Engs 104, W2010

Lecture 13

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Today

- Questions about Assignment 2
- Demonstration of Simplex Algorithm in Matlab
- Duality examples & properties
- Unimodularity

Primal

$$\max \quad c^T x$$
$$\text{s.t.} \quad Ax \le b$$
$$x \ge 0$$

Dual

$$\min \quad b^T y$$
$$\text{s.t.} \quad A^T y \ge c$$
$$y \ge 0$$

At optimal solutions, x & y

$$c^T x = b^T y$$

$$c^T x + (b - Ax)^T y = \cancel{b^T y + (c^T - y^T A^T)}$$
$$= c^T x + y^T (b - Ax) = y^T b + (c^T - y^T A) x$$

So $\quad y^T (b - Ax) = 0 = (c^T - y^T A) x$

$y^T(b - Ax) = 0$ means

either $(b - Ax)_i = 0$ or $y_i = 0$

$(c^T - y^T A)x = 0$ means

either $(c^T - y^T A)_i = 0$ or $x_i = 0$

Either $x_i = 0$ or dual equation i is equality <u>and</u>

either $y_i = 0$ or primal equation is equality.

Primal - Dual Diet Problem

$$\min \; c^T x \quad \leftarrow \text{cost of diet}$$

$$\text{s.t.} \quad Ax \geq b \quad \leftarrow \text{diet meets MDR constraints}$$

$$x \geq 0$$

$$\max(-c^T x) \qquad\qquad \min \; -b^T y$$

$$\text{s.t.} \quad -Ax \leq -b \quad \Longrightarrow \quad -A^T y \geq -c$$

$$x \geq 0 \qquad\qquad y \geq 0$$

Primal Dual

$$\min \; -b^T y \qquad\Longrightarrow\qquad \max \; b^T y$$

$$\text{s.t.} \quad -A^T y \ge -c \qquad\qquad A^T y \le c$$

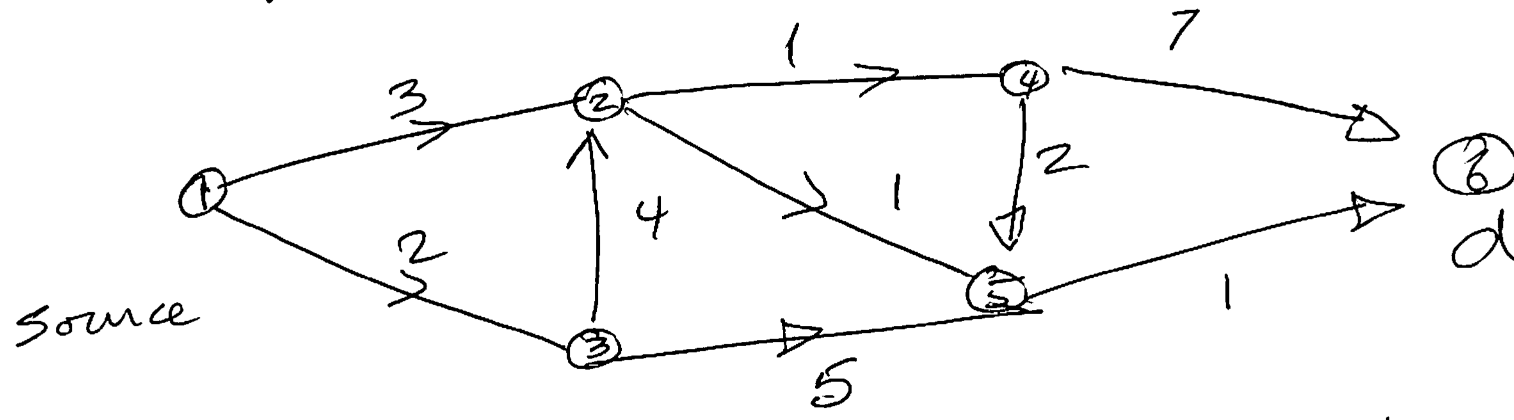
$$y \ge 0 \qquad\qquad y \ge 0$$

Interpretation: x is your allocation of food products to meet MDR's.

y is vitamin seller, y_i is charge for ith nutrient.

Max $b^T y$ = profit but $A^T y \le c$ means undercut food product cost.

Max flow problem:



"network of pipes"

c_{ij} = capacity of pipe from i to j

Problem: maximize flow from source to destination while

a) respecting capacities

b) respecting conservation of "mass"

Note: if x_{ij} is the flow from node i to node j

want/require

$$0 \le x_{ij} \le c_{ij}$$ and

$$\sum_i x_{ij} = \sum_k x_{jk}$$ conservation law

$$\max \sum_i x_{1i} = \max \sum_j x_{jn}$$

where n is destination.

$$\max \quad \sum \text{ flows out of source}$$

s.t.
$$\begin{bmatrix} I \\ N \\ -N \end{bmatrix} x \le \begin{bmatrix} C \\ 0 \\ 0 \end{bmatrix}$$

$$x \ge 0$$

N has 1 row for each non-terminal node i

+1 for edges entering i

−1 for edges leaving i

A "cut" of a network is a partition of nodes $S \cup T$ = all nodes,

$S \cap T = \phi$ source $\in S$, terminal/destination $\in T$

The cost of a cut is

$$\sum_{\substack{i \in S \\ j \in T}} c_{ij} \quad \text{where edges } ij \text{ exist}$$

Max flow = min cut by duality

Check online for details (& LP.pdf)

Facts about linear programs (augmented form)

(1) $Ax = b$, $A \in E^{m \times n}$ $\quad m \leq n$

$x \geq 0$

If x has more than m nonzero coordinates, say $K > m$ then the K columns of A corresponding to nonzero elements of x are linearly <u>dependent</u>. Can express one column in terms of others. Increase or decrease that x_j until some other $x_i = 0$.

Any x for which

$$Ax = b \qquad x \geq 0$$

is called <u>feasible.</u>

An x which has m nonzeros is called <u>basic</u>.

If $Ax = b$, $x \geq 0$ and x basic, call x a basic feasible solution.

<u>Theorem</u>: Solution to LP occurs at a Basic Feasible Solution. (BFS)

(B) Simplex Algorithm goes from one BFS to another decreasing cost.

Stops when no local change can decrease costs further

$\Longrightarrow$ local min at BFS

But LP's are convex so

local min $\Longrightarrow$ global min.

Simplex Algorithm finds global min BFS.

Implementation Considerations:

A "Cycling" – some BFS's have more than m zeros. Can cycle between BFS's without improving cost.

"Bland's Anti-Cycling"

B Testing for ≥ 0, ≤ 0 can be a problem due to rounding errors.

What you should know:

1. How to formulate LP's
2. How to put LP's into standard, augmented, canonical forms.
3. How to use LP solvers:
   a) Find initial BFS (Phase 1)
   b) Find optimal BFS (Phase 2)

Linear Programming with <u>Integer</u> Constraints

$$\left.\begin{aligned} &\min \ c^T x \\ &Ax = b \\ &x \geq 0 \\ &x \text{ integer} \end{aligned}\right\}$$

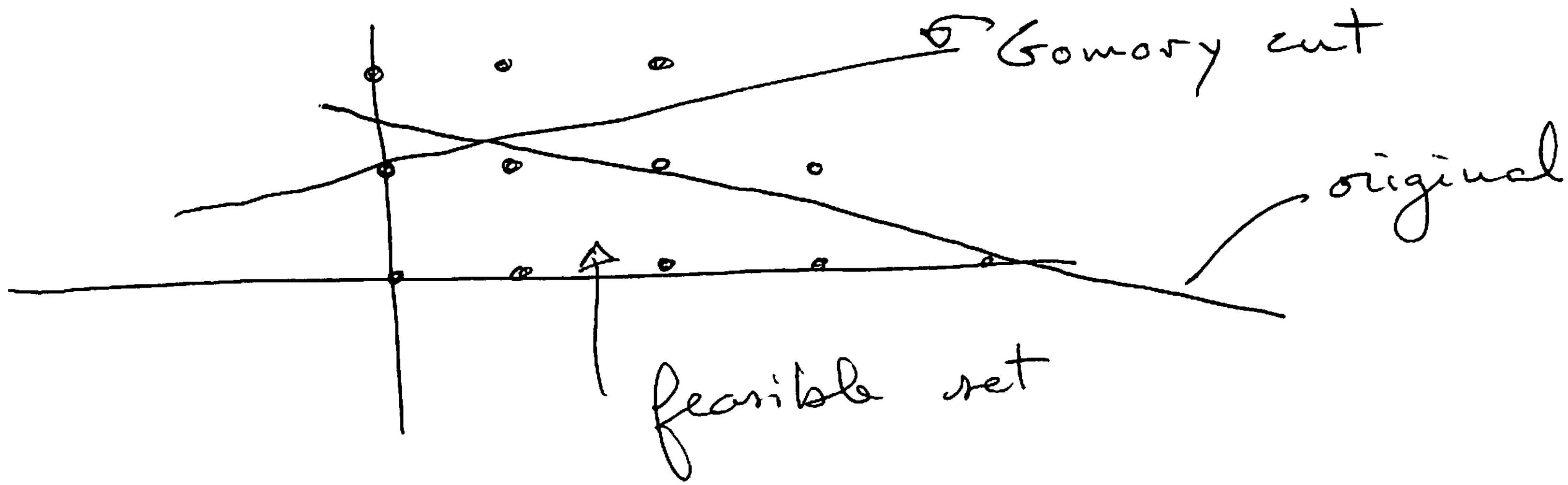
If x is # units for example.

(Not convex constraint) ⟶ x integer

Note solving LP without integer constraints and then "rounding" is not always the optimal solution

Integer LP's can be approximately solved by using "Gomory cuts"

min $c^T x$
$Ax = b$
$x \geq 0$
$\Longrightarrow$ Solve $\Longrightarrow$ Add new "cut" constraint $\Longrightarrow$ min $c^T x$
$A' x = b$
$x \geq 0$



Good news: Some LP's have integer solutions even if the LP is solved as a real valued LP.

$$A x = b \qquad x \text{ a BFS}$$

so

$$\underbrace{A_x x_n = b}_{m \times m}$$

A_x are selected columns of nonzero x's, x_n

$$(x_n)_j = \det(B_j)/\det(A_x)$$

Cramer's Rule

If A & b have integer entries and every m column submatrix of A, say A_s, has

$$\det(A_s) = \pm 1$$
$$\text{or} = 0$$

A is called "unimodular".

<u>Theorem</u> A unimodular & A, b integer

$\Longrightarrow$ BFS's are integer.

Example 3x3 assignment problem

$$
\begin{array}{c|c}
 & \begin{array}{ccccccccc} 14 & 15 & 16 & 24 & 25 & 26 & 34 & 35 & 36 \end{array} \\
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} &
\left[\begin{array}{ccccccccc}
1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1
\end{array}\right]
\end{array}
$$

Pick any 6 columns to get 6×6 matrix.

Determinant is 0 or ± 1.

$\Longrightarrow$ Simplex will compute integer BFS that is optimal.

3-Satisfiability Problem

$x_1, \ldots, x_n$ are Boolean Variables (T, F)

$\bar{x}_i$ is negation

$\cdot, \wedge$ is conjunction (and)

$+, \vee$ is disjunction (or)

~~$(x_i$, F~~

$$P = \underbrace{(x_1 + \bar{x}_2 + x_5) \cdot (x_2 + \bar{x}_4 + x_5) \cdot \ldots\ldots}_{\text{Conjunction of disjunctions with 3 terms that are } x \text{ or } \bar{x}}$$

$P = (x_1 + \bar{x}_2 + x_5)(x_2 + \bar{x}_4 + x_5) \dots$

Is there an assignment of T, F to x_i so that P is T?

Observe: P is T if all disjunctions are T.

Let $x_i = 0$ or 1

$\bar{x}_i = 1 - x_i \quad (= 0 \text{ or } 1)$

$x_i + \bar{x}_j + x_k \geq 1$ means T

So $P = T$ if

Integer LP
$$\left\{ \begin{array}{l} x_i + \bar{x}_j + x_k \geq 1 \\ \vdots \\ \text{list all disjunctions} \\ \text{and} \\ \quad x_i = 0 \text{ or } 1 \end{array} \right.$$

So we can solve 3-Sat if we can find <u>any</u> feasible solution to constraints. Does not matter what min is. Eg min $x_1 + x_2$ works.

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Combinatorial Optimization next

Complexity Hierarchy

Hardest — Intractable/Undecidable

Exponential

NP-Complete

Easiest — Solve in polynomial time