Set Theory

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Chapter 1

Foundations

The naive definition of a set is any collection of objects. Let $P(x)$ be a statement in mathematical logic. We can define a set X to be the collection of all objects which satisfy $P(x)$. That is,

$$X = \{x | P(x)\}$$

Discussion 1. *(Russell's Paradox) Let $X = \{x | x \notin X\}$. Is $X \in X$?*

Russell's Paradox tells us that the naive definition of a set is insufficient. We need a more nuanced definition. In Chapter 2, we will choose a small collection of axioms (known as ZFC) that will define what it means to be a set. In Chapter 1, we start with some fundamental concepts.

Definition 1 (Partial Order)**.** *We call a relation $\leq$ a* ***partial order*** *on a set X iff*

1. *r is reflexive: $\forall x \in X(x \leq x)$*

2. *r is anti-symmetric: $\forall x, y \in X((x \leq y \wedge y \leq x) \to x = y)$*

3. *r is transitive: $\forall x, y, z \in X((x \leq y \wedge y \leq z) \to x \leq z)$*

Problem 1. *Is $\leq$ a partial order on $\mathbb{Z}^+$?*

Problem 2. *Is $|$ (divides) a partial order on $\mathbb{Z}^+$?*

Problem 3. *Is $\subset$ a partial order on $\mathcal{P}(x)$ (the powerset of x) for any given set x?*

Problem 4. *Determine if a set can have more than one partial ordering by either finding a set with two partial orders or proving the statement false.*

Definition 2 (Linear Order)**.** *Let P be a set. Let $\leq$ be a partial order on P. We call P a* ***linearly ordered set*** *(and refer to $\leq$ as a linear order) if the following statement is true:*

$$\forall x, y \in P(x \leq y \vee y \leq x).$$

In other words, every element of the set P is comparable with every other element of the set.

Definition 3 (Strict Linear Order)**.** *Let X be a set. Let $<$ be a partial order on X. We say $(X, <)$ is a* ***strict linear order*** *if*

$$\forall x, y \in X(x = y \vee x < y \vee y < x).$$

Problem 5. *Which of the following relations from above are linear orders?*

1. $\leq$ *on* $\mathbb{Z}$

2. $|$ *on* $\mathbb{Z}^+$

3. $\subset$ *on* $\mathcal{P}(X)$

Definition 4 (Function)**.** *Let X and Y be sets. We say $f \subset X \times Y$ is a* ***function*** *from X to Y if,*

1. $\forall x \in X \exists y \in Y((x, y) \in f)$

2. $\forall x \in X \forall y_1, y_2 \in Y((x, y_1), (x, y_2) \in f \to y_1 = y_2)$

Definition 5 (Ordered Pair)**.** *For all x and y we define the* ***Ordered Pair*** *of x and y as $\{\{x\}, \{x, y\}\}$ and we denote it as (x, y).*

Notice $\{x, x\} = \{x\}$. So the ordered pair

$$(x, x) = \{\{x\}, \{x, x\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}.$$

Problem 6. *Write the following sets in ordered pair notation:*

1. $\{\{1\}, \{1, 2\}\}$

2. $\{\{1\}\}$

3. $\{\{\{1\}\}\}$

Problem 7. *Write the following ordered pairs in set notation:*

1. $(1, 3)$

2. $(1, 1)$

3. $((3, 4), (1, 2))$

Definition 6 (Well Ordering)**.** *A* ***well-ordered*** *set X is a linearly ordered set where every non-empty subset of X has a minimal element. That is,*

$$\forall Y \subset X (Y \neq \emptyset \to Y \text{ has a minimum element}).$$

Definition 7 (Strict Well Ordering)**.** *Let $(X, <)$ be a strict linear order. We say $(X, <)$ is a* ***strict well ordering*** *if*

$$\forall Y \subset X (Y \neq \emptyset \to Y \text{ has a minimum element}).$$

Problem 8. *Let $(X, <)$ be a strict well-ordering. Let $Y \subset X$ such that $Y \neq \emptyset$. Is Y a strictly well-ordered set?*

Problem 9. *Prove or disprove the following statement: Every finite linear ordered set is well-ordered.*

Chapter 2

Axioms and Transitivity

We discovered in Chapter 1 that we needed a deeper definition of a set. Now it's time define our 9 axioms of ZFC.

Axiom 1 (Set Existence)**.** *There is a set. That is,*

$$\exists x(x = x).$$

Axiom 2 (Extensionality)**.** *Two sets x and y are equal if they have the same elements. That is,*

$$\forall x, y((x = y) \leftrightarrow \forall z(z \in x \leftrightarrow z \in y)).$$

Theorem 1. *For any two sets x and y, $(x = y) \leftrightarrow (x \subset y \wedge y \subset x)$.*

Axiom 3 (Pairing)**.** *If x and y are sets, then $\{x, y\}$ is a set. That is,*

$$\forall x, y \exists z(z = \{x, y\}).$$

Definition 8 (Union/Intersection)**.** *We define the **union** and **intersection** of a set x as follows,*

$$\bigcup x = \{z | \exists y \in x(z \in y)\}$$

and

$$\bigcap x = \{z | \forall y \in x(z \in y)\}.$$

Axiom 4 (Union)**.** *For any set* x, $\bigcup x$ *is a set. That is,*

$$\forall x \exists z (z = \bigcup x).$$

Question 1. *Let* x *and* y *be sets. Is* $x \cup y$ *a set?*

Problem 10. *Prove or disprove the following statement. Let* X *and* Y *be sets. Then,*

$$(Y \cup X \subset X \cap Y) \leftrightarrow (X = Y).$$

Problem 11. *Prove or disprove the following statement. For all sets* X *and* Y,

$$[(X - Y) \cup (Y - X)] \cap [X \cap Y] = \emptyset.$$

Axiom 5 (Schema of Separation)**.** *Given any set* x *and any condition, then the sub-collection of all elements of* x *which satisfy that condition is also a set. That is,*

$$\forall x \forall \phi \exists z (z = \{y \in x | \phi(y)\}.$$

Question 2. *Let* x *and* y *be sets. Is* $x \cap y$ *a set?*

Problem 12. *The empty set is a set.*

Theorem 2. *There is no set containing all sets. That is,*

$$\neg \exists X \forall Y (Y \in X).$$

Problem 13. *Prove that* $(x, y) = (w, z)$ *iff* $x = w$ *and* $y = z$. *That is,*

$$((x, y) = (w, z)) \leftrightarrow (x = w \wedge y = z).$$

Definition 9 (Successor)**.** *Let* x *be a set. We define the* ***successor*** *of* x, *denoted* $S(x)$, *as follows,*

$$S(x) = x \cup \{x\}.$$

Problem 14. *Why is* $S(x)$ *a set?*

Problem 15. *Given a set* y, *express* $S(y)$, $S^2(y)$, *and* $S^3(y)$.

Axiom 6 (Infinity)**.** *There exist an infinite set. That is,*

$$\exists X(\emptyset \in X \wedge \forall x \in X(S(x) \in X)).$$

.

Axiom 7 (Power Set)**.** *Given any set x, the powerset of x is a set. That is,*

$$\forall x \exists y \forall z (z \in y \leftrightarrow z \subset x).$$

Problem 16. *Show that the collection of all finite ordinals is a set. That is, prove $\{x | x\ is\ a\ finite\ ordinal\}$ is a set.*

Axiom 8 (Replacement)**.** *If θ is a well-defined function, then for any set w the domain of, the range of $Theta$ on w is a set (ie. The range of the a function is a set). That is,*

$$\forall x, y, z(\theta(x, y) \wedge \theta(x, z) \leftrightarrow y = z) \to (\forall w \exists s(s = \{y | \exists x \in w(\theta(x, y))\})).$$

Chapter 3

Ordinals

Definition 10 (Transitive)**.** *A set x is* ***transitive*** *if and only if for all $y \in x (y \subset x)$.*

Problem 17. *Determine if the following sets are Transitive:*

1. $\mathbb{N}$
2. $\mathcal{P}(\mathbb{N})$
3. $\{\{\}, \{\{\}\}\}$,

Problem 18. *Prove or disprove the following statement:*
X is transitive if and only if for all $x \in X (\bigcup x \subset X)$.

Definition 11 (Ordinal)**.** *An* ***ordinal*** *is a set α which is transitive and strictly well-ordered by ϵ.*

Problem 19. *Let $X = \{0, 1, 2, 4\}$. Is X an ordinal?*

Problem 20. *Let $X = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}\}$. Is X an ordinal?*

Lemma 1. *Let X be an ordinal. Let $x \in X$. Then x is also an ordinal.*

Problem 21. *Prove that $\omega + \omega$ is a set.*

Problem 22. *Let X be a transitive set. Prove the following statements:*

1. $\forall y \in X (y \cap X = \emptyset \to y = \emptyset)$

2. $\forall a, x, y \in X(a \cap X = \{x, y\} \to a = \{x, y\})$

3. $\forall a, x, y \in X(a \cap X = (x, y) \to a = (x, y))$

4. $\forall a, x \in X(a \cap X = \bigcup x \to a = \bigcup x)$

5. $\forall a \in X(a \cap X \; is \; a \; relation \to \; a \; is \; a \; relation)$

6. $\forall a \in X(a \cap X \; is \; a \; function \to \; a \; is \; a \; function)$

Definition 12 (Initial Segment)**.** *Let $(X, <)$ be a linearly ordered set. Let $A \subset X$. We say A is an* ***initial segment*** *of X if*

$$\forall a \in A \forall x \in X(x < a \to x \in A).$$

.

Theorem 3. *Let α be an ordinal. Let A be an initial segment of α. Then A is an ordinal.*

Definition 13 (ON)**.** *Let ON be the collection of all ordinals.*

Lemma 2. *For each $\alpha \in ON, \alpha \notin \alpha$.*

Theorem 4. *Let α and β be ordinals. Then $\alpha = \beta$ or ($\alpha < \beta$ or $\beta < \alpha$).*

Problem 23. *Let A be an initial segment of an ordinal α such that $A \neq \alpha$. Prove $A \in \alpha$.*

Problem 24. *Prove or disprove the following statement: ON is transitive.*

Problem 25. *Let X be a set of ordinals. Prove $\bigcup X \in ON$.*

Definition 14 (Transitive Closure)**.** *Let X be a set. We define the* ***Transitive Closure*** *of X, denoted by $TC(X)$ to be*

$$TC(X) = X \cup \bigcup X \cup \bigcup \bigcup X \cup ...$$

Problem 26. *Find $TC(X)$ where $X = \{0, 3, \{5, 7\}\}$.*

Theorem 5. *Let X be a set. Then $TC(X)$ is transitive.*

Theorem 6. *For each set Y, if $X \in Y$ and Y is transitive, then $TC(X) \subset Y$.*

Problem 27. *Let X be transitive. Show $S(x)$ and $\mathcal{P}(X)$ are transitive.*

Problem 28. *Suppose for each $i \in I$, x_i is transitive. Show $\bigcup_{i\in I} x_i$ is transitive.*

Axiom 9 (Regularity)**.** *Every set X has a least element with respect to ϵ. That is,*

$$\forall X(X \neq \emptyset \to \exists y \in X(X \cap y = \emptyset)).$$

We call y the ϵ-minimal element of X

Problem 29. *Let $X = \{3, 4, \{2\}, \{1, \{2\}\}, \{1, 4, \{2\}\}\}$. Which elements of X are minimal with respect to ϵ?*

Problem 30. *Prove the following statement:*

$$\forall X((X \neq \emptyset \wedge X \textit{ is transitive}) \to \emptyset \in X).$$

WHERE SHOULD WE DEFINE LIM AND SUCC?

Problem 31. *Let $\alpha \in LIM$. Find $\bigcup \alpha$.*

Problem 32. *Let $\alpha \in SUCC$. Find $\bigcup(\alpha + 1)$.*

Chapter 4

Mathematical Universe

Definition 15 (V_α)**.** *Let $V_0 = \emptyset$. If α is an ordinal, $V_{\alpha+1} = P(V_\alpha)$. If a is a limit ordinal, $V_\alpha = \bigcup_{\beta<\alpha} V_\beta$*

Definition 16 (Rank)**.** *For each $X \in \bigcup_{\alpha\in ON} V_\alpha$ we define the **rank** of X (denoted by rank(X)) as the least α such that $X \subset V_\alpha$*

Problem 33. *Find the following ranks:*

1. *rank($\emptyset$)*
2. *rank(23)*
3. *rank(ω)*
4. *rank($\omega + 2$)*

Lemma 3. *For each $\alpha, \beta \in ON$*

1. *V_α is a set*
2. *V_α is transitive*
3. *$\alpha < \beta \rightarrow V_\alpha \subset V_\beta$*
4. *$\alpha < \beta \rightarrow V_\alpha \in V_\beta$*

Lemma 4. *Let $x \in y$ such that both rank(x) and rank(y) exists. Then $rank(x) < rank(y)$.*

Lemma 5. *For every ordinal α, the rank of α is α. That is,*

$$\forall \alpha \in ON(rank(\alpha) = \alpha).$$

Lemma 6. *For each set x, $x \subset \bigcup_{\alpha \in ON} V_\alpha$ implies $x \in \bigcup_{\alpha \in ON} V_\alpha$*

Lemma 7. *For each set x, x is transitive implies $x \in \bigcup_{\alpha \in ON} V_\alpha$*

Theorem 7. *For each set x, there exists $\alpha \in ON$ such that $x \in V_\alpha$. That is,*

$$\forall x \exists \alpha (x \in V_\alpha)$$

Problem 34. *Let A, B be sets. Let $f : A \to B$ be injective. Let $g : B \to A$ be injective. Prove there exists $\hat{A} \subset A$ such that $a \in \hat{A}$ if and only if when we continually take*

$$a, g^{-1}(a), f^{-1}(g^{-1}(a)), g^{-1}(f^{-1}(g^{-1}(a))), ...$$

we get a finite sequence that ends in an element of B.

Problem 35. *Let A, B be sets. Let $f : A \to B$ be injective. Let $g : B \to A$ be injective. Let $\hat{A}$ be defined as in the previous problem. Let $h : A \to B$ be defined by*

$$h(a) = \begin{cases} g^{-1}(a) & a \in \hat{A} \\ f(a) & a \notin \hat{A}. \end{cases}$$

Prove that h is one-to-one and onto.

Definition 17 (Bijection and Injective)**.** *Let X and Y be sets.*

1. $|X| = |Y| \leftrightarrow \exists f : X \to Y$ *such that f is a **bijection**.*
2. $|X| \leq |Y| \leftrightarrow \exists f : X \to Y$ *such that f is **injective**.*

Lemma 8. *If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$*

Definition 18 (Countable and Uncountable)**.** *Let X be a set.*

1. *We say X is* ***countable*** *if $|X| \leq \omega$.*

2. *We say X is* ***uncountable*** *if X is not countable.*

Problem 36. *Prove the following:*

1. *$\mathbb{Q}$ is countable.*

2. *$\mathbb{R}$ is uncountable.*

3. $\forall X(|X| < |\mathcal{P}(X)|)$

4. $\forall x \exists y(|x| < |y|)$

5. *$|x| \leq |y| \rightarrow \exists f : y \rightarrow x$ such that f is onto.*

Lemma 9. *Let $X = \{Q_n | n \in \omega\}$ be a family of countable sets. Then $\bigcup X$ is countable.*

Chapter 5

Ordinal Arithmetic

Definition 19 (Ordinal Addition)**.** *We define the **ordinal addition** of $\alpha+\beta$ as a copy of α followed by a copy of β.*

$$\underbrace{\underbrace{\bullet\quad\bullet\quad\bullet\cdots}_{\alpha}\quad\underbrace{\bullet\quad\bullet\quad\bullet\cdots}_{\beta}}_{\alpha+\beta}$$

Problem 37. *Draw $\alpha+\beta$ where $\alpha=\omega$ and $\beta=1$. Simplify if possible.*

Problem 38. *Draw $\alpha+\beta$ where $\alpha=1$ and $\beta=\omega$. Simplify if possible.*

Question 3. *What is the difference between the previous 2 problems?*

Problem 39. *Find $\bigcup(\omega)$.*

Problem 40. *Let $\alpha \in LIM$. Find $\bigcup(\alpha)$.*

Problem 41. *Find $\bigcup(\omega+1)$.*

Problem 42. *Let $\alpha \in ON$. Find $\bigcup(\alpha+1)$.*

Definition 20 (Addition)**.** *We define the **addition** of ordinals with the following properties. For every $\alpha,\beta \in ON$,*

1. $\alpha+0=\alpha$

2. $\alpha + (\beta + 1) = (\alpha + \beta) + 1$

3. *If* $\beta \in LIM$*, then* $\alpha + \beta = \bigcup_{\gamma<\beta} (\alpha + \gamma)$.

Problem 43. *Draw* $\alpha + \beta$*. Simplify if possible.*

1. $\alpha = 2,\ \beta = 4$

2. $\alpha = \omega,\ \beta = \omega + 1$

3. $\alpha = \omega + 1,\ \beta = \omega$

Problem 44. *For every* $\alpha \in ON$*, show that* $0 + \alpha = \alpha$.

Question 4. *Let* $\alpha, \beta, \delta \in ON$ *such that* $\beta < \delta$*. Decide if the following are true or false. If true, prove it. If false, provide a counter-example.*

1. $\alpha + \beta < \alpha + \delta$

2. $\alpha + \beta \leq \alpha + \delta$

3. $\beta + \alpha < \delta + \alpha$

4. $\beta + \alpha \leq \delta + \alpha$

Lemma 10. *Let* $\alpha, \beta \in ON$*. Suppose* $\beta \in LIM$*. Prove* $\alpha + \beta \in LIM$.

Lemma 11. *Let* $\alpha, \beta, \gamma, \delta \in ON$*. Prove* $\bigcup_{\delta<(\beta+\gamma)} (\alpha + \delta) = \bigcup_{\beta \leq \delta < (\beta+\gamma)} (\alpha + \delta)$.

Lemma 12. *Let* $\beta \leq \delta < \beta + \gamma$*. Prove that there exists* $\mu \in [0, \gamma)$ *such that* $\delta = \beta + \mu$.

Theorem 8. *Let* $\alpha, \beta, \gamma \in ON$*. Prove* $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$*. (Hint: Use Transfinite Induction).*

Definition 21 (Ordinal Multiplication)**.** *We define the* ***Ordinal Multiplication*** *of* $\alpha \cdot \beta$ *as* β*-many copies of* α*. The following image may help you visualize* $\alpha \cdot \beta$

$$\underbrace{\underbrace{\bullet\ \bullet\ \bullet \cdots}_{\alpha} \quad \underbrace{\bullet\ \bullet\ \bullet \cdots}_{\alpha} \quad \underbrace{\bullet\ \bullet\ \bullet \cdots}_{\alpha} \quad \cdots}_{\beta \textit{ many copies of } \alpha}$$

Definition 22 (Multiplication)**.** *We define the **multiplication** of ordinals with the following properties. For all $\alpha, \beta \in ON$:*

1. $\alpha \cdot 0 = 0$
2. $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$
3. *if* $\beta \in LIM$, *then* $\alpha \cdot \beta = \bigcup_{\gamma < \beta} (\alpha \cdot \gamma)$

Problem 45. *Draw $\alpha \cdot \beta$. Simplify if possible.*

1. $\alpha = 2, \beta = 3$
2. $\alpha = \omega, \beta = 2$
3. $\alpha = 2, \beta = \omega$
4. $\alpha = \omega, \beta = \omega_1 + 1$
5. $\alpha = \omega_1 + 1 = \beta = \omega$

Problem 46. *Prove for every $\alpha \in ON, 0 \cdot \alpha = 0$.*

Question 5. *Let $\alpha, \beta, \delta \in ON$ such that $\beta < \delta$. Decide if the following are true or false. If true prove it. If false provide a counter-example.*

1. $\alpha \cdot \beta < \alpha \cdot \delta$
2. $\alpha \cdot \beta \leq \alpha \cdot \delta$
3. $\beta \cdot \alpha < \delta \cdot \alpha$
4. $\beta \cdot \alpha \leq \delta \cdot \alpha$

Question 6. *Let $\alpha, \beta \in ON$ with $\beta \in LIM$, then must $(\alpha \cdot \beta) \in LIM$?*

Question 7. *If $\alpha, \beta \in ON$ and $\beta \in LIM$, then must $(\beta \cdot \alpha) \in LIM$?*

Lemma 13. *Let $\alpha, \beta, \gamma \in ON$. Then $\bigcup\limits_{\delta<(\beta+\gamma)} (\alpha \cdot \delta) = \bigcup\limits_{\beta \leq \delta<(\beta+\gamma)} (\alpha \cdot \delta)$.*

Problem 47. *Simplify the following expressions:*

1. $2 \cdot (\omega + 1)$
2. $(2 \cdot \omega) + (2 \cdot 1)$
3. $(\omega + 1) \cdot 2$
4. $(\omega \cdot 2) + (1 \cdot 2)$

Theorem 9. *Let $\alpha, \beta, \gamma \in ON$. Then $\alpha \cdot (\beta + \gamma) = (\alpha \cdot \beta) + (\alpha \cdot \gamma)$.*

Theorem 10. *Let $\alpha, \beta, \gamma \in ON$ Then $\alpha \cdot (\beta \cdot \gamma) = (\alpha \cdot \beta) \cdot \gamma$.*

Definition 23 (Exponentiation)**.** *We define the **exponentiation** of ordinals with the following properties. For all $\alpha, \beta \in ON$:*

1. $\alpha^0 = 1$
2. $\alpha^{\beta+1} = \alpha^\beta \cdot \alpha$
3. *If $\beta \in LIM$, then $\alpha^\beta = \bigcup\limits_{\gamma<\beta} (\alpha^\gamma)$.*

Problem 48. *List the following in increasing order. Indicate which are infact equal.*

$$\omega^{\omega_1},\ 3^\omega,\ \omega^\omega,\ \omega_1 + \omega_2,\ \omega,\ \omega_1,\ \omega + \omega_1,\ \omega^3,\ \omega \cdot 3,\ \omega \cdot \omega_1,\ \omega_1 \cdot \omega$$

Lemma 14. *Let $\alpha \in ON$. Then $1 \cdot \alpha = \alpha \cdot 1 = \alpha$.*

Problem 49. *Let $\alpha = \omega_1$ and $\beta = \omega_1 + \omega$. Can we find a $\mu_1, \mu_2 \in ON$ such that,*

1. $\alpha + \mu_1 = \beta$

2. $\mu_2 + \alpha = \beta$?

Lemma 15. *Let $\alpha, \beta \in ON$ such that $\alpha < \beta$. Then there exists a $\gamma \in ON$ such that $\alpha + \gamma = \beta$.*

Lemma 16. *Let $\alpha, \beta, \delta \in ON$ such that $\alpha + \beta = \alpha + \gamma$. Then $\beta = \gamma$.*

Problem 50. *Let $\alpha, \beta, \delta \in ON$ such that $\beta + \alpha = \gamma + \alpha$. Is it necessarily the case that $\beta = \gamma$?*

Problem 51. *Let $\alpha, \beta, \delta \in ON$ For each of the following is it necessarily the case that $\beta = \gamma$?*

1. $\alpha \cdot \beta = \alpha \cdot \gamma$
2. $\beta \cdot \alpha = \gamma \cdot \alpha$

Lemma 17. *Let $0 < \alpha \leq \beta$ be ordinals. Then there exists ordinals δ and γ such that $\gamma < \alpha$ and $\beta = \alpha \cdot \delta + \gamma$.*

Problem 52. *For the following find $\alpha + \beta$.*

1. $\alpha = \omega$, $\beta = \omega_1 + 1$
2. $\alpha = \omega \cdot 2$, $\beta = \omega \cdot \omega$
3. $\alpha = \omega_1$, $\beta = \omega_1 \cdot 2$

Lemma 18. *Let $\beta \in LIM$ and $\delta \in ON$ such that $\delta < \beta$. Then $\beta = \bigcup_{\delta \leq \gamma < \beta} (\gamma)$.*

Lemma 19. *Let $\alpha, \beta, \delta \in ON$ such that $\beta \in LIM$ and $\delta < \beta$. Then $\alpha + \beta = \bigcup_{\gamma < \beta} (\alpha + \gamma) = \bigcup_{\delta \leq \gamma < \beta} (\alpha + \gamma)$.*

Lemma 20. *Let $\alpha, \beta \in ON$ such that $\alpha \in LIM$. Then*

$$(\alpha + \beta = \beta) \leftrightarrow (\beta \geq \alpha \cdot \omega).$$

Definition 24 (Continuous)**.** *A class function (ie. A function from a class to a class) $f : ON \to ON$ is **continuous** if,*

1. *f is non-decreasing.*

2. $f(\alpha) = sup\{f(\beta)|\beta < \alpha\}$ *for each* $\alpha \in ON$.

Problem 53. *Let* $\alpha \in ON$. *Which of the following functions are continuous?*

1. $f(a) = \alpha + \omega$
2. $f(a) = \omega + \alpha$
3. $f(a) = \alpha \cdot \omega$
4. $f(a) = \omega \cdot \alpha$

Lemma 21. *Let* $f : ON \to ON$ *be strictly increasing. Let* $\alpha \in ON$. *Then* $f(\alpha) \geq \alpha$. *(Hint: By way of contradiction, let* α *be the least bad guy. i.e. let* α *be the least such that* $f(\alpha) < \alpha$.

Lemma 22. *Let* $(\alpha_n)_{n\in\omega}$ *be a non-decreasing sequence. Let* f *be non-decreasing function. Let* $\beta = sup\{\alpha_n|n \in \omega\}$. *Then,*

$$sup\{f(\alpha_n)|n \in \omega\} = sup\{f(\gamma)|\gamma < \beta\}.$$

Problem 54. *For each* $\alpha \in ON$, *let* $f_1(\alpha) = \omega_1 + \alpha$. *Find* $\beta \in ON$ *such that* $f_1(\beta) = \beta$.

Theorem 11. *Let* f *be a strictly increasing continuous function. Show that* f *has fixed point. (i.e.* $\exists\beta \in ON(f(\beta) = \beta)$*).*

(Hint: Construct $(\alpha_n)_{n\in\omega}$. *Consider the two cases of* $\beta = sup\{\alpha_n|n \in \omega\}$ *,* $\beta \in LIM$ *or* $\beta \notin LIM$*).*

Problem 55. *For each* $\alpha \in ON$, *let* $f_2(\alpha) = \omega_1 \cdot \alpha$. *Find* $\beta > \omega_1^\omega$ *such that* $f_2(\beta) = \beta$.

Theorem 12. *Let* f *be a strictly increasing function which is continuous. Then* f *has many arbitrarily high many fixed points. That is,*

$$\forall\alpha \in ON\exists\beta > \alpha(f(\beta) = \beta).$$

Chapter 6

The Axiom of Choice

Definition 25 (Axiom of Choice)**.** *The **Axiom of Choice**, denoted AC, states that for every non-empty family $\{x_i | i \in I\}$ of non-empty sets,*

$$\Pi_{i \in I}(x_i)$$

is non-empty.

Definition 26 (Well-Ordering Principle)**.** *The **Well-Ordering Principle**, denoted WOP, states: For every non-empty set X, there exists an ordinal α and a function $f : \alpha \to X$ such that f is a bijection.*

Definition 27 (Chain)**.** *Let $\mathbb{P}$ be order. Let $\mathbb{C} \subset \mathbb{P}$. We call $\mathbb{C}$ a **chain** if for all $x, y \in \mathbb{C}$, x and y are comparable.*

WE NEED A DEFINITION FOR COMPARABLE

Definition 28 (Zorn's Lemma)**.** ***Zorn's Lemma**, denoted ZL, states: Let $\mathbb{P}$ be a non-empty partial order such that every chain has an upper-bound. Then $\mathbb{P}$ has a maximal element.*

Lemma 23. *Let $\{x_i | i \in I\}$ be a non-empty family of non-empty sets. Let*

$$\mathbb{P} = \{f | dom(f) \subset I \land \forall i \in dom f(f(i) \in X_i).$$

Show that $\mathbb{P}$ ordered by inclusion is a partial order.

Lemma 24. *From above. Let $\mathbb{C}$ be a chain in $\mathbb{P}$. Show $\bigcup \mathbb{C} \in \mathbb{P}$.*

Theorem 13. *$ZL \to AC$.*

Chapter 7

Cardinal Arithmetic

Definition 29 (Cardinality)**.** *For any set X, we say $|X|$ denotes the* ***cardinality*** *of X. Let X and Y be sets.*

1. *If there exists $f : X \to Y$ such that f is injective, then $|X| \leq |Y|$.*
2. *If there exists $f : X \to Y$ such that f is surjective, then $|X| \geq |Y|$.*
3. *If there exists $f : X \to Y$ such that f is bijective, then $|X| = |Y|$.*
4. $(|X| \leq |Y| \wedge |X| \neq |Y|) \to (|X| < |Y|)$.

Definition 30 (Cardinal)**.** *Let $\kappa \in ON$. We say κ is a* ***cardinal*** *if for each $\alpha < \kappa$, $|\alpha| < |\kappa|$.*

Problem 56. *Which of the following are cardinals? If an ordinal is not a cardinal, then find a bijection between α and $|\alpha|$.*

1. 4
2. ω
3. $\omega + 1$
4. $\omega \cdot \omega$
5. ω_1
6. $\omega_1 + 1$

7. $\omega_1 + \omega$

Definition 31 (Cardinal Addition)**.** *Let κ and λ be cardinals. We define* ***cardinal addition*** *as*

$$\kappa \oplus \lambda = |\kappa + \lambda|.$$

Problem 57. *Evaluate the following.*

1. $1 \oplus 2$

2. $1 \oplus \omega$

3. $\omega \oplus 1$

Definition 32 (Cardinal Multiplication)**.** *Let κ, λ be cardinals. We define* ***Cardinal Multiplication*** *as*

$$\kappa \otimes \lambda = |\kappa \cdot \lambda|.$$

Problem 58. *Evaluate the following.*

1. $2 \otimes \omega$

2. $\omega \otimes 2$

Problem 59. *Evaluate the following.*

1. $\omega \oplus \omega$

2. $\omega \otimes \omega$

3. $\omega \oplus \omega_1$

4. $\omega_1 \oplus \omega$

5. $\omega_1 \otimes \omega_1$

6. $\omega_1 \oplus \omega_1$

Problem 60. *Find bijections between the following pairs of ordinals.*

1. ω *and* $\omega + \omega$

2. ω_1 *and* $\omega_1 + \omega_1$

Lemma 25. *For every cardinal* κ, $\kappa \oplus \kappa = \kappa$.

Lemma 26. *Let* κ *and* λ *be cardinals. Suppose* $\kappa \geq \lambda$. *Then* $\kappa \oplus \lambda = \kappa$ *(i.e.* $\kappa \oplus \lambda = max\{\kappa, \lambda\}$*).*

HINT: *For all ordinals* α *and* β, *if* $\alpha \leq \beta$ *then* $|\alpha| \leq |\beta|$.

Lemma 27. *For all cardinals* κ *and* λ, $\kappa \oplus \lambda = \lambda \oplus \kappa$.

Problem 61. *For all* $\alpha \in ON$, *there exists a unique* $\beta \in LIM$ *and a unique* $n \in \omega$, *such that* $\alpha = \beta + n$. *That is,*

$$\forall \alpha \in ON \exists ! \beta \in LIM \exists ! n \in \omega (\alpha = \beta + n).$$

Definition 33 (Diagonal)**.** *For each* $\gamma \in ON$, *we define the* ***diagonal*** D_γ *as*

$$D_\gamma = \{(\alpha, \beta) | \alpha + \beta = \gamma\}.$$

Problem 62. *Show that* $\bigcup_{\delta \omega_1} D_\delta = \omega_1 \times \omega_1$. *That is,*

$$\forall \alpha, \beta \in \omega_1 \exists \gamma < \omega_1, (\alpha, \beta) \in D_\gamma$$

.

Problem 63. *Give graphical depictions of the following diagonals as subsets of* $\omega_1 \times \omega_1$.

1. D_4
2. D_ω
3. $D_{\omega+5}$
4. $D_{\omega+\omega}$
5. $D_{\omega+\omega+\omega}$
6. $D_{\omega \cdot \omega}$

Problem 64. *For each* $\gamma < \omega_1$, $\bigcup_{\delta < \gamma} D_\delta \subset \gamma \times \gamma$.

Definition 34. *We define* $\trianglelefteq$ *on* $\omega_1 \times \omega_1$ *as follows,*

$$((\alpha_1, \beta_1) \trianglelefteq (\alpha_2, \beta_2)) \leftrightarrow ((\alpha_1 + \beta_1 < \alpha_2 + \beta_2) \vee (\alpha_1 + \beta_1 = \alpha_2 + \beta_2 \wedge \alpha \leq \alpha_2))$$

Problem 65. *Show* $\trianglelefteq$ *is well-ordering on* $\omega_1 \times \omega_1$*.*

Question 8. *What does* $\trianglelefteq$ *have do with the* D_γ*'s.*

Problem 66. *Show for each* $\gamma < \omega_1$, $|\bigcup\limits_{\delta<\gamma} D_\delta| < \omega_1$*.*

Problem 67. *Show* $\omega_1 \times \omega_1 = |\bigcup\limits_{\delta<\omega_1} D_\delta| = \omega_1$*.*

Lemma 28. *For every cardinal* κ, $\kappa \otimes \kappa = \kappa$*.*

Lemma 29. *Let* κ *and* λ *be cardinals. Suppose* $\lambda \leq \kappa$*. Then* $\kappa \otimes \lambda = \lambda \otimes \kappa = \kappa$*.*

Lemma 30. *Let* κ *and* λ *be cardinals. Then* $\kappa \oplus \lambda = max\{\kappa, \lambda\}$*.*

Definition 35 (κ^+)**.** *Let* κ *be a cardinal. We define* κ^+ *as the smallest cardinal* α *such that* $\alpha > \beta$*. That is,*

$$\kappa^+ = min\{\alpha | \alpha \text{ is a cardinal} \wedge \alpha > \kappa\}.$$

Definition 36 (ω_α)**.** *We define each* ω_α *by transfinite recursion as follows:*

1. $\omega_0 = \omega$
2. $\omega_{\alpha+1} = (\omega_\alpha)^+$
3. $\gamma \in LIM \rightarrow \omega_\gamma = sup\{\omega_\alpha | \alpha < \gamma\}$

Definition 37 (Successor and Limit Cardinals)**.** κ *is a* ***successor cardinal*** *if* $\kappa = \lambda^+$ *for some cardinal* λ*.* κ *is a* ***limit cardinal*** *if* $\kappa \geq \omega$ *and* κ *is not a successor ordinal.*

Lemma 31. *Let* $\alpha, \beta \in ON$*. Let* κ *be a cardinal. Then,*

1. ω_α *is a cardinal.*

2. $\forall \kappa \geq \omega(\kappa \text{ is a cardinal} \rightarrow \exists \alpha(\kappa = \omega_\alpha))$.

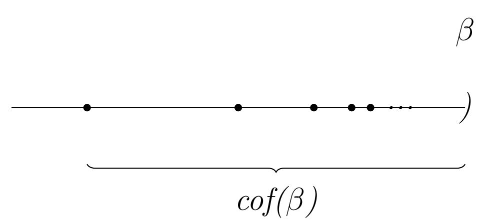
3. $\alpha < \beta \rightarrow \omega_\alpha < \omega_\beta$.

4. ω_α *is a limit cardinal* $\leftrightarrow \alpha \in LIM$.

5. ω_α *is a successor cardinal* $\leftrightarrow \alpha \in SUCC$.

Definition 38 (Cofinal Map)**.** *Let $\alpha, \beta \in ON$. Let $f : \alpha \rightarrow \beta$. We say f is a **cofinal map** if $f[\alpha]$ is unbounded in β.*

Definition 39 (Cofinality)**.** *Let $\beta \in ON$. We define the **cofinality** of β, written as $cof(\beta)$, as the least ordinal α such that there exists a cofinal map $f : \alpha \rightarrow \beta$.*



Lemma 32. *Let $\beta \in ON$.*

1. $cof(\beta) \leq \beta$.

2. $cof(\beta + 1) = 1$.

Problem 68. *Find the cofinality of each of the following ordinals.*

1. 4

2. ω

3. $\omega + \omega$

4. $\omega \cdot \omega$

5. ω_1

6. $\omega_1 + \omega$

7. $\omega_2 + \omega_1$

8. $\omega_2 \cdot \omega_1$

9. ω_ω

Lemma 33. *Let $\beta \in ON$. Then there exists a strictly increasing cofinal map $f : cof(\beta) \to \beta$.*

Lemma 34. *Let $\beta \in ON$. Then $cof(cof(\beta)) = cof(\beta)$.*

Definition 40 (Regular)**.** *Let $\alpha \in ON$. We say α is **regular** if $cof(\alpha) = \alpha$.*

Lemma 35. *Let $\alpha \in ON$. If α is regular, then α is a cardinal.*

Problem 69. *Let $\alpha \in ON$ such that α is an infinite cardinal. Is it necessarily the case that α is regular?*

Problem 70. *Is ω regular?*

Problem 71. *If ω_1 regular?*

Chapter 8

Clubs and Stationary Sets

Definition 41 (Club Sets)**.** *Let $\mu \in LIM$. Let $C < \mu$.*

1. *C is **closed** if $\forall B \subset C(B$ is bdd. in $C \to sup(B) \in C)$.*

2. *C is **unbdd.** in μ if $\forall \alpha < \mu \exists \beta \in C(\beta > \alpha)$.*

*C is called a **club set** in μ if C is closed and unbdd. in μ. Note "clubness" is dependent on mu. C may be a club in μ_0, but not be a club in μ_1 where $\mu_0 \neq \mu_1$.*

Problem 72. *Let $A \subset \omega$ such that A is finite. Is A necessarily a club in ω?*

Problem 73. *Let $A \subset \omega$ such that A is infinite. Is A necessarily a club in ω?*

Problem 74. *Let $A = \omega_1 \cap SUCC$. Is A necessarily a club in ω_1?*

Problem 75. *Let $A = \omega_1 \cap LIM$. Is A necessarily a club in ω_1?*

Problem 76. *Let $A, B \subset \omega_1$ be unbdd. in ω_1. Is $A \cap B$ necessarily unbdd. in ω_1?*

Problem 77. *Let $A, B \subset \omega_1$ be club in ω_1. Is $A \cap B$ necessarily club in ω_1?*

Lemma 36. *Let κ be a cardinal. Let $C_1, C_2 \subset \kappa$ be club in κ. Then $C_1 \cap C_2$ is club in ω_1.*

Lemma 37. *Let $\{C_n | n \in \omega\}$ be a countable family of clubs in ω_1. Then $\bigcap_{n\in\omega} C_n$ is club in ω_1.*

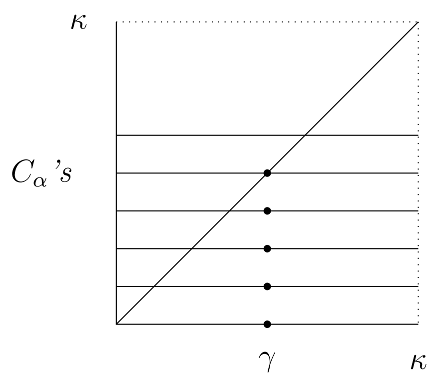
Problem 78. *Let $\{C_\alpha | \alpha < \kappa\}$ be a family of κ-many clubs in κ is $\bigcap_{\alpha \in \kappa} C_\alpha$ necessarily club in κ?*

Lemma 38. *Let $\lambda < cof(\kappa)$ where κ is a cardinal. Let $\{C_\alpha | \alpha < \lambda\}$ be a family of clubs in κ. Then $\bigcap_{\alpha < \lambda} C_\alpha$ is club in κ.*

Lemma 39. *Let κ be regular. Let $\{C_\alpha | \alpha \subset \kappa\}$ be a family of clubs in κ. Let*

$$D = \{\gamma < \kappa | \forall \alpha \leq \gamma (\gamma \in C_\alpha)\}.$$

Then D is club in κ.



Chapter 9

Tree

Chapter 10

Comments

$CommentsFromJustice$

1. Before Discussion 1, should we define set builder notation? Or is that expected knowledge? I can see it both ways. It's technically the first thing they'll see, so it might be beneficial to remind them.

2. Before the definition of Partial order, should we define relation? Similar to my first question about what's expected.

3. Should we combine certain problems, such as problems 1-3 into one problem with 3 parts?

4. We should specify if and iff relationships. I second this –

5. (Problem 5): Is extending functions really important? I ran into this in the first semester, I omitted it because I wasn't really sure if it had significance.

6. Can I remove the question from the definition of linear order. It seems worthwhile to me to keep it?

7. I still need to define well-ordering problems (might e-mail eric with this), but do I need to define least ordering as part of foundations?

8. I'm about to start the Axioms section, but how should they be ordered? Is there a preference. If not, I can complete it pretty quickly.

9. In certain cases, it may be better to wait to define things until their relevant axioms, things such as Ordinal, Transitive, or Finite sets

$$CommentsFromJusticeAboutAxiomssection$$

1. very first line... What should go here? It's a discussion?
2. do we need a more formal definition for the pairing axiom? like from the book?
3. problem 14 confuses me... did I write this correctly? If so, send me the solution :) It's bothering me..
4. Transitive and Finite, should these be introduced in this document? perhaps in the "foundations" section? For now, they are omitted
5. Any other items/problems that we think we should add to the Axioms section?

$$CommentsFromJusticeAboutOrdinalssection$$

1. Someone double check problem 24 for me (that it's all written correctly and makes sense)
2. where does the axiom of regularity live? (in axioms, ordinals?)
3. Rank: Should it live in ordinals?
4. I see eric put a note about Transfinite induction. Should that be defined. If so, where?
5. Axiom of regularity. I included two definitions. Should I have added both? or just "Every set X has a least element with respect to epsilon". I was confused what epsilon was, so I added the formal definition
6. At the very end I have questions regaurding LIM and SUCC. I'm not quite sure where to define those.

Comments Left From First Edit

The overarching notes at the beginning and the comment notes at the end. Have not yet been addressed.

1. Added note to code to rephrase the question about Russell's Paradox in Chapter 1.
2. **For Justice** In Chapter 1 the definition for Ordered Pair has "newline" in the code? What is its purpose?
3. In Chapter 2, do we know what if we want the worded definition, the mathematical logic, or both for each axiom?
4. In Chapter 3 there are two Problems that state "Prove or disprove the following statement:" One of them switches to a new line, while the other doesn't. Which way do we prefer?
5. At the end of Chapter 3 There are some questions $SUCC$ and LIM. that need a home.
6. Also at the end of Chapter 3, the large written in question about sizes of V_α has not been placed and still needs a home.
7. Chapter 4 has stuff in it, but needs to be looked through for order of problems and anything missing.
8. At some point (before Ordinal Arithmetic) we need to define $\omega, \omega_1, ..., \omega_\alpha$.
9. In AC we now have a definition for Chain, but we need a definition for Comparable.

$CommentsFromJusticeAboutMathematicalUniverse$

1. Problem 34. Did I write it correctly?
2. do we need to define Omega?

Eric Comments From Editing

1. When doing definitions (especially near the beginning), do we want the definition in words, mathematical logic, or both?
2. The definition of rank at the beginning of Mathematical Universe needs to be addressed.
3. Transfinite Induction is needed by Chapter 4.
4. Less than or equal to symbol is broken.

PostStanleyEditsComments

1. Changed definition wording for the following: Ordinal Addition, Ordinal Multiplication, Ordinal Exponentiation, κ^+, ω_α. Please read these to see if they are worded appropriately.