Definition 1. ***Set builder notation -*** *A way of defining sets in formal mathematical logic.*
Example: Let P(x) be a statement. We can define a set Y by writing

$$Y = \{x|P(x)\}$$

Meaning Y is the set consisting of elements that make p(x) a true statement

Potential instructor note: If sets are simply collections of objects, consider the following: Let $Y = \{x|x \notin x\}$. Note that $Y \in Y \to Y \notin Y$, and also $Y \notin Y \to Y \in Y$. Thus $Y \in Y \iff Y \notin Y$. This is clearly a contradiction. Thus, the need for a better definition of set arises...

Definition 2. ***Relation -*** *Let P be a set. Let $R \subset P \times P$. We say that R is a relation on P ($\leq$ and $\geq$ are examples of relations on $\mathbb{R}$).*

Definition 3. ***Partial Order -*** *Let P be a set. Let $\leq$ be a relation on P. We call P a partially ordered set if $\forall x, y, z \in P$ the following axioms hold:*

1. *reflexivity: $x \leq x$*
2. *anti-symmetry: $(x \leq y \wedge y \leq x) \to x = y$*
3. *transitivity: $(x \leq y \wedge y \leq z) \to x \leq z$*

Note: we can also state that $\leq$ is a partial order on P if the axioms hold. Two ways of saying the same thing.

Potential instructor note: It may help to draw a "relation" for lack of better terms, to the definition of equivalence relations. Assuming you're students have learned equivalence relations, it can be help-full to compare/contrast the definitions

Problem 1. *Determine if the following relations are partial orders on their respective sets*

1. *$\leq$ on $\mathbb{Z}$*
2. *$|$ on $\mathbb{Z}^+$*
3. *$\subset$ on $\mathcal{P}(X)$*

Note: $\mathcal{P}(X)$ is the power set.

Problem 2. *Determine if a set can have more than one partial ordering by either finding a set with two partial orders or proving the statement false.*

Problem 3. *Determine if the Lexicographical ordering is a partial order on the set of all "words"*

Note: Words in this case means any string of letters. "abbazdfg" is a word.

Definition 4. ***Linear Order -*** *Let P be a set. Let $\leq$ be a partial order on P. We call P a linearly ordered set (and refer to $\leq$ as a linear order) if the following statement is true:*

$$\forall x, y \in P(x \leq y \vee y \leq x)$$

In other words, every element of the set P is comparable with every other element of the set.

Problem 4. *Which of the following relations from problem 1 are linear orders?*

1. $\leq$ *on* $\mathbb{Z}$
2. $|$ *on* $\mathbb{Z}^+$
3. $\subset$ *on* $\mathcal{P}(X)$

Definition 5. ***Partial Function -*** *Consider the definition of a traditional function f from X to Y which has two parts:*

1. $\forall x \in X \exists y \in Y((x, y) \in f)$
2. $\forall x \in X \forall y_1, y_2 \in Y((x, y_1), (x, y_2) \in f \rightarrow y_1 = y_2)$

A Partial function on the other hand, only requires the second statement to hold true, thus the only requirement for a partial function is the following:

$$\forall x \in X \forall y_1, y_2 \in Y((x, y_1), (x, y_2) \in f \rightarrow y_1 = y_2)$$

Potential instructor note: For me it was easier to think about partial functions as basically a function where the whole domain doesn't need to me mapped to an output, only a subset of the domain. This English definition might help other students grasp the logic-based mathematical definition.

Problem 5. *Let X and Y be sets. Let $\mathcal{F}$ be the family of partial functions from X to Y. Let $f, g \in \mathcal{F}$. We say $f \leq g$ if $dom_g \subset dom_f$ and $f(dom_g) = g$. Is $\leq$ a partial order on $\mathcal{F}$?*

Potential instructor note: This is the start of the Axioms chapter. It may be useful to describe that the goal is to build the system called ZF. Again it might be useful to reintroduce Russel's paradox as this will give a good idea as to why we need to have a stricter definition of "set"

Definition 6. ***Axiom of extensionality -*** *Let X and Y be sets. We say two sets are equal if the following statement holds true:*

$$X = Y \iff \forall a(a \in x \iff a \in y)$$

Problem 6. *Prove or disprove the following theorem:*

$$\forall x \forall y(x = y \iff x \subset y \wedge y \subset x)$$

Definition 7. ***Pairing Axiom -*** $\forall x \forall y \exists z(x \in z \wedge y \in z \wedge \forall w(w \in z \rightarrow (w = x \vee w = y)))$

Potential instructor note: It can be difficult to look at the axioms in mathematical logic because the elements themselves are sets. This could confuse some students since most similar logic involves building sets from other objects (AKA set builder notation). It may be worth mentioning that the elements in the logic are sets themselves.

Definition 8. ***Ordered Pair -*** $\forall x \forall y((x, y) = \{\{x\}, \{x, y\}\})$

Note: $\forall x(\{x, x\} = \{x\})$

Problem 7. *Write the following sets in ordered pair notation:*

1. $\{\{1\}, \{1, 2\}\}$
2. $\{\{1, 2\}, \{1\}\}$
3. $\{\{1, 2\}, \{1\}\}$
4. $\{\{1\}\}$
5. $\{\{\{1\}\}\}$

Problem 8. *Write the following ordered pairs in set notation:*

1. $(1, 3)$
2. $(1, 1)$
3. $((3, 4), (1, 2))$

Definition 9. *Let x be a set.*
$\bigcup x = \{z | \exists y \in x(z \in y)\}$
$\bigcap x = \{z | \forall y \in x(z \in y)\}$

Example: $\bigcap \{[0, n] | n \in \mathbb{Z}^+\} = [0, 1]$

Definition 10. ***Union Axiom -*** $\forall x \exists z(z = \bigcup x)$

Problem 9. *Let X and Y be sets. Prove that $X \cup Y$ is a set*

Problem 10. *Prove or disprove the following statement:* $(Y \cup X \subset X \cap Y) \leftrightarrow (x = y)$

Definition 11. ***Axiom of separation -*** $\forall x \forall \theta \exists z\ (z = \{y \in x \mid \theta(y)\})$

Problem 11. *Prove or disprove the following statement:* $[(X - Y) \cup (Y - X)] \cap [X \cap Y] = \emptyset$

Problem 12. *Prove or disprove the following statement:* $((x, y) = (w, z)) \leftrightarrow (x = w \wedge y = z)$

Problem 13. *Prove or disprove the following statement:* $\neg \exists X \forall Y (Y \in X)$

Potential instructor note: It might be worth stating what this actually means. This question essentially is stating "The set of all sets is not a set". This can be a little confusing to look at in mathematical logic form.

Definition 12. ***Successors -*** *Let x be a set. The successor of x is $S(x) = x \cup \{x\}$*

Problem 14. *Is $S(X)$ a set?*

Definition 13. ***Axiom of Empty Set -*** *There exist an empty set. We denote it with $\emptyset$*

Definition 14. ***Finite -*** *A set is finite if there is no bijection between it and one of its proper subsets.*

Potential instructor note: Depending on the level of your students you may need to define/reintroduce the definition of bijection here. For this document, I will assume it is common knowledge.

Problem 15. *Determine if the following sets are finite or infinite:*

1. $\mathbb{R} \setminus \mathbb{N}$
2. *The set of all words*
3. $\{x \mid 2x + 300 = 107\}$

Definition 15. ***Transitive -*** *A set is transitive if and only if $\forall y \in x(y \subset x)$*

Problem 16. *Determine if the following sets are Transitive:*

1. $\mathbb{N}$
2. $\mathcal{P}(\mathbb{N})$
3. $\{M\}$
4. $\{\{\}, \{\{\}\}\},$

Problem 17. *Prove or disprove the following statement:*
X is transitive if and only if $\forall x \in X(\bigcup x \subset X)$

Definition 16. ***Ordinal -*** *An ordinal is a set α which is transitive and strictly well-ordered by ϵ*

Problem 18. *Let $X = \{0, 1, 2, 4\}$. Is X an ordinal?*

Problem 19. *Prove the following statement: Let X be an ordinal. Let $\mathbb{B} \in X$. Then $\mathbb{B}$ is also an ordinal.*

Definition 17. ***Axiom of Infinity -*** $\exists X(\emptyset \in X \wedge \forall x \in X(S(x) \in X))$

Problem 20. *Show that the collection of all finite ordinals is a set. That is, prove*
$\{x \mid x \text{ is finite ordinal}\}$ is a set.

Definition 18. ***Axiom of Power Set -*** $\forall x \exists y \forall z (z \in y \leftrightarrow z \subset x)$

Definition 19. ***Axiom of Replacement -*** *Let θ be a formula so that $\forall x, y, z(\theta(x, y) \wedge \theta(x, z) \leftrightarrow y = z$, then $\forall w \exists s(s = \{y \mid \exists x \in w(\theta(x, y))\}$*

Potential instructor note: It might be a good exercise to show an example proving that $\omega + \omega$ is a set using the axioms mentioned so far. Use $\theta(n, y) \leftrightarrow y = S^n(w)$

Problem 21. *Let X be a transitive set. Prove the following statements:*

1. $\forall y \in X(y \cap X = \emptyset \to y = \emptyset)$
2. $\forall a, x, y \in X(a \cap X = \{x, y\} \to a = \{x, y\})$
3. $\forall a, x, y \in X(a \cap X = (x, y) \to a = (x, y))$
4. $\forall a, x \in X(a \cap X = \bigcup x \to a = \bigcup x)$
5. $\forall a \in X(a \cap X$ *Is a relation* $\to$ *a is a relation*$)$
6. $\forall a \in X(a \cap X$ *Is a function* $\to$ *a is a function*$)$

Potential instructor note: *It's worth mentioning that one way to prove something is a relation is to simply prove it's a subset of the set cross itself. Ex: Proving r is relation on Y, you would prove $r \subset Y \times Y$*