ENGS 104, W 2010

Lecture 11

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<u>Today</u>

Linear Programming Basics.

See LP.pdf on BBoard for additional Presentation.

Motivation:

Consider

$$\min \quad c^T x \qquad x \in \mathbb{R}^n$$

$$\text{such that} \quad Ax = b$$

$Ax = b$ has $0, 1, \infty$ # of solutions

If 1 solution, <u>not</u> interesting.
If ∞ solutions, linear programming (LP)

Augmented ~~Standard~~ form LP:

$$\min_{x} \quad c^T x$$

Subject to (such that) $\quad Ax = b$

$\quad x \geq 0$

Augmented "~~Standard~~" form LP

$c \in E^n$ is cost vector

$x \in E^n$ is variable vector

$A \in E^{m \times n}$, $b \in E^m$ define constraints

$x \geq 0$ positivity constraints

Example

$$c = \begin{bmatrix} 2 \\ 1 \\ 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 3 & 7 \\ 2 & 1 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 11 \\ 3 \end{bmatrix}$$

$$\begin{array}{ll} \min & 2x_1 + x_2 + 5x_3 \\ \text{s.t.} & x_1 + 3x_2 + 7x_3 = 11 \\ & 2x_1 + x_2 = 3 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

Terminology

The set of x which satisfy

$$\left.\begin{array}{l} Ax = b \\ x \geq 0 \end{array}\right\}$$ is called the "feasible set".

Note if x, y are feasible & $0 \leq \alpha \leq 1$

$z = \alpha x + (1-\alpha) y$ is <u>also</u> feasible!

Check $Az = b$ and $z \geq 0$.

So the feasible set is a convex set.

Also $c^T z = \alpha c^T x + (1-\alpha) c^T y$

so the objective function to be minimized is also convex.

From previous discussions, such problems have only global minima, so <u>no</u> purely local mins exist. This is <u>good</u>!

<u>Examples</u> See LP.pdf

1. The Diet Problem
2. The assignment problem
3. The transportation problem
4. Other examples

Reducing problems to Augmented ~~Standard~~ LP form:

1. unconstrained variables?

Suppose $\min_{x} \; c^T x$

s.t. $Ax = b$

$x \in E^n$?

Let $x = y - z$, $y, z \in E^n$

$y, z \geq 0$

Let $\omega = \begin{bmatrix} y \\ z \end{bmatrix} \in E^{2n}$

$$\min \ [c^T, -c^T]\, \omega$$

$$\text{s.t.} \quad [A \ \ -A]\, \omega = b$$

$$\omega \geq 0$$

what if $x_i \leq 0$? $\rightarrow$ rewrite with $-x_i$ (≥ 0)

what if $x_i \geq d_i$ $\rightarrow$ rewrite with $x_i - d_i = y_i \geq 0$

etc.

Example

$$\min \quad c_1 x_1 + c_2 x_2$$

such that

$$a_1 x_1 + a_2 x_2 \le b_1$$

$$x_1 \ge d_1$$

x_2 unconstrained

Note $\quad a_1 x_1 + a_2 x_2 \le b_1$

$\iff \quad a_1 x_1 + a_2 x_2 + z_1 = b_1$

where $z_1 \ge 0$.

Examples

① $\min \|Ax-b\|_1$ 1 norm

② $\min \|Ax-b\|_\infty$ ∞ norm

In ① $|y| = \min \; u+v$

such that $u-v=y$

$u \geq 0, \; v \geq 0$

In ② $\|y\|_\infty = \min \; \alpha$

s.t. $\alpha \geq 0$

$-\alpha \leq y_i \leq \alpha$

Standard form LP

$$\max \quad c^T x$$
$$\text{s.t.} \quad Ax \le b$$
$$x \ge 0$$

Cannonical Form LP

$$\max \quad c^T x$$
$$\text{s.t.} \quad Ax \le b$$

Augmented form

$$\max \quad c^T x$$
$$\text{s.t.} \quad Ax = b, \quad x \ge 0$$

Bottom Line

Very many problems with "linear"-type constraints can be put into $\left\{\begin{array}{l}\text{Augmented} \\ \text{Standard} \\ \text{Cannonical}\end{array}\right\}$ Form through the introduction of additional variables!

$\|\cdot\|_\infty$, $\|\cdot\|_1$, etc

Standard, Canonical & Augmented Forms are all equivalent.

Can mechanically convert one to the other.

Question: how to convert

$Ax = b$ to $Ax \leq b$?

Answer $\quad Ax = b$ iff $\begin{cases} Ax \leq b \\ -Ax \leq -b \end{cases}$

How to convert $\min c^T x$ to $\max c^T x$?

Answer: $\min c^T x = -\max -c^T x$

Let's do examples on HW2!

"Duality"

Consider

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & Ax \le b \\ & x \ge 0 \end{aligned}$$

$$c^T x = \min_{y \geq 0} \; c^T x + (b - Ax)^T y$$

if $b \geq Ax$

$$-\infty = \min_{y \geq 0} \; c^T x + (b - Ax)^T y$$

if some $b_i < (Ax)_i$

ie if $b \ngeq Ax$

So

$$\max_{x\geq 0}\ \min_{y\geq 0}\ c^T x + (b-Ax)^T y = \max_{\substack{Ax\leq b\\ x\geq 0}} c^T x$$

Assume $$\underbrace{\max_{x\geq 0}\ \min_{y\geq 0}\ \sim\ =\ \min_{y\geq 0}\ \max_{x\geq 0}\ \sim}_{\text{"saddle point"}}$$

$$\min_{y\geq 0}\ \max_{x\geq 0}\ c^T x + y^T(b-Ax)$$

$$\min_{y \geq 0} \max_{x \geq 0} \quad c^T x + y^T(-Ax+b)$$

$$= \min_{y \geq 0} \max_{x \geq 0} \quad y^T b + (c^T - y^T A) x$$

$$\max_{x \geq 0} \quad y^T b + (c^T - y^T A) x = \infty$$

$$\text{if} \quad c^T \not\leq y^T A$$

$$= y^T b$$

$$\text{if} \quad c^T \leq y^T A$$

So

$$\min_{y \geq 0} \max_{x \geq 0} \; y^T b + (c^T - y^T A) x$$

$$= \min_{y \geq 0} \; y^T b$$

$$\text{s.t.} \quad c^T \leq y^T A$$

$$= \left.\begin{aligned} \min \quad & b^T y \\ & A^T y \geq c \\ & y \geq 0 \end{aligned}\right\} \text{Dual LP}$$

Primal

$$\max \quad c^T x$$

$$\text{s.t.} \quad Ax \le b$$

$$x \ge 0$$

Dual

$$\min \quad b^T y$$

$$\text{s.t.} \quad A^T y \ge c^T$$

$$y \ge 0$$

Dual of Dual is Primal!

Linear Programs are types of optimization problems.

There are a variety of methods to solve LP's.

⇒ 1. Simplex Algorithm
2. Ellipsoid Methods
3. Interior Point Methods