The control of a gantry

AAE 364L

In this experiment we will design a controller for a gantry or crane. Without a controller the pendulum of crane will swing for a long time. The idea is to use control to stop the swinging of the pendulum.

This experiment consists of a cart with mass M on a one dimensional track and a pendulum swinging from the cart. The pendulum is in the downward position. The cart is driven by a force from a servo motor. The position of the cart is denoted by $x_c(t)$, and the Voltage to the servo motor is denoted by $v(t)$. The angle between the pendulum and its resting position is denoted by α; see Figure 1. The nonlinear equations of motion are given by

$$\begin{aligned}
&\left((M_c+M_p)I_p+M_cM_pl_p^2+M_p^2l_p^2\sin(\alpha)^2\right)\ddot{x}_c+\left(I_p+M_pl_p^2\right)B_{eq}\dot{x}_c\\
=\;& M_pl_pB_p\cos(\alpha)\dot{\alpha}+\left(M_p^2l_p^3+I_pM_pl_p\right)\sin(\alpha)\dot{\alpha}^2+\left(I_p+M_pl_p^2\right)F_c+M_p^2l_p^2g\cos(\alpha)\sin(\alpha);\\
&\left((M_c+M_p)I_p+M_cM_pl_p^2+M_p^2l_p^2\sin(\alpha)^2\right)\ddot{\alpha}+(M_c+M_p)B_p\dot{\alpha}\\
=\;& M_pl_p\cos(\alpha)B_{eq}\dot{x}_c-(M_c+M_p)M_pgl_p\sin(\alpha)-M_p^2l_p^2\sin(\alpha)\cos(\alpha)\dot{\alpha}^2-M_pl_p\cos(\alpha)F_c;\\
F_c\;=\;&\frac{\eta_gK_g\eta_mK_t\left(vr_{mp}-K_gK_m\dot{x}_c\right)}{R_mr_{mp}^2}.
\end{aligned}\tag{0.1}$$

Here F_c is the force on the cart. The notation and values are given in the table in the Appendix. These values are given also given in the MATLAB file "setup_lab_ip02_spg.m" posted on the Web page for the course. So you will not have to calculate any of these values for the Lab. The linearized equations of motion are given by

$$\begin{aligned}
\ddot{x}_c &= \frac{-(I_p+M_pl_p^2)B_{eq}\dot{x}_c+M_pl_pB_p\dot{\alpha}+M_p^2l_p^2g\alpha+(I_p+M_pl_p^2)F_c}{(M_c+M_p)I_p+M_cM_pl_p^2}\\
\ddot{\alpha} &= \frac{M_pl_pB_{eq}\dot{x}_c-(M_c+M_p)B_p\dot{\alpha}-(M_c+M_p)M_pgl_p\alpha-M_pl_pF_c}{(M_c+M_p)I_p+M_cM_pl_p^2}\\
F_c &= \frac{\eta_gK_g\eta_mK_t\left(vr_{mp}-K_gK_m\dot{x}_c\right)}{R_mr_{mp}^2}.
\end{aligned}\tag{0.2}$$

Now let us convert the linear equations to a state space model of the form

$$\dot{x}=Ax+Bv.$$

Here A is a 4×4 matrix, B is a column vector of length 4 and the input v is the voltage to

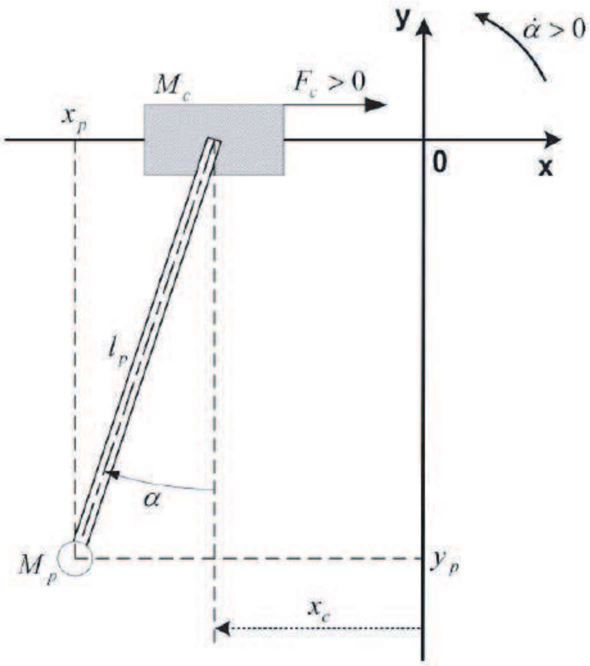

Figure 1 The Gantry

the motor. To convert the system in (0.2) we used the following state variables:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_c \\ \alpha \\ \dot{x}_c \\ \dot{\alpha} \end{bmatrix}. \tag{0.3}$$

By consulting the values in the table at the end of the section, we computed A and B, that is,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1.5216 & -11.6513 & 0.0049 \\ 0 & -26.1093 & 26.8458 & -0.0841 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 \\ 0 \\ 1.5304 \\ -3.5261 \end{bmatrix}. \tag{0.4}$$

These matrices A and B are contained in the MATLAB setup file on the Web page, so you do not have to retype A and B in MATLAB. The eigenvalues of A are given by $\{0, -11.3975, -0.1689 + 4.8040i, -0.1689 - 4.8040i\}$. So A only is marginally stable.

Let us compute the transfer function from the voltage v to the output x_c. Since $x_1 = x_c$, the corresponding output matrix C is given by

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}. \tag{0.5}$$

By using $ss2tf$ in MATLAB, we see that the transfer function from the voltage v to the output x_c is given by

$$\frac{X_c(s)}{V(s)} = C(sI - A)^{-1}B = \frac{1.53s^2 + 0.1114s + 34.59}{s^4 + 11.74s^3 + 26.96s^2 + 263.4s}. \tag{0.6}$$

Notice that $\frac{X_c(s)}{V(s)}$ has a pole at the origin. Moreover, the poles of $\frac{X_c(s)}{V(s)}$ are the eigenvalues of A. Finally, it is noted $\frac{X_c(s)}{V(s)}$ is a fourth order system.

Now let us compute the transfer function from the voltage v to the angle α. Since $x_2 = \alpha$, the corresponding output matrix C is given by

$$C = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}. \tag{0.7}$$

By using $ss2tf$ in MATLAB, we see that the transfer function from the voltage v to the angle α is given by

$$\frac{\alpha(s)}{V(s)} = C(sI - A)^{-1}B = \frac{-3.526s}{s^3 + 11.74s^2 + 26.96s + 263.4}. \tag{0.8}$$

Finally, it is noted $\frac{\alpha(s)}{V(s)}$ is a third order system.

Let x_e denote the position of the end of the pendulum along the track. The position of the end of the pendulum is given by $x_e = x_c + L_p \sin(\alpha)$ where L_p is the length of the pendulum; see Figure 1. (Notice that l_p is the distance to the center of mass of the pendulum and L_p is the distance to the end of the pendulum.) For small angles $x_e \approx x_c + L_p\alpha$. Since $x_1 = x_c$, $x_2 = \alpha$ and $L_p = 0.6413$ m, the corresponding output matrix C is given by

$$C = \begin{bmatrix} 1 & 0.6413 & 0 & 0 \end{bmatrix}. \tag{0.9}$$

By using $ss2tf$ in MATLAB we see that the transfer function from the voltage v to the end of the pendulum x_e is given by

$$\frac{X_e(s)}{V(s)} = C(sI - A)^{-1}B = \frac{-0.7311s^2 + 0.1114s + 34.59}{s^4 + 11.74s^3 + 26.96s^2 + 263.4s}. \tag{0.10}$$

Finally, it is noted that this is a fourth order system.

1 Part (i): The natural frequency of the pendulum

In this part we will experimentally determine the natural frequency of the pendulum. For the moment assume that the cart is fixed to the origin and does not move. However, the pendulum is free to swing. So assume that the cart is fixed. Then the equation of motion for the pendulum is given by

$$\left(I_p + M_p l_p^2\right)\ddot{\alpha} + M_p l_p g \sin(\alpha) = 0. \tag{1.1}$$

The equation of motion is nonlinear. Recall that for small angles, $\sin(\alpha) \approx \alpha$. Using this we see that the linearized equation of motion is determined by

$$\left(I_p + M_p l_p^2\right) \ddot{\alpha} + M_p l_p g \alpha = 0. \tag{1.2}$$

Hence the linear equation of motion for the pendulum is given by

$$\ddot{\alpha} + \omega_p^2 \alpha = 0 \quad \text{where} \quad \omega_p = \sqrt{\frac{M_p l_p g}{I_p + M_p l_p^2}}. \tag{1.3}$$

Here ω_p is the natural frequency of the pendulum. In other words, the pendulum will oscillate at ω_p for small angles. For example, assume that the initial condition are given by $\alpha(0) = \alpha_0$ and $\dot{\alpha} = 0$. The the solution to the linear differential equation in (1.3) is given by

$$\alpha(t) = \alpha_0 \cos(\omega_p t). \tag{1.4}$$

By consulting the table for our experiment (or the corresponding MATLAB setup file) we see that

$$\omega_p = 4.7543 \ rad/s. \tag{1.5}$$

So let us verify this number experimentally. Hold the cart still and start the pendulum off at a small angle $\alpha(0) = \alpha_0$ and $\dot{\alpha} = 0$. Record the output of the angle. Then measure the period τ. We expect the period to be approximately

$$\tau = \frac{2\pi}{\omega_p} = 1.3216 \ s. \tag{1.6}$$

1.1 The Lab steps to experimentally determine ω_p

(i) Start the WinCon server, open MATLAB from the WinCon server. In MATLAB change the directory to $D : \backslash aae364L \backslash labgantry \backslash section\#$.
Run the setup file "setup_lab_ip02_spg.m".

(ii) Open up labgantrywp in Simulink and hit build.

(iii) In the WinCon server window, open the plot for the angle, under the tab update ⇒ buffer; set to 20 seconds.

(iv) Hit the start button in WinCon.

(v) Secure the cart. Move the pendulum about 15 degrees and let it swing.

(vi) After the pendulum swings about ten revolutions hit stop.

(vii) Save the angle in MATLAB, that is, go to: File ⇒ Save ⇒ save as Mat file. Take this Mat file with you.

1.2 In your lab report include the following under Part (i):

(a) Hand in your experimental plot of the angle.

(b) Hand in your calculation of the frequency ω_p that you determined from the experiment.

2 Part (ii): The resonance of ω_n

In this part we will determine the natural frequency of the entire system with the cart and pendulum. Then we will excite the cart at its natural frequency to demonstrate resonance. In other words, we will show that a small sinusoid at the natural frequency will lead to a large amplitude in the response.

In this part the cart is free to move. Consider the linearized equations of motion in (0.2). Recall that the transfer function from the voltage v to the angle α is given by the following third order system:

$$G_\alpha(s) = \frac{\alpha(s)}{V(s)} = \frac{-3.526s}{s^3 + 11.74s^2 + 26.96s + 263.4} = \frac{-3.526s}{(s-\lambda)(s^2 + 2\zeta\omega_n s + \omega_n^2)}.$$

Compute the natural frequency ω_n and the damping ratio ζ for this transfer function $G_\alpha(s)$. Observe that one can also compute the natural frequency ω_n and the damping ratio ζ from the eigenvalues of A given in (0.4). The eigenvalues of A are 0, -11.3975 and $-0.1689 \pm 4.8040i$. So using the complex eigenvalues we have

$$(s + 0.1689 + 4.8040i)(s + 0.1689 - 4.8040i) = s^2 + 2\zeta\omega_n s + \omega_n^2.$$

Recall that one can also determine the damping ratio and natural frequency directly from the Bode plot of $G_\alpha(s)$. So plot the Bode plot of $G_\alpha(s)$ and determine the natural frequency ω_n from the bode plot.

To demonstrate resonance we will excite the system with a sinusoid of the form

$$v(t) = A\sin(\omega t). \tag{2.1}$$

In this case, the steady state response of the linear system is given by

$$\alpha_{ss}(t) = A|G_\alpha(\imath\omega)|\sin(\omega t + \angle G_\alpha(\imath\omega)). \tag{2.2}$$

So as the frequency $\omega \approx \omega_n$, the pendulum should start to oscillate wildly. Recall that the actual system is nonlinear. However, our linear approximation will demonstrate resonance.

2.1 Pre-Lab for resonance. Due at the beginning of the lab experiment. You will not be allowed to run the lab experiment with out a complete pre-lab.

(i) Hand in your values for the damping ratio ζ and natural frequency ω_n that you calculated.

(ii) Hand in the Bode plot of G_α. Determine $|G_\alpha(\imath\omega)|$ from the Bode plot for $\omega = 3$ rad/s, ω_n, and 7 rad/s.

2.2 The Lab steps to demonstrate resonance

(i) In the same directory, open up labgantryres in Simulink and hit build. In MATLAB command window, type $X\_MAX = 0.6; X\_MIN = -0.6;$.

(ii) In the WinCon server window, open the plot for the angle, under the tab update $\Rightarrow$ buffer; set to 40 seconds.

(iii) Place the cart at the left side of the track. Set the sinusoid to $3\sin(\omega t)$, that is, set the amplitude of the sinusoid to be 3. Run the experiment for ω varying from 3 rad/s to ω_n and then 7 rad/s.

(iv) Hit the start button in WinCon. After about 30 seconds hit the stop button.

(v) Save the angle plot in MATLAB for $\omega = 3$ rad/s, $\omega = \omega_n$ and $\omega = 7$ rad/s, that is, go to: File $\Rightarrow$ Save $\Rightarrow$ save as Mat file. Take this Mat file with you.

2.3 In your lab report include the following under Part (ii):

(a) Hand in the values for ω_n and ζ that you calculated.

(b) Notice that $\omega_n \approx \omega_p$. In two short sentences or less explain why $\omega_n \approx \omega_p$.

(c) Hand in the Bode plot of G_α. Hand in your value for the natural frequency ω_n that you computed from the Bode plot. In two short sentences or less explain how you computed ω_n from the Bode plot.

(d) Hand in your plots of the angle from the experiment for $\omega = 3$ rad/s, $\omega = \omega_n$ and $\omega = 7$ rad/s. What happens when the frequency $\omega = \omega_n$?

(e) Compute $|G_\alpha(\imath\omega)|$ from the Bode plot of G_α and from the experimental results for $\omega = 3$ rad/s, $\omega = \omega_n$ and $\omega = 7$ rad/s.

3 Part (iii): Pole placement

In this section we will use pole placement to control the oscillations of the pendulum. Consider the state space system given by

$$\dot{x} = Ax + Bv \quad \text{and} \quad y = Cx. \tag{3.1}$$

The input is v, the output is y and the vector x is the state. Here A is a $n \times n$ matrix, B is a column vector of length n, the state x is a vector of length n and C is the row vector of length n given by

$$\begin{aligned} A &= \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, \quad B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \\ C &= \begin{bmatrix} c_1 & c_2 & \cdots & c_n \end{bmatrix}. \end{aligned} \tag{3.2}$$

Recall that the solution to the state space system in (3.1) is given by

$$\begin{aligned} x(t) &= e^{At}x(0) + \int_0^t e^{A(t-\sigma)}Bv(\sigma)d\sigma \\ y(t) &= Ce^{At}x(0) + \int_0^t Ce^{A(t-\sigma)}Bv(\sigma)d\sigma. \end{aligned} \tag{3.3}$$

The transfer function from the input v to the output y is given by

$$\frac{Y(s)}{V(s)} = G(s) = C(sI - A)^{-1}B. \tag{3.4}$$

Recall that the characteristic polynomial for A is defined by $\det[sI - A]$ where det denotes the determinant. In particular, the the zeros of the characteristic polynomial $\det[sI - A]$ are the eigenvalues of A. By using the appropriate minors to compute the inverse of $sI - A$, it follows that the transfer function

$$G(s) = C(sI - A)^{-1}B = \frac{q(s)}{\det[sI - A]} \tag{3.5}$$

where q is a polynomial of degree at most $n-1$. In particular, the poles of $G(s)$ are contained in the set of all eigenvalues for A. Finally, it is noted that the open loop system G is stable if all the eigenvalues of A are contained in the open left half plane $\{s : \Re s < 0\}$.

We say that K is a *state feedback gain or state feedback vector* if K is a row vector of length n, that is,

$$K = \begin{bmatrix} k_1 & k_2 & \cdots & k_n \end{bmatrix}. \tag{3.6}$$

The reference signal r associated with the state feedback vector K is given by

$$v(t) = r(t) - Kx(t) = r(t) - \sum_{j=1}^{n} k_j x_j(t). \tag{3.7}$$

Substituting this v into $\dot{x} = Ax + Bv$ yields the following state variable feedback system:

$$\dot{x} = (A - BK)\,x + Br \quad \text{and} \quad y = Cx. \tag{3.8}$$

The solution to the state space feedback system in (3.8) is given by

$$\begin{aligned} x(t) &= e^{(A-BK)t}x(0) + \int_0^t e^{(A-BK)(t-\sigma)}Br(\sigma)d\sigma \\ y(t) &= Ce^{(A-BK)t}x(0) + \int_0^t Ce^{(A-BK)(t-\sigma)}Br(\sigma)d\sigma. \end{aligned} \tag{3.9}$$

The transfer function from the reference signal r to the output y is given by

$$\frac{Y(s)}{R(s)} = G_K(s) = C\left(sI - (A - BK)\right)^{-1}B. \tag{3.10}$$

Notice that the closed loop transfer function $G_K(s)$ depends upon the choice of the feedback gain K. The characteristic polynomial for $A - BK$ is defined by $\det[sI - (A - BK)]$. By using the appropriate minors to compute the inverse of $sI - (A - BK)$, it follows that the closed loop transfer function from the reference signal r to the output y is given by

$$G_K(s) = C(sI - (A - BK))^{-1}B = \frac{q(s)}{\det[sI - (A - BK)]} \tag{3.11}$$

where q is a polynomial of degree at most $n - 1$. In particular, the poles of the closed loop system $G_K(s)$ are contained in the set of all eigenvalues for $A - BK$. Finally, it is noted that the closed loop system G_K is stable if all the eigenvalues of $A - BK$ are contained in the open left half plane $\{s : \Re s < 0\}$.

Recall that the poles of the closed loop system depend upon the choice of the feedback gain K. The eigenvalues of the closed loop system in (3.8) are determined by the eigenvalues of $A-BK$. Our feedback strategy is to pick the state feedback gain K to place the eigenvalues of $A - BK$ at n specified locations $\{\lambda_j\}_1^n$ in the open left hand plane $\{s : \Re s < 0\}$. In this case, the closed loop system G_K is stable. Moreover, if the reference signal r set to zero, then the solution of the feedback system (3.8) converges to zero, that is,

$$x(t) = e^{(A-BK)t}x(0) \to 0.$$

So by choosing the state feedback vector appropriately, one can force the state to zero. Moreover, by placing the eigenvalues of $A - BK$ at n specified locations $\{\lambda_j\}_1^n$ in the open left hand plane $\{s : \Re s < 0\}$ one can dictate how the state x converges to zero. Finally, it is noted that in order to implement state variable feedback, one needs an actuator or an instrument to to read all the states $\{x_j\}_1^n$.

Consider the state space system in (3.1) or a pair of matrices $\{A, B\}$ where A is a $n \times n$ matrix and B is a column vector of length n. The controllability matrix W associated with the pair $\{A, B\}$ is the $n \times n$ matrix defined by

$$W = \begin{bmatrix} B & AB & A^2B & A^3B & \cdots & A^{n-1}B \end{bmatrix}. \tag{3.12}$$

The state space system in (3.1) or the pair $\{A, B\}$ is *controllable* if its controllability matrix W is invertible, or equivalently, the rank of W equals n. We will use the following result to control the pendulum swing in our gantry problem.

THEOREM 3.1 *Consider the state space system*

$$\dot{x} = Ax + Bv \tag{3.13}$$

where A is a $n \times n$ matrix and B is a column vector of length n. Assume that the pair $\{A, B\}$ is controllable. Let $\{\lambda_j\}_1^n$ any set of distinct complex numbers. Then there exists a state feedback vector K such that $\{\lambda_j\}_1^n$ are the eigenvalues of $A - BK$. In particular, the poles of the feedback system G_K are contained in $\{\lambda_j\}_1^n$, that is,

$$G_K(s) = C(sI - (A - BK))^{-1}B = \frac{q(s)}{(s - \lambda_1)(s - \lambda_2)\cdots(s - \lambda_n)}. \tag{3.14}$$

Finally, this state feedback gain K can be computed by using the place command in MATLAB, that is,

$$K = \text{place}\left(A, B, \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}\right). \tag{3.15}$$

The Gantry Design Problem. If one hits the pendulum, then the pendulum will oscillate for a long period of time. Your Prelab assignment is to design a controller to make the pendulum stop oscillating in 2.2 seconds or less. Bring you control design to the Lab. Then the TA will tap the pendulum to see if your design works.

One method to design a feedback controller is to consider what happens for transfer functions F of the form

$$F(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}. \tag{3.16}$$

Let PO denote the percent overshoot and t_s the settling time due to a step. Then it is well known that if the natural frequency ω_n and ζ are given by

$$\zeta = \frac{|\ln(PO/100)|}{\sqrt{\ln(PO/100)^2 + \pi^2}} \quad \text{and} \quad \omega_n = \frac{4}{t_s\zeta}. \tag{3.17}$$

Then the response of the transfer function F due to a step has the desired percent overshoot PO and settling time t_s. For our problem let us choose $PO = 5$ and $t_s = 2.2$ s. Hence

$$\zeta = 0.6901 \quad \text{and} \quad \omega_n = 2.6346\ rad/s. \tag{3.18}$$

So if $\zeta = 0.6901$ and $\omega_n = 2.6346$ rad/s for the transfer function F in (3.16), then the response due to a step will have a 5% percent overshoot and a settling time of 2.2 seconds. The poles λ_1 and λ_2 of F corresponding to $\zeta = 0.6901$ and $\omega_n = 2.6346$ rad/s are given by solving for the roots of

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = s^2 + 3.6364s + 6.9413 = (s - \lambda_1)(s - \lambda_2).$$

Therefore the poles of F corresponding to $\zeta = 0.6901$ and $\omega_n = 2.6346$ rad/s are given by

$$\lambda_1 = -1.8182 + 1.9067i \quad \text{and} \quad \lambda_2 = -1.8182 - 1.9067i. \tag{3.19}$$

Notice that our state space linear approximation for the cart and pendulum in (0.2) is fourth order. Moreover, the pair $\{A, B\}$ in (0.4) is controllable. Now let $\{\lambda_j\}_1^4$ be any four distinct complex numbers in the open left half plane $\{s : \Re s < 0\}$. Then one can use the place command in MATLAB to find a state feedback gain K such that $\{\lambda_j\}_1^4$ are the eigenvalues of $A - BK$, that is,

$$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix} = \text{place}\,(A, B, \begin{bmatrix} \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \end{bmatrix}) \tag{3.20}$$

For this choice of K the closed loop system is given by

$$G_K(s) = C(sI - (A - BK))^{-1}B = \frac{q(s)}{(s - \lambda_1)(s - \lambda_2)(s - \lambda_3)(s - \lambda_4)}. \tag{3.21}$$

Here $C = \begin{bmatrix} c_1 & c_2 & c_3 & c_4 \end{bmatrix}$ can be any matrix. For example, C can be the matrix any one of the matrices in (0.5), (0.7) or (0.9).

Notice that the closed loop transfer function in (3.21) is not in the same form as F in (3.16). However, we can use state feedback to find a state gain vector $k = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$

to place the poles $\{\lambda_j\}_1^4$ of G_K at any four locations that we want. So our design procedure is to place two poles at $\lambda_1 = -1.8182 + 1.9067i$ and $\lambda_2 = -1.8182 - 1.9067i$ which achieve a 5% overshoot and 2.2 seconds settling time for F. Then place the other two poles λ_3 and λ_4 on the real line far away from $-1.8182 \pm 1.9067i$. In this way we hope to mimic the response of the transfer function F due to a step. In the first part our experiment we will not be exciting the cart by a step, we will just tap the pendulum to see how it responds.

For your pre-lab use the place command in MATLAB to place four poles $\{\lambda_j\}_1^4$ where $\lambda_1 = -1.8182 + 1.9067i$ and $\lambda_2 = -1.8182 - 1.9067i$ while λ_3 and λ_4 are on the real line and far away from $-1.8182 \pm 1.9067i$. Now go to the course Web page and load the following MATLAB files in your computer

- setup_lab_ip02_spg.m
- setup_ip01_2_configuration.m
- setup_sp_configuration.m
- SPG_ABCD_eqns.m
- d_ip02_spg_pp.m
- s_spg_pp.mdl

Run the MATLAB file "setup_lab_ip02_spg.m" and open the Simulink file "s_spg_pp.mdl". The Simulink model is displayed in Figure 2. In this Simulink file set the amplitude in the position set point block to zero. All the variables that you need to run the simulation are now loaded in the computer. The computer contains all the values in the Table in the Appendix. In MATLAB set the following initial conditions:

$$X0 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \pi/2 \end{bmatrix}. \tag{3.22}$$

This sets the initial conditions $x_c(0) = 0$ m, $\dot{x}_c(0) = 0$ m/s, $\alpha(0) = 0$ rad and the initial angle angular velocity to $\dot{\alpha}(0) = \pi/2$ rad/s. These initial conditions are set to simulate someone hitting the pendulum with $\dot{\alpha}(0) = \pi/2$ rad/s and $\alpha(0) = 0$ rad. Here we want to see how our controller will respond when the pendulum is struck with an angular velocity of $\dot{\alpha}(0) = \pi/2$ rad/s while the pendulum is a rest. To simulate the pendulum with no control set $K = \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix}$ in MATLAB. Then run the Simulink file and save the angle plot to hand in later. Make sure that the amplitude in the position set point block is set equal to zero.

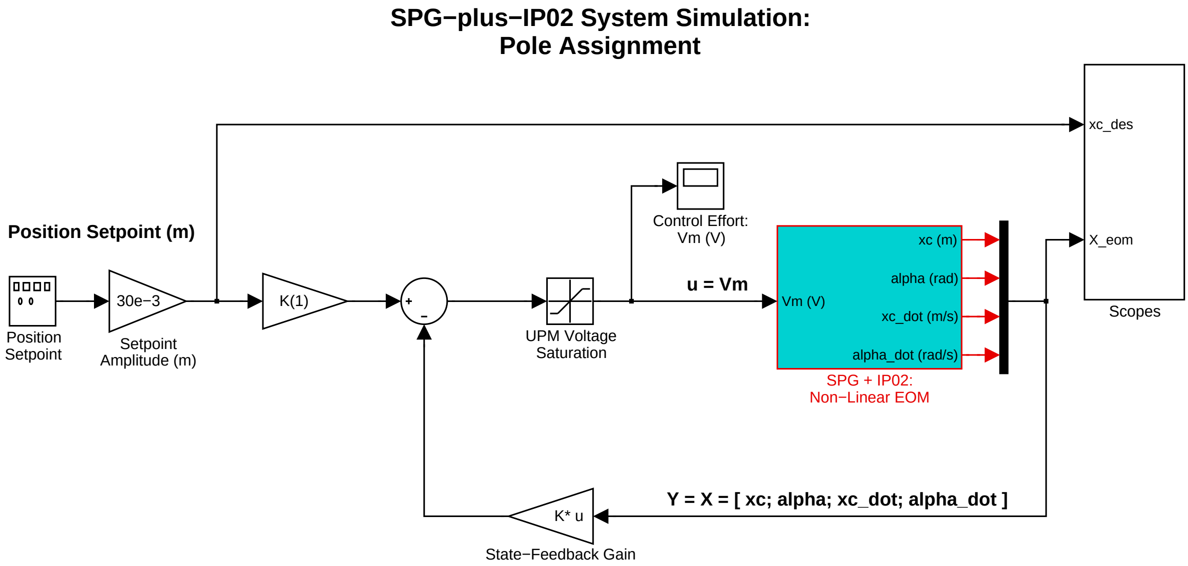

Figure 2: The closed loop nonlinear model.

Now you can use this Simulink file to design a feedback control with the place command. To accomplish this choose the poles λ_3 and λ_4 that you deem appropriate. Then in MATLAB type

$$K = \text{place}\left(A, B, \left[\begin{array}{cccc} -1.8182 + 1.9067i & -1.8182 - 1.9067i & \lambda_3 & \lambda_4 \end{array}\right]\right). \tag{3.23}$$

Then K will be a vector of length 4 which is the state feedback gain used in the Simulink file. Now run the Simulink file and check out the various scopes to see if you like the response. Then vary $\{\lambda_j\}_1^4$ until you achieve the response that meets the design objective. You do not have to choose $\lambda_1 = -1.8182+1.9067i$ and $\lambda_2 = -1.8182-1.9067i$. This is just a suggestion. You can choose $\{\lambda_j\}_1^4$ to have any value that you want as long as the angle stops moving in under 2.2 seconds. Save the angle plot for your best values of K. Then plot this angle on the same plot with $K = 0$. Bring your best value of the state feedback gain K to the Lab with you. For your pre-lab also bring your plot of the angle α for your best feedback gain and no feedback $K = 0$ on the same plot. Finally, it is noted that K cannot be infinitely large, that is, $|K(j)| \leq 200$ for all j.

3.1 Pre-Lab for pole placement. Due at the beginning of the lab experiment. You will not be allowed to run the lab experiment with out a complete pre-lab.

(i) Hand in your best values for the state gain K that you computed from Simulink.

(ii) Hand in the graph of the angle α for your best choice of the feedback gain K and the graph of the angle α for no control $K = 0$ on the same plot for 15 seconds.

3.2 The Lab steps to pole placement

(i) In MATLAB, enter your best values of K from your Simulink design, that is, set

$$K = \begin{bmatrix} K(1) & K(2) & K(3) & K(4) \end{bmatrix}.$$

(ii) In the same directory, open up labgantrypp in Simulink and hit build.

(iii) In the WinCon server window, open the plot for the angle, under the tab update $\Rightarrow$ buffer; set to 15 seconds.

(iv) Hit the start button in WinCon. Have the TA tap the pendulum. After about 10 seconds hit the stop button.

(v) If you are not happy with your response you can change the values of K, and run the experiment again. Note that you can only do this because it can be done cheaply.

(vi) For your best run, save the angle in MATLAB, that is, go to: File $\Rightarrow$ Save $\Rightarrow$ save as Mat file. Take this Mat file with you.

3.3 In your lab report include the following under Part (iii):

(a) Hand in the best values for the state gain K that you calculated using Simulink, along with the the time it takes for the pendulum to stop swinging after it is given initial angular velocity $\dot{\alpha}(0) = \pi/2$ rad/s.

(b) Hand in the best values for the state gain K that you used in the experiment.

(c) From the plot of the angle from the experiment, determine the initial angular velocity after you tapped the pendulum. Using the best values for the state gain K that you used in the experiment, simulate your system in Simulink with this initial condition of angular velocity. Hand in your plot of the angle from your best experiment. Compare that with the plot of the angle from your simulation using your best values for the state gain K.

(d) Is your simulation accurate? Use your simulation to estimate the time it takes for the pendulum to stop swinging after given initial angular velocity $\dot{\alpha}(0) = \pi/2$ rad/s.

4 Integral control of the gantry

In this section we will add an integral control to move the pendulum to various positions on the track. Our design objective is to move the cart and pendulum 0.5 meters on the track and have the pendulum stop moving as soon as possible. The idea behind our design is to incorporate an integral controller in our pole placement method. To this end, consider the four dimensional state variable system in (0.3). Now let us define a new state variable

$$x_5 = \int_0^t (u(\sigma) - x_1(\sigma))\, d\sigma \quad \text{or equivalently} \quad \dot{x}_5 = u - x_1. \tag{4.1}$$

Here u is the new input. Recall that x_1 is the position of the cart. In our problem u will be a constant. So x_5 will be the integral between the reference signal u and the position of the cart. Using $\dot{x}_5 = u - x_1$, we may write a new state space system as

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \\ \dot{x}_5 \end{bmatrix} = \begin{bmatrix} A & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ 0 \end{bmatrix} v + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \tag{4.2}$$

Now let us define the new state matrix A_i and B_i, that is,

$$\begin{aligned} A_i &= \begin{bmatrix} A & \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\ \begin{bmatrix} -1 & 0 & 0 & 0 \end{bmatrix} & 0 \end{bmatrix} \quad \text{and} \quad B_i = \begin{bmatrix} B \\ 0 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ 0 \end{bmatrix} \\ x &= \begin{bmatrix} x_c \\ \alpha \\ \dot{x}_c \\ \dot{\alpha} \\ \int (u - x_c) \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}. \end{aligned}$$

Our new state space system is given by

$$\dot{x} = A_i x + B_i v + Du. \tag{4.3}$$

Using the matrices A and B in (0.4) one can easily construct A_i, B_i and D in MATLAB. Recall that you already have A and B in MATLAB once you run “setup_lab_ip02_spg.m”. So in MATLAB A_i and B_i are given by

$$\begin{aligned} A_i &= [A, \text{zeros}(4,1); -1, 0, 0, 0, 0] \\ B_i &= [B; 0]\,. \end{aligned}$$

Finally, it is noted that A_i is a 5×5 matrix and B_i is a column vector of length 5.

It turns out that the pair $\{A_i, B_i\}$ is controllable. So one can use the place command in MATLAB to place the eigenvalues of $A_i - B_i K$ at any five distinct locations $\{\lambda_j\}_1^5$ in the open left half plane $\{s : \Re s < 0\}$. Substituting $v = -Kx$ into the new state variable system (4.3), we arrive at

$$\dot{x} = (A_i - B_i K)\, x + Du. \tag{4.4}$$

Now choose u to be the step $u(t) = u_0$ for all $t \geq 0$. Because all the eigenvalues of A are in the open left half plane, the state space system in (4.4) will move to steady state, that is, $\dot{x} = 0$ in steady state. In particular, $0 = \dot{x}_5 = u_0 - x_1$. In other words, in steady state $x_1 = u_0$, and the cart will move to the position u_0 on the track.

Now we would like to design a controller to move the gantry to position $u_0 = 0.5$ meters on the track and have the system settle down in under 2.2 seconds. To accomplish this choose five poles $\{\lambda_j\}_1^5$ that you deem appropriate. Then in MATLAB type

$$K = \text{place}\,(A_i, B_i, [\ \lambda_1 \quad \lambda_2 \quad \lambda_3 \quad \lambda_4 \quad \lambda_5\ ]). \tag{4.5}$$

Then K will be a vector of length 5 which is the state feedback gain used in the Simulink file. One could use $\lambda_1 = -1.8 + 1.9i$ and $\lambda_2 = -1.8 - 1.9i$ and choose the three remaining poles far away to try to maintain a 5% overshoot and 2.2 seconds settling time. However, this choice of λ_1 and λ_2 is not necessary and you may find a much better choice for your poles. Your design problem is to try to design a controller which will move the pendulum and the cart 0.5 meters down the track and have the pendulum settle down as fast as possible. You can use the Simulink file "aae364gantry2.mdl" on the website or simply rebuild the Simulink file "s_spg_pp.mdl" to match that given in Figure 2. To test your design, set the step to 1 and slider gain to 0.5 meters. Then choose the poles $\{\lambda_j\}_1^5$ that you think will work and run place in MATLAB. Finally, run the Simulink file in Figure 3. Keep changing the closed loop poles $\{\lambda_j\}_1^5$ until you find the state feedback gain which matches the design specifications. Bring your best gain matrix K to the Lab.

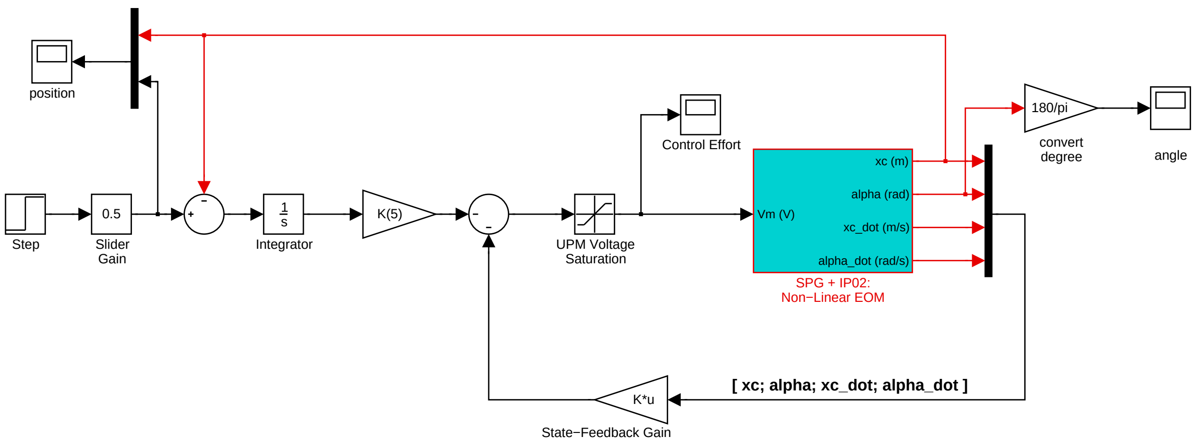

Figure 3: The closed loop nonlinear model.

4.1 Pre-Lab for the integral controller. Due at the beginning of the lab experiment. You will not be allowed to run the lab experiment with out a complete pre-lab.

(i) Hand in your best values for the state gain K that you computed from Simulink.

(ii) Hand in the graph of the cart position x_c and the angle α for your best choice of the feedback gain K, along with the settling time of α.

4.2 The Lab steps to integral control

(i) In MATLAB command window, enter your best values of K from your Simulink design, that is, set
$$K = \begin{bmatrix} K(1) & K(2) & K(3) & K(4) & K(5) \end{bmatrix}.$$
Make sure X_MAX is set to 0.6.

(ii) In the same directory, open up labgantryint in Simulink and hit build.

(iii) In the WinCon server window, open the plot for the cart position and angle, under the tab update ⇒ buffer; set to 15 seconds.

(iv) Set the step to 1 and the slider gain A to 0.5. Put the cart and pendulum at the left end of the track. Hit the start button in WinCon.

(v) If you are not happy with your response you can change the values of K, and run the experiment again.

(vi) For your best run, save the cart position and angle in MATLAB, that is, go to: File ⇒ Save ⇒ save as Mat file. Take this Mat file with you.

(vii) You can now use the slider gain to move the cart. Place the cart in the middle of the track. Now start the experiment. Then move the slider gain to various values between $[-0.2, 0.2]$. Watch the cart move to the position corresponds to the value of the slider gain. You can even tap the pendulum from time to time to see if the cart still moves to the desired position and stops the pendulum from swinging.

4.3 In your lab report include the following under Part (iii):

(a) Hand in the best values for the state gain K that you calculated using Simulink, along with settling time of the pendulum.

(b) Hand in the best values for the state gain K that you used in the experiment, along with settling time of the pendulum.

(c) Simulate your system using the best values for the state gain K that you used in the experiment. Hand in your plots of the cart position from the experiment and the simulation on the same figure. In a separate figure, hand in your plots of angle from the experiment and the simulation.

5 Appendix: The notation for the pendulum and cart

Symbol	Description	Value	Unit
R_m	motor armature resistance	2.6	Ω
L_m	motor armature inductance	0.18	mH
K_t	motor torque constant	0.00767	$N.m/A$
η_m	motor efficiency	100%	%
K_m	back-electromotive-force(EMF) constant	0.00767	$V.s/rad$
J_m	rotor moment of inertia	3.9×10^{-7}	$kg.m^2$
K_g	planetary gearbox ratio	3.71	
η_g	planetary gearbox efficiency	100%	%
M_{c2}	cart mass	0.57	kg
M_w	cart weight mass	0.37	kg
M_c	total cart weight mass including motor inertia	1.0731	kg
B_{eq}	viscous damping at motor pinion	5.4000	$N.s/m$
L_t	track length	0.990	m
T_c	cart travel	0.814	m
P_r	rack pitch	1.664×10^{-3}	$m/tooth$
r_{mp}	motor pinion radius	6.35×10^{-3}	m
N_{mp}	motor pinion number of teeth	24	
r_{pp}	position pinion radius	0.01482975	m
N_{pp}	position pinion number of teeth	56	
K_{EP}	cart encoder resolution	2.275×10^{-5}	$m/count$
M_p	long pendulum mass with T-fitting	0.230	kg
M_{pm}	medium pendulum mass with T-fitting	0.127	kg
L_p	long pendulum length from pivot to tip	0.6413	m
L_{pm}	medium pendulum length from pivot to tip	0.3365	m
l_p	long pendulum length: pivot to center of mass	0.3302	m
l_{pm}	medium pendulum length: pivot to center of mass	0.1778	m
J_p	long pendulum moment of inertia $\circlearrowright$ center of mass	7.88×10^{-3}	$kg.m^2$
J_{pm}	medium pendulum moment of inertia $\circlearrowright$ center of mass	1.20×10^{-3}	$kg.m^2$
B_p	viscous damping at pendulum axis	0.0024	$N.m.s/rad$
g	gravitational constant	9.81	m/s^2