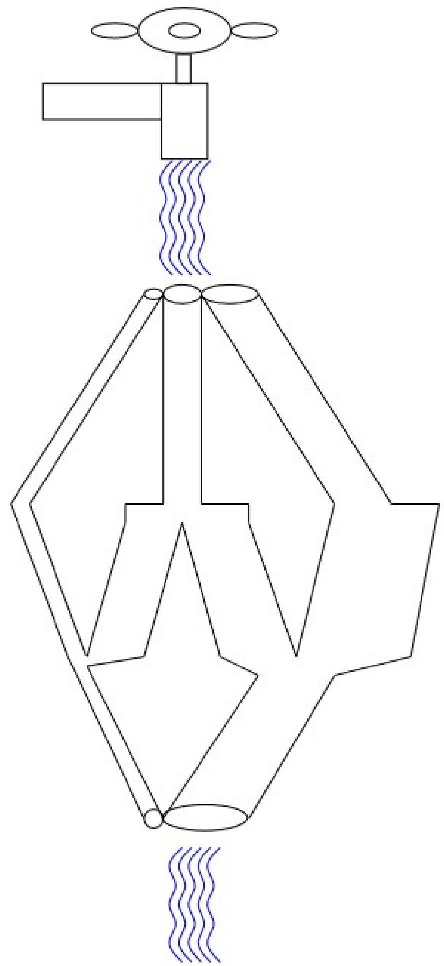
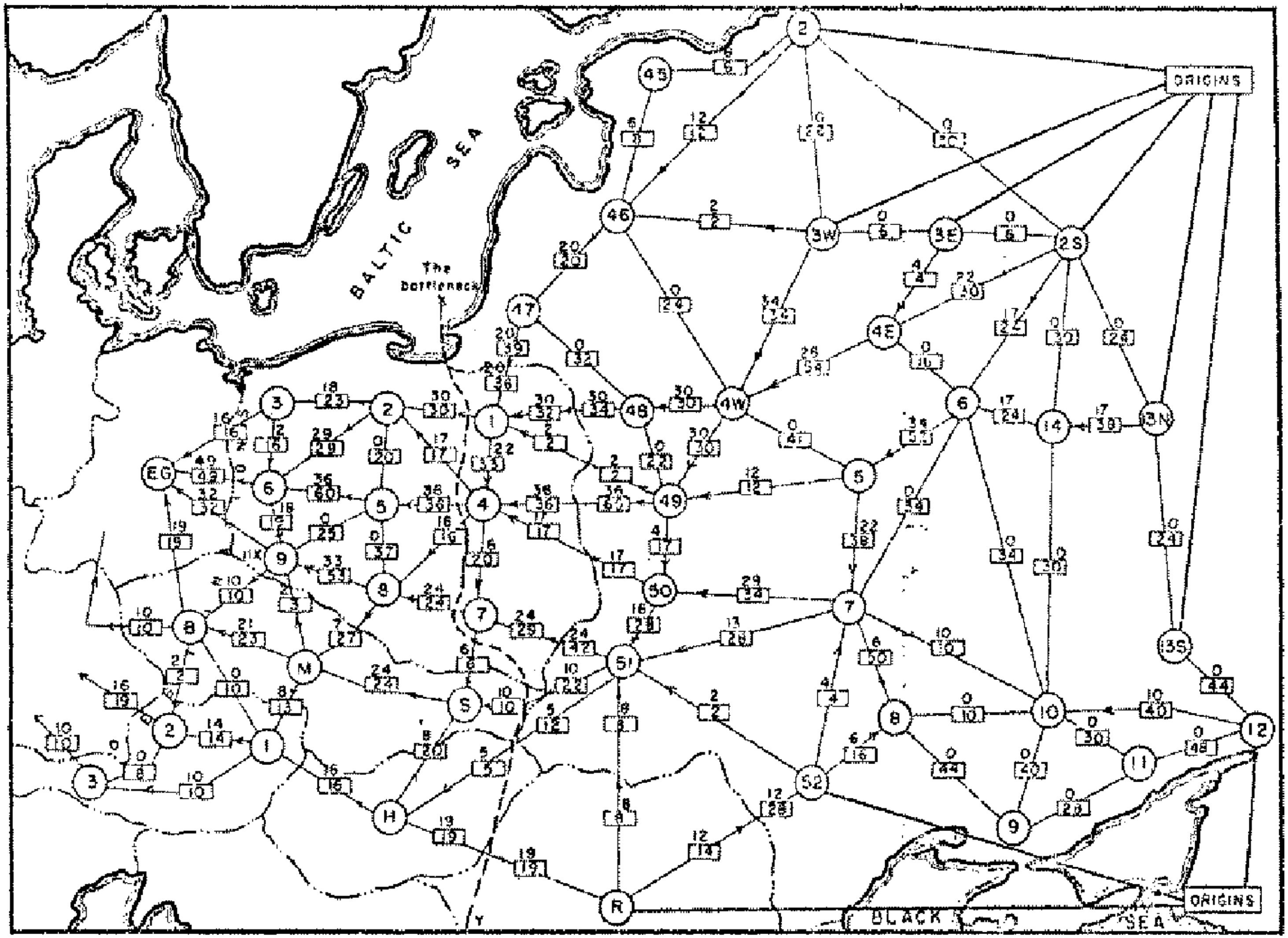
Lec 4. Cut and Flow: Fundamentals



Soviet rail network, 1955



Reference: *On the history of the transportation and maximum flow problems.*
Alexander Schrijver in Math Programming, 91: 3, 2002.

Maximum flow and minimum cut

Max flow and min cut.

- Two very rich algorithmic problems.
- Cornerstone problems in combinatorial optimization.
- Beautiful mathematical duality.

https://en.wikipedia.org/wiki/Maximum_flow_problem

Nontrivial applications / reductions.

- Data mining.
- Open-pit mining.
- Project selection.
- Airline scheduling.
- Bipartite matching.
- Baseball elimination.
- Image segmentation.
- Network connectivity.
- Network reliability.
- Distributed computing.
- Egalitarian stable matching.
- Security of statistical data.
- Network intrusion detection.
- Multi-camera scene reconstruction.
- Many many more . . .

Outline

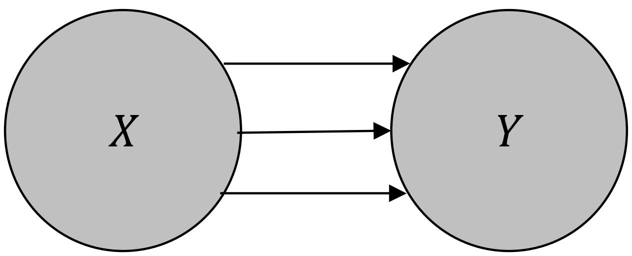
- Minimum cut
- Skew-symmetric transshipment (TS)
- Residual (Augmenting) graph
- Elementary augmentations
- Feasibility test

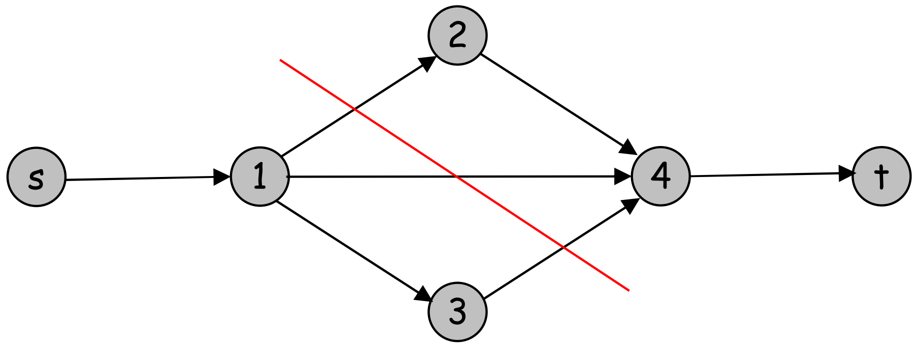
1. Minimum Cut

Cuts

$D = (V, A)$: a simple digraph (no parallel arcs via merging)

Def. For any disjoint $X, Y \subseteq V$, $A(X, Y) \coloneqq \{(u, v) \in A : u \in X, v \in Y\}$





Def. For any $U \subseteq V$,

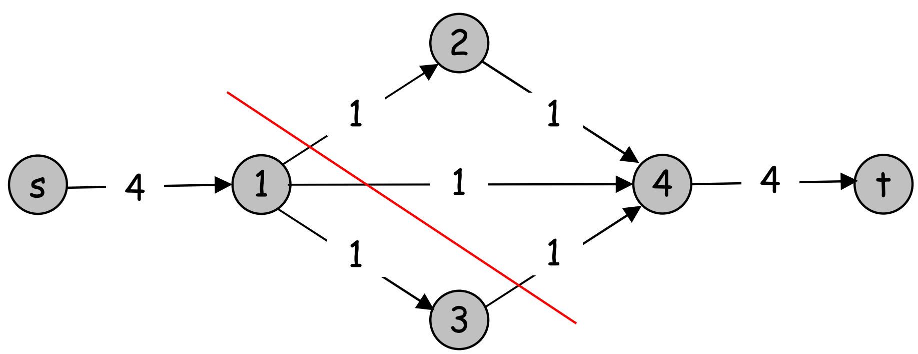
$$\delta^{out}(U) \coloneqq A(U, V \setminus U),$$
$$\delta^{in}(U) \coloneqq A(V \setminus U, U) = \delta^{out}(V \setminus U),$$
$$A[U] \coloneqq \{(u, v) \in A : u \in U, v \in U\}$$

Cut capacity

Notation: For any $x \in \mathbb{R}^A$, $x(B) \coloneqq \sum_{a \in B} x(a)$

$c \in \mathbb{R}^A$: an edge-capacity function (vector)

Def. The cut-capacity of U is $c(\delta^{out}(U))$

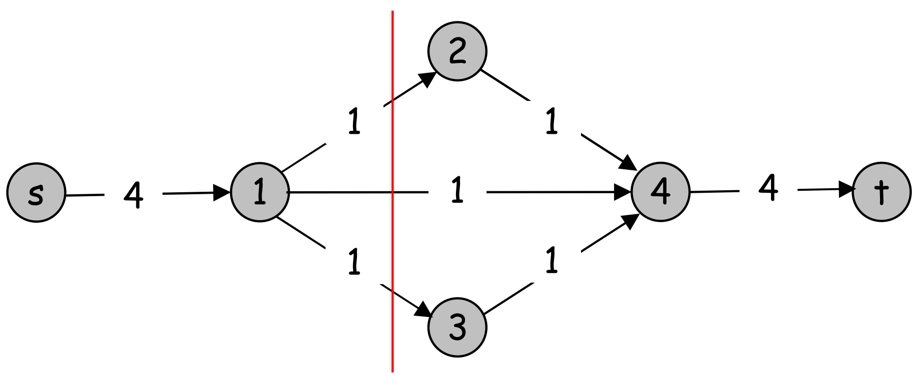


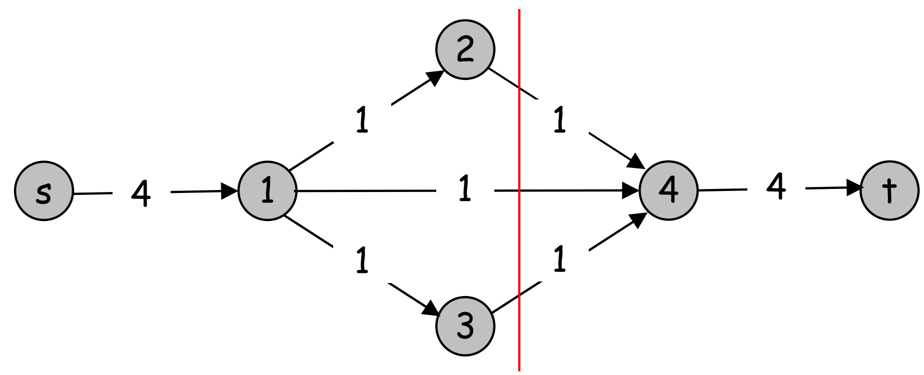
Minimum s-t cut

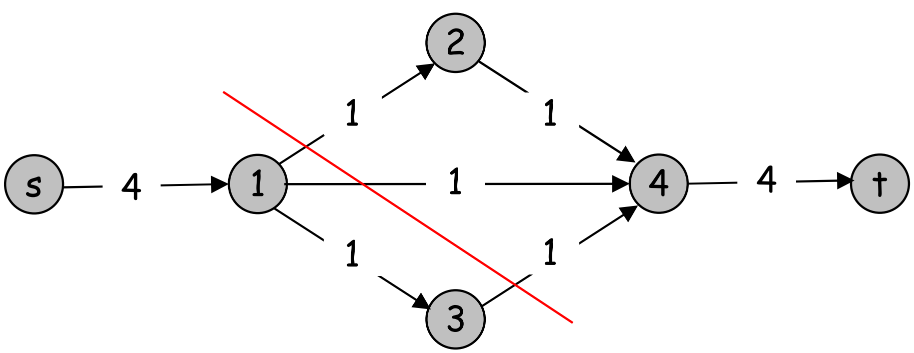
$s, t \in V$

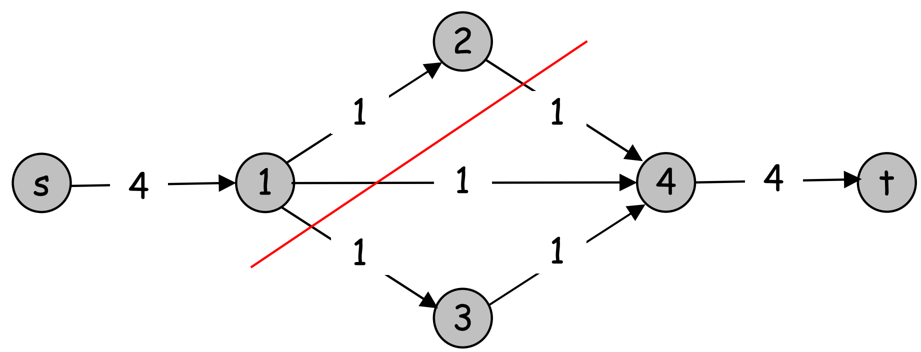
Def. An s-t cut is $\delta^{out}(U)$ for some $s \in U \subseteq V \setminus \{t\}$.

Min s-t cut problem: Find an s-t cut of minimum cut-capacity.









Computational hardness of Min-Cut

analogy of shortest path

NP-complete for arbitrary "capacity": Max-cut is NP-complete

Assumption: each edge $a \in A$ has a "mirror" (i.e. reverse edge) a^{-1} (if not, add it with $c(a^{-1}) = 0$ without changing the solution)

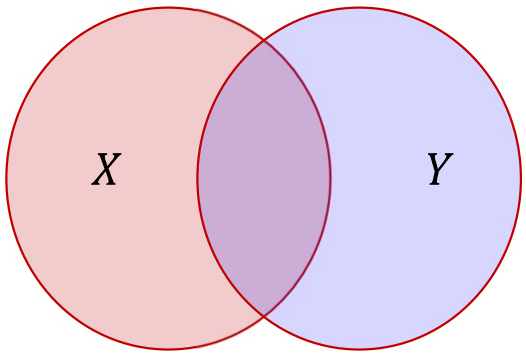
- forward edge: the direction with larger capacity
- backward edge: the direction with smaller capacity

Def. c is a flow network capacity if $c(a) + c(a^{-1}) \geq 0$ for each $a \in A$

Min-cut is polynomial if c is a flow network capacity.

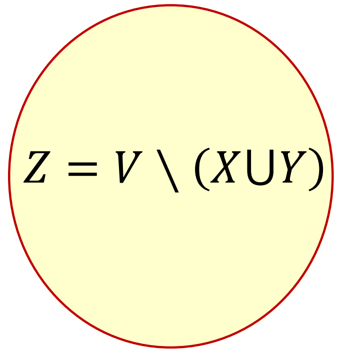
The cut equality

$$\left[\chi_{\delta^{out}(X)} + \chi_{\delta^{out}(Y)}\right] - \left[\chi_{\delta^{out}(X\cap Y)} + \chi_{\delta^{out}(X\cup Y)}\right] = \chi_{A(X\setminus Y, Y\setminus X)} + \chi_{A(Y\setminus X, X\setminus Y)}$$



Pf.

$$\begin{aligned}
\delta^{out}(X) &= A(X,Z) \cup A(X, Y\setminus X)\\
&= A(X,Z) \cup A(X\cap Y, Y\setminus X) \cup A(X\setminus Y, Y\setminus X)\\
\delta^{out}(Y) &= A(Y,Z) \cup A(X\cap Y, X\setminus Y) \cup A(Y\setminus X, X\setminus Y)\\
\delta^{out}(X\cap Y) &= A(X\cap Y, Z) \cup A(X\cap Y, Y\setminus X) \cup A(X\cap Y, X\setminus Y)\\
\delta^{out}(X\cup Y) &= A(X\cup Y, Z)
\end{aligned}$$



$$\chi_{A(X,Z)} + \chi_{A(Y,Z)} = \chi_{A(X\cap Y,Z)} + \chi_{A(X\cup Y,Z)}$$

Submodularity of cut capacities

$$\left[c\left(\delta^{out}(X)\right) + c\left(\delta^{out}(Y)\right)\right] - \left[c\left(\delta^{out}(X \cap Y)\right) + c\left(\delta^{out}(X \cup Y)\right)\right]$$
$$= c\left(A(X \setminus Y, Y \setminus X)\right) + c\left(A(Y \setminus X, X \setminus Y)\right)$$

Thm. If c is a flow network capacity, then $c\left(\delta^{out}(X)\right)$ is submodular:

$$c\left(\delta^{out}(X)\right) + c\left(\delta^{out}(Y)\right) \geq c\left(\delta^{out}(X \cap Y)\right) + c\left(\delta^{out}(X \cup Y)\right)$$

- Min-cut: minimizing submodular function
- more efficient algorithm via max-flow

2. Skew-Symmetric Transshipment

Skew-symmetry transshipment

$D = (V, A)$: a simple bidirected graph with $|A| = 2m$
$x \in \mathbb{R}^A$

Def. x is skew-symmetric if $x(a^{-1}) = -x(a)$ for each $a \in A$

- magnitude simply represents the amount
- sign simply represents the direction

$$A^+(x) \coloneqq \{a \in A : x(a) > 0\}$$

Skew-symmetry is preserved by linear combination.

Excess and deficit

Def. $x\left(\delta^{in}(v)\right)$ and $x(\delta^{out}(v))$ are the excess and deficit of v

$$x\left(\delta^{in}(v)\right) = x\big(\delta^{in}(v) \cap A^{+}(x)\big) - x\big(\delta^{out}(v) \cap A^{+}(x)\big) = -x\big(\delta^{out}(v)\big)$$

- v is excessive (resp, deficient, balanced) $\Leftrightarrow$
  $x\left(\delta^{in}(v)\right)$ is positive (resp, negative, 0).
- total (net) excess $= 0$, i.e. $\sum_{v \in V} x(\delta^{in}(v)) = 0$

Modularity. $\sum_{v \in U} x(\delta^{in}(v)) = x(A[U]) + x\left(\delta^{in}(U)\right) = x\left(\delta^{in}(U)\right)$

excess of U = sum of excesses of nodes in U

b-Transshipment (b-TS)

Def. If $x(\delta^{in}(v)) = b(v)$ for $v \in V$, then x is called a b-TS

$$\sum_{v \in V} b(v) = \sum_{v \in V} x(\delta^{in}(v)) = 0$$

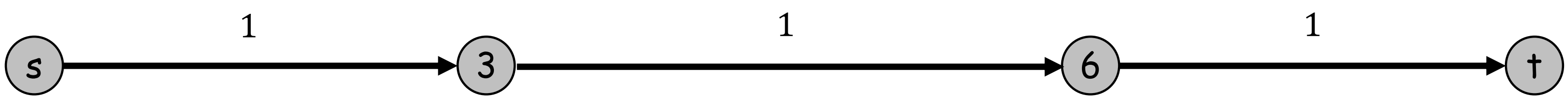
Def. x is called a *circulation* if all nodes are balanced (i.e. a 0-TS)

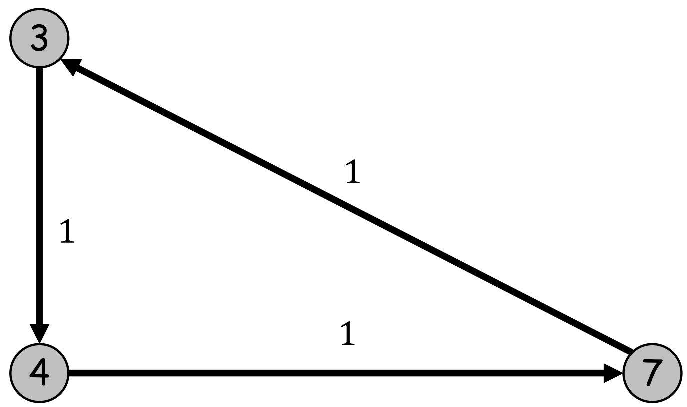
Def. The (absolute) value of a b-TS x: $|x| := (1/2)\|b\|_1$
= total excesses of excessive nodes,
= total deficits of deficient nodes.

Elementary skew-symmetric TS

P: a path or nontrivial circuit in A

χ^P: elementary TS along P given by $\chi^P(a) = 1$ (resp, -1, 0) if $a \in P$ (resp., $a \in P^{-1}$, $a \notin P \cup P^{-1}$)



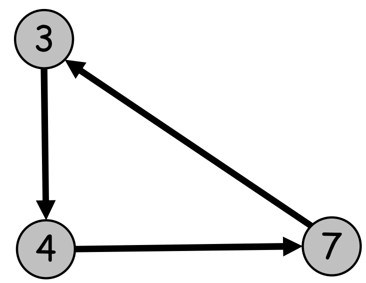


Elementary TS

TS = positive combinations of "paths & circuits"

χ_P along a path or circuit P in A

- $\chi_P(a) = 1$ and $\chi_P(a^{-1}) = -1$, for each $a \in P$;
- $\chi_P(a) = 0$ for each $a \notin P \cup P^{-1}$.



Elementary decomposition

TS = positive combinations of "paths & circuits"

In $O(nm)$ time, we may find

- a collection $\mathcal{P}$ of paths in $A^+(x)$ from deficient nodes to excessive nodes,
- a collection $\mathcal{C}$ of circuits in $A^+(x)$,
- and positive scaling factor $\varepsilon(P)$ for each $P \in \mathcal{P} \cup \mathcal{C}$

s.t. $|\mathcal{P}| + |\mathcal{C}| \leq |A^+(x)| \leq m$ and $x = \sum_{P \in \mathcal{P} \cup \mathcal{C}} \varepsilon(P) \chi^P$.

Moreover, if x is integer-valued, so are all $\varepsilon(P)$.

$|x| = \sum_{P \in \mathcal{P}} \varepsilon(P)$

(Linear) Cost

$\ell \in \mathbb{R}^A$: skew-symmetric edge price (length)

$$\text{cost}(x) := \sum_{a\in A} \ell(a)x(a) = 2\sum_{a\in A^+(x)} \ell(a)x(a)$$

- for each $a \in A$, $\ell(a^{-1})x(a^{-1}) = \ell(a)x(a)$

- Given an elementary decomposition $x = \sum_{P\in\mathcal{P}\cup\mathcal{C}} \varepsilon(P)\chi_P$,

$$cost(x) = 2\sum_{P\in\mathcal{P}\cup\mathcal{C}} \varepsilon(P)\ell(P)$$

- Cost is linear:

$$cost(x \pm y) = cost(x) \pm cost(y)$$

Adjusted price and cost

p: a node price function
ℓ_p: p-adjusted edge price (length); also skew-symmetric

Claim. After the adjustment, the cost of any b-TS x drops by a constant $2\sum_{v\in V} p(v)b(v)$.

Pf. True for every elementary TS.

Coro. Costs of circulations are invariant to the price adjustment.

Rounding of a fractional b-TS

Given a b-TS x where b is integral, comput in $O(mn)$ time a b-TS x' s.t.
(1) $x'(a) \in \{\lfloor x(a) \rfloor, \lceil x(a) \rceil\}$ for each edge $a \in A$,
(2) x' has the same or lower cost than x

F: the set of fractional edges

Fact. If $a \in F$ then $a^{-1} \in F$.

Fact. F contains a circuit.

Pf. If $(u, v) \in F$ then for some $w \neq u$, $(w, v) \in F$ hence $(v, w) \in F$.

Rounding along a fractional augmenting circuit

Find a circuit $C \subseteq F$ in $O(n)$ time, and by symmetry assume $\ell(C) \leq 0$.
Let

$$\varepsilon \coloneqq \min_{a \in C}\left(\lceil x(a) \rceil - x(a)\right), x' \coloneqq x + \varepsilon \chi_C$$

- $x' \coloneqq x + \varepsilon \chi_C$ is also a b-TS.
- For each $a \in A$, $\lfloor x(a) \rfloor \leq x'(a) \leq \lceil x(a) \rceil$.
- For each "bottleneck" $a \in C$ with $\varepsilon = \lceil x(a) \rceil - x(a)$,
  - $x'(a) = \lceil x(a) \rceil$ and $x'(a^{-1}) = \lfloor x(a^{-1}) \rfloor$
  - both a and a^{-1} become integral

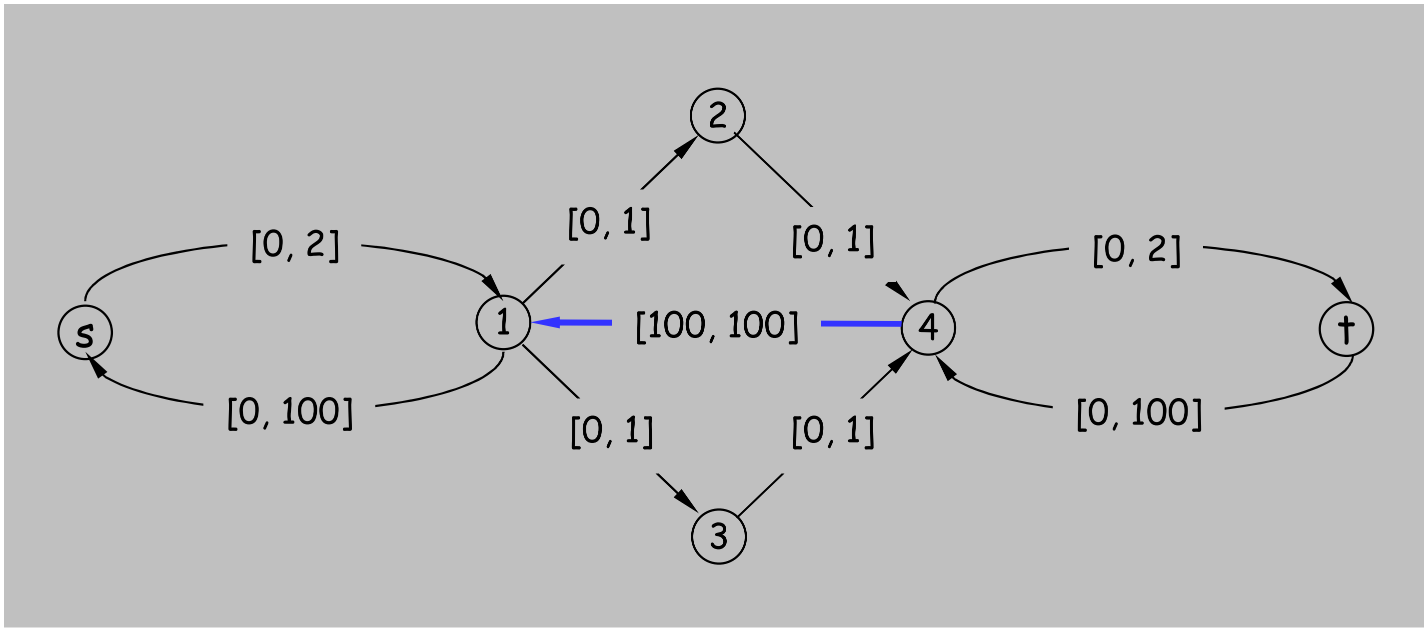
Rounding algorithm

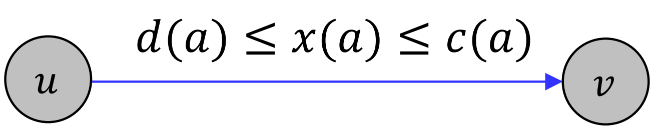
```
initialize F;
while F is nonempty
    find a non-positive circuit C in F;
    compute ε along C;
    update x and F along C;
return x
```

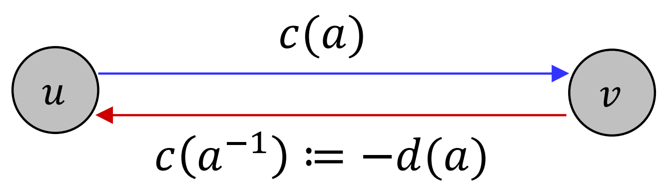
at most m rounding iterations, and each iteration takes $O(n)$ time

Capacitated flow network

Feasibility constraints: $d \le x \le c$







x: a skew-symmetric TS in a *capacitated* flow network: $D = (V, A;\ c)$

Def. x is called a pseudoflow under c if $x \le c$

Preflow and flow: additional excess constraints

x: a pseudoflow in a flow network $D = (V, A;\ c)$

Def. x is called an s-preflow under c if only s may be deficient

- s is called the source node.

Def. x is called an {s,t}-flow under c if all nodes except s, t are balanced.

- For any s-t separator U, $x(\delta^{out}(s)) = x(\delta^{out}(U))$
- As an s–t flow, its value is $val(x) := x\left(\delta^{out}(s)\right) = x\left(\delta^{in}(t)\right)$.

Extreme pseudoflows and and max-flows

Def. A pseudoflow x is said to be extreme if it has minimum cost among all pseudoflows with the same deficits.

- An extreme b-TS is a min-cost b-TS.
- Adjustment by node prices preserves the extremeness.

Max flow problem. Find an s-t flow of maximum value.
Min flow problem. Find an s-t flow of minimum value.

a min s-t flow $\Leftrightarrow$ a max t-s flow

Integrality theorem

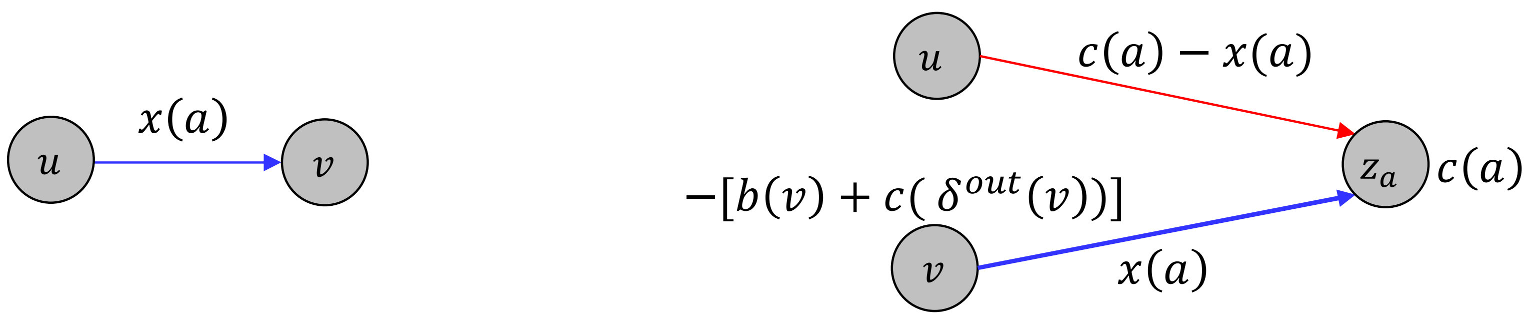
Thm. If c is integral and there is an s-t flow under c, then there exists an integral max s-t flow under c.

Thm. If b and c are integral and there is a b-TS under c, then there exists an integral min-cost b-TS under c.

Nonnegative capacitated TS ⇒ nonnegative uncapacitated TP

Construct D' and from D via edge splitting [Fulkerson 1960]:

- For each $a = (u, v) \in A$ with $c(a) > 0$,
  - add a node z_a with $b'(z_a) := c(a)$,
  - split a into a free (u, z_a), and (v, z_a) of price $\ell(a)$
- For each $v \in V$, $b'(v) \coloneqq -[b(v) + c(\delta^{out}(v))]$



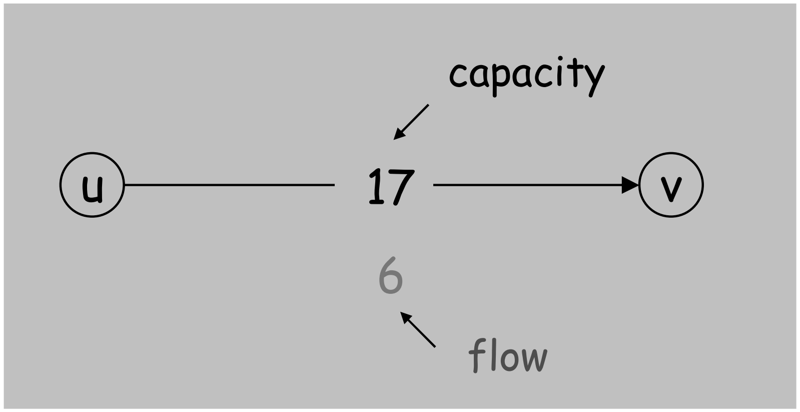
min-cost b-TS in D ↔ min-cost b'-TP in D'

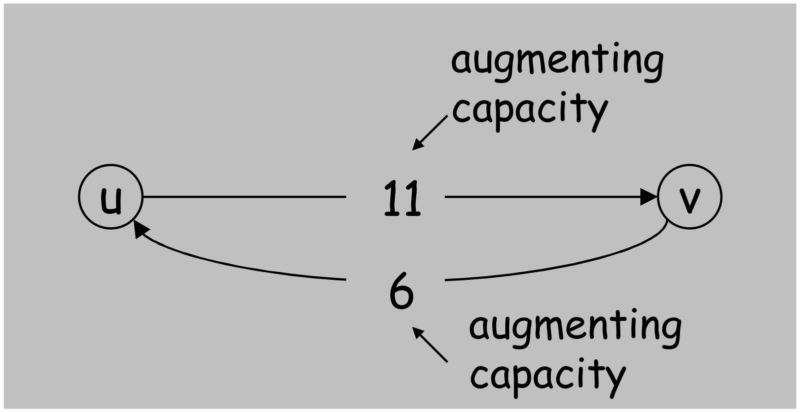
Remark. The reduction preserves integrality.

3. Residual (Augmenting) Graph

Residual (augmenting) graph

- x: a pseudoflow under c
- residual capacity: $c - x \geq 0$
- $a \in A$ is saturated if $x(a) = c(a)$, residual if $x(a) < c(a)$
- $A_x \coloneqq A^+(c - x)$: the set of residual edges
  - if $c \geq 0$, then $A^+(x) \subseteq A_x^{-1}$
- residue graph $D_x = (V, A_x)$





Sum and difference of pseudoflows

x, z: pseudoflows under c
y: a pseudoflow under $c - x$

- $x + y$ is a pseudoflow under c.
- $z - x$ is a pseudoflow under $c - x$ and $A^{+}(z - x) \subseteq A_x \cap A_z^{-1}$

Sum and difference of pseudoflows

x, z: b-TS's under c
y: a circulation under $c - x$

- $x + y$ is a b-TS under c.
- $z - x$ is a circulation under $c - x$

Thm. y is a min-cost circulation under $c - x \Leftrightarrow x + y$ is a min-cost b-TS under c

Thm. z is a min-cost b-TS under $c \Leftrightarrow z - x$ is a min-cost circulation under $c - x$

Algorithmic implication for min-cost TS

Phase 1. find a b-TS $x \leq c$

Phase 2. compute a min-cost circulation $y \leq c - x$

Phase 3. return $x + y$

Remark: Phase 1 is reduced to max-flow subject to nonnegative capacity

Sum and difference of flows

x, z: s-t flows under c
y: s-t flow under $c - x$

- $x + y$ is a s-t flow under c, and $val(x + y) = val(x) + val(y)$
- $z - x$ is a s-t flow under $c - x$, and $val(z - x) = val(z) - val(x)$
- $A^+(z - x) \subseteq A_x \cap A_z^{-1}$

Pf. For any $a \in A^+(z - x)$,

$$x(a) < z(a) \le c(a) \Rightarrow a \in A_x$$

$$z(a^{-1}) < x(a^{-1}) \le c(a^{-1}) \Rightarrow a^{-1} \in A_z \Rightarrow a \in A_z^{-1}$$

Thm. y is a max s-t flow under $c - x \Leftrightarrow x + y$ is a max s-t flow under c

Thm. z is a max s-t flow under $c \Leftrightarrow z - x$ is a max s-t flow under $c - x$

Algorithmic implication for max flow

Phase 1. find an s-t flow $x \leq c$
Phase 2. compute a max s-t flow $y \leq c - x$
Phase 3. return $x + y$

Remark: 2 computations of max-flow subject to nonnegative capacity!
Phase 1 is reduced to max-flow subject to nonnegative capacity
Phase 2 is a max-flow subject to nonnegative capacity

4. Elementary Augmentations

Elementary augmentation of a pseudoflow

x: a pseudoflow under c

P: a path or circuit in A_x

- bottleneck edge-set P^*: the edge with the smallest (residual) capacity
- bottleneck capacity $\Delta_x(P)$:= (residual) capacity of the bottleneck edge

Def. For any $0 < \varepsilon \le \Delta_x(P)$, $x + \varepsilon\chi_P$ is called an elementary augmentation of x along P. If $\varepsilon = \Delta_x(P)$, denote $x + \varepsilon\chi_P$ by $x \oplus P$.

- $P^{-1} \subseteq A_{x+\varepsilon\chi_P} \subseteq A_x \cup P^{-1}$ and $A_{x \oplus P} = (A_x \setminus P^*) \cup P^{-1}$
- $cost(x + \varepsilon\chi_P) = cost(x) + 2\varepsilon\ell(P)$
- If x is an s-t flow and P is an s-t path, then $x + \varepsilon\chi_P$ is also an s-t flow.

Augmenting path theorem for max flow

[Ford-Fulkerson 1955]

Thm. An s-t flow x is a max s-t flow under $c \Leftrightarrow A_x$ has no s-t path.

Pf. We show contrapositive.

$\Rightarrow$ For any s-t path P in A_x,

$$val(x \oplus P) = val(x) + \Delta_x(P) > val(x).$$

$\Leftarrow$ Assume x is not maximum and z is a max s-t flow.

- $z - x$ is a s-t flow under $c - x$ of positive value.
- An elementary decomposition of $z - x$ contains an s-t path

$$P \subseteq A^+(z - x) \subseteq A_x.$$

Augmenting circuit theorem for extreme pseudoflow

[Tolstoı 1930], [Ford and Fulkerson 1962], [Klein 1967]

Thm. A pseudoflow x is extreme $\Leftrightarrow$ A_x has no negative circuit.

Pf. We show contrapositive. Suppose that x is a b-TS.

$\Rightarrow$ For any negative circuit C in D_x,

$$cost(x \oplus C) = cost(x) + 2\Delta_x(C)\ell(C) < cost(x)$$

$\Leftarrow$ Assume x is not extreme and z is an extreme b-TS.

- $z - x$ is a circulation under $c - x$ of negative cost.
- An elementary decomposition of $z - x$ contains a negative circuit

$$C \subseteq A^+(z - x) \subseteq A_x.$$

Difference between extreme pseudoflows

x: an extreme pseudoflows under c
z: an min-cost b-TS under c

Thm. All circuits in $A^+(z-x)$ have 0 price (length).

Pf. Otherwise, for a negative circuit C, $x \oplus C$ has smaller cost than x; for a positive circuit C, $z \oplus C^{-1}$ has smaller cost than z.

Coro. Given an elementary decomposition $z - x = \sum_{P \in \mathcal{P} \cup \mathcal{C}} \varepsilon(P)\chi_P$,

$$cost(z-x) = 2\sum_{P\in\mathcal{P}} \varepsilon(P)\ell(P)$$

Growth of extreme pseudoflows

[Jewell 1958], [Iri 1960], [Busacker and Gowen 1961]

x: an extreme pseudoflow
P: a shortest path in A_x

Thm. For any $0 < \varepsilon \leq \varepsilon_x(P)$, $x + \varepsilon\chi_P$ is also an extreme pseudoflow.

Pf. For any extreme pseudoflow z with the same deficits as $x + \varepsilon\chi_P$,

$$cost(z) - cost(x) = cost(z - x) \geq 2\varepsilon\ell(P) = cost(x + \varepsilon\chi_P) - cost(x).$$

Thus, $cost(z) \geq cost(x + \varepsilon\chi_P)$.

MFMC Duality [Ford-Fulkerson 1954]

Assumption: (1) there is an s-t flow, and (2) no uncapacitated s-t path

Thm. max-flow value = min-cut capacity.

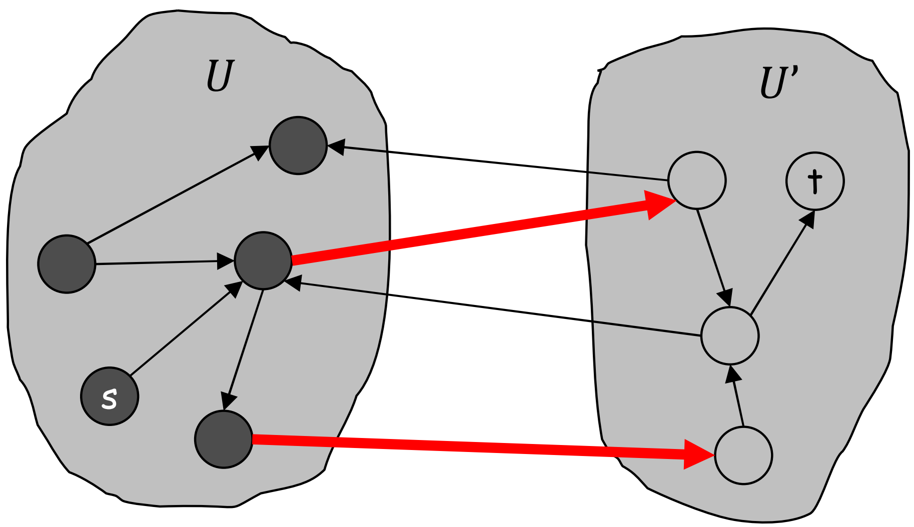
Pf. Let f be a max-flow.

($\leq$) for any s-t cut U, $val(f) = f(\delta^{out}(U)) \leq c(\delta^{out}(U))$

($=$) Each $a \in \delta^{out}(U)$ is saturated (i.e $a \notin A_f$) hence $f(a) = c(a) < \infty$. Thus,

$$val(f) = f\big(\delta^{out}(U)\big) = c\big(\delta^{out}(U)\big).$$

vertices reachable from s in D_f

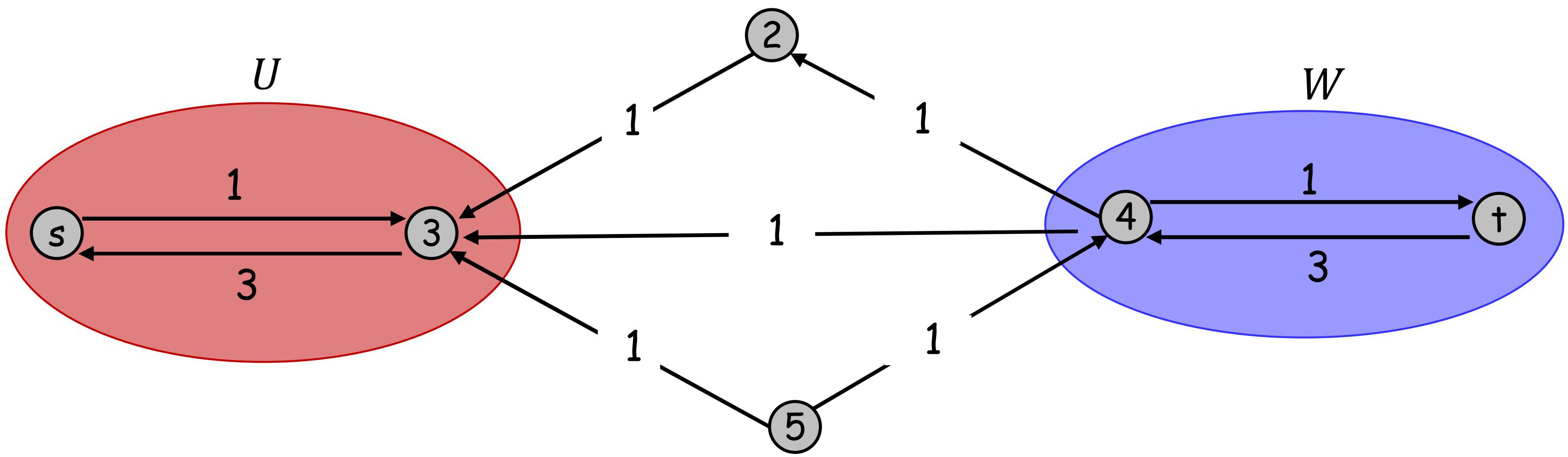


original network

Construction of min-cuts from max-flow

f: a maximum s-t flow. There is no s-t path in D_f.

- U: the set of vertices reachable from s in D_f.
- W: the set of vertices which can reach t in D_f.



Thm. U (resp., $V \backslash W$) is the minimal (resp., maximal) s-t cut with minimum cut-capacity.

Exercise: How about other s-t separators with minimum cut capacity?

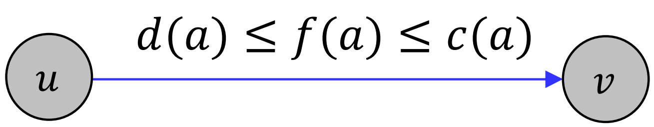
Optimality gap

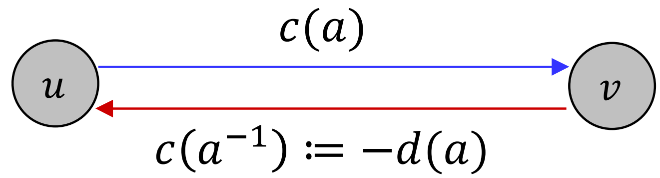
f: an s-t flow
f^*: a max s-t flow

Thm. $val(f^*) - val(f) =$ min s-t cut capacity of D_f.

Pf. $val(f^*) - val(f) = val(f^* - f)$, and $f^* - f$ is a max s-t flow in D_f.

Flow with demands





Thm. max s-t flow value with demands
= min of $c\left(\delta^{out}(U)\right) - d\left(\delta^{in}(U)\right)$ over all s-t separators U.

Thm. min s-t flow value with demands
= max of $d\left(\delta^{out}(U)\right) - c\left(\delta^{in}(U)\right)$ over all s-t separators U.

Coro. When $c = \infty$, min s-t flow value with demands
= max of $d\left(\delta^{out}(U)\right)$ over all s-t separators U with $\delta^{in}(U) = \emptyset$.

5. Feasibility Test of b-TS

Feasibility test of b-TS

Given $D = (V, A; c)$ and $b \in \mathbb{R}^V$ with $b(V) = 0$, decide whether there is a b-TS under c, and if so find one.

Reduction to max-flow subject to non-negative edge capacity

Disposal of negative capacities (demands)

Idea: shift negative capacities (demands) of edges to demands of nodes

"Edge demand" d: for each forward-backward pair $a, a^{-1} \in A$, let

$$d(a) = -c(a^{-1}), d(a^{-1}) = c(a^{-1}).$$

Step 1. Define the residual "edge capacity" $c' := c - d$: for each forward-backward pair $a, a^{-1} \in A$,

$$c(a) - d(a) = c(a) + c(a^{-1}) > 0,$$
$$c(a^{-1}) - d(a^{-1}) = 0.$$

Step 2. Define the residual "node demand" $b' : b'(v) := b(v) - d\left(\delta^{in}(v)\right)$

x is a b-TS under $c \Leftrightarrow x - d$ is a b'-TS under $c' = c - d$.

Remark. The reduction preserves integrality

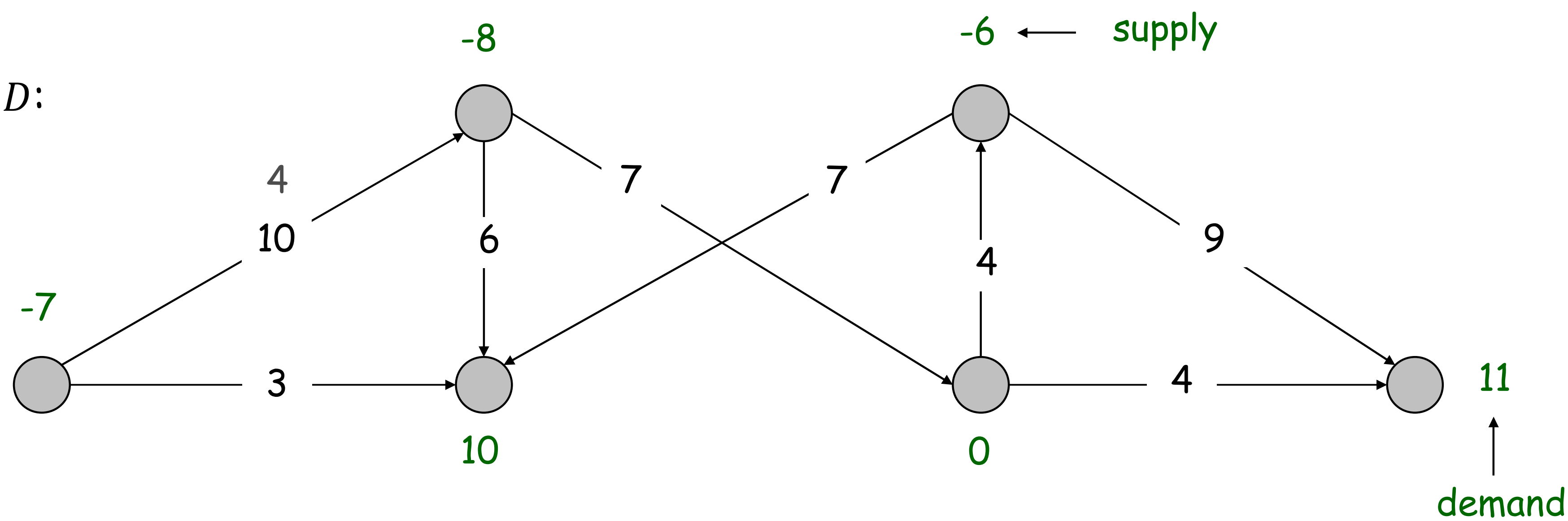
Reduction to nonnegative max-flow

Idea: translate supplies/demands on nodes to edge capacities

$D' = (V, A; c')$; $c' \geq 0$; $b' \in \mathbb{R}^V$ with $b'(V) = 0$

$$S := \{v \in V : b'(v) < 0\},$$
$$T := \{v \in V : b'(v) > 0\}.$$

D:

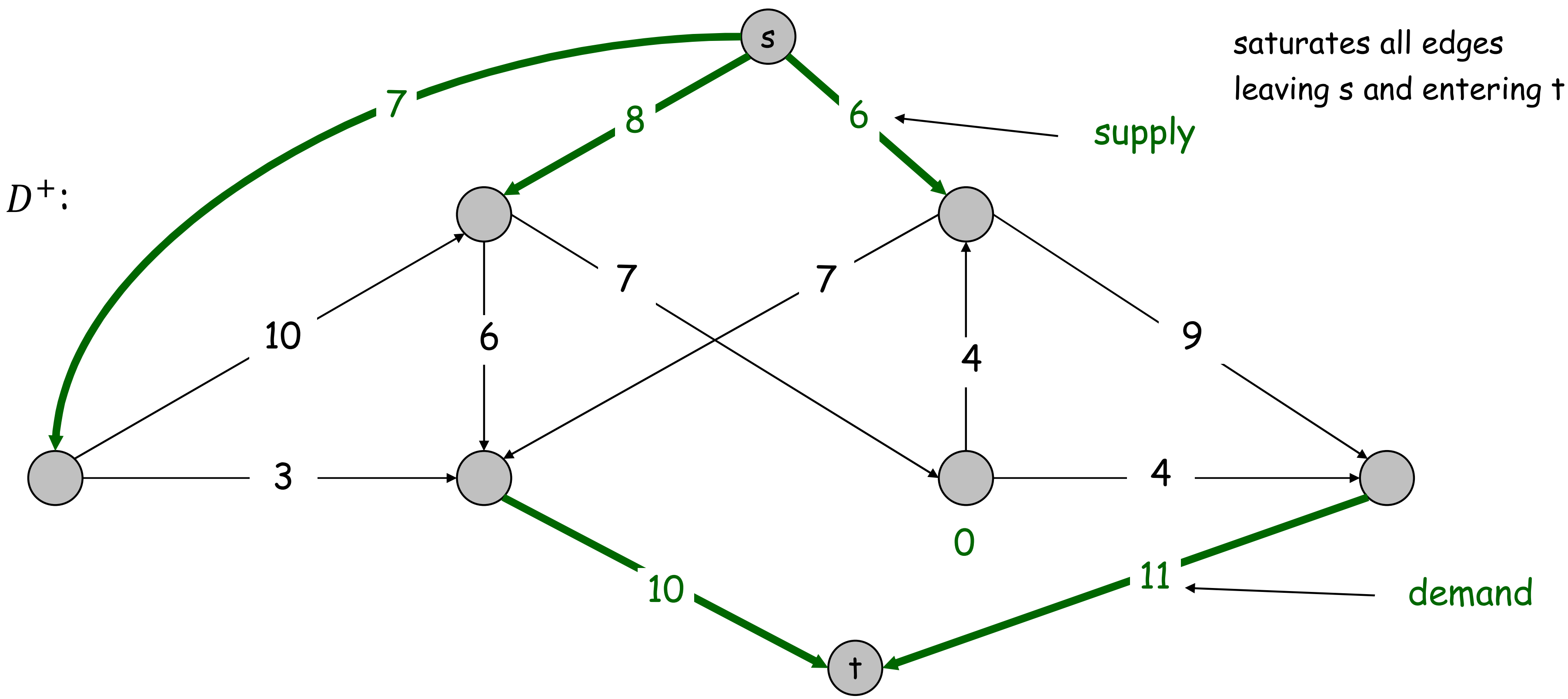


Reduction to nonnegative max-flow

Construct an extended flow network D^+ from D'

- add new source s and sink t.
- $\forall\, u \in S$, add edges $(s, u), (u, s)$ with capacity $-b'(u)$ and 0 resp.
- $\forall\, v \in T$, add edges $(v, t), (t, v)$ with capacity $b'(v)$ and 0 resp.

A b'-TS in D' $\leftrightarrow$ a source-saturating max s-t flow in D^+



Characterization of feasibility

Thm. D' has a b'-TS under c'
$\Leftrightarrow c'\big(\delta^{out}(U)\big) \geq -b'(U)\ \forall U \subseteq V$
$\Leftrightarrow c'\Big(\delta^{in}(U)\Big) \geq b'(U)\ \forall U \subseteq V.$

Pf. In D^+, $\{s\}$ has cut-capacity $-b'(S)$, and $U \cup \{s\}$ has cut-capacity

$$
\begin{aligned}
&-b'(S \setminus U) + b'(T \cap U) + c'(\delta^{out}(U)) \\
&= -b'(S) + b'(S \cap U) + b'(T \cap U) + c'(\delta^{out}(U)) \\
&= -b'(S) + b'(U) + c'(\delta^{out}(U))
\end{aligned}
$$

Thus, the min cut-capacity $= -b'(S)$ iff $c'\big(\delta^{out}(U)\big) \geq -b'(U)$.

$$c'\Big(\delta^{in}(U)\Big) = c'\big(\delta^{out}(V \setminus U)\big) \geq -b'(V \setminus U) = b'(U)$$

Characterization of feasibility

Thm. D has a b-TS under c
$\Leftrightarrow c\left(\delta^{out}(U)\right) \geq -b(U)\ \forall U \subseteq V$
$\Leftrightarrow c\left(\delta^{in}(U)\right) \geq b(U)\ \forall U \subseteq V.$

Pf.

$$c'\left(\delta^{in}(U)\right) = c\left(\delta^{in}(U)\right) - d\left(\delta^{in}(U)\right)$$
$$b'(U) = b(U) - d\left(\delta^{in}(U)\right)$$

Thus,

$$c'\left(\delta^{in}(U)\right) \geq b'(U) \Leftrightarrow c\left(\delta^{in}(U)\right) \geq b(U)$$

Hoffman's Circulation Theorem

Thm [Hoffman 1960] D has a circulation under c
$\Leftrightarrow c\left(\delta^{out}(U)\right) \geq 0 \; \forall U \subseteq V$
$\Leftrightarrow c\left(\delta^{in}(U)\right) \geq 0 \; \forall U \subseteq V$

Feasibility test of flow

Reduction to circulation: D' Identifing s and t in D to get D'

an s-t flow in $D \Leftrightarrow$ a circulation in D'

- find a circulation in D', if there is any, and then break up s and t.

Thm. There exists an s-t flow in $D \Leftrightarrow$ for any $U \subseteq V$ not separating s and t, $c\big(\delta^{out}(U)\big) \geq 0$.

Pf. $\Rightarrow c\big(\delta^{out}(U)\big) \geq f\big(\delta^{out}(U)\big) = 0$

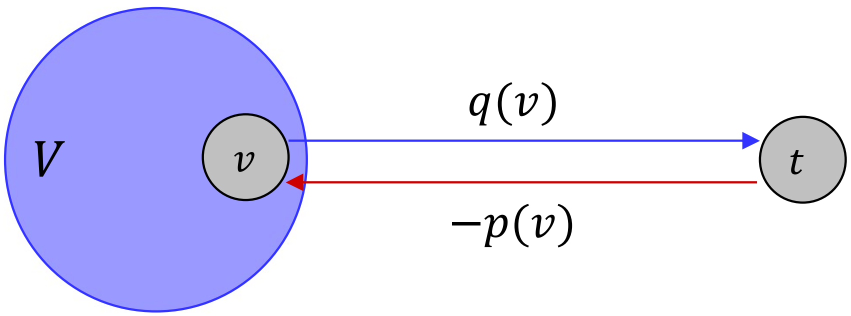
$\Leftarrow$ By Hoffman's Circulation Theorem, there exists a circulation in D'

Feasibility test of bounded TS

Given $D = (V, A; c)$ and $p, q \in \mathbb{R}^V$ with $p \leq q$, decide whether there exists a b-TS under c with $p \leq b \leq q$, and if so find one.

Reduction to circulation: Construct an extended network D^+ from D:

- add new node t.
- $\forall v \in V$, add edges $(v, t), (t, v)$ with capacity $q(v)$ and $-p(v)$ resp.



a b-TS in D with $p \leq b \leq q$ $\leftrightarrow$ a circulation in D^+

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Linking of bounded TS

Thm. D has a b-TS under c with $p \leq b \leq q$
$\Leftrightarrow c\left(\delta^{in}(U)\right) \geq \max\{p(U), -q(V \setminus U)\} \ \forall U \subseteq V$
$\Leftrightarrow D$ has a b-TS under c with $b \geq p$ and a b'-TS under c with $b' \leq q$

Pf. For any $U \subseteq V$,

- cut-capacity of U in D^+ : $c\left(\delta^{in}(U)\right) - p(U)$
- cut-capacity of $U \cup \{t\}$ in D^+: $c\left(\delta^{in}(U)\right) + q(V \setminus U)$

Follows from Hoffman's Circulation Theorem

Thm. If p, q and c are integral and D has a b-TS x under c with exists with $p \leq b \leq q$, then x can be taken integer

Summary

- Elementary decomposition and augmentation
- Residual graph
- Integrality theorem and rounding
- Min-cut via max-flow
- Feasibility via max-flow