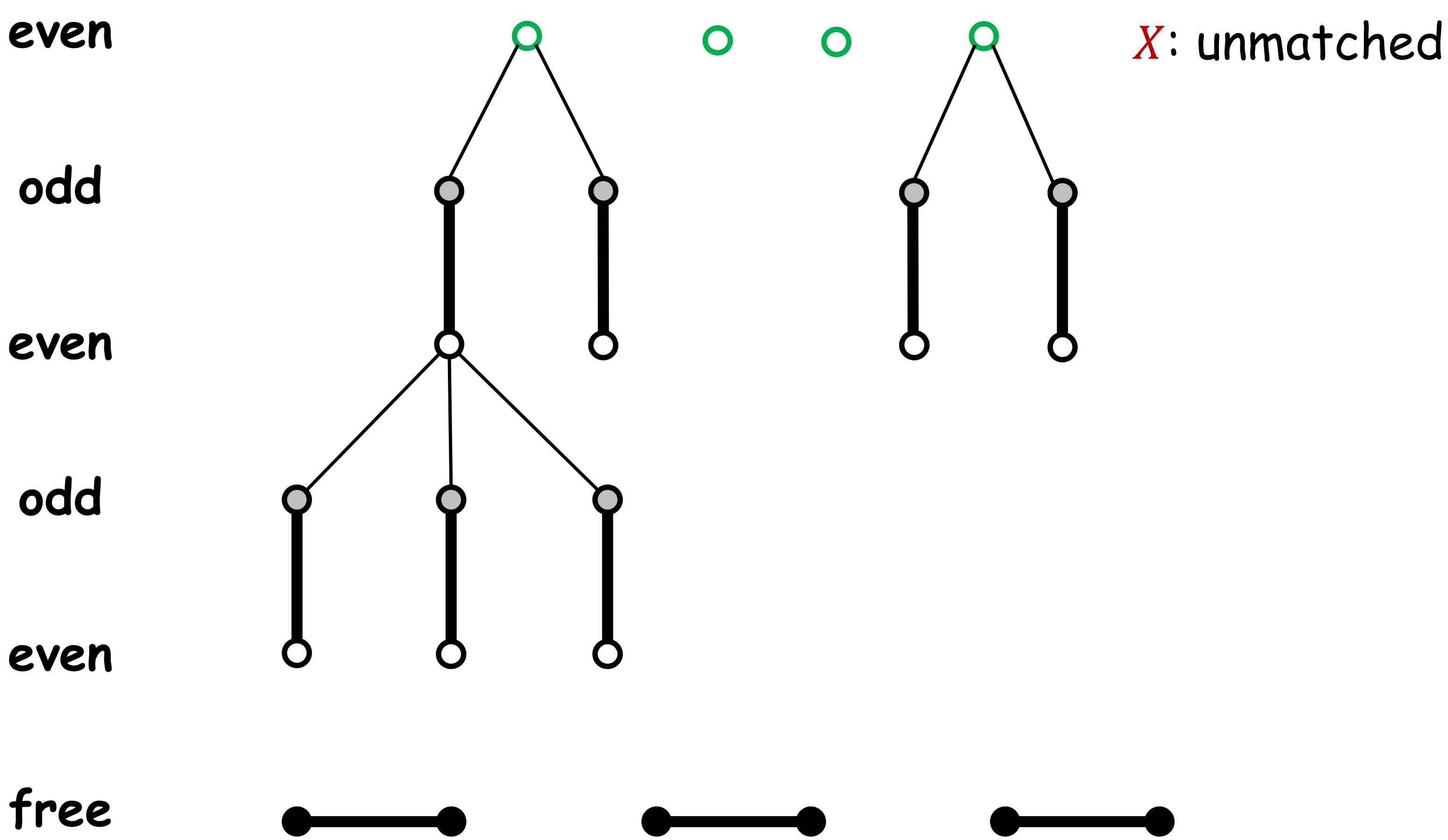
# $M$-alternating forest

- a spanning forest $(V, F)$ in $G$ with $M \subseteq F$
- each tree either has exactly one unmatched node (root of the tree), or is a single edge in $M$
- all root-to-leaf paths are $M$-alternating and even



# Parity of nodes

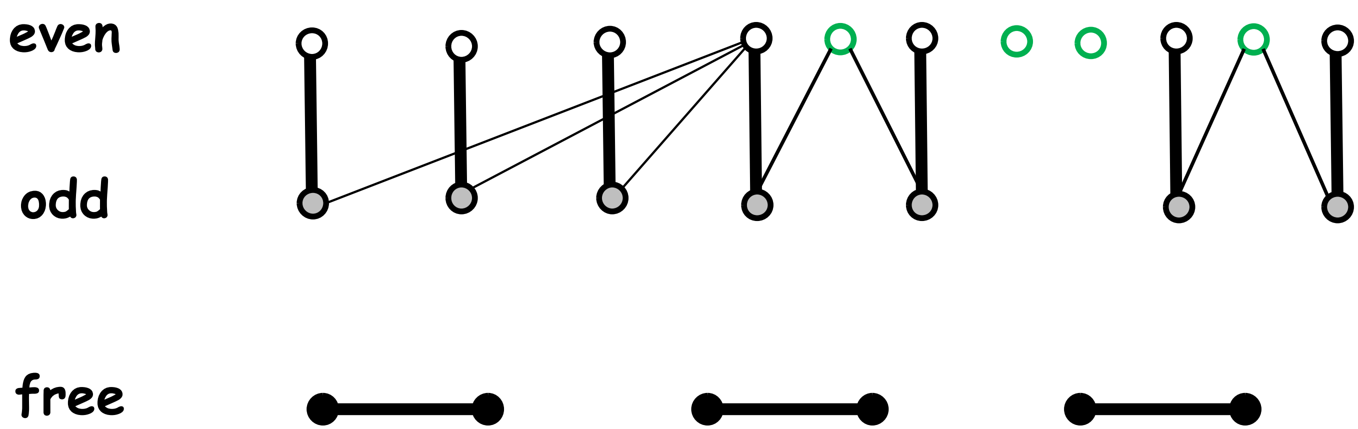
$even(F)$ := $\{v \in V \mid F$ contains an even-length $X - v$ path$\}$
$odd(F)$ := $\{v \in V \mid F$ contains an odd-length $X - v$ path$\}$
$free(F)$ := $\{v \in V \mid F$ contains no $X - v$ path$\}$.



# Simple test for maximality

- no edge between $even(F)$ and $even(F) \cup free(F)$
- Claim. $M$ is a maximum matching and $odd(F)$ is a Tutte-Berge set.

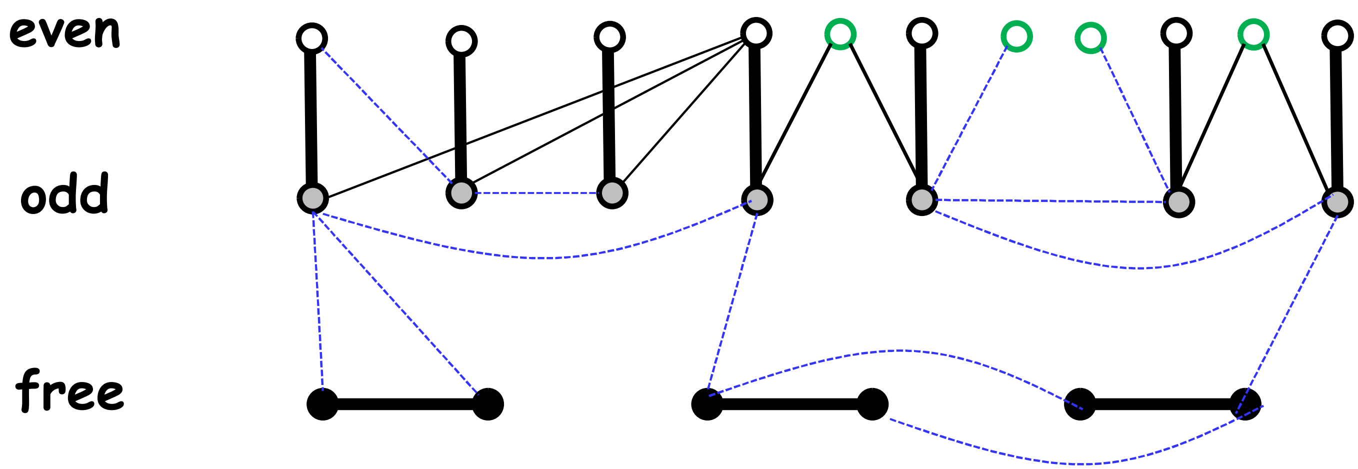
re-layout of the $M$-alternating forest



# A certificate for optimality

- no edge between $even(F)$ and $even(F) \cup free(F)$
- Claim. $M$ is a maximum matching and $odd(F)$ is a Tutte-Berge set.

the entire graph $G$

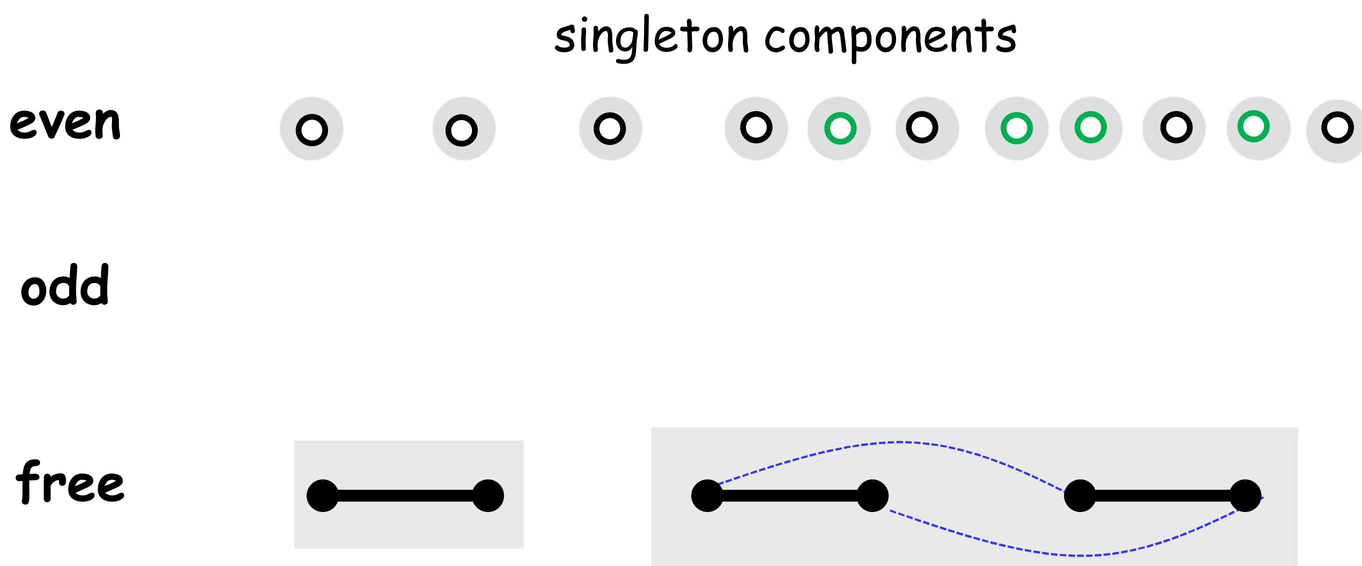


# A certificate for optimality

- no edge between $even(F)$ and $even(F) \cup free(F)$
- Claim. $M$ is a maximum matching and $odd(F)$ is a Tutte-Berge set.

component structure of $G - odd(F)$

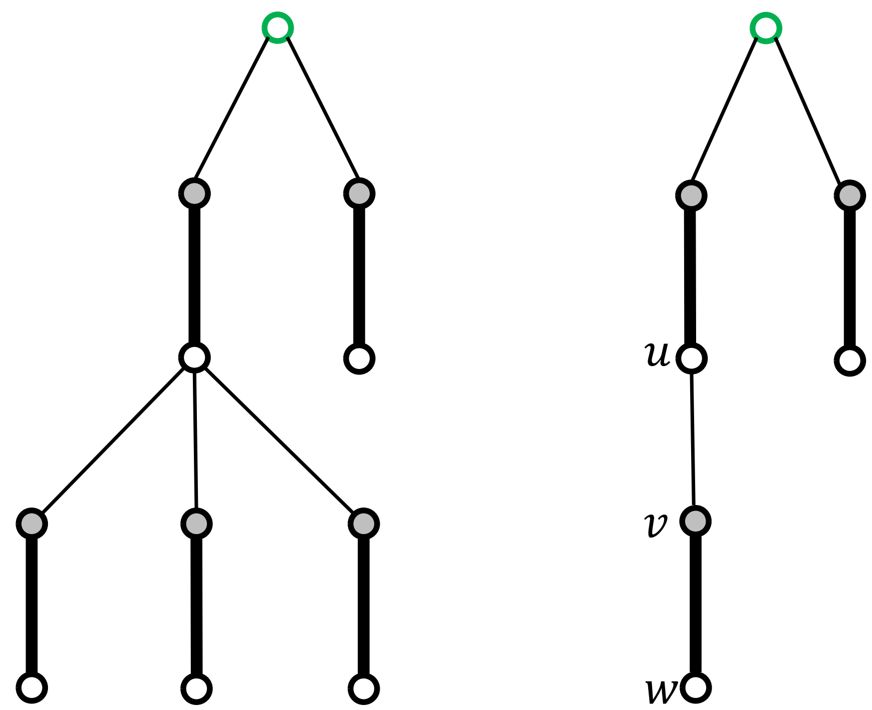
deficiency of $odd(F)$: $|even(F)| - |odd(F)| = |X|$




each component of free nodes has a perfect matching and has even size.

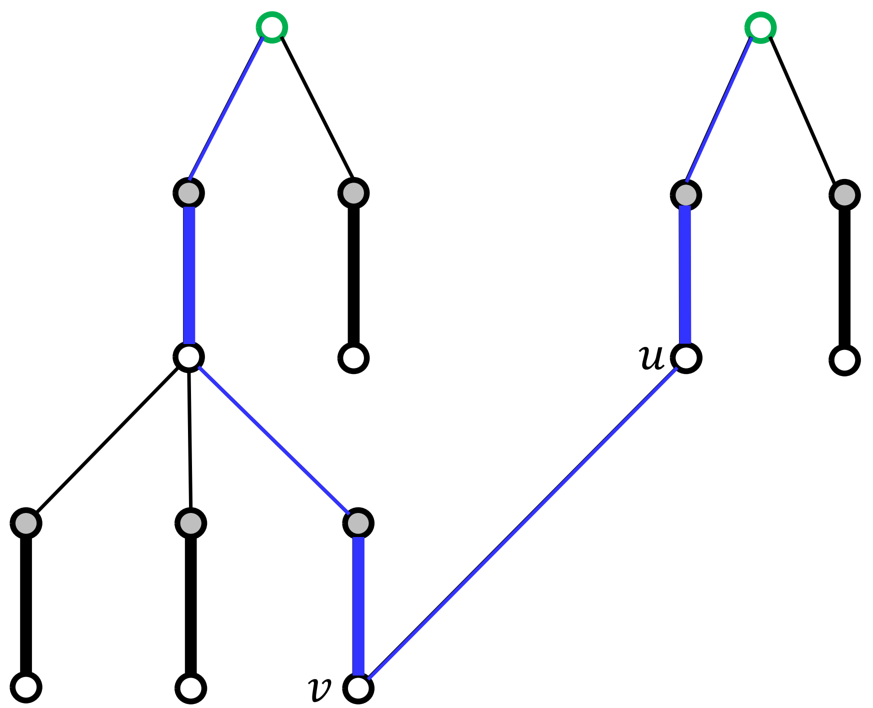
# Growth of the forest

- $e = (u, v)$ with $u \in even(F)$ and $v \in free(F)$
- $F \leftarrow F \cup \{e\}$; afterwards, $v \in odd(F)$, its mate $w \in even(F)$



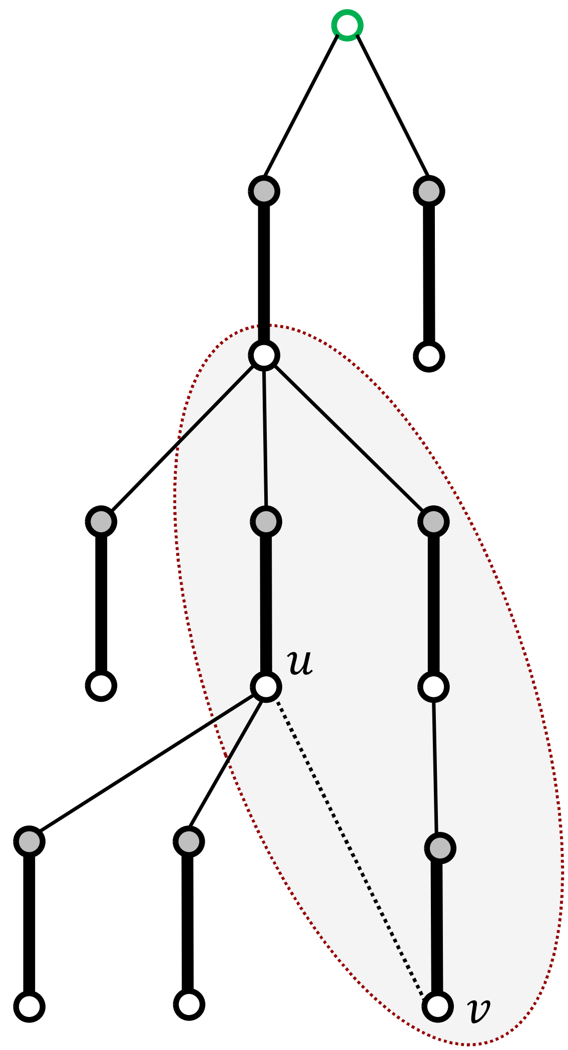
# Composition of an augmenting path

- $e = (u, v)$ with $u, v \in even(F)$ and across two tree components
- an augmenting path through $e$



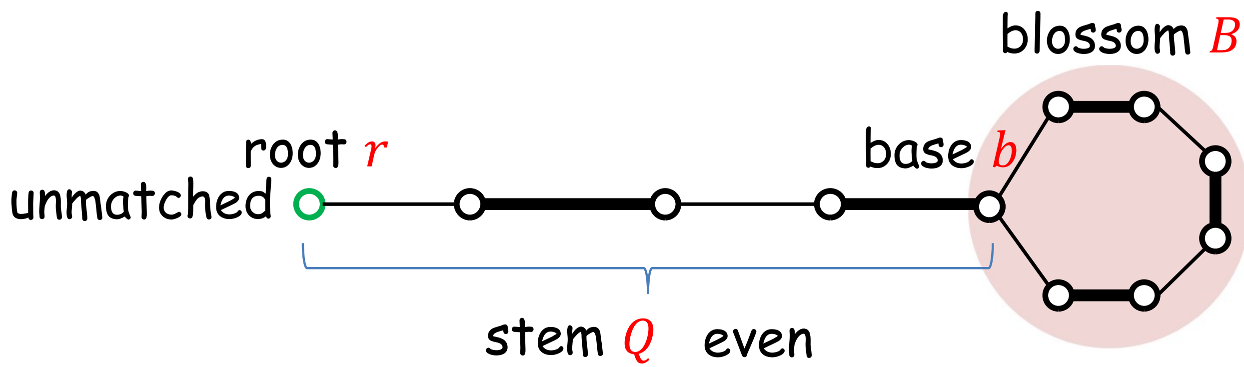
# The remaining challenging case

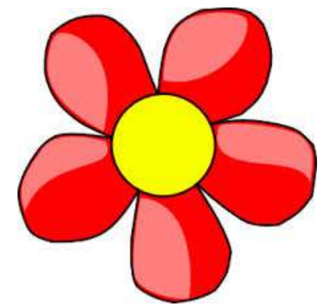
- $e = (u, v)$ with $u, v \in even(F)$ but in the same tree component



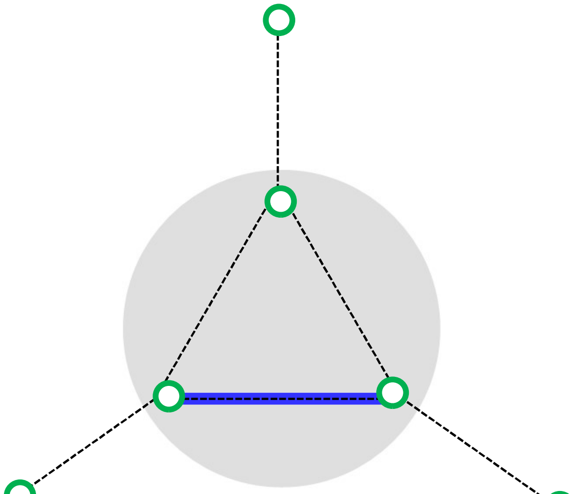
# 3. Blossom Shrinking

# Blossom, flower

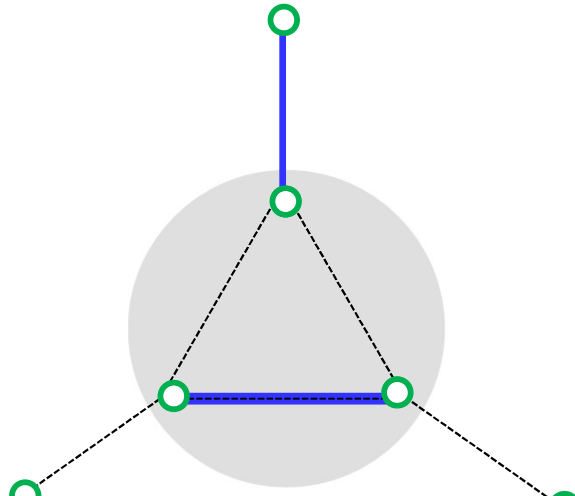


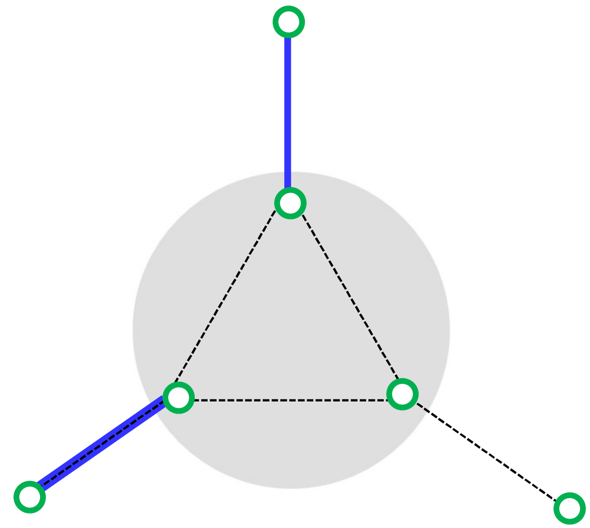
# Which is a blossom?



Yes

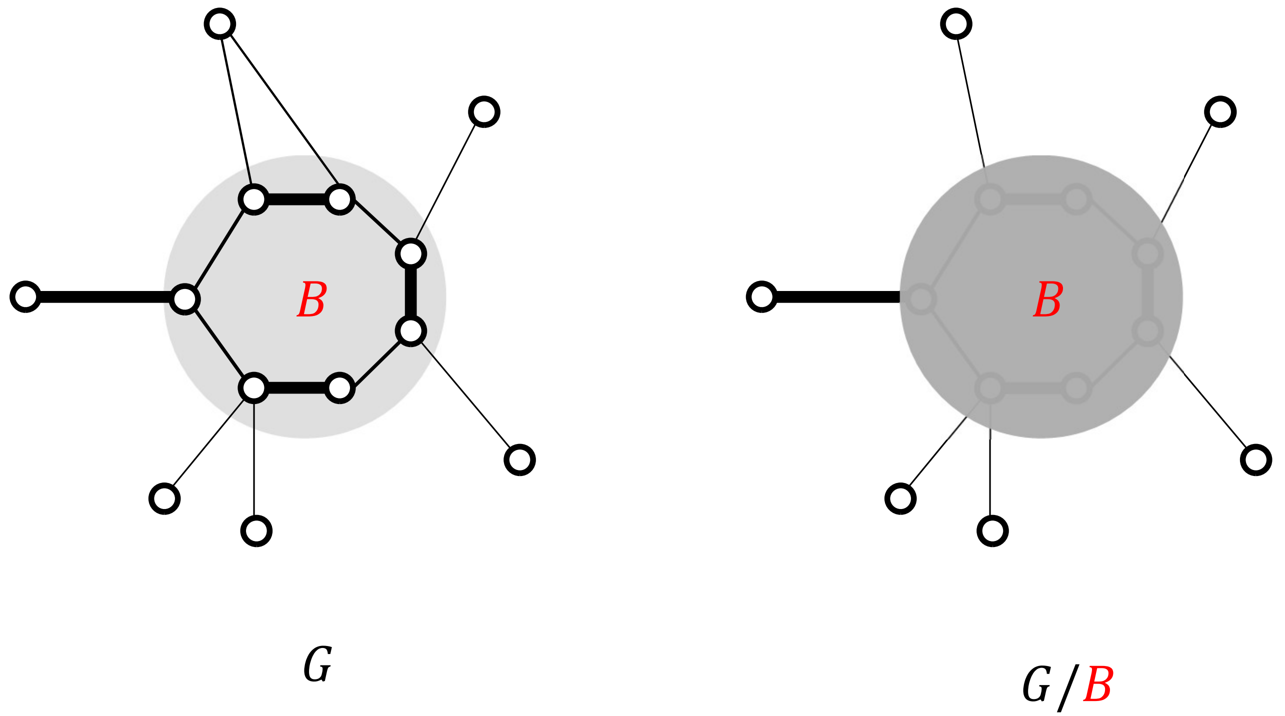


No



No

# Blossom shrinking




$$|V/B| = |V| - (|B| - 1)$$
$$|M/B| = |M| - (|B| - 1)/2$$
$$|V/B| - 2|M/B| = |V| - 2|M|$$

# Growth/Optimality preserving

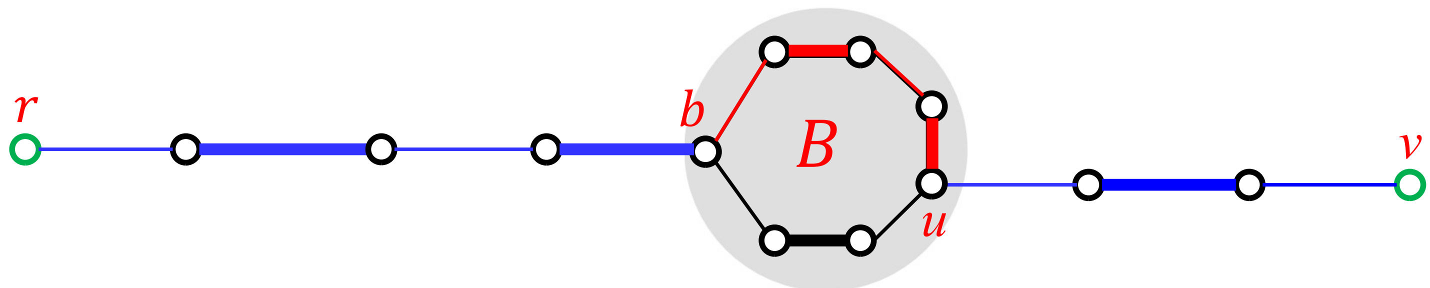
Thm: $G$ has an $M$-augmenting path $\Leftrightarrow$ $G/B$ has an $M/B$-augmenting path.

Thm: $M$ is a max. matching in $G$ $\Leftrightarrow$ $M/B$ is a max. matching in $G/B$.

# (⇐) Parity-preserving lift of a path

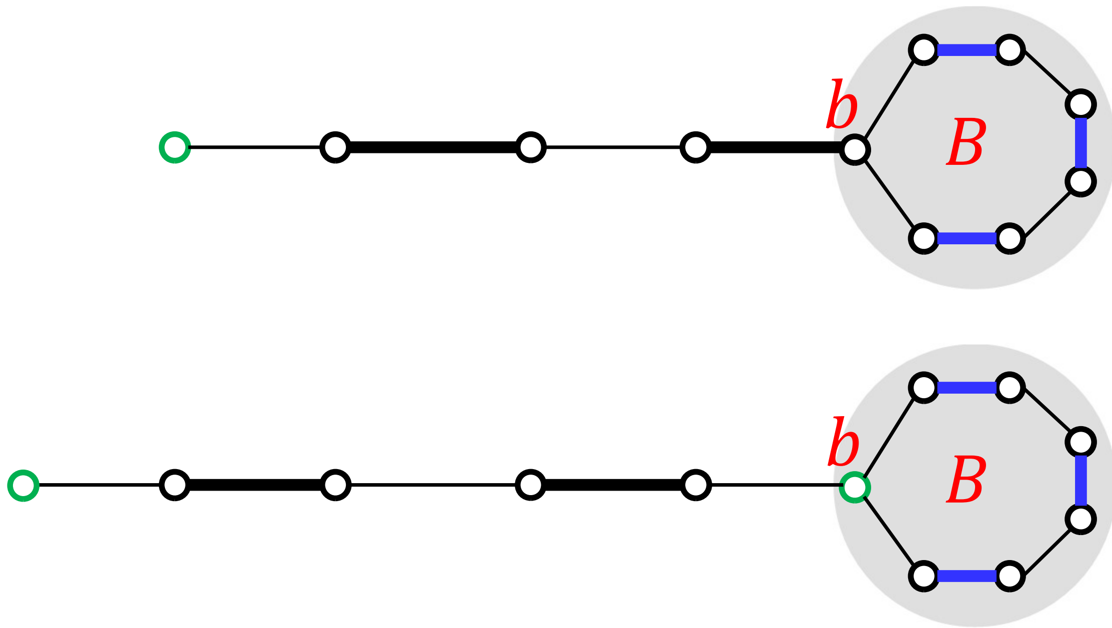
Parity-preserving lift of a path $P$ in $G/B$ to a path $P^+$ in $G$:

- if $P$ does not go through $B$, then $P^+ \coloneqq P$;
- otherwise, $P^+ \coloneqq P \cup$ the even $u - b$ path within the blossom




If $P$ is $M/B$-augmenting in $G/B$, then $P^+$ is $M$-augmenting in $G$

# (⇐) Lift of a matching

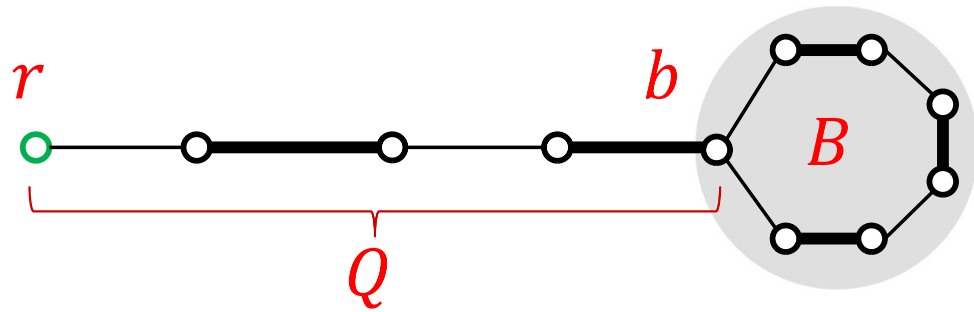


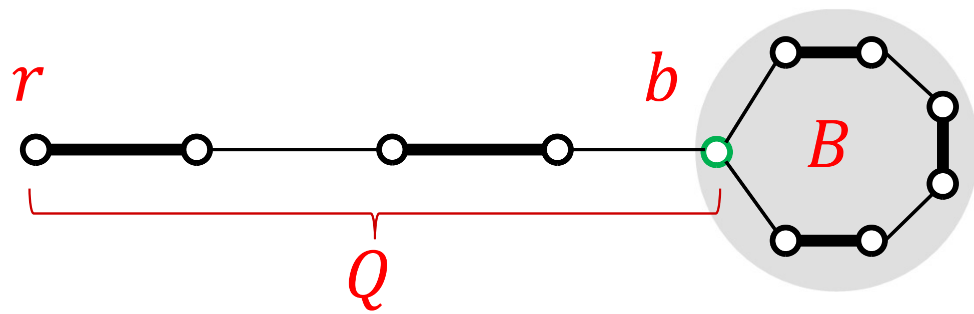

Lift a matching $N$ in $G/B$ to $N^+$ in $G$: add $(|B|-1)/2$ edges in $B$

- $|N^+| - |N| = (|B|-1)/2$
- if $|N| > |M/B|$, then $|N^+| > |M/B| + (|B|-1)/2 = |M|$

Question. If $N$ is a max. matching $G/B$, is $N^+$ also a max. matching in $G$?

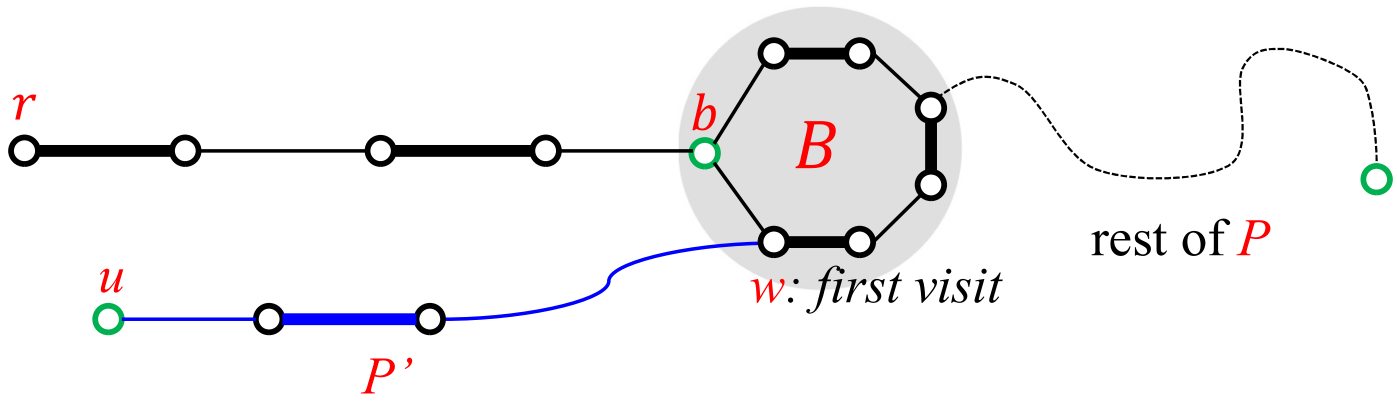
# (⇒) Reduction to blossom without stem




$$|M| = |M \oplus Q|$$
$$|M/B| = |(M/B) \oplus Q| = |(M \oplus Q)/B|$$

# (⇒) Augmenting path for contracted matching




∃ an $M$-augmenting path in $G$

⇒ ∃ an $M \oplus Q$-augmenting path $P$ in $G$

⇒ ∃ an $(M \oplus Q)/B$-augmenting path $P'$ in $G/B$

⇒ ∃ an $M/B$-augmenting path in $G/B$

# 4. Blossom Algorithm for Augmenting-Path

# Contract & Conquer

Initialization: $F \leftarrow M$;

while there exists an edge between $even(F)$ and $even(F) \cup free(F)$

find such edge $e = (u, v)$ with $u \in even(F)$;

Case 1: $v \in free(F)$

$\Rightarrow$ Extend the forest;

Case 2: $v \in even(F)$ and is in the same tree as $u$

$\Rightarrow$ Shrink a blossom;

Case 3: $v \in even(F)$ and is in another tree than $u$

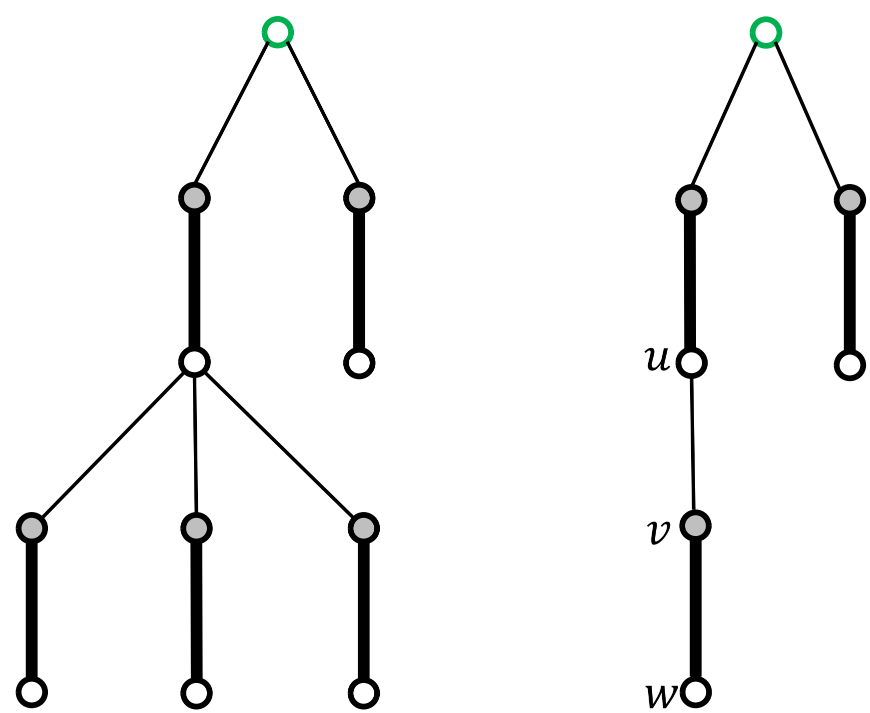
$\Rightarrow$ Lift an augmenting path iteratively, and return this path;

return null. // $M$ is maximum matching

# Case 1: extending forest

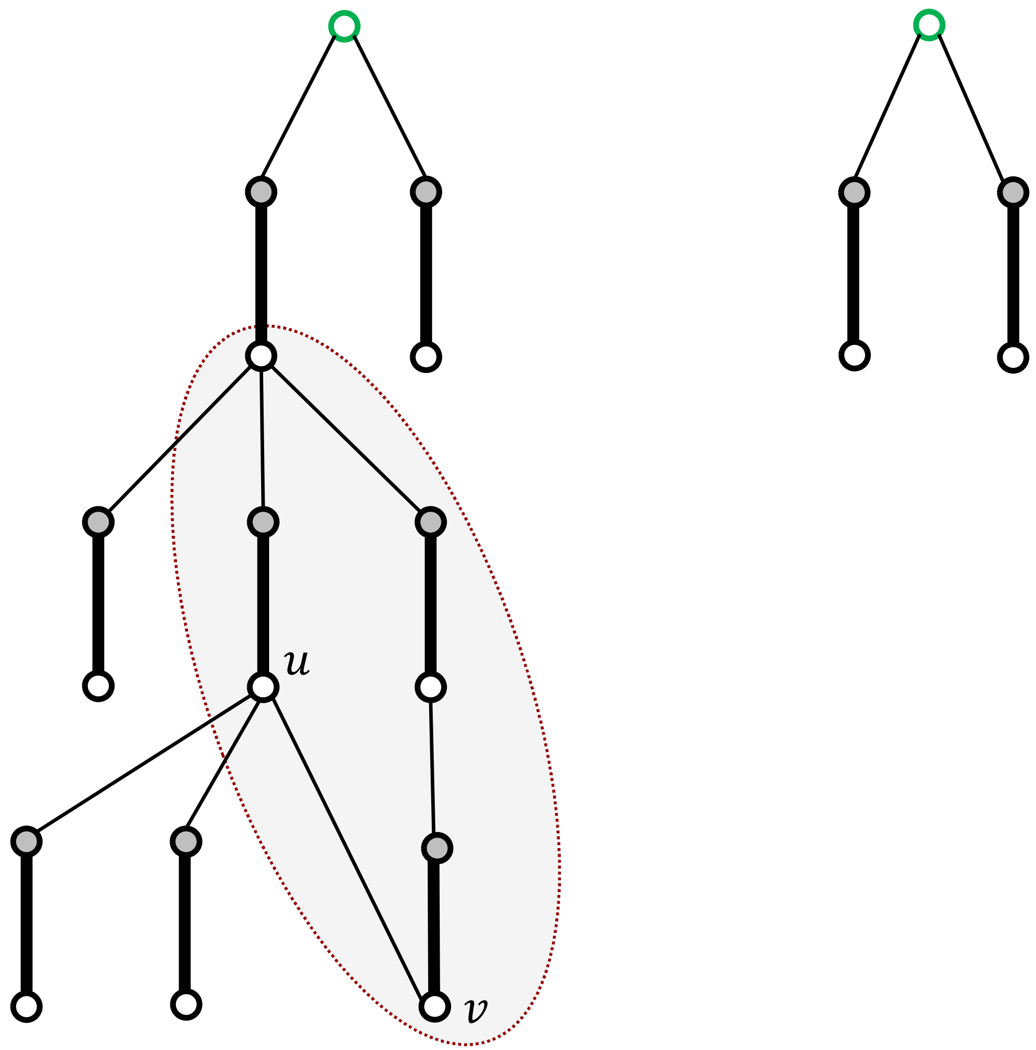
$F \leftarrow F \cup \{e\}$;

[ afterwards, $v \in odd(F)$, its mate $w \in even(F)$ ]



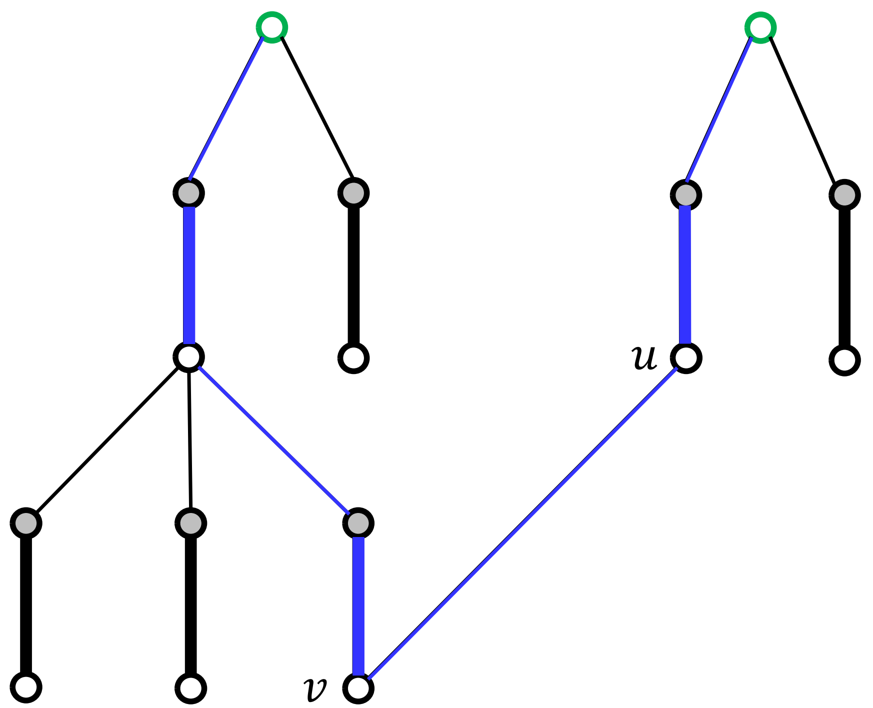
# Case 2: shrinking blossom

- A blossom $B$ is found.
- Shrink the blossom $B$ in $G$ and the forest [afterwards, $B \in even(F)$]

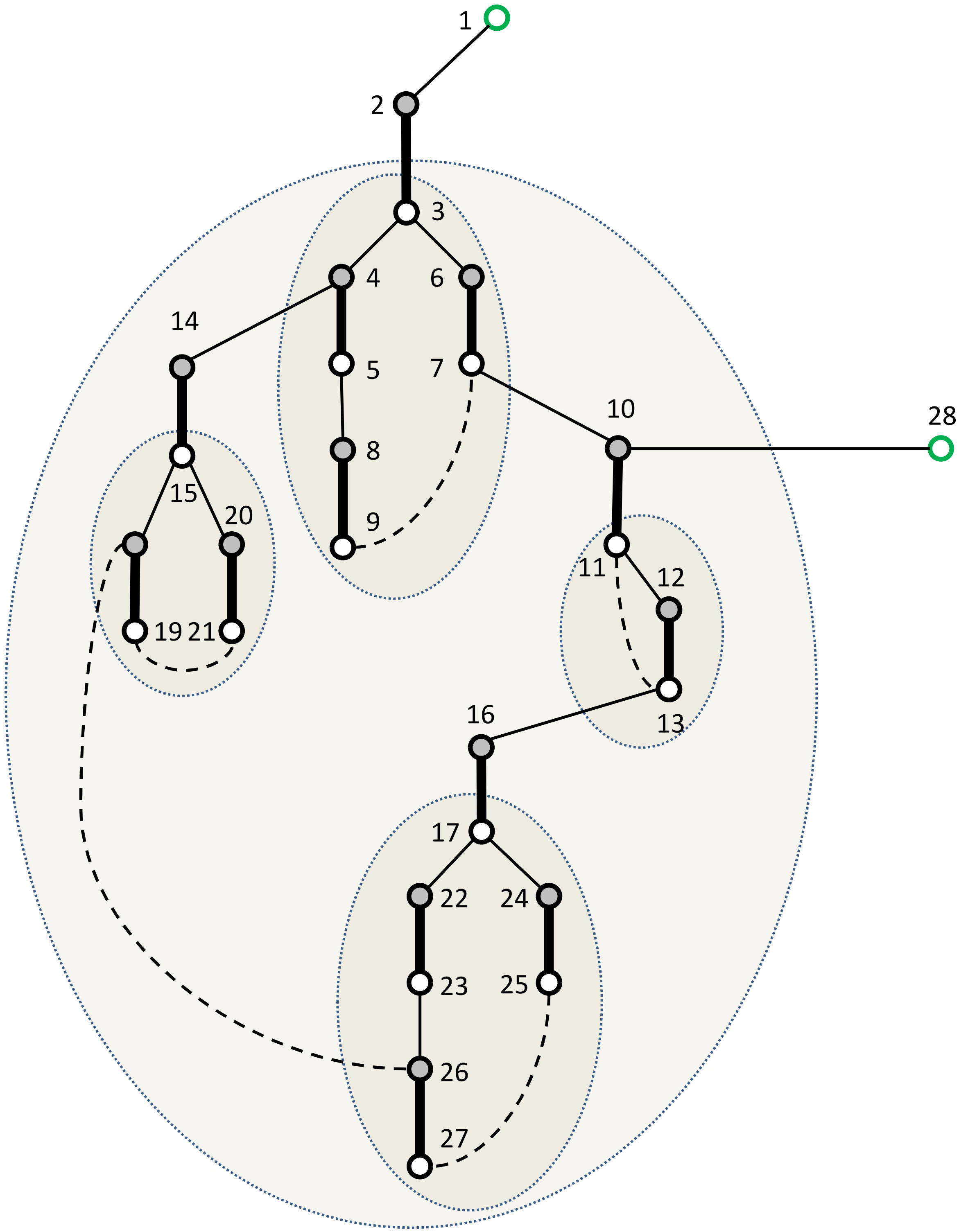


# Case 3: lifting an augmenting path

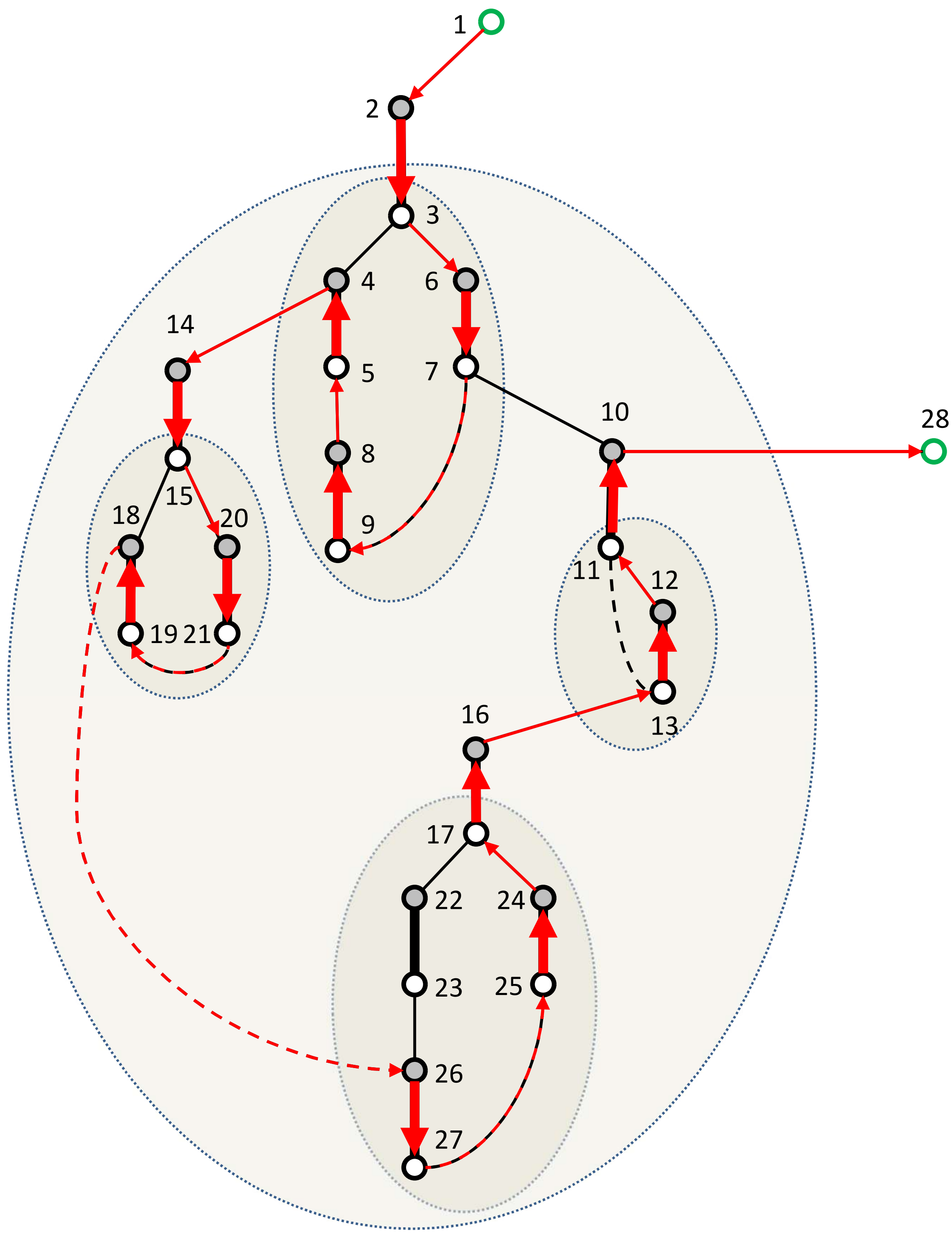
- An augmenting path $P$ is found
- Lift $P$ by expanding all blossoms in reverse order of shrinking
- Return the lifted path $P$



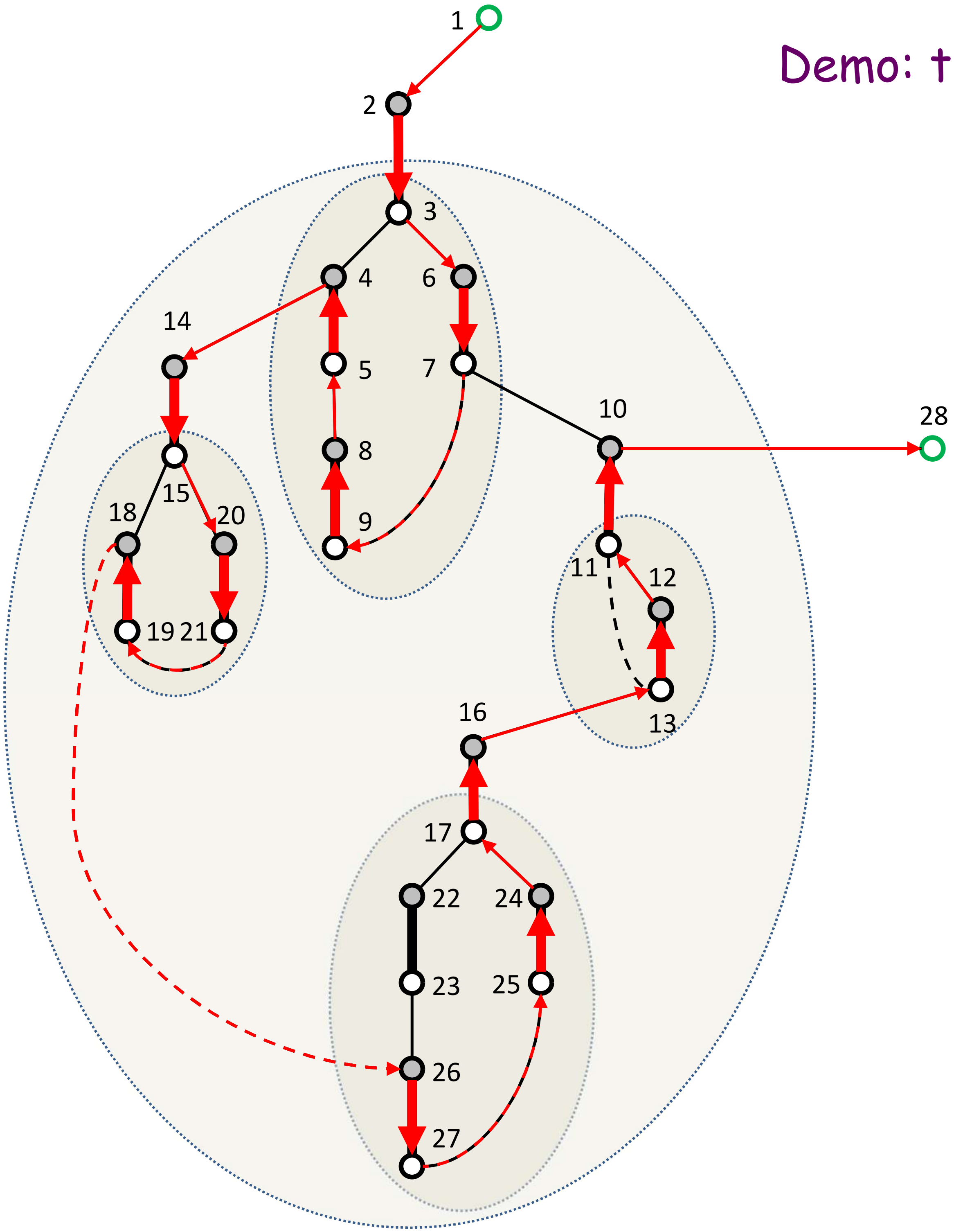
Demo: evolution of alternating forest



# Demo: lift of augmenting path



# Demo: the augmenting path



# Invariant properties

- Odd and free vertices are original nodes.
- Even vertices are produced by a sequence of blossoms, and each contain an odd number of original nodes.

# Number of iterations

**Thm.** The number of iterations $\leq |V|/2 + |M| \leq |V|$

**Pf.** Evolution of the "potential" $|V| + |free(F)|$:

- Initially, $|V| + 2|M|$
- In both Case 1 and Case 2, decreases by at least 2
  - Case 1 (forest extension): $|V|$ same, $|free(F)|$ drops by 2
  - Case 2 (blossom shrink): $|V|$ drops by $|B| - 1 \geq 2$, $|free(F)|$ same

# Running time

- For each $v \in even(F) \cup free(F)$, maintain a neighbor $u \in even(F)$ if there is any (for testing the while-condition)

- $O(n^2)$-time in total:
  - Case 1 (forest extension): $O(n)$ per iteration, total $O(n^2)$
  - Case 2 (blossom shrinking): $O(|B|n)$ per iteration, total $O(n^2)$
    - sum of $|B| \leq$ twice the decrease in # of vertices
  - Case 3 (path lift): $O(n)$

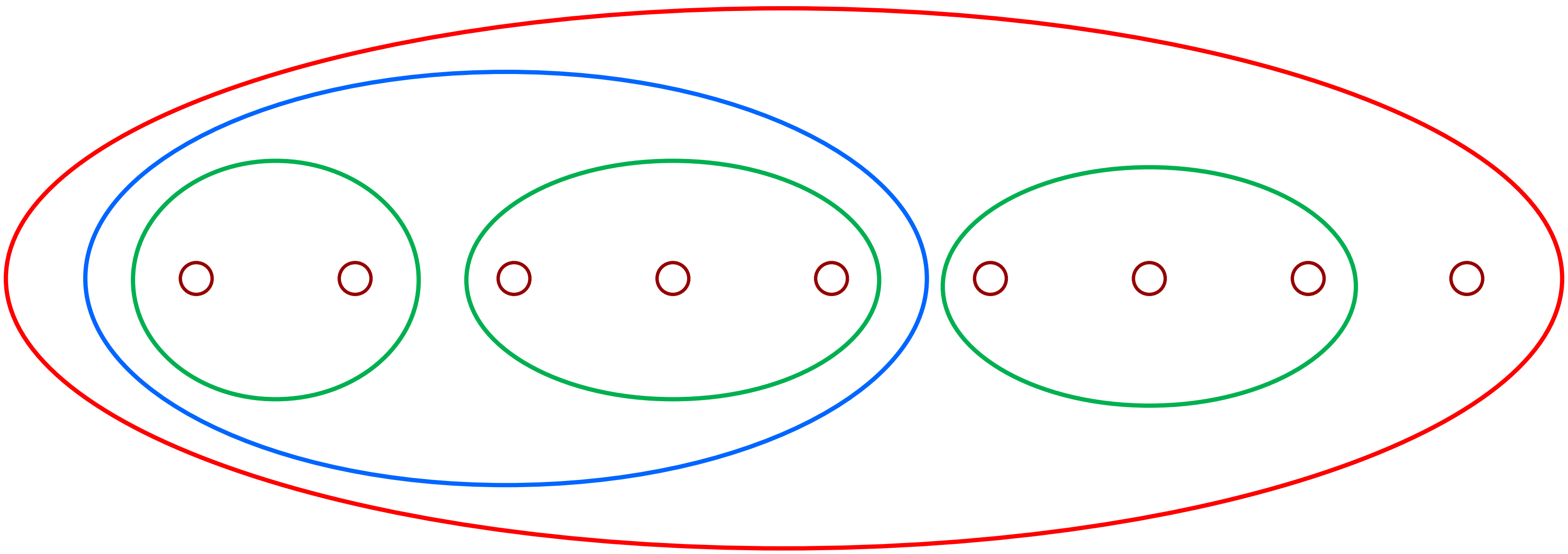
Implication: $O(n^3)$-time algorithm for max matching

- Speed-up with special data structure: $O(mn)$ time
- Fastest-known (Micali and Vazirani 1980): $O(mn^{1/2})$ time

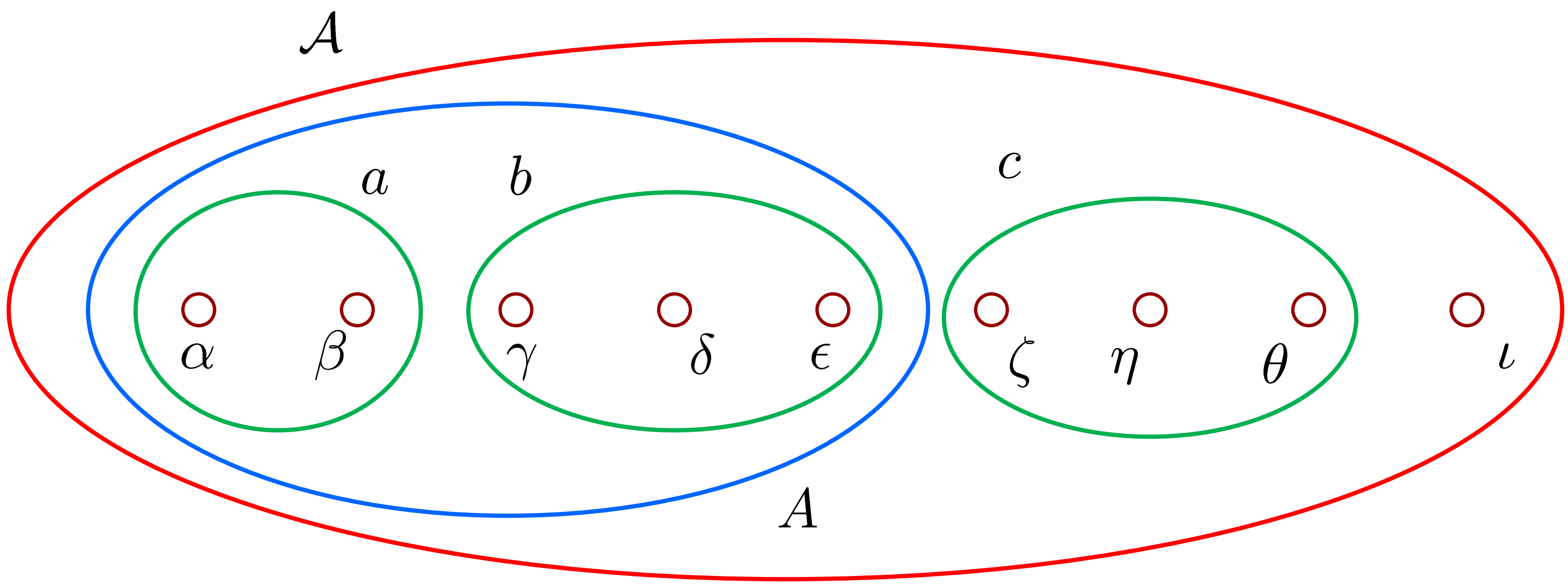
# Nested (Laminar, Tree) family

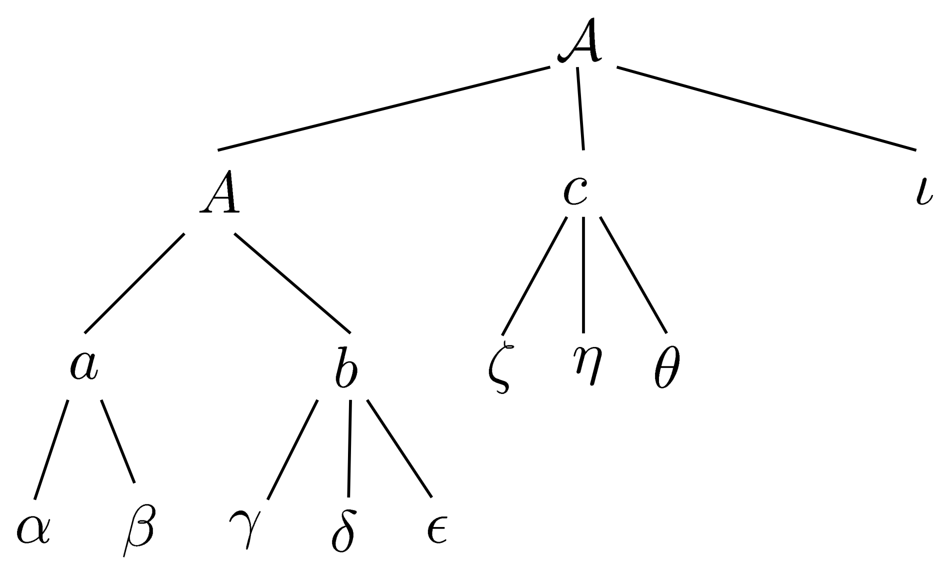
$\Omega$: a collection of subsets of a set $V$

Def. $\Omega$ is called nested if for all $S, T \in \Omega$: $S \subseteq T$ or $T \subseteq S$ or $S \cap T = \emptyset$.



# Tree representation of nested family



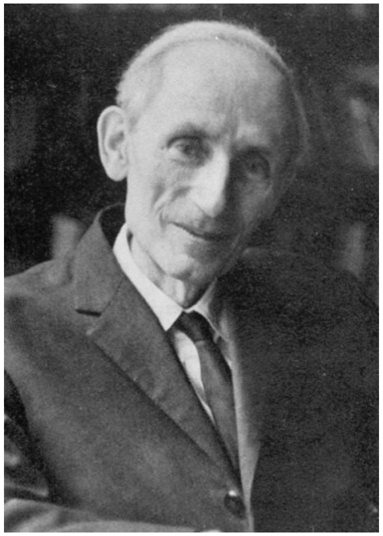



# Nested family of blossoms

size of a blossom = number of children

total size of blossoms
= total number of edges
< total number of vertices
< twice total number of leaves (nodes)

# 5. Edmonds-Gallai Decomposition



# Node classification from a maximum matching

- $M$: a maximum matching
- $F$: the final $M$-alternating forest output by the Blossom Algorithm
- Classification of $V$: $S = odd(F)$, $T = free(F)$, $R =$ the rest

Thm. $S$ is a Tutte-Berge set.

- $B_1, B_2, \ldots, B_k$: sequence of contracted blossoms
- $G_i = (V_i, E_i)$: the graph obtained after contracting $B_1, B_2, \ldots, B_i$
- $M_i$: the matching obtained after contracting $B_1, B_2, \ldots, B_i$
- $G_0 = (V_0, E_0) = (V, E), M_0 = M$
- $o(G_k - S) - |S| = |V_k| - 2|M_k|$

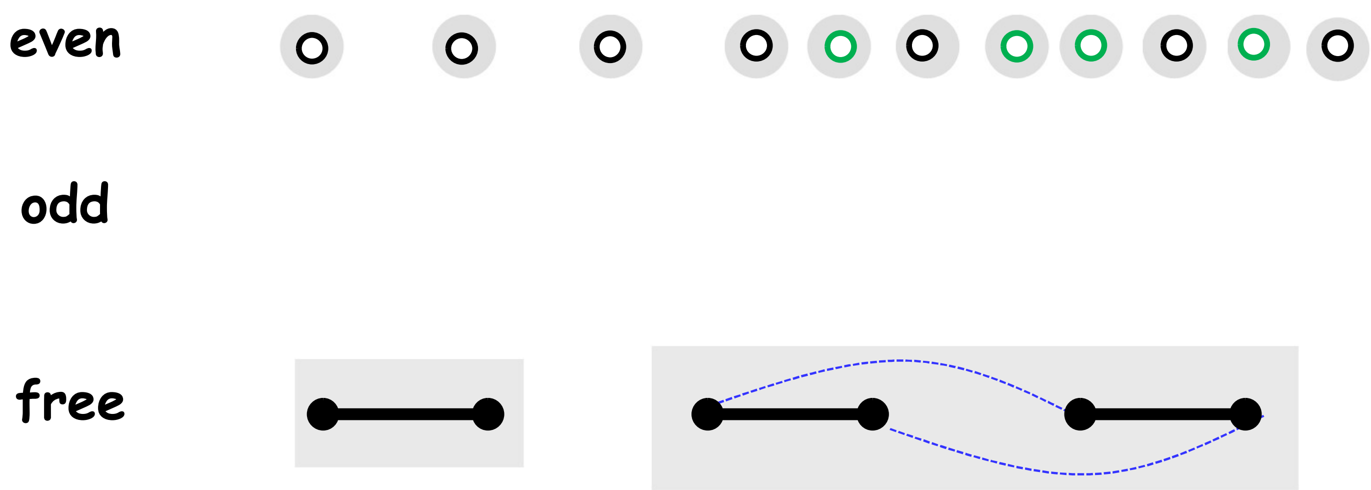
Claim. $o(G_i - S)$ and $|V_i| - 2|M_i|$ are invariant with $i$.

Pf. Unshrinking $B_i$ increases the size of the component containing $B_i$ by an even number $|B_i| - 1$. The parity of all components are preserved.

# Tutte-Berge decomposition from $S$

Component structure of $G - S$ from $G_k - S$:

Each even vertex is expanded to an odd component




Each component of free nodes has a perfect matching

# Invariance of node classification

Thm. $R$ is exactly the set of inessential nodes. In addition, each (odd) component $Q$ of $G[R]$ is factor-critical.

Pf. Each $v \notin R$ is essential due to the Tutte-Berge decomposition.
For each $v \in R$, there is an even $M$-alternating path in $G$ from $X$ to $v$:

- $P$: the path in $F$ from $X$ to either $v$ or the blossom containing $v$,
- Lift $P$ by unshrinking the blossoms in the reverse order

Each $v \in Q$ is inessential and missed by a max matching, which must contain a perfect matching of $Q - v$.

$S = N(R)$: Edmonds-Gallai set
$T = V \backslash N[R]$