LINEAR PROGRAMMING

A Concise Introduction

Thomas S. Ferguson

Contents

LINEAR PROGRAMMING

1. Introduction.

A linear programming problem may be defined as the problem of *maximizing or minimizing a linear function subject to linear constraints*. The constraints may be equalities or inequalities. Here is a simple example.

Find numbers x_1 and x_2 that maximize the sum $x_1 + x_2$ subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, and

$$\begin{aligned} x_1 + 2x_2 &\leq 4 \\ 4x_1 + 2x_2 &\leq 12 \\ -x_1 + x_2 &\leq 1 \end{aligned}$$

In this problem there are two unknowns, and five constraints. All the constraints are inequalities and they are all linear in the sense that each involves an inequality in some linear function of the variables. The first two constraints, $x_1 \geq 0$ and $x_2 \geq 0$, are special. These are called *nonnegativity constraints* and are often found in linear programming problems. The other constraints are then called the *main constraints*. The function to be maximized (or minimized) is called the *objective function*. Here, the objective function is $x_1 + x_2$.

Since there are only two variables, we can solve this problem by graphing the set of points in the plane that satisfies all the constraints (called the constraint set) and then finding which point of this set maximizes the value of the objective function. Each inequality constraint is satisfied by a half-plane of points, and the constraint set is the intersection of all the half-planes. In the present example, the constraint set is the five-sided figure shaded in Figure 1.

We seek the point (x_1, x_2), that achieves the maximum of $x_1 + x_2$ as (x_1, x_2) ranges over this constraint set. The function $x_1 + x_2$ is constant on lines with slope -1, for example the line $x_1 + x_2 = 1$, and as we move this line further from the origin up and to the right, the value of $x_1 + x_2$ increases. Therefore, we seek the line of slope -1 that is farthest from the origin and still touches the constraint set. This occurs at the intersection of the lines $x_1 + 2x_2 = 4$ and $4x_1 + 2x_2 = 12$, namely, $(x_1, x_2) = (8/3, 2/3)$. The value of the objective function there is $(8/3) + (2/3) = 10/3$.

Exercises 1 and 2 can be solved as above by graphing the feasible set.

It is easy to see in general that the objective function, being linear, always takes on its maximum (or minimum) value at a corner point of the constraint set, provided the

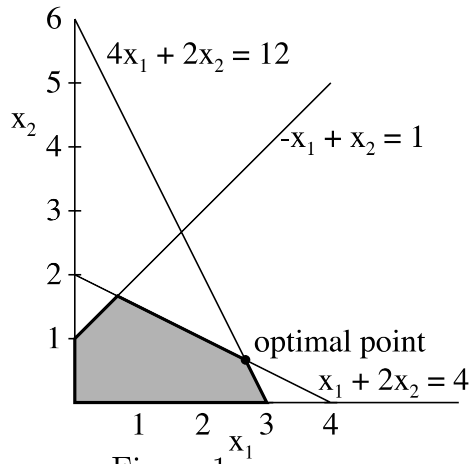

Figure 1.

constraint set is bounded. Occasionally, the maximum occurs along an entire edge or face of the constraint set, but then the maximum occurs at a corner point as well.

Not all linear programming problems are so easily solved. There may be many variables and many constraints. Some variables may be constrained to be nonnegative and others unconstrained. Some of the main constraints may be equalities and others inequalities. However, two classes of problems, called here the *standard maximum problem* and the *standard minimum problem*, play a special role. In these problems, all variables are constrained to be nonnegative, and all main constraints are inequalities.

We are given an m-vector, $\boldsymbol{b} = (b_1, \ldots, b_m)^\mathsf{T}$, an n-vector, $\boldsymbol{c} = (c_1, \ldots, c_n)^\mathsf{T}$, and an $m \times n$ matrix,

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}$$

of real numbers.

The Standard Maximum Problem: Find an n-vector, $\boldsymbol{x} = (x_1, \ldots, x_n)^\mathsf{T}$, to maximize

$$\boldsymbol{c}^\mathsf{T}\boldsymbol{x} = c_1x_1 + \cdots + c_nx_n$$

subject to the constraints

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &\le b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &\le b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &\le b_m \end{aligned} \qquad (\text{or } \boldsymbol{Ax} \le \boldsymbol{b})$$

and

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0 \qquad (\text{or } \boldsymbol{x} \ge \boldsymbol{0}).$$

The Standard Minimum Problem: Find an m-vector, $\boldsymbol{y} = (y_1, \ldots, y_m)$, to minimize

$$\boldsymbol{y}^{\mathsf{T}}\boldsymbol{b} = y_1 b_1 + \cdots + y_m b_m$$

subject to the constraints

$$\begin{aligned} y_1 a_{11} + y_2 a_{21} + \cdots + y_m a_{m1} &\geq c_1 \\ y_1 a_{12} + y_2 a_{22} + \cdots + y_m a_{m2} &\geq c_2 \\ &\vdots \\ y_1 a_{1n} + y_2 a_{2n} + \cdots + y_m a_{mn} &\geq c_n \end{aligned} \qquad (\text{or } \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} \geq \boldsymbol{c}^{\mathsf{T}})$$

and

$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0 \qquad (\text{or } \boldsymbol{y} \geq \boldsymbol{0}).$$

Note that the main constraints are written as $\leq$ for the standard maximum problem and $\geq$ for the standard minimum problem. The introductory example is a standard maximum problem.

We now present examples of four general linear programming problems. Each of these problems has been extensively studied.

Example 1. The Diet Problem. There are m different types of food, $F_1, \ldots, F_m$, that supply varying quantities of the n nutrients, $N_1, \ldots, N_n$, that are essential to good health. Let c_j be the minimum daily requirement of nutrient, N_j. Let b_i be the price per unit of food, F_i. Let a_{ij} be the amount of nutrient N_j contained in one unit of food F_i. The problem is to supply the required nutrients at minimum cost.

Let y_i be the number of units of food F_i to be purchased per day. The cost per day of such a diet is

$$b_1 y_1 + b_2 y_2 + \cdots + b_m y_m. \tag{1}$$

The amount of nutrient N_j contained in this diet is

$$a_{1j} y_1 + a_{2j} y_2 + \cdots + a_{mj} y_m$$

for $j = 1, \ldots, n$. We do not consider such a diet unless all the minimum daily requirements are met, that is, unless

$$a_{1j} y_1 + a_{2j} y_2 + \cdots + a_{mj} y_m \geq c_j \qquad \text{for } j = 1, \ldots, n. \tag{2}$$

Of course, we cannot purchase a negative amount of food, so we automatically have the constraints

$$y_1 \geq 0, y_2 \geq 0, \ldots, y_m \geq 0. \tag{3}$$

Our problem is: minimize (1) subject to (2) and (3). This is exactly the standard minimum problem.

Example 2. The Transportation Problem. There are I ports, or production plants, $P_1, \ldots, P_I$, that supply a certain commodity, and there are J markets, $M_1, \ldots, M_J$, to which this commodity must be shipped. Port P_i possesses an amount s_i of the commodity ($i = 1, 2, \ldots, I$), and market M_j must receive the amount r_j of the commodity ($j = 1, \ldots, J$). Let b_{ij} be the cost of transporting one unit of the commodity from port P_i to market M_j. The problem is to meet the market requirements at minimum transportation cost.

Let y_{ij} be the quantity of the commodity shipped from port P_i to market M_j. The total transportation cost is

$$\sum_{i=1}^{I} \sum_{j=1}^{J} y_{ij} b_{ij}. \tag{4}$$

The amount sent from port P_i is $\sum_{j=1}^{J} y_{ij}$ and since the amount available at port P_i is s_i, we must have

$$\sum_{j=1}^{J} y_{ij} \le s_i \qquad \text{for } i = 1, \ldots, I. \tag{5}$$

The amount sent to market M_j is $\sum_{i=1}^{I} y_{ij}$, and since the amount required there is r_j, we must have

$$\sum_{i=1}^{I} y_{ij} \ge r_j \qquad \text{for } j = 1, \ldots, J. \tag{6}$$

It is assumed that we cannot send a negative amount from P_I to M_j, we have

$$y_{ij} \ge 0 \qquad \text{for } i = 1, \ldots, I \text{ and } j = 1, \ldots, J. \tag{7}$$

Our problem is: minimize (4) subject to (5), (6) and (7).

Let us put this problem in the form of a standard minimum problem. The number of y variables is IJ, so $m = IJ$. But what is n? It is the total number of main constraints. There are $n = I + J$ of them, but some of the constraints are $\ge$ constraints, and some of them are $\le$ constraints. In the standard minimum problem, all constraints are $\ge$. This can be obtained by multiplying the constraints (5) by -1:

$$\sum_{j=1}^{J} (-1) y_{ij} \ge -s_j \qquad \text{for } i = 1, \ldots, I. \tag{5$'$}$$

The problem "minimize (4) subject to (5$'$), (6) and (7)" is now in standard form. In Exercise 3, you are asked to write out the matrix $\boldsymbol{A}$ for this problem.

Example 3. The Activity Analysis Problem. There are n activities, $A_1, \ldots, A_n$, that a company may employ, using the available supply of m resources, $R_1, \ldots, R_m$ (labor hours, steel, etc.). Let b_i be the available supply of resource R_i. Let a_{ij} be the amount of resource R_i used in operating activity A_j at unit intensity. Let c_j be the net value to the company of operating activity A_j at unit intensity. The problem is to choose the intensities which the various activities are to be operated to maximize the value of the output to the company subject to the given resources.

Let x_j be the intensity at which A_j is to be operated. The value of such an activity allocation is

$$\sum_{j=1}^{n} c_j x_j. \tag{8}$$

The amount of resource R_i used in this activity allocation must be no greater than the supply, b_i; that is,

$$\sum_{j=1} a_{ij} x_j \le b_i \qquad \text{for } i = 1, \ldots, m. \tag{9}$$

It is assumed that we cannot operate an activity at negative intensity; that is,

$$x_1 \ge 0, x_2 \ge 0, \ldots, x_n \ge 0. \tag{10}$$

Our problem is: maximize (8) subject to (9) and (10). This is exactly the standard maximum problem.

Example 4. The Optimal Assignment Problem. There are I persons available for J jobs. The value of person i working 1 day at job j is a_{ij}, for $i = 1, \ldots, I$, and $j = 1, \ldots, J$. The problem is to choose an assignment of persons to jobs to maximize the total value.

An assignment is a choice of numbers, x_{ij}, for $i = 1, \ldots, I$, and $j = 1, \ldots, J$, where x_{ij} represents the proportion of person i's time that is to be spent on job j. Thus,

$$\sum_{j=1}^{J} x_{ij} \le 1 \qquad \text{for } i = 1, \ldots, I \tag{11}$$

$$\sum_{i=1}^{I} x_{ij} \le 1 \qquad \text{for } j = 1, \ldots, J \tag{12}$$

and

$$x_{ij} \ge 0 \qquad \text{for } i = 1, \ldots, I \text{ and } j = 1, \ldots, J. \tag{13}$$

Equation (11) reflects the fact that a person cannot spend more than 100% of his time working, (12) means that only one person is allowed on a job at a time, and (13) says that no one can work a negative amount of time on any job. Subject to (11), (12) and (13), we wish to maximize the total value,

$$\sum_{i=1}^{I} \sum_{j=1}^{J} a_{ij} x_{ij}. \tag{14}$$

This is a standard maximum problem with $m = I + J$ and $n = IJ$.

Terminology.

The function to be maximized or minimized is called the **objective function**.

A vector, $\boldsymbol{x}$ for the standard maximum problem or $\boldsymbol{y}$ for the standard minimum problem, is said to be **feasible** if it satisfies the corresponding constraints.

The set of feasible vectors is called the **constraint set**.

A linear programming problem is said to be **feasible** if the constraint set is not empty; otherwise it is said to be **infeasible**.

A feasible maximum (resp. minimum) problem is said to be **unbounded** if the objective function can assume arbitrarily large positive (resp. negative) values at feasible vectors; otherwise, it is said to be **bounded**. Thus there are three possibilities for a linear programming problem. It may be bounded feasible, it may be unbounded feasible, and it may be infeasible.

The **value** of a bounded feasible maximum (resp, minimum) problem is the maximum (resp. minimum) value of the objective function as the variables range over the constraint set.

A feasible vector at which the objective function achieves the value is called **optimal**.

All Linear Programming Problems Can be Converted to Standard Form. A linear programming problem was defined as maximizing or minimizing a linear function subject to linear constraints. All such problems can be converted into the form of a standard maximum problem by the following techniques.

A minimum problem can be changed to a maximum problem by multiplying the objective function by -1. Similarly, constraints of the form $\sum_{j=1}^{n} a_{ij}x_j \ge b_i$ can be changed into the form $\sum_{j=1}^{n}(-a_{ij})x_j \le -b_i$. Two other problems arise.

(1) *Some constraints may be equalities.* An equality constraint $\sum_{j=1}^{n} a_{ij}x_j = b_i$ may be removed, by solving this constraint for some x_j for which $a_{ij} \neq 0$ and substituting this solution into the other constraints and into the objective function wherever x_j appears. This removes one constraint and one variable from the problem.

(2) *Some variable may not be restricted to be nonnegative.* An unrestricted variable, x_j, may be replaced by the difference of two nonnegative variables, $x_j = u_j - v_j$, where $u_j \ge 0$ and $v_j \ge 0$. This adds one variable and two nonnegativity constraints to the problem.

Any theory derived for problems in standard form is therefore applicable to general problems. However, from a computational point of view, the enlargement of the number of variables and constraints in (2) is undesirable and, as will be seen later, can be avoided.

Exercises.

1. Consider the linear programming problem: Find y_1 and y_2 to minimize $y_1 + y_2$ subject to the constraints,

$$\begin{aligned} y_1 + 2y_2 &\geq 3 \\ 2y_1 + y_2 &\geq 5 \\ y_2 &\geq 0. \end{aligned}$$

Graph the constraint set and solve.

2. Find x_1 and x_2 to maximize $ax_1 + x_2$ subject to the constraints in the numerical example of Figure 1. Find the value as a function of a.

3. Write out the matrix $\boldsymbol{A}$ for the transportation problem in standard form.

4. Put the following linear programming problem into standard form. Find x_1, x_2, x_3, x_4 to maximize $x_1 + 2x_2 + 3x_3 + 4x_4 + 5$ subject to the constraints,

$$\begin{aligned} 4x_1 + 3x_2 + 2x_3 + x_4 &\leq 10 \\ x_1 \qquad\quad - x_3 + 2x_4 &= 2 \\ x_1 + x_2 + x_3 + x_4 &\geq 1, \end{aligned}$$

and

$$x_1 \geq 0, x_3 \geq 0, x_4 \geq 0.$$

2. Duality.

To every linear program there is a dual linear program with which it is intimately connected. We first state this duality for the standard programs. As in Section 1, $\boldsymbol{c}$ and $\boldsymbol{x}$ are n-vectors, $\boldsymbol{b}$ and $\boldsymbol{y}$ are m-vectors, and $\boldsymbol{A}$ is an $m \times n$ matrix. We assume $m \geq 1$ and $n \geq 1$.

Definition. *The dual of the standard maximum problem*

$$\begin{array}{l} \textit{maximize } \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \\ \textit{subject to the constraints } \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \textit{ and } \boldsymbol{x} \geq 0 \end{array} \tag{1}$$

is defined to be the standard minimum problem

$$\begin{array}{l} \textit{minimize } \boldsymbol{y}^{\mathsf{T}}\boldsymbol{b} \\ \textit{subject to the constraints } \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} \geq \boldsymbol{c}^{\mathsf{T}} \text{ and } \boldsymbol{y} \geq 0 \end{array} \tag{2}$$

Let us reconsider the numerical example of the previous section: Find x_1 and x_2 to maximize $x_1 + x_2$ subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, and

$$\begin{array}{rcrcr} x_1 & + & 2x_2 & \leq & 4 \\ 4x_1 & + & 2x_2 & \leq & 12 \\ -x_1 & + & x_2 & \leq & 1. \end{array} \tag{3}$$

The dual of this standard maximum problem is therefore the standard minimum problem: Find y_1, y_2, and y_3 to minimize $4y_1 + 12y_2 + y_3$ subject to the constraints $y_1 \geq 0$, $y_2 \geq 0$, $y_3 \geq 0$, and

$$\begin{array}{rcrcrcr} y_1 & + & 4y_2 & - & y_3 & \geq & 1 \\ 2y_1 & + & 2y_2 & + & y_3 & \geq & 1. \end{array} \tag{4}$$

If the standard minimum problem (2) is transformed into a standard maximum problem (by multiplying $\boldsymbol{A}$, $\boldsymbol{b}$, and $\boldsymbol{c}$ by -1), its dual by the definition above is a standard minimum problem which, when transformed to a standard maximum problem (again by changing the signs of all coefficients) becomes exactly (1). Therefore, *the dual of the standard minimum problem (2) is the standard maximum problem (1). The problems (1) and (2) are said to be duals.*

The general standard maximum problem and the dual standard minimum problem may be simultaneously exhibited in the display:

$$\begin{array}{c|cccc|c} & x_1 & x_2 & \cdots & x_n & \\ \hline y_1 & a_{11} & a_{12} & \cdots & a_{1n} & \leq b_1 \\ y_2 & a_{21} & a_{22} & \cdots & a_{2n} & \leq b_2 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ y_m & a_{m1} & a_{m2} & \cdots & a_{mn} & \leq b_m \\ \hline & \geq c_1 & \geq c_2 & \cdots & \geq c_n & \end{array} \tag{5}$$

Our numerical example in this notation becomes

$$\begin{array}{c|cc|l} & x_1 & x_2 & \\ \hline y_1 & 1 & 2 & \le 4 \\ y_2 & 4 & 2 & \le 12 \\ y_3 & -1 & 1 & \le 1 \\ \hline & \ge 1 & \ge 1 & \end{array} \tag{6}$$

The relation between a standard problem and its dual is seen in the following theorem and its corollaries.

Theorem 1. *If* $\boldsymbol{x}$ *is feasible for the standard maximum problem (1) and if* $\boldsymbol{y}$ *is feasible for its dual (2), then*

$$\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \le \boldsymbol{y}^{\mathsf{T}}\boldsymbol{b}. \tag{7}$$

Proof.

$$\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \le \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{y}^{\mathsf{T}}\boldsymbol{b}.$$

The first inequality follows from $\boldsymbol{x} \ge \boldsymbol{0}$ and $\boldsymbol{c}^{\mathsf{T}} \le \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A}$. The second inequality follows from $\boldsymbol{y} \ge \boldsymbol{0}$ and $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$. ∎

Corollary 1. *If a standard problem and its dual are both feasible, then both are bounded feasible.*

Proof. If $\boldsymbol{y}$ is feasible for the minimum problem, then (7) shows that $\boldsymbol{y}^{\mathsf{T}}\boldsymbol{b}$ is an upper bound for the values of $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ for $\boldsymbol{x}$ feasible for the maximum problem. Similarly for the converse. ∎

Corollary 2. *If there exists feasible* $\boldsymbol{x}^*$ *and* $\boldsymbol{y}^*$ *for a standard maximum problem (1) and its dual (2) such that* $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^* = \boldsymbol{y}^{*\mathsf{T}}\boldsymbol{b}$, *then both are optimal for their respective problems.*

Proof. If $\boldsymbol{x}$ is any feasible vector for (1), then $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} \le \boldsymbol{y}^{*\mathsf{T}}\boldsymbol{b} = \boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}^*$. which shows that $\boldsymbol{x}^*$ is optimal. A symmetric argument works for $\boldsymbol{y}^*$. ∎

The following fundamental theorem completes the relationship between a standard problem and its dual. It states that the hypothesis of Corollary 2 are always satisfied if one of the problems is bounded feasible. The proof of this theorem is not as easy as the previous theorem and its corollaries. We postpone the proof until later when we give a constructive proof via the simplex method. (The simplex method is an algorithmic method for solving linear programming problems.) We shall also see later that this theorem contains the Minimax Theorem for finite games of Game Theory.

The Duality Theorem. *If a standard linear programming problem is bounded feasible, then so is its dual, their values are equal, and there exists optimal vectors for both problems.*

There are three possibilities for a linear program. It may be feasible bounded (f.b.), feasible unbounded (f.u.), or infeasible (i). For a program and its dual, there are therefore nine possibilities. Corollary 1 states that three of these cannot occur: If a problem and its dual are both feasible, then both must be bounded feasible. The first conclusion of the Duality Theorem states that two other possiblities cannot occur. If a program is feasible bounded, its dual cannot be infeasible. The x's in the accompanying diagram show the impossibilities. The remaining four possibilities can occur.

Standard Maximum Problem

		f.b.	f.u.	i.
	f.b.		x	x
Dual	f.u.	x	x	
	i.	x		

(8)

As an example of the use of Corollary 2, consider the following maximum problem. Find x_1, x_2, x_2, x_4 to maximize $2x_1 + 4x_2 + x_3 + x_4$, subject to the constraints $x_j \ge 0$ for all j, and

$$\begin{aligned} x_1 + 3x_2 \qquad\quad + x_4 &\le 4 \\ 2x_1 + x_2 \qquad\qquad\quad &\le 3 \\ x_2 + 4x_3 + x_4 &\le 3. \end{aligned} \tag{9}$$

The dual problem is found to be: find y_1, y_2, y_3 to minimize $4y_1 + 3y_2 + 3y_3$ subject to the constraints $y_i \ge 0$ for all i, and

$$\begin{aligned} y_1 + 2y_2 \qquad\quad &\ge 2 \\ 3y_1 + y_2 + y_3 &\ge 4 \\ 4y_3 &\ge 1 \\ y_1 \qquad + y_3 &\ge 1. \end{aligned} \tag{10}$$

The vector $(x_1, x_2, x_3, x_4) = (1, 1, 1/2, 0)$ satisfies the constraints of the maximum problem and the value of the objective function there is $13/2$. The vector $(y_1, y_2, y_3) = (11/10, 9/20, 1/4)$ satisfies the constraints of the minimum problem and has value there of $13/2$ also. Hence, both vectors are optimal for their respective problems.

As a corollary of the Duality Theorem we have

The Equilibrium Theorem. *Let $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ be feasible vectors for a standard maximum problem (1) and its dual (2) respectively. Then $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal if, and only if,*

$$y_i^* = 0 \qquad \text{for all } i \text{ for which } \textstyle\sum_{j=1}^n a_{ij}x_j^* < b_i \tag{11}$$

and

$$x_j^* = 0 \qquad \text{for all } j \text{ for which } \textstyle\sum_{i=1}^m y_i^* a_{ij} > c_j \tag{12}$$

Proof. If: Equation (11) implies that $y_i^* = 0$ unless there is equality in $\sum_j a_{ij}x_j^* \le b_i$. Hence

$$\sum_{i=1}^m y_i^* b_i = \sum_{i=1}^m y_i^* \sum_{j=1}^n a_{ij}x_j^* = \sum_{i=1}^m \sum_{j=1}^n y_i^* a_{ij} x_j^*. \tag{13}$$

Similarly Equation (12) implies

$$\sum_{i=1}^{m}\sum_{j=1}^{n} y_i^* a_{ij} x_j^* = \sum_{j=1}^{n} c_j x_j^* . \tag{14}$$

Corollary 2 now implies that $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal.

Only if: As in the first line of the proof of Theorem 1,

$$\sum_{j=1}^{n} c_j x_j^* \le \sum_{i=1}^{m}\sum_{j=1}^{n} y_i^* a_{ij} x_j^* \le \sum_{i=1}^{m} y_i^* b_i . \tag{15}$$

By the Duality Theorem, if $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal, the left side is equal to the right side so we get equality throughout. The equality of the first and second terms may be written as

$$\sum_{j=1}^{n}\left(c_j - \sum_{i=1}^{m} y_i^* a_{ij}\right) x_j^* = 0. \tag{16}$$

Since $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are feasible, each term in this sum is nonnegative. The sum can be zero only if each term is zero. Hence if $\sum_{i=1}^{m} y_i^* a_{ij} > c_j$, then $x_j^* = 0$. A symmetric argument shows that if $\sum_{j=1}^{n} a_{ij} x_j^* < b_i$, then $y_i^* = 0$. ∎

Equations (11) and (12) are sometimes called the **complementary slackness conditions**. They require that a strict inequality (a slackness) in a constraint in a standard problem implies that the complementary constraint in the dual be satisfied with equality.

As an example of the use of the Equilibrium Theorem, let us solve the dual to the introductory numerical example. Find y_1, y_2, y_3 to minimize $4y_1 + 12y_2 + y_3$ subject to $y_1 \ge 0$, $y_2 \ge 0$, $y_3 \ge 0$, and

$$\begin{aligned} y_1 + 4y_2 - y_3 &\ge 1 \\ 2y_1 + 2y_2 + y_3 &\ge 1 . \end{aligned} \tag{17}$$

We have already solved the dual problem and found that $x_1^* > 0$ and $x_2^* > 0$. Hence, from (12) we know that the optimal $\boldsymbol{y}^*$ gives equality in both inequalities in (17) (2 equations in 3 unknowns). If we check the optimal $\boldsymbol{x}^*$ in the first three main constraints of the maximum problem, we find equality in the first two constraints, but a strict inequality in the third. From condition (11), we conclude that $y_3^* = 0$. Solving the two equations,

$$\begin{aligned} y_1 + 4y_2 &= 1 \\ 2y_1 + 2y_2 &= 1 \end{aligned}$$

we find $(y_1^*, y_2^*) = (1/3, 1/6)$. Since this vector is feasible, the "if" part of the Equilibrium Theorem implies it is optimal. As a check we may find the value, $4(1/3) + 12(1/6) = 10/3$, and see it is the same as for the maximum problem.

In summary, if you conjecture a solution to one problem, you may solve for a solution to the dual using the complementary slackness conditions, and then see if your conjecture is correct.

Interpretation of the dual. In addition to the help it provides in finding a solution, the dual problem offers advantages in the interpretation of the original, primal problem. In practical cases, the dual problem may be analyzed in terms of the primal problem.

As an example, consider the diet problem, a standard minimum problem of the form (2). Its dual is the standard maximum problem (1). First, let us find an interpretation of the dual variables, $x_1, x_2, \ldots, x_n$. In the dual constraint,

$$\sum_{j=1}^{n} a_{ij}x_j \leq b_i, \tag{18}$$

the variable b_i is measured as price per unit of food, F_i, and a_{ij} is measured as units of nutrient N_j per unit of food F_i. To make the two sides of the constraint comparable, x_j must be measured in of price per unit of food F_i. (This is known as a **dimensional analysis**.) Since c_j is the amount of N_j required per day, the objective function, $\sum_1^n c_j x_j$, represents the total price of the nutrients required each day. Someone is evidently trying to choose vector $\boldsymbol{x}$ of prices for the nutrients to maximize the total worth of the required nutrients per day, subject to the constraints that $\boldsymbol{x} \geq \mathbf{0}$, and that the total value of the nutrients in food F_i, namely, $\sum_{j=1}^n a_{ij}x_j$, is not greater than the actual cost, b_i, of that food.

We may imagine that an entrepreneur is offering to sell us the nutrients without the food, say in the form of vitamin or mineral pills. He offers to sell us the nutrient N_j at a price x_j per unit of N_j. If he wants to do business with us, he would choose the x_j so that price he charges for a nutrient mixture substitute of food F_i would be no greater than the original cost to us of food F_i. This is the constraint, (18). If this is true for all i, we may do business with him. So he will choose $\boldsymbol{x}$ to maximize his total income, $\sum_1^n c_j x_j$, subject to these constraints. (Actually we will not save money dealing with him since the duality theorem says that our minimum, $\sum_1^m y_i b_i$, is equal to his maximum, $\sum_1^n c_j x_j$.) The optimal price, x_j, is referred to as the **shadow price** of nutrient N_j. Although no such entrepreneur exists, the shadow prices reflect the actual values of the nutrients as shaped by the market prices of the foods, and our requirements of the nutrients.

Exercises.

1. Find the dual to the following standard minimum problem. Find y_1, y_2 and y_3 to minimize $y_1 + 2y_2 + y_3$, subject to the constraints, $y_i \geq 0$ for all i, and

$$\begin{aligned} y_1 - 2y_2 + y_3 &\geq 2 \\ -y_1 + y_2 + y_3 &\geq 4 \\ 2y_1 + y_3 &\geq 6 \\ y_1 + y_2 + y_3 &\geq 2. \end{aligned}$$

2. Consider the problem of Exercise 1. Show that $(y_1, y_2, y_3) = (2/3, 0, 14/3)$ is optimal for this problem, and that $(x_1, x_2, x_3, x_4) = (0, 1/3, 2/3, 0)$ is optimal for the dual.

3. Consider the problem: Maximize $3x_1+2x_2+x_3$ subject to $x_1 \geq 0$, $x_2 \geq 0$, $x_3 \geq 0$, and

$$\begin{aligned} x_1 - x_2 + \ x_3 &\leq 4 \\ 2x_1 + x_2 + 3x_3 &\leq 6 \\ -x_1 \qquad\quad + 2x_3 &\leq 3 \\ x_1 + x_2 + \ x_3 &\leq 8. \end{aligned}$$

(a) State the dual minimum problem.

(b) Suppose you suspect that the vector $(x_1, x_2, x_3) = (0, 6, 0)$ is optimal for the maximum problem. Use the Equilibrium Theorem to solve the dual problem, and then show that your suspicion is correct.

4. (a) State the dual to the transportation problem.

(b) Give an interpretation to the dual of the transportation problem.

3. The Pivot Operation.

Consider the following system of equations.

$$\begin{array}{rcrcrcl} 3y_1 & + & 2y_2 & & & = & s_1 \\ y_1 & - & 3y_2 & + & 3y_3 & = & s_2 \\ 5y_1 & + & y_2 & + & y_3 & = & s_3 \end{array} \tag{1}$$

This expresses *dependent* variables, s_1, s_2 and s_3 in terms of the *independent* variables, y_1, y_2 and y_3. Suppose we wish to obtain y_2, s_2 and s_3 in terms of y_1, s_1 and y_3. We solve the first equation for y_2,

$$y_2 = \tfrac{1}{2}s_1 - \tfrac{3}{2}y_1,$$

and substitute this value of y_2 into the other equations.

$$\begin{array}{rcrcrcl} y_1 & - & 3(\frac{1}{2}s_1 - \frac{3}{2}y_1) & + & 3y_3 & = & s_2 \\ 5y_1 & + & (\frac{1}{2}s_1 - \frac{3}{2}y_1) & + & y_3 & = & s_3\,. \end{array}$$

These three equations simplified become

$$\begin{array}{rcrcrcl} -\frac{3}{2}y_1 & + & \frac{1}{2}s_1 & & & = & y_2 \\ \frac{11}{2}y_1 & - & \frac{3}{2}s_1 & + & 3y_3 & = & s_2 \\ \frac{7}{2}y_1 & + & \frac{1}{2}s_1 & + & y_3 & = & s_3. \end{array} \tag{2}$$

This example is typical of the following class of problems. We are given a system of n linear functions in m unknowns, written in matrix form as

$$\boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} = \boldsymbol{s}^{\mathsf{T}} \tag{3}$$

where $\boldsymbol{y}^{\mathsf{T}} = (y_1, \ldots, y_m)$, $\boldsymbol{s}^{\mathsf{T}} = (s_1, \ldots, s_n)$, and

$$\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}.$$

Equation (3) therefore represents the system

$$\begin{array}{ccccccccc} y_1a_{11} & + & \cdots & + & y_ia_{i1} & + & \cdots & + & y_ma_{m1} & = & s_1 \\ \vdots & & & & \vdots & & & & & & \vdots \\ y_1a_{1j} & + & \cdots & + & y_ia_{ij} & + & \cdots & + & y_ma_{mj} & = & s_j \\ \vdots & & & & \vdots & & & & & & \vdots \\ y_1a_{1n} & + & \cdots & + & y_ia_{in} & + & \cdots & + & y_ma_{mn} & = & s_n. \end{array} \tag{4}$$

In this form, $s_1, \ldots, s_n$ are the dependent variables, and $y_1, \ldots, y_m$ are the independent variables.

Suppose that we want to interchange one of the dependent variables for one of the independent variables. For example, we might like to have $s_1, \ldots, s_{j-1}, y_i, s_{j+1}, \ldots, s_n$ in terms of $y_1, \ldots, y_{i-1}, s_j, y_{i+1}, \ldots, y_m$, with y_i and s_j interchanged. This may be done if and only if $a_{ij} \neq 0$. If $a_{ij} \neq 0$, we may take the jth equation and solve for y_i, to find

$$y_i = \frac{1}{a_{ij}}(-y_1 a_{1j} - \cdots - y_{i-1}a_{(i-1)j} + s_j - y_{i+1}a_{(i+1)j} - \cdots - y_m a_{mj}). \tag{5}$$

Then we may substitute this expression for y_i into the other equations. For example, the kth equation becomes

$$y_1\left(a_{1k} - \frac{a_{ik}a_{1j}}{a_{ij}}\right) + \cdots + s_j\left(\frac{a_{ik}}{a_{ij}}\right) + \cdots + y_m\left(a_{mk} - \frac{a_{ik}a_{mj}}{a_{ij}}\right) = s_k. \tag{6}$$

We arrive at a system of equations of the form

$$\begin{array}{ccccccccc} y_1\hat{a}_{11} & + & \cdots & + & s_j\hat{a}_{i1} & + & \cdots + & y_m\hat{a}_{m1} & = & s_1 \\ \vdots & & & & \vdots & & & & & \vdots \\ y_1\hat{a}_{1j} & + & \cdots & + & s_j\hat{a}_{ij} & + & \cdots + & y_m\hat{a}_{mj} & = & y_i \\ \vdots & & & & \vdots & & & & & \vdots \\ y_1\hat{a}_{1n} & + & \cdots & + & s_j\hat{a}_{in} & + & \cdots + & y_m\hat{a}_{mn} & = & s_n. \end{array} \tag{7}$$

The relation between the $\hat{a}_{ij}$'s and the a_{ij}'s may be found from (5) and (6).

$$\begin{aligned} \hat{a}_{ij} &= \frac{1}{a_{ij}} \\ \hat{a}_{hj} &= -\frac{a_{hj}}{a_{ij}} \quad \text{for } h \neq i \\ \hat{a}_{ik} &= \frac{a_{ik}}{a_{ij}} \quad \text{for } k \neq j \\ \hat{a}_{hk} &= a_{hk} - \frac{a_{ik}a_{hj}}{a_{ij}} \quad \text{for } k \neq j \text{ and } h \neq i. \end{aligned}$$

Let us mechanize this procedure. We write the original $m \times n$ matrix $\boldsymbol{A}$ in a display with y_1 to y_m down the left side and s_1 to s_n across the top. This display is taken to represent the original system of equations, (3). To indicate that we are going to interchange y_i and s_j, we circle the entry a_{ij}, and call this entry *the pivot.* We draw an arrow to the new display with y_i and s_j interchanged, and the new entries of $\hat{a}_{ij}$'s. The new display, of course, represents the equations (7), which are equivalent to the equations (4).

$$\begin{array}{c|ccccc} & s_1 & \cdots & s_j & \cdots & s_n \\ \hline y_1 & a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ y_i & a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ y_m & a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \\ \hline \end{array} \quad \longrightarrow \quad \begin{array}{c|ccccc} & s_1 & \cdots & y_i & \cdots & s_n \\ \hline y_1 & \hat{a}_{11} & \cdots & \hat{a}_{1j} & \cdots & \hat{a}_{1n} \\ \vdots & \vdots & & \vdots & & \vdots \\ s_j & \hat{a}_{i1} & \cdots & \hat{a}_{ij} & \cdots & \hat{a}_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ y_m & \hat{a}_{m1} & \cdots & \hat{a}_{mj} & \cdots & \hat{a}_{mn} \\ \hline \end{array}$$

In the introductory example, this becomes

$$\begin{array}{c|ccc} & s_1 & s_2 & s_3 \\ \hline y_1 & 3 & 1 & 5 \\ y_2 & 2 & -3 & 1 \\ y_3 & 0 & 3 & 1 \end{array} \quad \longrightarrow \quad \begin{array}{c|ccc} & y_2 & s_2 & s_3 \\ \hline y_1 & -3/2 & 11/2 & 7/2 \\ s_1 & 1/2 & -3/2 & 1/2 \\ y_3 & 0 & 3 & 1 \end{array}$$

(Note that the matrix $\boldsymbol{A}$ appearing on the left is the transpose of the numbers as they appear in equations (1).) We say that we have *pivoted* around the circled entry from the first matrix to the second.

The pivoting rules may be summarized in symbolic notation:

$$\boxed{\begin{array}{cc} p & r \\ c & q \end{array}} \quad \longrightarrow \quad \boxed{\begin{array}{cc} 1/p & r/p \\ -c/p & q-(rc/p) \end{array}}$$

This signifies: The pivot quantity goes into its reciprocal. Entries in the same row as the pivot are divided by the pivot. Entries in the same column as the pivot are divided by the pivot and changed in sign. The remaining entries are reduced in value by the product of the corresponding entries in the same row and column as themselves and the pivot, divided by the pivot.

We pivot the introductory example twice more.

$$\begin{array}{c|ccc} & y_2 & s_2 & s_3 \\ \hline y_1 & -3/2 & 11/2 & 7/2 \\ s_1 & 1/2 & -3/2 & 1/2 \\ y_3 & 0 & 3 & 1 \end{array} \quad \longrightarrow \quad \begin{array}{c|ccc} & y_2 & s_2 & y_3 \\ \hline y_1 & -3/2 & -5 & -7/2 \\ s_1 & 1/2 & -3 & -1/2 \\ s_3 & 0 & 3 & 1 \end{array} \quad \longrightarrow \quad \begin{array}{c|ccc} & y_2 & y_1 & y_3 \\ \hline s_2 & .3 & -.2 & .7 \\ s_1 & 1.4 & -.6 & 1.6 \\ s_3 & -.9 & .6 & -1.1 \end{array}$$

The last display expresses y_1, y_2 and y_3 in terms of s_1, s_2 and s_3. Rearranging rows and columns, we can find $\boldsymbol{A}^{-1}$.

$$\begin{array}{c|ccc} & y_1 & y_2 & y_3 \\ \hline s_1 & -.6 & 1.4 & 1.6 \\ s_2 & -.2 & .3 & .7 \\ s_3 & .6 & -.9 & -1.1 \end{array} \qquad \text{so} \qquad \boldsymbol{A}^{-1} = \begin{pmatrix} -.6 & 1.4 & 1.6 \\ -.2 & .3 & .7 \\ .6 & -.9 & -1.1 \end{pmatrix}$$

The arithmetic may be checked by checking $\boldsymbol{A}\boldsymbol{A}^{-1} = \boldsymbol{I}$.

Exercise. Solve the system of equations $\boldsymbol{y}^\mathsf{T}\boldsymbol{A} = \boldsymbol{s}^\mathsf{T}$,

$$\begin{array}{c|cccc} & s_1 & s_2 & s_3 & s_4 \\ \hline y_1 & 1 & 1 & 0 & 2 \\ y_2 & 0 & 2 & -1 & 1 \\ y_3 & -1 & 1 & 0 & 3 \\ y_4 & 1 & -2 & 2 & 1 \end{array}$$

for y_1, y_2, y_3 and y_4, by pivoting, in order, about the entries

(1) first row, first column (circled)
(2) second row, third column (the pivot is -1)
(3) fourth row, second column (the pivot turns out to be 1)
(4) third row, fourth column (the pivot turns out to be 3).

Rearrange rows and columns to find $\boldsymbol{A}^{-1}$, and check that the product with $\boldsymbol{A}$ gives the identity matrix.

4. The Simplex Method

The Simplex Tableau. Consider the standard minimum problem: Find $\boldsymbol{y}$ to minimize $\boldsymbol{y}^\mathsf{T}\boldsymbol{b}$ subject to $\boldsymbol{y} \geq \boldsymbol{0}$ and $\boldsymbol{y}^\mathsf{T}\boldsymbol{A} \geq \boldsymbol{c}^\mathsf{T}$. It is useful conceptually to transform this last set of inequalities into equalities. For this reason, we add *slack variables*, $\boldsymbol{s}^\mathsf{T} = \boldsymbol{y}^\mathsf{T}\boldsymbol{A} - \boldsymbol{c}^\mathsf{T} \geq \boldsymbol{0}$. The problem can be restated: Find $\boldsymbol{y}$ and $\boldsymbol{s}$ to minimize $\boldsymbol{y}^\mathsf{T}\boldsymbol{b}$ subject to $\boldsymbol{y} \geq \boldsymbol{0}$, $\boldsymbol{s} \geq \boldsymbol{0}$ and $\boldsymbol{s}^\mathsf{T} = \boldsymbol{y}^\mathsf{T}\boldsymbol{A} - \boldsymbol{c}^\mathsf{T}$.

We write this problem in a tableau to represent the linear equations $\boldsymbol{s}^\mathsf{T} = \boldsymbol{y}^\mathsf{T}\boldsymbol{A} - \boldsymbol{c}^\mathsf{T}$.

$$\begin{array}{c|cccc|c}
 & s_1 & s_2 & \cdots & s_n & \\
\hline
y_1 & a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
y_2 & a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
y_m & a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \\
\hline
1 & -c_1 & -c_2 & \cdots & -c_n & 0
\end{array}$$

The Simplex Tableau

The last column represents the vector whose inner product with $\boldsymbol{y}$ we are trying to minimize.

If $-\boldsymbol{c} \geq \boldsymbol{0}$ and $\boldsymbol{b} \geq \boldsymbol{0}$, there is an obvious solution to the problem; namely, the minimum occurs at $\boldsymbol{y} = \boldsymbol{0}$ and $\boldsymbol{s} = -\boldsymbol{c}$, and the minimum value is $\boldsymbol{y}^\mathsf{T}\boldsymbol{b} = 0$. This is feasible since $\boldsymbol{y} \geq \boldsymbol{0}$, $\boldsymbol{s} \geq \boldsymbol{0}$, and $\boldsymbol{s}^\mathsf{T} = \boldsymbol{y}^\mathsf{T}\boldsymbol{A} - \boldsymbol{c}$, and yet $\sum y_i b_i$ cannot be made any smaller than 0, since $\boldsymbol{y} \geq \boldsymbol{0}$, and $\boldsymbol{b} \geq \boldsymbol{0}$.

Suppose then we cannot solve this problem so easily because there is at least one negative entry in the last column or last row. (exclusive of the corner). Let us pivot about a_{11} (suppose $a_{11} \neq 0$), including the last column and last row in our pivot operations. We obtain this tableau:

$$\begin{array}{c|cccc|c}
 & y_1 & s_2 & \cdots & s_n & \\
\hline
s_1 & \hat{a}_{11} & \hat{a}_{12} & \cdots & \hat{a}_{1n} & \hat{b}_1 \\
y_2 & \hat{a}_{21} & \hat{a}_{22} & \cdots & \hat{a}_{2n} & \hat{b}_2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
y_m & \hat{a}_{m1} & \hat{a}_{m2} & \cdots & \hat{a}_{mn} & \hat{b}_m \\
\hline
1 & -\hat{c}_1 & -\hat{c}_2 & \cdots & -\hat{c}_n & \hat{v}
\end{array}$$

Let $\boldsymbol{r} = (r_1, \ldots, r_n) = (y_1, s_2, \ldots, s_n)$ denote the variables on top, and let $\boldsymbol{t} = (t_1, \ldots, t_m) = (s_1, y_2, \ldots, y_m)$ denote the variables on the left. The set of equations $\boldsymbol{s}^\mathsf{T} = \boldsymbol{y}^\mathsf{T}\boldsymbol{A} - \boldsymbol{c}^\mathsf{T}$ is then equivalent to the set $\boldsymbol{r}^\mathsf{T} = \boldsymbol{t}^\mathsf{T}\hat{\boldsymbol{A}} - \hat{\boldsymbol{c}}$, represented by the new tableau. Moreover, the objective function $\boldsymbol{y}^\mathsf{T}\boldsymbol{b}$ may be written (replacing y_1 by its value in terms of s_1)

$$\begin{aligned}
\sum_{i=1}^{m} y_i b_i &= \frac{b_1}{a_{11}} s_1 + (b_2 - \frac{a_{21}b_1}{a_{11}})y_2 + \cdots + (b_m - \frac{a_{m1}b_1}{a_{11}})y_2 + \frac{c_1 b_1}{a_{11}} \\
&= \boldsymbol{t}^\mathsf{T}\hat{\boldsymbol{b}} + \hat{v}
\end{aligned}$$

This is represented by the last column in the new tableau.

We have transformed our problem into the following: Find vectors, $\boldsymbol{y}$ and $\boldsymbol{s}$, to minimize $\boldsymbol{t}^{\mathsf{T}}\hat{\boldsymbol{b}}$ subject to $\boldsymbol{y} \geq \boldsymbol{0}$, $\boldsymbol{s} \geq 0$ and $\boldsymbol{r} = \boldsymbol{t}^{\mathsf{T}}\hat{\boldsymbol{A}} - \hat{\boldsymbol{c}}$ (where $\boldsymbol{t}^{\mathsf{T}}$ represents the vector, $(s_1, y_2, \ldots, y_m)$, and $\boldsymbol{r}^{\mathsf{T}}$ represents $(y_1, s_2, \ldots, s_n)$). This is just a *restatement* of the original problem.

Again, if $-\hat{\boldsymbol{c}} \geq \boldsymbol{0}$ and $\hat{\boldsymbol{b}} \geq \boldsymbol{0}$, we have the obvious solution: $\boldsymbol{t} = \boldsymbol{0}$ and $\boldsymbol{r} = -\hat{\boldsymbol{c}}$ with value $\hat{v}$.

It is easy to see that this process may be continued. This gives us a method, admittedly not very systematic, for finding the solution.

The Pivot Madly Method. *Pivot madly until you suddenly find that all entries in the last column and last row (exclusive of the corner) are nonnegative Then, setting the variables on the left to zero and the variables on top to the corresponding entry on the last row gives the solution. The value is the lower right corner.*

This same "method" may be used to solve the dual problem: Maximize $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ subject to $\boldsymbol{x} \geq \boldsymbol{0}$ and $\boldsymbol{Ax} \leq \boldsymbol{b}$. This time, we add the slack variables $\boldsymbol{u} = \boldsymbol{b} - \boldsymbol{Ax}$. The problem becomes: Find $\boldsymbol{x}$ and $\boldsymbol{u}$ to maximize $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$ subject to $\boldsymbol{x} \geq \boldsymbol{0}$, $\boldsymbol{u} \geq \boldsymbol{0}$, and $\boldsymbol{u} = \boldsymbol{b} - \boldsymbol{Ax}$. We may use the same tableau to solve this problem if we write the constraint, $\boldsymbol{u} = \boldsymbol{b} - \boldsymbol{Ax}$ as $-\boldsymbol{u} = \boldsymbol{Ax} - \boldsymbol{b}$.

$$
\begin{array}{c|cccc|c}
 & x_1 & x_2 & \cdots & x_n & -1 \\
\hline
-u_1 & a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
-u_2 & a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
-u_m & a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \\
\hline
 & -c_1 & -c_2 & \cdots & -c_n & 0
\end{array}
$$

We note as before that if $-\boldsymbol{c} \geq \boldsymbol{0}$ and $\boldsymbol{b} \geq \boldsymbol{0}$, then the solution is obvious: $\boldsymbol{x} = \boldsymbol{0}$, $\boldsymbol{u} = \boldsymbol{b}$, and value equal to zero (since the problem is equivalent to minimizing $-\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}$).

Suppose we want to pivot to interchange u_1 and x_1 and suppose $a_{11} \neq 0$. The equations

$$
\begin{aligned}
-u_1 &= a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n - b_1 \\
-u_2 &= a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n - b_2 \quad \text{etc.}
\end{aligned}
$$

become

$$
\begin{aligned}
-x_1 &= \frac{1}{a_{11}}u_1 + \frac{a_{12}}{a_{11}}x_2 + \frac{a_{1n}}{a_{11}}x_n - \frac{b_1}{a_{11}} \\
-u_2 &= -\frac{a_{21}}{a_{11}}u_1 + \left(a_{22} - \frac{a_{21}a_{12}}{a_{11}}\right)x_2 + \cdots \quad \text{etc.}
\end{aligned}
$$

In other words, the same pivot rules apply!

$$
\boxed{\begin{array}{cc} p & r \\ c & q \end{array}} \longrightarrow \boxed{\begin{array}{cc} 1/p & r/p \\ -c/p & q - (rc/p) \end{array}}
$$

If you pivot until the last row and column (exclusive of the corner) are nonnegative, you can find the solution to the dual problem and the primal problem at the same time.

In summary, you may write the simplex tableau as

$$
\begin{array}{c|cccc|c}
 & x_1 & x_2 & \cdots & x_n & -1 \\
\hline
y_1 & a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\
y_2 & a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\
\vdots & \vdots & \vdots & & \vdots & \vdots \\
y_m & a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \\
\hline
1 & -c_1 & -c_2 & \cdots & -c_n & 0
\end{array}
$$

If we pivot until all entries in the last row and column (exclusive of the corner) are nonnegative, then the value of the program and its dual is found in the lower right corner. The solution of the minimum problem is obtained by letting the y_i's on the left be zero and the y_i's on top be equal to the corresponding entry in the last row. The solution of the maximum problem is obtained by letting the x_j's on top be zero, and the x_j's on the left be equal to the corresponding entry in the last column.

Example 1. Consider the problem: Maximize $5x_1 + 2x_2 + x_3$ subject to all $x_j \geq 0$ and

$$
\begin{array}{rcrcrl}
x_1 & + & 3x_2 & - & x_3 & \leq 6 \\
 & & x_2 & + & x_3 & \leq 4 \\
3x_1 & + & x_2 & & & \leq 7.
\end{array}
$$

The dual problem is : Minimize $6y_1 + 4y_2 + 7y_3$ subject to all $y_i \geq 0$ and

$$
\begin{array}{rcrcrl}
y_1 & & & + & 3y_3 & \geq 5 \\
3y_1 & + & y_2 & + & y_3 & \geq 2 \\
-y_1 & + & y_2 & & & \geq 1.
\end{array}
$$

The simplex tableau is

$$
\begin{array}{c|ccc|c}
 & x_1 & x_2 & x_3 & \\
\hline
y_1 & 1 & 3 & -1 & 6 \\
y_2 & 0 & 1 & 1 & 4 \\
y_3 & 3 & 1 & 0 & 7 \\
\hline
 & -5 & -2 & -1 & 0
\end{array}\ .
$$

If we pivot once about the circled points, interchanging y_2 with x_3, and y_3 with x_1, we arrive at

$$
\begin{array}{c|ccc|c}
 & y_3 & x_2 & y_2 & \\
\hline
y_1 & & & & 23/3 \\
x_3 & & & & 4 \\
x_1 & & & & 7/3 \\
\hline
 & 5/3 & 2/3 & 1 & 47/3
\end{array}\ .
$$

From this we can read off the solution to both problems. The value of both problems is $47/3$. The optimal vector for the maximum problem is $x_1 = 7/3$, $x_2 = 0$ and $x_3 = 4$. The optimal vector for the minimum problem is $y_1 = 0$, $y_2 = 1$ and $y_3 = 5/3$.

The Simplex Method is just the pivot madly method with the crucial added ingredient that tells you which points to choose as pivots to approach the solution systematically.

Suppose after pivoting for a while, one obtains the tableau

$$\begin{array}{c|c|c} & \boldsymbol{r} & \\ \hline \boldsymbol{t} & \boldsymbol{A} & \boldsymbol{b} \\ \hline & -\boldsymbol{c} & v \end{array}$$

where $\boldsymbol{b} \geq 0$. Then one immediately obtains a feasible point for the maximum problem (in fact an extreme point of the constraint set) by letting $\boldsymbol{r} = \boldsymbol{0}$ and $\boldsymbol{t} = \boldsymbol{b}$, the value of the program at this point being v.

Similarly, if one had $-\boldsymbol{c} \geq \boldsymbol{0}$, one would have a feasible point for the minimum problem by setting $\boldsymbol{r} = -\boldsymbol{c}$ and $\boldsymbol{t} = \boldsymbol{0}$.

Pivot Rules for the Simplex Method. We first consider the case where we have already found a feasible point for the maximum problem.

Case 1: $\boldsymbol{b} \geq \boldsymbol{0}$. *Take any column with last entry negative, say column j_0 with $-c_{j_0} < 0$. Among those i for which $a_{i,j_0} > 0$, choose that i_0 for which the ratio $b_i/a_{i,j_0}$ is smallest. (If there are ties, choose any such i_0.) Pivot around a_{i_0,j_0}.*

Here are some examples. In the tableaux, the possible pivots are circled.

$$\begin{array}{c|cccc|c} & r_1 & r_2 & r_3 & r_4 & \\ \hline t_1 & 1 & -1 & 3 & 1 & 6 \\ t_2 & 0 & 1 & 2 & 4 & 4 \\ t_3 & 3 & 0 & 3 & 1 & 7 \\ \hline & -5 & -1 & -4 & 2 & 0. \end{array} \qquad \begin{array}{c|ccc|c} & r_1 & r_2 & r_3 & \\ \hline t_1 & 5 & 2 & 1 & 1 \\ t_2 & 1 & 0 & -1 & 0 \\ t_3 & 3 & -1 & 4 & 2 \\ \hline & -1 & -2 & -3 & 0 \end{array}.$$

What happens if you cannot apply the pivot rule for case 1? First of all it might happen that $-c_j \geq 0$ for all j. Then you are finished — you have found the solution. The only other thing that can go wrong is that after finding some $-c_{j_0} < 0$, you find $a_{i,j_0} \leq 0$ for all i. If so, then *the maximum problem is unbounded feasible.* To see this, consider any vector, $\boldsymbol{r}$, such that $r_{j_0} > 0$, and $r_j = 0$ for $j \neq j_0$. Then $\boldsymbol{r}$ is feasible for the maximum problem because

$$t_i = \sum_{j=1}^{n}(-a_{i,j})r_j + b_i = -a_{i,j_0}r_{j_0} + b_i \geq 0$$

for all i, and this feasible vector has value $\sum c_j r_j = c_{j_0} r_{j_0}$, which can be made as large as desired by making r_{j_0} sufficiently large.

Such pivoting rules are good to use because:
1. After pivoting, the $\boldsymbol{b}$ column stays nonnegative, so you still have a feasible point for the maximum problem.
2. The value of the new tableau is never less than that of the old (generally it is greater).

Proof of 1. Let the new tableau be denoted with hats on the variables. We are to show $\hat{b}_i \ge 0$ for all i. For $i = i_0$ we have $\hat{b}_{i_0} = b_{i_0}/a_{i_0,j_0}$ still nonnegative, since $a_{i_0,j_0} > 0$. For $i \ne i_0$, we have

$$\hat{b}_i = b_i - \frac{a_{i,j_0} b_{i_0}}{a_{i_0,j_0}}.$$

If $a_{i,j_0} \le 0$, then $\hat{b}_i \ge b_i \ge 0$ since $a_{i,j_0} b_{i_0}/a_{i_0,j_0} \le 0$. If $a_{i,j_0} > 0$, then by the pivot rule, $b_i/a_{i,j_0} \ge b_{i_0}/a_{i_0,j_0}$, so that $\hat{b}_i \ge b_i - b_i = 0$.

Proof of 2. $\hat{v} = v - (-c_{j_0})(b_{i_0}/a_{i_0,j_0}) \ge v$, because $-c_{j_0} < 0$, $a_{i_0,j_0} > 0$, and $b_{i_0} \ge 0$.

These two properties imply that if you keep pivoting according to the rules of the simplex method and the value keeps getting greater, then because there are only a finite number of tableaux, you will eventually terminate either by finding the solution or by finding that the problem is unbounded feasible. In the proof of 2, note that v increases unless the pivot is in a row with $b_{i_0} = 0$. So the simplex method will eventually terminate unless we keep pivoting in rows with a zero in the last column.

Example 2. Maximize $x_1 + x_2 + 2x_3$ subject to all $x_i \ge 0$ and

$$\begin{array}{rcl} x_2 + 2x_3 & \le & 3 \\ -x_1 + 3x_3 & \le & 2 \\ 2x_1 + x_2 + x_3 & \le & 1. \end{array}$$

We pivot according to the rules of the simplex method, first pivoting about row three column 2.

$$\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & \\ \hline y_1 & 0 & 1 & 2 & 3 \\ y_2 & -1 & 0 & 3 & 2 \\ y_3 & 2 & 1 & 1 & 1 \\ \hline & -1 & -1 & -2 & 0 \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_1 & y_3 & x_3 & \\ \hline y_1 & -2 & -1 & 1 & 2 \\ y_2 & -1 & 0 & 3 & 2 \\ x_2 & 2 & 1 & 1 & 1 \\ \hline & 1 & 1 & -1 & 1 \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_1 & y_3 & y_2 & \\ \hline y_1 & & & & 4/3 \\ x_3 & & & & 2/3 \\ x_2 & & & & 1/3 \\ \hline & 2/3 & 1 & 1/3 & 5/3 \end{array}$$

The value for the program and its dual is $5/3$. The optimal vector for the maximum problem is given by $x_1 = 0$, $x_2 = 1/3$ and $x_3 = 2/3$. The optimal vector for the minimum problem is given by $y_1 = 0$, $y_2 = 1/3$ and $y_3 = 1$.

Case 2: Some b_i are negative. Take the first negative b_i, say $b_k < 0$ (where $b_1 \ge 0, \ldots, b_{k-1} \ge 0$). Find any negative entry in row k, say $a_{k,j_0} < 0$. (The pivot will be in column j_0.) Compare $b_k/a_{k,j_0}$ and the $b_i/a_{i,j_0}$ for which $b_i \ge 0$ and $a_{i,j_0} > 0$, and choose i_0 for which this ratio is smallest (i_0 may be equal to k). You may choose any such i_0 if there are ties. Pivot on a_{i_0,j_0}.

Here are some examples. In the tableaux, the possible pivots according to the rules for Case 2 are circled.

	r_1	r_2	r_3	r_4	
t_1	2	3	1	0	2
t_2	−1	−1	−2	2	−1
t_3	1	−1	0	−1	−2
	1	−1	0	2	0

	r_1	r_2	r_3	
t_1	1	−1	0	0
t_2	−1	−2	−3	−2
t_3	2	2	1	1
	1	0	−2	0

What if the rules of Case 2 cannot be applied? The only thing that can go wrong is that for $b_k < 0$, we find $a_{kj} \geq 0$ for all j. If so, then *the maximum problem is infeasible*, since the equation for row k reads

$$-t_k = \sum_{j=1}^{n} a_{kj} r_j - b_k.$$

For all feasible vectors ($\boldsymbol{t} \geq \mathbf{0}$, $\boldsymbol{r} \geq \mathbf{0}$), the left side is negative or zero, and the right side is positive.

The objective in Case 2 is to get to Case 1, since we know what to do from there. The rule for Case 2 is likely to be good because:
1. The nonnegative b_i stay nonnegative after pivoting, and
2. b_k has become no smaller (generally it gets larger).

Proof of 1. Suppose $b_i \geq 0$, so $i \neq k$. If $i = i_0$, then $\hat{b}_{i_0} = b_{i_0}/a_{i_0,j_0} \geq 0$. If $i \neq i_0$, then

$$\hat{b}_i = b_i - \frac{a_{i,j_0}}{ai_0, j_0} b_{i_0}.$$

Now $b_{i_0}/a_{i_0,j_0} \geq 0$. Hence, if $a_{i,j_0} \leq 0$, then $\hat{b}_i \geq b_i \geq 0$. and if $a_{i,j_0} > 0$, then $b_{i_0}/a_{i_0,j_0} \leq b_i/a_{i,j_0}$, so that $\hat{b}_i \geq b_i - b_i = 0$.

Proof of 2. If $k = i_0$, then $\hat{b}_k = b_k/a_{k,j_0} > 0$ (both $b_k < 0$ and $a_{k,j_0} < 0$). If $k \neq i_0$, then

$$\hat{b}_k = b_k - \frac{a_{k,j_0}}{a_{i_0,j_0}} b_{i_0} \geq b_k,$$

since $b_{i_0}/a_{i_0,j_0} \geq 0$ and $a_{k,j_0} < 0$.

These two properties imply that if you keep pivoting according to the rule for Case 2, and b_k keeps getting larger, you will eventually get $b_k \geq 0$, and be one coefficient closer to having all $b_i \geq 0$. Note in the proof of 2, that if $b_{i_0} > 0$, then $\hat{b}_k > b_k$.

Example 3. Minimize $3y_1 - 2y_2 + 5y_3$ subject to all $y_i \geq 0$ and

$$\begin{aligned} - \ y_2 + 2y_3 &\geq 1 \\ y_1 \qquad + \ y_3 &\geq 1 \\ 2y_1 - 3y_2 + 7y_3 &\geq 5. \end{aligned}$$

We pivot according to the rules for Case 2 of the simplex method, first pivoting about row two column one.

$$
\begin{array}{c|ccc|c}
 & x_1 & x_2 & x_3 & \\
\hline
y_1 & 0 & 1 & 2 & 3 \\
y_2 & -1 & 0 & -3 & -2 \\
y_3 & 2 & 1 & 7 & 5 \\
\hline
 & -1 & -1 & -5 & 0
\end{array}
\longrightarrow
\begin{array}{c|ccc|c}
 & y_2 & x_2 & x_3 & \\
\hline
y_1 & 0 & 1 & 2 & 3 \\
x_1 & -1 & 0 & 3 & 2 \\
y_3 & 2 & 1 & 1 & 1 \\
\hline
 & -1 & -1 & -2 & 2
\end{array}
$$

Note that after this one pivot, we are now in Case 1. The tableau is identical to the example for Case 1 except for the lower right corner and the labels of the variables. Further pivoting as is done there leads to

$$
\begin{array}{c|ccc|c}
 & y_2 & y_3 & x_1 & \\
\hline
y_1 & & & & 4/3 \\
x_3 & & & & 2/3 \\
x_2 & & & & 1 \\
\hline
 & 2/3 & 1 & 1/3 & 11/3
\end{array}
$$

The value for both programs is $11/3$. The solution to the minimum problem is $y_1 = 0$, $y_2 = 2/3$ and $y_3 = 1$. The solution to the maximum problem is $x_1 = 0$, $x_2 = 1/3$ and $x_3 = 2/3$.

The Dual Simplex Method. The simplex method has been stated from the point of view of the maximum problem. Clearly, it may be stated with regard to the minimum problem as well. In particular, if a feasible vector has already been found for the minimum problem (i.e. the bottom row is already nonnegative), it is more efficient to improve this vector with another feasible vector than it is to apply the rule for Case 2 of the maximum problem.

We state the simplex method for the minimum problem.

Case 1: $-\boldsymbol{c} \geq \mathbf{0}$. Take any row with the last entry negative, say $b_{i_0} < 0$. Among those j for which $a_{i_0,j} < 0$, choose that j_0 for which the ratio $-c_j/a_{i_0,j}$ is closest to zero. Pivot on a_{i_0,j_0}.

Case 2: Some $-c_j$ are negative. Take the first negative $-c_j$, say $-c_k < 0$ (where $-c_1 \geq 0, \ldots, -c_{k-1} \geq 0$). Find any positive entry in column k, say $a_{i_0,k} > 0$. Compare $-c_k/a_{i_0,k}$ and those $-c_j/a_{i_0,j}$ for which $-c_j \geq 0$ and $a_{i_0,j} < 0$, and choose j_0 for which this ratio is closest to zero. (j_0 may be k). Pivot on a_{i_0,j_0}.

Example:

$$
\begin{array}{c|ccc|c}
 & x_1 & x_2 & x_3 & \\
\hline
y_1 & -2 & -1 & -2 & -1 \\
y_2 & 2 & 3 & 5 & 2 \\
y_3 & -3 & 4 & -1 & 1 \\
\hline
 & 1 & 2 & 3 & 0
\end{array}
\longrightarrow
\begin{array}{c|ccc|c}
 & y_1 & x_2 & x_3 & \\
\hline
x_1 & & & & 1/2 \\
y_2 & & & & 1 \\
y_3 & & & & 5/2 \\
\hline
 & 1/2 & 3/2 & 2 & -1/2
\end{array}
$$

If this example were treated as Case 2 for the maximum problem, we might pivot about the 4 or the 5. Since we have a feasible vector for the minimum problem, we apply Case 1 above, find a unique pivot and arrive at the solution in one step.

Exercises. For the linear programming problems below, state the dual problem, solve by the simplex (or dual simplex) method, and state the solutions to both problems.

1. Maximize $x_1 - 2x_2 - 3x_3 - x_4$ subject to the constraints $x_j \geq 0$ for all j and

$$\begin{aligned} x_1 - x_2 - 2x_3 - x_4 &\leq 4 \\ 2x_1 + x_3 - 4x_4 &\leq 2 \\ -2x_1 + x_2 + x_4 &\leq 1. \end{aligned}$$

2. Minimize $3y_1 - y_2 + 2y_3$ subject to the constraints $y_i \geq 0$ for all i and

$$\begin{aligned} 2y_1 - y_2 + y_3 &\geq -1 \\ y_1 + 2y_3 &\geq 2 \\ -7y_1 + 4y_2 - 6y_3 &\geq 1. \end{aligned}$$

3. Maximize $-x_1 - x_2 + 2x_3$ subject to the constraints $x_j \geq 0$ for all j and

$$\begin{aligned} -3x_1 + 3x_2 + x_3 &\leq 3 \\ 2x_1 - x_2 - 2x_3 &\leq 1 \\ -x_1 + x_3 &\leq 1. \end{aligned}$$

4. Minimize $5y_1 - 2y_2 - y_3$ subject to the constraints $y_i \geq 0$ for all i and

$$\begin{aligned} -2y_1 + 3y_3 &\geq -1 \\ 2y_1 - y_2 + y_3 &\geq 1 \\ 3y_1 + 2y_2 - y_3 &\geq 0. \end{aligned}$$

5. Minimize $-2y_2 + y_3$ subject to the constraints $y_i \geq 0$ for all i and

$$\begin{aligned} -y_1 - 2y_2 &\geq -3 \\ 4y_1 + y_2 + 7y_3 &\geq -1 \\ 2y_1 - 3y_2 + y_3 &\geq -5. \end{aligned}$$

6. Maximize $3x_1 + 4x_2 + 5x_3$ subject to the constraints $x_j \geq 0$ for all j and

$$\begin{aligned} x_1 + 2x_2 + 2x_3 &\leq 1 \\ -3x_1 + x_3 &\leq -1 \\ -2x_1 - x_2 &\leq -1. \end{aligned}$$

5. Generalized Duality.

We consider the general form of the linear programming problem, allowing some constraints to be equalities, and some variables to be unrestricted ($-\infty < x_j < \infty$).

The General Maximum Problem. Find x_j, $j = 1, \ldots, n$, to maximize $\boldsymbol{x}^T\boldsymbol{c}$ subject to

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \quad \text{for } i = 1, \ldots, k$$
$$\sum_{j=1}^{n} a_{ij}x_j = b_i \quad \text{for } i = k+1, \ldots, m$$

and

$$x_j \ge 0 \quad \text{for } j = 1, \ldots, \ell$$
$$x_j \text{ unrestricted} \quad \text{for } j = \ell+1, \ldots, n.$$

The dual to this problem is

The General Minimum Problem. Find y_i, $i = 1, \ldots, m$, to minimize $\boldsymbol{y}^T\boldsymbol{b}$ subject to

$$\sum_{i=1}^{m} y_i a_{ij} \ge c_j \quad \text{for } j = 1, \ldots, \ell$$
$$\sum_{i=1}^{m} y_i a_{ij} = c_j \quad \text{for } j = \ell+1, \ldots, n$$

and

$$y_i \ge 0 \quad \text{for } i = 1, \ldots, k$$
$$y_i \text{ unrestricted} \quad \text{for } i = k+1, \ldots, m.$$

In other words, a strict equality in the constraints of one program corresponds to an unrestricted variable in the dual.

If the general maximum problem is transformed into a standard maximum problem by

1. replacing each equality constraint, $\sum_j a_{ij}x_j = b_i$, by two inequality constraints, $\sum_j a_{ij}x_j =\le b_i$ and $\sum_j(-a_{ij})x_j \le -b_i$, and

2. replacing each unrestricted variable, x_j, by the difference of two nonnegative variables, $x_j = x'_j - x''_j$ with $x'_j \ge 0$ and $x''_j \ge 0$,

and if the dual general minimum problem is transformed into a standard minimum problem by the same techniques, the transformed standard problems are easily seen to be duals by the definition of duality for standard problems. (Exercise 1.)

Therefore the theorems on duality for the standard programs in Section 2 are valid for the general programs as well. The Equilibrium Theorem seems as if it might require special attention for the general programming problems since some of the y_i^* and x_j^* may be negative. However, it is also valid. (Exercise 2.)

Solving General Problems by the Simplex Method. The simplex tableau and pivot rules may be used to find the solution to the general problem if the following modifications are introduced.

1. Consider the general minimum problem. We add the slack variables, $\boldsymbol{s}^{\mathsf{T}} = \boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} - \boldsymbol{c}^{\mathsf{T}}$, but the constraints now demand that $s_1 \geq 0, \ldots, s_\ell \geq 0, s_{\ell+1} = 0, \ldots, s_n = 0$. If we pivot in the simplex tableau so that $s_{\ell+1}$, say, goes from the top to the left, it becomes an independent variable and may be set equal to zero as required. Once $s_{\ell+1}$ is on the left, it is immaterial whether the corresponding $\hat{b}_i$ in the last column is negative or positive, since this $\hat{b}_i$ is the coefficient multiplying $s_{\ell+1}$ in the objective function, and $s_{\ell+1}$ is zero anyway. In other words, once $s_{\ell+1}$ is pivoted to the left, we may delete that row — we will never pivot in that row and we ignore the sign of the last coefficient in that row.

This analysis holds for all equality constraints: pivot $s_{\ell+1}, \ldots, s_n$ to the left and delete. This is equivalent to solving each equality constraint for one of the variables and substituting the result into the other linear forms.

2. Similarly, the unconstrained y_i, $i = k+1, \ldots, m$, must be pivoted to the top where they represent dependent variables. Once there we do not care whether the corresponding $-\hat{c}_j$ is positive or not, since the unconstrained variable y_i may be positive or negative. In other words, once y_{k+1}, say, is pivoted to the top, we may delete that column. (If you want the value of y_{k+1} in the solution, you may leave that column in, but do not pivot in that column — ignor the sign of the last coefficient in that column.)

After all equality constraints and unrestricted variables are taken care of, we may pivot according to the rules of the simplex method to find the solution.

Similar arguments apply to the general maximum problem. The unrestricted x_i may be pivoted to the left and deleted, and the slack variables corresponding to the equality constraints may be pivoted to the top and deleted.

3. What happens if the above method for taking care of equality constraints cannot be made? It may happen in attempting to pivot one of the s_j for $j \geq \ell + 1$ to the left, that one of them, say s_α, cannot be so moved without simultaneously pivoting some s_j for $j < \alpha$ back to the top, because all the possible pivot numbers in column α are zero, except for those in rows labelled s_j for $j \geq \ell + 1$. If so, column α represents the equation

$$s_\alpha = \sum_{j \geq \ell+1} y_j \hat{a}_{j,\alpha} - \hat{c}_\alpha.$$

This time there are two possibilities. If $\hat{c}_\alpha \neq 0$, *the minimum problem is infeasible* since all s_j for $j \geq \ell + 1$ must be zero. The original equality constraints of the minimum problem were inconsistent. If $\hat{c}_\alpha = 0$, equality is automatically obtained in the above equation and column α may be removed. *The original equality constraints contained a redundency.*

A dual argument may be given to show that if it is impossible to pivot one of the unrestricted y_i, say y_β, to the top (without moving some unrestricted y_i from the top back to the left), then the maximum problem is infeasible, unless perhaps the corresponding last entry in that row, $\hat{b}_\beta$, is zero. If $\hat{b}_\beta$ is zero, we may delete that row as being redundant.

4. In summary, the general simplex method consists of three stages. In the first stage, all equality constraints and unconstrained variables are pivoted (and removed if desired). In the second stage, one uses the simplex pivoting rules to obtain a feasible solution for the problem or its dual. In the third stage, one improves the feasible solution according to the simplex pivoting rules until the optimal solution is found.

Example 1. Maximize $5x_2 + x_3 + 4x_4$ subject to the constraints $x_1 \geq 0$, $x_2 \geq 0$, $x_4 \geq 0$, x_3 unrestricted, and

$$\begin{array}{rcrcrcrcl} -x_1 & + & 5x_2 & + & 2x_3 & + & 5x_4 & \leq & 5 \\ & & 3x_2 & & & + & x_4 & = & 2 \\ -x_1 & & & + & x_3 & + & 2x_4 & = & 1. \end{array}$$

In the tableau, we put arrows pointing up next to the variables that must be pivoted to the top, and arrows pointing left above the variables to end up on the left.

$$\begin{array}{c|cccc|c} & x_1 & x_2 & \overleftarrow{x_3} & x_4 & \\ \hline y_1 & -1 & 5 & 2 & 5 & 5 \\ \uparrow y_2 & 0 & 3 & 0 & 1 & 2 \\ \uparrow y_3 & -1 & 0 & 1 & 2 & 1 \\ \hline & 0 & -5 & -1 & -4 & 0 \end{array} \longrightarrow \begin{array}{c|cccc|c} & x_1 & x_2 & \uparrow y_3 & x_4 & \\ \hline y_1 & 1 & 5 & -2 & 1 & 3 \\ \uparrow y_2 & 0 & 3 & 0 & 1 & 2 \\ \overleftarrow{x_3} & -1 & 0 & 1 & 2 & 1 \\ \hline & -1 & -5 & 1 & -2 & 1 \end{array}$$

After deleting the third row and the third column, we pivot y_2 to the top, and we have found a feasible solution to the maximum problem. We then delete y_2 and pivot according to the simplex rule for Case 1.

$$\longrightarrow \begin{array}{c|ccc|c} & x_1 & x_2 & \uparrow y_2 & \\ \hline y_1 & 1 & 2 & -1 & 1 \\ x_4 & 0 & 3 & 1 & 2 \\ \hline & -1 & 1 & 2 & 5 \end{array} \longrightarrow \begin{array}{c|ccc|c} & y_1 & x_2 & \uparrow y_2 & \\ \hline x_1 & 1 & 2 & & 1 \\ x_4 & 0 & 3 & & 2 \\ \hline & 1 & 3 & 1 & 6 \end{array}$$

After one pivot, we have already arrived at the solution. The value of the program is 6, and the solution to the maximum problem is $x_1 = 1$, $x_2 = 0$, $x_4 = 2$ and (from the equality constraint of the original problem) $x_3 = -2$.

Solving Matrix Games by the Simplex Method. Consider a matrix game with $n \times m$ matrix $\boldsymbol{A}$. If Player I chooses a mixed strategy, $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $\sum_1^n x_i = 1$ and $x_i \geq 0$ for all i, he wins at least λ on the average, where $\lambda \leq \sum_{i=1}^n x_i a_{ij}$ for $j = 1, \ldots, m$. He wants to choose $x_1, \ldots, x_n$ to maximize this minimum amount he wins. This becomes a linear programming problem if we add λ to the list of variables chosen by I. The problem becomes: Choose $x_1, \ldots, x_n$ and λ to maximize λ subject to $x_1 \geq 0, \ldots, x_n \geq 0$, λ unrestricted, and

$$\lambda - \sum_{i=1}^n x_i a_{ij} \leq 0 \qquad \text{for } j = 1, \ldots, n, \text{ and}$$

$$\sum_{i=1}^n x_i = 1.$$

This is clearly a general minimum problem.

Player II chooses $y_1, \ldots, y_m$ with $y_i \geq 0$ for all i and $\sum_1^m y_i = 1$, and loses at most μ, where $\mu \geq \sum_{j=1}^m a_{ij} y_j$ for all i. The problem is therefore: Choose $y_1 \ldots, y_m$ and μ to minimize μ subject to $y_1 \geq 0, \ldots, y_m \geq 0$, μ unrestricted, and

$$\mu - \sum_{j=1}^m a_{ij} y_j \geq 0 \qquad \text{for } i = 1, \ldots, n, \text{ and}$$

$$\sum_{j-1}^m y_j = 1.$$

This is exactly the dual of Player I's problem. These problems may be solved simultaneously by the simplex method for general programs.

Note, however, that if $\boldsymbol{A}$ is the game matrix, it is the negative of the transpose of $\boldsymbol{A}$ that is placed in the simplex tableau:

	x_1	x_2	$\cdots$	x_n	$\overleftarrow{\lambda}$	
y_1	$-a_{11}$	$-a_{21}$	$\cdots$	$-a_{n1}$	1	0
$\vdots$	$\vdots$				$\vdots$	$\vdots$
y_m	$-a_{1m}$	$-a_{2m}$	$\cdots$	$-a_{nm}$	1	0
$\uparrow\mu$	1	1	$\cdots$	1	0	1
	0	0	$\cdots$	0	-1	0

Example 2. Solve the matrix game with matrix

$$\boldsymbol{A} = \begin{pmatrix} 1 & 4 & 0 & -1 \\ 2 & -1 & 3 & 1 \\ -1 & 1 & 0 & 3 \end{pmatrix}.$$

The tableau is:

	x_1	x_2	x_3	$\overleftarrow{\lambda}$	
y_1	-1	-2	1	1	0
y_2	-4	1	-1	1	0
y_3	0	-3	0	1	0
y_4	1	-1	-3	1	0
$\uparrow\mu$	1	1	1	0	1
	0	0	0	-1	0

We would like to pivot to interchange λ and μ. Unfortunately, as is always the case for solving games by this method, the pivot point is zero. So we must pivot twice, once to move μ to the top and once to move λ to the left. First we interchange y_3 and λ and delete the λ row. Then we interchange x_3 and μ and delete the μ column.

	x_1	x_2	x_3	y_3	
y_1	-1	1	1	-1	0
y_2	-4	4	-1	-1	0
y_4	1	2	-3	-1	0
$\uparrow\mu$	1	1	1	0	1
	0	-3	0	1	0

$\longrightarrow$

	x_1	x_2	y_3	
y_1	-2	0	-1	-1
y_2	-3	5	-1	1
y_4	4	5	-1	3
x_3	1	1	0	1
	0	-3	1	0

Now, we use the pivot rules for the simplex method until finally we obtain the tableau, say

	y_4	y_2	y_1	
y_3				3/7
x_2				16/35
x_1				2/7
x_3				9/35
	13/35	8/35	14/35	33/35

Therefore, the value of the game is 33/35. The unique optimal strategy for Player I is (2/7,16/35,9/35). The unique optimal strategy for Player II is (14/35,8/35,0,13/35).

Exercises.

1. (a) Transform the general maximum problem to standard form by (1) replacing each equality constraint, $\sum_j a_{ij}x_j = b_i$ by the two inequality constraints, $\sum_j a_{ij}x_j \le b_i$ and $\sum_j(-a_{ij})x_j \le -b_i$, and (2) replacing each unrestricted x_j by $x'_j - x''_j$, where x'_j and x''_j are restricted to be nonnegative.

(b) Transform the general minimum problem to standard form by (1) replacing each equality constraint, $\sum_i y_i a_{ij} = c_j$ by the two inequality constraints, $\sum_i y_i a_{ij} \le c_j$ and $\sum_i y_j(-a_{ij}) \le -c_j$, and (2) replacing each unrestricted y_i by $y'_i - y''_i$, where y'_i and y''_i are restricted to be nonnegative.

(c) Show that the standard programs (a) and (b) are duals.

2. Let $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ be feasible vectors for a general maximum problem and its dual respectively. Assuming the Duality Theorem for the general problems, show that $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal if, and only if,

$$\begin{aligned} y_i^* &= 0 \qquad \text{for all } i \text{ for which } \textstyle\sum_{j=1}^n a_{ij}x_j^* < b_j \\ &\text{and} \\ x_j^* &= 0 \qquad \text{for all } j \text{ for which } \textstyle\sum_{i=1}^m y_i^* a_{ij} > c_j. \end{aligned} \tag{$*$}$$

3. (a) State the dual for the problem of Example 1.

(b) Find the solution of the dual of the problem of Example 1.

4. Solve the game with matrix, $\boldsymbol{A} = \begin{pmatrix} 1 & 0 & -2 & 1 \\ -1 & 2 & 3 & 0 \\ -2 & -3 & 4 & -3 \end{pmatrix}$. First pivot to interchange λ and y_1. Then pivot to interchange μ and x_2, and continue.

6. Cycling.

Unfortunately, the rules of the simplex method as stated give no visible improvement if b_r, the constant term in the pivot row, is zero. In fact, it can happen that a sequence of pivots, chosen according to the rules of the simplex method, leads you back to the original tableau. Thus, if you use these pivots you could cycle forever.

Example 1. Maximize $3x_1 - 5x_2 + x_3 - 2x_4$ subject to all $x_i \ge 0$ and

$$\begin{aligned} x_1 - 2x_2 - x_3 + 2x_4 &\le 0 \\ 2x_1 - 3x_2 - x_3 + \ \ x_4 &\le 0 \\ x_3 \qquad\quad &\le 1 \end{aligned}$$

If you choose the pivots very carefully, you will find that in six pivots according the the rules of the simplex method you will be back with the original tableau. Exercise 1 shows how to choose the pivots for this to happen.

A modification of the simplex method that avoids cycling. Given two n-dimensional vectors $\mathbf{r}$ and $\mathbf{s}$, $\mathbf{r} \neq \mathbf{s}$, we say that $\mathbf{r}$ is *lexographically less than* $\mathbf{s}$, in symbols, $\mathbf{r} \prec \mathbf{s}$, if in the first coordinate in which $\mathbf{r}$ and $\mathbf{s}$ differ, say $r_1 = s_1, \ldots, r_{k-1} = s_{k-1}$, $r_k \neq s_k$, we have $r_k < s_k$.

For example, $(0, 3, 2, 3) \prec (1, 2, 2, -1) \prec (1, 3, -100, -2)$.

The following procedure may be used to avoid cycling. Add a unit matrix to the right of the b-column of the simplex tableau, and an m-vector $(c_{n+1}, \ldots, c_{n+m})$, initially $\mathbf{0}$, to the right of the value variable v. This produces a modified simplex tableau:

$$\begin{array}{c|cccc|ccccc} & x_1 & x_2 & \cdots & x_n & & & & & \\ \hline y_1 & a_{11} & a_{12} & \cdots & a_{1n} & b_1 & 1 & 0 & \ldots & 0 \\ y_2 & a_{21} & a_{22} & \cdots & a_{2n} & b_2 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & & & & \vdots \\ y_m & a_{m1} & a_{m2} & \cdots & a_{mn} & b_m & 0 & 0 & \ldots & 1 \\ \hline & -c_1 & -c_2 & \cdots & -c_n & v & c_{n+1} & c_{n+2} & \ldots & c_{n+m} \end{array}$$

Think of the right side of the tableau as vectors, $\mathbf{b}_1 = (b_1, 1, 0, \ldots, 0), \ldots, \mathbf{b}_m = (b_m, 0, 0, \ldots, 1)$, and $\mathbf{v} = (v, c_{n+1}, \ldots, c_{n+m})$, and use the simplex pivot rules as before, but replace the b_i by the $\mathbf{b}_i$ and use the lexographic ordering.

MODIFIED SIMPLEX RULE: We suppose all $b_i \ge 0$. Find $c_s < 0$. Among all r such that $a_{rs} > 0$, pivot about that a_{rs} such that $\mathbf{b}_r / a_{rs}$ is lexographically least.

We note that the vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are linearly independent, that is,

$$\sum_{i=1}^{m} \alpha_i \mathbf{b}_i = \mathbf{0} \quad \text{implies} \quad \alpha_i = 0 \quad \text{for} \quad i = 1, 2, \ldots, m.$$

We will be pivoting about various elements $a_{rs} \neq 0$ and it is important to notice that these vectors stay linearly independent after pivoting. This is because if $\mathbf{b}_1, \ldots, \mathbf{b}_m$ are linearly independent, and if after pivoting, the new $\mathbf{b}$-vectors become

$$(*) \qquad \begin{aligned} \mathbf{b}'_r &= (1/a_{rs})\mathbf{b}_r \quad \text{and, for} \quad i \neq r, \\ \mathbf{b}'_i &= \mathbf{b}_i - (a_{is}/a_{rs})\mathbf{b}_r, \end{aligned}$$

then $\sum_{i=1}^m \alpha_i \mathbf{b}'_i = \mathbf{0}$ implies that

$$\sum_{i=1}^m \alpha_i \mathbf{b}'_i = \sum_{i\neq r} \alpha_i \mathbf{b}_i - \sum_{i\neq r} (\alpha_i a_{is}/a_{rs})\mathbf{b}_r + (\alpha_r/a_{rs})\mathbf{b}_r = \mathbf{0}$$

which implies that $\alpha_i = 0$ for $i = 1, \ldots, m$.

We state the basic properties of the modified simplex method which show that cycling does not occur. Note that we start with all $\mathbf{b}_i \succ \mathbf{0}$, that is, the first non-zero coordinate of each $\mathbf{b}_i$ is positive.

Properties of the modified simplex rule:

(1) If all $c_s \geq 0$, a solution has already been found.

(2) If for $c_s < 0$, all $a_{rs} \leq 0$, then the problem is unbounded.

(3) After pivoting, the $\mathbf{b}'_i$ stay lexographically positive, as is easily checked from equations (*).

(4) After pivoting, the new $\mathbf{v}'$ is lexographically greater than $\mathbf{v}$, as is easily seen from $\mathbf{v}' = \mathbf{v} - (c_s/a_{rs})\mathbf{b}_r$.

Thus, for the Modified Simplex Rule, the value $\mathbf{v}$ always increases lexographically with each pivot. Since there are only a finite number of simplex tableaux, and since $\mathbf{v}$ increases lexographically with each pivot, the process must eventually stop, either with an optimal solution or with an unbounded solution.

This finally provides us with a proof of the duality theorem! The proof is constructive. Given a bounded feasible linear programming problem, use of the Modified Simplex Rule will eventually stop at a tableau, from which the solution to the problem and its dual may be read out. Moreover the value of the problem and its dual are the same.

In practical problems, the Modified Simplex Rule is never used. This is partly because cycling in practical problems is rare, and so is not felt to be needed. But also there is a very simple modification of the simplex method, due to Bland ("New finite pivoting rules for the simplex method", *Math. of Oper. Res.* **2**, 1977, pp. 103-107), that avoids cycling, and is yet very easy to include in a linear programming algorithm. This is called the **Smallest-Subscript Rule**: If there is a choice of pivot columns or a choice of pivot rows, select the row (or column) with the x variable having the lowest subscript, or if there are no x variables, with the y variable having the lowest subscript.

Exercises.

1. (a) Set up the simplex tableau for Example 1.

(b) Pivot as follows.
1. Interchange x_1 and y_1.
2. Interchange x_2 and y_2.
3. Interchange x_3 and x_1.
4. Interchange x_4 and x_2.
5. Interchange y_1 and x_3.
6. Interchange y_2 and x_4.

Now, note you are back where you started.

2. Solve the problem of Example 1 by pivoting according to the modified simplex rule.

3. Solve the problem of Example 1 by pivoting according to the smallest subscript rule.

7. Four problems with nonlinear objective functions solved by linear methods.

1. Constrained Games. Find x_j for $j = 1, \ldots, n$ to maximize

$$\min_{1\le i\le p}\left[\sum_{j=1}^{n} c_{ij}x_j\right] \tag{1}$$

subject to the constraints as in the general maximum problem

$$\begin{aligned}\sum_{j=1}^{n} a_{ij}x_j &\le b_i \qquad \text{for } i = 1, \ldots, k\\ \sum_{j=1}^{n} a_{ij}x_j &= b_i \qquad \text{for } i = k+1, \ldots, n\end{aligned} \tag{2}$$

and

$$\begin{aligned} x_j &\ge 0 \qquad \text{for } j = 1, \ldots, \ell\\ (x_j &\text{ unrestricted} \qquad \text{for } j = \ell+1, \ldots, n)\end{aligned} \tag{3}$$

This problem may be considered as a generalization of the matrix game problem from player I's viewpoint, and may be transformed in a similar manner into a general linear program, as follows. Add λ to the list of unrestricted variables, subject to the constraints

$$\lambda - \sum_{j=1}^{n} c_{ij}x_j \le 0 \qquad \text{for } i = 1, \ldots, p \tag{4}$$

The problem becomes: maximize λ subject to the constraints (2), (3) and (4).

Example 1. The General Production Planning Problem (See Zukhovitskiy and Avdeyeva, "Linear and Convex Programming", (1966) W. B. Saunders pg. 93) There are n activities, $A_1, \ldots, A_n$, a company may employ using the available supplies of m resources, $R_1, \ldots, R_m$. Let b_i be the available supply of R_i and let a_{ij} be the amount of R_i used in operating A_j at unit intensity. Each activity may produce some or all of the p different parts that are used to build a complete product (say, a machine). Each product consists of N_1 parts #1,... ,N_p parts $\#p$. Let c_{ij} denote the number of parts $\#i$ produced in operating A_j at unit intensity. The problem is to choose activity intensities to maximize the number of complete products built.

Let x_j be the intensity at which A_j is operated, $j = 1, \ldots, n$. Such a choice of intensities produces $\sum_{j=1}^{n} c_{ij}x_j$ parts $\#i$, which would be required in building $\sum_{j=1}^{n} c_{ij}x_j/N_i$ complete products. Therefore, such an intensity selection may be used to build

$$\min_{1\le i\le p}\left[\sum_{j=1}^{n} c_{ij}x_j/N_i\right] \tag{5}$$

complete products. The amount of R_i used in this intensity selection must be no greater than b_i,

$$\sum_{j=1}^{n} a_{ij}x_j \le b_i \qquad \text{for } i = 1, \ldots, m \tag{6}$$

and we cannot operate at negative intensity

$$x_j \ge 0 \qquad \text{for } j = 1, \ldots, n \tag{7}$$

We are required to maximize (5) subject to (6) and (7). When $p = 1$, this reduces to the Activity Analysis Problem.

Exercise 1. In the general production planning problem, suppose there are 3 activities, 1 resource and 3 parts. Let $b_1 = 12$, $a_{11} = 2$, $a_{12} = 3$ and $a_{13} = 4$, $N_1 = 2$, $N_2 = 1$, and $N_3 = 1$, and let the c_{ij} be as given in the matrix below. (a) Set up the simplex tableau of the associated linear program. (b) Solve. (Ans. $x_1 = x_2 = x_3 = 4/3$, value = 8.)

$$\mathbf{C} = \begin{array}{r} \\ \text{part \#1} \\ \text{part \#2} \\ \text{part \#3} \end{array} \begin{array}{c} \begin{array}{ccc} A_1 & A_2 & A_3 \end{array} \\ \begin{pmatrix} 2 & 4 & 6 \\ 2 & 1 & 3 \\ 1 & 3 & 2 \end{pmatrix} \end{array}$$

2. Minimizing the sum of absolute values. Find y_i for $i = 1, \ldots, m$, to minimize

$$\sum_{j=1}^{p} \Big| \sum_{i=1}^{m} y_i b_{ij} - b_j \Big| \tag{8}$$

subject to the constraints as in the general minimum problem

$$\begin{aligned} \sum_{i=1}^{m} y_i a_{ij} &\ge c_j \qquad \text{for } j = 1, \ldots, \ell \\ \sum_{i=1}^{m} y_i a_{ij} &= c_j \qquad \text{for } j = \ell + 1, \ldots, n \end{aligned} \tag{9}$$

and

$$\begin{aligned} y_i &\ge 0 \qquad \text{for } i = 1, \ldots, k \\ (y_i \text{ unrestricted} &\qquad \text{for } i = k + 1, \ldots, m.) \end{aligned} \tag{10}$$

To transform this problem to a linear program, add p more variables, $y_{m+1}, \ldots, y_{m+p}$, where y_{m+j} is to be an upper bound of the jth term of (8), and try to minimize $\sum_{j=1}^{p} y_{m+j}$. The constraints

$$\Big| \sum_{i=1}^{m} y_i b_{ij} - b_j \Big| \le y_{m+j} \qquad \text{for } j = 1, \ldots, p$$

are equivalent to the $2p$ linear constraints

$$\begin{aligned} \sum_{i=1}^{m} y_i b_{ij} - b_j &\le y_{m+j} \qquad \text{for } j = 1, \ldots, p \\ -\sum_{i=1}^{m} y_i b_{ij} + b_j &\le y_{m+j} \qquad \text{for } j = 1, \ldots, p \end{aligned} \tag{11}$$

The problem becomes minimize $\sum_{j=1}^{p} y_{m+j}$ subject to the constraints (9), (10) and (11). In this problem, the constraints (11) imply that $y_{m+1}, \ldots, y_{m+p}$ are nonnegative, so it does not matter whether these variables are restricted to be nonnegative or not. Computations are generally easier if we leave them unrestricted.

Example 2. It is desired to find an m-dimensional vector, $\boldsymbol{y}$, whose average distance to p given hyperplanes

$$\sum_{i=1}^{m} y_i b_{ij} = b_j \qquad \text{for } j = 1, \ldots, p$$

is a minimum. To find the (perpendicular) distance from a point $(y_1^0, \ldots, y_m^0)$ to the plane $\sum_{i=1}^{m} y_i b_{ij} = b_j$, we normalize this equation by dividing both sides by $d_j = (\sum_{i=1}^{m} b_{ij}^2)^{1/2}$. The distance is then $|\sum_{i=1}^{m} y_i^0 b'_{ij} - b'_j|$, where $b'_{ij} = b_{ij}/d_j$ and $b'_j = b_j/d_j$. Therefore, we are searching for $y_1, \ldots, y_m$ to minimize

$$\frac{1}{p} \sum_{j=1}^{p} \Big| \sum_{i=1}^{m} y_i b'_{ij} - b'_j \Big|.$$

There are no constraints of the form (9) or (10) to the problem as stated.

Exercise 2. Consider the problem of finding y_1 and y_2 to minimize $|y_1 + y_2 - 1| + |2y_1 - y_2 + 1| + |y_1 - y_2 - 2|$ subject to the constraints $y_1 \ge 0$ and $y_2 \ge 0$. (a) Set up the simplex tableau of the associated linear program. (b) Solve. (Ans. $(y_1, y_2) = (0, 1)$, value $= 3$.)

3. Minimizing the maximum of absolute values. Find $y_1, \ldots, y_m$ to minimize

$$\max_{1 \le j \le p} \Big| \sum_{i=1}^{m} y_i b_{ij} - b_j \Big| \tag{12}$$

subject to the general constraints (9) and (10). This objective function combines features of the objective functions of 1. and 2. A similar combination of methods transforms this problem to a linear program. Add μ to the list of unrestricted variables subject to the constraints

$$\Big| \sum_{i=1}^{m} y_i b_{ij} - b_j \Big| \le \mu \qquad \text{for } j = 1, \ldots, p$$

and try to minimize μ. These p constraints are equivalent to the following $2p$ linear constraints

$$\begin{aligned} \sum_{i=1}^{m} y_i b_{ij} - b_j &\le \mu \qquad \text{for } j = 1, \ldots, p \\ -\sum_{i=1}^{m} y_i b_{ij} + b_j &\le \mu \qquad \text{for } j = 1, \ldots, p \end{aligned} \tag{13}$$

The problem becomes: minimize μ subject to (9), (10) and (13).

Example 3. Chebyshev Approximation. Given a set of p linear affine functions in m unknowns,

$$\psi_j(y_1, \ldots, y_m) = \sum_{i=1}^{m} y_i b_{ij} - b_j \qquad \text{for } j = 1, \ldots, p, \tag{14}$$

find a point, $(y_1^0, \ldots, y_m^0)$, for which the maximum deviation (12) is a minimum. Such a point is called a Chebyshev point for the system (14), and is in some sense a substitute for the notion of a solution of the system (l4) when the system is inconsistent. Another substitute would be the point that minimizes the total deviation (8). If the functions (l4) are normalized so that $\sum_{i=1}^{m} b_{ij}^2 = 1$ for all j, then the maximum deviation (12) becomes the maximum distance of a point $\boldsymbol{y}$ to the planes

$$\sum_{i=1}^{m} y_i b_{ij} = b_j \qquad \text{for } j = 1, \ldots, p.$$

Without this normalization, one can think of the maximum deviation as a "weighted" maximum distance. (See Zukhovitskiy and Avdeyeva, pg. 191, or Stiefel "Note on Jordan Elimination, Linear Programming, and Tchebycheff Approximation" *Numerische Mathematik* **2** (1960), 1-17.

Exercise 3. Find a Chebyshev point (*unnormalized*) for the system

$$\begin{aligned} \psi_1(y_1, y_2) &= y_1 + y_2 - 1 \\ \psi_2(y_1, y_2) &= 2y_1 - y_2 + 1 \\ \psi_3(y_1, y_2) &= y_1 - y_2 - 2 \end{aligned}$$

by (a) setting up the associated linear program, and (b) solving. (Ans. $y_1 = 0$, $y_2 = -1/2$, value $= 3/2$.)

4. Linear Fractional Programming. (Charnes and Cooper, "Programming with linear fractional functionals", *Naval Research Logistics Quarterly* **9** (1962), 181-186.) Find $\boldsymbol{x} = (x_1, \ldots, x_n)^{\mathsf{T}}$ to maximize

$$\frac{\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x} + \alpha}{\boldsymbol{d}^{\mathsf{T}}\boldsymbol{x} + \beta} \tag{15}$$

subject to the general constraints (2) and (3). Here, $\boldsymbol{c}$ and $\boldsymbol{d}$ are n-dimensional vectors and α and β are real numbers. To avoid technical difficulties we make two assumptions: that the constraint set is nonempty and bounded, and that the denominator $\boldsymbol{d}^{\mathsf{T}}\boldsymbol{x}+\beta$ is strictly positive throughout the constraint set.

Note that the objective function remains unchanged upon multiplication of numerator and denominator by any number $t>0$. This suggests holding the denominator fixed say $(\boldsymbol{d}^{\mathsf{T}}\boldsymbol{x}+\beta)t=1$, and trying to maximize $\boldsymbol{c}^{\mathsf{T}}\boldsymbol{x}t+\alpha t$. With the change of variable $\boldsymbol{z}=\boldsymbol{x}t$, this becomes embedded in the following linear program. Find $\boldsymbol{z}=(z_1,\ldots,z_n)$ and t to maximize

$$\boldsymbol{c}^{\mathsf{T}}\boldsymbol{z}+\alpha t$$

subject to the constraints

$$\begin{aligned} &\boldsymbol{d}^{\mathsf{T}}\boldsymbol{z}+\beta t=1 \\ &\sum_{j=1}^{n} a_{ij}z_j \le b_i t \qquad \text{for } i=1,\ldots,k \\ &\sum_{j=1}^{n} a_{ij}z_j = b_i t \qquad \text{for } i=k+1,\ldots,m \end{aligned}$$

and

$$\begin{aligned} t &\ge 0 & & \\ z_j &\ge 0 & &\text{for } j=1,\ldots,\ell \\ z_j &\text{ unrestricted} & &\text{for } j=\ell+1,\ldots,n. \end{aligned}$$

Every value achievable by a feasible $\boldsymbol{x}$ in the original problem is achievable by a feasible $(\boldsymbol{z},t)$ in the linear program, by letting $t=(\boldsymbol{d}^{\mathsf{T}}\boldsymbol{x}+\beta)^{-1}$ and $\boldsymbol{z}=\boldsymbol{x}t$. Conversely, every value achievable by a feasible $(\boldsymbol{z},t)$ with $t>0$ in the linear program is achievable by a feasible $\boldsymbol{x}$ in the original problem by letting $\boldsymbol{x}=\boldsymbol{z}/t$. But t cannot be equal to zero for any feasible $(\boldsymbol{z},t)$ of the linear program, since if $(\boldsymbol{z},0)$ were feasible, $\boldsymbol{x}+\delta\boldsymbol{z}$ would be feasible for the original program for all $\delta>0$ and any feasible $\boldsymbol{x}$, so that the constraint set would either be empty or unbounded. (One may show t cannot be negative either, so that it may be taken as one of the unrestricted variables if desired.) Hence, a solution of the linear program always leads to a solution of the original problem.

Example 4. Activity analysis to maximize rate of return. There are n activities $A_1,\ldots,A_n$ a company may employ using the available supply of m resources $R_1,\ldots,R_m$. Let b_i be the available supply of R_i and let a_{ij} be the amount of R_i used in operating A_j at unit intensity. Let c_j be the net return to the company for Operating A_j at unit intensity, and let d_j be the time consumed in operating A_j at unit intensity. Certain other activities not involving $R_1,\ldots,R_m$ are required of the company and yield net return α at a time consumption of β. The problem is to maximize the rate of return (15) subject to the restrictions $\boldsymbol{A}\boldsymbol{x}\le\boldsymbol{b}$ and $\boldsymbol{x}\ge\boldsymbol{0}$. We note that the constraint set is nonempty if $\boldsymbol{b}\ge\boldsymbol{0}$, that it is generally bounded (for example if $a_{ij}>0$ for all i and j), and that $\boldsymbol{d}^{\mathsf{T}}\boldsymbol{x}+\beta$ is positive on the constraint set if $\boldsymbol{d}\ge\boldsymbol{0}$ and $\beta>0$.

Exercise 4. There are two activities and two resources. There are 2 units of R_1 and 3 units of R_2 available. Activity A_1 operated at unit intensity requires 1 unit of R_1 and 3 units of R_2, yields a return of 2, and uses up 1 time unit. Activity A_2 operated at unit intensity requires 3 units of R_1 and 2 units of R_2, yields a return of 3, and uses up 2 time units. It requires 1 time unit to get the whole procedure started at no return. (a) Set up the simplex tableau of the associated linear program. (b) Find the intensities that maximize the rate of return. (Ans. A1 at intensity 5/7, A2 at intensity 3/7, maximal rate = 19/18.)

8. The Transportation Problem.

The transportation problem was described and discussed in Section 1. Briefly, the problem is to find numbers y_{ij} of amounts to ship from Port P_i to Market M_j to minimize the total transportation cost,

$$\sum_{i=1}^{I}\sum_{j=1}^{J} y_{ij} b_{ij}, \tag{1}$$

subject to the nonnegativity constraints, $y_{ij} \ge 0$ for all i and j, the supply constraints,

$$\sum_{j=1}^{J} y_{ij} \le s_i \qquad \text{for } i = 1, \ldots, I \tag{2}$$

and the demand constraints,

$$\sum_{i=1}^{I} y_{ij} \ge r_j \qquad \text{for } j = 1, \ldots, J \tag{3}$$

where s_i, r_j, b_{ij} are given nonnegative numbers.

The dual problem is to find numbers $u_1, \ldots, u_I$ and $v_1, \ldots, v_J$, to maximize

$$\sum_{j=1}^{J} r_j v_j - \sum_{i=1}^{I} s_i u_i \tag{4}$$

subject to the constraints $v_j \ge 0$, $u_i \ge 0$, and

$$v_j - u_i \le b_{ij} \qquad \text{for all } i \text{ and } j\,. \tag{5}$$

Clearly, the transportation problem is not feasible unless supply is at least as great as demand:

$$\sum_{i=1}^{I} s_i \ge \sum_{j=1}^{J} r_j. \tag{6}$$

It is also easy to see that if this inequality is satisfied, then the problem is feasible; for example, we may take $y_{ij} = s_i r_j / \sum_{i=1}^{I} s_i$ (at least if $\sum s_i > 0$. Since the dual problem is obviously feasible ($u_i = v_j = 0$), condition (6) is necessary and sufficient for both problems to be bounded feasible.

Below, we develop a simple algorithm for solving the transportation problem under the assumption that the total demand is equal to the total supply. This assumption can easily be avoided by the device of adding to the problem a dummy market, called *the dump*, whose requirement is chosen to make the total demand equal to the total supply, and such that all transportation costs to the dump are zero. Thus we assume

$$\sum_{i=1}^{I} s_i = \sum_{j=1}^{J} r_j. \tag{7}$$

Under this assumption, the inequalities in constraints (2) and (3) must be satisfied with equalities, because if there were a strict inequality in one of the constraints (2) or (3), then summing (2) over i and (3) over j would produce a strict inequality in (7). Thus we may replace (2) and (3) with

$$\sum_{j=1}^{J} y_{ij} = s_i \qquad \text{for } i = 1, \ldots, I \tag{8}$$

and

$$\sum_{i=1}^{I} y_{ij} = r_j \qquad \text{for } j = 1, \ldots, J. \tag{9}$$

Hence, in order to solve both problems, it is sufficient (and necessary) to find feasible vectors, y_{ij} and u_i and v_j, that satisfy the equilibrium conditions. Due to the equalities in the constraints (8) and (9), the equilibrium conditions reduce to condition that for all i and j

$$y_{ij} > 0 \quad \text{implies that} \quad v_j - u_i = b_{ij}. \tag{10}$$

This problem can, of course, be solved by the simplex method. However, the simplex tableau for this problem invoves an IJ by $I+J$ constraint matrix. Instead, we use a more efficient algorithm that may be described easily in terms of the I by J *transportation array.*

	M_1	M_2	$\ldots$	M_J	
P_1	b_{11} y_{11}	b_{12} y_{12}	$\ldots$	b_{1J} y_{1J}	s_1
P_2	b_{21} y_{21}	b_{22} y_{22}	$\ldots$	b_{2J} y_{2J}	s_2
$\vdots$	$\vdots$	$\vdots$		$\vdots$	
P_I	b_{I1} y_{I1}	b_{I2} y_{I2}	$\ldots$	b_{IJ} y_{IJ}	s_I
	r_1	r_2		r_J	

(11)

The following is an example of a transportation array.

	M_1	M_2	M_3	M_4	
P_1	4	7	11	3	5
P_2	7	5	6	4	7
P_3	1	3	4	8	8
	2	9	4	5	

(12)

This indicates there are three ports and four markets. Port P_1 has 5 units of the commodity, P_2 has 7 units, and P_3 has 8 units. Markets M_1, M_2, M_3 and M_4 require 2, 9, 4 and 5 units of the commodity respectively. The shipping costs per unit of the commodity are entered in the appropriate squares. For example, it costs 11 per unit sent from P_1 to M_3. We are to enter the y_{ij}, the amounts to be shipped from P_i to M_j, into the array.

The algorithm consists of three parts.

1. Find a basic feasible shipping schedule, y_{ij}.
2. Test for optimality.
3. If the test fails, find an improved basic feasible shipping schedule, and repeat 2.

1. Finding an initial basic feasible shipping schedule, y_{ij}. We describe the method informally first using the example given above. Choose any square, say the upper left corner, $(1,1)$, and make y_{11} as large as possible subject to the constraints. In this case, y_{11} is chosen equal to 2. We ship all of M_1's requirements from P_1. Thus, $y_{21} = y_{31} = 0$. We choose another square, say $(1,2)$, and make y_{12} as large as possible subject to the constraints. This time, $y_{12} = 3$, since there are only three units left at P_1. Hence, $y_{13} = y_{14} = 0$. Next, choose square $(2,2)$, say, and put $y_{22} = 6$, so that M_2 receives all of its requirements, 3 units from P_1 and 6 units from P_2. Hence, $y_{32} = 0$. One continues in this way until all the variables y_{ij} are determined. (This way of choosing the basic feasible shipping schedule is called the Northwest Corner Rule.)

	M_1	M_2	M_3	M_4	
P_1	4 2	7 3	11	3	5
P_2	7	5 6	6 1	4	7
P_3	1	3	4 3	8 5	8
	2	9	4	5	

(13)

We have circled those variables, y_{ij}, that have appeared in the solution for use later in finding an improved basic feasible shipping schedule. Occasionally, it is necessary to circle a variable, y_{ij}, that is zero. The general method may be stated as follows.

A1. (a) Choose any available square, say (i_0, j_0), specify $y_{i_0 j_0}$ as large as possible subject to the constraints, and circle this variable.

(b) Delete from consideration whichever row or column has its constraint satisfied, but not both. If there is a choice, do not delete a row (column) if it is the last row (resp. column) undeleted.

(c) Repeat (a) and (b) until the last available square is filled with a circled variable, and then delete from consideration both row and column.

We note two properties of this method of choosing a basic feasible shipping schedule.

Lemma 1. *(1) There are exactly $I+J-1$ circled variables. (2) No submatrix of r rows and c columns contains more than $r+c-1$ circled variables.*

Proof. One row or column is deleted for each circled variable entered xcept for the last one for which a row and a column are deleted. Since there are $I+J$ rows and columns, there must be $I+J-1$ circled variables, proving (1). A similar argument proves (2) by concentrating on the r rows and c columns, and noting that at most one circled variable is deleted from the submatrix for each of its rows and columns, and the whole submatrix is deleted when $r+c-1$ of its rows and columns are deleted. ∎

The converse of Lemma 1 is also valid.

Lemma 2. *Any feasible set of circled variables satisfying (1) and (2) of Lemma 1 is obtainable by this algorithm.*

Proof. Every row and column has at least one circled variable since deleting a row or column with no circled variable would give a submatrix contradicting (3). From (2), since there are $I+J$ rows and columns but only $I+J-1$ circled variables, there is at least one row or column with exactly one circled variable. We may choose this row or column first in the algorithm. The remaining $(I-1)\times J$ or $I\times(J-1)$ subarray satisfies the same two properties, and another row or column with exactly one circled variable may be chosen next in the algorithm. This may be continued until all circled variables have been selected. ∎

To illustrate the circling of a zero and the use of (b) of algorithm A1, we choose a different order for selecting the squares in the example above. We try to find a good initial solution by choosing the squares with the smallest transportation costs first. This is called the **Least-Cost Rule**.

The smallest transportation cost is in the lower left square. Thus $y_{31}=2$ is circled and M_1 is deleted. Of the remaining squares, 3 is the lowest transportation cost and we might choose the upper right corner next. Thus, $y_{14}=5$ is circled and we may delete either P_1 or M_4, but not both, according to rule (b). Say we delete P_1. Next $y_{32}=6$ is circled and P_3 is deleted. Of the ports, only P_2 remains, so we circle $y_{22}=3$, $y_{23}=4$ and $y_{24}=0$.

4	7	11	3 5	5
7	5 3	6 4	4 0	7
1 2	3 6	4	8	8
2	9	4	5	

(14)

2. Checking for optimality. Given a feasible shipping schedule, y_{ij}, we propose to use the equilibrium theorem to check for optimality. This entails finding feasible u_i and v_j that satisfy the equilibrium conditions (10).

One method is to solve the equations

$$v_j - u_i = b_{ij} \tag{15}$$

for all (i,j)-squares containing circled variables (whether or not the variables are zero). There are $I + J - 1$ circled variables and so $I + j - 1$ equations in $I + J$ unknowns. Therefore, one of the variables may be fixed, say equal to zero, and the equations may be used to solve for the other variables. Some of the u_i or v_j may turn out to be negative, but this does not matter. One can always add a large positive constant to all the u_i and v_j to make them positive without changing the values of $v_j - u_i$.

Once the u_i and v_j have been determined to satisfy equilibrium, feasibility must be checked by checking that $v_j - u_i \le b_{ij}$ for all (i,j)-squares without circled variables. In summary,

A2. (a) Set one of the v_j or u_i to zero, and use (15) for squares containing circled variables to find all the v_j and u_i.

(b) Check feasibility, $v_j - u_i \le b_{ij}$, for the remaining squares. If feasible, the solution is optimal for the problem and its dual.

We must prove that part (a) of A2 can always be carried out. That is, we must show that when one of the variables, v_j or u_i, is set equal to zero, there exists a unique solution to (15) for the (i,j)-squares containing circled variables. As in the proof of Lemma 2, we may find a row (or column) with exactly one circled variable. The u_i (resp. v_j) corresponding to this row (resp. column) may be found uniquely once the rest of the u_i and v_j have been determined. We delete this row (resp. column) and note that the same is also true of the reduced array. Continuing in like manner, we may reduce the array to a single element. Either dual variable corresponding to this square may be fixed and the rest of the variables found uniquely. The initial fixed variable may be adjusted to give any preassigned u_i or v_j the value zero. ■

Let us check the feasible shipping schedule (14) for optimality.
First solve for the u_i and v_j. We put $u_2 = 0$ because that allows us to solve quickly for $v_2 = 5$, $v_3 = 6$, and $v_4 = 4$. (Generally, it is a good idea to start with a $u_i = 0$ (or $v_j = 0$) for which there are many circled variables in the ith row (jth column).) Knowing $v_4 = 4$ allows us to solve for $u_1 = 1$. Knowing $v_2 = 5$ allows us to solve for $u_3 = 2$, which allows us to solve for $v_1 = 3$. We write the v_j variables across the top of the array and u_i along the left.

	3	5	6	4	
1	4	7	11	3 5	5
0	7	5 3	6 4	4 0	7
2	1 2	3 6	4	8	8
	2	9	4	5	

(16)

Then, check feasibility of the remaining six squares. The upper left square satisfies the constraint $v_j - u_i \le b_{ij}$, since $3 - 1 = 2 \le 4$. Similarly, all the squares may be seen to satisfy the constraints, and hence the above gives the solution to both primal and dual problems. The optimal shipping schedule is as noted, and the value is $\sum\sum y_{ij}b_{ij} = 2\cdot 1 + 6\cdot 3 + 3\cdot 5 + 4\cdot 6 + 0\cdot 4 + 5\cdot 3 = 74$. As a check, $\sum v_j r_j - \sum u_i s_i = 95 - 21 = 74$.

Notice that if b_{21} were equal to 2 instead of 7, then the same shipping schedule may be obtained from the least cost rule, yet it would not be optimal, because $v_1 - u_2 > b_{21} = 2$.

3. The improvement routine. Suppose we have tested a feasible shipping schedule for optimality by computing the dual variables u_i and v_j to satisfy equilibrium and found that the dual variables are not feasible. Take any square (i_0, j_0) for which $v_{j_0} - u_{i_0} > b_{i_0 j_0}$. We would like to ship some amount of the commodity from port P_{i_0} to market M_{j_0}, say $y_{i_0 j_0} = \theta$. But if we add θ to that square, we must subtract and add θ to other squares containing circled variables to keep the constraints satisfied.

Consider the above example with b_{21} changed to 2. Then feasibility is not satisfied for square $(2,1)$. We would like to add θ to square $(2,1)$. This requires subtracting θ from squares $(3,1)$ and $(2,2)$, and adding θ to square $(3,2)$.

4	7	11	3 5	5
7 $+\theta$	5 $-\theta$ 3	6 4	4 0	7
1 $-\theta$ 2	3 $+\theta$ 6	4	8	8
2	9	4	5	

(17)

We choose θ as large as possible, bearing in mind that negative shipments are not allowed. This means we choose θ to be the minimum of the y_{ij} in the squares in which we are subtracting θ. In the example, $\theta = 2$.

At least one of the y_{ij} is put to, or remains at, 0. Remove any one such from the circled variables and circle $y_{i_0 j_0}$. Then perform the check for optimality. This is done for the example where it is seen that the new schedule is optimal, with value 72.

	2	5	6	4	
1	4	7	11	3 5	5
0	2 2	5 1	6 4	4 0	7
2	1	3 8	4	8	8
	2	9	4	5	

(18)

In summary,

A3. (a) Choose any square (i,j) with $v_j - u_i > b_{ij}$, set $y_{ij} = \theta$, but keep the constraints satisfied by subtracting and adding θ to appropriate circled variables.

(b) Choose θ to be the minimum of the variables in the squares in which θ is subtracted.

(c) Circle the new variable and remove from the circled variables one of the variables from which θ was subtracted that is now zero.

Note that if all non-circled variables y_{ij} are given any values, we may solve for the circled variables one at a time using (8) and (9) by choosing a row or column with one circled variable first, and continuing in like manner until all variables are specified. Therefore, step (a) of A3 may always be carried out.

We must show that step (c) of A3 leaves a set of circled variables abtainable by the method of A1 of the algorithm so that A2 and A3 may be repeated. To see this, note that when θ is circled as a new variable, the set of variables depending on θ is contained in a square subarray of say k rows and columns containing $2k$ circled variables, with 2 circled variables in each row and column. The other $I + J - 2k$ rows and columns with circled variables can be deleted one at a time as in the proof of Lemma 2. If any one of the remaining circled variables may also be deleted one row or column at a time. ■

Let us see what improvement is made in the transportation cost by this procedure. The cost before the change minus the cost afterward may be computed as $\sum\sum y_{ij}b_{ij} - \sum\sum y'_{ij}b_{ij} = \theta\sum^{+} b_{ij} - \theta\sum^{-} b_{ij}$, where $\sum^{+}$ is the sum over the squares to which θ has been added, and $\sum^{-}$ is the sum over the squares to which θ has been subtracted. Since $b_{ij} = v_j - u_i$ for all those squares except the new one, call it (i_0, j_0), we may write this improvement in the cost as

$$\theta(v_{j_0} - u_{i_0} - b_{i_0 j_0}) - \theta\overset{+}{\sum}(v_i - u_j) + \theta\overset{-}{\sum}(v_j - u_i) = \theta(v_{j_0} - u_{i_0} - b_{i_0 j_0}) \qquad (19)$$

since each v_j and u_i that is added is also subtracted. This is never negative, and it is strictly positive if $\theta > 0$. That is the improvement routine gives a strict improvement when $\theta > 0$.

It may happen that $\theta = 0$, in which case the improvement routine gives no improvement, but changes the basic shipping schedule into a new one from which one can hope for an improvement or even an optimality check.

Exercises. Solve the transportation problems with the following arrays.

1.

1	4	2
2	6	2
3	1	

2.

1	3	5	8	6
2	5	6	7	5
1	2	3	5	

3.

10	8	9	15
5	2	3	20
6	7	4	30
7	6	8	35
25	25	50	

4.

4	3	4	2	4
8	6	7	5	4
6	4	5	3	6
7	5	6	4	4
2	7	3	6	

9. Solutions to Exercises.

Solutions to Exercises of Section 1.

1. The graph of the problem is

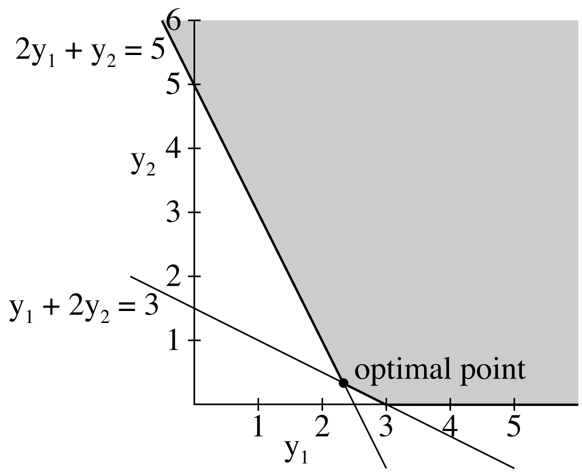

Figure 1.

The constraint set is shaded. The objective function, $y_1 + y_2$, has slope -1. As we move a line of slope -1 down, the last place it touches the constraint set is at the intersection of the two lines, $2y_1 + y_2 = 5$ and $y_1 + 2y_2 = 3$. The point of intersection, namely $(2\frac{1}{3}, \frac{1}{3})$, is the optimal vector.

2. For a less than -1, the optimal vector is $(0,1)$. For a from -1 to $\frac{1}{2}$, the optimal vector is $(\frac{2}{3}, \frac{5}{3})$. For a from $\frac{1}{2}$ to 2, the optimal vector is $(\frac{8}{3}, \frac{8}{3})$. For a from 0 to $\frac{1}{2}$, the optimal vector is $(3,0)$. Therefore, the value is

$$\text{Value} = \begin{cases} 1 & \text{for } a \le -1 \\ (2a+5)/3 & \text{for } -1 \le a \le 1/2 \\ (8a+2)/3 & \text{for } 1/2 \le a \le 2 \\ 3a & \text{for } a \ge 2. \end{cases}$$

3. Let $\boldsymbol{y}^T = (y_{11}, \ldots, y_{1J}, y_{21}, \ldots, y_{2J}, \;\cdots\; , y_{I1}, \ldots, y_{IJ})$. Then the main constraints

may be written as $\boldsymbol{y}^{\mathsf{T}}\boldsymbol{A} \le \boldsymbol{c}^{\mathsf{T}}$, where

$$\boldsymbol{A} = \begin{pmatrix}
-1 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
-1 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
-1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
0 & -1 & \cdots & 0 & 1 & 0 & \cdots & 0 \\
0 & -1 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & -1 & \cdots & 0 & 0 & 0 & \cdots & 1 \\
\vdots & & & \vdots & \vdots & & & \vdots \\
0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & -1 & 0 & 1 & \cdots & 0 \\
\vdots & & \ddots & \vdots & \vdots & & \ddots & \vdots \\
0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 1
\end{pmatrix}$$

and

$$\boldsymbol{c}^{\mathsf{T}} = (-s_1, -s_2, \ldots, -s_I, r_1, r_2, \ldots, r_J).$$

4. Since x_2 is unrestricted, we must replace x_2, wherever it occurs, by $u_2 - v_2$ with $u_2 \ge 0$ and $v_2 \ge 0$. The second main constraint is an equality so it must be replaced by two inequalities. The problem in standard form becomes:

Maximize $x_1 + 2u_2 - 2v_2 + 3x_3 + 4x_4$ subject to

$$\begin{array}{rcrcrcrcrcr}
4x_1 & + & 3u_2 & - & 3v_2 & + & 2x_3 & + & x_4 & \le & 10 \\
x_1 & & & & & - & x_3 & + & 2x_4 & \le & 2 \\
-x_1 & & & & & + & x_3 & - & 2x_4 & \le & -2 \\
-x_1 & - & u_2 & + & v_2 & - & x_3 & - & x_4 & \le & -1,
\end{array}$$

and

$$x_1 \ge 0, u_2 \ge 0, v_2 \ge 0, x_3 \ge 0, x_4 \ge 0.$$

Solutions to Exercises of Section 2.

1. Find x_1, x_2, x_3 and x_4 to maximize $2x_1 + 4x_2 + 6x_3 + 2x_4$, subject to

$$\begin{aligned} x_1 - x_2 - 2x_3 + x_4 &\le 1 \\ -2x_1 + x_2 \qquad\quad + x_4 &\le 2 \\ x_1 + x_2 + \ x_3 + x_4 &\le 1 \end{aligned}$$

and

$$x_1 \ge 0, x_2 \ge 0, x_3 \ge 0, x_4 \ge 0.$$

2. We check that $\boldsymbol{y} = (\frac{2}{3}, 0, \frac{14}{3})$ and $\boldsymbol{x} = (0, \frac{1}{3}, \frac{2}{3}, 0)$ are feasible for their respective problems, and that they have the same value. Clearly, $\boldsymbol{y} \ge 0$ and $\boldsymbol{x} \ge 0$. Substituting $\boldsymbol{y}$ into the main constraints of Exercise 1, we find $\frac{2}{3} + \frac{14}{3} = \frac{16}{3} \ge 2$, $-\frac{2}{3} + \frac{14}{3} = 4 \ge 4$, $\frac{4}{3} + \frac{14}{3} = 6 \ge 6$, and $\frac{2}{3} + \frac{14}{3} = \frac{16}{3} \ge 2$, so $\boldsymbol{y}$ is feasible. Similarly substituting $\boldsymbol{x}$ into the main constraints of the solution of Exercise 1, we find $-\frac{1}{3} + \frac{4}{3} = 1 \le 1$, $\frac{1}{3} \le 2$, and $\frac{1}{3} + \frac{2}{3} = 2 \le 2$, so $\boldsymbol{x}$ is feasible. The value of $\boldsymbol{y}$ is $\frac{2}{3} + \frac{14}{3} = \frac{16}{3}$, and the value of $\boldsymbol{x}$ is $\frac{4}{3} + \frac{12}{3} = \frac{16}{3}$. Since these are equal, both are optimal.

3. (a) Find y_1, y_2, y_3 and y_4 to minimize $4y_1 + 6y_2 + 3y_3 + 8y_4$, subject to

$$\begin{aligned} y_1 + 2y_2 - \ y_3 + y_4 &\ge 3 \\ -y_1 + \ y_2 \qquad\quad + y_4 &\ge 2 \\ y_1 + 3y_2 + 2y_3 + y_4 &\ge 1 \end{aligned}$$

and

$$y_1 \ge 0, y_2 \ge 0, y_3 \ge 0, y_4 \ge 0.$$

(b) If $\boldsymbol{x} = (0, 6, 0)$ is optimal for the maximum problem, then the strict inequality in the first, third and fourth main constraints implies that $y_1 = y_3 = y_4 = 0$. Also, $x_2 > 0$ implies that there is equality in the second constraint in the minimum problem. Solving this gives $y_2 = 2$. After checking that $\boldsymbol{y} = (0, 2, 0, 0)$ is feasible, we conclude it is optimal.

4. (a) Find $\boldsymbol{x} = (x_1, \ldots, x_I, x_{I+1}, \ldots, x_{I+J})$ to maximize $-\sum_{i=1}^{I} s_i x_i + \sum_{j=1}^{J} r_j x_{I+j}$ subject to

$$-x_i + x_{I+j} \le b_{ij} \qquad \text{for all } i \text{ and } j$$

and

$$x_i \ge 0 \qquad \text{for all } i.$$

(b) Since b_{ij} is measured in cost, say in dollars, per unit of the commodity, the main constraints show that the x_i are also measured in dollars per unit of the commodity. The objective function is therefore measured in dollars. (This is the dimensional analysis.) We may imagine an independent entrepreneur who offers to buy the commodities at port P_i

for x_i dollars per unit, and to sell them at market M_j for x_{I+j} dollars per unit. If the min constraints are satisfied, we can essentially get all our goods transported from the ports to the markets at no increase in cost, so we would accept the offer. Naturally he will choose the x_i to maximize his profit, which is just the objective function, $-\sum_{i=1}^{I} s_i x_i + \sum_{j=1}^{J} r_j x_{I+j}$. Thus, x_i for $1 \le i \le I$ represents the shadow value of a unit of the commodity at port P_i, and x_{I+j} for $1 \le j \le J$ represents the shadow value of a unit of the commodity at market M_j.

Solution to the Exercise of Section 3.

$$
\begin{array}{c|cccc|}
 & s_1 & s_2 & s_3 & s_4 \\ \hline
y_1 & 1 & 1 & 0 & 2 \\
y_2 & 0 & 2 & -1 & 1 \\
y_3 & -1 & 1 & 0 & 3 \\
y_4 & 1 & -2 & 2 & 1 \\ \hline
\end{array}
\longrightarrow
\begin{array}{c|cccc|}
 & y_1 & s_2 & s_3 & s_4 \\ \hline
s_1 & 1 & 1 & 0 & 2 \\
y_2 & 0 & 2 & -1 & 1 \\
y_3 & 1 & 2 & 0 & 5 \\
y_4 & -1 & -3 & 2 & -1 \\ \hline
\end{array}
\longrightarrow
\begin{array}{c|cccc|}
 & y_1 & s_2 & y_2 & s_4 \\ \hline
s_1 & 1 & 1 & 0 & 2 \\
s_3 & 0 & -2 & -1 & -1 \\
y_3 & 1 & 2 & 0 & 5 \\
y_4 & -1 & 1 & 2 & 1 \\ \hline
\end{array}
$$

$$
\longrightarrow
\begin{array}{c|cccc|}
 & y_1 & y_4 & y_2 & s_4 \\ \hline
s_1 & 2 & -1 & -2 & 1 \\
s_3 & -2 & 2 & 3 & 1 \\
y_3 & 3 & -2 & -4 & 3 \\
s_2 & -1 & 1 & 2 & 1 \\ \hline
\end{array}
\longrightarrow
\begin{array}{c|cccc|}
 & y_1 & y_4 & y_2 & y_3 \\ \hline
s_1 & 1 & -\frac{1}{3} & -\frac{2}{3} & -\frac{1}{3} \\
s_3 & -3 & \frac{8}{3} & \frac{13}{3} & -\frac{1}{3} \\
s_4 & 1 & -\frac{2}{3} & -\frac{4}{3} & \frac{1}{3} \\
s_2 & -2 & \frac{5}{3} & \frac{10}{3} & -\frac{1}{3} \\ \hline
\end{array}
=
\begin{array}{c|cccc|}
 & y_1 & y_2 & y_3 & y_4 \\ \hline
s_1 & 1 & -\frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\
s_2 & -2 & \frac{10}{3} & -\frac{1}{3} & -\frac{5}{3} \\
s_3 & -3 & \frac{13}{3} & -\frac{1}{3} & \frac{8}{3} \\
s_4 & 1 & -\frac{4}{3} & \frac{1}{3} & -\frac{2}{3} \\ \hline
\end{array}
$$

Therefore,

$$
\boldsymbol{A}^{-1} = \frac{1}{3}\begin{pmatrix} 3 & -2 & -1 & -1 \\ -6 & 10 & -1 & -5 \\ -9 & 13 & -1 & 8 \\ 3 & -4 & 1 & -2 \end{pmatrix}
$$

Solutions to the Exercises of Section 4.

1. Minimize $4y_1 + 2y_2 + y_3$ subject to

$$\begin{array}{rcrcrcr} y_1 & + & 2y_2 & - & 2y_3 & \geq & 1 \\ -y_1 & + & & + & y_3 & \geq & -2 \\ -2y_1 & + & y_2 & & & \geq & -3 \\ -y_1 & - & 4y_2 & + & y_3 & \geq & -1 \end{array} \qquad \text{and} \qquad y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$$

Pivoting according to the simplex rules gives

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & \\ \hline y_1 & 1 & -1 & -2 & -1 & 4 \\ y_2 & \textcircled{2} & 0 & 1 & -4 & 2 \\ y_3 & -2 & 1 & 0 & 1 & 1 \\ \hline & -1 & 2 & 3 & 1 & 0 \end{array} \longrightarrow \begin{array}{c|cccc|c} & y_2 & x_2 & x_3 & x_4 & \\ \hline y_1 & -\frac{1}{2} & -1 & -\frac{5}{2} & \textcircled{1} & 3 \\ x_1 & \frac{1}{2} & 0 & \frac{1}{2} & -2 & 1 \\ y_3 & 1 & 1 & 1 & -3 & 3 \\ \hline & \frac{1}{2} & 2 & \frac{7}{2} & -1 & 1 \end{array}$$

$$\longrightarrow \begin{array}{c|cccc|c} & y_2 & x_2 & x_3 & y_1 & \\ \hline x_4 & & & & & 3 \\ x_1 & & & & & 7 \\ y_3 & & & & & 12 \\ \hline & 0 & 1 & 1 & 1 & 4 \end{array}$$

From this we see that the value for both programs is 4, the solution to the minimum problem is $y_1 = 1$, $y_2 = 0$, $y_3 = 0$, and the solution to the maximum problem is $x_1 = 7$, $x_2 = 0$, $x_3 = 0$, $x_4 = 3$.

2. Maximize $-x_1 + 2x_2 + x_3$ subject to

$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & 7x_3 & \leq & 3 \\ -x_1 & & & + & 4x_3 & \leq & -1 \\ -2y_1 & + & y_2 & & & \geq & -3 \\ -y_1 & - & 4y_2 & + & y_3 & \geq & -1 \end{array} \qquad \text{and} \qquad y_1 \geq 0, y_2 \geq 0, y_3 \geq 0.$$

$$\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & \\ \hline y_1 & 2 & 1 & -7 & 3 \\ y_2 & \textcircled{-1} & 0 & 4 & -1 \\ y_3 & 1 & 2 & -6 & 2 \\ \hline & 1 & -2 & -1 & 0 \end{array} \longrightarrow \begin{array}{c|ccc|c} & y_2 & x_2 & x_3 & \\ \hline y_1 & 2 & -1 & 1 & 1 \\ x_1 & -1 & 0 & 4 & 1 \\ y_3 & 1 & \textcircled{2} & -2 & 1 \\ \hline & 1 & -2 & 3 & -1 \end{array} \longrightarrow \begin{array}{c|ccc|c} & y_2 & y_3 & x_3 & \\ \hline y_1 & & & & 3/2 \\ x_1 & & & & 1 \\ x_2 & & & & 1/2 \\ \hline & 2 & 1 & 1 & 0 \end{array}$$

The value is zero, the solution to the minimum problem is $y_1 = 0$, $y_2 = 2$, $y_3 = 1$, and the solution to the maximum problem is $x_1 = 1$, $x_2 = 1/2$, $x_3 = 0$.

3.

$$\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & \\ \hline y_1 & -3 & 3 & 1 & 3 \\ y_2 & 2 & -1 & -2 & 1 \\ y_3 & -1 & 0 & 1 & 1 \\ \hline & 1 & 1 & -2 & 0 \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_1 & x_2 & y_3 & \\ \hline y_1 & -2 & 3 & -1 & 2 \\ y_2 & 0 & -1 & 2 & 3 \\ x_3 & -1 & 0 & 1 & 1 \\ \hline & -1 & 1 & 2 & 2 \end{array}$$

The first column shows that the maximum problem is unbounded feasible.

4.

$$\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & \\ \hline y_1 & -2 & 2 & 3 & 5 \\ y_2 & 0 & -1 & 2 & -2 \\ y_3 & 3 & 1 & -1 & -1 \\ \hline & 1 & -1 & 0 & 0 \end{array} \longrightarrow \begin{array}{c|ccc|c} & x_1 & x_2 & y_3 & \\ \hline y_1 & 7 & 5 & 3 & 2 \\ y_2 & 6 & 1 & 2 & -4 \\ x_3 & -3 & -1 & -1 & 1 \\ \hline & 1 & -1 & 0 & 0 \end{array}$$

The second row shows that the maximum problem is infeasible.

5. Using the Dual Simplex Method,

$$\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & \\ \hline y_1 & -1 & 4 & 2 & 0 \\ y_2 & -2 & 1 & -3 & -2 \\ y_3 & 0 & 7 & 1 & 1 \\ \hline & 3 & 1 & 5 & 0 \end{array} \longrightarrow \begin{array}{c|ccc|c} & y_2 & x_2 & x_3 & \\ \hline y_1 & & & & 1 \\ x_1 & & & & 1 \\ y_3 & & & & 1 \\ \hline & \frac{3}{2} & \frac{5}{2} & \frac{1}{2} & -3 \end{array}$$

The value is -3, the solution to the minimum problem is $y_1 = 0,\ y_2 = 3/2,\ y_3 = 0$, and the solution to the maximum problem is $x_1 = 1,\ x_2 = 0,\ x_3 = 0$.

6. Using the Dual Simplex Method , case 2,

$$\begin{array}{c|ccc|c} & x_1 & x_2 & x_3 & \\ \hline y_1 & 1 & 2 & 2 & 1 \\ y_2 & -3 & 0 & 1 & -1 \\ y_3 & -2 & -1 & 0 & -1 \\ \hline & -3 & -4 & -5 & 0 \end{array} \longrightarrow \begin{array}{c|ccc|c} & y_1 & x_2 & x_3 & \\ \hline x_1 & & & & 1 \\ y_2 & & & & 2 \\ y_3 & & & & 1 \\ \hline & 3 & 2 & 1 & 3 \end{array}$$

The value is 3, the solution to the minimum problem is $y_1 = 3,\ y_2 = 0,\ y_3 = 0$, and the solution to the maximum problem is $x_1 = 1,\ x_2 = 0,\ x_3 = 0$.

Solutions to the Exercises of Section 5.

1. (a) Maximize $\sum_{j=1}^{\ell} c_j x_j + \sum_{j=\ell+1}^{n} c_j x_j' + \sum_{j=\ell+1}^{n} (-c_j) x_j''$ subject to

$$\sum_{j=1}^{\ell} a_{ij}x_j + \sum_{j=\ell+1}^{n} a_{ij}x_j' + \sum_{j=\ell+1}^{n} (-a_{ij})x_j'' \le b_i \quad \text{for } i = 1,\ldots,k$$
$$\sum_{j=1}^{\ell} a_{ij}x_j + \sum_{j=\ell+1}^{n} a_{ij}x_j' + \sum_{j=\ell+1}^{n} (-a_{ij})x_j'' \le b_i \quad \text{for } i = k+1,\ldots,m$$
$$\sum_{j=1}^{\ell} a_{ij}x_j + \sum_{j=\ell+1}^{n} (-a_{ij})x_j' + \sum_{j=\ell+1}^{n} a_{ij}x_j'' \le b_i \quad \text{for } i = k+1,\ldots,m$$

and

$$x_j \ge 0 \quad \text{for } j = 1,\ldots,\ell$$
$$x_j' \ge 0 \text{ and } x_j'' \ge 0 \quad \text{for } j = \ell+1,\ldots,n$$

(b) Minimize $\sum_{i=1}^{k} b_i y_i + \sum_{i=k+1}^{m} b_i y_i' + \sum_{i=k+1}^{m} (-b_i) y_i''$ subject to

$$\sum_{i=1}^{k} y_i a_{ij} + \sum_{i=k+1}^{m} y_j' a_{ij} + \sum_{i=k+1}^{m} y_i''(-a_{ij}) \ge c_j \quad \text{for } j = 1,\ldots,\ell$$
$$\sum_{i=1}^{k} y_i a_{ij} + \sum_{i=k+1}^{m} y_j' a_{ij} + \sum_{i=k+1}^{m} y_i''(-a_{ij}) \ge c_j \quad \text{for } j = \ell+1,\ldots,n$$
$$\sum_{i=1}^{k} y_i a_{ij} + \sum_{i=k+1}^{m} y_j'(-a_{ij}) + \sum_{i=k+1}^{m} y_i'' a_{ij} \ge c_j \quad \text{for } j = \ell+1,\ldots,n$$

and

$$y_i \ge 0 \quad \text{for } i = 1,\ldots,k$$
$$y_i' \ge 0 \text{ and } y_i'' \ge 0 \quad \text{for } i = k+1,\ldots,m$$

(c) It is easy to check that these standard programs are dual.

2. One can mimic the proof of the Equilibrium Theorem for Standard Problems. Let $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ be feasible for the General Maximum Problem and its dual. If the condition (*) is satisfied, then

$$\sum_{i=1}^{m} y_i^* b_i = \sum_{i=1}^{m} y_i^* (\sum_{j=1}^{n} a_{ij}x_j^*) = \sum_{i=1}^{m}\sum_{j=1}^{n} y_i^* a_{ij} x_j^*$$

since $y_i^* = 0$ whenever $b_i \neq \sum_{j=1}^n a_{ij}x_j^*$. Similarly,

$$\sum_{j=1}^n c_j x_j^* = \sum_{j=1}^n (\sum_{i=1}^m y_i^* a_{ij}) x_j^* = \sum_{i=1}^m \sum_{j=1}^n y_i^* a_{ij} x_j^*.$$

Since the values are equal, both $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal.

Now suppose $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$ are optimal. By the Duality Theorem, their values are equal, that is, $\sum_j c_j x_j^* = \sum_i y_i^* b_i$. But

$$\sum_j c_j x_j \leq \sum_j \sum_i y_i a_{ij} x_j \leq \sum_i y_i b_i$$

for all feasible $\boldsymbol{x}$ and $\boldsymbol{y}$. So for $\boldsymbol{x}^*$ and $\boldsymbol{y}^*$, we get equality throughout. Therefore,

$$\sum_j (c_j - \sum_i a_{ij} y_i) x_j^* = 0$$

and since $c_j \leq \sum_i a_{ij} y_i^*$ for all j, we must have that $x_j^* = 0$ whenever $c_j < \sum_i a_{ij} y_i^*$. Similarly, $y_i^* = 0$ whenever $\sum_j a_{ij} x_j^* < b_i$, as was to be shown.

3. (a) Minimize $5y_1 + 2y_2 + y_3$ subject to

$$\begin{array}{rcrcrcl} -y_1 & & & - & y_3 & \geq & 0 \\ 5y_1 & + & 3y_2 & & & \geq & 5 \\ 2y_1 & & & + & y_3 & = & 1 \\ 5y_1 & + & y_2 & + & 2y_3 & \geq & 4 \end{array} \qquad \text{and} \qquad y_1 \geq 0,\ y_2 \text{ and } y_3 \text{ unrestricted.}$$

(b) From the display in Example 1, we see that the optimal values are $y_1 = 1$, and $y_2 = 1$. Then solving the equality constraint for y_3, we find $y_3 = 1 - 2y_1 = -1$.

4. We first pivot to interchange y_1 and λ and delete the λ row.

	x_1	x_2	x_3	λ	
y_1	-1	1	2	1	0
y_2	0	-2	3	1	0
y_3	2	-3	-4	1	0
y_4	-1	0	3	1	0
$\uparrow\mu$	1	1	1	0	1
	0	0	0	-1	0

$\longrightarrow$

	x_1	x_2	x_3	y_1	
y_2	1	-3	1	-1	0
y_3	3	-4	-6	-1	0
y_4	0	-1	1	-1	0
$\uparrow\mu$	1	1	1	0	1
	-1	1	2	1	0

Then we interchange x_2 and μ, delete the μ row, and pivot once according to the rules of the simplex method:

$\longrightarrow$

	x_1	x_3	y_1	
y_2	4	4	-1	3
y_3	7	-2	-1	4
y_4	1	2	-1	1
x_2	1	1	0	1
	-2	1	1	-1

$\longrightarrow$

	y_3	x_3	y_1	
y_2				5/7
x_1				4/7
y_4				3/7
x_2				3/7
	2/7	3/7	5/7	1/7

The value of the game is $1/7$. The optimal strategy for Player I is $(4/7, 3/7,\ 0\ )$, and the optimal strategy for Player II is $(5/7,\ 0\ , 2/7,\ 0\ )$.

Solutions to the Exercises of Section 6.

1. Pivoting according to the instructions gives

$$\begin{array}{c|cccc|c} & x_1 & x_2 & x_3 & x_4 & \\ \hline y_1 & 1 & -2 & -1 & 2 & 0 \\ y_2 & 2 & -3 & -1 & 1 & 0 \\ y_3 & 0 & 0 & 1 & 0 & 1 \\ \hline & -3 & 5 & -1 & 2 & 0 \end{array} \longrightarrow \begin{array}{c|cccc|c} & y_1 & x_2 & x_3 & x_4 & \\ \hline x_1 & 1 & -2 & -1 & 2 & 0 \\ y_2 & -2 & 1 & 1 & -3 & 0 \\ y_3 & 0 & 0 & 1 & 0 & 1 \\ \hline & 3 & -1 & -4 & 8 & 0 \end{array} \longrightarrow \begin{array}{c|cccc|c} & y_1 & y_2 & x_3 & x_4 & \\ \hline x_1 & -3 & 2 & 1 & -4 & 0 \\ x_2 & -2 & 1 & 1 & -3 & 0 \\ y_3 & 0 & 0 & 1 & 0 & 1 \\ \hline & 1 & 1 & -3 & 5 & 0 \end{array}$$

$$\longrightarrow \begin{array}{c|cccc|c} & y_1 & y_2 & x_1 & x_4 & \\ \hline x_3 & -3 & 2 & 1 & -4 & 0 \\ x_2 & 1 & -1 & -1 & 1 & 0 \\ y_3 & 3 & -2 & -1 & 4 & 1 \\ \hline & -8 & 7 & 3 & -7 & 0 \end{array} \longrightarrow \begin{array}{c|cccc|c} & y_1 & y_2 & x_1 & x_2 & \\ \hline x_3 & 1 & -2 & -3 & 4 & 0 \\ x_4 & 1 & -1 & -1 & 1 & 0 \\ y_3 & -1 & 2 & 3 & -4 & 1 \\ \hline & -1 & 0 & -4 & 7 & 0 \end{array}$$

$$\longrightarrow \begin{array}{c|cccc|c} & x_3 & y_2 & x_1 & x_2 & \\ \hline y_1 & 1 & -2 & -3 & 4 & 0 \\ x_4 & -1 & 1 & 2 & 3 & 0 \\ y_3 & 1 & 0 & 0 & 0 & 1 \\ \hline & 1 & -2 & -7 & 11 & 0 \end{array} \longrightarrow \begin{array}{c|cccc|c} & x_3 & x_4 & x_1 & x_2 & \\ \hline y_1 & -1 & 2 & 1 & -2 & 0 \\ y_2 & -1 & 1 & 2 & 3 & 0 \\ y_3 & 1 & 0 & 0 & 0 & 1 \\ \hline & -1 & 2 & -3 & 5 & 0 \end{array}$$

and we are back where we started with just some columns interchanged.

2.

$$\begin{array}{c|cccc|cccc} & x_1 & x_2 & x_3 & x_4 & & & & \\ \hline y_1 & 1 & -2 & -1 & 2 & 0 & 1 & 0 & 0 \\ y_2 & 2 & -3 & -1 & 1 & 0 & 0 & 1 & 0 \\ y_3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline & -3 & 5 & -1 & 2 & 0 & 0 & 0 & 0 \end{array} \longrightarrow \begin{array}{c|cccc|cccc} & y_2 & x_2 & x_3 & x_4 & & & & \\ \hline y_1 & -.5 & -.5 & -.5 & 1.5 & 0 & 1 & -.5 & 0 \\ x_1 & .5 & -1.5 & -.5 & .5 & 0 & 0 & .5 & 0 \\ y_3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline & 1.5 & .5 & -2.5 & 3.5 & 0 & 0 & 1.5 & 0 \end{array}$$

$$\longrightarrow \begin{array}{c|cccc|cccc} & y_2 & x_2 & y_3 & x_4 & & & & \\ \hline y_1 & -.5 & -.5 & .5 & 1.5 & .5 & 1 & -.5 & .5 \\ x_1 & .5 & -1.5 & .5 & .5 & .5 & 0 & .5 & .5 \\ x_3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ \hline & 1.5 & .5 & 2.5 & 3.5 & 2.5 & 0 & 1.5 & 2.5 \end{array}$$

The value is 2.5, the solution to the maximum problem is $\boldsymbol{x} = (.5, 0, 1, 0)$, and the solution to the minimum problem is $\boldsymbol{y} = (0, 1.5, 2.5)$.

3. The first four pivots are the same as for Problem 1. Then

$$
\begin{array}{c|cccc|c}
 & y_1 & y_2 & x_1 & x_2 & \\
\hline
x_3 & 1 & -2 & -3 & 4 & 0 \\
x_4 & 1 & -1 & -1 & 1 & 0 \\
y_3 & -1 & 2 & 3 & -4 & 1 \\
\hline
 & -1 & 0 & -4 & 7 & 0
\end{array}
\longrightarrow
\begin{array}{c|cccc|c}
 & y_1 & y_2 & y_3 & x_2 & \\
\hline
x_3 & 0 & 0 & 1 & 0 & 1 \\
x_4 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
x_1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & -\frac{4}{3} & \frac{1}{3} \\
\hline
 & -\frac{7}{3} & \frac{8}{3} & \frac{4}{3} & \frac{5}{3} & \frac{4}{3}
\end{array}
\longrightarrow
\begin{array}{c|cccc|c}
 & x_4 & y_2 & y_3 & x_2 & \\
\hline
x_3 & 0 & 0 & 1 & 0 & 1 \\
y_1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
x_1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} \\
\hline
 & \frac{7}{2} & \frac{3}{2} & \frac{5}{2} & \frac{1}{2} & \frac{5}{2}
\end{array}
$$

This is the solution found in Problem 2, but with some columns interchanged and some rows interchanged.

Solutions to the Exercises of Section 6.

1. Pivoting according to the instructions gives

$$
\begin{array}{c|cccc|c}
 & x_1 & x_2 & x_3 & x_4 & \\
\hline
y_1 & 1 & -2 & -1 & 2 & 0 \\
y_2 & 2 & -3 & -1 & 1 & 0 \\
y_3 & 0 & 0 & 1 & 0 & 1 \\
\hline
 & -3 & 5 & -1 & 2 & 0
\end{array}
\longrightarrow
\begin{array}{c|cccc|c}
 & y_1 & x_2 & x_3 & x_4 & \\
\hline
x_1 & 1 & -2 & -1 & 2 & 0 \\
y_2 & -2 & 1 & 1 & -3 & 0 \\
y_3 & 0 & 0 & 1 & 0 & 1 \\
\hline
 & 3 & -1 & -4 & 8 & 0
\end{array}
\longrightarrow
\begin{array}{c|cccc|c}
 & y_1 & y_2 & x_3 & x_4 & \\
\hline
x_1 & -3 & 2 & 1 & -4 & 0 \\
x_2 & -2 & 1 & 1 & -3 & 0 \\
y_3 & 0 & 0 & 1 & 0 & 1 \\
\hline
 & 1 & 1 & -3 & 5 & 0
\end{array}
$$

$$
\longrightarrow
\begin{array}{c|cccc|c}
 & y_1 & y_2 & x_1 & x_4 & \\
\hline
x_3 & -3 & 2 & 1 & -4 & 0 \\
x_2 & 1 & -1 & -1 & 1 & 0 \\
y_3 & 3 & -2 & -1 & 4 & 1 \\
\hline
 & -8 & 7 & 3 & -7 & 0
\end{array}
\longrightarrow
\begin{array}{c|cccc|c}
 & y_1 & y_2 & x_1 & x_2 & \\
\hline
x_3 & 1 & -2 & -3 & 4 & 0 \\
x_4 & 1 & -1 & -1 & 1 & 0 \\
y_3 & -1 & 2 & 3 & -4 & 1 \\
\hline
 & -1 & 0 & -4 & 7 & 0
\end{array}
$$

$$
\longrightarrow
\begin{array}{c|cccc|c}
 & x_3 & y_2 & x_1 & x_2 & \\
\hline
y_1 & 1 & -2 & -3 & 4 & 0 \\
x_4 & -1 & 1 & 2 & 3 & 0 \\
y_3 & 1 & 0 & 0 & 0 & 1 \\
\hline
 & 1 & -2 & -7 & 11 & 0
\end{array}
\longrightarrow
\begin{array}{c|cccc|c}
 & x_3 & x_4 & x_1 & x_2 & \\
\hline
y_1 & -1 & 2 & 1 & -2 & 0 \\
y_2 & -1 & 1 & 2 & 3 & 0 \\
y_3 & 1 & 0 & 0 & 0 & 1 \\
\hline
 & -1 & 2 & -3 & 5 & 0
\end{array}
$$

and we are back where we started with just some columns interchanged.

2.

$$
\begin{array}{c|cccc|cccc}
 & x_1 & x_2 & x_3 & x_4 & & & & \\
\hline
y_1 & 1 & -2 & -1 & 2 & 0 & 1 & 0 & 0 \\
y_2 & 2 & -3 & -1 & 1 & 0 & 0 & 1 & 0 \\
y_3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
\hline
 & -3 & 5 & -1 & 2 & 0 & 0 & 0 & 0
\end{array}
\longrightarrow
\begin{array}{c|cccc|cccc}
 & y_2 & x_2 & x_3 & x_4 & & & & \\
\hline
y_1 & -.5 & -.5 & -.5 & 1.5 & 0 & 1 & -.5 & 0 \\
x_1 & .5 & -1.5 & -.5 & .5 & 0 & 0 & .5 & 0 \\
y_3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
\hline
 & 1.5 & .5 & -2.5 & 3.5 & 0 & 0 & 1.5 & 0
\end{array}
$$

$$
\longrightarrow
\begin{array}{c|cccc|cccc}
 & y_2 & x_2 & y_3 & x_4 & & & & \\
\hline
y_1 & -.5 & -.5 & .5 & 1.5 & .5 & 1 & -.5 & .5 \\
x_1 & .5 & -1.5 & .5 & .5 & .5 & 0 & .5 & .5 \\
x_3 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\
\hline
 & 1.5 & .5 & 2.5 & 3.5 & 2.5 & 0 & 1.5 & 2.5
\end{array}
$$

The value is 2.5, the solution to the maximum problem is $\boldsymbol{x} = (.5, 0, 1, 0)$, and the solution to the minimum problem is $\boldsymbol{y} = (0, 1.5, 2.5)$.

3. The first four pivots are the same as for Problem 1. Then

$$
\begin{array}{c|cccc|c}
 & y_1 & y_2 & x_1 & x_2 & \\
\hline
x_3 & 1 & -2 & -3 & 4 & 0 \\
x_4 & 1 & -1 & -1 & 1 & 0 \\
y_3 & -1 & 2 & 3 & -4 & 1 \\
\hline
 & -1 & 0 & -4 & 7 & 0
\end{array}
\longrightarrow
\begin{array}{c|cccc|c}
 & y_1 & y_2 & y_3 & x_2 & \\
\hline
x_3 & 0 & 0 & 1 & 0 & 1 \\
x_4 & \frac{2}{3} & -\frac{1}{3} & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
x_1 & -\frac{1}{3} & \frac{2}{3} & \frac{1}{3} & -\frac{4}{3} & \frac{1}{3} \\
\hline
 & -\frac{7}{3} & \frac{8}{3} & \frac{4}{3} & \frac{5}{3} & \frac{4}{3}
\end{array}
\longrightarrow
\begin{array}{c|cccc|c}
 & x_4 & y_2 & y_3 & x_2 & \\
\hline
x_3 & 0 & 0 & 1 & 0 & 1 \\
y_1 & \frac{3}{2} & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\
x_1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{3}{2} & \frac{1}{2} \\
\hline
 & \frac{7}{2} & \frac{3}{2} & \frac{5}{2} & \frac{1}{2} & \frac{5}{2}
\end{array}
$$

This is the solution found in Problem 2, but with some columns interchanged and some rows interchanged.

Solutions to the Exercises of Section 8.

1. The Northwest Corner Rule and the Least-Cost Rule lead to the same feasible shipping schedule. Solve for the dual variables and find that the constraint $v_j - u_i \le b_{ij}$ is not satisfied for $i = 1,\ j = 2$. Add θ to that square, and add and subtract θ from other squares keeping the constraints satisfied. The transporation cost can be improved by taking $\theta = 1$.

1 2	4 	2
2 1	6 1	2
3	1	

$\longrightarrow$

	2	6	
1	1 $-\theta$ 2	4 $+\theta$	2
0	2 $+\theta$ 1	6 $-\theta$ 1	2
	3	1	

$\longrightarrow$

	1	4	
0	1 1	4 1	2
-1	2 2	6 	2
	3	1	

This leads to a transportation array that satisfies the optimality condition. The value of the optimal shipping schedule is $1+4+4=9$. As a check, $\sum v_j r_j - \sum u_i s_i = 7+2 = 9$.

2. The Least-Cost Rule, which gives the same result no matter how the rows and columns are rearranged, leads to success on the first try. The value is $1+6+15+35 = 57$.

	1	3	5	6	
0	1 1	3 2	5 3	8 	6
-1	2 	5 	6 0	7 5	5
	1	2	3	5	

3. After finding the Least-Cost shipping schedule and the dual varaibles, we see that the constraint $v_j - u_i \le b_{ij}$ is not satisfied for $i = 2,\ j = 3$. This leads to an improvement by adding $\theta = 5$ to that square. The new schedule is optimal, with value 535.

	7	6	8	
-1	10 	8 	9 15	15
4	5 	2 $-\theta$ 20	3 $+\theta$	20
4	6 	7 	4 30	30
0	7 25	6 $+\theta$ 5	8 $-\theta$ 5	35
	25	25	50	

$\longrightarrow$

	7	6	7	
-2	10 	8 	9 15	15
4	5 	2 15	3 5	20
3	6 	7 	4 30	30
0	7 25	6 10	8 	35
	25	25	50	

4. After finding an initial shipping schedule using the Least-Cost Rule, it takes two more applications of the improvment routine with $\theta = 1$ in each to arrive at the optimal schedule.

	8	6	7	5	
3	4 $+\theta$	3	4	2 $-\theta$ 4	4
0	8 $-\theta$ 2	6	7 2	5	4
2	6	4 $-\theta$ 4	5	3 $+\theta$ 2	6
1	7	5 $+\theta$ 3	6 $-\theta$ 1	4	4
	2	7	3	6	

	8	7	7	6	
4	4 $+\theta$ 1	3	4	2 $-\theta$ 3	4
0	8 $-\theta$ 1	6	7 3	5 $+\theta$	4
3	6	4 3	5	3 3	6
2	7	5 4	6	4	4
	2	7	3	6	

	7	6	7	5	
3	4 2	3	4	2 2	4
0	8	6	7 3	5 1	4
2	6	4 3	5	3 3	6
1	7	5 4	6	4	4
	2	7	3	6	

Related Texts

[1] Vašek Chvátal, *Linear Programming*, (1983) W. H. Freeman & Co.

[2] G. B. Dantzig, *Linear Programming and Extensions*, (1963) Princeton University Press.

[3] David Gale, *The Theory of Linear Economic Models*, (1960) McGraw-Hill.

[4] Samuel Karlin, *Mathematical Methods and Theory in Games, Programming and Economics*, vol. 1, (1959) Addison-Wesley.

[5] James K. Strayer, *Linear Programming and Applications*, (1989) Springer-Verlag.