Part 3

a)

$$\pi(p|x) = \frac{\pi(x|p)\cdot\pi(p)}{\pi(x)} = \frac{\pi(x|p)\cdot\pi(p)}{\int_0^1 \pi(x|p)\cdot\pi(p)}$$

$$= \frac{\cancel{\binom{n}{k}} p^k \cdot (1-p)^{n-k} \cdot \frac{p^{\alpha-1}(1-p)^{\beta-1}}{\cancel{B(\alpha,\beta)}}}{\int_0^1 \cancel{\binom{n}{k}} p^k \cdot (1-p)^{n-k} \cdot \frac{p^{\alpha-1}(1-p)^{\beta-1}}{\cancel{B(\alpha,\beta)}}\, dp}$$

$$= \frac{p^{k+\alpha-1}\cdot(1-p)^{n-k+\beta-1}}{\int_0^1 p^{k+\alpha-1}(1-p)^{n-k+\beta-1}\, dp} = \frac{p^{k+\alpha-1}\cdot(1-p)^{n-k+\beta-1}}{B(k+\alpha, n-k+\beta)} = \mathrm{Beta}(k+\alpha, n-k+\beta)$$

Also gilt $\alpha' = \alpha+k$ und $\beta' = \beta+n-k$

b) $\mathrm{Beta}(p;1,1) = \frac{p^0(1-p)^0}{1} = \frac{1}{1} = 1$

c) $$\theta_{MAP} = \underset{\theta\in[0,1]}{\arg\max}\{p(\theta|n,k)\} = \underset{\theta\in[0,1]}{\arg\max}\{p(k|n,\theta)\cdot p(\theta)\}$$

$$= \underset{\theta\in[0,1]}{\arg\max}\{\mathrm{Beta}(\alpha+k,\beta+n-k)\}$$

$$= \underset{\theta\in[0,1]}{\arg\max}\left\{\frac{\theta^{\alpha+k-1}\cdot(1-\theta)^{\beta+n-k-1}}{B(\alpha+k,\beta+n-k)}\right\} = \underset{\theta\in[0,1]}{\arg\max}\left\{\theta^{\alpha+k-1}(1-\theta)^{\beta+n-k-1}\right\}$$

$$\frac{d}{d\theta} \; \underbrace{\theta^{\alpha+k-1}}_{u} \cdot \underbrace{(1-\theta)^{\beta+n-k-1}}_{v}$$

$$= \underbrace{(\alpha+k-1) \cdot \theta^{\alpha+k-2}}_{u'} \cdot \underbrace{(1-\theta)^{\beta+n-k-1}}_{v} + \underbrace{(\beta+n-k-1) \cdot (1-\theta)^{\beta+n-k-2} \cdot (-1)}_{v'} \cdot \underbrace{\theta^{\alpha+k-1}}_{u}$$

0 setzen

$$(\alpha+k-1) \cdot \hat{\theta}^{\alpha+k-2} \cdot (1-\hat{\theta})^{\beta+n-k-1} - (\beta+n-k-1) \cdot (1-\hat{\theta})^{\beta+n-k-2} \cdot \theta^{\alpha+k-1} = 0$$

$$(\alpha+k-1) \cdot \hat{\theta}^{\alpha+k-2} \cdot (1-\hat{\theta})^{\beta+n-k-1} = (\beta+n-k-1) \cdot (1-\hat{\theta})^{\beta+n-k-2} \cdot \theta^{\alpha+k-1}$$

$$(\alpha+k-1) \cdot 1 \cdot (1-\hat{\theta}) = (\beta+n-k-1) \quad 1 \quad \theta$$

$$(\alpha+k-1) - (\alpha\hat{\theta} + k\hat{\theta} - \hat{\theta}) = \beta\hat{\theta} + n\hat{\theta} \cancel{-k}\hat{\theta} - \hat{\theta}$$

$$\alpha+k-1 - \alpha\hat{\theta} + \hat{\theta} = \beta\hat{\theta} + n\hat{\theta} - \hat{\theta}$$

$$\alpha+k-1 = \beta\hat{\theta} + n\hat{\theta} - 2\hat{\theta} + \alpha\hat{\theta} \qquad |:\theta$$

$$\frac{\alpha+k-1}{\hat{\theta}} = \beta + n - 2 + \alpha$$

$$\hat{\theta} = \frac{\alpha+k-1}{n+\beta+\alpha-2}$$