(3)

a) $$C\left(x; \frac{\mu}{\sigma^2}, \frac{1}{\sigma^2}\right) = \sqrt{\frac{1}{\sigma^2 2\pi}} \cdot \exp\left(-\frac{\mu^2\sigma^2}{\sigma^4 \cdot 2}\right) \cdot \exp\left(\frac{\mu}{\sigma^2} \cdot x - \frac{1}{\sigma^2} \cdot \frac{x^2}{2}\right)$$

$$= \sqrt{\frac{1}{2\pi\sigma^2}} \cdot \exp\left(\frac{\mu}{\sigma^2} \cdot x - \frac{1}{\sigma^2} \cdot \frac{x^2}{2} - \frac{\mu^2}{\sigma^2 \cdot 2}\right)$$

$$= \sqrt{\frac{1}{2\pi\sigma^2}} \cdot \exp\left(\frac{2}{2}\frac{\mu}{\sigma^2} \cdot x - \frac{1}{\sigma^2} \cdot \frac{x^2}{2} - \frac{\mu^2}{\sigma^2 \cdot 2}\right)$$

$$= \sqrt{\frac{1}{2\pi\sigma^2}} \cdot \exp\left(\frac{1}{2\sigma^2} \cdot \left(2\mu x - x^2 - \mu^2\right)\right)$$

$$= \sqrt{\frac{1}{2\pi\sigma^2}} \cdot \exp\left(-\frac{1}{2\sigma^2} \cdot \left(-2\mu x + x^2 + \mu^2\right)\right)$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \cdot \exp\left(-\frac{1}{2\sigma^2} \cdot \left(x - \mu\right)^2\right) = \mathcal{N}(x; \mu, \sigma^2)$$

b) $$\mathcal{N}\left(x; \frac{J}{\rho}, \frac{1}{\rho}\right) = \frac{1}{\sqrt{2\pi \cdot \frac{1}{\rho}}} \cdot \exp\left(-\frac{1}{2} \cdot \frac{\left(x - \frac{J}{\rho}\right)^2}{\frac{1}{\rho}}\right)$$

$$= \sqrt{\frac{\rho}{2\pi}} \cdot \exp\left(-\frac{\rho}{2} \cdot \left(x^2 - 2x\frac{J}{\rho} + \left(\frac{J}{\rho}\right)^2\right)\right)$$

$$= \sqrt{\frac{\rho}{2\pi}} \cdot \exp\left(-\frac{\rho x^2}{2} + \frac{2xJ\rho}{2\rho} - \frac{\rho \cdot J^2}{2\rho^2}\right)$$

$$= \sqrt{\frac{\rho}{2\pi}} \cdot \exp\left(-\frac{\rho x^2}{2} + x \cdot J - \frac{J^2}{2\rho}\right)$$

$$= \sqrt{\frac{\rho}{2\pi}} \cdot \exp\left(-\frac{J^2}{2\rho}\right) \cdot \exp\left(J \cdot x - \rho \cdot \frac{x^2}{2}\right) = C(x; J, \rho)$$

c) $$C(x; J_1, \rho_1) = \sqrt{\frac{\rho_1}{2\pi}} \cdot \exp\left(-\frac{J_1^2}{2\rho_1}\right) \cdot \exp\left(J_1 \cdot x - \rho_1 \cdot \frac{x^2}{2}\right)$$

$$C(x; J_2, \rho_2) = \sqrt{\frac{\rho_2}{2\pi}} \cdot \exp\left(-\frac{J_2^2}{2\rho_2}\right) \cdot \exp\left(J_2 \cdot x - \rho_2 \cdot \frac{x^2}{2}\right)$$

$$C(x; J_1 + J_2, \rho_1 + \rho_2) = \sqrt{\frac{\rho_1 + \rho_2}{2\pi}} \cdot \exp\left(-\frac{(J_1 + J_2)^2}{2 \cdot (\rho_1 + \rho_2)}\right) \cdot \exp\left((J_1 + J_2) \cdot x - (\rho_1 + \rho_2) \cdot \frac{x^2}{2}\right)$$

$$\mathcal{N}\left(\frac{J_1}{\rho_1}; \frac{J_2}{\rho_2}, \frac{1}{\rho_1} + \frac{1}{\rho_2}\right) = \frac{1}{\sqrt{2\pi \cdot \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)}} \cdot \exp\left(-\frac{1}{2} \cdot \frac{\left(\frac{J_1}{\rho_1} - \frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1} + \frac{1}{\rho_2}}\right)$$

d) $C(x, J_1, \rho_1) \cdot C(x; J_2, \rho_2)$

$$= \sqrt{\frac{\rho_1}{2\pi}} \cdot \exp\left(-\frac{J_1^2}{2\rho_1}\right) \cdot \exp\left(J_1 \cdot x - \rho_1 \cdot \frac{x^2}{2}\right) \cdot \sqrt{\frac{\rho_2}{2\pi}} \cdot \exp\left(-\frac{J_2^2}{2\rho_2}\right) \cdot \exp\left(J_2 \cdot x - \rho_2 \cdot \frac{x^2}{2}\right)$$

$$= \frac{\sqrt{\rho_1 \cdot \rho_2}}{2\pi} \cdot \exp\left(-\frac{J_1^2}{2\rho_1} - \frac{J_2^2}{2\rho_2} + J_1 \cdot x - \rho_1 \cdot \frac{x^2}{2} + J_2 \cdot x - \rho_2 \cdot \frac{x^2}{2}\right)$$

$C(x; J_1 + J_2, \rho_1 + \rho_2) \cdot N\left(\frac{J_1}{\rho_1}; \frac{J_2}{\rho_2}, \frac{1}{\rho_1} + \frac{1}{\rho_2}\right)$

$$= \sqrt{\frac{\rho_1 + \rho_2}{2\pi}} \cdot \exp\left(-\frac{(J_1 + J_2)^2}{2 \cdot (\rho_1 + \rho_2)}\right) \cdot \exp\left((J_1 + J_2) \cdot x - (\rho_1 + \rho_2) \cdot \frac{x^2}{2}\right) \cdot$$

$$\frac{1}{\sqrt{2\pi \cdot \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)}} \cdot \exp\left(-\frac{1}{2} \cdot \frac{\left(\frac{J_1}{\rho_1} - \frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1} + \frac{1}{\rho_2}}\right)$$

$$= \frac{\sqrt{\rho_1 + \rho_2}}{2\pi \cdot \sqrt{\frac{1}{\rho_1} + \frac{1}{\rho_2}}} \cdot \exp\left(-\frac{(J_1 + J_2)^2}{2 \cdot (\rho_1 + \rho_2)} + (J_1 + J_2) \cdot x - (\rho_1 + \rho_2) \cdot \frac{x^2}{2} - \frac{1}{2} \cdot \frac{\left(\frac{J_1}{\rho_1} - \frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1} + \frac{1}{\rho_2}}\right)$$

Zuerst zeigen wir die Gleichheit der Faktoren vor dem exp. Dazu multiplizieren wir mit 1.

$$\frac{\sqrt{\rho_1 \cdot \rho_2}}{2\pi} = \frac{\sqrt{\rho_1 \cdot \rho_2}}{2\pi} \cdot \frac{\sqrt{\frac{1}{\rho_1} + \frac{1}{\rho_2}}}{\sqrt{\frac{1}{\rho_1} + \frac{1}{\rho_2}}} = \frac{\sqrt{\rho_1 \cdot \rho_2 \cdot \left(\frac{1}{\rho_1} + \frac{1}{\rho_2}\right)}}{2\pi \cdot \sqrt{\frac{1}{\rho_1} + \frac{1}{\rho_2}}}$$

$$= \frac{\sqrt{\frac{\rho_1 \cdot \rho_2}{\rho_1} + \frac{\rho_1 \cdot \rho_2}{\rho_2}}}{2\pi \cdot \sqrt{\frac{1}{\rho_1} + \frac{1}{\rho_2}}}$$

$$= \frac{\sqrt{\rho_2 + \rho_1}}{2\pi \cdot \sqrt{\frac{1}{\rho_1} + \frac{1}{\rho_2}}}$$ ✓

Also gilt noch zu zeigen

$$-\frac{J_1^2}{2\rho_1} - \frac{J_2^2}{2\rho_2} + J_1 \cdot x - \rho_1 \cdot \frac{x^2}{2} + J_2 \cdot x - \rho_2 \cdot \frac{x^2}{2}$$

$$\overset{?}{=} -\frac{(J_1+J_2)^2}{2\cdot(\rho_1+\rho_2)} + (J_1+J_2)\cdot x - (\rho_1+\rho_2)\cdot\frac{x^2}{2} - \frac{1}{2}\cdot\frac{\left(\frac{J_1}{\rho_1}-\frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1}+\frac{1}{\rho_2}}$$

Es gilt

$$J_1\cdot x - \rho_1 \cdot \frac{x^2}{2} + J_2 \cdot x - \rho_2 \cdot \frac{x^2}{2} = (J_1+J_2)\cdot x - (\rho_1+\rho_2)\cdot\frac{x^2}{2}$$

Somit gilt noch zu zeigen

$$-\frac{(J_1+J_2)^2}{2\cdot(\rho_1+\rho_2)} - \frac{1}{2}\cdot\frac{\left(\frac{J_1}{\rho_1}-\frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1}+\frac{1}{\rho_2}} \overset{?}{=} -\frac{J_1^2}{2\rho_1} - \frac{J_2^2}{2\rho_2}$$

Es gilt

$$-\frac{(J_1+J_2)^2}{2\cdot(\rho_1+\rho_2)} - \frac{1}{2}\cdot\frac{\left(\frac{J_1}{\rho_1}-\frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1}+\frac{1}{\rho_2}} = -\frac{1}{2}\left(\frac{(J_1+J_2)^2}{\rho_1+\rho_2} + \frac{\left(\frac{J_1}{\rho_1}-\frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1}+\frac{1}{\rho_2}}\right)$$

$$-\frac{J_1^2}{2\rho_1} - \frac{J_2^2}{2\rho_2} = -\frac{1}{2}\left(\frac{J_1^2}{\rho_1}+\frac{J_2^2}{\rho_2}\right)$$

Also gilt nur noch zu zeigen

$$\frac{(J_1+J_2)^2}{\rho_1+\rho_2} + \frac{\left(\frac{J_1}{\rho_1}-\frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1}+\frac{1}{\rho_2}} \overset{?}{=} \frac{J_1^2}{\rho_1}+\frac{J_2^2}{\rho_2}$$

Durch Ausmultiplizieren und Multiplikation mit 1 erhalten wir

$$\frac{(J_1+J_2)^2}{\rho_1+\rho_2} + \frac{\left(\frac{J_1}{\rho_1} - \frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1}+\frac{1}{\rho_2}}$$

$$= \frac{J_1^2 + 2J_1J_2 + J_2^2}{\rho_1+\rho_2} + \frac{\left(\frac{J_1}{\rho_1}\right)^2 - 2\frac{J_1 J_2}{\rho_1 \rho_2} + \left(\frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1}+\frac{1}{\rho_2}}$$

$$= \frac{J_1^2 + 2J_1J_2 + J_2^2}{\rho_1+\rho_2} + \frac{\left(\frac{J_1}{\rho_1}\right)^2 - 2\frac{J_1 J_2}{\rho_1 \rho_2} + \left(\frac{J_2}{\rho_2}\right)^2}{\frac{1}{\rho_1}+\frac{1}{\rho_2}} \cdot \frac{\rho_1\cdot\rho_2}{\rho_1\cdot\rho_2}$$

$$= \frac{J_1^2 + 2J_1J_2 + J_2^2}{\rho_1+\rho_2} + \frac{\frac{J_1^2\rho_2}{\rho_1} - 2J_1J_2 + \frac{J_2^2\cdot\rho_1}{\rho_2}}{\rho_1+\rho_2}$$

$$= \frac{J_1^2 + J_2^2 + \frac{J_1^2\cdot\rho_2}{\rho_1} + \frac{J_2^2\cdot\rho_1}{\rho_2}}{\rho_1+\rho_2}$$

$$= \frac{J_1^2\left(1+\frac{\rho_2}{\rho_1}\right) + J_2^2\left(1+\frac{\rho_1}{\rho_2}\right)}{\rho_1+\rho_2}$$

Also gilt zu zeigen

$$\frac{J_1^2\left(1+\frac{\rho_2}{\rho_1}\right) + J_2^2\left(1+\frac{\rho_1}{\rho_2}\right)}{\rho_1+\rho_2} \overset{?}{=} \frac{J_1^2}{\rho_1} + \frac{J_2^2}{\rho_2}$$

Durch Multiplizieren mit 1 auf die Rechten Seite gilt:

$$\frac{J_1^2}{\rho_1} + \frac{J_2^2}{\rho_2} = \frac{J_1^2}{\rho_1} \cdot \frac{1 + \frac{\rho_2}{\rho_1}}{1 + \frac{\rho_2}{\rho_1}} + \frac{J_2^2}{\rho_2} \cdot \frac{1 + \frac{\rho_1}{\rho_2}}{1 + \frac{\rho_1}{\rho_2}}$$

$$= \frac{J_1^2 \left(1 + \frac{\rho_2}{\rho_1}\right)}{\rho_1 + \rho_2} + \frac{J_2^2 \left(1 + \frac{\rho_1}{\rho_2}\right)}{\rho_2 + \rho_1} = \frac{J_1^2 \left(1 + \frac{\rho_2}{\rho_1}\right) + J_2^2 \left(1 + \frac{\rho_1}{\rho_2}\right)}{\rho_1 + \rho_2}$$

Damit ist die Gleichheit bewiesen