Definition 1. ***Set builder notation -*** *A way of defining sets in formal mathematical logic.*
Example: Let P(x) be a statement. We can define a set Y by writing

$$Y = \{x|P(x)\}$$

Meaning Y is the set consisting of elements that make p(x) a true statement

Potential instructor note: If sets are simply collections of objects, consider the following: Let $Y = \{x|x \notin x\}$. Note that $Y \in Y \to Y \notin Y$, and also $Y \notin Y \to Y \in Y$. Thus $Y \in Y \iff Y \notin Y$. This is clearly a contradiction. Thus, the need for a better definition of set arises...

Definition 2. ***Relation -*** *Let P be a set. Let $R \subset P \times P$. We say that R is a relation on P ($\leq$ and $\geq$ are examples of relations on $\mathbb{R}$).*

Definition 3. ***Partial Order -*** *Let P be a set. Let $\leq$ be a relation on P. We call P a partially ordered set if $\forall x, y, z \in P$ the following axioms hold:*

1. *reflexivity: $x \leq x$*
2. *anti-symmetry: $(x \leq y \wedge y \leq x) \to x = y$*
3. *transitivity: $(x \leq y \wedge y \leq z) \to x \leq z$*

Note: we can also state that $\leq$ is a partial order on P if the axioms hold. Two ways of saying the same thing.

Potential instructor note: It may help to draw a "relation" for lack of better terms, to the definition of equivalence relations. Assuming you're students have learned equivalence relations, it can be help-full to compare/contrast the definitions

Problem 1. *Determine if the following relations are partial orders on their respective sets*

1. *$\leq$ on $\mathbb{Z}$*
2. *$|$ on $\mathbb{Z}^+$*
3. *$\subset$ on $\mathcal{P}(X)$*

Note: $\mathcal{P}(X)$ is the power set.

Problem 2. *Determine if a set can have more than one partial ordering by either finding a set with two partial orders or proving the statement false.*

Problem 3. *Determine if the Lexicographical ordering is a partial order on the set of all "words"*

Note: Words in this case means any string of letters. "abbazdfg" is a word.

Definition 4. ***Linear Order -*** *Let P be a set. Let $\leq$ be a partial order on P. We call P a linearly ordered set (and refer to $\leq$ as a linear order) if the following statement is true:*

$$\forall x, y \in P(x \leq y \vee y \leq x)$$

In other words, every element of the set P is comparable with every other element of the set.

Problem 4. *Which of the following relations from problem 1 are linear orders?*

1. $\leq$ *on* $\mathbb{Z}$
2. $|$ *on* $\mathbb{Z}^+$
3. $\subset$ *on* $\mathcal{P}(X)$

Definition 5. ***Partial Function -*** *Consider the definition of a traditional function f from X to Y which has two parts:*

1. $\forall x \in X \exists y \in Y((x, y) \in f)$
2. $\forall x \in X \forall y_1, y_2 \in Y((x, y_1), (x, y_2) \in f \to y_1 = y_2)$

A Partial function on the other hand, only requires the second statement to hold true, thus the only requirement for a partial function is the following:

$$\forall x \in X \forall y_1, y_2 \in Y((x, y_1), (x, y_2) \in f \to y_1 = y_2)$$

Potential instructor note: For me it was easier to think about partial functions as basically a function where the whole domain doesn't need to me mapped to an output, only a subset of the domain. This English definition might help other students grasp the logic-based mathematical definition.

Problem 5. *Let X and Y be sets. Let $\mathcal{F}$ be the family of partial functions from X to Y. Let $f, g \in \mathcal{F}$. We say $f \leq g$ if $dom_g \subset dom_f$ and $f(dom_g) = g$. Is $\leq$ a partial order on $\mathcal{F}$?*