Set Theory

Adrienne Stanley, Justice Adams, and Eric Nichols

February 19, 2019

Chapter 1

Foundations

Discussion 1. *Let $P(x)$ be a statement (eg. x is a bunny) such that $X = \{x|P(x)\}$ is the set of all sets such that $P(x)$ is true (eg. if $P(x)$ says "x is a bunny", then X is the set of all x's such that x is a bunny).*

Question 1 (Russell's Paradox)**.** *Let $Y = \{x|x \notin x\}$. Does this make sense? What does this mean about mathematics?*

Definition 1 (Partial Order)**.** *We call a relation r a* ***partial order*** *iff*

1. *r is reflexive: xrx*
2. *r is anti-symmetric: $(xry \wedge yrx) \to x = y$*
3. *r is transitive: $(xry \wedge yrz) \to xrz$*

Problem 1. *Is $\leq$ a partial order on $\mathbb{Z}^+$?*

Problem 2. *If $|$ (division) a partial order on $\mathbb{Z}^+$?*

Problem 3. *Is $\subset$ a partial order on $\mathcal{P}(x)$ (the powerset of x)?*

Problem 4. *Determine if a set can have more than one partial ordering by either finding a set with two partial orders or proving the statement false.*

Problem 5. *Is lex. (lexicographic ordering) a partial order on all "words"?*

Note: Words in this case means any string of letters. "abbazdfg" is a word.

Definition 2 (Linear Order)**.** *Let P be a set. Let $\leq$ be a partial order on P. We call P a linearly ordered set (and refer to $\leq$ as a linear order) if the following statement is true:*

$$\forall x, y \in P(x \leq y \vee y \leq x)$$

In other words, every element of the set P is comparable with every other element of the set.

Definition 3 (Strict Linear Order)**.** *Let X b a set. Let $<$ be a partial order on X. We say $(X, <)$ is a **strict linear order** if*

$$\forall x, y \in X(x = y \vee x < y \vee y < x)$$

Problem 6. *Which of the following relations from above are linear orders?*

1. $\leq$ *on* $\mathbb{Z}$
2. $|$ *on* $\mathbb{Z}^+$
3. $\subset$ *on* $\mathscr{P}(X)$

Definition 4. ***Partial Function -*** *Consider the definition of a traditional function f from X to Y which has two parts:*

1. $\forall x \in X \exists y \in Y((x, y) \in f)$
2. $\forall x \in X \forall y_1, y_2 \in Y((x, y_1), (x, y_2) \in f \rightarrow y_1 = y_2))$

A Partial function on the other hand, only requires the second statement to hold true, thus the only requirement for a partial function is the following:

$$\forall x \in X \forall y_1, y_2 \in Y((x, y_1), (x, y_2) \in f \rightarrow y_1 = y_2))$$

Problem 7. *Let X and Y be sets. Let $\mathscr{F}$ be the family of partial functions from X to Y. Let $f, g \in \mathscr{F}$. We say $f \leq g$ if $dom_g \subset dom_f$ and $f(dom_g) = g$. Is $\leq$ a partial order on $\mathscr{F}$?*

Definition 5. ***Ordered Pair -*** $\forall x \forall y((x, y) = \{\{x\}, \{x, y\}\})$

Note: $\forall x(\{x, x\} = \{x\})$

Problem 8. *Write the following sets in ordered pair notation:*

1. $\{\{1\},\{1,2\}\}$
2. $\{\{1,2\},\{1\}\}$
3. $\{\{1,2\},\{1\}\}$
4. $\{\{1\}\}$
5. $\{\{\{1\}\}\}$

Problem 9. *Write the following ordered pairs in set notation:*

1. $(1,3)$
2. $(1,1)$
3. $((3,4),(1,2))$

Definition 6 (Well Ordering)**.** *A well-ordered set X is a linearly ordered set if*

$$\forall X \subset Y (X \neq \emptyset \rightarrow X has a minimum element)$$

Definition 7 (Strict Well Ordering)**.** *Let $(X,<)$ be a strict linear order. We say $(X,<)$ is a* ***strict well ordering*** *if*

$$\forall Y \subset X (Y \neq \emptyset \rightarrow Y has a minimum element)$$

Problem 10. *Let $(X,<)$ be a strict well-ordering. Let $Y \subset X | Y \neq \emptyset$ Is Y a strictly well-ordered set?*

Problem 11. *Prove or disprove the following statement: Every finite linear ordered set is well-ordered*

Chapter 2

Axioms and Transitivity

Discussion 2 (ZFC). ***WHAT IS IT? WHAT MAKES IT COOL? SOMETHING ABOUT VARIABLES***

Axiom 1 (Extensionality).

$$\forall x, y((x = y) \leftrightarrow \forall z(z \in x \leftrightarrow z \in y))$$

It can be difficult to look at the axioms in mathematical logic because the elements themselves are sets. This could confuse some students since most similar logic involves building sets from other objects (AKA set builder notation). It may be worth mentioning that the elements in the logic are sets themselves.

Theorem 1. $(x = y) \leftrightarrow (x \subset y \wedge y \subset x)$

Axiom 2 (Pairing).

$$\forall x, y \exists z(z = \{x, y\})$$

Definition 8 (Union/Intersection). *We define the* ***union*** *and* ***intersection*** *of the set x as follows,*

$$\bigcup x = \{z | \exists y \in x(z \in y)\}$$

$$\bigcap x = \{z | \forall y \in x(z \in y)\}$$

Axiom 3 (Union).

$$\forall x \exists z(z = \bigcup x)$$

Question 2. *Let x and y be sets. Is $x \cup y$ a set?*

Problem 12. *Prove or disprove the following statement:*

$$(Y \cup X \subset X \cap Y) \leftrightarrow (x = y)$$

Problem 13. *Prove or disprove the following statement:*

$$[(X - Y) \cup (Y - X)] \cap [X \cap Y] = \emptyset$$

Problem 14. *Prove or disprove the following statement:*

$$(X \cap Y) \subset (X \cup Y) \leftrightarrow \forall b((b \in X \cap b \in Y) \rightarrow (b \in X \cup b \in Y))$$

IS THE ABOVE PROBLEM WRITTEN CORRECTLY?!

Axiom 4 (Schema of Separation)**.**

$$\forall x \forall \Phi \exists z (z = \{y \in x | \Phi y\}$$

Question 3. *Let x and y be sets. Is $x \cap y$ a set?*

Theorem 2.

$$\neg \exists X \forall Y (Y \in X)$$

(ie. there is no set containing all sets).

Problem 15. *Prove* $((x, y) = (w, z)) \leftrightarrow (x = w \wedge y = z)$

Definition 9 (Successor)**.** *Let x be a set. We define the* ***successor*** *of x, denoted $S(x)$, as follows,*

$$S(x) = x \cup \{x\}$$

Problem 16. *Why is $S(x)$ a set?*

Problem 17. *Write y, $S(y)$, $S^2(y)$, and $S^3(y)$.*

Axiom 5 (Empty Set)**.** *There exist an empty set. We denote it with $\emptyset$*

Axiom 6 (Infinity)**.** $\exists X (\emptyset \in X \wedge \forall x \in X (S(x) \in X))$

Axiom 7 (Power Set)**.** $\forall x \exists y \forall z (z \in y \leftrightarrow z \subset x)$

Axiom 8 (Replacement)**.** *Let θ be a formula so that* $\forall x, y, z(\theta(x, y) \wedge \theta(x, z) \leftrightarrow y = z$, *then* $\forall w \exists s (s = \{y \mid \exists x \in w(\theta(x, y))\}$

Chapter 3

Ordinals

Definition 10 (Transitive)**.** *A set is transitive if and only if* $\forall y \in x(y \subset x)$

Problem 18. *Determine if the following sets are Transitive:*

1. $\mathbb{N}$
2. $\mathcal{P}(\mathbb{N})$
3. $\{\{\},\{\{\}\}\},$

Problem 19. *Prove or disprove the following statement:*
X *is transitive if and only if* $\forall x \in X(\bigcup x \subset X)$

Definition 11 (Ordinal)**.** *An ordinal is a set* α *which is transitive and strictly well-ordered by* ϵ

Problem 20. *Let* $X = \{0, 1, 2, 4\}$*. Is* X *an ordinal?*

Problem 21. *Let* $X = \{\theta, \{\theta\}, \{\{\theta\}\}\}$*. Is* X *an ordinal?*

Problem 22. *Prove the following statement: Let* X *be an ordinal. Let* $\mathbb{B} \in X$*. Then* $\mathbb{B}$ *is also an ordinal.*

Problem 23. *Show that the collection of all finite ordinals is a set. That is, prove*

$$\{x | x \text{ is a finite ordinal}\}$$

is a set.

Problem 24. *Prove that* $\omega + \omega$ *is a set*

Problem 25. *Let* X *be a transitive set. Prove the following statements:*

1. $\forall y \in X(y \cap X = \emptyset \to y = \emptyset)$
2. $\forall a, x, y \in X(a \cap X = \{x, y\} \to a = \{x, y\})$
3. $\forall a, x, y \in X(a \cap X = (x, y) \to a = (x, y))$
4. $\forall a, x \in X(a \cap X = \bigcup x \to a = \bigcup x)$
5. $\forall a \in X(a \cap X$ *is a relation* $\to a$ *is a relation*$)$
6. $\forall a \in X(a \cap X$ *is a function* $\to a$ *is a function*$)$

Definition 12 (Initial Segment)**.** *Let* $(X, <)$ *be a linearly ordered set. Let* $A \subset X$*. We say* A *is an initial segment if*

$$\forall a \in A \forall x \in X(x < a \to x \in A)$$

Theorem 3. *Let* α *be an ordinal. Let* A *be an initial segment of* α*. Then* A *is an ordinal*

Definition 13 (ON)**.** *Let* ON *be the collection of all ordinals*

Lemma 1. *For each* $\alpha \in ON, \alpha \notin \alpha$

Theorem 4. *Let* α *&* β *be ordinals. Then* $\alpha = \beta$ *or* $(\alpha < \beta$ *or* $\beta < \alpha)$

Problem 26. *Let* A *be an initial segment of* α *such that* $A \neq \alpha$*. Prove* $A \in \alpha$

Problem 27. *Prove or disprove the following statement:* ON *is transitive*

Problem 28. *Let* X *be a set of ordinals. Prove* $\bigcup X \in ON$

Definition 14 (Transitive Closure)**.** *Let* X *be a set. We define the Transitive closure of* X*, denoted by* $TC(X)$ *to be*

$$TC(X) = x \cup \bigcup X \cup \bigcup \bigcup X \cup ...$$

Problem 29. *Find* $TC(X)$ *where* $X = \{0, 3, \{5, 7\}\}$

Theorem 5. *Let X be a set. Then $TC(X)$ is transitive*

Theorem 6. $\forall Y(x \in Y \wedge Y$ *is transitive* $\rightarrow TC(X) \subset Y)$

Problem 30. *Let X be transitive. Show $S(x)$ & $P(X)$ are transitive.*

Problem 31. $\forall i \in I(x_i$ *is transitive*$)$. *Show* $\bigcup_{i\in I} x_i$ *is transitive*

Definition 15 (Rank)**.** *Let $V_0 = \emptyset$. If α is an ordinal, $V_{\alpha+1} = P(V_\alpha)$. If a is a limit ordinal, $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$*

$\forall X \in \bigcup_{\alpha\in ON} V_\alpha$ *we define rank X (denoted by rank(X)) as the smallest α such that $X \subset V_\alpha$*

Problem 32. *Find the following ranks:*

1. *rank($\emptyset$)*
2. *rank(23)*
3. *rank(ω)*
4. *rank($\omega + 2$)*

Lemma 2. $\forall \alpha, \beta \in ON$

1. *V_α is a set*
2. *V_α is transitive*
3. $\alpha < \beta \rightarrow V_\alpha \subset V_\beta$
4. $\alpha < \beta \rightarrow V_\alpha \in V_\beta$

Axiom 9 (Regularity)**.** *Every set X has a least element with respect to ϵ. More formally,*

$$\forall x(x \neq \emptyset \rightarrow \exists y \in x(x \cap y = \emptyset))$$

y is called the ϵ-minimal element of x

Problem 33. *Let $X = \{3, 4, \{2\}, \{1, \{2\}\}, \{1, 4, \{2\}\}\}$. Which elements of X are minimal with respect to ϵ?*

Problem 34. *Prove the following statement:*

$$\forall X((X \neq \emptyset \wedge X \textit{ is transitive}) \to \emptyset \in X)$$

Lemma 3. *Let $x \in y$ such that both rank(x) and rank(y) exists. Then rank(x) $<$ rank(y)*

Lemma 4. $\forall \alpha \in ON(\textit{rank}(\alpha) = \alpha)$

Lemma 5. $\forall x(x \subset \bigcup_{\alpha \in ON} V_\alpha \to x \in \bigcup_{\alpha \in ON} V_\alpha)$

Lemma 6. $\forall x(x \textit{ is transitive } \to x \in \bigcup_{\alpha \in ON} V_\alpha)$

Theorem 7. $\forall x \exists \alpha \in ON(x \in V_\alpha)$

WHERE SHOULD WE DEFINE LIM AND SUCC?

Problem 35. *Let $\alpha \in LIM$. Find $\bigcup \alpha$*

Problem 36. *Let $\alpha \in SUCC$. Find $\bigcup(\alpha + 1)$*

Chapter 4

Mathematical Universe

Chapter 5

Ordinal Arithmetic

Definition 16 $(\alpha + \beta)$**.** *We define $\alpha + \beta$ as a copy of α followed bu a copy of β.*

$$\underbrace{\underbrace{\bullet \quad \bullet \quad \bullet \cdots}_{\alpha} \quad \underbrace{\bullet \quad \bullet \quad \bullet \cdots}_{\beta}}_{\alpha + \beta}$$

Problem 37. *Simplify and draw $\alpha + \beta$ where $\alpha = \omega$ and $\beta = 1$*

Problem 38. *Simplify and draw $\alpha + \beta$ where $\alpha = 1$ and $\beta = \omega$*

Question 4. *What is the difference between the previous 2 problems?*

Definition 17 (Addition)**.** $\forall \alpha, \beta \in ON$*:*

1. $\alpha + 0 = \alpha$
2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$
3. *if* $\beta \in LIM$*, then* $\alpha + \beta = \bigcup_{\gamma < \beta} (\alpha + \gamma)$

Problem 39. *Simplify and draw $\alpha + \beta$.*

1. $\alpha = 2$, $\beta = 4$
2. $\alpha = \omega$, $\beta = \omega + 1$

3. $\alpha = \omega + 1$, $\beta = \omega$

Problem 40. *Find* $\bigcup(\omega)$

Problem 41. *Let* $\alpha \in LIM$. *Find* $\bigcup(\alpha)$.

Problem 42. *Find* $\bigcup(\omega + 1)$.

Problem 43. *Let* $\alpha \in ON$. *Find* $\bigcup(\alpha + 1)$

Question 5. *What is the first Limit Ordinal after* ω*?* ***I FEEL THIS SHOULD HAVE BE PLACED EARLIER IN THE SEQUENCE, ALSO WE NEED TO DEFINE ON, LIM, AND SUCC****.*

Problem 44. *For every* $\alpha \in ON$, *show that* $0 + \alpha = \alpha$.

Question 6. *Let* $\alpha, \beta, \delta \in ON$ *such that* $\beta < \delta$. *Decide if the following are true or false. If true prove it. If false provide a counter-example.*

1. $\alpha + \beta < \alpha + \delta$
2. $\alpha + \beta \leq \alpha + \delta$
3. $\beta + \alpha < \delta + \alpha$
4. $\beta + \alpha \leq \delta + \alpha$

Lemma 7. *Let* $\alpha, \beta \in ON$. *Suppose* $\beta \in LIM$. *Prove* $\alpha + \beta \in LIM$.

Lemma 8. *Let* $\alpha, \beta, \gamma, \delta \in ON$. *Prove* $\displaystyle\bigcup_{\delta < (\beta + \gamma)} (\alpha + \delta) = \bigcup_{\beta \leq \delta < (\beta + \gamma)} (\alpha + \delta)$.

Lemma 9. *Let* $\beta \leq \delta < \beta + \gamma$. *Prove that there exists* $\mu \in [0, \gamma)$ *such that* $\delta = \beta + \mu$.

Theorem 8. *Let* $\alpha, \beta, \gamma \in ON$. *Prove* $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$. *(Hint: Use Transfinite Induction).*

Definition 18 $(\alpha \cdot \beta)$**.** *We define* $\alpha \cdot \beta$ *as* β*-many copies of* α.

Example 1. $\omega \cdot 2 = \omega + \omega$ *and* $2 \cdot \omega = \omega$.

Definition 19 (Multiplication)**.** $\forall \alpha, \beta \in ON$*:*

1. $\alpha \cdot 0 = 0$

2. $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$

3. *if* $\beta \in LIM$, *then* $\alpha \cdot \beta = \bigcup_{\gamma < \beta} (\alpha \cdot \gamma)$

Problem 45. *Simplify and draw* $\alpha \cdot \beta$

1. $\alpha = 2, \beta = 3$

2. $\alpha = \omega, \beta = \omega_1 + 1$

3. $\alpha = \omega_1 + 1 = \beta = \omega$

Problem 46. *Prove* $\forall \alpha \in ON (0 \cdot \alpha = 0)$.

Question 7. *Let* $\alpha, \beta, \delta \in ON$ *such that* $\beta < \delta$. *Decide if the following are true or false. If true prove it. If false provide a counter-example.*

1. $\alpha \cdot \beta < \alpha \cdot \delta$

2. $\alpha \cdot \beta \leq \alpha \cdot \delta$

3. $\beta \cdot \alpha < \delta \cdot \alpha$

4. $\beta \cdot \alpha \leq \delta \cdot \alpha$

Lemma 10. *Let* $\alpha, \beta \in ON$ *with* $\beta \in LIM$, *then* $(\alpha \cdot \beta) \in LIM$.

Question 8. *If* $\alpha, \beta \in ON$ *and* $\beta \in LIM$, *then must* $(\beta \cdot \alpha) \in LIM$?

Lemma 11. *Let* $\alpha, \beta, \gamma \in ON$. *Then* $\bigcup_{\delta < (\beta + \gamma)} (\alpha \cdot \delta) = \bigcup_{\beta \leq \delta < (\beta + \gamma)} (\alpha \cdot \delta)$.

Problem 47. *Simplify the following expressions:*

1. $2 \cdot (\omega + 1)$

2. $(2 \cdot \omega) + (2 \cdot 1)$

3. $(\omega + 1) \cdot 2$

4. $(\omega \cdot 2) + (1 \cdot 2)$

Theorem 9. *Let $\alpha, \beta, \gamma \in ON$. Then $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.*

Theorem 10. $\forall \alpha, \beta, \gamma \in ON[\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma]$

Definition 20 (Exponentiation)**.** *$\forall \alpha, \beta \in ON$:*

1. $\alpha^0 = 1$
2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$
3. *if $\beta \in LIM$, then* $\alpha^\beta = \bigcup_{\gamma<\beta}(\alpha^\gamma)$

Problem 48. *List the following in increasing order. Indicate which are infact equal.*

$$\omega^{\omega_1}, 3^\omega, \omega^\omega, \omega_1 + \omega_2, \omega, \omega_1, \omega + \omega_1, \omega^3, \omega \cdot 3, \omega \cdot \omega_1, \omega_1 \cdot \omega$$

Problem 49. *Let $\alpha \in ON$. Then $1 \cdot \alpha = \alpha \cdot 1 = \alpha$.*

Problem 50. *Let $\alpha = \omega_1$ and $\beta = \omega_1 + \omega$. Can we find a $\mu_1, \mu_2 \in ON$ such that,*

1. $\alpha + \mu_1 = \beta$
2. $\mu_2 + \alpha = \beta$

Lemma 12. *Let $\alpha, \beta \in ON$ such that $\alpha < \beta$. Then there exists a $\gamma \in ON$ such that $\alpha + \gamma = \beta$.*

Lemma 13. *Let $\alpha, \beta, \delta \in ON$ such that $\alpha + \beta = \alpha + \gamma$. Then $\beta = \gamma$.*

Problem 51. *Let $\alpha, \beta, \delta \in ON$ such that $\beta + \alpha = \gamma + \alpha$. Is it necessarily the case that $\beta = \gamma$?*

Problem 52. *Let $\alpha, \beta, \delta \in ON$ For each of the following is it necessarily the case that $\beta = \gamma$?*

1. $\alpha \cdot \beta = \alpha \cdot \gamma$
2. $\beta \cdot \alpha = \gamma \cdot \alpha$

Lemma 14. *Let $0 < \alpha \leq \beta$ be ordinals. Then there exists ordinals δ and γ such that $\gamma < \alpha$ and $\beta = \alpha \cdot \delta + \gamma$.*

Problem 53. *For the following find $\alpha + \beta$.*

1. $\alpha = \omega$, $\beta = \omega_1 + 1$
2. $\alpha = \omega \cdot 2$, $\beta = \omega \cdot \omega$
3. $\alpha = \omega_1$, $\beta = \omega_1 \cdot 2$

Lemma 15. *Let $\beta \in LIM$ and $\delta \in ON$ such that $\delta < \beta$. Then $\beta = \bigcup_{\delta \leq \gamma < \beta} (\gamma)$.*

Lemma 16. *Let $\alpha, \beta, \delta \in ON$ such that $\beta \in LIM$ and $\delta < \beta$. Then $\alpha + \beta = \bigcup_{\gamma < \beta} (\alpha + \gamma) = \bigcup_{\delta \leq \gamma < \beta} (\alpha + \gamma)$*

Lemma 17. *Let $\alpha, \beta \in \omega$ such that $\alpha \in LIM$. Then*

$$(\alpha + \beta = \beta) \leftrightarrow (\beta \geq \alpha \cdot \omega)$$

.

Definition 21 (Continuous)**.** *A class function (ie. A function from a class to a class) $f : ON \to ON$ is* ***continuous*** *if,*

1. *f is non-decreasing*
2. *$f(\alpha) = sup\{f(\beta) | \beta < \alpha\}$*

Problem 54. *Which of the following functions are continuous?*

1. $f_1 = \alpha + \omega$
2. $f_2 = \omega + \alpha$
3. $f_3 = \alpha \cdot \omega$
4. $f_4 = \omega \cdot \alpha$

Lemma 18. *Let $f : ON \to ON$ be strictly increasing. Let $\alpha \in ON$. Then $f(\alpha) \geq \alpha$. (Hint: BWOC let α be the least bad guy. i.e. let α be the least such that $f(\alpha) < \alpha$.*

Lemma 19. *Let* $(\alpha_n)_{n\in\omega}$ *be a non-decreasing sequence. Let* f *be non-decreasing function. Let* $\beta = sup\{\alpha_n | n \in \omega\}$. *Then,*

$$sup\{f(\alpha_n)|n \in \omega\} = sup\{f(\gamma)|\gamma < \beta\}$$

INSTRUCTORS NOTE: *This theorem holds for any sequence. The use of non-decreasing in this lemma is to help guide the proof of the following theorem.*

Problem 55. *Let* $f_1(\alpha) = \omega_1 + \alpha$. *Find* $\beta \in ON$ *such that* $f_1(\beta) = \beta$.

Theorem 11. *Let* f *be a strictly increasing continuous function. Show that* f *has fixed point. (i.e.* $\exists\beta \in ON(f(\beta) = \beta)$*).*

(Hint: Construct $(\alpha_n)_{n\in\omega}$. *Consider the two cases of* $\beta = sup\{\alpha_n|n \in \omega\}$ *,* $\beta \in LIM$ *or* $\beta \notin LIM$*).*

Problem 56. *Let* $f_2(\alpha) = \omega_1 \cdot \alpha$. *Find* $\beta > \omega_1^\omega$ *such that* $f_2(\beta) = \beta$.

Theorem 12. *Let* f *be a strictly increasing function which is continuous. Then* f *has many arbitrarily high fixed points.*

$$\forall\alpha \in ON\exists\beta > \alpha(f(\beta) = \beta)$$

Chapter 6

The Axiom of Choice

Definition 22 (Axiom of Choice)**.** *The* ***Axiom of Choice****, denoted AC, states that for every non-empty family $\{x_i | i \in I\}$ of non-empty sets,*

$$\Pi_{i \in I}(x_i)$$

is non-empty.

Definition 23 (Well-Ordering Principle)**.** *For every non-empty set X, there exists an ordinal α and a function $f : \alpha \to X$ such that f is a bijection*

Definition 24 (Chain)**.** ***WE NEED A FORMAL DEFINITION***

Definition 25 (Zorn's Lemma)**.** ***Zorn's Lemma****, denoted ZL, states: Let $\mathbb{P}$ be a non-empty partial order such that every chain has an upper-bound. Then $\mathbb{P}$ has a maximal element.*

Lemma 20. *Let $\{x_i | i \in I\}$ be a non-empty family of non-empty sets. Let*

$$\mathbb{P} = \{f | dom(f) \subset I \wedge \forall i \in dom(f)(f(i) \in X_i$$

Show that $\mathbb{P}$ ordered by inclusion is a partial order.

Lemma 21. *From above. Let $\mathbb{C}$ be a chain in $\mathbb{P}$. Show $\bigcup \mathbb{C} \in \mathbb{P}$.*

Theorem 13. *$ZL \to AC$.*

Chapter 7

Cardinal Arithmetic

Chapter 8

Clubs and Stationary Sets

Chapter 9

Tree

Chapter 10

Comments

Comments

1. We should specify if and iff relationships.
2. The following are in notes, but not really used: Comparable, Compatible, Chain, Anti-Chain, Minimal/Minimum, Maximal/Maximum.

CommentsFromJustice

1. Before Discussion 1, should we define set builder notation? Or is that expected knowledge? I can see it both ways. It's technically the first thing they'll see, so it might be beneficial to remind them.
2. Before the definition of Partial order, should we define relation? Similar to my first question about what's expected.
3. Should we combine certain problems, such as problems 1-3 into one problem with 3 parts?
4. We should specify if and iff relationships. I second this –
5. (Problem 5): Is extending functions really important? I ran into this in the first semester, I omitted it because I wasn't really sure if it had significance.
6. Can I remove the question from the definition of linear order. It seems worthwhile to me to keep it?

7. I still need to define well-ordering problems (might e-mail eric with this), but do I need to define least ordering as part of foundations?

8. I'm about to start the Axioms section, but how should they be ordered? Is there a preference. If not, I can complete it pretty quickly.

9. In certain cases, it may be better to wait to define things until their relevant axioms, things such as Ordinal, Transitive, or Finite sets

$CommentsFromJusticeAboutAxiomssection$

1. very first line... What should go here? It's a discussion?

2. do we need a more formal definition for the pairing axiom? like from the book?

3. problem 14 confuses me... did I write this correctly? If so, send me the solution :) It's bothering me..

4. Transitive and Finite, should these be introduced in this document? perhaps in the "foundations" section? For now, they are omitted

5. Any other items/problems that we think we should add to the Axioms section?

$CommentsFromJusticeAboutOrdinalssection$

1. Someone double check problem 24 for me (that it's all written correctly and makes sense)

2. where does the axiom of regularity live? (in axioms, ordinals?)

3. Rank: Should it live in ordinals?

4. I see eric put a note about Transfinite induction. Should that be defined. If so, where?

5. Axiom of regularity. I included two definitions. Should I have added both? or just "Every set X has a least element with respect to epsilon". I was confused what epsilon was, so I added the formal definition

6. At the very end I have questions regaurding LIM and SUCC. I'm not quite sure where to define those.