AAE 421 Notes

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Contents

Chapter 1

Introduction

HI!

(How's it going?)

1.1 Ingredients

Dynamics	Modelling
Aerospace vehicles	Applications
State space concepts	System analysis and control design
Linear algebra	Mathematical tools
MATLAB	Computational tools

1.2 MATLAB

Introduce yourself to MATLAB

```
>> lookfor
>> help
>> quit
```

Learn the representation, addition and multiplication of real and complex numbers.

1.3 Some general concepts

1.3.1 Aircraft

Recall that the main components of an aircraft are as follows.

Fuselage
Engine(s)
Main wing(s)
Horizontal tail, vertical tail
Control surfaces: elevators, ailerons, rudder

1.3.2 Systems and control

dynamical system: A system which evolves with time, e.g, an aerospace vehicle.

inputs: control; disturbance: throttle, elevator, ailerons, rudder; wind

outputs: performance; measured

state

equilibrium

stability

feedback. feedback control, open loop control
manual control, automatic control

Chapter 2

State space representation of dynamical systems

A dynamical system is a system which evolves with time. Examples include aerospace vehicles, cars, motorcycles, chemical plants, electrical circuits, and structures. Dynamical systems are not restricted to engineering. For example, they occur in biology and economics. The evolution of a species is a dynamical system. So is the economy of a country.

Differential equations prove to be well suited to the mathematical modelling of dynamical systems. A representation of a dynamical system which consists of a bunch of first order ordinary differential equations is sometimes called a state space representation of the system. Let us look at some examples.

2.1 Linear examples

2.1.1 A first example

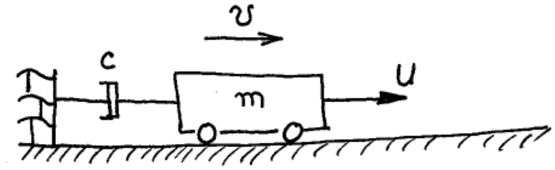


Figure 2.1: First example

Consider a small cart of mass m which is constrained to move along a horizontal line. The cart is subject to viscous friction with damping coefficient c; it is also subject to an input force which can be represented by a real scalar variable $u(t)$ where the real scalar variable t represents time. Let the real scalar variable $v(t)$ represent the speed of the cart at time t; we will regard this as the output of the cart. Applying Newton's second law in a horizontal direction, the motion of the cart can be described by the following first order ordinary differential equation (ODE):

$$m\dot{v}(t) = -cv(t) + u(t)$$

Introducing $x := v$ and rearranging the above equation results in

$$\begin{aligned} \dot{x} &= ax + bu \\ y &= x \end{aligned}$$

where $a := -c/m < 0$ and $b = 1/m$.

2.1.2 The unattached mass

Consider a small cart of mass m which is constrained to move without friction along a horizontal line. It is also subject to an input force which can be represented by a real scalar variable $u(t)$. Here we are interested in the horizontal displacement $q(t)$ of the cart from a fixed point on its line of motion; we will regard $y = q$ as the output of the cart.

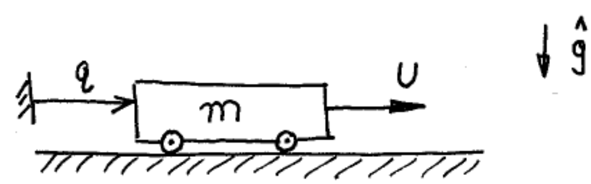


Figure 2.2: The unattached mass

Application of Newton's second law to the unattached mass illustrated in Figure 2.2 results in

$$m\ddot{q} = u$$

Introducing the state variables,

$$x_1 := q\,; \qquad x_2 := \dot{q}$$

yields the following state space description:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u/m \\ y &= x_1 \end{aligned}$$

2.1.3 Excited spring-mass-damper

Consider a system which can be modelled as a simple mechanical system consisting of a body of mass m attached to a base via a linear spring of spring constant k and linear dashpot with damping coefficient c. The base is subject to an acceleration u which we will regard as the input to the system. As output y, we will consider the force transmitted to the mass

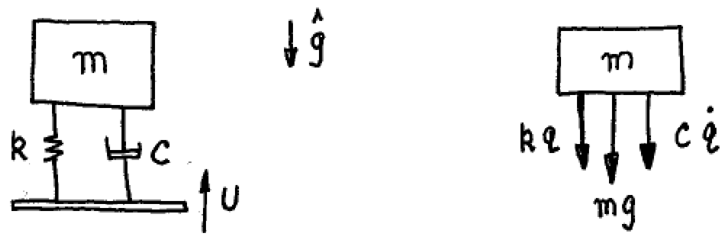


Figure 2.3: Spring-mass-damper with exciting base

from the spring-damper combination. Letting q be the deflection of the spring applying and Newton's second law in the vertical direction, the motion of the system can be described by

$$m(\ddot{q} + u) = -c\dot{q} - kq - mg$$

where g is the gravitational acceleration constant; also, $y = -kq - c\dot{q}$. Introducing $x_1 := q$, $x_2 := \dot{q}$ results in the following state space description:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(k/m)x_1 - (c/m)x_2 - u - g \\ y &= -kx_1 - cx_2 \end{aligned}$$

2.1.4 A simple structure

Consider a structure consisting of two floors. The scalar variables q_1 and q_2 represent the lateral displacement of the floors from their nominal positions. If we apply Newton's second

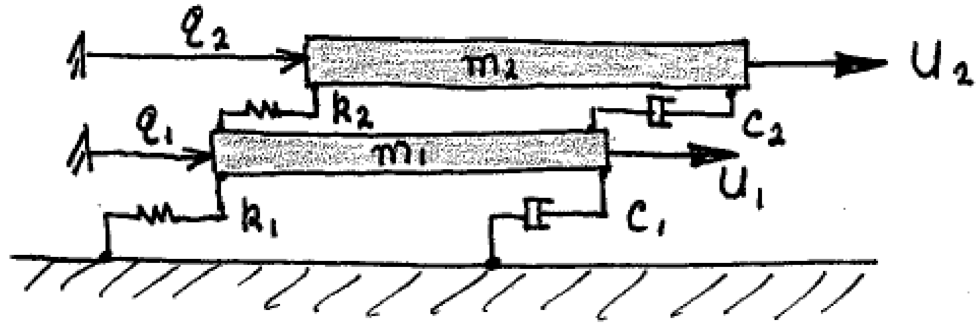


Figure 2.4: A simple structure

law in the horizontal direction to each floor, we obtain the following two coupled differential equations:

$$\begin{array}{ccccccccccc} m_1\ddot{q}_1 & + & (c_1+c_2)\dot{q}_1 & + & (k_1+k_2)q_1 & - & c_2\dot{q}_2 & - & k_2q_2 & = & u_1 \\ m_2\ddot{q}_2 & - & c_2\dot{q}_1 & - & k_2q_1 & + & c_2\dot{q}_2 & + & k_2q_2 & = & u_2 \end{array}$$

Here u_2 is a control input resulting from a force applied to the second floor and u_1 is a disturbance input resulting from a force applied to the first floor. We have not considered any outputs here. With

$$x_1 = q_1 \qquad x_2 = \dot{q}_1 \qquad x_3 = q_2 \qquad x_4 = \dot{q}_2\,,$$

we obtain

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k_1+k_2}{m_1}x_1 - \frac{c_1+c_2}{m_1}x_2 + \frac{k_2}{m_1}x_3 + \frac{c_2}{m_1}x_4 + \frac{1}{m_1}u_1 \\
\dot{x}_3 &= x_4 \\
\dot{x}_4 &= \frac{k_2}{m_2}x_1 + \frac{c_2}{m_2}x_2 - \frac{k_2}{m_2}x_3 - \frac{c_2}{m_2}x_4 + \frac{1}{m_2}u_2
\end{aligned}$$

Thus, four state variable were needed to describe this system.

We could also choose

$$x_1 = q_1 \qquad x_2 = q_2 \qquad x_3 = \dot{q}_1 \qquad x_4 = \dot{q}_2\,,$$

to obtain

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{k_1+k_2}{m_1}x_1 + \frac{k_2}{m_1}x_2 - \frac{c_1+c_2}{m_1}x_3 + \frac{c_2}{m_1}x_4 + \frac{1}{m_1}u_1 \\
\dot{x}_4 &= \frac{k_2}{m_2}x_1 - \frac{k_2}{m_2}x_2 + \frac{c_2}{m_2}x_3 - \frac{c_2}{m_2}x_4 + \frac{1}{m_2}u_2
\end{aligned}$$

2.2 Nonlinear examples

2.2.1 Planar solid pendulum

Here u is a torque applied to the pendulum and the output y is the angle the pendulum makes with a vertical line.

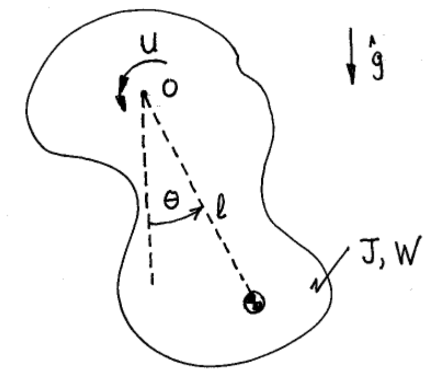


Figure 2.5: Planar solid pendulum

Looking at the moments about the support axis and equating that to rate of change of angular momentum about that axis results in

$$J\ddot{\theta} + Wl\sin\theta = u$$

Introducing

$$x_1 := \theta\ ; \qquad x_2 := \dot{\theta}$$

results in

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1 + b_2 u\\ y &= x_1\end{aligned}$$

where $a := Wl/J > 0$ and $b_2 = 1/J$. Due to the nonlinear term $\sin x_1$, this is considered a nonlinear system.

Discuss: equilibrium, stability, instability, restoring moment, non-restoring moment.

2.2.2 Body in central force motion

$$\begin{array}{ccccccl}\ddot{r} & - & r\omega^2 & + & g(r) & = 0\\ r\dot{\omega} & + & 2\dot{r}\omega & & & = 0\end{array}$$

Figure 2.6: Body in central force motion

For the simplest situation in orbit mechanics (a "satellite" orbiting YFHB)

$$g(r) = \mu/r^2 \qquad \mu = GM$$

where G is the universal constant of gravitation and M is the mass of YFHB. Let

$$x_1 = r, \quad x_2 = \dot{r}, \quad x_3 = \omega$$

2.2.3 The Lunicycle

The Lunicycle system illustrated in Figure 2.7 consists of a rider on top of a single wheel. The rider applies a torque u to the wheel. This torque is about the axis which is perpendicular to the wheel and passes through the wheel center O. It is clockwise when positive. We can describe the motion of this system with the coordinates p and θ where p is the horizontal displacement of the wheel center and θ is the angle between the rider and a vertical line.

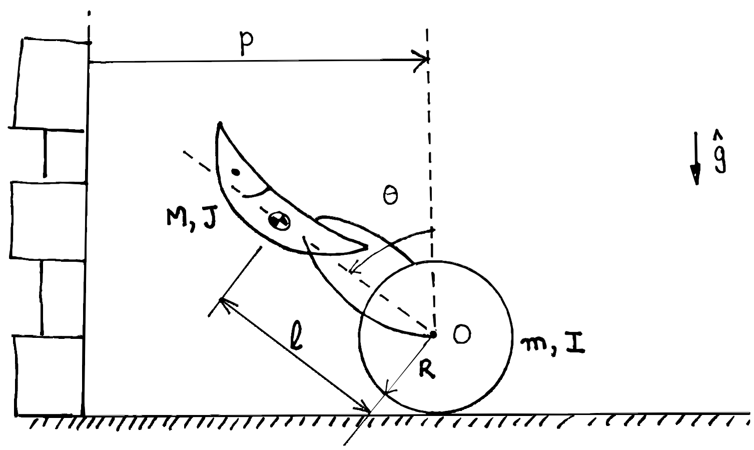


Figure 2.7: Lunicycle

Assuming that the vertical plane is a plane of symmetry for the wheel and rider, the motion of this system can be described by

$$\boxed{\begin{array}{rcrcrcl} (I/R^2+M+m)\,\ddot{p} & - & (Ml\cos\theta)\,\ddot{\theta} & + & Ml\sin\theta\dot{\theta}^2 & = & u/R \\ -(Ml\cos\theta)\,\ddot{p} & + & (J+Ml^2)\,\ddot{\theta} & - & Mgl\sin\theta & = & u \end{array}}$$

where m is the mass of the wheel,

M is the mass of the rider,

I is the moment of inertia of the wheel about the axis perpendicular to the wheel and through the wheel center,

J is the moment of inertia of the rider about a horizontal axis through its mass center,

l is distance from wheel center to mass center of rider,

R is radius of wheel, and

g is the gravitational acceleration constant.

2.3 General representation

All of the preceding systems can be described by a bunch of first order ordinary differential equations of the form

$$\boxed{\begin{array}{rcl} \dot{x}_1 & = & F_1(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\ \dot{x}_2 & = & F_2(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \\ & \vdots & \\ \dot{x}_n & = & F_n(x_1, x_2, \ldots, x_n, u_1, u_2, \ldots, u_m) \end{array}}$$

Such a description of a dynamical system is called a state space description; the real scalar variables, $x_i(t)$ are called the state variables; the real scalar variables, $u_i(t)$ are called the input variables and the real scalar variable t is called the time variable. If the system has outputs, they are described by

$$\boxed{\begin{array}{rcl} y_1 & = & H_1(x_1, \ldots, x_n,\ u_1, \ldots, u_m) \\ y_2 & = & H_2(x_1, \ldots, x_n,\ u_1, \ldots, u_m) \\ & \vdots & \\ y_p & = & H_p(x_1, \ldots, x_n,\ u_1, \ldots, u_m) \end{array}}$$

where the real scalar variables, $y_i(t)$ are called the output variables

When a system has no inputs or the inputs are fixed at some constant values, the system is described by

$$\boxed{\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, x_2, \ldots, x_n) \\ \dot{x}_2 & = & f_2(x_1, x_2, \ldots, x_n) \\ & \vdots & \\ \dot{x}_n & = & f_n(x_1, x_2, \ldots, x_n) \end{array}}$$

2.3.1 Higher order ODE descriptions

Here, we demonstrate a general procedure for converting a bunch of differential equations of any order into a bunch of first order differential equations.

Single equation. Consider a dynamical system described by a single n^{th}- order differential equation of the form

$$\boxed{F(q, \dot{q}, \ldots, q^{(n)}, u) = 0}$$

where $q(t)$ is a real scalar and $q^{(n)} := \frac{d^n q}{dt^n}$. To obtain an equivalent state space description, we proceed as follows.

• First solve for the highest order derivative $q^{(n)}$ of q as a function of $q, \dot{q}, \ldots, q^{(n-1)}$ and u to obtain something like:

$$q^{(n)} = a(q, \dot{q}, \ldots, q^{(n-1)}, u)$$

- Now introduce state variables,

$$\begin{aligned} x_1 &:= q \\ x_2 &:= \dot{q} \\ &\vdots \\ x_n &:= q^{(n-1)} \end{aligned}$$

- This yields the following state space description:

$$\boxed{\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= a(x_1, x_2, \ldots, x_n, u) \end{aligned}}$$

Example 1 Consider a system described by

$$8q\frac{d^2q}{dt^2} + 2\frac{d^3q}{dt^3} + 6\frac{dq}{dt} - 2q^3 = 4u\,.$$

Solving for $\frac{d^3q}{dt^3}$, the highest derivative of q, we obtain

$$\frac{d^3q}{dt^3} = -4q\frac{d^2q}{dt^2} - 3\frac{dq}{dt} + q^3 + 2u\,.$$

With state variables $x_1 := q$, $x_2 := \dot{q}$, and $x_3 := \ddot{q}$, we obtain the following state space representation:

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1^3 - 3x_2 - 4x_1x_3 + 2u\,. \end{aligned}$$

Multiple equations. (Recall the simple structure and the lunacycle.) Consider a dynamical system described by N scalar differential equations in N variables:

$$\begin{aligned} F_1(q_1, \dot{q}_1, \ldots, q_1^{(n_1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2)},\ \ldots,\ q_N, \dot{q}_N, \ldots,\ q_N^{(n_N)},\ u_1, u_2, \ldots, u_m\ ) &= 0 \\ F_2(q_1, \dot{q}_1, \ldots, q_1^{(n_1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2)},\ \ldots,\ q_N, \dot{q}_N, \ldots,\ q_N^{(n_N)},\ u_1, u_2, \ldots, u_m\ ) &= 0 \\ &\vdots \\ F_N(q_1, \dot{q}_1, \ldots, q_1^{(n_1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2)},\ \ldots,\ q_N, \dot{q}_N, \ldots,\ q_N^{(n_N)},\ u_1, u_2, \ldots, u_m\ ) &= 0 \end{aligned}$$

where $t, q_1(t), q_2(t), \ldots, q_N(t)$ are real scalars. Note that $q_i^{(n_i)}$ is the highest order derivative of q_i which appears in the above equations.

To obtain a first order state space description, we assume that one can uniquely solve for the highest order derivatives, $q_1^{(n_1)},\ q_2^{(n_2)},\ \ldots,\ q_N^{(n_N)}$, to obtain:

$$
\begin{aligned}
q_1^{(n_1)} &= a_1(\, q_1, \dot{q}_1, \ldots q_1^{(n_1-1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2-1)},\ \ldots,\ q_N, \dot{q}_N, \ldots, q_N^{(n_N-1)},\ u_1, u_2, \ldots, u_m\,) \\
q_2^{(n_2)} &= a_2(\, q_1, \dot{q}_1, \ldots q_1^{(n_1-1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2-1)},\ \ldots,\ q_N, \dot{q}_N, \ldots, q_N^{(n_N-1)},\ u_1, u_2, \ldots, u_m\,) \\
&\ \vdots \\
q_N^{(n_N)} &= a_N(\, q_1, \dot{q}_1, \ldots q_1^{(n_1-1)},\ q_2, \dot{q}_2, \ldots, q_2^{(n_2-1)},\ \ldots,\ q_N, \dot{q}_N, \ldots, q_N^{(n_N-1)},\ u_1, u_2, \ldots, u_m\,)
\end{aligned}
$$

Now let

$$
\begin{array}{llll}
x_1 := q_1 & x_2 := \dot{q}_1 & \ldots & x_{n_1} := q_1^{(n_1-1)} \\
x_{n_1+1} := q_2, & x_{n_1+2} := \dot{q}_2 & \ldots & x_{n_1+n_2} := q_2^{(n_2-1)} \\
 & & \vdots & \\
x_{n_1+\ldots+n_{N-1}+1} := q_N & x_{n_1+\ldots+n_{N-1}+2} := \dot{q}_N & \ldots & x_n := q_N^{(n_N-1)}
\end{array}
$$

where

$$n := n_1 + n_2 + \ldots + n_N$$

to obtain

$$
\boxed{
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= x_3 \\
&\ \vdots \\
\dot{x}_{n_1-1} &= x_{n_1} \\
\dot{x}_{n_1} &= a_1(x_1, x_2, \ldots, x_n,\ u_1, u_2, \ldots, u_m) \\
\dot{x}_{n_1+1} &= x_{n_1+2} \\
&\ \vdots \\
\dot{x}_{n_1+n_2} &= a_2(x_1, x_2, \ldots, x_n,\ u_1, u_2, \ldots, u_m) \\
&\ \vdots \\
\dot{x}_n &= a_N(x_1, x_2, \ldots, x_n,\ u_1, u_2, \ldots, u_m)
\end{aligned}
}
$$

Example 2

$$
\begin{aligned}
\ddot{q}_1 + \dot{q}_2 + 2q_1 &= 0 \\
-\ddot{q}_1 + \dot{q}_1 + \dot{q}_2 + 4q_2 &= 0
\end{aligned}
$$

The highest order derivatives of q_1 and q_2 appearing in these equations are $\ddot{q}_1$ and $\dot{q}_2$, respectively. Solving for $\ddot{q}_1$ and $\dot{q}_2$, we obtain

$$
\begin{aligned}
\ddot{q}_1 &= -q_1 + \frac{1}{2}\dot{q}_1 + 2q_2 \\
\dot{q}_2 &= -q_1 - \frac{1}{2}\dot{q}_1 - 2q_2\,.
\end{aligned}
$$

Introducing state variables $x_1 := q_1$, $x_2 := \dot{q}_1$, and $x_3 = q_2$ we obtain the following state space description:

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -x_1 + \frac{1}{2}x_2 + 2x_3 \\
\dot{x}_3 &= -x_1 - \frac{1}{2}x_2 - 2x_3
\end{aligned}
$$

Example 3 Consider the system described by

$$
\begin{aligned}
\dot{q}_1 + 2\ddot{q}_2 - q_2 &= 0 \\
3\ddot{q}_2 + \dot{q}_1 - \dot{q}_2 &= 0\,.
\end{aligned}
$$

The highest order derivatives of q_1 and q_2 appearing in these equations are $\dot{q}_1$ and $\ddot{q}_2$, respectively. Solving for these highest order derivatives, we obtain

$$
\begin{aligned}
\dot{q}_1 &= 3q_2 - 2\dot{q}_2 \\
\ddot{q}_2 &= -q_2 + \dot{q}_2\,.
\end{aligned}
$$

Introducing state variables $x_1 = q_1$, $x_2 = q_2$, and $x_3 = \dot{q}_2$, we obtain the following state space representation:

$$
\begin{aligned}
\dot{x}_1 &= 3x_2 - 2x_3 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= -x_2 + x_3\,.
\end{aligned}
$$

Exercises

Exercise 1 Obtain a state space representation of the following system.

$$\frac{d^3q}{dt^3} + 2\ddot{q} + 3\dot{q} + 4q = 0$$

Exercise 2 Obtain a state space representation of the following system.

$$\frac{d^4q}{dt^4} + \dot{q}\cos q + \cos\ddot{q} + 421 = 0$$

Exercise 3 Obtain a state space representation of the following system.

$$(1+q^3)\frac{d^3q}{dt^3} + qe^{\dot{q}} + \ddot{q} = 0$$

Exercise 4 By appropriate definition of state variables, obtain a first order state space description of the following systems.

(a)

$$\frac{d^3q}{dt^3} + 5\frac{d^2q}{dt^2} + q\frac{dq}{dt} = 0$$

(b)

$$\frac{d^3q}{dt^3} + q^2\frac{d^2q}{dt} + \frac{dq}{dt} + 5q^3 = 0$$

Exercise 5 Consider a system described by a single n^{th}-order linear differential equation of the form

$$q^{(n)} + a_{n-1}q^{(n-1)} + \ldots + a_1\dot{q} + a_0 q = 0$$

where $q(t)$ is a real scalar and $q^{(n)} := \frac{d^n q}{dt^n}$. By appropriate definition of state variables, obtain a first order state space description of this system.

Exercise 6 Obtain a state space representation of the following system.

$$\begin{aligned} \ddot{q}_1 + \dot{q}_2 + q_1^3 &= 0 \\ \ddot{q}_1 - \dot{q}_2 + q_2^3 &= 0 \end{aligned}$$

Exercise 7 Obtain state space representations of the systems described by

(a)

$$\begin{aligned} \ddot{q}_1 + \dot{q}_2 + 5q_1 &= 0 \\ \ddot{q}_2 + \dot{q}_1 + 6q_2 &= 0 \end{aligned}$$

(b)

$$\begin{aligned}\ddot{q}_1 + \ddot{q}_2 + 4q_2 &= 0\\ \ddot{q}_1 - \ddot{q}_2 + 2q_1 &= 0\end{aligned}$$

(c)

$$\begin{aligned}\ddot{q}_1 + \dot{q}_1 + q_1 q_2 &= 0\\ (1+q_2^2)\dot{q}_2 + \dot{q}_1 y_2 &= 0\end{aligned}$$

Exercise 8 Obtain a state space representation of the following system.

$$\begin{aligned}\frac{d^3 q_1}{dt^3} + \sin(\dot{q}_1 - \dot{q}_2) + q_1^3 &= 0\\ \frac{d^3 q_2}{dt^3} + \cos(q_2 - q_1) + \cos q_2 &= 0\end{aligned}$$

Exercise 9 Obtain a state space representation of the following system.

$$\begin{aligned}\ddot{q}_1 + \dot{q}_1 + q_2^3 &= 0\\ \ddot{q}_2 + \dot{q}_2 + q_1^3 &= 0\end{aligned}$$

Exercise 10 Obtain a state space representation of the following system.

$$\begin{aligned}\ddot{q}_1 + \ddot{q}_2 + q_1 - q_2 &= 0\\ \ddot{q}_1 - \ddot{q}_2 + q_1 + q_2 &= 0\end{aligned}$$

Exercise 11 Obtain a state space representation of the following system.

$$\begin{aligned}\ddot{q}_1 + \dot{q}_2 + q_1^3 &= 0\\ \ddot{q}_1 - \dot{q}_2 + q_2^3 &= 0\end{aligned}$$

2.4 Vectors

A scalar is a real or a complex number. In this section all the definitions and results are given for real scalars. However, they also hold for complex scalars; to get the results for complex scalars, simply replace ‘real’ with ‘complex’.

Consider any positive integer n, that is, $n = 1, 2, \ldots$. A real n-vector x is an *ordered* n*-tuple* of real numbers, $x_1, x_2, \ldots, x_n$. This is usually written as follows:

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{or} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{or} \qquad x = (x_1,\, x_2,\, \cdots,\, x_n)$$

The real numbers $x_1, x_2, \ldots, x_n$ are called the (scalar) components of x; x_i is called the i-th component.

Addition. The addition of any two n-vectors x and y yields another n-vector $x + y$ which is defined by:

$$x + y \; = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} + \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} := \begin{bmatrix} x_1 + y_1 \\ x_2 + y_2 \\ \vdots \\ x_n + y_n \end{bmatrix}.$$

Zero vector: The zero n-vector is the n-vector whose components are all zero, that is,

$$0 := \left.\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}\right\} n \text{ times}\,.$$

Note that we are using the same symbol, 0, for a zero scalar and a zero vector.

The negative of x:

$$-x := \begin{bmatrix} -x_1 \\ -x_2 \\ \vdots \\ -x_n \end{bmatrix}$$

Properties of addition

(a) (Commutative). For each pair x and y of n-vectors,

$$x + y = y + x\,.$$

(b) (Associative). For each triplet x, y, z of n-vectors,

$$(x + y) + z = x + (y + z)\,.$$

(c) There is an n-vector 0 such that for every n-vector x,

$$x + 0 = x\,.$$

(d) For each n-vector x there is an n-vector $-x$ such that

$$x + (-x) = 0\,.$$

Scalar multiplication. The multiplication of an n-vector x by a scalar α yields an n-vector αx which is defined by:

$$\alpha x \;=\; \alpha \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \;:=\; \begin{bmatrix} \alpha x_1 \\ \alpha x_2 \\ \vdots \\ \alpha x_n \end{bmatrix}\,.$$

Properties of scalar multiplication

(a) For each scalar α and pair x, y of n-vectors,

$$\alpha(x + y) = \alpha x + \alpha y\,.$$

(b) For each pair of scalars α, β and n-vector x,

$$(\alpha + \beta)x = \alpha x + \beta x\,.$$

(c) For each pair of scalars α, β, and n-vector x,

$$\alpha(\beta x) = (\alpha\beta)x\,.$$

(d) For each n-vector x,

$$1x = x\,.$$

Subtraction of a vector y from a vector x is defined by:

$$x - y := x + (-y)$$

Hence, for any two n-vectors x and y,

$$x - y \;=\; \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} - \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = \begin{bmatrix} x_1 - y_1 \\ x_2 - y_2 \\ \vdots \\ x_n - y_n \end{bmatrix}$$

2-vectors and pictures. Any 2-vector can be represented in a plane by a point or a directed line segment.

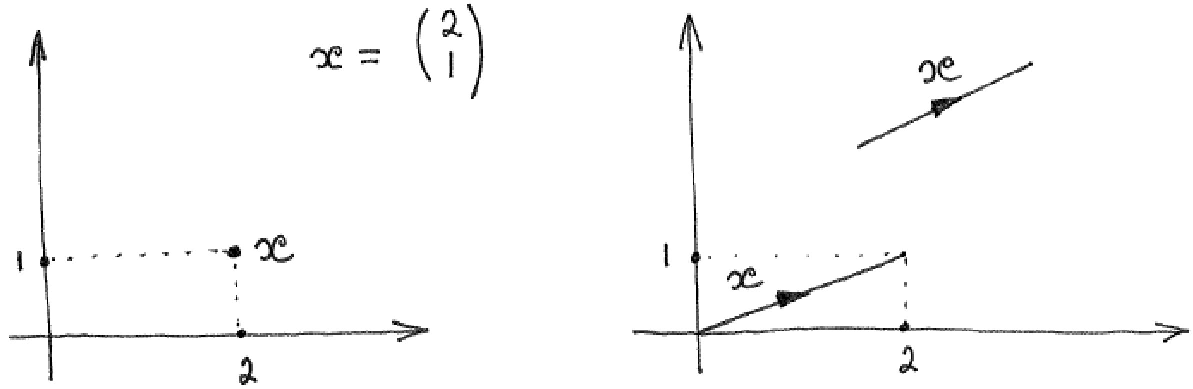


Figure 2.8: 2-vectors

Derivatives. Suppose $x(\cdot)$ is a function of a real variable t where $x(t)$ is an n-vector. Then

$$\dot{x} := \frac{dx}{dt} := \begin{bmatrix} \frac{dx_1}{dt} \\ \\ \frac{dx_2}{dt} \\ \\ \vdots \\ \\ \frac{dx_n}{dt} \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \\ \\ \vdots \\ \\ \dot{x}_n \end{bmatrix}$$

MATLAB. Representation of real and complex vectors
Addition and subtraction of vectors
Multiplication of a vector by a scalar

2.5 Vector representation of dynamical systems

Recall the general descriptions of dynamical systems given in Section 2.3. We define the state (vector) x as the vector with components, $x_1, x_2, \cdots, x_n$, that is,

$$x := \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

If the system has inputs, we define the input (vector) u as the vector with components, $u_1, u_2, \cdots, u_m$, that is,

$$u := \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}.$$

If the system has outputs, we define the output (vector) y as the vector with components, $y_1, y_2, \cdots, y_p$, that is,

$$y := \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}.$$

We introduce the vector valued functions F and H defined by

$$F(x,u) := \begin{bmatrix} F_1(x_1, x_2, \ldots, x_n, \; u_1, u_2, \cdots, u_m) \\ F_2(x_1, x_2, \ldots, x_n, \; u_1, u_2, \cdots, u_m) \\ \vdots \\ F_n(x_1, x_2, \ldots, x_n, \; u_1, u_2, \cdots, u_m) \end{bmatrix}$$

and

$$H(x,u) := \begin{bmatrix} H_1(x_1, x_2, \ldots, x_n, \; u_1, u_2, \cdots, u_m) \\ H_2(x_1, x_2, \ldots, x_n, \; u_1, u_2, \cdots, u_m) \\ \vdots \\ H_p(x_1, x_2, \ldots, x_n, \; u_1, u_2, \cdots, u_m) \end{bmatrix}$$

respectively.

The general representation of a continuous-time dynamical system can be compactly described by the following equations:

$$\boxed{\begin{array}{rcl} \dot{x} & = & F(x,u) \\ y & = & H(x,u) \end{array}} \tag{2.1}$$

where $x(t)$ is an n-vector, $u(t)$ is an m-vector, $y(t)$ is a p-vector and the real variable t is the time variable. The first equation above is a first order vector differential equation and is called the state equation. The second equation is called the output equation.

When a system has no inputs of the input vector vector is constant, it can be described by

$$\dot{x} = f(x)\,.$$

A system described by the above equations is called autonomous or time-invariant because the right-hand sides of the equations do not depend explicitly on time t. For the first part of the course, we will only concern ourselves with these systems.

However, one can have a system containing time-varying parameters. In this case the system might be described by

$$\begin{aligned} \dot{x} &= F(t, x, u) \\ y &= H(t, x, u) \end{aligned}$$

that is, the right-hand sides of the differential equation and/or the output equation depend explicitly on time. Such a system is called non-autonomous or time-varying. We will look at them later.

2.6 Solutions and equilibrium states

Consider a system described by

$$\dot{x} = f(x) \tag{2.2}$$

A solution of this system is any continuous function $x(\cdot)$ satisfying $\dot{x}(t) = f(x(t))$ for all t.

An equilibrium solution is the simplest type of solution; it is constant for all time, that is, it satisfies

$$x(t) \equiv x^e$$

for some fixed state vector x^e. The state x^e is called an equilibrium state.

Since an equilibrium solution must satisfy the above differential equation, all equilibrium states must satisfy the equilibrium condition:

$$\boxed{f(x^e) = 0}$$

or, in scalar terms,

$$\begin{aligned} f_1(x_1^e, x_2^e, \ldots, x_n^e) &= 0 \\ f_2(x_1^e, x_2^e, \ldots, x_n^e) &= 0 \\ &\vdots \\ f_n(x_1^e, x_2^e, \ldots, x_n^e) &= 0 \end{aligned}$$

Conversely, if a state x^e satisfies the above equilibrium condition, then there is a solution satisfying $x(t) \equiv x^e$; hence x^e is an equilibrium state.

Note that solving the equilibrium conditions involves solving n equations in n unknowns, $x_1^e, \cdots, x_n^e$.

Example 4 Spring mass damper. With $u = 0$ this system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(k/m)x_1 - (c/m)x_2 - g \end{aligned}$$

Hence equilibrium states are given by:

$$\begin{aligned} x_2^e &= 0 \\ -(k/m)x_1^e - (c/m)x_2^e - g &= 0 \end{aligned}$$

This results in a single equilibrium state given by:

$$x^e = 0 = \begin{bmatrix} 0 \\ -mg/k \end{bmatrix}$$

Example 5 The unattached mass. With $u = 0$ this system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{aligned}$$

Hence

$$x^e = \begin{bmatrix} x_1^e \\ 0 \end{bmatrix}$$

where x_1^e is arbitrary. Here we have an infinite number of equilibrium states.

Example 6 Planar pendulum. With $u = 0$,

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -a\sin x_1\end{aligned}$$

The the equilibrium condition yields:

$$x_2^e = 0 \qquad \text{and} \qquad \sin(x_1^e) = 0\,.$$

Hence, all equilibrium states are of the form

$$x^e = \begin{bmatrix} k\pi \\ 0 \end{bmatrix}$$

where k is an arbitrary integer. Physically,there are only two distinct equilibrium states

$$x^e = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \text{and} \qquad x^e = \begin{bmatrix} \pi \\ 0 \end{bmatrix}$$

Higher order ODEs

An equilibrium solution of the higher order ODE

$$F(q, \dot{q}, \ldots, q^{(n)}) = 0$$

is simply a constant solution, that is, it satisfies

$$q(t) = q^e \qquad \text{for all} \quad t$$

for some q^e which we call an equilibrium value of q. Clearly, q^e is given by

$$F(q^e, 0, \ldots, 0) = 0 \tag{2.3}$$

For the state space description of this system introduced earlier, all equilibrium states are given by

$$x^e = \begin{bmatrix} q^e \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

where q^e solves (2.3).

Multiple higher order ODEs

Equilibrium solutions

$$q_i(t) = q_i^e \qquad \text{for all } t \text{ and for } i = 1, 2, \ldots, m\,.$$

Hence

$$\begin{aligned}
F_1(q_1^e, 0, \ldots,\; q_2^e, 0, \ldots, \ldots,\; q_m^e, \ldots, 0\,) &= 0 \\
F_2(q_1^e, 0, \ldots,\; q_2^e, 0, \ldots, \ldots,\; q_m^e, \ldots, 0\,) &= 0 \\
&\vdots \\
F_m(q_1^e, 0, \ldots,\; q_2^e, 0, \ldots, \ldots,\; q_m^e, \ldots, 0\,) &= 0
\end{aligned} \tag{2.4}$$

For the state space description of this system introduced earlier, all equilibrium states are given by

$$x^e = \begin{bmatrix} q_1^e \\ 0 \\ \vdots \\ q_2^e \\ 0 \\ \vdots \\ \vdots \\ q_m^e \\ \vdots \\ 0 \end{bmatrix}$$

where $q_1^e, q_2^e, \ldots, q_m^e$ solve (2.4).

Example 7 Central force motion in inverse square gravitational field

$$\begin{array}{ccccl}
\ddot{r} & - & r\omega^2 & + & \mu/r^2 = 0 \\
r\dot{\omega} & + & 2\dot{r}\omega & & = 0
\end{array}$$

Equilibrium solutions

$$r(t) \equiv r^e\,, \qquad \omega(t) \equiv \omega^e$$

Hence,

$$\dot{r}, \ddot{r}, \dot{\omega} = 0$$

This yields

$$\begin{array}{ccl}
-r^e(\omega^e)^2 & + & \mu/(r^e)^2 = 0 \\
0 & & = 0
\end{array}$$

Thus there are infinite number of equilibrium solutions given by:

$$\omega^e = \pm\sqrt{\mu/(r^e)^3}$$

where r^e is arbitrary. Note that, for this state space description, an equilibrium state corresponds to a circular orbit.

2.6.1 Controlled equilibrium states and trim conditions

Consider now a system with inputs described by

$$\dot{x} = F(x, u) \tag{2.5}$$

Suppose the input is constant and equal to u^e, that is, $u(t) \equiv u^e$. Then the resulting system is described by

$$\dot{x}(t) = F(x(t), u^e)$$

The equilibrium states x^e of this system are given by $F(x^e, u^e) = 0$. This leads to the following definition.

A state x^e is a controlled equilibrium state *of the system, $\dot{x} = F(x, u)$, if there is a constant input u^e such that*

$$\boxed{F(x^e, u^e) = 0}$$

Consider now a system with output y given by $y = H(x, u)$. If x^e is a controlled equilibrium state corresponding to constant input u^e, then the corresponding constant output y^e is given by

$$\boxed{y^e = H(x^e, u^e)}$$

Sometimes, we refer to the triple (x^e, y^e, u^e) as a trim condition for the system.

Example 8 (*Controlled pendulum*)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin(x_1) + u \\ y &= x_1 \end{aligned}$$

Any state of the form

$$x^e = \begin{bmatrix} x_1^e \\ 0 \end{bmatrix}$$

is a controlled equilibrium state. The corresponding constant input is $u^e = \sin(x_1^e)$ and the corresponding output is $y^e = x_1^e$. Note that the system does not have any equilibrium states when $|u^e| > 1$.

Exercises

Exercise 12 Find all the equilibrium states (or state that none exist) for each of the following systems.

(a)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + 1 \end{aligned}$$

(b)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= 0\end{aligned}$$

(c)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= 1\end{aligned}$$

(d)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_1 - x_1^3\end{aligned}$$

Exercise 13 Obtain all the equilibrium states of the following system.

$$\begin{aligned}\dot{x}_1 &= x_1(1-x_2)\\ \dot{x}_2 &= x_2(1-x_1)\end{aligned}$$

2.7 Numerical simulation

2.7.1 MATLAB

```
>> help ode45

 ODE45  Solve non-stiff differential equations, medium order method.
    [T,Y] = ODE45('F',TSPAN,Y0) with TSPAN = [T0 TFINAL] integrates the
    system of differential equations y' = F(t,y) from time T0 to TFINAL with
    initial conditions Y0.  'F' is a string containing the name of an ODE
    file.  Function F(T,Y) must return a column vector.  Each row in
    solution array Y corresponds to a time returned in column vector T.  To
    obtain solutions at specific times T0, T1, ..., TFINAL (all increasing
    or all decreasing), use TSPAN = [T0 T1 ... TFINAL].

    [T,Y] = ODE45('F',TSPAN,Y0,OPTIONS) solves as above with default
    integration parameters replaced by values in OPTIONS, an argument
    created with the ODESET function.  See ODESET for details.  Commonly
    used options are scalar relative error tolerance 'RelTol' (1e-3 by
    default) and vector of absolute error tolerances 'AbsTol' (all
    components 1e-6 by default).

    [T,Y] = ODE45('F',TSPAN,Y0,OPTIONS,P1,P2,...) passes the additional
    parameters P1,P2,... to the ODE file as F(T,Y,FLAG,P1,P2,...) (see
    ODEFILE).  Use OPTIONS = [] as a place holder if no options are set.

    It is possible to specify TSPAN, Y0 and OPTIONS in the ODE file (see
    ODEFILE).  If TSPAN or Y0 is empty, then ODE45 calls the ODE file
    [TSPAN,Y0,OPTIONS] = F([],[],'init') to obtain any values not supplied
    in the ODE45 argument list.  Empty arguments at the end of the call list
    may be omitted, e.g. ODE45('F').

    As an example, the commands

        options = odeset('RelTol',1e-4,'AbsTol',[1e-4 1e-4 1e-5]);
        ode45('rigidode',[0 12],[0 1 1],options);

    solve the system y' = rigidode(t,y) with relative error tolerance 1e-4
    and absolute tolerances of 1e-4 for the first two components and 1e-5
    for the third.  When called with no output arguments, as in this
    example, ODE45 calls the default output function ODEPLOT to plot the
    solution as it is computed.

    [T,Y,TE,YE,IE] = ODE45('F',TSPAN,Y0,OPTIONS) with the Events property in
    OPTIONS set to 'on', solves as above while also locating zero crossings
```

```
    of an event function defined in the ODE file.  The ODE file must be
    coded so that F(T,Y,'events') returns appropriate information.  See
    ODEFILE for details.  Output TE is a column vector of times at which
    events occur, rows of YE are the corresponding solutions, and indices in
    vector IE specify which event occurred.

    See also ODEFILE and
        other ODE solvers:  ODE23, ODE113, ODE15S, ODE23S, ODE23T, ODE23TB
        options handling:   ODESET, ODEGET
        output functions:   ODEPLOT, ODEPHAS2, ODEPHAS3, ODEPRINT
        odefile examples:   ORBITODE, ORBT2ODE, RIGIDODE, VDPODE

>>
```

2.7.2 SIMULINK

2.8 Exercises

Chapter 3

Equations of motion

Before developing the equations of motion of an aircraft, we look at some reasonably simple systems which contain aerodynamic forces and obtain their equations of motion. To do this, we need to look at how aerodynamic forces are modelled.

3.1 Aerodynamic forces and trim

Consider a rigid body in motion relative to the air. The aerodynamic force system on the body is a complicated distributed force system which is distributed over the surface of the body.

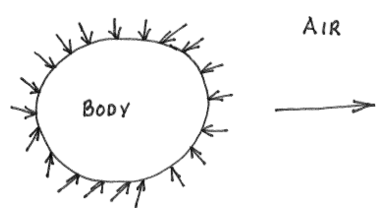


Figure 3.1: In the wind

However, if we choose any point in the body, the total aerodynamic force system on the body is equivalent to a single force $\bar{F}$ acting at this point and a couple of some moment $\bar{M}$; see Figure 3.2. The total aerodynamic force $\bar{F}$ is the resultant of the distributed aerodynamic force system and is independent of the chosen point. The total aerodynamic moment $\bar{M}$ depends on the point and is the moment resultant of the distributed aerodynamic force system about that point.

If the moment $\bar{M}$ is zero for some point C, then the original aerodynamic force system is equivalent to a single force $\bar{F}$ placed at that point and we will call that point a **center of pressure (cp)**; see Figure 3.3. Actually, the total aerodynamic moment about any point on the line of action of the force placed at C is zero. Thus, there is a whole line of points for which the total aerodynamic moment is zero. Thus the center of pressure is not unique. From basic mechanics, we know that a cp will exist provided $\bar{F}$ is nonzero and perpendicular to $\bar{M}$.

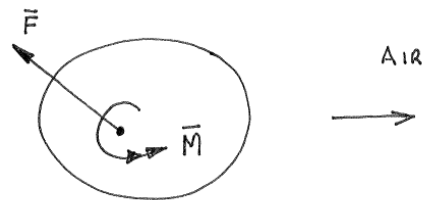


Figure 3.2: Equivalent aerodynamic force system

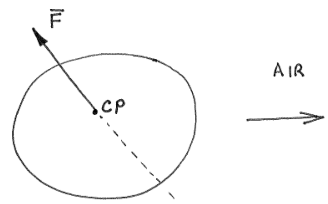


Figure 3.3: Center of pressure

Examples Figures 3.4, 3.5 and 3.6 illustrate some examples.

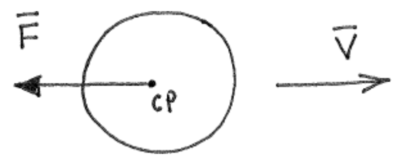


Figure 3.4: Sphere

Not that, for the flat plate, the center of pressure depends on the orientation of the plate relative to the velocity vector $\bar{V}$; there is no point which is a center of pressure for every orientation of the body relative to $\bar{V}$. This is usually the case for most bodies. Thus, we cannot use center of pressure as a reference point on a body.

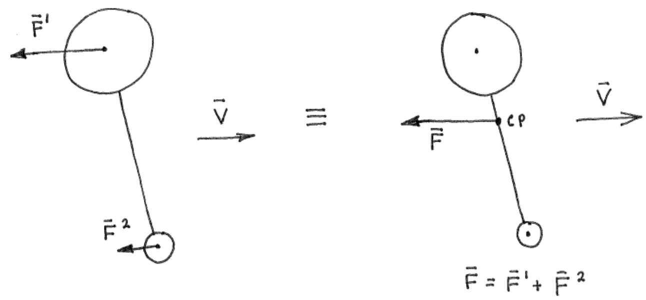


Figure 3.5: Two spheres

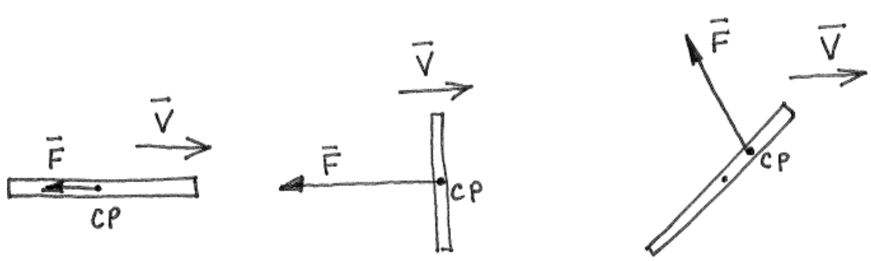


Figure 3.6: Flat plate

Power. Relative to a reference frame fixed in the air, the power of the aerodynamic forces on a rigid body may be expressed as

$$Power = \bar{F} \cdot \bar{V} + \bar{M} \cdot \bar{\omega}$$

where $\bar{V}$ is the velocity of a body-fixed moment reference point and $\bar{\omega}$ is the angular velocity of the body. So, if the moment reference point is a center of pressure, then

$$Power = \bar{F} \cdot \bar{V}$$

where $\bar{V}$ is the velocity of the center of pressure. Since the aerodynamic forces dissipate the energy of the body, the power is negative. So, we must have

$$\bar{F} \cdot \bar{V} < 0,$$

that is the angle between the total force $\bar{F}$ and $\bar{V}$ must be greater than 90 degrees; see Figure 3.7

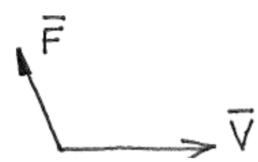


Figure 3.7: $\bar{F}$ and $\bar{V}$

Drag. The component of the force $\bar{F}$ that is parallel to the air velocity $\bar{V}$ is called the drag (force) and is denoted by $\bar{D}$. From the above considerations of power, we conclude that *the drag force must always opposite to* $\bar{V}$. Thus, for a body in translation, we have

$$Power = -DV$$

where D is the magnitude of the drag force and is simply called the drag. Thus, the power dissipated by the aerodynamic forces equals the product of the drag D and the airspeed V.

3.1.1 Uniform translation of a rigid body in air

Consider now a rigid body which is in *translation with constant velocity* $\bar{V}$ *relative to the "free air."* Thus, relative to the free air, every point in the body moves with velocity $\bar{V}$. We will refer to V, the magnitude of $\bar{V}$ as the airspeed.

For a given body, the aerodynamic force $\bar{F}$ and moment $\bar{M}$ depend on the air density ρ, the airspeed V and the orientation of the velocity $\bar{V}$ relative to the body. In addition, $\bar{M}$ depends on the point under consideration.

It is useful to introduce the dynamic pressure associated with the airflow. This is defined by

$$\boxed{\bar{q} = \frac{\rho V^2}{2}} \tag{3.1}$$

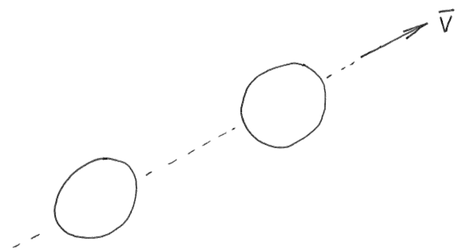


Figure 3.8: Rigid body in uniform translation

Note that this quantity has the dimensions of pressure. (Normally a symbol with an overhead bar indicates a vector quantity; this is an exception; we use the bar to distinguish this quantity from q, the pitch rate of an aircraft.)

For a fixed orientation of the velocity $\bar{V}$ relative to the body, $\bar{F}$ and $\bar{M}$ usually depend *linearly* on the dynamic pressure. So, we may express $\bar{F}$ and $\bar{M}$ as,

$$\bar{F} = \bar{q}S\bar{C}_F \qquad \text{and} \qquad \bar{M} = \bar{q}S\bar{c}\bar{C}_M \tag{3.2}$$

where S is some reference area associated with the body and $\bar{c}$ is some reference length associated with the body. The dimensionless vectors $\bar{C}_F$ and $\bar{C}_M$ depend only on the orientation of the velocity $\bar{V}$ relative to the body.

3.1.2 Some trim conditions

We regard a body to be trimmed or in a trim condition if is translating with a constant velocity relative to an inertial reference frame. So, the external forces on a trimmed body must satisfy the following necessary conditions.

$$\Sigma\bar{F} = 0 \qquad \text{and} \qquad \Sigma\bar{M} = 0$$

Free fall or gliding near flat earth. Here the body is only subject to weight and aerodynamic forces. Since the weight acts through the mass center (cm) of the body, it follows from the moment trim condition that, the total aerodynamic moment about the mass center must be zero, that is

$$\bar{M} = 0\,.$$

Hence *the mass center is a center of pressure*. From the force trim condition, the total aerodynamic force $\bar{F}$ is simply the opposite of the weight force, that is

$$\bar{F} = -\bar{W}\,;$$

see Figures 3.9 and 3.10.

Powered flight without thrust moment. Consider now a body subject to a thrust force $\bar{T}$ in addition to aerodynamic forces and weight. Consider first the situation in which the

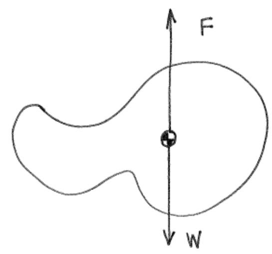


Figure 3.9: Free fall or gliding

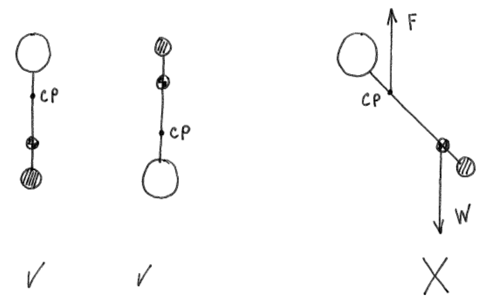


Figure 3.10: More free fall

line of action of the thrust vector $\bar{T}$ passes through the mass center of the body. Then, as in free fall, the aerodynamic moment about mass center must be zero, that is

$$\bar{M} = 0\,.$$

Hence, as in free fall, *the mass center is a center of pressure*; see Figure 3.11. Also,

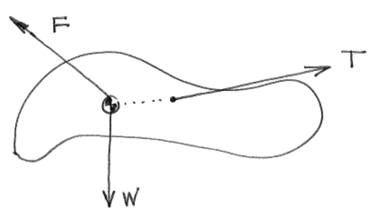


Figure 3.11: Powered flight without thrust moment.

$$\bar{F} + \bar{W} + \bar{T} = \bar{0}.$$

Powered flight with thrust moment. Consider now the situation in which the line of action of the thrust vector $\bar{T}$ does not pass through the mass center of the body. In this case, the thrust vector has a moment $\bar{M}_T$ about the mass center.

Since the external force system on the body is equivalent to the force system consisting of the three forces: weight $\bar{W}$ acting at mass center, thrust $\bar{T}$ and the total aerodynamic force $\bar{F}$ acting at a center of pressure, we have two possibilities:

1) All three forces are parallel and in same plane. See Figure 3.12.

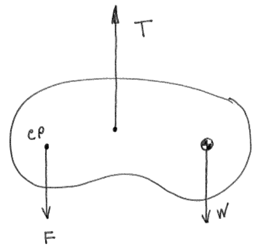
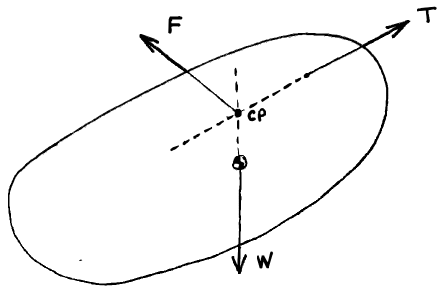


Figure 3.12: Powered flight with thrust moment: $M_T \neq 0$



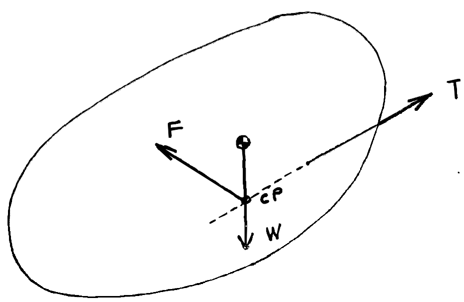


Figure 3.13: Powered flight with thrust moment: $M_T \neq 0$

2) All three forces intersect at a common point.
See Figure 3.13.

In either of the above cases, we also have

$$\bar{F} + \bar{W} + \bar{T} = \bar{0}.$$

3.2 Bodies in drag

In this section, we look at the motion of some simple bodies which are subject to an aerodynamic drag force $\bar{D}$. If $\bar{V}$ is the velocity of the body relative to the air, then the direction of the drag force $\bar{D}$ is exactly opposite to $\bar{V}$. Mathematically, we can express this as

$$\boxed{\bar{D} = -D\frac{\bar{V}}{V}}$$

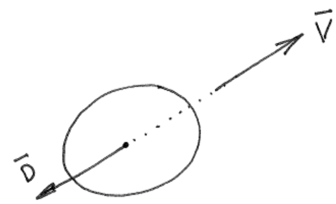


Figure 3.14: A body in drag

where D is the magnitude of the drag force. We simply refer to D as the **drag**. As a consequence of previous remarks, for a fixed orientation of the velocity $\bar{V}$ relative to the body, the drag D depends linearly on the dynamic pressure

$$\bar{q} = \frac{\rho V^2}{2}$$

where ρ is the air density. Usually D is represented by the following relationship

$$D = \bar{q} S C_D$$

where S is a reference area associated with the body. For example, for a road vehicle such as a car or a motorcycle, S is the frontal area of the vehicle. The dimensionless scalar C_D is called the **coefficient of drag**. For a fixed body, C_D only depends on the orientation of the velocity relative to the body. We can also express the above relationship as

$$\boxed{D = \kappa V^2}$$

where

$$\kappa = \rho S C_D / 2\,.$$

Thus, vectorially, drag can be represented by

$$\bar{D} = -\kappa V \bar{V} \qquad \text{where} \quad V = |\bar{V}|\,.$$

3.2.1 Simple oscillator in drag

Consider the simple vertical spring-mass system subject to an upwind of speed w; see Figure 3.15. The location of the mass can be described by q, the deflection of the spring from its free length. To obtain the drag on the mass, we have to consider the velocity $\bar{V}$ of the mass *relative to the air*; this is given by $\bar{V} = (\dot{q} + w)\hat{e}_3$ where $\hat{e}_3 = \hat{g}$ is a unit vector in the downward vertical direction. Thus the drag on the mass is given by

$$D = \kappa |w + \dot{q}|^2$$

where $\kappa = \frac{\rho S C_D}{2}$. Consider now the free-body-diagram (FBD) of the mass shown in Figure 3.16. Here we represent the drag force by the signed scalar $\tilde{D}$. Since the drag force is opposite to $\bar{V}$, the sign of $\tilde{D}$ is opposite to that of $\dot{q} + w$; thus it is given by

$$\tilde{D} = D\,\mathrm{sgm}\,(\dot{q} + w) = \kappa |w + \dot{q}|(w + \dot{q})$$

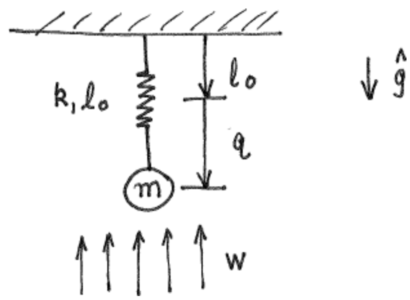


Figure 3.15: Oscillator in drag

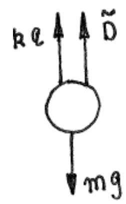


Figure 3.16: FBD of oscillator in drag

Application of Newton's second law in the downward vertical direction yields:

$$m\ddot{q} = -\tilde{D} - kq + mg$$

Hence, the motion of the system is described by the following nonlinear differential equation:

$$\boxed{m\ddot{q} + \kappa|w+\dot{q}|(w+\dot{q}) + kq - mg = 0}$$

Clearly, this is a nonlinear system.

To obtain equilibrium positions, we set $\dot{q}$ and $\ddot{q}$ to zero and $q = q^e$ to obtain

$$\kappa w^2 + kq^e - mg = 0\,..$$

Thus, there is a unique equilibrium position and it is given by

$$q^e = \frac{mg - \kappa w^2}{k}\,.$$

Note that q^e could be negative.

To obtain a state space description, let $x_1 = q$ and $x_2 = \dot{q}$ to obtain

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(k/m)x_1 - (\kappa/m)|x_2{+}w|(x_2{+}w) + g
\end{aligned}$$

This simple example illustrates that, when determining aerodynamic forces, one must consider the velocity of the body relative to the air and this velocity is not necessarily the same as the velocity of the body relative to the earth. If there is no wind (air is at rest relative to earth) then, the velocity of a body relative to the air is the same as the velocity of the body relative to the earth.

3.2.2 The cannonball: ballistics in drag

This example introduces some notation commonly used in considering the motion of aerospace vehicles. Consider a cannonball of mass m in flight in a vertical plane near the surface of the earth. Assume the undisturbed air is still, that is, no wind. We model the resultant aerodynamic force on the ball as a drag force of magnitude D acting opposite to the velocity $\bar{V}$ of the cannonball relative to the earth.

Choosing a reference point O fixed in the earth, the location of the cannonball can be described by the two scalars p and h which are the horizontal range and altitude, respectively, of the cannonball. We describe the velocity $\bar{V}$ of the cannonball relative to the earth by two scalars V and γ where V is the speed of the cannonball relative to the earth and the flight path angle γ is the angle between the velocity $\bar{V}$ and the positive horizontal direction; see Figure 3.17. We now show that the motion of the ball is governed by the following equations:

$$\boxed{\begin{array}{rcl} m\dot{V} &=& -mg\sin\gamma - D \\ V\dot{\gamma} &=& -g\cos\gamma \end{array}} \tag{3.3}$$

and

$$\boxed{\begin{array}{rcl} \dot{p} &=& V\cos\gamma \\ \dot{h} &=& V\sin\gamma \end{array}} \tag{3.4}$$

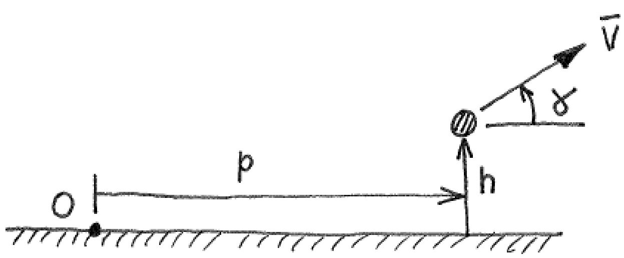


Figure 3.17: Cannonball

First introduce reference frame **e** fixed in the ground as illustrated in Figure 3.18. This is

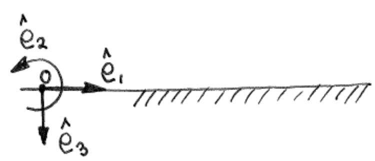


Figure 3.18: Ground frame

the usual earth fixed reference frame used in aircraft analysis; note that $\hat{e}_3$ is in the downward vertical direction and $\hat{e}_2$ is perpendicular to the plane of motion. Then the position of the ball can be described by

$$\bar{r} = p\,\hat{e}_1 - h\,\hat{e}_3\,.$$

Hence the velocity of the ball is given by

$$\bar{V} = \frac{{}^e d}{dt}(\bar{r}) = \frac{{}^e d}{dt}(p\,\hat{e}_1 - h\,\hat{e}_3) = \dot{p}\hat{e}_1 - \dot{h}\hat{e}_3\,.$$

We can also express the velocity as

$$\bar{V} = V\cos\gamma\,\hat{e}_1 - V\sin\gamma\,\hat{e}_3\,.$$

Comparing the above two expressions for $\bar{V}$ yields the kinematic relationships:

$$\begin{aligned}\dot{p} &= V\cos\gamma\\ \dot{h} &= V\sin\gamma\end{aligned}$$

Now we apply $\Sigma\bar{F} = m\bar{a}$ to the ball. As illustrated in the free body diagram (FBD), there are only two forces acting on the ball, namely, its weight force $\bar{W}$ and the drag force $\bar{D}$. Considering the reference frame $\mathbf{e}$ fixed in the earth as inertial, application of $\Sigma\bar{F} = m\bar{a}$

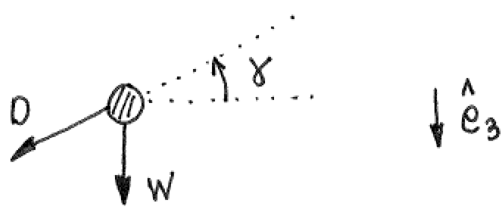


Figure 3.19: Free body diagram of the cannonball

to the ball yields

$$m\frac{{}^e d\bar{V}}{dt} = \bar{W} + \bar{D}\,. \tag{3.5}$$

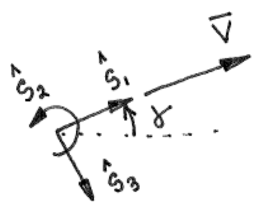


Figure 3.20: Reference frame $\mathbf{s}$

To obtain scalar components of the above vector equation, we introduce a reference frame which is defined by the motion of the ball: let

$$\mathbf{s} = (\hat{s}_1,\ \hat{s}_2,\ \hat{s}_3)\,,$$

where $\hat{s}_1$ is the unit vector in the direction of $\bar{V}$, $\hat{s}_3$ is the unit vector which is 90 degrees clockwise from $\hat{s}_1$, and $\hat{s}_2 = \hat{e}_2$. Then

$$\bar{V} = V\hat{s}_1 \qquad \text{and} \qquad {}^e\bar{\omega}^s = \dot{\gamma}\hat{s}_2\,,$$

where ${}^e\bar{\omega}^s$ is the angular velocity of reference frame $\mathbf{s}$ in $\mathbf{e}$. Application of the basic kinematic equation (BKE) between frames $\mathbf{s}$ and $\mathbf{e}$ yields

$$\frac{{}^e d\bar{V}}{dt} \quad = \quad \frac{{}^s d\bar{V}}{dt} + {}^e\bar{\omega}^s \times \bar{V} \quad = \quad \dot{V}\hat{s}_1 - V\dot{\gamma}\hat{s}_3$$

Resolving the forces into components relative to **s**, we have

$$\begin{aligned}\bar{D} &= \;= -D\,\hat{s}_1 \\ \bar{W} &= -mg\sin\gamma\,\hat{s}_1 + mg\cos\gamma\,\hat{s}_3\,.\end{aligned}$$

Taking components of $\Sigma\bar{F} = m\bar{a}$ relative to **s** yields:

$$\begin{aligned}\hat{s}_1 : \qquad m\dot{V} &= -mg\sin\gamma - D \\ \hat{s}_3 : \qquad -mV\dot{\gamma} &= mg\cos\gamma\end{aligned}$$

These equations result in (3.3). ■

For the above system, the equations of motion (EOMs) consisted of four first order differential equations. We could have obtained two second order differential equations in the coordinates p and h; however, they are not as "nice" as (3.3)-(3.4).

Trim Conditions. Considering $D = \kappa V^2$, we have

$$\begin{aligned}-mg\sin\gamma - \kappa V^2 &= 0 \\ mg\cos\gamma &= 0\end{aligned}$$

Hence

$$\gamma^e = -\pi/2 \qquad \text{and} \qquad V^e = \sqrt{mg/\kappa}$$

that is, the ball is falling vertically downward with speed $\sqrt{mg/\kappa}$.

3.2.3 A simple weathercock

The weathercock illustrated in Figure 3.21 consists of a ball which is attached to an earth-fixed point O via a slender rod. It is free to rotate in a *horizontal plane* about a vertical axis through O. It is subject to a wind of speed w which is coming from a fixed direction (as seen from earth.)

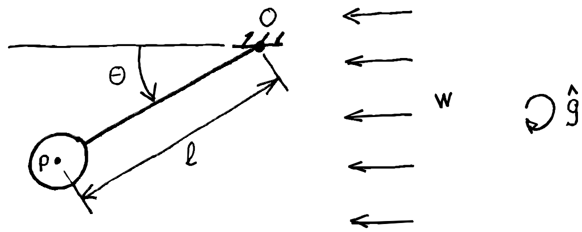


Figure 3.21: A simple weathercock (top view)

We let θ denote the angle between the rod and a line parallel to the wind; it is positive when counterclockwise as shown. We will show that the motion of the weathercock is governed by the following nonlinear second order differential equation:

$$\boxed{J\ddot{\theta} + \kappa l^2 V\dot{\theta} + \kappa l w V \sin\theta = 0} \tag{3.6}$$

with

$$V = \sqrt{w^2 + 2wl\sin\theta\dot{\theta} + l^2\dot{\theta}^2} \qquad \text{and} \qquad \kappa = \rho S C_D/2$$

where J is the moment of inertia of the weathercock about its axis of rotation, l is the distance from the center P of the ball to O, and w is the speed of the wind relative to the earth.

We will model the aerodynamic forces on the weathercock as a drag force $\bar{D}$ acting at the center of the ball. The drag is given by

$$D = \frac{1}{2}\rho S C_D V^2$$

where ρ is the air density, S is a reference area associated with ball and the dimensionless constant C_D is the coefficient of drag of the ball. The direction of the drag is opposite to $\bar{V}$, the velocity of the center P of the ball relative to the air.

Velocity relative to air. To figure out the magnitude and velocity of $\bar{V}$, the velocity of P relative to the air, we first introduce reference frames e and b fixed in the earth and in the weathercock, respectively; see Figure 3.22. Since ${}^{e}\bar{v}^{P} = {}^{e}\bar{v}^{air} + {}^{air}\bar{v}^{P}$ we see that

$$\bar{V} = {}^{air}\bar{v}^{P} = {}^{e}\bar{v}^{P} - {}^{e}\bar{v}^{air},$$

that is, the velocity of P relative to the air is the velocity of P relative relative to the earth minus the velocity of the air relative to the earth. Clearly, ${}^{e}\bar{v}^{air} = -w\hat{e}_1$; and from basic kinematics we have ${}^{e}\bar{v}^{P} = l\dot{\theta}\,\hat{b}_3$. Hence

$$\bar{V} = w\,\hat{e}_1 + l\dot{\theta}\,\hat{b}_3\,.$$

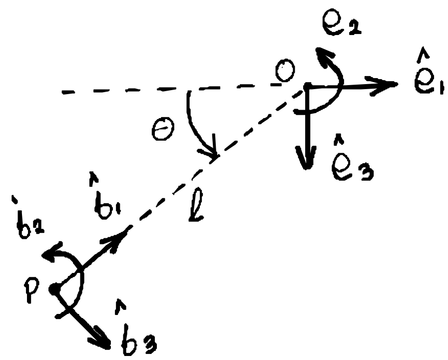


Figure 3.22: Reference frames

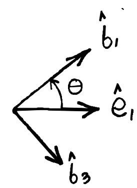


Figure 3.23: $\hat{e}_1 = C_\theta \hat{b}_1 + S_\theta \hat{b}_3$

Noting that $\hat{e}_1 = C_\theta \hat{b}_1 + S_\theta \hat{b}_3$, we obtain the following expression for $\bar{V}$ expressed in terms of the body frame:

$$\bar{V} = wC_\theta \hat{b}_1 + (wS_\theta + l\dot{\theta})\hat{b}_3 \,.$$

Hence,

$$V^2 = (wC_\theta)^2 + (wS_\theta + l\dot{\theta})^2 = w^2 + 2wlS_\theta\dot{\theta} + l^2\dot{\theta}^2$$

and, the speed of the ball relative to the air is given by

$$V = \sqrt{w^2 + 2wlS_\theta\dot{\theta} + l^2\dot{\theta}^2} \,.$$

To obtain information of the direction of $\bar{V}$, let α be the angle between $\bar{V}$ and $\hat{b}_1$ as shown in Figure 3.24. We call α the angle of attack of the weathercock.

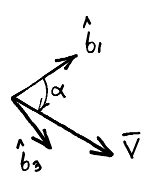


Figure 3.24: Angle of attack α

Then,

$$\bar{V} = VC_\alpha \hat{b}_1 + VS_\alpha \hat{b}_3 \,.$$

Using our previous expression for $\bar{V}$, we now obtain that

$$VC_\alpha = wC_\theta \qquad \text{and} \qquad VS_\alpha = wS_\theta + l\dot{\theta} \,.$$

Consider now the free body diagram of the weathercock. If we apply

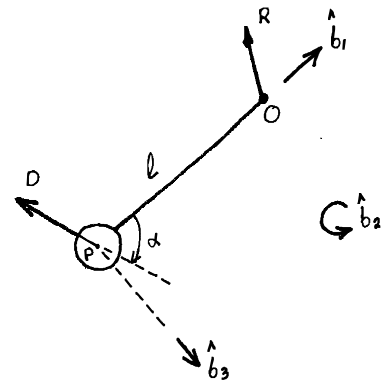


Figure 3.25: FBD (partial) of weathercock

$$\Sigma \bar{M}^O = \dot{\bar{H}}^O$$

with the earth as the inertial frame, then

$$\bar{H}^O = J\dot{\theta}\hat{b}_2$$

and we obtain

$$\hat{e}_2 : \qquad -DlS_\alpha = J\ddot{\theta}\,.$$

Recalling our expression for D, this yields

$$J\ddot{\theta} + \kappa l V^2 S_\alpha = 0$$

with $\kappa = \rho S C_D/2$. Substitute in our expression for VS_α to yield

$$J\ddot{\theta} + \kappa l^2 V\dot{\theta} + \kappa l w V S_\theta = 0\,.$$

3.3 Lift, drag and pitching moment

So far we have only considered situations in which the aerodynamic force system was modelled by drag. Here we look at lift and pitching moment for a body undergoing planar motion.

Consider a rigid body such as an aircraft which has a plane of symmetry; see Figure 3.26.

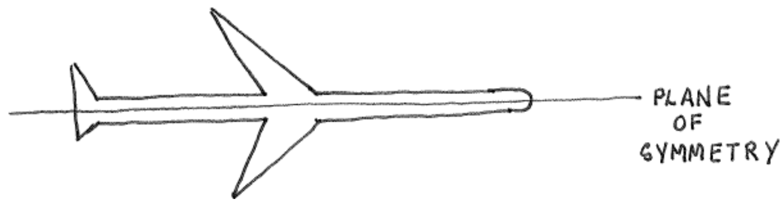


Figure 3.26: Body with symmetry plane

Suppose the body is in *uniform translation with constant velocity* $\bar{V}$ *relative to the "free air."* Thus, relative to the free air, every point in the body moves with velocity $\bar{V}$; see Figure 3.27. We will assume that the plane of symmetry of the body and $\bar{V}$ are parallel.

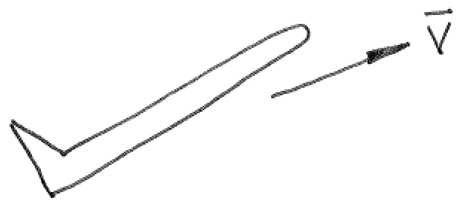


Figure 3.27: Body with symmetry plane in uniform translation

If we choose any reference point in the symmetry plane of the body, then the total aerodynamic force system on the body is equivalent to the total aerodynamic force $\bar{F}$ acting at this point and a couple of some moment $\bar{M}$. The single force lies in the symmetry plane of the body and is independent of the reference point. The moment vector $\bar{M}$ is perpendicular to the symmetry plane and depends on the reference point.

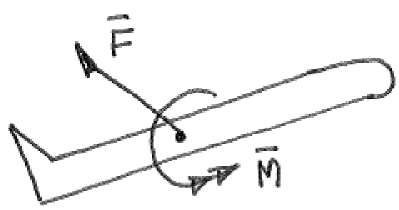


Figure 3.28: Equivalent aerodynamic force system

Since the original aerodynamic force is equivalent to a planar force system, then, provided $\bar{F}$ is not zero, there is a point about which the pitching moment is zero. We refer to such a point as a center of pressure (cp). So, the original force system is equivalent to a single force $\bar{F}$ placed at C. Actually, the pitching moment about any point on the line of action of the force placed at a center of pressure is zero. Thus, there is a whole line of points about which the pitching moment is zero. We refer to this line as the cp line.

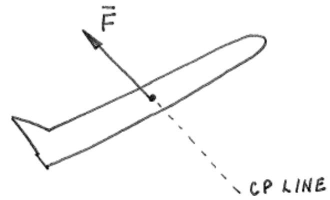


Figure 3.29: A single equivalent force and cp line

Lift and drag. The aerodynamic force $\bar{F}$ is commonly resolved into components perpendicular and parallel to the velocity vector $\bar{V}$. These components are called the lift force $\bar{L}$ and the drag force $\bar{D}$, respectively. The drag force is always in the direction opposite to the velocity $\bar{V}$ and its magnitude is denoted by the scalar D which is usually referred to as the drag. The lift force is represented by the scalar L which is usually referred to as the lift; the lift L is considered positive if the force is "upward" (that is, 90° clockwise from $\bar{V}$). For an aircraft, L is usually positive. However for a racecar like a formula one car, L is usually negative and is sometimes referred to as a downforce.

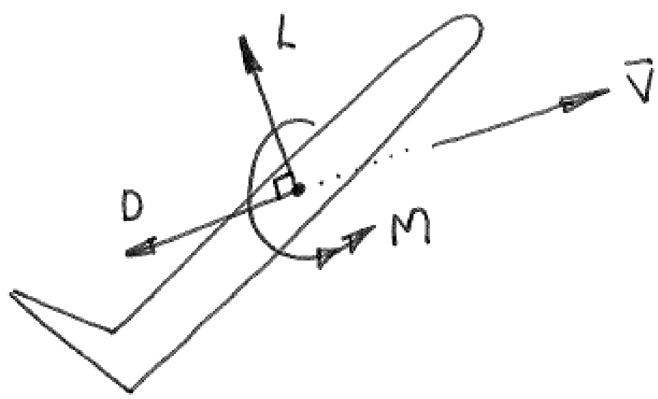


Figure 3.30: Lift, drag and pitching moment

Pitching moment. The moment vector $\bar{M}$ can be represented by a scalar M which is usually referred to as the pitching moment; a positive value of M corresponds to a "nose-up" pitching moment.

In general, L, D and M depend on the shape of the body, the air density, the magnitude of $\bar{V}$ and the orientation of $\bar{V}$ relative to the body. In addition, M depends on the point under consideration.

Angle of attack. To specify the orientation of the velocity $\bar{V}$ relative to the body, we first introduce a reference line fixed in the plane of symmetry of the body. We sometimes call this the zero-α line. The angle of attack α is the angle between the zero-α line and the velocity $\bar{V}$. It is considered positive as illustrated. We refer to V, the magnitude of $\bar{V}$, as the airspeed. When we refer to *the* cp of the body, we will be referring to the point (if any) on the zero-α line that is a cp.

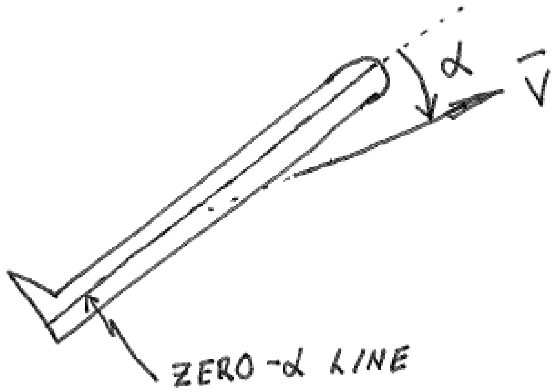


Figure 3.31: Angle of attack α

Body fixed aerodynamic components. We can also resolve the total aerodynamic force into components relative to a body fixed reference frame. To this end, we choose a reference frame

$$\mathbf{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$$

which is fixed in the body where

- $\hat{b}_1$ is along the zero-α line in the plane of symmetry of the body and pointing forward,
- $\hat{b}_2$ is perpendicular to the plane of symmetry of the body and pointing starboard,
- $\hat{b}_3$ is in the plane of symmetry of the body and pointing downward.

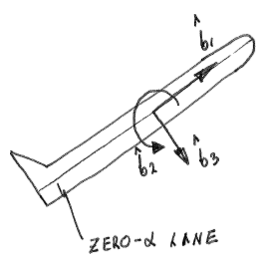


Figure 3.32: Body-fixed frame

If we let X and Z be the components of the total aerodynamic force $\bar{F}$ with respect to the body fixed 1-axis and 3-axis, respectively, then

$$\bar{F} = X\hat{b}_1 + Z\hat{b}_3 \qquad \text{and} \qquad \bar{M} = M\hat{b}_3\,. \tag{3.7}$$

To relate the above components to lift and drag, we note that

$$\bar{F} = \bar{D} + \bar{L}$$

and, resolving $\bar{L}$ and $\bar{D}$ into body fixed components (see Figure 3.34), we obtain:

$$\bar{D} = -D\cos\alpha\,\hat{b}_1 - D\sin\alpha\,\hat{b}_3 \qquad \text{and} \qquad \bar{L} = L\sin\alpha\,\hat{b}_1 - L\cos\alpha\,\hat{b}_3\,.$$

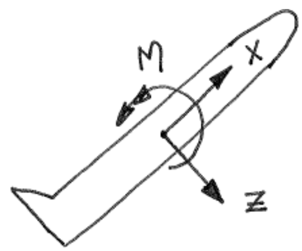


Figure 3.33: Body-fixed aerodynamic components

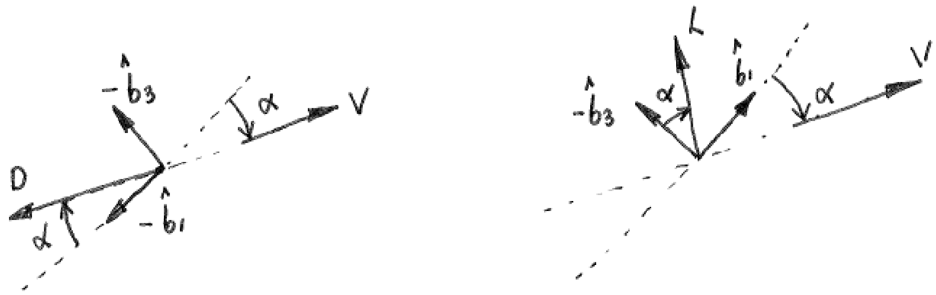


Figure 3.34: Lift, drag and body-fixed frame.

Hence

$$\boxed{X = L\sin\alpha - D\cos\alpha \qquad \text{and} \qquad Z = -L\cos\alpha - D\sin\alpha} \tag{3.8}$$

When α *is small* we obtain the following approximations:

$$X \approx -D \qquad \text{and} \qquad Z \approx -L\,. \tag{3.9}$$

3.4 Dimensionless aerodynamic coefficients

We now introduce the following dimensionless coefficients:

Lift coefficient: $C_L := \dfrac{L}{\bar{q}S}$

Drag coefficient: $C_D := \dfrac{D}{\bar{q}S}$

Pitching moment coefficient: $C_M := \dfrac{M}{\bar{q}S\bar{c}}$

where

$$\bar{q} = \frac{1}{2}\rho V^2$$

is called the dynamic pressure with ρ being the air density; S is some reference area and $\bar{c}$ is some reference length associated with the body. For *air at sea level*,

$$\boxed{\rho = 2.377 \times 10^{-3}\,\text{slugs/ft}^3}$$

Experimentally, for a given body (under static conditions) it seems that over a range of speeds V and air densities ρ, *the dimensionless aerodynamic coefficients depend only on the angle of attack* α. Rearranging the above equations, we obtain

$$\boxed{\begin{array}{rcr} L & = & \bar{q}SC_L \\ \\ D & = & \bar{q}SC_D \\ \\ M & = & \bar{q}S\bar{c}C_M \end{array}}$$

Recalling the body-fixed components X and Z of the aerodynamic force, we can also introduce the following dimensionless coefficients

$$C_X := \frac{X}{\bar{q}S} \qquad \text{and} \qquad C_Z := \frac{Z}{\bar{q}S} \tag{3.10}$$

Then

$$X = \bar{q}SC_X \qquad \text{and} \qquad Z = \bar{q}SC_Z \tag{3.11}$$

Also, from relationship (3.8), we have

$$\boxed{C_X = C_L \sin\alpha - C_D \cos\alpha \qquad \text{and} \qquad C_Z = -C_L \cos\alpha - C_D \sin\alpha} \tag{3.12}$$

3.4.1 Variation of dimensionless coefficients with α

Lift coefficient C_L. In the useful range of an airfoil (before stall), C_L is an increasing function of α. If we define

$$\boxed{C_{L_\alpha} := \frac{dC_L}{d\alpha}}$$

then, before stall, we have $C_{L_\alpha} > 0$. After stall, C_L decreases with increasing α. A useful (pre-stall) approximation for the dependence of C_L on α is the linear approximation given by

$$\boxed{C_L = C_{L_0} + C_{L_\alpha}\alpha} \tag{3.13}$$

where C_{L_0} and C_{L_α} are constants with $C_{L_\alpha} > 0$. We will also use the above relationship for a complete aircraft.

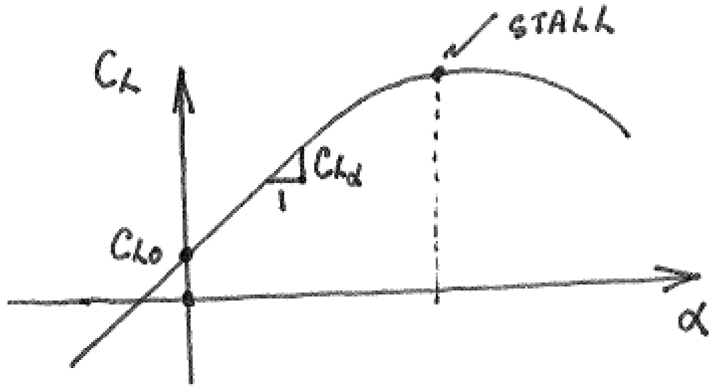


Figure 3.35: Lift coefficient vs α

Drag coefficient C_D. A useful relationship for modeling the drag coefficient is given by the drag polar relationship

$$\boxed{C_D = C_{DM} + k(C_L - C_{L_{DM}})^2} \tag{3.14}$$

where C_{DM} is the minimum value of C_D, $C_{L_{DM}}$ is the value of C_L at the minimum value of C_D and k is a positive constant.

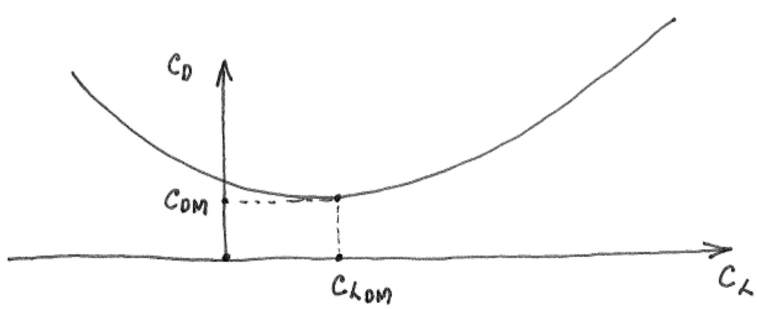


Figure 3.36: Drag coefficient vs lift coefficient

We will use the above relationships for any body with a plane of symmetry; in particular, we will use them for a complete aircraft.

Pitching moment coefficient C_M. Letting

$$\boxed{C_{M_\alpha} := \frac{dC_M}{d\alpha}}$$

a useful approximation is one in which C_{M_α} is constant. This yields the linear approximation given by

$$\boxed{C_M = C_{M_0} + C_{M_\alpha}\alpha} \tag{3.15}$$

where C_{M_0} and C_{M_α} are constants. The values and the signs of C_{M_0} and C_{M_α} depend on the location of the moment point on the body.

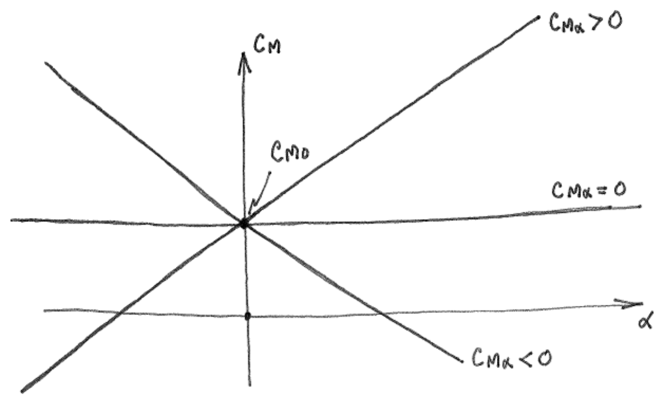


Figure 3.37: Pitching moment coefficient versus α

3.5 Variation of pitching moment over body

Here we investigate the manner in which the pitching moment M varies over a body. Consider the body whose plane of symmetry is illustrated in Figure 3.38. By definition, M is zero for any point on the cp-line. If the total aerodynamic force $\bar{F}$ points upward as illustrated, then, its moment about any point forward of the cp-line is clockwise and its moment about any point behind the cp-line is counterclockwise. Hence $M < 0$ for any point forward of the cp-line and $M > 0$ for any point behind the cp-line.

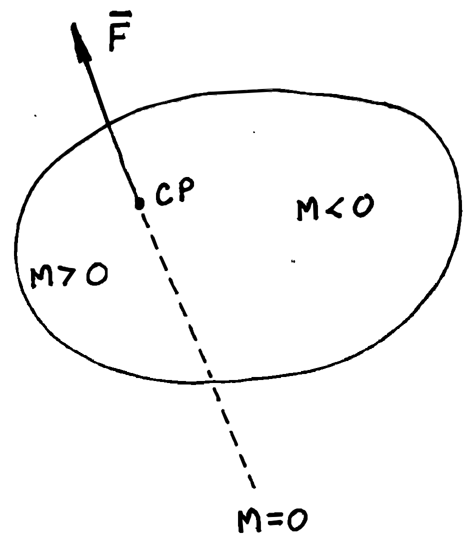


Figure 3.38: Variation of M over body

Thus, we have the following equivalences:

$$M > 0 \quad \Longleftrightarrow \quad (P \text{ is behind cp line})$$

$$M = 0 \quad \Longleftrightarrow \quad (P \text{ is on cp line})$$

$$M < 0 \quad \Longleftrightarrow \quad (P \text{ is forward of cp line})$$

or

$$C_M > 0 \quad \Longleftrightarrow \quad (P \text{ is behind cp line})$$

$$C_M = 0 \quad \Longleftrightarrow \quad (P \text{ is on cp line})$$

$$C_M < 0 \quad \Longleftrightarrow \quad (P \text{ is forward of cp line})$$

Consider any two points R and P in the plane of symmetry of the body. We will obtain a relationship between the pitching moment M at point P and the pitching moment M^R at point R. To this end, we choose a reference frame

$$\mathbf{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$$

which is fixed in the body where

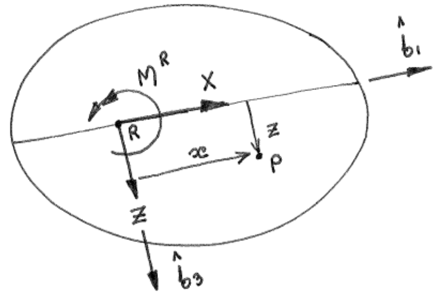


Figure 3.39:

$\hat{b}_1$ is along the zero-α line in the plane of symmetry of the body and pointing forward,
$\hat{b}_2$ is perpendicular to the plane of symmetry of the body and pointing starboard,
$\hat{b}_3$ is in the plane of symmetry of the body and pointing downward.

Then,

$$\bar{F} = X\hat{b}_1 + Z\hat{b}_3 \qquad \text{and} \qquad \bar{M}^R = M^R\hat{b}_2$$

where $\bar{M}^R$ is the pitching moment about R. Let x, z denote the coordinates of P relative to point R, that is, the position vector from R to P is

$$x\hat{b}_1 + z\hat{b}_3\,.$$

Then, the pitching moment $\bar{M}$ about point P is given by

$$\bar{M} = \bar{M}^R - (x\hat{b}_1 + z\hat{b}_3) \times (X\hat{b}_1 + Z\hat{b}_3)\,.$$

Carrying out the above cross product yields $\bar{M} = M\hat{b}_2$ where

$$M = M^R + Zx - Xz \tag{3.16}$$

Recalling the expressions for X and Z in terms of L and D, we obtain that

$$M = M^R - (L\cos\alpha + D\sin\alpha)x + (D\cos\alpha - L\sin\alpha)z\,.$$

Dividing the above expression by $\bar{q}S\bar{c}$ we obtain that C_M, the pitching moment coefficient at P is given by

$$C_M = C_M^R + C_Z\frac{x}{\bar{c}} - C_X\frac{z}{\bar{c}} \tag{3.17}$$

or

$$\boxed{C_M = C_{M^R} - (C_L\cos\alpha + C_D\sin\alpha)\tfrac{x}{\bar{c}} + (C_D\cos\alpha - C_L\sin\alpha)\tfrac{z}{\bar{c}}} \tag{3.18}$$

Computation of x^{cp}. At a center of pressure we have

$$M = 0 \qquad \text{or} \qquad C_M = 0\,.$$

Hence

$$M^R + Zx - Xz = 0.$$

The above equation describes a line and along this line $M = 0$. This is the cp line. We are usually interested in the center of pressure on the zero-α reference line. Let x^{cp} denote the x coordinate of this point (Figure 3.40). Assuming that the reference point R lies on the zero-α line, and noting that $z = 0$ for the cp on the zero-α line, we have

$$M^R + Zx^{cp} = 0 \tag{3.19}$$

Hence, provided $Z \neq 0$, the location of the cp on the reference line is given by

$$x^{cp} = \frac{M^R}{-Z} = \frac{C_{M^R}}{-C_Z}\,\bar{c}\,. \tag{3.20}$$

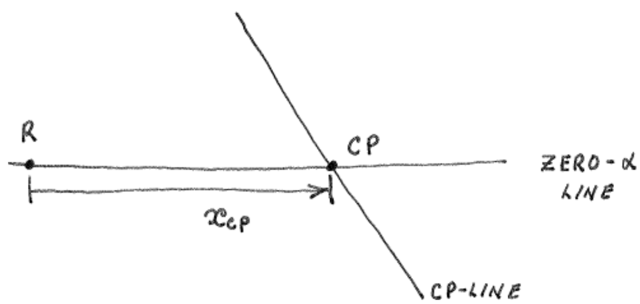


Figure 3.40: x^{cp} and cp line

Considering any point in the body, we now obtain that

$$M = M^R + Zx - Xz = M^R + Zx^{cp} + Z(x - x^{cp}) - Xz = Z(x - x^{cp}) - Xz,$$

that is,

$$M = (x - x^{cp})Z - Xz\,. \tag{3.21}$$

Assume that

$$Z < 0\,.$$

This is equivalent to saying that the total aerodynamic force points upward and is usually the case for trimmed flight. Then, as one proceeds forward along the reference line, the pitching moment becomes more negative, that is, more nose down. This agrees with our previous conclusions.

3.5.1 Aerodynamic center

We define an aerodynamic center (ac) to be any point in the plane of symmetry of the body for which the derivative of the pitching moment with respect to the angle of attack is zero, that is,

$$\frac{dM}{d\alpha} = 0\,. \tag{3.22}$$

Noting that

$$\frac{dM}{d\alpha} = \bar{q}S\bar{c}C_{M_\alpha},$$

an ac is also characterized by

$$C_{M_\alpha} = 0\,.$$

For an aircraft, this is usually called a neutral point (np).

Now let us examine how $\frac{dM}{d\alpha}$ varies over the plane of symmetry. Let M^{ac} be the pitching moment at an aerodynamic center. Then

$$\frac{dM^{ac}}{d\alpha} = 0\,.$$

Consider now any other point P in the plane of symmetry of the body. If M is the pitching moment at this point then, $M\hat{b}_3 = \bar{M}$ where

$$\bar{M} = \bar{M}^{ac} + \bar{r} \times \bar{F}.$$

In the above equation, $\bar{M}^{ac} = M^{ac}\hat{b}_3$ while $\bar{r}$ is the position vector from the point P to the ac and $\bar{F}$ is the total aerodynamic force. Differentiating the above equation with respect to α and noting that $\frac{{}^b d\bar{M}^{ac}}{d\alpha} = 0$, we obtain that

$$\frac{{}^b d\bar{M}}{d\alpha} = \bar{r} \times \frac{{}^b d\bar{F}}{d\alpha}\,.$$

Thus, if we place the vector $\frac{{}^b d\bar{F}}{d\alpha}$ at an aerodynamic center, then the vector $\frac{{}^b d\bar{M}}{d\alpha}$ is the moment of $\frac{{}^b d\bar{F}}{d\alpha}$ about point P. Since $\frac{{}^b d\bar{M}}{d\alpha} = \frac{dM}{d\alpha}\hat{b}_3$, it follows that $\frac{dM}{d\alpha}$ is zero for every point along the line of action of $\frac{{}^b d\bar{F}}{d\alpha}$. We call this line the ac-line. see Figure 3.41.

Now assume that $\frac{{}^b d\bar{F}}{d\alpha}$ is pointing upward as illustrated in Figure 3.41. This is the case for an aircraft under the usual trim conditions. In this case we can conclude that

$$\frac{dM}{d\alpha} > 0 \qquad \Longleftrightarrow \qquad (P \text{ is behind ac line})$$

$$\frac{dM}{d\alpha} = 0 \qquad \Longleftrightarrow \qquad (P \text{ is on ac line})$$

$$\frac{dM}{d\alpha} < 0 \qquad \Longleftrightarrow \qquad (P \text{ is forward of ac line})$$

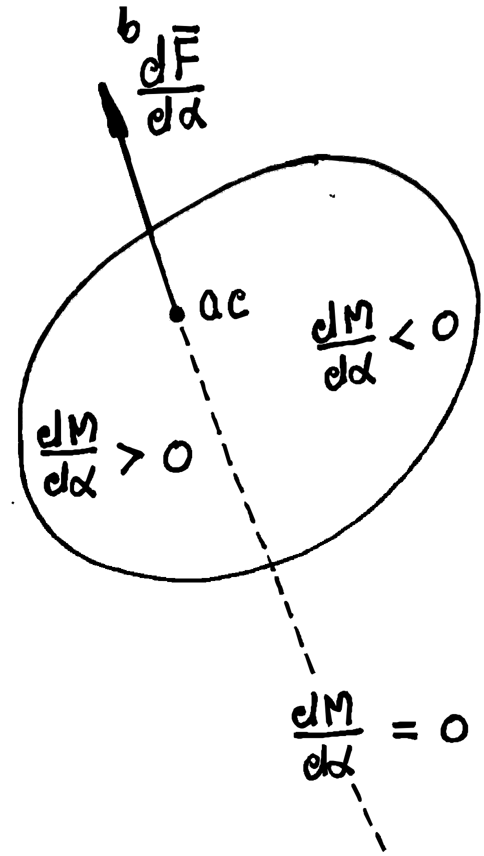


Figure 3.41: ac-line and variation of $\frac{dM}{d\alpha}$ over plane of symmetry

or

$$C_{M_\alpha} > 0 \quad \Longleftrightarrow \quad (P \text{ is behind ac line})$$

$$C_{M_\alpha} = 0 \quad \Longleftrightarrow \quad (P \text{ is on ac line})$$

$$C_{M_\alpha} < 0 \quad \Longleftrightarrow \quad (P \text{ is forward of ac line})$$

If we express $\bar{F}$ as

$$\bar{F} = X\hat{b}_1 + Z\hat{b}_3\,,$$

then

$$\frac{{}^b d\bar{F}}{d\alpha} = \frac{dX}{d\alpha}\hat{b}_1 + \frac{dZ}{d\alpha}\hat{b}_3$$

Hence, $\frac{{}^b d\bar{F}}{d\alpha}$ is pointing upward if and only if

$$\frac{dZ}{d\alpha} < 0\,.$$

This is usually the case for trimmed flight.

Computation of x^{ac}. As before consider any two points R and P in the plane of symmetry of the body. Differentiating the expression (3.16) with respect to α, we obtain

$$\frac{dM}{d\alpha} = \frac{dM^R}{d\alpha} + \frac{dZ}{d\alpha}x - \frac{dX}{d\alpha}z \tag{3.23}$$

Hence, the location of aerodynamic centers are given by

$$\frac{dM^R}{d\alpha} + \frac{dZ}{d\alpha}x - \frac{dX}{d\alpha}z = 0 \tag{3.24}$$

The above equation describes a line and along this line $dM/d\alpha = 0$. This is the ac line. We are interested in the location of the ac on the zero-α line, that is $z = 0$. *Let x^{ac} denote the x coordinate of the ac which is on the zero-α line* (3.42); then

$$\frac{dM^R}{d\alpha} + \frac{dZ}{d\alpha}x^{ac} = 0; \tag{3.25}$$

hence the location of this ac is given by

$$x^{ac} = -\frac{dM^R}{d\alpha} \Big/ \frac{dZ}{d\alpha} = -\frac{C_{M^R_\alpha}}{C_{Z_\alpha}}\bar{c}\,. \tag{3.26}$$

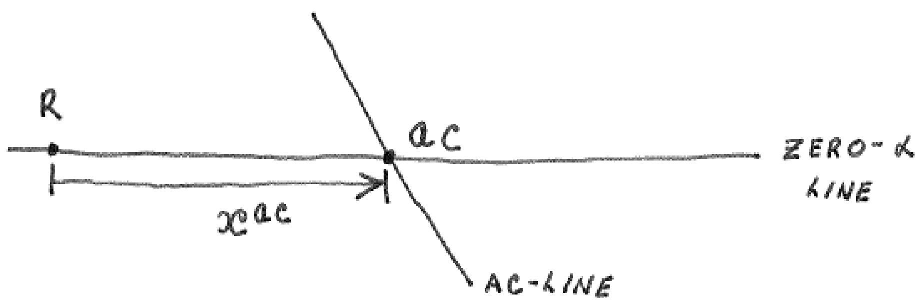


Figure 3.42: x^{ac} and ac line.

Considering any point P in the body, it now follows that

$$\begin{aligned}\frac{dM}{d\alpha} &= \frac{dM^R}{d\alpha} + \frac{dZ}{d\alpha}x - \frac{dX}{d\alpha}z \\ &= \frac{dM^R}{d\alpha} + \frac{dZ}{d\alpha}x^{ac} + \frac{dZ}{d\alpha}(x - x^{ac}) - \frac{dX}{d\alpha}z \\ &= \frac{dZ}{d\alpha}(x - x^{ac}) - \frac{dX}{d\alpha}z\end{aligned}$$

that is,

$$\frac{dM}{d\alpha} = \frac{dZ}{d\alpha}(x - x^{ac}) - \frac{dX}{d\alpha}z\,. \tag{3.27}$$

Motion of cp. Consider again the cp point on the reference line whose position is given by $x = x^{cp}$. We will investigate how this point changes with α. Differentiating equation (3.19) with respect to α yields

$$\frac{dM^R}{d\alpha} + \frac{dZ}{d\alpha}x^{cp} + Z\frac{dx^{cp}}{d\alpha} = 0$$

that is,

$$\frac{dM^{cp}}{d\alpha} + Z\frac{dx^{cp}}{d\alpha} = 0\,.$$

Hence, provided Z is nonzero,

$$\frac{dx^{cp}}{d\alpha} = -\frac{dM^{cp}}{d\alpha}/Z \tag{3.28}$$

So, assuming $Z < 0$, we can conclude that

$\frac{dM^{cp}}{d\alpha} > 0 \quad \left(C_{M^{cp}_\alpha} > 0\right) \quad \Longleftrightarrow$ (cp moves forward with increasing α.)

$\frac{dM^{cp}}{d\alpha} < 0 \quad \left(C_{M^{cp}_\alpha} < 0\right) \quad \Longleftrightarrow$ (cp moves backward with increasing α.)

3.5.2 Approximations

For small α and small z,

$$Z \approx -L \qquad \text{and} \qquad Zx - Xz \approx -Lx\,.$$

We now obtain the following approximate expressions for the pitching moment and pitching moment coefficient at P:

$$M = M^R - Lx \tag{3.29}$$

and

$$\boxed{C_M = C_{M^R} - C_L\frac{x}{\bar{c}}} \tag{3.30}$$

respectively; see Figure 3.43. The above relationships are useful if we are given aerodynamic data relative to a point R and, for analysis purposes, we need aerodynamic data relative to another point. So, for positive lift, the pitching moment decreases as we go forward along

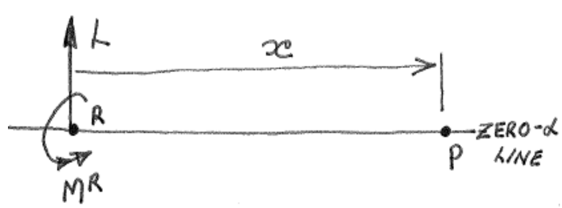


Figure 3.43: Pitching moment approximation

the reference line.

Letting $x = x^{cp}$ and $C_M = 0$, the location of the center of pressure (cp) along the zero-α line is given by

$$\boxed{x^{cp} = \frac{C_{M^R}}{C_L}\,\bar{c}} \tag{3.31}$$

In general, the location of this cp depends on α. So, for any other point

$$C_M = \frac{(x^{cp} - x)}{\bar{c}}\,C_L$$

Considering points close to the zero α line, we have the following approximate relationship:

$$\frac{dM}{d\alpha} = \frac{dM^R}{d\alpha} - x\frac{dL}{d\alpha}$$

or

$$\boxed{C_{M_\alpha} = C_{M^R_\alpha} - C_{L_\alpha}\frac{x}{\bar{c}}} \tag{3.32}$$

Letting $x = x^{ac}$ and $C_{M_\alpha} = 0$, the approximate location of the aerodynamic center on the reference line is given by

$$x^{ac} = \frac{dM^R}{d\alpha}\Big/\frac{dL}{d\alpha}\ .$$

or

$$\boxed{x^{ac} = \frac{C_{M^R_\alpha}}{C_{L_\alpha}}\,\bar{c}} \tag{3.33}$$

So, for any other point

$$C_{M_\alpha} = \frac{(x^{ac} - x)}{\bar{c}}\,C_{L\alpha} \tag{3.34}$$

3.6 Trim

3.6.1 Horizontal cruising aircraft

Let M_T be the moment of the thrust vector about the mass center of the aircraft. We consider two cases.

$M_T = 0$: In this case, the center of pressure must be at the mass center. Also,

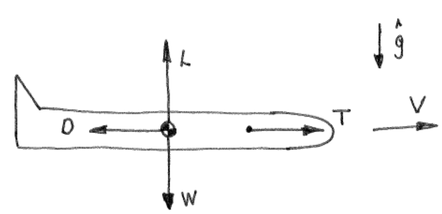


Figure 3.44: Horizontal cruising: no thrust moment

$$L \approx W \qquad \text{and} \qquad T \approx D$$

The reason that approximations are used in the above expressions is that the thrust vector may not be horizontal.

$M_T \neq 0$

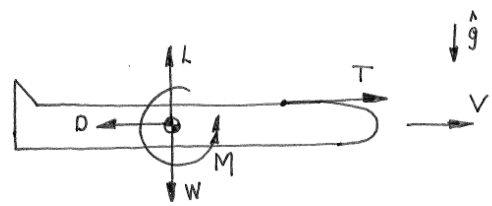


Figure 3.45: Horizontal cruising: with thrust moment

$$M + M_T = 0 \qquad \text{and} \qquad L \approx W \qquad \text{and} \qquad T \approx D$$

Gliding or free fall. The center of pressure must be at the mass center.

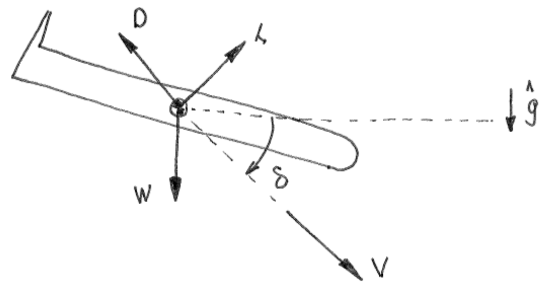


Figure 3.46: Gliding

Since the three forces $\bar{W}, \bar{L}$ and $\bar{D}$ must be in equilibrium, we must have $\tan\delta = D/L$; see Figure 3.47. Hence

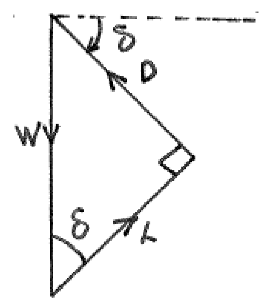


Figure 3.47: Force equilibrium for gliding

$$\tan\delta = \frac{C_D}{C_L}$$

From this we see that the equilibrium δ depends only on the equilibrium angle of attack and is independent of speed.

3.7 Longitudinal static stability

Consider an aircraft or any body in trimmed longitudinal flight, that is, relative to the air, the aircraft is translating parallel to its plane of symmetry with constant velocity. We say that this trim condition has the property of longitudinal static stability (LSS) if

$$\frac{dM}{d\alpha} < 0$$

or, equivalently,

$$\boxed{C_{M_\alpha} < 0}$$

where M is the pitching moment about the mass center (cm)of the body. Sometimes the quantity $-\frac{dM}{d\alpha}$ is called the pitch stiffness.

3.7.1 Static margin

From previous discussions, LSS means that, under usual flight conditions, the cm should be forward of the ac line. Recall that for any point close to to the zero-α line, we have

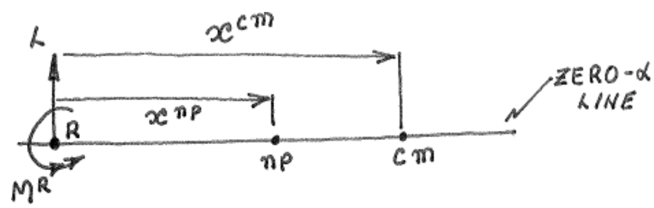


Figure 3.48: x^{cm} and x^{np}

$$C_M = C_{M^R} - C_L \frac{x}{\bar{c}}$$

where x is the displacement along the zero-α line of the point from the reference point R. Hence

$$C_{M_\alpha} = C_{M^R_\alpha} - C_{L_\alpha} \frac{x}{\bar{c}}$$

Considering the neutral point (aerodynamic center) we have $x = x^{np}$ and $C_{M_\alpha} = 0$; hence

$$0 = C_{M^R_\alpha} - C_{L_\alpha} \frac{x^{np}}{\bar{c}}$$

which results in

$$C_{M^R_\alpha} = C_{L_\alpha} \frac{x^{np}}{\bar{c}}$$

Considering the center of mass, we have $x = x^{cm}$ and

$$C_{M_\alpha} = C_{M^R_\alpha} - C_{L_\alpha} \frac{x^{cm}}{\bar{c}}$$

Substituting the above expression for $C_{M^R_\alpha}$ results in the following expression for C_{M_α} at the c.m.

$$C_{M_\alpha} = -\frac{(x^{cm} - x^{np})}{\bar{c}} C_{L\alpha} \tag{3.35}$$

where x^{cm} and x^{np} are the respective displacements displacement along the $\hat{b}_1$-axis of the mass center and neutral point (aerodynamic center) relative to the reference point on the $\hat{b}_1$-axis.

Recall that, under usual flight conditions, $C_{L_\alpha} > 0$. Hence, if the mass center is forward of the neutral point, that is, $x^{cm} - x^{cp}$ is positive, then C_{M_α} is negative and the pitching moment is a restoring moment. If the mass center is behind the neutral point, that is, $x^{cm} - x^{np}$ is negative, then C_{M_α} is positive and the pitching moment is a "destabilizing moment". As a result of this, we introduce the static margin defined by

$$\boxed{\text{static margin} = \frac{x^{cm} - x^{np}}{\bar{c}}}$$

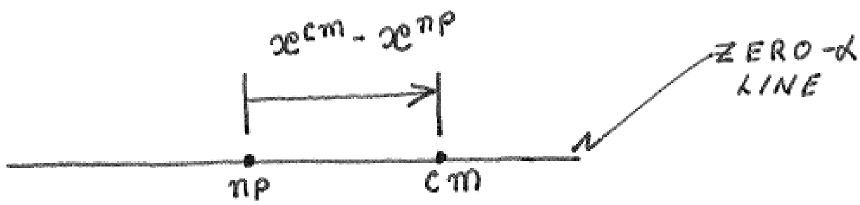


Figure 3.49: Static margin

cm is forward of np	positive static margin	$C_{M_\alpha} < 0$	restoring moment
cm is behind np	negative static margin	$C_{M_\alpha} > 0$	"destabilizing moment"

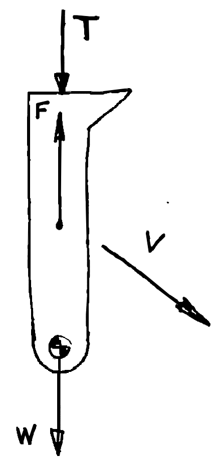


Figure 3.50: Trim for a very “nose-heavy” aircraft

3.7.2 The stabilizing effects of horizontal tail

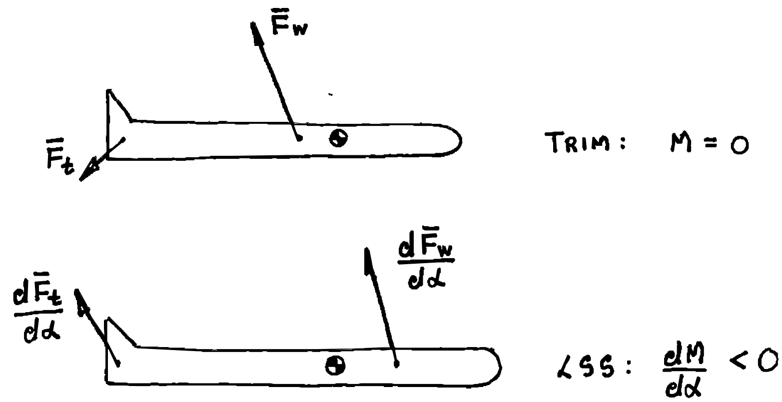


Figure 3.51: Stabilizing effects of tail

The tail will move the overall neutral point of the aircraft rearwards. If the "lift" at the tail is negative, it will move the overall center of pressure of the aircraft forwards; hence the center of mass can be further forward.

3.8 Elevator

Consider an aircraft in longitudinal flight. Suppose that the thrust moment about the cm is zero. Then under trim conditions, the aerodynamic pitching moment M about the cm must be zero and, hence,

$$C_M = 0\,,$$

where C_M is the pitching moment coefficient about the cm. Recall that, for a fixed vehicle configuration, C_M depends only on angle of attack α. Hence, for a fixed vehicle configuration, the trim value of the angle of attack is fixed. Note that the above condition is independent of the speed of the aircraft. To be able to vary this trim value of angle of attack, we need to change the shape of the vehicle. Note that, if an aircraft is flying horizontally, changing the angle of attack is equivalent to changing the pitch of the aircraft. Usually, in horizontal cruise, we also want the the aircraft to be horizontal. One way to vary the trim angle of attack is to use an elevator.

An elevator is simply a movable part of the horizontal tail; it can be made to rotate about an axis parallel to the pitch axis of the aircraft. We let el be the angle of the elevator(s) relative to the aircraft where el is considered positive as shown, that is, with the trailing edge down; see Figure 3.52.

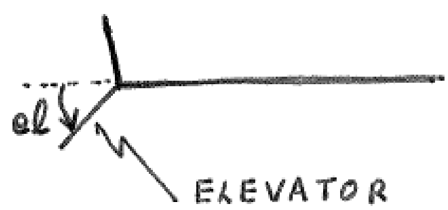


Figure 3.52: Elevator and el

Define

$$\boxed{C_{M_{el}} = \frac{\partial C_M}{\partial el}}$$

this is an example of a control derivative. Since the main effect of a positive elevator deflection is to decrease the overall pitching moment, we must have

$$\boxed{C_{M_{el}} < 0}$$

For a given elevator deflection el, the trim value of α is given by

$$C_M(\alpha, el) = 0\,.$$

To understand how the trim value of α changes with el, we differentiate the above expression with respect to el to obtain

$$\frac{\partial C_M}{\partial \alpha}\frac{\partial \alpha}{\partial el} + \frac{\partial C_M}{\partial el} = 0\,,$$

that is,

$$C_{M_\alpha}\frac{\partial \alpha}{\partial el} + C_{M_{el}} = 0\,.$$

This yields:

$$\frac{\partial \alpha}{\partial el} = -\frac{C_{M_{el}}}{C_{M_\alpha}}\,. \tag{3.36}$$

If the vehicle is LSS, then $C_{M_\alpha} < 0$; hence

$$\frac{\partial \alpha}{\partial el} < 0\,,$$

that is, *the trim value of α decreases with positive elevator deflection.* We also see that if we make C_{M_α} more negative, we reduce the effectiveness of the elevator and the vehicle is less maneuverable.

If we consider C_{M_α} and $C_{M_{el}}$ to be constant, then we obtain the following linear expression for C_M.

$$\boxed{C_M(\alpha, el) = C_{M_0} + C_{M_\alpha}\alpha + C_{M_{el}}el} \tag{3.37}$$

where C_{M_0} is a constant.

In general, the lift coefficient C_L increases with C_L, that is $C_{L_{el}} > 0$, where

$$\boxed{C_{L_{el}} = \frac{\partial C_L}{\partial el}\,.}$$

If we consider C_{L_α} and $C_{L_{el}}$ to be constant, then, we obtain the following linear relationship:

$$\boxed{C_L(\alpha, el) = C_{L_0} + C_{L_\alpha}\alpha + C_{L_{el}}el} \tag{3.38}$$

where C_{L_0} is a constant.

If we model C_D with the drag polar relationship

$$\boxed{C_D = C_{DM} + k(C_L - C_{L_{DM}})^2} \tag{3.39}$$

then C_D depends on el as a result of the dependency of C_L on el.

3.9 A general weathercock

Recall first the simple weathercock from Section 3.2.3. Its motion is governed by

$$J\ddot{\theta} = M$$

where the aerodynamic moment M is given by

$$M = -\kappa l w V \sin\theta - \kappa l^2 V \dot{\theta}$$

with

$$V = \sqrt{w^2 + 2wl\sin\theta\dot{\theta} + l^2\dot{\theta}^2} \qquad \text{and} \qquad \kappa = \rho S C_D/2\,.$$

Notice that M *depends on* $\dot{\theta}$ *in addition to* θ.

When $\dot{\theta}$ is small, we have the following approximations:

$$V \approx w$$

and the moment M can be approximated by

$$M = -\kappa l V^2 \sin\theta - \kappa l^2 V \dot{\theta}\,.$$

Recall that the dynamic pressure is given by

$$\bar{q} = \frac{1}{2}\rho V^2\,;$$

hence

$$\kappa l V^2 = \bar{q} S l C_D \qquad \text{and} \qquad \kappa l^2 V = \frac{\bar{q} S l^2 C_D}{V}$$

If we introduce the moment coefficient,

$$C_M = \frac{M}{\bar{q} S l}$$

then,

$$C_M = -C_D \sin\theta - \frac{l}{2V}(2C_D)\,\dot{\theta} \qquad \text{and} \qquad M = \bar{q} S l C_M\,.$$

Thus, we can express C_M as

$$C_M = C_M(\theta) + \frac{l}{2V} C_{M_{\dot{\theta}}}\dot{\theta} \tag{3.40}$$

where

$$C_M(\theta) = -C_D \sin\theta \qquad \text{and} \qquad C_{M_{\dot{\theta}}} = -2C_D\,.$$

Note the form of the dependency of C_M on $\dot{\theta}$.

Imagine now a general weathercock or some rigid body which is constrained to pivot or rotate about a fixed vertical axis and is subject to a wind of speed V. The pivot axis is perpendicular to the plane of symmetry of the weathercock and passes through this plane

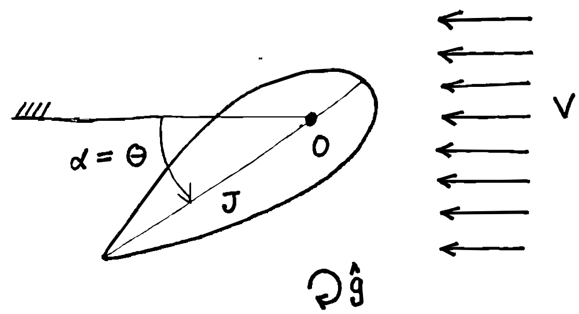


Figure 3.53: A weathercock

at point O. We can describe the orientation of the weathercock with the angle θ. Note that θ is also the angle of attack α of body relative to the wind.

The rotational motion of the body about the pivot axis is determined by the aerodynamic moment M on the body about the pivot axis. The motion of the weathercock is described by:

$$J\ddot{\alpha} = M \tag{3.41}$$

where the aerodynamic moment M is evaluated relative to O and J is the moment of inertia of the weathercock about the pivot axis.

Recall the aerodynamic force system on a rigid body. For non steady state body motion, we need to add a "correction term" to the statically derived coefficients. The most important correction is to the moment coefficient C_M. A useful model of the "dynamic moment coefficient" is given by

$$\underbrace{C_M(\alpha, \dot{\alpha})}_{dynamic} = \underbrace{C_M(\alpha)}_{static} + \frac{\bar{c}}{2V} C_{M_{\dot{\alpha}}} \dot{\alpha} \tag{3.42}$$

where $C_{M_{\dot{\alpha}}} \leq 0$. Note that $C_{M_{\dot{\alpha}}}$ is not $\frac{\partial C_M}{\partial \dot{\alpha}}$; it is given by

$$C_{M_{\dot{\alpha}}} = \frac{2V}{\bar{c}} \frac{\partial C_M}{\partial \dot{\alpha}}$$

Thus, the "dynamic moment" is given by

$$\underbrace{M(\alpha, \dot{\alpha})}_{dynamic} = \bar{q}S\bar{c}C_M(\alpha) + \frac{\bar{q}S\bar{c}^2}{2V} C_{M_{\dot{\alpha}}} \dot{\alpha}$$

Using our expression for M, we obtain the following equation of motion

$$J\ddot{\alpha} - \frac{\bar{q}S\bar{c}^2}{2V} C_{M_{\dot{\alpha}}} \dot{\alpha} - \bar{q}S\bar{c}C_M(\alpha) = 0\,. \tag{3.43}$$

where the pitching moment terms are evaluated relative to O and J is the moment of inertia of the airfoil about the pivot axis.

Equilibrium position(s) From the equation of motion, it should be clear that an equilibrium angle of attack α^e is determined by

$$C_M(\alpha^e) = 0$$

If we consider

$$C_M(\alpha) = C_{M^R}(\alpha) - C_L(\alpha)\frac{x}{\bar{c}},$$

then, the equilibrium angle of attack is given by

$$C_{M^R}(\alpha^e) - C_L(\alpha^e)\frac{x}{\bar{c}} = 0$$

Stability. Later we will show that the stability of this system about an equilibrium position depends only on the sign of the pitch stiffness

$$-\frac{dM}{d\alpha}(\alpha^e)$$

at that position; here $M(\alpha) = \bar{q}S\bar{c}C_M(\alpha)$ is the static pitching moment about O. Specifically we will show the following:

positive pitch stiffness,	$\frac{dM}{d\alpha}(\alpha^e) < 0$:	stable
negative pitch stiffness,	$\frac{dM}{d\alpha}(\alpha^e) > 0$:	unstable

Hence

$C_{M_\alpha}(\alpha^e) < 0$:	stable
$C_{M_\alpha}(\alpha^e) > 0$:	unstable

Recalling that

$$C_{M_\alpha} = \frac{x^{ac} - x}{\bar{c}} C_{L\alpha}$$

and $C_{L\alpha}(\alpha^e) > 0$, we see that the location of the aerodynamic center relative to O determines stability.

ac is behind O	$x^{ac} < x$	stable
ac is forward of O	$x^{ac} > x$	unstable

3.10 Exercises

Exercise 14 The following data refers to a model of a medium-sized transport aircraft at low speed flight conditions and is taken from [1]. Consider an aircraft whose lift coefficient depends linearly on α with

$$C_{L_0} = 0.2\,, \qquad C_{L_\alpha} = 0.085\,, \mathrm{deg}^{-1}$$

and whose drag coefficient is given by the drag polar relationship with

$$C_{DM} = 0.016\,, \qquad k = 0.042\,, \qquad C_{L_{DM}} = 0\,.$$

For a fixed elevator setting, the static pitching moment coefficient about some reference point R is given by

$$C_{M^R} = C_{M_0^R} + C_{M_\alpha^R}\,\alpha$$

with

$$C_{M_0^R} = 0.05\,, \qquad C_{M_\alpha^R} = -0.022\,\mathrm{deg}^{-1}\,.$$

(a) Plot C_L versus α.

(b) Plot C_D versus C_L.

(c) Plot C_L/C_D versus α.

Consider any point P which is located on the zero-α line of the aircraft. Let x be the displacement (measured positive in the forward direction) of P from R. For

$$x = 0.25\bar{c} \qquad x = 0 \qquad x = -0.2588\bar{c} \qquad x = -0.5\bar{c}.$$

plot the following graphs.

(d) The (exact) pitching moment coefficient vs α.

(e) The approximate pitching moment coefficient vs α.

Also,

(f) Determine $x^{ac}/\bar{c}$.

(g) What is $C_{M^{ac}}$?.

(h) Plot $x^{cp}/\bar{c}$ versus α.

(i) Plot $C_{M_\alpha^{cp}}$ versus α.

Exercise 15 Suppose the aircraft of the last exercise is pivoted about a vertical line perpendicular to its crossection and

$$\begin{array}{rcll} S &=& 2170\,\mathrm{ft}^2 & \\ \bar{c} &=& 17.5\,\mathrm{ft} & \\ J_2 &=& 4.1\times 10^6\,\mathrm{slug}\cdot\mathrm{ft}^2 & \\ C_{M_{\dot\alpha}} &=& -22\,\mathrm{rad}^{-1} & \qquad\text{(CM-alphadot)} \\ V &=& 400\,\mathrm{ft/sec} & \end{array}$$

Using MATLAB, numerically simulate this system for the four different values of x given in the last exercise. (Use approximate moment relationship.) Plot your results.

Exercise 16 The following data is representative of a small single piston-engine general aviation airplane such as a **Cessna 182** under a cruise configuration; it is taken from [2]. Before stall, the lift coefficient depends linearly on α with

$$C_{L_0} = 0.307\,, \qquad C_{L_\alpha} = 4.41\,\mathrm{rad}^{-1}$$

and the drag coefficient is given by the drag polar relationship with

$$C_{DM} = 0.0223\,, \qquad k = 0.0554\,, \qquad C_{L_{DM}} = 0\,.$$

For a fixed elevator setting, the static pitching moment coefficient about some reference point R is given by

$$C_{M^R} = C_{M_0^R} + C_{M_\alpha^R}\,\alpha$$

with

$$C_{M_0^R} = 0.04\,, \qquad C_{M_\alpha^R} = -0.613\,\mathrm{rad}^{-1}\,.$$

(a) Plot C_L versus α.

(b) Plot C_D versus C_L.

(c) Plot C_L/C_D versus α.

(d) Plot C_X versus α

(e) Plot C_Z versus α.

Suppose the mass center (cm) of the aircraft is located close the zero-α line of the aircraft. Let x^{cm} be the displacement (measured positive in the forward direction) of the cm from R. For

$$x^{cm} = 0.25\bar{c}, \qquad x^{cm} = 0, \qquad x^{cm} = -0.1390, \bar{c} \qquad x^{cm} = -0.25\bar{c}.$$

plot the following graphs.

(f) The (exact) pitching moment coefficient vs α.

(g) The approximate pitching moment coefficient vs α.

Also,

(h) Determine $x^{np}/\bar{c}$.

(i) What is $C_{M^{np}}$?

(j) Plot $x^{cp}/\bar{c}$ versus α.

(k) Plot $C_{M_\alpha^{cp}}$ versus α.

Suppose the thrust has no moment about the cm, consider the x^{cm} as a variable and let α^e be the trim value of α.

(l) Plot α^e versus $x^{cm}/\bar{c}$.

(m) Plot $C_L(\alpha^e)$ versus $x^{cm}/\bar{c}$.

(n) Plot $C_D(\alpha^e)$ versus $x^{cm}/\bar{c}$.

(o) Plot $C_L(\alpha^e)/C_D(\alpha^e)$ versus $x^{cm}/\bar{c}$.

Exercise 17 Answer the following questions for a typical aircraft under typical flight conditions (before stall).

(a) C_L always increases with increasing α. **True** **False**

(b) C_D always increases with increasing α **True** **False**

(c) C_M always increases with increasing α. **True** **False**

(d) Aircraft is longitudinally statically stable when C_{M_α} is positive at mass center. **True** **False**

(e) Aircraft is longitudinally statically stable when mass center is forward of neutral point. **True** **False**

Bibliography

[1] Stevens, B.L., and Lewis, F.L., *Aircraft Control and Simulation*, Wiley, 1992.

[2] Roskam, J., *Airplane Flight Dynamics and Automatic Flight Controls*, Design, Analysis and Research Corporation, 1995.

Chapter 4

Aircraft longitudinal dynamics

In this section we consider a special, but very important, type of aircraft motion called a longitudinal motion. We assume that, the aircraft is rigid, has a plane of symmetry and the aircraft moves so that its plane of symmetry is vertical and is fixed relative to the earth. So every point on the aircraft moves in a fixed vertical plane which is parallel to the plane of symmetry of the aircraft. In other words, the aircraft is flying "wings level" in a vertical plane.

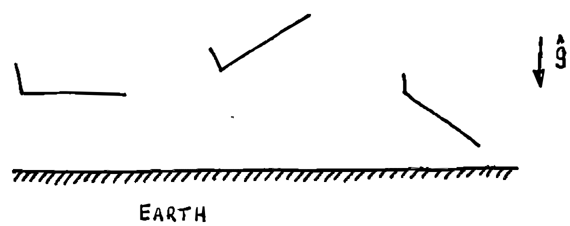


Figure 4.1: Longitudinal motion

4.1 Kinematics

The motion of a rigid aircraft relative to the earth can be completely described by specifying *the position of the* mass center (cm) *of the aircraft and the orientation of the aircraft.* In longitudinal motion, the mass center of the aircraft moves in a fixed vertical plane corresponding to the aircraft plane of symmetry. Here will model the surface of the earth as a plane; this is referred to as a flat earth model.

To discuss the motion of the aircraft relative to the earth, we choose an arbitrary point O in the symmetry plane and fixed in the earth as the origin of our position vectors; see Figure 4.2. We introduce a reference frame

$$\mathbf{e} = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$$

fixed in the earth where $\hat{e}_1$ and $\hat{e}_3$ are in the plane of symmetry of the aircraft with $\hat{e}_1$ horizontal and $\hat{e}_3$ vertically downward; $\hat{e}_2$ is perpendicular to the plane of symmetry of the

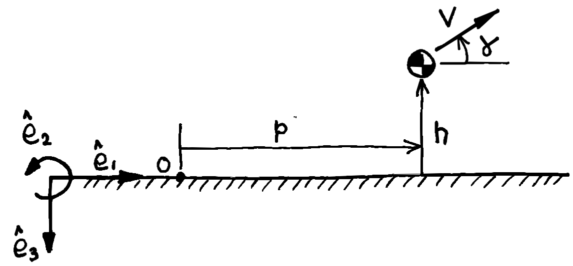


Figure 4.2: Earth fixed frame and mass center kinematics

aircraft. The position of the aircraft mass center relative to the origin O can be expressed as

$$\bar{r} = p\hat{e}_1 - h\hat{e}_3$$

where p is called the horizontal range and h is the altitude or height of the aircraft. If we let $\bar{V}$ be the velocity of the mass center of the aircraft relative to earth then,

$$\bar{V} = \dot{p}\hat{e}_1 - \dot{h}\hat{e}_3$$

Oftentimes, $\bar{V}$ is described by its magnitude V, which is the speed of the cm relative to earth, and the flight path angle γ, which is the angle between the velocity vector $\bar{V}$ and the horizontal. It is considered positive when it corresponds to a positive rotation about $\hat{e}_2$, that is, it is considered positive when counterclockwise. Noting that

$$\bar{V} = V\cos\gamma\,\hat{e}_1 - V\sin\gamma\,\hat{e}_3\,,$$

we obtain the following differential equations which are sometimes called navigation equations:

$$\boxed{\begin{array}{rcl} \dot{p} &=& V\cos\gamma \\ \dot{h} &=& V\sin\gamma \end{array}} \tag{4.1}$$

To describe the orientation of the aircraft, we introduce a reference frame

$$\mathbf{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$$

fixed in the aircraft with

$\hat{b}_1$ is along the zero-α line and pointing forward
$\hat{b}_2$ perpendicular to the plane of symmetry of the aircraft and pointing starboard
$\hat{b}_3$ in the plane of symmetry of the aircraft and pointing downward

The *orientation* of the aircraft relative to earth can be described by the pitch angle θ, where a positive value of θ corresponds to "aircraft nose up". The angular velocity of the aircraft relative to the earth is thus given by

$${}^{e}\bar{\omega}^{b} = q\hat{b}_2 = q\hat{e}_2$$

where

$$q := \dot{\theta} \tag{4.2}$$

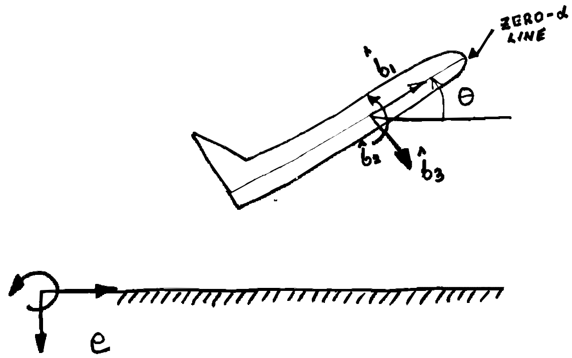


Figure 4.3: Aircraft fixed frame and pitch angle

is called the pitch rate.

Sometimes $\bar{V}$ is expressed in components relative to the aircraft frame, that is,

$$\bar{V} = u\,\hat{b}_1 + w\,\hat{b}_3 \tag{4.3}$$

where u and w are the scalar components of $\bar{V}$ relative to $\mathbf{b}$. In this case,

$$V = (u^2 + w^2)^{\frac{1}{2}}\,.$$

Since

$$\begin{aligned} \hat{b}_1 &= \cos\theta\,\hat{e}_1 - \sin\theta\,\hat{e}_3 \\ \hat{b}_3 &= \sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_3 \end{aligned}$$

we obtain the following expression for $\bar{V}$ expressed in the $\mathbf{e}$ frame:

$$\bar{V} = (u\cos\theta + w\sin\theta)\hat{e}_1 + (-u\sin\theta + w\cos\theta)\hat{e}_3$$

This yields the following navigation equations:

$$\boxed{\begin{aligned} \dot{p} &= u\cos\theta + w\sin\theta \\ \dot{h} &= u\sin\theta - w\cos\theta \end{aligned}} \tag{4.4}$$

4.2 Forces

4.2.1 Gravity

Assuming the aircraft is "close to the earth", the total gravitational attraction of the earth on the aircraft is equivalent to a single force

$$\bar{W} = W\,\hat{e}_3$$

which is in the downward vertical direction and has point of application at the mass center of the aircraft. Its magnitude W is called the weight of the aircraft and is given by

$$W = mg$$

where m is the mass of the aircraft and g is the gravitational acceleration constant of the earth:

$$g = 9.8\,\mathrm{m/s^2} = 32.2\,\mathrm{ft/sec^2}\,.$$

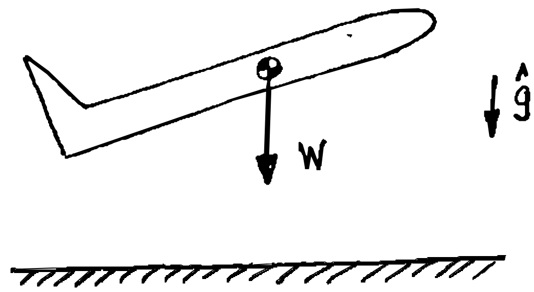


Figure 4.4: Weight

4.2.2 Aerodynamic forces and moments

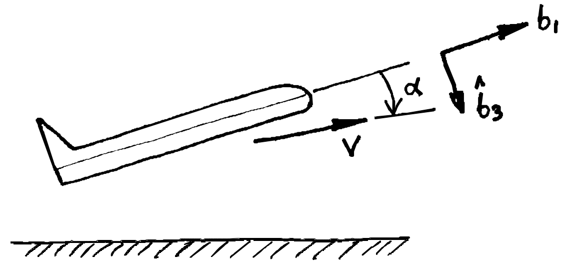


Figure 4.5: Angle of attack

Suppose $\bar{V}$ is the velocity of the mass center of the aircraft relative to the air and V is the magnitude of $\bar{V}$. We call V the airspeed. The angle of attack α is the angle between $\bar{V}$ and a line fixed in the symmetry plane of the aircraft. It is considered positive when $\bar{V}$ is clockwise of the aircraft with the aircraft being viewed from the starboard (right) side. Throughout this development, we assume that there is no wind; in this case, the velocity of the aircraft relative to the air is the same as its velocity relative to the earth. If there is a wind of velocity $\bar{V}^a$ then the velocity of the aircraft relative to the earth is $\bar{V} + \bar{V}^a$. Aerodynamic forces depend on the motion of the aircraft relative to the air.

If we express $\bar{V}$ in components relative to the previously introduced aircraft frame, that is,

$$\bar{V} = u\,\hat{b}_1 + w\,\hat{b}_3$$

where u and w are the scalar components of $\bar{V}$ relative to b, then

$$u = V\cos\alpha\,, \qquad w = V\sin\alpha$$

and

$$V = (u^2 + v^2)^{\frac{1}{2}}\,, \qquad \alpha = \tan^{-1}\left(\frac{w}{u}\right)$$

If we choose any point in the symmetry plane of the aircraft, the total aerodynamic force system on the aircraft is equivalent to a single force $\bar{F}$ acting at this point and a couple of some moment $\bar{M}$. The single force lies in the symmetry plane and is independent of

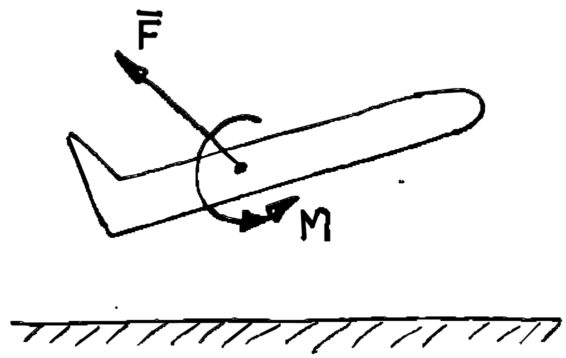


Figure 4.6: Aerodynamic force and moment

the reference point. The moment $\bar{M}$ is called the pitching moment and is perpendicular the symmetry plane; letting

$$\bar{M} = M\hat{b}_2$$

a positive value of the scalar M corresponds to a "nose-up" pitching moment.

The force $\bar{F}$ is sometimes resolved into components perpendicular and parallel to the velocity vector $\bar{V}$. these components are called the lift force $\bar{L}$ and the drag force $\bar{D}$, respectively. Thus,

$$\bar{F} = \bar{L} + \bar{D}\,.$$

The drag force always has direction opposite to $\bar{V}$ and its magnitude is denoted by D. The lift force is represented by the scalar L, where L is considered positive if the force is "upward" (that is 90^o counterclockwise from $\bar{V}$).

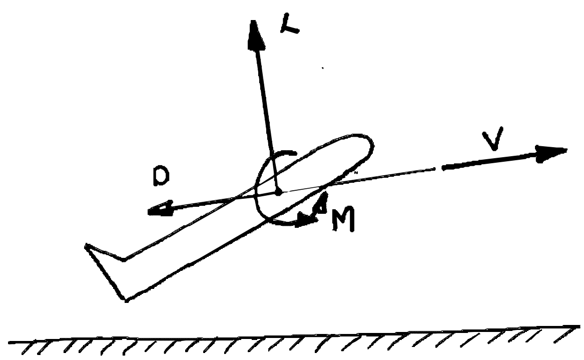


Figure 4.7: Lift and drag

Sometimes $\bar{F}$ is resolved into components relative to the aircraft fixed frame, in this case we have

$$\bar{F} = X\hat{b}_1 + Z\hat{b}_3\,.$$

Since

$$\begin{aligned} \bar{D} &= -D\cos\alpha\,\hat{b}_1 - D\sin\alpha\,\hat{b}_3 \\ \bar{L} &= L\sin\alpha\,\hat{b}_1 - L\cos\alpha\,\hat{b}_3\,, \end{aligned}$$

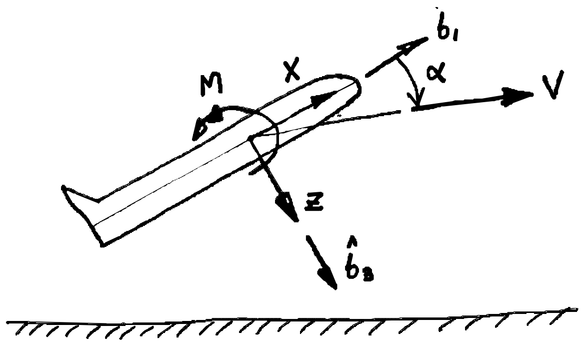


Figure 4.8: X and Z

we obtain the following relationships:

$$\begin{aligned} X &= -D\cos\alpha + L\sin\alpha \\ Z &= -D\sin\alpha - L\cos\alpha\,. \end{aligned}$$

We now introduce the following non-dimensionalized coefficients:

Lift coefficient:	$C_L := \frac{L}{\bar{q}S}$
Drag coefficient:	$C_D := \frac{L}{\bar{q}S}$
Pitching moment coefficient:	$C_M := \frac{M}{\bar{q}S\bar{c}}$

where

$$\bar{q} = \frac{1}{2}\rho V^2$$

is called the dynamic pressure with ρ being the air density; S is some reference area; and $\bar{c}$ is some reference length. Experimentally, for a given aircraft with control surfaces in a fixed configuration and under static conditions, it seems that these coefficients depend only on the angle of attack α. Rearranging the above equations, we obtain

$$\boxed{\begin{aligned} L &= \bar{q}SC_L \\ D &= \bar{q}SC_D \\ M &= \bar{q}S\bar{c}C_M \end{aligned}}$$

For uniform translation, we have already studied the dependency of the above coefficients on α.

We also introduce the non-dimensionalized coefficients,

$$C_X := \frac{X}{\bar{q}S} \qquad \text{and} \qquad C_Z := \frac{Z}{\bar{q}S}\,. \tag{4.5}$$

Rearranging the above equations, we obtain

$$\boxed{X = \bar{q}SC_X \qquad \text{and} \qquad Z = \bar{q}SC_Z}$$

Also,

$$\boxed{\begin{array}{rcl} C_X &=& -C_D\cos\alpha + C_L\sin\alpha \\ C_Z &=& -C_D\sin\alpha - C_L\cos\alpha\,. \end{array}}$$

Effect of mass center location on pitching moment

In deriving the equations of motion of an aircraft, one needs the pitching moment of the aerodynamic force system about the mass center. Suppose the aerodynamic data is specified relative to another reference point R on the aircraft and M^R is the pitching moment about R. Let x and z denote the coordinates of the cm relative to R, that is, the position of the cm relative to R is

$$x\hat{b}_1 + z\hat{b}_3$$

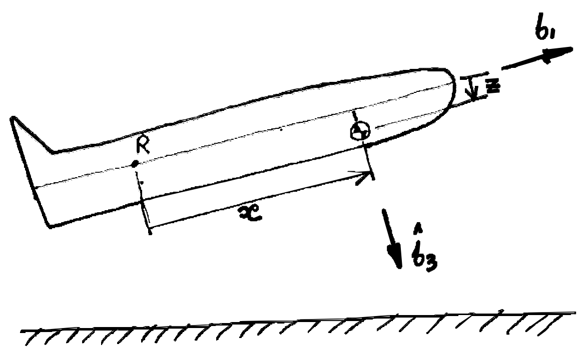


Figure 4.9: Effect of cm location on M

Then, as we have already seen, pitching moment coefficient C_M about the cm is given by

$$C_M = C_M^R + C_Z\frac{x}{\bar{c}} - C_X\frac{z}{\bar{c}}$$

where

$$C_X = C_L\sin\alpha - C_D\cos\alpha \qquad \text{and} \qquad C_Z = -C_L\cos\alpha - C_D\sin\alpha$$

Suppose that for small α and z, we approximate the above expression by

$$\boxed{C_M = C_{M^R} - C_L\frac{x}{\bar{c}}} \tag{4.6}$$

Effect of angular rates on aerodynamic coefficients

For non steady state aircraft motion, we need to add "correction terms" to the statically derived coefficients. To this end, we modify the lift coefficient to obtain the dynamic lift coefficient

$$\underbrace{C_L(\alpha, el, \dot{\alpha}, q)}_{dynamic} = \underbrace{C_L(\alpha, el)}_{static} + \frac{\bar{c}}{2V} C_{L_{\dot{\alpha}}} \dot{\alpha} + \frac{\bar{c}}{2V} C_{L_q} q \tag{4.7}$$

where $C_{L_{\dot{\alpha}}}$ and C_{L_q} are constants. Note that

$$C_{L_{\dot{\alpha}}} = \frac{2V}{\bar{c}} \frac{\partial C_L}{\partial \dot{\alpha}}, \qquad C_{L_q} = \frac{2V}{\bar{c}} \frac{\partial C_L}{\partial q}$$

If we consider C_{L_α} and $C_{L_{el}}$ to be constant, then, the total (dynamic) lift coefficient is given by

$$\boxed{C_L = C_{L_0} + C_{L_\alpha} \alpha + C_{L_{el}} el + \frac{\bar{c}}{2V} C_{L_{\dot{\alpha}}} \dot{\alpha} + \frac{\bar{c}}{2V} C_{L_q} q}$$

where C_{L_0} is a constant.

We also modify the pitching moment coefficient to obtain the dynamic pitching moment coefficient

$$\underbrace{C_M(\alpha, el, \dot{\alpha}, q)}_{dynamic} = \underbrace{C_M(\alpha, el)}_{static} + \frac{\bar{c}}{2V} C_{M_{\dot{\alpha}}} \dot{\alpha} + \frac{\bar{c}}{2V} C_{M_q} q \tag{4.8}$$

where

$$C_{M_{\dot{\alpha}}} \leq 0 \qquad \text{and} \qquad C_{M_q} \leq 0$$

are constants. Note that

$$C_{M_{\dot{\alpha}}} = \frac{2V}{\bar{c}} \frac{\partial C_M}{\partial \dot{\alpha}}, \qquad C_{M_q} = \frac{2V}{\bar{c}} \frac{\partial C_M}{\partial q}$$

If we consider $C_{M^R_\alpha}$ and $C_{M^R_{el}}$ to be constant at the reference point then, the total (dynamic) pitching moment coefficient about the mass center is given by

$$\boxed{C_M = C_{M^R_0} + C_{M^R_\alpha} \alpha + C_{M^R_{el}} el + \frac{\bar{c}}{2V} C_{M^R_{\dot{\alpha}}} \dot{\alpha} + \frac{\bar{c}}{2V} C_{M^R_q} q - C_L \frac{x}{\bar{c}}}$$

where $C_{M^R_0}$ is a constant.

For now, *we will not modify the drag coefficient.* We will compute C_D using the static lift coefficient $C_L(\alpha, el)$ in the drag polar relationship. Thus

$$\boxed{C_D = C_{DM} + k(C_L(\alpha, el) - C_{L_{DM}})^2} \tag{4.9}$$

where $C_L(\alpha, el) = C_{L_0} + C_{L_\alpha} \alpha + C_{L_{el}} el$.

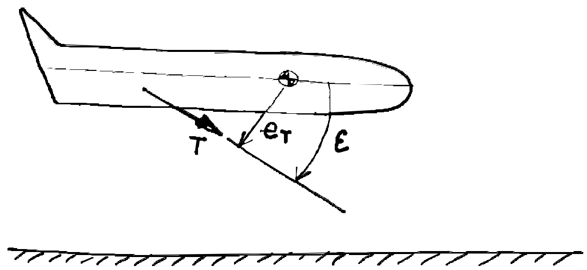


Figure 4.10: Thrust force

4.2.3 Thrust force

We assume that the aircraft propulsion system generates a thrust T where the the direction of the thrust force $\bar{T}$ is fixed relative to the aircraft and lies in the aircraft symmetry plane. We let the angle between the thrust force and the fixed "zero-α" line in the aircraft be ϵ where ϵ is measured positive in the same direction as α. If the line of action of the thrust force does not pass through the mass center of the aircraft, this force will exert a moment $M_T\hat{b}_2$ about the mass center where

$$M_T = e_T T\,, \tag{4.10}$$

where the magnitude of e_T is the distance from the line of action of the thrust force to the mass center and the sign of e_T is determined by the right-hand rule. In general the thrust depends on the throttle setting th and the airspeed V.

4.3 Equations of motion

We shall choose reference frame **e** fixed in the earth as inertial.

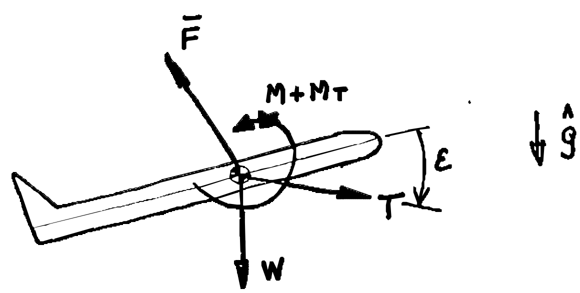


Figure 4.11: Free body diagram of aircraft

Assuming that the pitching moment $\bar{M}$ is evaluated relative to the mass center of the aircraft, we can apply $\Sigma\bar{M} = \dot{\bar{H}}$ at the mass center to obtain

$$\boxed{J_2\,\ddot{\theta} = M + M_T} \tag{4.11}$$

where J_2 is the moment of inertia of the aircraft about a line passing through the mass center and perpendicular to the plane of symmetry.

We will assume that the air is stationary relative to the earth; hence, $\bar{V}$ is the velocity of the mass center of the aircraft relative to the earth and the inertial acceleration of the mass center is $\frac{^e d\bar{V}}{dt}$. Applying $\Sigma\bar{F} = m\bar{a}$ to the mass center of the aircraft we obtain

$$\boxed{m\frac{^e d\bar{V}}{dt} = \bar{F} + \bar{W} + \bar{T}} \tag{4.12}$$

4.3.1 Body fixed components

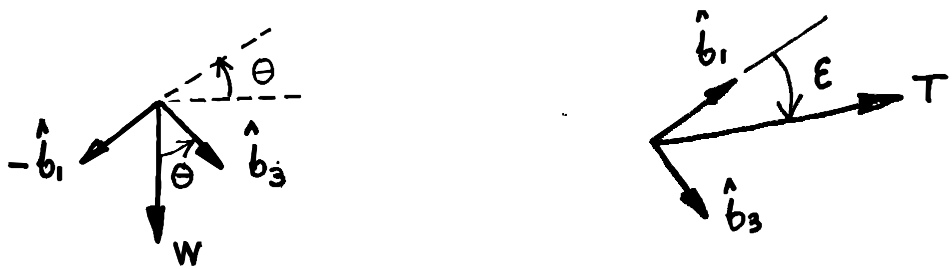


Figure 4.12: Resolving forces into body-fixed components

Considering components relative to the body fixed frame and applying the BKE and recalling that $\bar{V} = u\hat{b}_1 + w\hat{b}_2$, we have

$$\begin{aligned}\frac{^e d\bar{V}}{dt} &= \frac{^b d\bar{V}}{dt} + {^e\bar{\omega}^b} \times \bar{V} \quad = \quad \dot{u}\,\hat{b}_1 + \dot{w}\,\hat{b}_3 + (\dot{\theta}\,\hat{b}_2) \times (u\,\hat{b}_1 + w\,\hat{b}_3) \\ &= (\dot{u} + \dot{\theta}w)\,\hat{b}_1 + (\dot{w} - \dot{\theta}u)\,\hat{b}_3\,.\end{aligned}$$

Expressing the forces relative to the body fixed frame, we obtain

$$\begin{aligned}\bar{F} &= X\,\hat{b}_1 + Z\,\hat{b}_3 \\ \bar{W} &= -W\sin\theta\,\hat{b}_1 + W\cos\theta\,\hat{b}_3 \\ \bar{T} &= T\cos\epsilon\,\hat{b}_1 + T\sin\epsilon\,\hat{b}_3\end{aligned}$$

Hence, taking components of $\Sigma\bar{F} = m\bar{a}$ relative to the body fixed frame yields:

$$\begin{aligned}m(\dot{u} + \dot{\theta}w) &= X + X_T - W\sin\theta \\ m(\dot{w} - \dot{\theta}u) &= Z + Z_T + W\cos\theta\end{aligned}$$

with

$$X_T = T\cos\epsilon\,, \qquad\qquad Z_T = T\sin\epsilon\,.$$

Recalling the moment equation and the definition of pitch rate q, the longitudinal motion of an aircraft can be described by the following set of four first order differential equations:

$$\boxed{\begin{aligned}m\dot{u} &= -mqw + X + X_T - W\sin\theta \\ m\dot{w} &= \quad mqu + Z + Z_T + W\cos\theta \\ \dot{\theta} &= q \\ J_2\dot{q} &= M + M_T\end{aligned}} \tag{4.13}$$

This is a state space representation with state variables u, w, θ, q.

4.3.2 Stability axes components

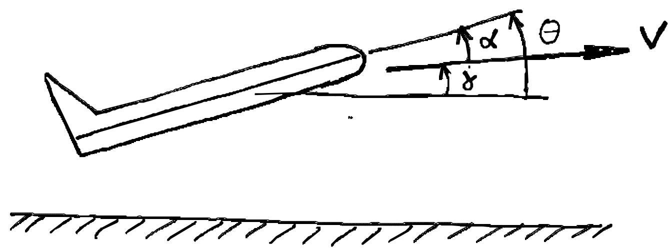


Figure 4.13: Angles

Recall the definition of the flight path angle γ. We have the following relationship

$$\theta = \gamma + \alpha\,.$$

Hence

$$\gamma = \theta - \alpha\,.$$

Introduce the stability reference frame

$$\mathbf{s} = (\hat{s}_1, \hat{s}_2, \hat{s}_3)$$

where $\hat{s}_1$ in the direction of $\bar{V}$, $\hat{s}_2$ is in the same direction as $\hat{b}_2$ and $\hat{s}_3$ is in the plane of symmetry and perpendicular to $\hat{s}_1$. With this reference frame, the velocity can be simply

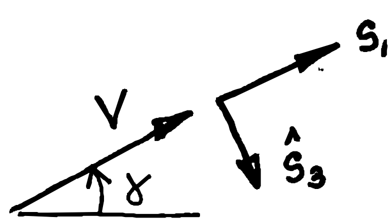


Figure 4.14: Stability reference frame

expressed as

$$\bar{V} = V\hat{s}_1$$

Using the BKE we obtain that

$$\begin{aligned} \frac{{}^e d\bar{V}}{dt} &= \frac{{}^s d\bar{V}}{dt} + {}^e\bar{\omega}^s \times \bar{V} \quad = \quad \dot{V}\hat{s}_1 + (\dot{\gamma}\hat{s}_2) \times (V\hat{s}_1) \\ &= \dot{V}\hat{s}_1 - \dot{\gamma}V\hat{s}_3\,. \end{aligned}$$

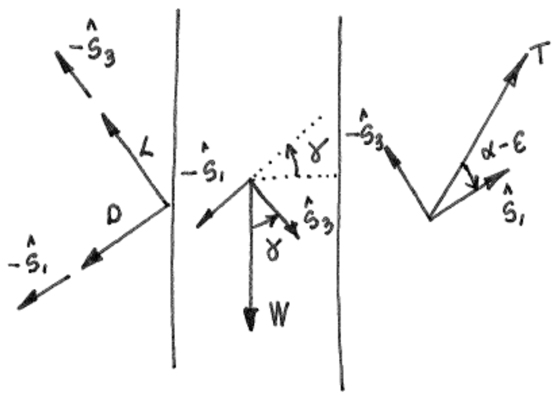


Figure 4.15: Resolving forces into components relative to stability frame

Also,

$$\begin{aligned}
\bar{L} &= -L\hat{s}_3 \\
\bar{D} &= -D\hat{s}_1 \\
\bar{W} &= -W\sin\gamma\,\hat{s}_1 + W\cos\gamma\,\hat{s}_3 \\
\bar{T} &= T\cos(\alpha-\epsilon)\,\hat{s}_1 - T\sin(\alpha-\epsilon)\,\hat{s}_3
\end{aligned}$$

Considering components of $\Sigma\bar{F} = m\bar{a}$ relative to $\hat{s}_1$ and $\hat{s}_3$, using the moment equation and recalling the definition of pitch rate q yields the following set of four first order differential equations:

$$\boxed{\begin{aligned}
m\dot{V} &= -D - W\sin\gamma + T\cos(\alpha-\epsilon) \\
mV\dot{\gamma} &= L - W\cos\gamma + T\sin(\alpha-\epsilon) \\
\dot{\theta} &= q \\
J_2\dot{q} &= M + M_T
\end{aligned}} \tag{4.14}$$

with $\alpha = \theta - \gamma$. This is a state space representation with state variables V, γ, θ, q. If we append horizontal range p and height h as state variables we need to append the navigation equations:

$$\begin{aligned}
\dot{p} &= V\cos\gamma \\
\dot{h} &= V\sin\gamma
\end{aligned} \tag{4.15}$$

Equations (4.14) and (4.15) yield a six-dimensional state space description in the state variables $V, \gamma, \theta, q, p, h$.

Quite often γ is replaced with α to yield the following state space representation

$$\boxed{\begin{aligned}
m\dot{V} &= -D - W\sin\gamma + T\cos(\alpha-\epsilon) \\
mV\dot{\alpha} &= -L + W\cos\gamma - T\sin(\alpha-\epsilon) + mVq \\
\dot{\theta} &= q \\
J_2\dot{q} &= M + M_T
\end{aligned}} \tag{4.16}$$

with $\gamma = \theta - \alpha$. Now the state variables are V, α, θ, q. If we include the navigation equations, the state variables are $V, \alpha, \theta, q, p, h$.

4.3.3 Trim conditions

Here we consider steady state flying conditions corresponding to a constant velocity $\bar{V}^e$ and a constant pitch angle θ^e. This results in a constant speed V^e, a constant flight path angle γ^e, zero pitch rate ($q = 0$) and a constant angle of attack $\alpha^e = \theta^e - \gamma^e$.

Considering state equations (4.14) or (4.16), the steady state flying conditions are

$$\begin{aligned} -D - W\sin\gamma^e + T\cos(\alpha^e - \epsilon) &= 0 \\ L - W\cos\gamma^e + T\sin(\alpha^e - \epsilon) &= 0 \\ M + M_T &= 0 \end{aligned} \tag{4.17}$$

For fixed elevator deflection and throttle setting, this yields three equations in the three unknowns V^e, γ^e, α^e. The pitch angle can be obtained from $\theta^e = \gamma^e + \alpha^e$.

When $M_T = 0$ the third equilibrium equation above reduces to $M = 0$. In this case, the equilibrium angle of attack α^e is completely determined by

$$C_M(\alpha^e, el) = 0\,. \tag{4.18}$$

In particular, α^e is independent of the aircraft speed. It only depends on the elevator deflection el.

Horizontal flight. Here $\gamma^e = 0$. Hence,

$$\begin{aligned} D - T\cos(\alpha^e - \epsilon) &= 0 \\ L - T\sin(\alpha^e - \epsilon) &= W \\ M + M_T &= 0 \end{aligned}$$

If one chooses a desired V^e, one can solve these three equations for α^e, the elevator deflection el and the throttle setting th needed for level flight at this speed. If $\alpha^e \approx \epsilon$, we obtain

$$\begin{aligned} L &\approx W \\ T &\approx D \end{aligned}$$

Hence,

$$\frac{C_L}{C_D} \approx \frac{W}{T}$$

Gliding. Here $T = 0$; hence $M_T = 0$ and the angle of attack is given by (4.18). Since

$$\begin{aligned} -D - W\sin(\gamma^e) &= 0 \\ L - W\cos(\gamma^e) &= 0 \end{aligned}$$

the flight path angle is given by

$$\tan\gamma^e = -\frac{C_D(\alpha^e)}{C_L(\alpha^e)}$$

The angle $-\gamma_e$ is called the gliding angle. The speed can be obtained from

$$(V^e)^2 = -\frac{2W\sin(\gamma^e)}{\rho S C_D(\alpha^e)}$$

4.3.4 Phugoid motion

Suppose that $M_T = 0$ and J_2 is very small. If we consider $J_2 = 0$, we must have $M = 0$ and hence

$$\alpha = \alpha^e$$

where $C_M(\alpha^e) = 0$. The motion of the aircraft is described

$$\begin{aligned} m\dot{V} &= -D - W\sin\gamma + T\cos(\alpha^e - \epsilon) \\ mV\dot{\gamma} &= L - W\cos\gamma + T\sin(\alpha^e - \epsilon) \end{aligned} \tag{4.19}$$

and

$$\theta = \gamma + \alpha^e\,.$$

4.4 Example: Cessna 182

The following data is representative of a small single piston-engine general aviation airplane such as a **Cessna 182**; it is taken from [2].

$$
\begin{aligned}
W &= 2,650\,\mathrm{lb}\\
S &= 174\,\mathrm{ft}^2\\
\bar{c} &= 4.9\,\mathrm{ft}\\
J_2 &= 1,346\,\mathrm{slug}\cdot\mathrm{ft}^2
\end{aligned}
$$

The lift coefficient depends linearly on α with

$$C_{L_0} = 0.307, \qquad C_{L_\alpha} = 4.41\,\mathrm{rad}^{-1}, \qquad C_{L_{el}} = 0.43\,\mathrm{rad}^{-1}\,.$$

The drag coefficient is given by the drag polar relationship with

$$C_{DM} = 0.0223\,, \qquad k = 0.0554\,, \qquad C_{L_{DM}} = 0\,.$$

The static pitching moment coefficient about some reference point R is given

$$C_{M^R} = C_{M_0^R} + C_{M_\alpha^R}\alpha + C_{M_{el}^R} el$$

with

$$C_{M_0^R} = 0.04 \qquad C_{M_\alpha^R} = -0.613\,\mathrm{rad}^{-1} \qquad C_{M_{el}} = -1.122\,\mathrm{rad}^{-1}$$

The aerodynamic coefficients for rate dependency are

$$
\begin{aligned}
C_{M_{\dot\alpha}} &= -7.27\,\mathrm{rad}^{-1}, \qquad C_{M_q} = -12.4\,\mathrm{rad}^{-1}\,,\\
C_{L_{\dot\alpha}} &= 1.7\,\mathrm{rad}^{-1}, \qquad C_{L_q} = 3.9\,\mathrm{rad}^{-1}\,,
\end{aligned}
$$

We model the thrust by

$$T = (550\ th)\ \eta/V$$

where $\eta = 0.7$ is the propeller efficiency and th is the engine output in horsepower. Note that 1 hp = 550 ft-lb/sec. Also,

$$\epsilon = 0 \qquad \text{and} \qquad e_T = 0\,.$$

Exercise 18 (Your first flight) Consider the model for the Cessna 182 aircraft given in the notes. Obtain a SIMULINK model of this vehicle with state variables $V, \alpha, \theta, q, p, h$ and "fly" your model to achieve the following trim conditions. (Consider $x^{cm} = 0$.)

(a) Gliding with $el = 0$.

(b) Horizontal level flight with

$$th = 100\ \mathrm{hp}$$

In this part, you need to vary the elevator position until the desired trim condition is achieved.

Illustrate your results with plots of the time histories of V, α (deg), γ (deg), and h; also plot the aircraft trajectory, that is, h versus p.

4.5 Example: transport aircraft

The following model of a medium-sized transport aircraft at low speed flight conditions is taken from [1].

$$\begin{aligned} W &= 162,000\,\text{lb} \\ S &= 2170\,\text{ft}^2 \\ \bar{c} &= 17.5\,\text{ft} \\ J_2 &= 4.1\times 10^6\,\text{slug}\cdot\text{ft}^2 \end{aligned}$$

The lift coefficient depends linearly on α with

$$C_{L_0} = 0.2 \qquad \text{and} \qquad C_{L_\alpha} = 0.085\,\text{deg}^{-1}\,.$$

The drag coefficient is given by the drag polar relationship with

$$C_{DM} = 0.016 \qquad k = 0.042 \qquad C_{L_{DM}} = 0\,.$$

The static pitching moment coefficient about some reference point R is given

$$C_{M^R} = C_{M_0^R} + C_{M_\alpha^R}\alpha + C_{M_{el}}el$$

with

$$C_{M_0^R} = 0.05 \qquad C_{M_\alpha^R} = -0.022\,\text{deg}^{-1} \qquad C_{M_{el}} = -0.016\,\text{deg}^{-1}$$

The pitching moment damping coefficients are

$$C_{M_{\dot\alpha}} = -6\,\text{rad}^{-1} \qquad C_{M_q} = -16\,\text{rad}^{-1}\,,$$

The thrust is given by

$$T = th(T_0 + T_V V) \qquad T_0 = 60,000\,\text{lb} \qquad T_V = -38\,\text{lb}\cdot\text{sec/ft}$$

and

$$\epsilon = 0 \qquad e_T = 2.0\,\text{ft}$$

Exercise 19 (Your first flight) Consider the model for the longitudinal motion of a transport aircraft given above. Determine the location of the neutral point. Using state variables $V, \alpha, \theta, q, p, h$ and using MATLAB, "fly" this model to answer the the next two items .

(a) With mass center $0.25\bar{c}$ forward of aerodynamic reference point R, that is,

$$x = -0.25\bar{c},$$

determine the elevator position necessary to achieve level flight with the following throttle settings.

$$th = 0.3 \qquad \text{and} \qquad th = 0.2\,.$$

(b) Using the the first elevator deflection and throttle setting from part (a), determine a mass center position which results in an unstable aircraft.

For parts (a) and (b), illustrate your results with plots of the time histories of V, α (deg), γ (deg), and h; also plot the aircraft trajectory, that is, h versus p.

Bibliography

[1] Stevens, B.L., and Lewis, F.L., *Aircraft Control and Simulation*, Wiley, 1992.

[2] Roskam, J., *Airplane Flight Dynamics and Automatic Flight Controls*, Design, Analysis and Research Corporation, 1995.

Chapter 5

Linear systems and linearization

In this section we look at the state space description of linear systems. We also present a systematic procedure for approximating a nonlinear system by a linear system. To study linear systems we need matrices.

5.1 Matrices

Recall that an $m \times n$ matrix A is an array of scalars consisting of m *rows* and n *columns*:

$$A = \underbrace{\left.\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}\right\}}_{n \text{ columns}} m \text{ rows}$$

The scalars a_{ij} are called the elements of A; the first index i indicates the row number and the second index j indicates the column number. If the scalars are real numbers, A is called a real matrix. Recall the following operations and definitions in matrix algebra:

Matrix addition: $A + B$

Multiplication of a matrix by a scalar: αA

Zero matrices: 0

Negative of the matrix A: $-A$

Matrix multiplication: AB

Some properties:

$$(AB)C = A(BC) \qquad \text{(associative)}$$

$$\text{In general,} \quad AB \neq BA \qquad \text{(non-commutative)}$$

$$\begin{aligned} A(B+C) &= AB + AC \\ (B+C)A &+ BA + CA \end{aligned}$$

$$\begin{aligned} A(\alpha B) &= \alpha AB \\ (\alpha A)B &= \alpha AB \end{aligned}$$

$n \times n$ identity matrix:

$$I := \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{bmatrix} = \text{diag}\,(1, 1, \dots, 1)$$

$$AI = IA = A$$

Inverse of a square $(n = m)$ matrix A: A^{-1}

$$AA^{-1} = A^{-1}A = I$$

Some properties:

$$\begin{aligned} (\alpha A)^{-1} &= \frac{1}{\alpha}A^{-1} \\ (AB)^{-1} &= B^{-1}A^{-1} \end{aligned}$$

Transpose of A: A^T

Some properties:

$$\begin{aligned} (A+B)^T &= A^T + B^T \\ (\alpha A)^T &= \alpha A^T \end{aligned}$$

$$(AB)^T = B^T A^T$$

Determinant of a square matrix A : $\det A$

Some properties:

$$\begin{aligned} \det(AB) &= \det(A)\det(B) \\ \det(A^T) &= \det(A) \end{aligned}$$

Note that, in general,

$$\begin{aligned} \det(A+B) &\neq \det(A)+\det(B) \\ \det(\alpha A) &\neq \alpha \det(A) \end{aligned}$$

Fact: A is invertible if and only if $\det A \neq 0$

Exercise 20 Compute the determinant and inverse of the following matrix.

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

MATLAB

Representation of real and complex matrices
Addition and multiplication of matrices
Multiplication of a matrix by a scalar
Matrix powers
Transpose of a matrix
Inverse of a matrix

```
>> help zeros

 ZEROS  All zeros.
        ZEROS(N) is an N-by-N matrix of zeros.
        ZEROS(M,N) or ZEROS([M,N]) is an M-by-N matrix of zeros.
        ZEROS(SIZE(A)) is the same size as A and all zeros.

>> help eye

 EYE    Identity matrix.
        EYE(N) is the N-by-N identity matrix.
        EYE(M,N) or EYE([M,N]) is an M-by-N matrix with 1's on
        the diagonal and zeros elsewhere.
        EYE(SIZE(A)) is the same size as A.

>> help det

 DET    Determinant.
        DET(X) is the determinant of the square matrix X.

>> help inv

 INV    Matrix inverse.
        INV(X) is the inverse of the square matrix X.
```

```
A warning message is printed if X is badly scaled or
nearly singular.
```

Exercise 21 Determine which of the following matrices are invertible. Obtain the inverses of those which are. Check your answers using MATLAB.

$$\text{(a)} \quad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \text{(c)} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{(d)} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Exercise 22 Determine which of the following matrices are invertible. Using MATLAB, check your answers and determine the inverse of those that are invertible.

$$\text{(a)} \quad \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{(b)} \quad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{(c)} \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

5.2 Linear time-invariant systems

A linear time invariant (LTI) system (without inputs) is described by

$$\boxed{\dot{x} = Ax}$$

where the n-vector $x(t)$ is the state vector at time t and the $n \times n$ matrix A is called the system matrix.

In scalar terms, such a system is described by

$$\begin{aligned}
\dot{x}_1 &= a_{11}x_1 + \ldots + a_{1n}x_n \\
&\vdots \\
\dot{x}_n &= a_{n1}x_1 + \ldots + a_{nn}x_n
\end{aligned}$$

where a_{ij} is the ij-th component of the matrix A.

Example 9 The unattached mass

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 0
\end{aligned}$$

Here

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Example 10 Damped linear oscillator

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\frac{k}{m}x_1 - \frac{c}{m}x_2
\end{aligned}$$

Here

$$A = \begin{bmatrix} 0 & 1 \\ -k/m & -c/m \end{bmatrix}$$

Example 11 B&B

$$\begin{array}{ccccccccccc}
m\ddot{q}_1 & + & c\dot{q}_1 & + & kq_1 & - & c\dot{q}_2 & - & kq_2 & = & 0 \\
m\ddot{q}_2 & - & c\dot{q}_1 & - & kq_1 & + & c\dot{q}_2 & + & kq_2 & = & 0
\end{array}$$

With $x_1 := q_1, x_2 := q_2, x_3 := \dot{q}_1$ and $x_4 := \dot{q}_2$, we have the following state space description:

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -\frac{k}{m}x_1 + \frac{k}{m}x_2 - \frac{c}{m}x_3 + \frac{c}{m}x_4 \\
\dot{x}_4 &= \frac{k}{m}x_1 - \frac{k}{m}x_2 + \frac{c}{m}x_3 - \frac{c}{m}x_4
\end{aligned}$$

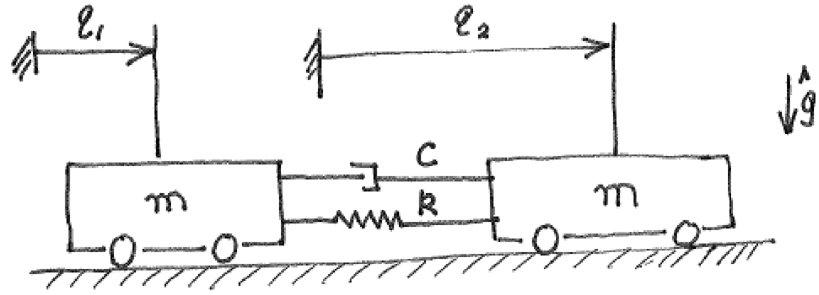


Figure 5.1: B&B

The system matrix associated with this state space description is given by

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m & k/m & -c/m & c/m \\ k/m & -k/m & c/m & -c/m \end{bmatrix}$$

5.3 Linearization

Why did people believe that the earth was flat? In this section, we see how to approximate a nonlinear system by a linear system.

5.3.1 Linearization and derivatives

Scalar functions of a scalar variable. Suppose f is a scalar valued function of a scalar variable x. Consider any x^* where f has a derivative at x^*. Then, for all x close to x^*, we can approximate $f(x)$ by

$$\boxed{f(x) \approx f(x^*) + f'(x^*)(x - x^*)} \tag{5.1}$$

where $f'(x^*)$ is the derivative of f at x^*. Note that the *approximation depends linearly on x.* This is illustrated in Figure 5.2.

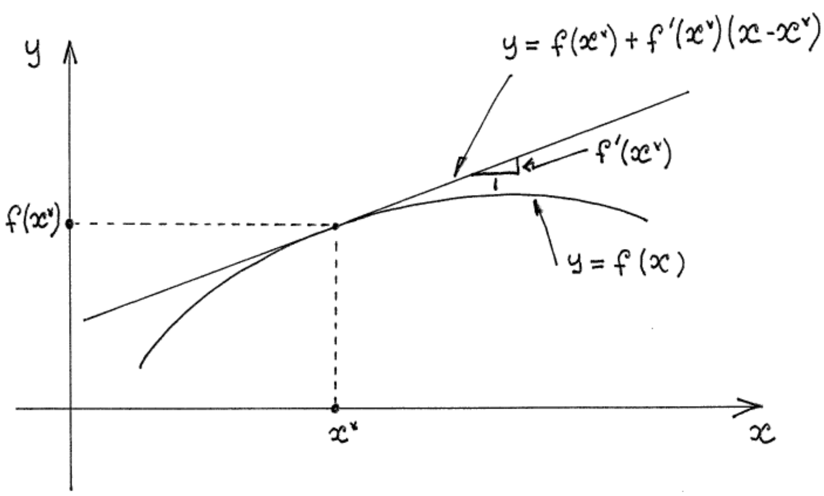


Figure 5.2: Linearization

To analytically demonstrate the above approximation, recall the definition of the derivative of f at x^*;

$$f'(x^*) = \lim_{x \to x^*} \frac{f(x) - f(x^*)}{x - x^*}.$$

Hence,

$$f'(x^*) = \frac{f(x) - f(x^*)}{x - x^*} + r(x - x^*) \qquad \text{where} \qquad \lim_{x \to x^*} r(x - x^*) = 0$$

Rearranging this results in

$$f(x) = f(x^*) + f'(x^*) + o(x - x^*)$$

where $o(x-x^*) = r(x-x^*)(x-x^*)$. So,

$$\lim_{x\to x^*} \frac{o(x-x^*)}{x-x^*} = 0$$

Let δx be the perturbation of x from x^*, that is,

$$\boxed{\delta x = x - x^*}$$

Then $x = x^* + \delta x$. So, when δx is small, we obtain the approximation:

$$\boxed{f(x^*{+}\delta x) \approx f(x^*) + f'(x^*)\delta x}$$

In particular, if $f(x^*) = 0$ then,

$$f(x^*{+}\delta x) \approx f'(x^*)\delta x$$

Example 12 If

$$f(x) = x^2$$

then

$$\begin{aligned} f(x^* + \delta x) &= (x^* + \delta x)^2 \\ &= (x^*)^2 + 2x^*\delta x + (\delta x)^2 \\ &= f(x^*) + f'(x^*)\delta x + o(\delta x) \end{aligned}$$

where $o(\delta x) = (\delta x)^2$. Note that $\lim_{\delta x\to 0} o(\delta x)/\delta x = 0$. Thus, for small δx we have

$$(x^* + \delta x)^2 \approx (x^*)^2 + 2x^*\delta x\,.$$

Example 13 Consider

$$f(x) = \sin x\,.$$

Here $f'(x) = \cos x$. Hence, for any x^*, we can approximate $\sin(x^*{+}\delta x)$ by

$$\sin(x^*{+}\delta x) \approx \sin x^* + \cos(x^*)\delta x$$

In particular, for $x^* = 0$ we obtain

$$\sin(\delta x) \approx \delta x$$

and for $x^* = \pi$, we obtain

$$\sin(\pi{+}\delta x) \approx -\delta x$$

Example 14

$$f(x) = |x|$$

Example 15

$$f(x) = x|x|$$

Scalar systems.

Example 16

$$\dot{x} = x - x^3$$

Scalar functions of a vector variable. Suppose f is a scalar valued function of several scalar variables $x_1, x_2, \cdots, x_n$. Considering any specific values, $x_1^*, x_2^*, \cdots, x_n^*$, of these variables, we have the following approximation when the variables are close to their specific values:

$$\boxed{f(x) \approx f(x^*) + \frac{\partial f}{\partial x_1}(x^*)(x_1 - x_1^*) + \frac{\partial f}{\partial x_2}(x^*)(x_2 - x_2^*) + \cdots + \frac{\partial f}{\partial x_n}(x^*)(x_n - x_n^*)}$$

where

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \qquad x^* = \begin{bmatrix} x_1^* \\ x_2^* \\ \vdots \\ x_n^* \end{bmatrix}$$

and

$\frac{\partial f}{\partial x_i}$ is the partial derivative of f with respect to x_i.

Letting

$$\delta x := x - x^*$$

we have $x = x^* + \delta x$ and we can write the above approximation as

$$\boxed{f(x^*+\delta x) \approx f(x^*) + \frac{\partial f}{\partial x_1}(x^*)\delta x_1 + \frac{\partial f}{\partial x_2}(x^*)\delta x_2 + \cdots + \frac{\partial f}{\partial x_n}(x^*)\delta x_n}$$

Let $\frac{\partial f}{\partial x}$ be the row matrix given by

$$\boxed{\frac{\partial f}{\partial x}(x^*) := \begin{bmatrix} \frac{\partial f}{\partial x_1}(x^*) & \frac{\partial f}{\partial x_2}(x^*) & \cdots & \frac{\partial f}{\partial x_n}(x^*) \end{bmatrix}}$$

The above approximation can now be written compactly as

$$f(x) \approx f(x^*) + \frac{\partial f}{\partial x}(x^*)(x - x^*)$$

or

$$f(x^* + \delta x) \approx f(x^*) + \frac{\partial f}{\partial x}(x^*)\delta x$$

In particular, when $f(x^*) = 0$, we have

$$f(x^* + \delta x) \approx \frac{\partial f}{\partial x}(x^*)\delta x$$

Compare these expressions to the corresponding expressions in the scalar case.

Example 17 Consider

$$f(x) = (1 + x_1)\cos x_2 - 1\,.$$

Linearization about $x^* = 0$ results in

$$\begin{aligned} f(\delta x) &\approx f(0) + \frac{\partial f}{\partial x_1}(0)\,\delta x_1 + \frac{\partial f}{\partial x_2}(0)\,\delta x_2 \\ &= 0 + (\cos x_2|_{x=0})\,\delta x_1 + (-\,(1+x_1)\sin x_2|_{x=0})\,\delta x_2 \\ &= \delta x_1\,. \end{aligned}$$

General case. Consider an m-vector-valued function f of an n-vector variable x. Suppose that each of the partial derivatives, $\frac{\partial f_i}{\partial x_j}(x^*)$ exist and are continuous about some x^*.

- The derivative of f (Jacobian of f) at x^* is the following $m \times n$ matrix:

$$\frac{\partial f}{\partial x}(x^*) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^*) & \frac{\partial f_1}{\partial x_2}(x^*) & \dots & \frac{\partial f_1}{\partial x_n}(x^*) \\ \frac{\partial f_2}{\partial x_1}(x^*) & \frac{\partial f_2}{\partial x_2}(x^*) & \dots & \frac{\partial f_2}{\partial x_n}(x^*) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_m}{\partial x_1}(x^*) & \frac{\partial f_m}{\partial x_2}(x^*) & \dots & \frac{\partial f_m}{\partial x_n}(x^*) \end{bmatrix}$$

that is,

$$\frac{\partial f}{\partial x}(x^*)_{ij} = \frac{\partial f_i}{\partial x_j}(x^*)$$

Some properties:

$$\begin{aligned} \tfrac{\partial(f+h)}{\partial x} &= \tfrac{\partial f}{\partial x} + \tfrac{\partial f}{\partial x} \\ \tfrac{\partial(\alpha f)}{\partial x} &= \alpha\tfrac{\partial f}{\partial x} \qquad\qquad (\alpha \text{ is a constant scalar}) \end{aligned}$$

$$\begin{array}{lrclrcl} \text{(Constant function:)} & f(x) & \equiv & c & \tfrac{\partial f}{\partial x}(x) & = & 0 \\ \text{(Linear function:)} & f(x) & = & Ax & \tfrac{\partial f}{\partial x}(x) & = & A \end{array}$$

Consider any vector x^*. Then, for any x 'close' to x^*, that is when $\delta x := x - x^*$ is 'small', we have

$$\boxed{f(x) \approx f(x^*) + \frac{\partial f}{\partial x}(x^*)(x - x^*)} \tag{5.2}$$

or

$$\boxed{f(x^*+\delta x) \approx f(x^*) + \frac{\partial f}{\partial x}(x^*)\delta x} \tag{5.3}$$

In particular, when $f(x^*) = 0$, we have

$$f(x^*+\delta x) \approx \frac{\partial f}{\partial x}(x^*)\delta x$$

5.3.2 Linearization of systems

Consider a system described by

$$\dot{x} = f(x)$$

and suppose x^e is an equilibrium state, that is,

$$f(x^e) = 0\,.$$

We wish to study the behavior of this system near the equilibrium state x^e. To this end, we introduce the perturbed state

$$\boxed{\delta x := x - x^e}$$

The evolution of the perturbed state is described by

$$\delta\dot{x} = f(x^e + \delta x)$$

When x is 'close' to x^e, that is when δx is 'small',

$$\begin{aligned} f(x^e + \delta x) &\approx f(x^e) + \frac{\partial f}{\partial x}(x^e)\delta x \\ &= \frac{\partial f}{\partial x}(x^e)\delta x \end{aligned}$$

This leads to the following definition:

The linearization of $\dot{x} = f(x)$ about x^e

$$\boxed{\delta\dot{x} = \underbrace{\frac{\partial f}{\partial x}(x^e)}_{A}\,\delta x}$$

where

$$\frac{\partial f}{\partial x}(x^e) := \begin{bmatrix} \frac{\partial f_1}{\partial x_1}(x^e) & \frac{\partial f_1}{\partial x_2}(x^e) & \dots & \frac{\partial f_1}{\partial x_n}(x^e) \\ \frac{\partial f_2}{\partial x_1}(x^e) & \frac{\partial f_2}{\partial x_2}(x^e) & \dots & \frac{\partial f_2}{\partial x_n}(x^e) \\ \vdots & \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1}(x^e) & \frac{\partial f_n}{\partial x_2}(x^e) & \dots & \frac{\partial f_n}{\partial x_n}(x^e) \end{bmatrix}$$

that is,

$$\frac{\partial f}{\partial x}(x^e)_{ij} = \frac{\partial f_i}{\partial x_j}(x^e)$$

• In terms of scalars, the linearization of the system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots, x_n) \\ \dot{x}_2 &= f_2(x_1, x_2, \dots, x_n) \\ &\vdots \\ \dot{x}_n &= f_n(x_1, x_2, \dots, x_n) \end{aligned}$$

about an equilibrium state x^e is given by

$$\boxed{\begin{array}{rcccccccc} \delta\dot{x}_1 &=& \frac{\partial f_1}{\partial x_1}(x^e)\delta x_1 &+& \frac{\partial f_1}{\partial x_2}(x^e)\delta x_2 &+& \ldots &+& \frac{\partial f_1}{\partial x_n}(x^e)\delta x_n \\ \delta\dot{x}_2 &=& \frac{\partial f_2}{\partial x_1}(x^e)\delta x_1 &+& \frac{\partial f_2}{\partial x_2}(x^e)\delta x_2 &+& \ldots &+& \frac{\partial f_2}{\partial x_n}(x^e)\delta x_n \\ &\vdots&&&&&&& \\ \delta\dot{x}_n &=& \frac{\partial f_n}{\partial x_1}(x^e)\delta x_1 &+& \frac{\partial f_n}{\partial x_2}(x^e)\delta x_2 &+& \ldots &+& \frac{\partial f_n}{\partial x_n}(x^e)\delta x_n \end{array}}$$

where

$$\delta x_i = x_i - x_i^e\,, \qquad i = 1, 2, \ldots, n$$

Example 18 (The simple planar pendulum.) Recall

$$\begin{array}{rcl} \dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& -(Wl/J)\sin x_1 - (c/J)x_2 \end{array}$$

Here,

$$\frac{\partial f}{\partial x} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -(Wl/J)\cos x_1 & -c/J \end{bmatrix}$$

Hence, linearization about any equilibrium state $(x_1^e\ ,\ x_2^e)$ results in

$$\delta\dot{x} = A\delta x$$

where

$$A = \begin{bmatrix} 0 & 1 \\ -(Wl/J)\cos x_1^e & -c/J \end{bmatrix}$$

For eqm. state $x^e = (0,\ 0)$ we obtain

$$A = \begin{bmatrix} 0 & 1 \\ -Wl/J & -c/J \end{bmatrix}$$

For eqm. state $x^e = (\pi,\ 0)$ we obtain

$$A = \begin{bmatrix} 0 & 1 \\ Wl/J & -c/J \end{bmatrix}$$

5.3.3 Implicit linearization

The previous section presented a formal definition of the linearizaton of a state space system. This is the way to go when starting from a higher order ODE description.

- It is usually easier to first linearize and then obtain a state space description than vice-versa.

Consider a system described by the single higher order differential equation:

$$F(q, \dot{q}, \ldots, q^{(n)}) = 0\,.$$

Suppose q^e is an equilibrium solution of this equation, that is,

$$F(q^e, 0, \ldots, 0) = 0$$

Let

$$\delta q = q - q^e\,.$$

When $\delta q, \delta\dot{q} \ldots, \delta q^{(n)}$ are small, we have

$$F(q, \dot{q}, \ldots, q^{(n)}) \approx \frac{\partial F}{\partial q}(*)\delta q + \frac{\partial F}{\partial \dot{q}}(*)\delta\dot{q} + \ldots + \frac{\partial F}{\partial q^{(n)}}(*)\delta q^{(n)}$$

where $* = q^e, 0, 0, \ldots, 0$. Hence, we define the linearization of the above system about an equilibrium solution q^e by

$$\boxed{\frac{\partial F}{\partial q}(*)\delta q \quad + \quad \frac{\partial F}{\partial \dot{q}}(*)\delta\dot{q} \quad + \quad \ldots \quad + \quad \frac{\partial F}{\partial q^{(n)}}(*)\delta q^{(n)} \quad = \quad 0}$$

where $* = q^e, 0, 0, \ldots, 0$. This is a higher order linear differential equation with constant coefficients. One can now obtain a state space description by introducing state variables

$$\delta x_1 = \delta q\,, \qquad \delta x_2 = \delta\dot{q}\,, \qquad \ldots \qquad \delta x_n = \delta q^{(n-1)}\,.$$

- From a practical viewpoint, it is usually easier to linearize each term in the equation separately.

To linearize a term, proceed as follows:

Determine which variables the term depends on.
For each variable, differentiate the term with respect to that variable, evaluate the derivative at the equilibrium conditions and multiply the derivative by the perturbation of the variable.
If a term is linear wrt a variable, then simply replace the variable with the perturbation of the variable.

Example 19 (The simple planar pendulum.) This system is described by

$$J\ddot{\theta} + c\dot{\theta} + Wl\sin\theta = 0\,.$$

Physically, it has two different equilibrium solutions: $\theta^e = 0, \pi$. We now linearize about $\theta = \theta^e$. Linearization of the first term yields $J\delta\ddot{\theta}$. Linearization of the second results in $c\delta\dot{\theta}$. Considering the final term we obtain $Wl\cos(\theta^e)\delta\theta$. Hence, linearization of this system about $\theta = \theta^e$ is given by

$$J\delta\ddot{\theta} + c\delta\dot{\theta} + Wl\cos(\theta^e)\delta\theta = 0\,.$$

For $\theta^e = 0$, we have

$$J\delta\ddot{\theta} + c\delta\dot{\theta} + Wl\delta\theta = 0\,.$$

and $\theta^e = \pi$ results in

$$J\delta\ddot{\theta} + c\delta\dot{\theta} - Wl\delta\theta = 0\,.$$

For multiple equations, see some of the next examples.

Example 20 Orbit mechanics

$$\begin{aligned} \ddot{r} - r\omega^2 + \mu/r^2 &= 0 \\ r\dot{\omega} + 2\dot{r}\omega &= 0 \end{aligned}$$

Linearization about equilibrium solutions corresponding to constant values r^e, ω^e of r and ω, respectively, results in:

$$\begin{aligned} \delta\ddot{r} - (\omega^e)^2\delta r - (2r^e\omega^e)\delta\omega - (2\mu/(r^e)^3)\delta r &= 0 \\ \dot{\omega}^e\delta r + r^e\delta\dot{\omega} + 2\omega^e\delta\dot{r} + 2\dot{r}^e\delta\omega &= 0 \end{aligned}$$

Using the relationship

$$(\omega^e)^2 = \mu/(r^e)^3$$

we obtain

$$\begin{aligned} \delta\ddot{r} - 3(\omega^e)^2\delta r - (2r^e\omega^e)\delta\omega &= 0 \\ r^e\delta\dot{\omega} + 2\omega^e\delta\dot{r} &= 0 \end{aligned}$$

Introducing state variables:

$$\delta x_1 := \delta r, \qquad \delta x_2 := \delta\dot{r} \qquad \delta x_3 := \delta\omega$$

we obtain a LTI system with system matrix

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 3(\omega^e)^2 & 0 & 2r^e\omega^e \\ 0 & -2\omega^e/r^e & 0 \end{bmatrix}$$

Example 21 (WC) Consider a water tank where A_t is the crossectional area of the tank; A_o is the crossectional area of the outlet pipe; h is the height of the water about the outlet pipe; ρ is the water density; v is the speed of the water leaving the tank; and q_{in} is the flow rate of the water into the tank.

We have

$$\frac{1}{2}\rho v^2 = \rho g h$$

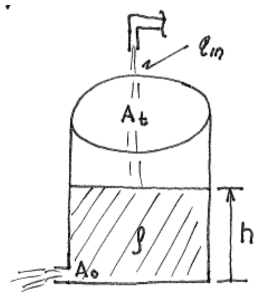


Figure 5.3: Body with symmetry plane in uniform translation

Figure 5.4: WC

and

$$q_{in} = A_o v + A_t \dot{h}\,.$$

Hence, the behavior of the WC can be described by

$$\boxed{A_t \dot{h} = -A_o\sqrt{2gh} + q_{in}} \tag{5.4}$$

Equilibrium values h^e of h are given by

$$0 = -A_o\sqrt{2gh^e} + q_{in}$$

that is,

$$h^e = q_{in}^2/2gA_o^2\,.$$

Linearization about $h = h^e$ results in

$$A_t\,\delta\dot{h} = -\left(A_o g/\sqrt{2gh^e}\right)\,\delta h\,,$$

which can also be written as

$$\boxed{(q_{in}A_t)\,\delta\dot{h} = -(A_o^2 g)\,\delta h}$$

Example 22 (Simple weathercock) Recall that the motion of the simple weathercock is governed by

$$J\ddot{\theta} + \kappa l^2 V\dot{\theta} + \kappa l w V \sin\theta = 0$$

with

$$V = \sqrt{w^2 + 2wl\sin\theta\dot{\theta} + l^2\dot{\theta}^2} \qquad \text{and} \qquad \kappa = \rho S C_D/2\,.$$

We now linearize about $\theta(t) \equiv \theta^e$. Since

$$\begin{aligned}
\frac{\partial(V\dot{\theta})}{\partial\theta}\,|^e &= \frac{\partial V}{\partial\theta}\dot{\theta}\,|^e = 0\\
\frac{\partial(V\dot{\theta})}{\partial\dot{\theta}}\,|^e &= \left[\frac{\partial V}{\partial\dot{\theta}}\dot{\theta} + V\right]^e = w\\
\frac{\partial(V\sin\theta)}{\partial\theta}\,|^e &= \left[\frac{\partial V}{\partial\theta}\sin\theta + V\cos\theta\right]^e = \cos\theta^e w\\
\frac{\partial(V\sin\theta)}{\partial\dot{\theta}}\,|^e &= \frac{\partial V}{\partial\dot{\theta}}\sin\theta\,|^e = 0\,,
\end{aligned}$$

linearization yields

$$J\delta\ddot{\theta} + (\kappa l^2 w)\,\delta\dot{\theta} + (\kappa l w^2 \cos\theta^e)\,\delta\theta = 0\,.$$

For $\theta^e = 0$, we obtain

$$J\delta\ddot{\theta} + (\kappa l^2 w)\,\delta\dot{\theta} + (\kappa l w^2)\,\delta\theta = 0\,.$$

For $\theta^e = \pi$, we have

$$J\delta\ddot{\theta} + (\kappa l^2 w)\,\delta\dot{\theta} - (\kappa l w^2)\,\delta\theta = 0\,.$$

With $\delta x_1 = \delta\theta$ and $\delta x_2 = \delta\dot{\theta}$, the A-matrices for the above two linear systems are respectively given by

$$A = \begin{bmatrix} 0 & 1 \\ -\kappa l w^2/J & -\kappa l^2 w/J \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 0 & 1 \\ \kappa l w^2/J & -\kappa l^2 w/J \end{bmatrix}.$$

Example 23 (Pendulum on cart) Recall that the motion of the pendulum on a cart can be described by

$$\begin{array}{rcrcl} (M+m)\ddot{y} - ml\cos\theta\,\ddot{\theta} & + & ml\sin\theta\dot{\theta}^2 & = & 0 \\ -ml\cos\theta\,\ddot{y} + ml^2\ddot{\theta} & + & mlg\sin\theta & = & 0 \end{array}$$

where M is the mass of the cart, m is the pendulum mass, l is distance from cart to pendulum mass, and g is the gravitational acceleration constant. The variables y and θ are the cart displacement and the pendulum angle, respectively. The equilibrium solutions are characterized by

$$\theta^e = 0 \text{ or } \pi \qquad \text{and} \qquad y^e \text{ is arbitrary.}$$

Linearizing about an equilibrium solution yields:

$$\begin{array}{rcrcl} (M+m)\,\delta\ddot{y} - (mld)\,\delta\ddot{\theta} & & & = & 0 \\ -(mld)\,\delta\ddot{y} + (ml^2)\,\delta\ddot{\theta} & + & (mlgd)\,\delta\theta & = & 0 \end{array}$$

where

$$d = \cos\theta^e = \begin{cases} 1 & \text{if} \quad \theta^e = 0 \\ -1 & \text{if} \quad \theta^e = \pi \end{cases}$$

To obtain a linearized state space description, we first simultaneously solve the above two equations for $\delta\ddot{y}$ and $\delta\ddot{\theta}$ to obtain

$$\begin{array}{rcl} \delta\ddot{y} & = & -\beta g\,\delta\theta \\ \delta\ddot{\theta} & = & -(1+\beta)(dg/l)\,\delta\theta \end{array}$$

where $\beta = m/M$. Letting

$$\delta x_1 = \delta y\,, \quad \delta x_2 = \delta\theta\,, \quad \delta x_3 = \delta\dot{y}\,, \quad \delta x_4 = \delta\dot{\theta}\,,$$

we obtain the state model $\delta\dot{x} = A\delta x$ where

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -\beta g & 0 & 0 \\ 0 & -(1+\beta)(dg/l) & 0 & 0 \end{bmatrix}$$

Considering the parameter values,

$$M = 1\,, \quad m = 1\,, \quad l = 1\,, \quad g = 1,$$

and $\theta^e = 0$, results in

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \end{bmatrix}$$

Checking our answer in MATLAB:

```
 A = linmod('pendcart',[0;0;0;0])
A =
         0   -2.0000         0         0
    1.0000         0         0         0
         0         0         0    1.0000
         0   -1.0000         0         0
```

The difference in our answer and that from MATLAB is due to a different ordering of state variables:

```
 [sizes xo states] = pendcart;
 states
states =
    'pendcart/thetadot'
    'pendcart/theta'
    'pendcart/y'
    'pendcart/ydot'
```

If we order our state variables in the same manner as MATLAB then we obtain the same A matrix as MATLAB.

Example 24 Two-link robotic manipulator

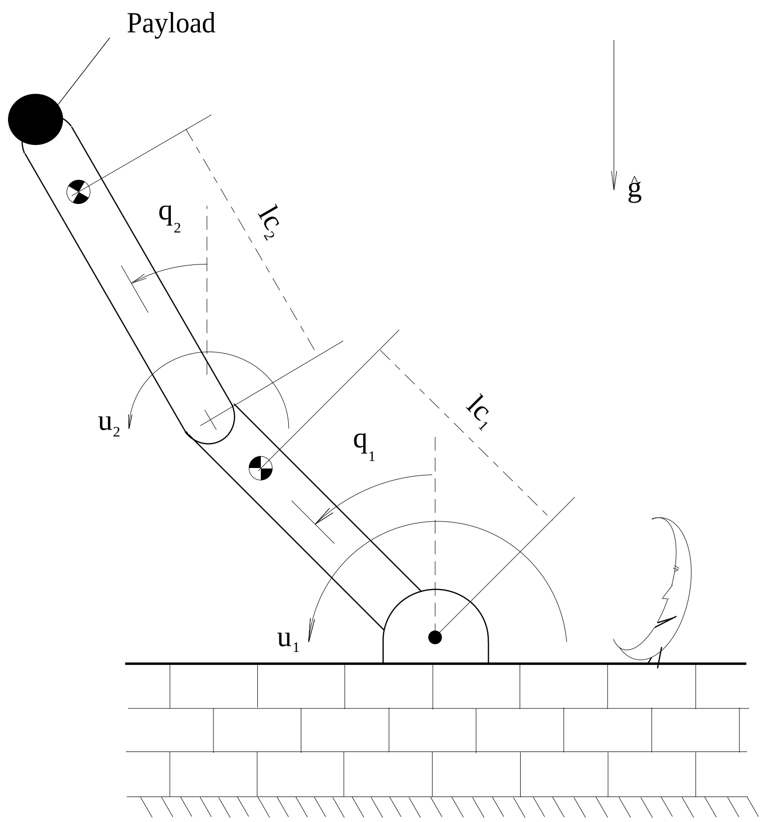


Figure 5.5: A simplified model of a two link manipulator

The coordinates q_1 and q_2 denote the angular location of the first and second links relative to the local vertical, respectively. The second link includes a payload located at its end. The masses of the first and the second links are m_1 and m_2, respectively. The moments of inertia of the first and the second links about their centers of mass are I_1 and I_2, respectively. The locations of the center of mass of links one and two are determined by lc_1 and lc_2, respectively; l_1 is the length of link 1. The equations of motion for the two arms are described by:

$$\begin{aligned}
&[m_1 lc_1^2 + m_2 l_1^2 + I_1]\ddot{q}_1 \quad +[m_2 l_1 lc_2 \cos(q_1 - q_2)]\ddot{q}_2 \quad +m_2 l_1 lc_2 \sin(q_1 - q_2)\dot{q}_2^2 - [m_1 lc_1 + m_2 l_1] g \sin(q_1) = 0 \\
&[m_2 l_1 lc_2 \cos(q_1 - q_2)]\ddot{q}_1 \quad +[m_2 lc_2^2 + I_2]\ddot{q}_2 \quad -m_2 l_1 lc_2 \sin(q_1 - q_2)\dot{q}_1^2 - m_2 g lc_2 \sin(q_2) = 0
\end{aligned}$$

The equilibrium solutions are given by:

Exercises

Exercise 23 Recall the two-degree-of-freedom spring-mass-damper system described by

$$\begin{aligned} m\ddot{q}_1 + C(\dot{q}_1 - \dot{q}_2) + K(q_1 - q_2) + c\dot{q}_1 + kq_1 &= 0 \\ m\ddot{q}_2 + C(\dot{q}_2 - \dot{q}_1) + K(q_2 - q_1) + c\dot{q}_2 + kq_2 &= 0 \end{aligned}$$

Obtain an A-matrix for a state space description of this system.

Exercise 24 Linearize each of the following systems about the zero equilibrium state.

(i)

$$\begin{aligned} \dot{x}_1 &= (1 + x_1^2)x_2 \\ \dot{x}_2 &= -x_1^3 \end{aligned}$$

(ii)

$$\begin{aligned} \dot{x}_1 &= \sin x_2 \\ \dot{x}_2 &= (\cos x_1)x_3 \\ \dot{x}_3 &= e^{x_1}x_2 \end{aligned}$$

Exercise 25

(a) Obtain all equilibrium states of the following system:

$$\begin{aligned} \dot{x}_1 &= 2x_2(1 - x_1) - x_1 \\ \dot{x}_2 &= 3x_1(1 - x_2) - x_2 \end{aligned}$$

(b) Linearize the above system about the zero equilibrium state.

Problem 1

(a) Obtain all equilibrium states of the following system.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= \sin(x_2 - x_1) - x_2 \end{aligned}$$

(b) Linearize the above system about a non-zero equilibrium state.

Exercise 26 For each of the following systems, linearize about each equilibrium solution and obtain the system A-matrix for a state space representation of these linearized systems.

(a)

$$\ddot{y} + (y^2 - 1)\dot{y} + y = 0 \,.$$

where $y(t)$ is a scalar.

(b)

$$\ddot{y} + \dot{y} + y - y^3 = 0$$

where $y(t)$ is a scalar.

(c)

$$\begin{aligned}(M+m)\ddot{y} + ml\,\ddot{\theta}\cos\theta - ml\,\dot{\theta}^2\sin\theta + ky &= 0\\ ml\ddot{y}\cos\theta + ml^2\,\ddot{\theta} + mgl\,\sin\theta &= 0\end{aligned}$$

where $y(t)$ and $\theta(t)$ are scalars.

(d)

$$\ddot{y} + 0.5\dot{y}|\dot{y}| + y = 0\,.$$

where $y(t)$ is a scalar.

Exercise 27

(a) Obtain all equilibrium solutions of the following system:

$$\begin{aligned}(\cos q_1)\ddot{q}_1 + (\sin q_1)\ddot{q}_2 \quad &+ \quad (\sin q_2)\dot{q}_1^2 + \sin q_2 = 0\\ -(\sin q_1)\ddot{q}_1 + (\cos q_1)\ddot{q}_2 \quad &+ \quad (\cos q_2)\dot{q}_2^2 + \sin q_1 = 0\end{aligned}$$

(b) Linearize the above system about its zero solution.

Exercise 28 (Simple pendulum in drag)

Figure 5.6: Pendulum in drag

Recall the simple pendulum in drag whose motion is described by

$$ml\ddot{\theta} + \kappa V(l\dot{\theta} - w\sin\theta) + mg\sin\theta = 0$$

where

$$V = \sqrt{l^2\dot{\theta}^2 + w^2 - 2lw\sin(\theta)\dot{\theta}} \qquad \text{with} \qquad \kappa = \frac{\rho S C_D}{2}$$

and
g is the gravitational acceleration constant of the earth,
S is the reference area associated with the ball,
C_D is the drag coefficient of the ball,
ρ is the air density.

(a) Obtain the equilibrium values θ^e of θ.

(b) Linearize the system about θ^e.

(c) Obtain an expression for the A matrix for a state space representation of the linearized system.

(d) Compare the behavior of the nonlinear system with that of the linearized system for the following parameters.

$$\begin{array}{lll} l = 0.1 \text{ m} & m = 10 \text{ grams} & g = 9.81 \text{ m/s}^2 \\ C_D = 0.2 & S = .01 \text{ m}^2 & \rho = 0.3809 \text{ kg/m}^3 \end{array}$$

Consider the following cases:

θ^e	w (m/s)
0	0
0	5
0	15
0	20
180°	20

Illustrate your results with time histories of $\delta\theta$ in degrees. Comment on your results.

Exercise 29 (a) Obtain all equilibrium states of the following system.

$$\begin{aligned} \dot{x}_1 &= 1 - e^{(x_1 - x_2)} \\ \dot{x}_2 &= x_1 \end{aligned}$$

(b) Linearize the above system about one of the equilibrium states found in part (a).

Chapter 6

Linearization of aircraft longitudinal dynamics

6.1 Linearization

In this chapter, we linearize the equations which describe the longitudinal dynamics of an aircraft. Recall that using the state variables, airspeed (V), angle of attack (α), pitch angle (θ) and pitch rate (q), the longitudinal dynamics of an aircraft can be described by

$$\begin{array}{rcl} m\dot{V} & = & -D - W\sin\gamma + T\cos(\alpha-\epsilon) \\ mV\dot{\alpha} & = & -L + W\cos\gamma - T\sin(\alpha-\epsilon) + mVq \\ \dot{\theta} & = & q \\ J_2\dot{q} & = & M + M_T \end{array} \tag{6.1}$$

where the flight path angle $\gamma = \theta - \alpha$. Suppose we are interested in the behavior of the aircraft around trim conditions corresponding to a constant airspeed V^e, constant angle of attack α^e and a constant pitch angle θ^e, that is,

$$V(t) \equiv V^e\,, \qquad \alpha(t) \equiv \alpha^e\,, \qquad \theta(t) \equiv \theta^e\,, \qquad q(t) \equiv 0\,.$$

Then the following equilibrium conditions must be satisfied:

$$\begin{array}{rcl} -D - W\sin\gamma^e + T\cos(\alpha^e-\epsilon) & = & 0 \\ -L + W\cos\gamma^e - T\sin(\alpha^e-\epsilon) & = & 0 \\ M + M_T & = & 0 \end{array}$$

where $\gamma^e = \theta^e - \alpha^e$. Introducing the perturbed variables:

$$\delta V := V - V^e\,, \qquad \delta\alpha := \alpha - \alpha^e\,, \qquad \delta\theta := \theta - \theta^e\,, \qquad \delta q := q\,,$$

linearization of the nonlinear differential equations (6.1) about the equilibrium conditions yields:

$$m\delta\dot{V} \;=\; -\frac{\partial D}{\partial V}\delta V - \frac{\partial D}{\partial \alpha}\delta\alpha - W\cos\gamma^e\delta\gamma + \frac{\partial T}{\partial V}\cos(\alpha^e-\epsilon)\delta V - T\sin(\alpha^e-\epsilon)\delta\alpha$$

$$
\begin{aligned}
mV^e\delta\dot{\alpha} &= -\frac{\partial L}{\partial V}\delta V - \frac{\partial L}{\partial \alpha}\delta\alpha - \frac{\partial L}{\partial q}\delta q - \frac{\partial L}{\partial \dot{\alpha}}\delta\dot{\alpha} - W\sin\gamma^e\delta\gamma - \frac{\partial T}{\partial V}\sin(\alpha^e-\epsilon)\delta V - T\cos(\alpha^e-\epsilon)\delta\alpha + mV^e\delta q \\
\delta\dot{\theta} &= \delta q \\
J_2\delta\dot{q} &= \frac{\partial M}{\partial V}\delta V + \frac{\partial M}{\partial \alpha}\delta\alpha + \frac{\partial M}{\partial q}\delta q + \frac{\partial M}{\partial \dot{\alpha}}\delta\dot{\alpha} + \frac{\partial M_T}{\partial V}\delta V
\end{aligned}
$$

where $\delta\gamma = \delta\theta - \delta\alpha$. Since the equilibrium conditions imply that

$$
\begin{aligned}
W\sin\gamma^e - T\cos(\alpha^e-\epsilon) &= -D \\
W\cos\gamma^e - T\sin(\alpha^e-\epsilon) &= L\,,
\end{aligned}
$$

we can rewrite the linearization as follows:

$$\begin{array}{rcl} \delta\dot{V} & = & (X_V + T_V\cos(\alpha^e - \epsilon))\,\delta V + X_\alpha \delta\alpha - g\cos\gamma^e \delta\theta \\ (V^e - Z_{\dot{\alpha}})\,\delta\dot{\alpha} & = & (Z_V - T_V\sin(\alpha^e - \epsilon))\,\delta V + Z_\alpha \delta\alpha - g\sin\gamma^e \delta\theta + (V^e + Z_q)q \\ \delta\dot{\theta} & = & \delta q \\ -M_{\dot{\alpha}}\delta\dot{\alpha} + \delta\dot{q} & = & (M_V + M_{T_V})\delta V + M_\alpha \delta\alpha + M_q \delta q \end{array}$$

where

$$T_V := \frac{1}{m}\frac{\partial T}{\partial V} \qquad \text{and} \qquad M_{T_V} := \frac{1}{J_2}\frac{\partial M_T}{\partial V}$$

while the **aerodynamic dimensional derivatives** are given by

$$\begin{array}{llll} X_V := -\dfrac{1}{m}\dfrac{\partial D}{\partial V} & X_\alpha := \dfrac{1}{m}\left(L - \dfrac{\partial D}{\partial \alpha}\right) & & \\ Z_V := -\dfrac{1}{m}\dfrac{\partial L}{\partial V} & Z_\alpha := -\dfrac{1}{m}\left(D + \dfrac{\partial L}{\partial \alpha}\right) & Z_q := -\dfrac{1}{m}\dfrac{\partial L}{\partial q} & Z_{\dot{\alpha}} := -\dfrac{1}{m}\dfrac{\partial L}{\partial \dot{\alpha}} \\ M_V := \dfrac{1}{J_2}\dfrac{\partial M}{\partial V} & M_\alpha := \dfrac{1}{J_2}\dfrac{\partial M}{\partial \alpha} & M_q := \dfrac{1}{J_2}\dfrac{\partial M}{\partial q} & M_{\dot{\alpha}} := \dfrac{1}{J_2}\dfrac{\partial M}{\partial \dot{\alpha}} \end{array}$$

Figure 6.1: Aerodynamic dimensional derivatives

Introducing the perturbed state vector

$$\delta x := \begin{bmatrix} \delta V \\ \delta\alpha \\ \delta\theta \\ \delta q \end{bmatrix}$$

the linearization can be written as $E\delta\dot{x} = \tilde{A}\delta x$ where

$$E := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & V^e - Z_{\dot{\alpha}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -M_{\dot{\alpha}} & 0 & 1 \end{bmatrix} \quad \text{and} \quad \tilde{A} := \begin{bmatrix} X_V + T_V\cos(\alpha^e - \epsilon) & X_\alpha & -g\cos\gamma^e & 0 \\ Z_V - T_V\sin(\alpha^e - \epsilon) & Z_\alpha & -g\sin\gamma^e & V^e + Z_q \\ 0 & 0 & 0 & 1 \\ M_V + M_{T_V} & M_\alpha & 0 & M_q \end{bmatrix}$$

Putting this linearization in standard form, we obtain

$$\boxed{\delta\dot{x} = A\delta x \qquad \text{with} \qquad A := E^{-1}\tilde{A}}$$

6.2 Dimensionless aerodynamic derivatives

Quite often, data for the linearized dynamics of an aircraft is specified in terms of dimensionless derivatives. The derivatives of the aerodynamic coefficients are usually made dimensionless in the following fashion.

	V	α	q	$\dot{\alpha}$
C_D	$C_{D_V} := V\dfrac{\partial C_D}{\partial V}$	$C_{D_\alpha} := \dfrac{\partial C_D}{\partial \alpha}$		
C_L	$C_{L_V} := V\dfrac{\partial C_L}{\partial V}$	$C_{L_\alpha} := \dfrac{\partial C_L}{\partial \alpha}$	$C_{L_q} := \dfrac{2V}{\bar{c}}\dfrac{\partial C_L}{\partial q}$	$C_{L_{\dot{\alpha}}} := \dfrac{2V}{\bar{c}}\dfrac{\partial C_L}{\partial \dot{\alpha}}$
C_M	$C_{M_V} := V\dfrac{\partial C_M}{\partial V}$	$C_{M_\alpha} := \dfrac{\partial C_M}{\partial \alpha}$	$C_{M_q} := \dfrac{2V}{\bar{c}}\dfrac{\partial C_M}{\partial q}$	$C_{M_{\dot{\alpha}}} := \dfrac{2V}{\bar{c}}\dfrac{\partial C_M}{\partial \dot{\alpha}}$

Figure 6.2: Aerodynamic dimensionless derivatives

Note that, in our model of the Cessna 182 and the transport aircraft, C_D, C_L and C_M were independent of V, hence C_{D_V}, C_{L_V} and C_{M_V} are all zero.

6.2.1 Dimensional derivatives in terms of dimensionless derivatives

Quite often, data for the linearized dynamics of an aircraft is specified in terms of the above dimensionless derivatives. Hence we need to relate the dimensional derivatives which appear in our linearization to the dimensionless derivatives. These relationships are given in the following table. To illustrate how these relationships are obtained, we will compute the expression for X_V. Recall that

$$L = \bar{q}SC_L\,, \qquad D = \bar{q}SC_D\,, \qquad M = \bar{q}S\bar{c}C_M\,,$$

where

$$\bar{q} = \frac{1}{2}\rho V^2\,.$$

Hence

$$\frac{\partial \bar{q}}{\partial V} = \rho V = \frac{2\bar{q}}{V}$$

and

$$\begin{aligned} X_V &= -\frac{1}{m}\frac{\partial D}{\partial V} \\ &= -\frac{1}{m}\left[\frac{\partial \bar{q}}{\partial V} S C_D + \bar{q} S \frac{\partial C_D}{\partial V}\right] \\ &= -\frac{1}{m}\left[\frac{2\bar{q}}{V} S C_D + \frac{\bar{q} S}{V} C_{D_V}\right] \\ &= -\frac{\bar{q} S}{m V}\left[2 C_D + C_{D_V}\right] \end{aligned}$$

$$X_V = -\frac{\bar{q}S}{mV}(2C_D + C_{D_V})$$

$$X_\alpha = \frac{\bar{q}S}{m}(C_L - C_{D_\alpha})$$

$$Z_V = -\frac{\bar{q}S}{mV}(2C_L + C_{L_V})$$

$$Z_\alpha = -\frac{\bar{q}S}{m}(C_D + C_{L_\alpha})$$

$$Z_q = -\frac{\bar{q}S\bar{c}}{2mV}C_{L_q}$$

$$Z_{\dot{\alpha}} = -\frac{\bar{q}S\bar{c}}{2mV}C_{L_{\dot{\alpha}}}$$

$$M_V = \frac{\bar{q}S\bar{c}}{J_2 V}(2C_M + C_{M_V})$$

$$M_\alpha = \frac{\bar{q}S\bar{c}}{J_2}C_{M_\alpha}$$

$$M_q = \frac{\bar{q}S\bar{c}^2}{2J_2 V}C_{M_q}$$

$$M_{\dot{\alpha}} = \frac{\bar{q}S\bar{c}^2}{2J_2 V}C_{M_{\dot{\alpha}}}$$

Figure 6.3: Dimensional derivatives and dimensionless derivatives

6.3 Phugoid motion approximation

Recall

$$\begin{aligned}
\delta\dot{V} &= (X_V + T_V\cos(\alpha^e - \epsilon))\,\delta V + X_\alpha\delta\alpha - g\cos\gamma^e\delta\theta && \text{(6.2a)}\\
(V^e - Z_{\dot{\alpha}})\,\delta\dot{\alpha} &= (Z_V - T_V\sin(\alpha^e - \epsilon))\,\delta V + Z_\alpha\delta\alpha - g\sin\gamma^e\delta\theta + (V^e + Z_q)\delta q && \text{(6.2b)}\\
\delta\dot{\theta} &= \delta q && \text{(6.2c)}\\
-M_{\dot{\alpha}}\delta\dot{\alpha} + \delta\dot{q} &= (M_V + M_{T_V})\delta V + M_\alpha\delta\alpha + M_q\delta q && \text{(6.2d)}
\end{aligned}$$

Let

$$M_\alpha = \epsilon^{-2}, \qquad \delta\tilde{\alpha} = \epsilon^{-1}\delta\alpha$$

Then $\delta\alpha = \epsilon\delta\tilde{\alpha}$ and second and fourth differential equations become

$$\begin{aligned}
\epsilon(V^e - Z_{\dot{\alpha}})\,\delta\dot{\tilde{\alpha}} &= (Z_V - T_V\sin(\alpha^e - \epsilon))\,\delta V + \epsilon Z_\alpha\delta\tilde{\alpha} - g\sin\gamma^e\delta\theta + (V^e + Z_q)\delta q && \text{(6.3)}\\
\epsilon\,(-\epsilon M_{\dot{\alpha}}\delta\dot{\alpha} + \delta\dot{q}) &= \epsilon(M_V + M_{T_V})\delta V + \delta\tilde{\alpha} + \epsilon M_q\delta q && \text{(6.4)}
\end{aligned}$$

Considering $\epsilon = 0$, these equations result in $\delta\tilde{\alpha} = 0$ and

$$(Z_V - T_V\sin(\alpha^e - \epsilon))\,\delta V - g\sin\gamma^e\delta\theta + (V^e + Z_q)\delta q = 0$$

Also $\delta\alpha = 0$. Recalling equations (6.2a) and (6.2c), we now obtain a second order system

$$\begin{aligned}
\delta\dot{V} &= (X_V + T_V\cos(\alpha^e - \epsilon))\,\delta V - g\cos\gamma^e\delta\theta && \text{(6.5a)}\\
(V^e + Z_q)\delta\dot{\theta} &= -(Z_V - T_V\sin(\alpha^e - \epsilon))\,\delta V + g\sin\gamma^e\delta\theta && \text{(6.5b)}
\end{aligned}$$

Exercises

Exercise 30 Compute the aerodynamic dimensionless derivatives and the thrust derivatives for the longitudinal dynamics of the transport aircraft given in the notes. Consider two cases:

(a)

$$x = -0.25\bar{c} \qquad th = 0.3$$

and the following trim conditions

$$el = -1.305 \text{ deg} \qquad V^e = 506 \text{ ft/sec} \qquad \alpha^e = 0.53 \text{ deg} \qquad \gamma^e = 0$$

(b)

$$x = 0.2\bar{c} \qquad th = 0.3$$

and the corresponding equilibrium conditions for level flight.

Exercise 31 Write a MATLAB program which computes the system A matrix for the linearization of the V, α, θ, q representation of aircraft longitudinal dynamics. The input data to the program consists of the aerodynamic dimensionless derivatives, the thrust derivatives, the trim conditions, and the general aircraft parameters. Use the program to compute A for the aircraft and the two sets of trim conditions of Exercise 30.

Exercise 32 Compare the behavior of the nonlinear and the linearized models of the transport aircraft of Exercise 30. For each of the two sets of conditions given in Exercise 30, simulate both models in MATLAB and compare the responses of $\delta V, \delta\alpha, \delta\theta$ for initial conditions close to equilibrium and not so close to equilibrium.

Chapter 7

Systems with inputs

Here we consider systems subject to inputs from sources external to the system. In general, one divides inputs into control inputs and disturbance inputs. In an aircraft, control inputs are the deflections of the control surfaces (elevator, ailerons, rudder) and the throttle setting. An example of a disturbance input is wind gust. In general, the control inputs to a system are those inputs one can use to influence system behavior to ones liking. Disturbance inputs also effect system behavior, but, one has no control over them.

7.1 Examples

Examples abound.

Example 25 (Controlled pendulum or single link manipulator) A controlled pendulum or single link manipulator can be described by

$$J\ddot{\theta} + Wl\sin\theta = u\,,$$

where the control input u is a torque applied to the pendulum. Introducing state variables $x_1 = \theta$ and $x_2 = \dot{\theta}$, this system has the following state space description:

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= (Wl/J)\sin x_1 + u/J\,.\end{aligned}$$

7.2 General description

A very general state space description of a system with inputs is as follows:

$$\begin{aligned}\dot{x}_1 &= F_1(x_1,\ldots,x_n,\,u_1,\ldots,u_m)\\ \dot{x}_2 &= F_2(x_1,\ldots,x_n,\,u_1,\ldots,u_m)\\ &\vdots\\ \dot{x}_n &= F_n(x_1,\ldots,x_n,\,u_1,\ldots,u_m)\end{aligned}$$

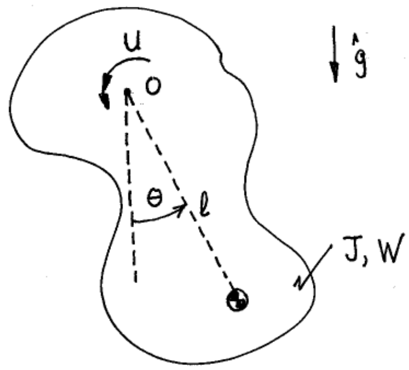


Figure 7.1: Single link manipulator

where the scalar variables $x_1(t), \ldots, x_n(t)$ are the state variables, the scalar variables $u_1(t), \ldots, u_m(t)$ are the input variables, and the scalar variable t represents 'time'. If we introduce the state (vector) and the input (vector),

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \text{and} \qquad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix},$$

respectively, then, in vector form, our general state space description of a system is given by

$$\boxed{\dot{x} = F(x, u)}$$

In this section, we are "lumping" all inputs into the single vector u.

7.3 Linear systems and linearization

7.3.1 Linear systems

Linear time-invariant (LTI) systems with n state variables and m input variables can be described by

$$\boxed{\dot{x} = Ax + Bu} \tag{7.1}$$

where A is a constant $n \times n$ matrix and B is a constant $n \times m$ matrix.

Example 26 (B&B)

$$\begin{aligned} \ddot{q}_1 + k(q_1 - q_2) &= u \\ \ddot{q}_2 - k(q_1 - q_2) &= 0 \end{aligned}$$

Figure 7.2: B&B

Introducing the usual state variables,

$$x_1 = q_1\,, \qquad x_2 = q_2\,, \qquad x_3 = \dot{q}_1\,, \qquad x_4 = \dot{q}_2\,,$$

we obtain the state equations:

$$\begin{aligned}
\dot{x}_1 &= x_3 \\
\dot{x}_2 &= x_4 \\
\dot{x}_3 &= -kx_1 + kx_2 + u \\
\dot{x}_4 &= kx_1 - kx_2
\end{aligned}$$

Here,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k & k & 0 & 0 \\ k & -k & 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

7.3.2 Linearization

Controlled equilibrium states

Consider a system described by $\dot{x} = F(x, u)$ and suppose the input is constant and equal to u^e, that is, $u(t) \equiv u^e$. Then the resulting system is described by

$$\dot{x}(t) = F(x(t), u^e)$$

The equilibrium states x^e of this system are given by $F(x^e, u^e) = 0$. This leads to the following definition.

Defn. *A state x^e is a* controlled equilibrium state *or* trim state *if there exists a constant input u^e such that*

$$\boxed{F(x^e, u^e) = 0}$$

Example 27 Single link manipulator

$$\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -(Wl/J)\sin(x_1) + u/J
\end{aligned}$$

Any state of the form

$$x^e = \begin{bmatrix} x_1^e \\ 0 \end{bmatrix}$$

is a controlled equilibrium state. The corresponding constant input is $u^e = Wl\sin(x_1^e)$. Note that, if $|u^e| > Wl$, this system does not have an equilibrium state.

For LTI systems, controlled equilibrium states satisfy

$$Ax^e + Bu^e = 0\,.$$

If A is invertible then,

$$x^e = -A^{-1}Bu^e$$

Linearization

Suppose x^e is a controlled equilibrium state corresponding to a constant input u^e. So, if $u(t) \equiv u^e$ and $x(0) = x^e$ then $x(t)$ will remain at its equilibrium value x^e for all time. We now wish to study the behavior of the system due to small perturbations in $x(0)$ and u from x^e and u^e, respectively. To this end, we introduce the perturbed state

$$\delta x := x - x^e$$

and the perturbed input

$$\delta u := u - u^e\,.$$

Since x^e is constant, we have $\delta\dot{x} = \dot{x} = F(x,u)$. Also $x = x^e + \delta x$ and $u = u^e + \delta u$. Hence

$$\delta\dot{x} = F(x^e + \delta x,\ u^e + \delta u)$$

Define

$$\frac{\partial F}{\partial x}(x^e,u^e) \quad := \quad \left[\frac{\partial F_i}{\partial x_j}(*)\right] \quad = \quad \begin{bmatrix} \frac{\partial F_1}{\partial x_1}(*) & \frac{\partial F_1}{\partial x_2}(*) & \dots & \frac{\partial F_1}{\partial x_n}(*) \\ \frac{\partial F_2}{\partial x_1}(*) & \frac{\partial F_2}{\partial x_2}(*) & \dots & \frac{\partial F_2}{\partial x_n}(*) \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial x_1}(*) & \frac{\partial F_n}{\partial x_2}(*) & \dots & \frac{\partial F_n}{\partial x_n}(*) \end{bmatrix}$$

$$\frac{\partial F}{\partial u}(x^e,u^e) \quad := \quad \left[\frac{\partial F_i}{\partial u_j}(*)\right] \quad = \quad \begin{bmatrix} \frac{\partial F_1}{\partial u_1}(*) & \frac{\partial F_1}{\partial u_2}(*) & \dots & \frac{\partial F_1}{\partial u_m}(*) \\ \frac{\partial F_2}{\partial u_1}(*) & \frac{\partial F_2}{\partial u_2}(*) & \dots & \frac{\partial F_2}{\partial u_m}(*) \\ \vdots & \vdots & & \vdots \\ \frac{\partial F_n}{\partial u_1}(*) & \frac{\partial F_n}{\partial u_2}(*) & \dots & \frac{\partial F_n}{\partial u_m}(*) \end{bmatrix}$$

where $* = (x^e, u^e)$. When x is 'close' to x^e and u is 'close' to u^e, that is, when δx and δu are 'small',

$$\begin{aligned} F(x^e + \delta x, u^e + \delta u) &\approx F(x^e, u^e) + \frac{\partial F}{\partial x}(x^e, u^e)\delta x + \frac{\partial F}{\partial u}(x^e, u^e)\delta u \\ &= \frac{\partial F}{\partial x}(x^e, u^e)\delta x + \frac{\partial F}{\partial u}(x^e, u^e)\delta u \end{aligned}$$

Hence,

$$\delta \dot{x} \approx \frac{\partial F}{\partial x}(x^e, u^e)\delta x + \frac{\partial F}{\partial u}(x^e, u^e)\delta u \,.$$

This leads to the following definition.

The linearization of $\dot{x} = F(x, u)$ about (x^e, u^e):

$$\boxed{\delta \dot{x} = A\,\delta x + B\,\delta u} \tag{7.2}$$

where

$$A = \frac{\partial F}{\partial x}(x^e, u^e) \qquad B = \frac{\partial F}{\partial u}(x^e, u^e)$$

Example 28 (Single link manipulator) The linearization of this system about any (x^e, u^e) is given by (7.2) with

$$A = \begin{bmatrix} 0 & 1 \\ -(Wl/J)\cos(x_1^e) & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1/J \end{bmatrix}$$

7.3.3 Term by term linearization

As we have seen before, it is usually easier to linearize higher order differential equations directly and term by term.

Example 29

$$(1 + y^2)\ddot{y} + (2 + u)\dot{y} + \cos(y)\sin(u) = 0\,.$$

7.4 Linearized longitudinal dynamics of an aircraft

Recall that using the state variables, airspeed (V), angle of attack (α), pitch angle (θ) and pitch rate (q), the longitudinal dynamics of an aircraft can be described by

$$\begin{aligned} m\dot{V} &= -D - W\sin\gamma + T\cos(\alpha - \epsilon) \\ mV\dot{\alpha} &= -L + W\cos\gamma - T\sin(\alpha - \epsilon) + mVq \\ \dot{\theta} &= q \\ J_2\dot{q} &= M + M_T \end{aligned} \tag{7.3}$$

where the flight path angle $\gamma = \theta - \alpha$. Neglecting disturbance inputs such as wind gusts, the longitudinal dynamics have two inputs, the control inputs:

$$\textsf{throttle setting} : th \qquad \text{and} \qquad \textsf{elevator deflection} : el$$

Suppose we are interested in the behavior of the aircraft around equilibrium conditions corresponding to a constant throttle setting th^e and a constant elevator setting el^e. Let V^e, α^e and θ^e be the corresponding equilibrium values of airspeed, angle of attack, and pitch angle. Of course, the corresponding equilibrium value of pitch rate is zero.

Introducing the perturbed variables:

$$\delta V = V - V^e\,, \qquad \delta\alpha = \alpha - \alpha^e\,, \qquad \delta\theta = \theta - \theta^e\,, \qquad \delta q = q\,,$$

and

$$\delta el = el - el^e\,, \qquad \delta th = th - th^e\,,$$

linearization of the nonlinear differential equations (7.3) about the equilibrium conditions yields:

$$\begin{aligned} m\delta\dot{V} &= \cdots\cdots \quad -\frac{\partial D}{\partial el}\,\delta el + \frac{\partial T}{\partial th}\cos(\alpha^e - \epsilon)\,\delta th \\ mV^e\delta\dot{\alpha} &= \cdots\cdots \quad -\frac{\partial L}{\partial el}\,\delta el - \frac{\partial T}{\partial th}\sin(\alpha^e - \epsilon)\,\delta th \\ \delta\dot{\theta} &= \cdots\cdots \\ J_2\delta\dot{q} &= \cdots\cdots \quad \frac{\partial M}{\partial el}\,\delta el + \frac{\partial M_T}{\partial th}\,\delta th \end{aligned}$$

The terms indicated by $\cdots\cdots$ are the right hand side terms we have obtained before.

We now introduce the following aerodynamic dimensional derivatives:

$$\boxed{X_{el} := -\frac{1}{m}\frac{\partial D}{\partial el} \qquad Z_{el} := -\frac{1}{m}\frac{\partial L}{\partial el} \qquad M_{el} := \frac{1}{J_2}\frac{\partial M}{\partial el}}$$

and let

$$\boxed{T_{th} := \frac{1}{m}\frac{\partial T}{\partial th} \qquad M_{T_{th}} := \frac{1}{J_2}\frac{\partial M_T}{\partial th}}$$

Note that, in the transport aircraft example, X_{el} and Z_{el} are both zero.

Introducing the perturbed state and input vectors,

$$\delta x := \begin{bmatrix} \delta V \\ \delta\alpha \\ \delta\theta \\ \delta q \end{bmatrix} \qquad \text{and} \qquad \delta u := \begin{bmatrix} \delta th \\ \delta el \end{bmatrix}$$

the linearization can be written as

$$E\delta\dot{x} = \tilde{A}\delta x + \tilde{B}\delta u \tag{7.4}$$

where the matrices E and $\tilde{A}$ are as defined previously while

$$\tilde{B} := \begin{bmatrix} T_{th}\cos(\alpha^e - \epsilon) & X_{el} \\ -T_{th}\sin(\alpha^e - \epsilon) & Z_{el} \\ 0 & 0 \\ M_{T_{th}} & M_{el} \end{bmatrix} \tag{7.5}$$

Putting this linearization in standard form, we obtain

$$\delta\dot{x} = A\delta x + B\delta u \qquad \text{where} \qquad A := E^{-1}\tilde{A} \quad \text{and} \quad B := E^{-1}\tilde{B}\,. \tag{7.6}$$

Chapter 8

Systems with inputs and outputs

$$\begin{array}{ccc} \text{OUTPUTS} & & \text{INPUTS} \\ \longleftarrow & \boxed{\text{SYSTEM}} & \longleftarrow \end{array}$$

Disturbance inputs, control inputs

Measured outputs; performance outputs

8.1 Examples

Examples abound.

8.2 General description

A very general state space description of an input-output system is given by

$$\dot{x} = F(x, u) \tag{8.1a}$$

$$y = H(x, u) \tag{8.1b}$$

where t is 'time', the m-vector $u(t)$ is the input at time t, the p-vector $y(t)$ is the output at time t and the n-vector $x(t)$ is the system state at time t. The functions F and H describe the system under consideration. In this section, we "lump" all the inputs into u and all the outputs into y.

8.3 Linear systems and linearization

8.3.1 Linear systems

All finite-dimensional, linear, time invariant, input-output systems can be described in state space form by

$$y \longleftarrow \boxed{\begin{array}{rcl} \dot{x} &=& Ax + Bu \\ y &=& Cx + Du \end{array}} \longleftarrow u$$

where A, B, C, D are matrices of appropriate dimensions.

8.3.2 Linearization

Recall the concept of a controlled equilibrium state for system (8.1a). We say that an output y^e is a controlled equilibrium output for system (8.1) if there is a constant input u^e so that the equations

$$\begin{aligned} F(x^e, u^e) &= 0 \\ H(x^e, u^e) &= y^e \end{aligned}$$

have a solution for some controlled equilibrium state x^e.

Linearization about (x^e, u^e): Suppose x^e is controlled equilibrium state for system (8.1a) and u^e is the corresponding input, that is, $F(x^e, u^e) = 0$. Then, the linearization of system (8.1) about (x^e, u^e) is given by

$$\begin{aligned} \delta\dot{x} &= A\,\delta x + B\,\delta u \\ \delta y &= C\,\delta x + D\,\delta u \end{aligned}$$

where the matrices A, B, C, D are given by

$$A = \frac{\partial F}{\partial x}(x^e, u^e) \qquad B = \frac{\partial F}{\partial u}(x^e, u^e) \qquad C = \frac{\partial H}{\partial x}(x^e, u^e) \qquad D = \frac{\partial H}{\partial u}(x^e, u^e)$$

Exercises

Exercise 33 Consider an input-output system with input u and output y described by

$$\begin{aligned} 2\ddot{y} - \cos(\theta)\,\ddot{\theta} \quad &+ \quad \sin(\theta)\,\dot{\theta}^2 &= u \\ -\cos(\theta)\,\ddot{y} + \ddot{\theta} \quad &+ \quad \sin(\theta) &= 0 \end{aligned}$$

where θ is another time dependent variable.

(a) Linearize (by hand) this system about equilibrium conditions corresponding to $u^e = 0$, $y^e = 1$ and $\theta^e = \pi$.

(b) Obtain the A, B, C, D matrices for a state space representation of the linearization found in part (a).

8.4 ⟵ Transfer functions ⟵

Here we look at another way to describe a LTI (linear time invariant) input-output system. A state space description of a system is a time-domain description. Here we look at what is sometimes called a frequency domain description, the transfer function. This is based on the Laplace Transform.

Laplace transform. Consider any signal (function of time) f. The Laplace Transform of f, which we denote by $\mathcal{L}(f)$ or $\hat{f}$ is a function of a complex variable s and is given by

$$\hat{f}(s) = \int_0^\infty e^{-st} f(t)\, dt$$

In the above definition, $f(t)$ could be a scalar, a vector or a matrix.

The following table provides the Laplace transforms of commmonly encountered functions which arise in the behavior of LTI systems.

$f(t)$	$\hat{f}(s)$
$e^{\alpha t}$	$\frac{1}{s-\alpha}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$te^{\alpha t}$	$\frac{1}{(s-\alpha)^2}$

We also have the following useful relationship.

$$\mathcal{L}(\dot{f})(s) = s\hat{f}(s) - f(0) \tag{8.2}$$

Transfer function matrix. Consider a single-input single-output (SISO) system with input u and output y. Its transfer function $\hat{G}$ is a function of a complex variable s and is defined by

$$\hat{G}(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$$

where all system initial conditions are assumed to be zero. Note that, because the system is linear, the above ratio is independent of the input. For a nonlinear system, the above ratio would depend on the input; so, we cannot define a transfer function for a nonlinear system. We can rewrite the above relationship as

$$\boxed{\hat{y}(s) = \hat{G}(s)\hat{u}(s)} \tag{8.3}$$

The above relationship allows us to define transfer functions for multi-input multi-output (MIMO) systems. We say that $\hat{G}$ is a transfer function (matrix) for a linear system if the above relationship holds for all inputs and zero initial conditions. Note that, for a MIMO system, $\hat{G}$ is a $p \times m$ matrix where p is the number of input components and m is the number of output components.

8.4.1 Transfer function from state space description

Taking the Laplace transform of

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\\ \\ x(0) &= x_0\end{aligned}$$

yields

$$\begin{aligned}s\hat{x}(s) - x_0 &= A\hat{x}(s) + B\hat{u}(s)\\ \hat{y}(s) &= C\hat{x}(s) + D\hat{u}(s)\end{aligned}$$

where $\hat{x}, \hat{y}, \hat{u}$ are the Laplace transforms of x, y, u, that is, $\hat{x} = \mathcal{L}(x)$ etc. Solving these equations for $\hat{y}$ in terms of x_0 and $\hat{u}$ yields

$$\hat{y}(s) = C(sI - A)^{-1}x_0 \quad + \quad \hat{G}(s)\hat{u}(s)$$

where the transfer function matrix $\hat{G}(s)$ is defined by

$$\boxed{\hat{G}(s) = C(sI - A)^{-1}B + D}$$

Zero initial state response: $x(0) = 0$. Here, the relationship between the Laplace transform of the input $\hat{u}$ and the Laplace transform of the output $\hat{y}$ is given by

$$\boxed{\hat{y}(s) = \hat{G}(s)\hat{u}(s)}$$

Sometimes this is represented by

$$\hat{y} \longleftarrow \boxed{\hat{G}} \longleftarrow \hat{u}$$

Example 30

$$\begin{aligned}\dot{x}(t) &= -x(t) + u(t)\\ y(t) &= x(t)\end{aligned}$$

Here

$$\hat{G}(s) = C(sI - A)^{-1}B + D = \frac{1}{s+1}$$

Example 31 Unattached mass

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= u\\ y &= x_1\end{aligned}$$

Here

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad D = 0$$

Hence

$$\begin{aligned} \hat{G}(s) &= C(sI - A)^{-1}B \\ &= \frac{1}{s^2} \end{aligned}$$

Example 32 (MIMO (Multi-input multi-output) B&B.) Here we obtain the transfer function matrix for a MIMO (multi-input multi-output) system. The system in Figure 8.1 is described by

$$\begin{aligned} m\ddot{q}_1 &= k(q_2 - q_1) + u_1 \\ m\ddot{q}_2 &= -k(q_2 - q_1) + u_2 \\ y_1 &= q_1 \\ y_2 &= q_2 \end{aligned}$$

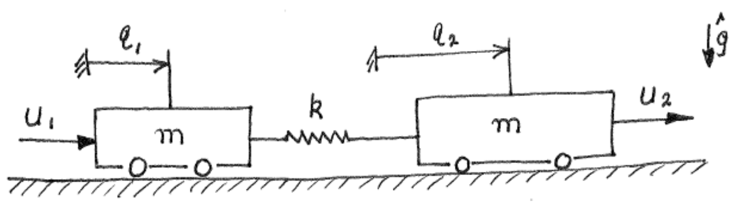


Figure 8.1: (Multi-input multi-output) B&B

With $x_1 = q_1, x_2 = q_2, x_3 = \dot{q}_1, x_4 = \dot{q}_2$, we have

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m & k/m & 0 & 0 \\ k/m & -k/m & 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1/m & 0 \\ 0 & 1/m \end{pmatrix}$$

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad D = 0$$

Hence

$$\begin{aligned} \hat{G}(s) &= C(sI - A)^{-1}B + D \\ &= \begin{pmatrix} \frac{ms^2+k}{ms^2(ms^2+2k)} & \frac{k}{ms^2(ms^2+2k)} \\ \frac{k}{ms^2(ms^2+2k)} & \frac{ms^2+k}{ms^2(ms^2+2k)} \end{pmatrix} \end{aligned}$$

• Here we present a more convenient way to compute transfer functions by hand. Taking the Laplace transform of the original second order differential equations with zero initial conditions, we get

$$\begin{array}{rcl} ms^2\hat{q}_1 &=& -k(\hat{q}_1-\hat{q}_2) \;+\; \hat{u}_1 \\ ms^2\hat{q}_2 &=& k(\hat{q}_1-\hat{q}_2) \;+\; \hat{u}_2 \\ \hat{y}_1 &=& \hat{q}_1 \\ \hat{y}_2 &=& \hat{q}_2 \end{array}$$

hence,

$$\begin{pmatrix} ms^2+k & -k \\ -k & ms^2+k \end{pmatrix}\begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix} = \begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix}$$

Solving, yields

$$\begin{array}{rcl} \hat{y} = \begin{pmatrix} \hat{q}_1 \\ \hat{q}_2 \end{pmatrix} &=& \dfrac{1}{\Delta(s)}\begin{pmatrix} ms^2+k & k \\ k & ms^2+k \end{pmatrix}\begin{pmatrix} \hat{u}_1 \\ \hat{u}_2 \end{pmatrix} \\ \Delta(s) &=& ms^2(ms^2+2k) \end{array}$$

So,

$$\hat{G}(s) = \begin{pmatrix} \frac{ms^2+k}{ms^2(ms^2+2k)} & \frac{k}{ms^2(ms^2+2k)} \\ \\ \frac{k}{ms^2(ms^2+2k)} & \frac{ms^2+k}{ms^2(ms^2+2k)} \end{pmatrix}$$

8.4.2 Rational functions

Recall the following formula for the inverse of a matrix M:

$$M^{-1} = \frac{1}{\det(M)}\operatorname{adj}(M)$$

where the ij-th component of the matrix $\operatorname{adj}(M)$ is given by

$$\operatorname{adj}(M)_{ij} = (-1)^{i+j}\det(\hat{M}_{ji})\,.$$

Here $\hat{M}_{ij}$ is the submatrix of M obtained by deleting the i-th row and the j-th column of M.

So,

$$(sI-A)^{-1} = \frac{1}{\Delta(s)}\operatorname{adj}(sI-A)$$

where

$$\Delta(s) = \det(sI-A)$$

is the characteristic polynomial of A and $\operatorname{adj}(sI-A)$ is the adjugate of $sI-A$. If A is $n\times n$, its characteristic polynomial is a monic polynomial of order n. Each element of $\operatorname{adj}(sI-A)$

is $\pm$ the determinant of an $(n-1)\times(n-1)$ submatrix of $sI-A$; hence each element of $\mathrm{adj}\,(sI-A)$ is a polynomial of order at most $n-1$.

We have shown that the transfer function of a system described by

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{8.4}$$

is given by

$$\hat{G}(s) = C(sI-A)^{-1}B + D\,.$$

If $D = 0$ then $\hat{G}(s) = C(sI-A)^{-1}B$; hence

$$\hat{G}_{ij}(s) = \frac{N_{ij}(s)}{\Delta(s)}$$

where N_{ij} is a polynomial of order less than or equal to $n-1$. So, if $D=0$, then $\hat{G}_{ij}$ is a rational function (ratio of two polynomials) and is strictly proper (order of numerator strictly less than order of denominator) . If D is nonzero

$$\hat{G}_{ij}(s) = \frac{N_{ij}(s)}{\Delta(s)}$$

where N_{ij} is a polynomial of order less than or equal to n. So, $\hat{G}_{ij}$ is a rational function and is proper (order of numerator less than or equal to order of denominator). So we have the following general conclusion:

If $\hat{G}$ is the transfer function of a system described by (8.4), then every element of $\hat{G}$ is a proper rational function.

Example 33

$$\begin{aligned} \hat{G}(s) &= \tfrac{s}{s^2+1} && \text{rational,} \quad \text{strictly proper} \\ \hat{G}(s) &= \tfrac{s^2}{s^2+1} && \text{rational,} \quad \text{proper} \\ \hat{G}(s) &= \tfrac{s^3}{s^2+1} && \text{rational,} \quad \text{not proper} \\ \hat{G}(s) &= \tfrac{e^{-s}}{s^2+1} && \text{not rational} \end{aligned}$$

Example 34 Consider the scalar system

$$\dot{y}(t) = ay(t-h) + u(t)$$

where $h > 0$ is a time delay. Taking the Laplace transform of the above equation with all initial conditions set to zero:

$$s\hat{y}(s) = ae^{-hs}\hat{y}(s) + \hat{u}(s)$$

Hence

$$\hat{G}(s) = \frac{1}{s - ae^{-hs}}$$

which is not a rational function.

8.4.3 Poles

A complex number λ is defined to be a pole of $\hat{G}$ if for some element $\hat{G}_{ij}$ of $\hat{G}$,

$$\lim_{s\to\lambda} \hat{G}_{ij}(s) = \infty\,.$$

When

$$\hat{G}(s) = C(sI-A)^{-1}B + D$$

it should be clear that if λ is a pole of $\hat{G}$ then λ is an eigenvalue of A. However, the converse is not always true.

Example 35

$\hat{G}(s)$	poles
$\frac{1}{s^2+3s+2}$	-1, -2
$\frac{s+1}{s^2+3s+2}$	-2
$\begin{pmatrix} \frac{1}{s} & \frac{e^{-s}}{s+1} \\ \frac{s^2}{s+1} & \frac{s+1}{s+2} \end{pmatrix}$	0, -1, -2

8.4.4 Zeros

Suppose $\hat{g}$ is a scalar transfer function. A complex number λ is defined to be a zero of $\hat{g}$ if

$$\hat{g}(\lambda) = 0\,.$$

8.4.5 Pole-zero-gain description

Suppose $p_1, \cdots, p_n$ are the poles and $z_1, \cdots, z_m$ are the zeros of a rational transfer scalar function $\hat{g}$. Then

$$\hat{g}(s) = k\frac{(s-z_1)\cdots(s-z_m)}{(s-p_1)\cdots(s-p_n)} \tag{8.5}$$

where the scalar is called the high frequency gain of the system.

8.4.6 Exercises

Exercise 34 Show that the transfer function of

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\alpha_0 x_1 - \alpha_1 x_2 + u \\ y &= \beta_0 x_1 + \beta_1 x_2 \end{aligned}$$

is given by

$$\hat{G}(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

Exercise 35 Show that the transfer function of

$$\begin{array}{rclcl} \dot{x}_1 & = & & -\alpha_0 x_2 & + & \beta_0 u \\ \dot{x}_2 & = & x_1 & -\alpha_1 x_2 & + & \beta_1 u \\ y & = & & x_2 & & \end{array}$$

is given by

$$\hat{G}(s) = \frac{\beta_1 s + \beta_0}{s^2 + \alpha_1 s + \alpha_0}$$

Exercise 36 Obtain the transfer function of the following system:

$$\begin{aligned} \ddot{q}_1 + q_1 - q_2 &= 0 \\ \dot{q}_2 + \dot{q}_1 + q_2 &= u \\ y &= q_1 \end{aligned}$$

Invariance of $\hat{G}$ under state transformations. Suppose one introduces a state transformation

$$x = T\xi$$

where T is square and invertible. Then,

$$\begin{aligned} \dot{\xi} &= \bar{A}\xi + \bar{B}u \\ y &= \bar{C}\xi + \bar{D}u \end{aligned}$$

with

$$\begin{array}{ll} \bar{A} = T^{-1}AT & \qquad \bar{B} = T^{-1}B \\ \bar{C} = CT & \qquad \bar{D} = D \end{array}$$

It can be readily verified that

$$\bar{C}(sI - \bar{A})^{-1}\bar{B} + \bar{D} = C(sI - A)^{-1}B + D$$

that is, the transfer function matrix is unaffected by a state transformation.

MATLAB

```
>> help ss2tf

 SS2TF  State-space to transfer function conversion.
        [NUM,DEN] = SS2TF(A,B,C,D,iu)  calculates the transfer function:

                    NUM(s)          -1
            H(s) = -------- = C(sI-A) B + D
                    DEN(s)
```

```
    of the system:
            .
            x = Ax + Bu
            y = Cx + Du

    from the iu'th input.  Vector DEN contains the coefficients of the
    denominator in descending powers of s.  The numerator coefficients
    are returned in matrix NUM with as many rows as there are
    outputs y.
```

```
>> sys1 = ss(A,B,C,D)
>> sys2 = tf(num,den)
>> sys2 = tf(sys1)
>> [z p k] = zpkdata(sys2,'v')
```

8.5 State space realization of transfer functions

Here we consider the following type of problem: Given a function $\hat{G}$ of a complex variable s, find constant matrices A, B, C, D (or determine that none exist) so that

$$\hat{G}(s) = C(sI - A)^{-1}B + D\,. \tag{8.6}$$

When the above relationship holds, we say that (A, B, C, D) is a finite-dimensional state space realization of $\hat{G}$. We will only consider SISO (scalar input scalar output) systems. So, $\hat{G}(s)$ is a scalar.

We have already seen that if $\hat{G}$ has a finite-dimensional state space realization, then it is rational and proper. Is the converse true? That is, does every proper rational $\hat{G}$ have a finite dimensional state space realization? This is answered in our next result.

Theorem 1 *Suppose $\hat{G}$ is a scalar valued function of a complex variable. Then $\hat{G}$ has a finite-dimensional state space realization if and only if $\hat{G}$ is rational and proper.*

Proof We have already seen the necessity of $\hat{G}$ being proper rational. We prove sufficiency in a later section by demonstrating specific realizations. ■

Minimal realizations. It should be clear from Section 8.4 that a given transfer function has an infinite number of state space realizations. One might expect that the dimension of every state space realization is the same. This is not even true as the next example illustrates. We say that a realization is a minimal realization if its dimension is less than or equal to that of any other realization. Clearly, all minimal realizations have the same dimension. Later on we will characterize minimal state space realizations.

Example 36 Consider a SISO system of dimension two described by

$$A = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \qquad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad D = 0$$

Here

$$\hat{G}(s) = C(sI - A)^{-1}B + D = \frac{1}{s+1}$$

This also has the following one dimensional realization

$$A = -1 \qquad B = 1 \qquad C = 1 \qquad D = 0$$

Example 37 Consider a SISO system of dimension two described by

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 1 \end{pmatrix} \qquad D = 0$$

Here

$$\hat{G}(s) = C(sI - A)^{-1}B + D = \frac{s-1}{s^2-1} = \frac{1}{s+1}$$

This also has the following one dimensional realization

$$A = -1 \qquad B = 1 \qquad C = 1 \qquad D = 0$$

‘

8.6 State space realization of of SISO transfer functions

8.6.1 Controllable canonical form realization

Suppose $\hat{G}$ is a proper rational scalar function of a complex variable s. If $\hat{G} \neq 0$ then, for some positive integer n, $\hat{G}(s)$ can be expressed as

$$\hat{G}(s) = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} \quad + \quad d\,. \tag{8.7}$$

Note that, in the above formulation, the order of the numerator is strictly less than the order of the denominator and the coefficient of the highest order term in the denominator is one. The following example illustrates how one can always put a proper rational function in the above form.

Example 38 Consider

$$\hat{G}(s) = \frac{4s^2 + 8s + 6}{2s^2 + s + 1}$$

Since the numerator has the same order as the denominator, we divide the numerator by the denominator to obtain that

$$4s^2 + 8s + 6 = 2(2s^2 + s + 1) + 6s + 4\,.$$

Hence

$$\hat{G}(s) = \frac{6s + 4}{2s^2 + s + 1} + 2$$

Finally we make the leading coefficient of the denominator equal to one:

$$\hat{G}(s) = \frac{3s + 2}{s^2 + 0.5s + 0.5} + 2\,.$$

When $\hat{G}$ is expressed as in (8.7), we will show that

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

where

$$\boxed{\begin{array}{ll} A = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -\alpha_0 & -\alpha_1 & -\alpha_2 & \dots & -\alpha_{n-2} & -\alpha_{n-1} \end{pmatrix}, & B = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{pmatrix} \\ C = \begin{pmatrix} \beta_0 & \beta_1 & \beta_2 & \dots & \beta_{n-2} & \beta_{n-1} \end{pmatrix}, & D = d \end{array}}$$

Remark 1 A matrix A with the structure indicated above is called a companion matrix or the matrix A is said to be in companion form. When the pair (A, B) have the structure given above, the pair are said to be in controllable canonical form. A companion matrix has ones on its super-diagonal and zeros everywhere else except for the last row which can be anything. Thus an $n \times n$ companion matrix is completely specified by specifying the n numbers which make up its last row.

Note that, in the above state space realization, the coefficients of the denominator polynomial in (8.7) determine A whereas the coefficients of the numerator polynomial determine C. The matrix B is the same for all realizations.

Proof of above claim. Letting

$$v(s) := \begin{pmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{pmatrix}$$

and

$$\Delta(s) := \alpha_0 + \alpha_1 s + \ldots + s^n$$

we have

$$(sI - A)v(s) = \begin{pmatrix} s & -1 & 0 & \ldots & 0 & 0 \\ 0 & s & -1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & -1 \\ \alpha_0 & \alpha_1 & \alpha_2 & \ldots & \alpha_{n-2} & s + \alpha_{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ s \\ \vdots \\ \vdots \\ s^{s-2} \\ s^{n-1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \Delta(s) \end{pmatrix} = \Delta(s)B$$

Hence, whenever s is not an eigenvalue of A,

$$(sI - A)^{-1}B = \frac{1}{\Delta(s)} v(s)$$

and

$$\begin{aligned} C(sI - A)^{-1}B &= \frac{1}{\Delta(s)} \begin{pmatrix} \beta_0 & \beta_1 & \ldots & \beta_{n-1} \end{pmatrix} \begin{pmatrix} 1 \\ s \\ \vdots \\ s^{n-1} \end{pmatrix} \\ &= \frac{\beta_0 + \beta_1 s + \cdots + \beta_{n-1} s^{n-1}}{\alpha_0 + \alpha_1 s + \cdots + \alpha_{n-1} s^{n-1} + s^n} \end{aligned}$$

So,

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

■

We will see later why this realization is called controllable.

8.6.2 Observable canonical form realization*

Suppose $\hat{G}(s)$ is a proper rational scalar function of a complex variable s. Then, for some positive integer n $\hat{G}$ can be expresssed as

$$\hat{G}(s) = \frac{\beta_{n-1}s^{n-1} + \cdots + \beta_1 s + \beta_0}{s^n + \alpha_{n-1}s^{n-1} + \cdots + \alpha_1 s + \alpha_0} \quad + \quad d$$

We will show that

$$\hat{G}(s) = C(sI - A)^{-1}B + D$$

where

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 & -\alpha_0 \\ 1 & 0 & \dots & 0 & -\alpha_1 \\ 0 & 1 & \ddots & 0 & -\alpha_2 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & 1 & -\alpha_{n-1} \end{pmatrix} \qquad B = \begin{pmatrix} \beta_0 \\ \beta_1 \\ \vdots \\ \vdots \\ \beta_{n-1} \end{pmatrix}$$

$$C = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 \end{pmatrix} \qquad D = d$$

• Noting that $C(sI - A)^{-1}B + D$ is a scalar and using the results of the previous section, we have

$$C(sI - A)^{-1}B + D = B^T(sI - A^T)^{-1}C^T + D = \hat{G}(s)$$

■

We will see later why this realization is called observable.

MATLAB

```
>> help tf2ss

 TF2SS  Transfer function to state-space conversion.
        [A,B,C,D] = TF2SS(NUM,DEN)  calculates the state-space
        representation:
                .
                x = Ax + Bu
                y = Cx + Du

        of the system:
                        NUM(s)
                H(s) = --------
                        DEN(s)

        from a single input.  Vector DEN must contain the coefficients of
        the denominator in descending powers of s.  Matrix NUM must
        contain the numerator coefficients with as many rows as there are
        outputs y.  The A,B,C,D matrices are returned in controller
        canonical form.  This calculation also works for discrete systems.
        To avoid confusion when using this function with discrete systems,
        always use a numerator polynomial that has been padded with zeros
        to make it the same length as the denominator.  See the User's
        guide for more details.
```

Example 39 Consider

$$\hat{G}(s) = \frac{s^2}{s^2+1}$$

This can be written as

$$\hat{G}(s) = \frac{-1}{s^2+1} + 1$$

hence a state space representation is given by

$$A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} -1 & 0 \end{pmatrix} \qquad D = 1$$

or,

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u \\ y &= -x_1 + u \end{aligned}$$

Matlab time:

```
>>  num =  [1 0 0]
>>  den =  [1 0 1]
>> [a,b,c,d]=tf2ss(num,den)

a =
     0    -1
     1     0

b =
     1
     0

c =
     0    -1

d =
     1
```

Example 40 Suppose we wish to obtain a state space realization of the input output system described by

$$\ddot{y} + 2\dot{y} + 3y = \dot{u} + 2u$$

where u and y are scalars.

8.6.3 Exercises

Exercise 37 The motion of a pendulum on a cart can be described by

$$\begin{array}{rcrcl} (M+m)\ddot{y} - ml\cos\theta\,\ddot{\theta} & + & ml\sin\theta\dot{\theta}^2 & = & u \\ -ml\cos\theta\,\ddot{y} + ml^2\ddot{\theta} & + & mlg\sin\theta & = & 0 \end{array}$$

where M is the mass of the cart, m is the pendulum mass, l is distance from cart to pendulum mass, and g is the gravitational acceleration constant. The variables y,θ and u are the cart displacement, the pendulum angle, and a force applied to the cart, respectively Consider this an input-output system with input u and output y.

(a) Linearize (by hand) this system about equilibrium conditions corresponding to $u^e = 0$, $y^e = 1$ and $\theta^e = \pi$.

(b) Obtain (by hand) the A, B, C, D matrices for a state space representation of the linearization found in part (b).

(c) Obtain (by hand) the transfer function of the linearized system.

(d) Determine the poles and zeros of the transfer function.

(e) Do parts (b)-(d) in MATLAB when all the parameters M, m, l and g equal one.

Exercise 38 Obtain a state space representation of the following transfer function.

$$\hat{G}(s) = \frac{(s-2)(s+2)}{s(s-4)(s+3)}$$

Exercise 39 Obtain a state space representation of the following input-output system.

$$\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = \frac{du}{dt} + u\,.$$

Exercise 40 Obtain the transfer function of the following system:

$$\begin{array}{rcl} \dot{x}_1 & = & -x_1 + x_2 \\ \dot{x}_2 & = & x_2 + x_3 \\ \dot{x_3} & = & u \\ y & = & x_1 + x_2 \end{array}$$

What are the poles and zeros of this transfer function?

8.7 General form of solution

Impulses. The unit impulse function or the Dirac delta function δ is a mathematical model of a signal which is "active" (nonzero) over a very short interval around $t = 0$ and whose integral over the interval is one. So,

$$\delta(t) = 0 \qquad \text{when} \qquad t \neq 0$$

and

$$\int_0^{0^+} \delta(t)dt = 1$$

An important property of δ is the following. Suppose f is any function of time (it could be scalar-valued, vector valued, or matrix valued), then

$$\int_0^{0^+} \delta(t)f(t)\,dt = f(0)\,.$$

(Don't confuse this δ with the δ used in linearization.)

Impulse response matrix and convolution integral: The output response of the linear time-invariant system

$$\begin{aligned}\dot{x}(t) &= Ax(t) + Bu(t)\\ y(t) &= Cx(t) + Du(t)\end{aligned}$$

subject to initial condition

$$x(0) = x_0$$

is unique and is given by

$$\boxed{y(t) = Ce^{At}x_0 + \int_0^t G(t-\tau)u(\tau)d\tau} \tag{8.8}$$

where the impulse response matrix G is defined by

$$\boxed{G(t) = Ce^{At}B + \delta(t)D}$$

To see this, we first show that

$$x(t) = e^{A(t)}x_0 + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau$$

Hence

$$\begin{aligned} y(t) &= Ce^{At}x_0 + \int_t^t Ce^{A(t-\tau)}Bu(\tau)d\tau + Du(t)\\ &= Ce^{At}x_0 + \int_0^t Ce^{A(t-\tau)}Bu(\tau) + \delta(t-\tau)Du(\tau)d\tau\\ &= Ce^{At}x_0 + \int_0^t G(t-\tau)u(\tau)\,d\tau \end{aligned}$$

Zero initial conditions response: $x(0) = 0$.

$$\boxed{y(t) = \int_0^t G(t-\tau)u(\tau)d\tau} \tag{8.9}$$

Sometimes this is represented by

$$y \longleftarrow \boxed{G} \longleftarrow u$$

This defines a linear map from the space of input functions to the space of output functions.

Example 41

$$\begin{aligned} \dot{x} &= -x + u \\ y &= x \end{aligned}$$

Here

$$\begin{aligned} G(t) &= Ce^{At}B + D \\ &= e^{-t} \end{aligned}$$

So, for zero initial state,

$$y(t) = \int_0^t e^{-(t-\tau)}u(\tau)\,d\tau$$

Example 42 Unattached mass

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned}$$

Here

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \qquad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad C = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad D = 0$$

Hence

$$e^{At} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

and

$$\begin{aligned} G(t) &= Ce^{At}B \\ &= t \end{aligned}$$

So,

$$y(t) = \int_0^t (t-\tau)u(\tau)\,d\tau$$

• Description (8.9) is a very general representation of a *linear input-output system*. It can also be used for *infinite-dimensional systems* such as *distributed parameter systems* and *systems with delays.*

Impulse response: Suppose $t_0 = 0$ and

$$u(t) = \delta(t)v$$

where v is constant m vector. Then, from (8.8),

$$y(t) = Ce^{At}x_0 + G(t)v$$

If the initial state x_0 is zero, we obtain

$$y(t) = G(t)v$$

From this we can see that G_{ij} is the zero initial state response of y_i when $u_j = \delta$ and all other inputs are zero, that is,

$$\left.\begin{array}{rcl} u_1(t) &=& 0 \\ &\vdots& \\ u_j(t) &=& \delta(t) \\ &\vdots& \\ u_m(t) &=& 0 \end{array}\right\} \quad \Longrightarrow \quad \left\{\begin{array}{rcl} y_1(t) &=& G_{1j}(t) \\ &\vdots& \\ y_p(t) &=& G_{pj}(t) \end{array}\right.$$

• So, for a SISO (single-input single-output) system, G is the output response due a unit impulse input and zero initial state.

Response to exponentials and sinusoids: Suppose $t_0 = 0$ and

$$u(t) = e^{\lambda t}v$$

where v is a constant m vector and λ is not an eigenvalue of A. Then

$$y(t) = \underbrace{Ce^{At}(x_0 - x_0^p)}_{\text{transient part}} + \underbrace{e^{\lambda t}\hat{G}(\lambda)v}_{\text{steady state part}}$$

where

$$x_0^p = (\lambda I - A)^{-1}Bv$$

and

$$\boxed{\hat{G}(\lambda) := C(\lambda I - A)^{-1}B + D}$$

If A is asymptotically stable, then the transient part goes to zero as t goes to infinity.

Exercises

Problem 2 (a) Obtain the transfer function of the following system with input u and output y:

$$\begin{aligned}
2\ddot{q}_1 + \ddot{q}_2 - q_1 - 2q_2 &= u \\
\ddot{q}_1 + 2\ddot{q}_2 - 2q_1 - q_2 &= 2u \\
y &= q_1 + q_2
\end{aligned}$$

(b) What are the poles and zeros of the transfer function?

Problem 3 For each of the following transfer functions, obtain a finite-dimensional state space representation (if one exists). If one does not exist, state why.

(a)

$$\hat{G}(s) = \frac{1}{2s(s-1)(s+1)}$$

(b)

$$\hat{G}(s) = \frac{s^2+1}{s^2-1}$$

(c)

$$\hat{G}(s) = \frac{s^3}{s^2-1}$$

(d)

$$\hat{G}(s) = \frac{e^{-2s}}{s^2-1}$$

Problem 4 Obtain a state space representation of the following input-output system.

$$\frac{d^3y}{dt^3} - 2\frac{d^2y}{dt^2} - \frac{dy}{dt} + 2y = \frac{d^3u}{dt^3} + \frac{du}{dt} + u\,.$$

Chapter 9

Behavior of linear systems

9.1 Introduction

In this chapter we discuss the behavior of linear time-invariant systems. Recall that such systems can be described by

$$\boxed{\dot{x} = Ax} \tag{9.1}$$

where the n-vector $x(t)$ is the system state at time t and A is a constant $n \times n$ matrix. Of course, the material in this chapter also applies to linearizations of nonlinear systems, that is, systems described by

$$\delta\dot{x} = A\delta x\,.$$

A solution of (9.1) is any continuous function $x(\cdot)$ which satisfies $\dot{x}(t) = Ax(t)$. The differential equation (9.1) has an infinite number of solutions, however, if one specifies an initial condition of the form:

$$\boxed{x(t_0) = x_0} \tag{9.2}$$

then, there is an unique solution which satisfies this initial condition and this solution is defined for all time t. We refer t_0 as an initial time and to x_0 as an initial state.

Since the system is time-invariant we can, wlog (without loss of generality), consider only zero initial time, that is, $t_0 = 0$. For suppose $x(\cdot)$ is the solution corresponding to

$$x(0) = x_0\,.$$

Then, the solution $\tilde{x}(\cdot)$ corresponding to $\tilde{x}(t_0) = x_0$ is given by $\tilde{x}(t) = x(t-t_0)$.

Some results needed for this chapter.

- For any two real or complex numbers a and b.

$$e^{(a+b)} = e^a e^b$$

- If θ is any real number and $\jmath = \sqrt{-1}$, then

$$e^{\jmath\theta} = \cos\theta + \jmath\sin\theta$$

and

$$|e^{\jmath\theta}| = 1\,.$$

- $$\frac{de^{\lambda t}}{dt} = \lambda e^{\lambda t}\,,$$

for any complex number λ.

9.2 Scalar systems

Consider a system described by

$$\boxed{\dot{x} = ax}$$

where $x(t)$ and the constant a are real scalars. All solutions are given by:

$$\boxed{x(t) = ce^{at}\ ; \qquad c = x_0 := x(0)}$$

(a) If $a < 0$ you decay with increasing age. (Stable)

(b) If $a > 0$ you grow with increasing age. (Unstable)

(c) If $a = 0$ you stay the same; every state is an equilibrium state. (Neutrally stable)

The qualitative behavior of this system is completely determined by the sign of a.

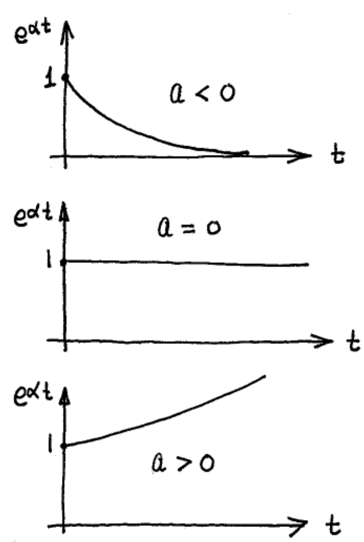


Figure 9.1: The three types of modes for real scalar systems

Example 43 (The first example) Here

$$a = -c/m < 0$$

Thus, all solutions are of the form $e^{-(c/m)t}x_0$; hence they decay exponentially.

Example 44 (Simple population dynamics)

$$\dot{x} = (b - d)x$$

where b is the birth rate and d is the death rate. If $b > d$, population increases exponentially.

Example 45 (Linearized WC) Recall that linearization of the WC about an equilibrium state is described by

$$(q_{in}A_t)\,\delta\dot{h} = -(A_o^2 g)\,\delta h$$

that is,

$$\delta\dot{h} = a\delta h \qquad \text{with} \qquad a = -A_o^2 g/(q_{in}A_t)$$

9.2.1 Exercises

Exercise 41 For each of the following scalar systems, state whether they are stable, unstable, or neutrally stable.

(♠)

$$\dot{x} = -2x$$

(♣)

$$\dot{x} = -10x$$

(♢)

$$\dot{x} = 4x$$

(♡)

$$\dot{x} = 0$$

Exercise 42 Determine the solution to

$$\dot{x} = 4x \qquad \text{subject to} \qquad x(0) = -5\,.$$

9.3 Second order systems

Examples: Simple oscillator, linearized pendulum; linearized weathercock.

Recall the simple spring-mass-damper (or simple oscillator) system described by

$$m\ddot{y} + c\dot{y} + ky = 0$$

Letting $a_2 := m$, $a_1 := c$ and $a_0 := k$ the simple oscillator is a specific example of the general second order system:

$$\boxed{a_2\ddot{y} + a_1\dot{y} + a_0 y = 0} \tag{9.3}$$

where $y(t)$ and the constants a_0, a_1, a_2 are real scalars with $a_2 \neq 0$.

Linearizing a second order nonlinear system,

$$F(\ddot{y}, \dot{y}, y) = 0$$

about an equilibrium value y^e of y results in

$$a_2\delta\ddot{y} + a_1\delta\dot{y} + a_0\delta y = 0 \tag{9.4}$$

where $\delta y = y - y^e$ and

$$a_0 = \left.\frac{\partial F}{\partial y}\right|^e, \qquad a_1 = \left.\frac{\partial F}{\partial \dot{y}}\right|^e, \qquad a_2 = \left.\frac{\partial F}{\partial \ddot{y}}\right|^e .$$

To discuss the behavior of (9.3), we need to determine the roots of the following second order characteristic polynomial associated with differential equation (9.3):

$$p(s) = a_2 s^2 + a_1 s + a_0 \,. \tag{9.5}$$

Let λ_1 and λ_2 be the roots of the above polynomial, that is, $p(\lambda_1) = 0$ and $p(\lambda_2) = 0$. Except in a very special case $(\lambda_1 = \lambda_2)$, all solutions to (9.3) are of the form

$$\boxed{y(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}} \tag{9.6}$$

where and c_1 and c_2 are arbitrary constants. These constants can be uniquely determined by the initial values of y and $\dot{y}$. The roots are given by

$$\lambda_1 = \frac{-a_1 - \sqrt{a_1^2 - 4a_0a_2}}{2a_2} \qquad \text{and} \qquad \lambda_2 = \frac{-a_1 + \sqrt{a_1^2 - 4a_0a_2}}{2a_2} \,. \tag{9.7}$$

Exercise 43 Show that every function y of the form given in (9.6) is a solution of the differential equation (9.3).

9.3.1 Real distinct roots

The roots λ_1 and λ_2 are real and distinct if and only if

$$a_1^2 - 4a_0a_2 > 0\,.$$

In this case, every solution is simply a linear combination of the two real exponential solutions

$$e^{\lambda_1 t} \qquad \text{and} \qquad e^{\lambda_2 t}\,.$$

We refer to the above two special solutions as modes or modal solutions of the system. Thus every solution is a linear combination of these two modes. We have already discussed such modes in scalar systems.

Example 46 Consider

$$\ddot{y} - y = 0\,.$$

The characteristic polynomial for this system is given by

$$p(s) = s^2 - 1 = (s+1)(s-1)\,.$$

Hence the two roots for this system are

$$\lambda = 1 \qquad \text{and} \qquad \lambda_2 = -1\,.$$

So, the general solution for this system is given by

$$y(t) = c_1 e^{-t} + c_2 e^t\,.$$

Consider now specific initial conditions:

$$y(0) = y_0 \qquad \text{and} \qquad \dot{y}(0) = \dot{y}_0,$$

where y_0 and $\dot{y}_0$ are specified numbers. letting $t = 0$, we must have

$$y_0 = c_1 + c_2\,.$$

Since $\dot{y}(t) = -c_1 e^{-t} + c_2 e^t$, letting $t = 0$ now yields:

$$\dot{y}_0 = -c_1 + c_2\,.$$

We can now solve for the two unknowns c_1 and c_2 to obtain

$$c_1 = (y_0 - \dot{y}_0)/2 \qquad \text{and} \qquad c_2 = (y_0 + \dot{y}_0)/2\,.$$

Thus, we obtain the following solutions corresponding to different initial conditions:

y_0	$\dot{y}_0$	$y(t)$
1	1	e^t
1	-1	e^{-t}
1	0	$(e^t + e^{-t})/2$

Example 47 Linearized inverted pendulum

$$J\ddot{\theta} + mgl\sin\theta = 0$$

Consider $\theta^e = \pi$. Linearization results in

$$J\delta\ddot{\theta} - mgl\delta\theta = 0$$

Here

$$p(s) = Js^2 - mgl$$

This polynomial has two real roots given by

$$\lambda_1 = -\sqrt{mgl/J} < 0 \qquad \text{and} \qquad \lambda_2 = \sqrt{mgl/J} > 0$$

9.3.2 Complex roots

The roots λ_1 and λ_2 are complex if and only if

$$a_1^2 - 4a_0a_2 < 0\,.$$

Clearly, if the roots are complex, they are distinct. In this case we can express the roots as

$$\lambda_1 = \alpha + \jmath\omega \qquad \text{and} \qquad \lambda_2 = \alpha - \jmath\omega \tag{9.8}$$

where $\jmath = \sqrt{-1}$ and the real numbers α and ω are given by

$$\alpha = -a_1/2a_2 \qquad \text{and} \qquad \omega = \frac{\sqrt{4a_0a_2 - a_1^2}}{2a_2}\,.$$

Note that λ_2 is the complex conjugate of λ_1. The constants c_1 and c_2 in the the general solution (9.6) are arbitrary complex numbers. Since $e^{\lambda_1 t}$ is the complex conjugate of $e^{\lambda_2 t}$, it follows that for real solutions y, the constant c_2 must be the complex conjugate of c_1.

Remark 2 In treatment of the systems under consideration here, it is common to describe the system by

$$\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n^2 y = 0\,,$$

with $\zeta < 1$. Thus,

$$a_0 = \omega_n^2\,, \qquad a_1 = 2\zeta\omega_n\,, \qquad a_2 = 1\,.$$

In this case,

$$\alpha = -\zeta\omega_n\,, \qquad \omega = \omega_n\sqrt{1-\zeta^2}\,.$$

To obtain an expression for solutions in terms of real numbers, we first note that

$$\begin{aligned} e^{\lambda_1 t} &= e^{(\alpha t + \jmath\omega t)} = e^{\alpha t} e^{\jmath\omega t} = e^{\alpha t}(\cos\omega t + \jmath \sin\omega t) \\ &= e^{\alpha t}\cos\omega t + \jmath e^{\alpha t}\sin\omega t\,. \end{aligned}$$

Since $e^{\lambda_2 t}$ is the complex conjugate of $e^{\lambda_1 t}$, we have

$$e^{\lambda_2 t} = e^{\alpha t}\cos\omega t - \jmath e^{\alpha t}\sin\omega t$$

Substitution of the above two expressions into (9.6) yields the following expression for the general solution to (9.3):

$$\boxed{y(t) = \tilde{c}_1 e^{\alpha t}\cos\omega t + \tilde{c}_2 e^{\alpha t}\sin\omega t} \tag{9.9}$$

where $\tilde{c}_1$ and $\tilde{c}_2$ are real constants. These constants are uniquely determined by the initial values of y and $\dot{y}$. Thus every solution is a linear combination of the real oscillatory modal solutions:

$$e^{\alpha t}\cos\omega t \qquad \text{and} \qquad e^{\alpha t}\sin\omega t\,.$$

To obtain further insight into the solutions for this case, introduce

$$Y = \sqrt{\tilde{c}_1^2 + \tilde{c}_2^2}$$

and let ϕ be the number between 0 and 2π which is uniquely defined by

$$\cos\phi = \tilde{c}_1/Y \qquad \text{and} \qquad \sin\phi = \tilde{c}_2/Y\,.$$

Then,

$$y(t) = Ye^{\alpha t}[\cos\omega t\cos\phi + \sin\omega t\sin\phi]\,.$$

Since

$$\cos\omega t\cos\phi + \sin\omega t\sin\phi = \cos(\omega t - \phi)\,,$$

we finally obtain that

$$\boxed{y(t) = Ye^{\alpha t}\cos(\omega t - \phi)}\,. \tag{9.10}$$

Thus, all solutions can be characterized by the single real oscillatory modal solution

$$\boxed{e^{\alpha t}\cos\omega t}$$

Based on the sign of α, there are three types of oscillatory modes:

$\alpha < 0$	stable
$\alpha = 0$	marginally stable
$\alpha > 0$	unstable

These are illustrated in Figure 9.2.

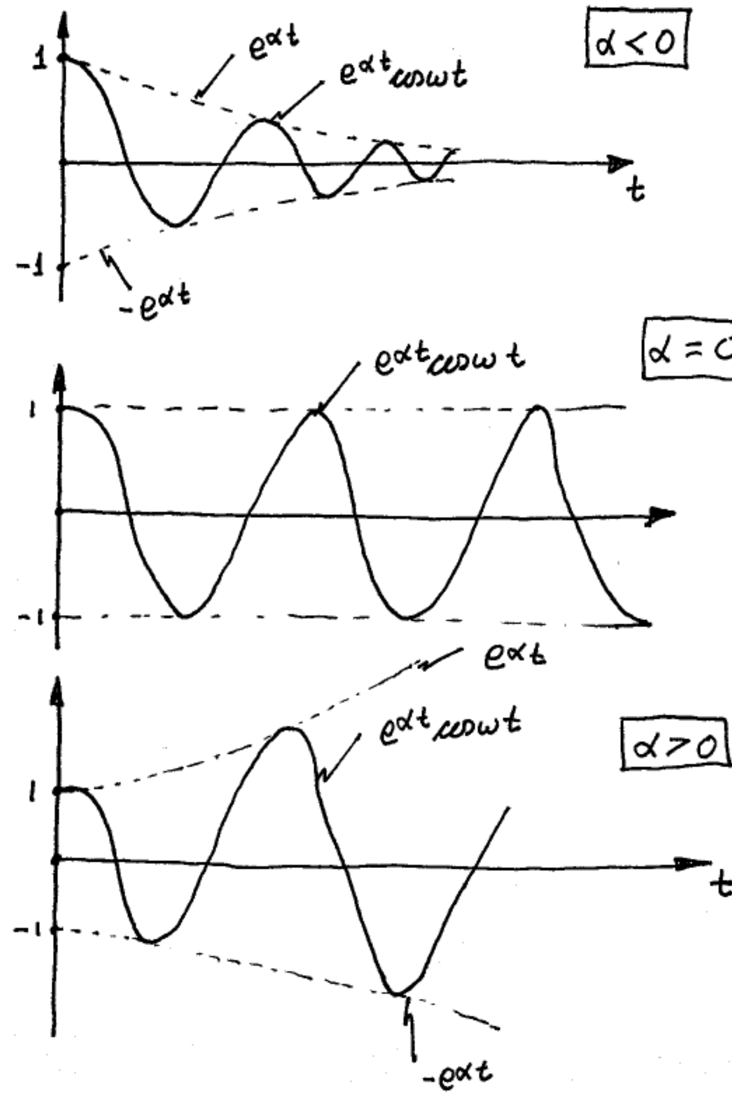


Figure 9.2: The three types of oscillatory modal solutions

Example 48 Consider

$$\ddot{y} - 4\dot{y} + 13y = 0 = 0\,.$$

The characteristic polynomial for this system is given by

$$p(s) = s^2 - 4s + 13\,.$$

Since (a MATLAB moment),

```
>>roots([1 -4 13])
ans =
   2.0000 + 3.0000i
   2.0000 - 3.0000i
```

the two roots for this system are

$$\lambda = 2 + 3\jmath \qquad \text{and} \qquad \lambda = 2 - 3\jmath\,.$$

So, the general solution for this system is given by

$$y(t) = c_1 e^{(2+3\jmath)t} + c_2 e^{(2-3\jmath)t}$$

or

$$y(t) = \tilde{c}_1 e^{2t}\cos 3t + \tilde{c}_2 e^{2t}\sin 3t\,.$$

Consider now specific initial conditions:

$$y(0) = y_0 \qquad \text{and} \qquad \dot{y}(0) = \dot{y}_0,$$

where y_0 and $\dot{y}_0$ are specified numbers. letting $t = 0$, we must have

$$y_0 = \tilde{c}_1\,.$$

Since

$$\dot{y}(t) = \tilde{c}_1 2e^{2t}\cos 3t - \tilde{c}_1 e^{2t} 3\sin 3t + \tilde{c}_2 2e^{2t}\sin 3t + \tilde{c}_2 e^{2t} 3\cos 3t\,,$$

letting $t = 0$ now yields:

$$\dot{y}_0 = 2\tilde{c}_1 + 3\tilde{c}_2\,.$$

We can now solve for the two unknowns $\tilde{c}_1$ and $\tilde{c}_2$ to obtain

$$\tilde{c}_1 = y_0 \qquad \text{and} \qquad \tilde{c}_2 = (\dot{y}_0 - 2y_0)/3\,.$$

Thus, we obtain the following solutions corresponding to different initial conditions:

y_0	$\dot{y}_0$	$y(t)$
1	2	$e^{2t}\cos 3t$
0	3	$e^{2t}\sin 3t$
1	5	$e^{2t}(\cos 3t + \sin 3t)$

Exercise 44 Determine the solutions to

$$\ddot{y} - 4y = 0$$

subject to the following sets of initial conditions

$$\begin{aligned} y(0) = 1 \quad &\text{and} \quad \dot{y}(0) = 1 \\ y(0) = 1 \quad &\text{and} \quad \dot{y}(0) = -1 \\ y(0) = 1 \quad &\text{and} \quad \dot{y}(0) = 0 \end{aligned}$$

Exercise 45 Determine the solution to

$$\ddot{y} + 4y = 0$$

subject to the initial conditions

$$y(0) = 1 \text{ and } \dot{y}(0) = 0\,.$$

Exercise 46 Determine the solution to

$$\ddot{y} + 2\dot{y} + 2y = 0$$

subject to the initial conditions

$$y(0) = 1 \text{ and } \dot{y}(0) = 0\,.$$

Exercise 47 What is the solution to

$$2\ddot{q} - 4\dot{q} + 4q = 0 \qquad \text{with} \quad q(0) = 1 \quad \text{and} \quad \dot{q}(0) = 0\,?$$

9.3.3 Stability

In this section assume

$$a_2 > 0\,.$$

Real roots.

The system is stable if both modes are stable, that is, $\lambda_1 < 0$ *and* $\lambda_2 < 0$.

The system is unstable if one of the modes is unstable, that is, either $\lambda_1 > 0$ *or* $\lambda_2 > 0$.

The system is neutrally unstable if one of the modes is stable and the other mode is neutrally stable that is, either $\lambda_1 = 0$ and $\lambda_2 < 0$ or vice versa.

Looking at the expressions for λ_1 and λ_2, we can conclude that the system is stable if and only if

$$a_0 > 0 \qquad \text{and} \qquad a_1 > 0\,,$$

neutrally stable if and only if

$$a_0 = 0 \qquad \text{and} \qquad a_1 > 0\,,$$

and unstable if and only if

$$a_0 < 0 \qquad \text{or} \qquad a_1 < 0\,.$$

Complex roots. We have the following conclusions for stability.

$\alpha = \Re(\lambda) < 0$	Stable
$\alpha = \Re(\lambda) = 0$	Neutrally stable
$\alpha = \Re(\lambda) > 0$	Unstable

Since $\alpha = -a_1/2$, we can conclude that the system is stable if and only if

$$a_1 > 0\,,$$

neutrally stable if

$$a_1 = 0\,,$$

and unstable if and only if

$$a_1 < 0\,.$$

Real and complex roots. Combining the results for the case of real and complex roots and noting that if $a_0 \leq 0$, the roots must be real, we can make the following stability conclusions for any second order system based on the coefficients a_0 and a_1 :

Both coefficients positive	$a_0 > 0$ and $a_1 > 0$	Stable
At least one coefficient negative	$a_0 < 0$ or $a_1 < 0$	Unstable
One coefficent zero and the other positive	$a_0 = 0$, $a_1 > 0$ or $a_0 > 0$, $a_1 = 0$	Neutrally stable

Stability conditions for second order systems

Example 49

$$m\ddot{y} + (k - m\Omega^2)y = 0$$

Example 50 (Linearized weathercock) Recall that linearization of the weathercock about the zero solution results in

$$J\delta\ddot{\theta} + (\kappa l^2 w)\,\delta\dot{\theta} + (\kappa l w^2)\,\delta\theta = 0\,.$$

Introducing $\beta = \kappa l/2J$, this description simplifies to

$$\delta\ddot{\theta} + (2\beta l w)\,\delta\dot{\theta} + (2\beta w^2)\,\delta\theta = 0\,.$$

This is a second order linear system with

$$a_1 = 2\beta l w \qquad \text{and} \qquad a_0 = 2\beta w^2\,.$$

For stability we require that both a_0 and a_1 be positive; if either one of these parameters is negative, the system is unstable. Hence, we can make the following stability conclusions:

$w > 0$	Stable
$w < 0$	Unstable

The roots $\lambda_{1,2}$ for this system will be complex if $a_1^2 < 4a_0$, that is, $\beta l^2 < 2$ or $\kappa l^3 < 4J$. In this case, we have

$$\lambda_{1,2} = \left(-\beta l \pm \jmath\sqrt{2\beta - \beta^2 l^2}\right) w \qquad \text{where} \qquad \beta = \frac{\kappa l}{2J}$$

Note that the roots depend linearly on w.

9.3.4 Experimental determination of ω and α

One can experimentally determine α and ω from a graph of $y(t) = Ye^{\alpha t}\cos(\omega t - \phi)$. Suppose that successive peak values of y occur at times t_1 and t_2 and let Y_1, Y_2 be the respective peak values; see Figure 9.3

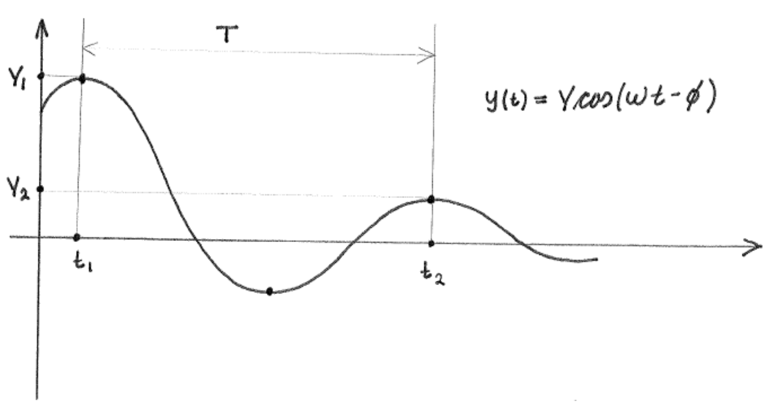


Figure 9.3: Experimental determination of α and ω

Let $T = t_2 - t_1$, that is T is the time between successive peaks. Since $T = 2\pi/\omega$, we have

$$\boxed{\omega = 2\pi/T} \tag{9.11}$$

One can also determine T by letting it equal to twice the time between successive zero crossings of $y(t)$.

Since Y_1 is the value of y at t_1 and Y_2 is the value of y at $t_2 = t_1 + T$, we have

$$Y_1 = Ye^{\alpha t_1} \qquad Y_2 = Ye^{\alpha(t_1+T)}\,.$$

Hence,

$$\frac{Y_2}{Y_1} = \frac{Ye^{\alpha(t_1+T)}}{Ye^{\alpha t_1}} = e^{\alpha T}$$

It now follows that

$$\boxed{\alpha = \frac{\ln\left(\frac{Y_2}{Y_1}\right)}{T}} \tag{9.12}$$

Exercises

Exercise 48 For each of the following second order systems, state whether they are stable, unstable, or neutrally stable. (No calculations are necessary)

(♠)

$$2\ddot{y} + 7\pi\dot{y} + 3y = 0$$

(♣)

$$203\ddot{y} - 340\dot{y} + 421y = 0$$

(♢)

$$6\ddot{y} + 7\dot{y} - 8y = 0$$

(♡)

$$-5\ddot{y} + 10\dot{y} + 20y = 0$$

(♠♠)

$$\ddot{y} + 204y = 0$$

(♠♣)

$$\ddot{y} + 251\dot{y} = 0$$

(♠♢)

$$\ddot{y} = 0$$

9.4 Eigenvalues and eigenvectors

Suppose A is a square $n \times n$ matrix.

DEFN. *A number λ is an* eigenvalue *of A if there is a nonzero vector v such that*

$$\boxed{Av = \lambda v}$$

The nonzero vector v is called an eigenvector *of A corresponding to λ.*

In the above definition, λ can be a real or complex number and v can be a real or complex vector.

So, an eigenvector is a nonzero n-vector with the property that there is a number λ such that the above relationship holds or, equivalently,

$$\boxed{(\lambda I - A)v = 0}$$

Since v is nonzero, the matrix $\lambda I - A$ must be singular; this is equivalent to

$$\boxed{\det(\lambda I - A) = 0}$$

• The characteristic polynomial of A:

$$\text{charpoly}(A) := \det(sI - A)$$

Note that $\det(sI - A)$ is an n-th order monic polynomial in s, that is, the coefficient of its highest order term s^n is one, so it must have the following form:

$$\det(sI - A) = a_0 + a_1 s + \ldots + s^n$$

• We conclude that a complex number λ is an eigenvalue of A if and only if it is a root of the n-th order characteristic polynomial of A. Hence, A has at least one distinct eigenvalue and at most n distinct eigenvalues. Suppose A has l *distinct* eigenvalues, $\lambda_1, \lambda_2, \ldots, \lambda_l$; since these are the distinct roots of the characteristic polynomial of A, we must have

$$\det(sI - A) = (s - \lambda_1)^{m_2}(s - \lambda_2)^{m_1} \cdots (s - \lambda_i)^{m_i}$$

and $m_1 + m_2 + \cdots + m_l = n$.

• The integer m_i is called the algebraic multiplicity of λ_i, that is, it is the multiplicity of the eigenvalue as a root of the characteristic polynomial.

• It should be clear that if v^1 and v^2 are any two eigenvectors corresponding to the same eigenvalue λ and ξ_1 and ξ_2 are any two numbers then (provided it is nonzero) $\xi_1 v^1 + \xi_2 v^2$ is also an eigenvector for λ. The set of eigenvectors corresponding to λ along with the zero vector is called the **eigenspace** of A associated with λ; it is simply the set of vectors v which satisfy $(\lambda I - A)v = 0$.

• It follows that A is invertible if and only if all its eigenvalues are nonzero.

Example 51 (Scalar systems) Consider $A = a$ where a is a scalar. Then

$$\det(sI - A) = s - a.$$

So a is the only eigenvalue of a.

Example 52 (Second order systems) Consider a system described by

$$a_2\ddot{y} + a_1\dot{y} + a_0 y = 0$$

Considering the usual state variables, this system is described by $\dot{x} = Ax$ where

$$A = \begin{pmatrix} 0 & 1 \\ -a_0/a_2 & -a_1/a_2 \end{pmatrix}$$

The characteristic polynomial of this A matrix is

$$\det(sI - A) = (a_0/a_2) + (a_1/a_2)s + s^2$$

Hence the eigenvalues of A are the roots of the polynomial

$$p(s) = a_0 + a_1 s + a_2 s^2$$

which is the characteristic polynomial associated with this system.

Example 53 (An eigenvalue-eigenvector calculation)

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

The characteristic polynomial of A:

$$\begin{aligned} \det(sI - A) &= \det \begin{pmatrix} s-3 & 1 \\ 1 & s-3 \end{pmatrix} \\ &= (s-3)(s-3) - 1 \\ &= s^2 - 6s + 8 \\ &= (s-2)(s-4) \end{aligned}$$

The roots of this polynomial yield *two distinct real eigenvalues*

$$\lambda_1 = 2 \qquad \text{and} \qquad \lambda_2 = 4\,.$$

To compute eigenvectors for λ_1, we use $(\lambda_1 I - A)v = 0$ to obtain

$$\begin{aligned} -v_1 + v_2 &= 0 \\ v_1 - v_2 &= 0 \end{aligned}$$

which is equivalent to

$$v_1 - v_2 = 0$$

So, only one linearly independent eigenvector for λ_1; let's take

$$v = v^1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

In a similar fashion, λ_2 has one linearly independent eigenvector; we take

$$v^2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

To check above calculations,

$$Av^1 = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \lambda_1 v^1$$

Similarly for λ_2.

Example 54

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

Example 55 (Another eigenvalue-eigenvector calculation)

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

The characteristic polynomial of A:

$$\begin{aligned} \det(sI - A) &= \det \begin{pmatrix} s-1 & -1 \\ 1 & s-1 \end{pmatrix} \\ &= (s-1)(s-1)+1 \\ &= s^2 - 2s + 2 \end{aligned}$$

Computing the roots of this polynomial yields *two distinct complex eigenvalues*

$$\lambda_1 = 1 + \jmath \qquad \text{and} \qquad \lambda_2 = 1 - \jmath\,.$$

To compute eigenvectors for λ_1, we use $(\lambda_1 I - A)v = 0$ to obtain

$$\begin{aligned} \jmath v_1 - v_2 &= 0 \\ v_1 + \jmath v_2 &= 0 \end{aligned}$$

which is equivalent to

$$\jmath v_1 - v_2 = 0$$

So, only one linearly independent eigenvector; let's take

$$v = v^1 = \begin{pmatrix} 1 \\ \jmath \end{pmatrix}$$

(We could also take $\begin{pmatrix} \jmath & 1 \end{pmatrix}^T$.) In a similar fashion, λ_2 has one linearly independent eigenvector; we take

$$v^2 = \begin{pmatrix} 1 \\ -\jmath \end{pmatrix}$$

To check above calculations,

$$Av^1 = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \jmath \end{pmatrix} = \begin{pmatrix} 1+\jmath \\ -1+\jmath \end{pmatrix} = (1+\jmath) \begin{pmatrix} 1 \\ \jmath \end{pmatrix} = \lambda_1 v^1$$

Similarly for λ_2.

Exercise 49 Compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$$

Exercise 50 Compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & -2 \end{pmatrix}$$

Exercise 51 Compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 3 \end{pmatrix}$$

Exercise 52 Compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 53 Compute the eigenvalues and eigenvectors of the following matrices.

(a)

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$$

(b)

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 4 & 0 & 3 & 0 \end{pmatrix}$$

Exercise 54 Obtain expressions for the eigenvalues and eigenvectors of

$$A = \begin{pmatrix} \alpha & \omega \\ -\omega & \alpha \end{pmatrix} \qquad \alpha, \omega \neq 0$$

9.4.1 Real A

Recall Examples 53 and 55. They illustrate the following facts.

If A is real, its genuine complex eigenvalues and eigenvectors occur in complex conjugate pairs.

To see this, suppose λ is an eigenvalue of A with eigenvector v. Then

$$Av = \lambda v$$

Taking complex conjugate of both sides of this equation yields

$$\bar{A}\bar{v} = \bar{\lambda}\bar{v}$$

Since A is real, $\bar{A} = A$; hence

$$A\bar{v} = \bar{\lambda}\bar{v}$$

that is, $\bar{\lambda}$ is an eigenvalue of A with eigenvector $\bar{v}$ ■

We also have the following observations.

Real eigenvectors for real eigenvalues.

Genuine complex eigenvectors for (genuine) complex eigenvalues.

9.4.2 MATLAB

```
>> help roots

 ROOTS  Find polynomial roots.
        ROOTS(C) computes the roots of the polynomial whose coefficients
        are the elements of the vector C. If C has N+1 components,
        the polynomial is C(1)*X^N + ... + C(N)*X + C(N+1).

>> help poly

 POLY   Characteristic polynomial.
        If A is an N by N matrix, POLY(A) is a row vector with
        N+1 elements which are the coefficients of the
        characteristic polynomial, DET(lambda*EYE(A) - A) .
        If V is a vector, POLY(V) is a vector whose elements are
        the coefficients of the polynomial whose roots are the
        elements of V . For vectors, ROOTS and POLY are inverse
        functions of each other, up to ordering, scaling, and
        roundoff error.
        ROOTS(POLY(1:20)) generates Wilkinson's famous example.


>> help eig

 EIG    Eigenvalues and eigenvectors.
        EIG(X) is a vector containing the eigenvalues of a square
        matrix X.

        [V,D] = EIG(X) produces a diagonal matrix D of
        eigenvalues and a full matrix V whose columns are the
        corresponding eigenvectors so that X*V = V*D.

        [V,D] = EIG(X,'nobalance') performs the computation with
        balancing disabled, which sometimes gives more accurate results
        for certain problems with unusual scaling.
```

9.4.3 Companion matrices

Recall the second order system described by

$$\ddot{y} + a_1\dot{y} + a_0 y = 0\,.$$

If we introduce the usual state variables, $x_1 = y$ and $x_2 = \dot{y}$ then, this system is described by $\dot{x} = Ax$ where A is the companion matrix given by

$$A = \begin{pmatrix} 0 & 1 \\ -a_0 & -a_1 \end{pmatrix}.$$

The characteristic polynomial of A is given by

$$\det(sI - A) = \begin{vmatrix} s & -1 \\ a_0 & s + a_1 \end{vmatrix} \;=\; s(s+a_1) + a_0 \;=\; s^2 + a_1 s + a_0\,.$$

Recall that this is the polynomial whose roots determine the behavior of this system.

Consider now a system described by an n-th order linear differential equation:

$$y^{(n)} + a_{n-1}y^{(n-1)} + \cdots + a_1\dot{y} + a_0 y = 0\,. \tag{9.13}$$

If we introduce the usual state variables, $x_1 = y_2\,, x_2 = \dot{y},\ \cdots,\ x_n = y^{(n-1)}$, then, this system is described by $\dot{x} = Ax$ where A is the $n \times n$ companion matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{pmatrix} \tag{9.14}$$

Fact 1 *The characteristic polynomial of the $n \times n$ companion matrix in (9.14) is given by*

$$p(s) = a_0 + a_1 s + \ldots + a_{n-1}s^{n-1} + s^n$$

The above fact allows us to obtain the characteristic polynomial of a companion matrix by inspection; we do not have to explicitly compute $\det(sI - A)$.

Exercise 55 Explicitly compute the characteristic polynomial of

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{pmatrix}$$

Exercise 56 Compute the eigenvalues and eigenvectors of the matrix

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

Exercise 57 What is the real 2×2 companion matrix with eigenvalues $1+\jmath$, $1-\jmath$?

9.5 System significance of eigenvectors and eigenvalues

Consider the following system and initial condition:

$$\dot{x} = Ax\,; \qquad x(0) = x_0 \tag{9.15}$$

The following result is fundamental to the behavior of linear systems. It is the reason eigenvalues and eigenvectors are important in considering the behavior of linear systems.

Suppose that v is an eigenvector of A and λ is its corresponding eigenvalue. Then the solution of (9.15) with initial state $x_0 = v$ is given by

$$\boxed{x(t) = e^{\lambda t}v}$$

This above special solution sometimes called a mode of the system or a modal solution of the differential equation.

Proof. Clearly $x(0) = v$. Differentiating the above expression for $x(t)$ and using $Av = \lambda v$ yields

$$\begin{aligned} \dot{x}(t) &= e^{\lambda t}\lambda v \\ &= e^{\lambda t}Av \\ &= A(e^{\lambda t}v) \\ &= Ax(t) \end{aligned}$$

∎

Since v is an eigenvector, $e^{\lambda t}v$ is also an eigenvector for each t, so, we can make the following conclusion:

Once an eigenvector, always an eigenvector!

Example 56 Recall example 53 in which

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

and $\lambda = 2$ is an eigenvalue with $v = (1,1)$ a corresponding eigenvalue. Hence, the initial state

$$x_0 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

results in the solution

$$x(t) = \begin{pmatrix} e^{2t} \\ e^{2t} \end{pmatrix}$$

What is the solution for

$$x_0 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}?$$

Example 57

$$m\ddot{q} + (k - m\Omega^2)q = 0\,.$$

with $k < m\Omega^2$.

Roughly speaking, the converse of the above main result is also true; specifically, if the above system has a nonzero solution of the form $e^{\lambda t}v$, then λ is an eigenvalue of A and v is a corresponding eigenvector. To see this note that $\dot{x} = Ax$ implies that

$$\lambda e^{\lambda t}v = A(e^{\lambda t}v)$$

Since $e^{\lambda t} \neq 0$, we obtain $Av = \lambda v$. Since $v \neq 0$, this last equation implies that λ is an eigenvalue of A and v is a corresponding eigenvector. ■

9.5.1 Real systems and complex eigenvalues

Suppose we are interested in the behavior of a 'real' system with real state x and real system matrix A and suppose λ is a (genuine) complex eigenvalue of A, that is,

$$Av = \lambda v$$

for some nonzero vector v. Since A is real and λ is complex, the eigenvector v is complex. The complex system can start with a complex initial state but the 'real' system cannot. So are the above results useful? They are, just wait.

Since A is real, $\bar{\lambda}$ is also and eigenvalue of A with corresponding eigenvector $\bar{v}$; hence the (complex) solution corresponding to $x_0 = \bar{v}$ is given by

$$e^{\bar{\lambda}t}\bar{v} = \overline{e^{\lambda t}v}$$

Let

$$v = u + \jmath w$$

where u and w are real vectors. Thus u is the real part of v and w is the imaginary part of v. Now note that

$$u = \frac{1}{2}(v + \bar{v})$$

Consider now the solution corresponding to the real initial initial state $x_0 = u$. Since the solution depends linearly on initial state, the solution for this real initial state is

$$\frac{1}{2}(e^{\lambda t}v + \overline{e^{\lambda t}v}\,) \quad = \quad \text{real part of } e^{\lambda t}v$$

Let

$$\boxed{\lambda = \alpha + \jmath\omega}$$

where α and ω are real numbers. Thus α is the real part of λ and ω is the imaginary part of λ. Then,

$$e^{\lambda t} = e^{(\alpha+\jmath\omega)t} = e^{\alpha t}e^{\jmath\omega t} = e^{\alpha t}(\cos\omega t + \jmath\sin\omega t)$$

and

$$\begin{aligned} e^{\lambda t}v &= e^{\alpha t}(\cos\omega t + \jmath \sin\omega t)(u + \jmath w) \\ &= [e^{\alpha t}\cos(\omega t)\, u - e^{\alpha t}\sin(\omega t)\, w\,] \;+\; \jmath\left(e^{\alpha t}\sin(\omega t)\, u + e^{\alpha t}\cos(\omega t)\, w\right) \end{aligned}$$

Since the solution due to $x_0 = u$ is the real part of $e^{\lambda t}v$, this solution is given by

$$x(t) = e^{\alpha t}\cos(\omega t)\, u - e^{\alpha t}\sin(\omega t)\, w$$

Similarly, one can show that the solution due to initial state $x_0 = w$ is the imaginary part of $e^{\lambda t}v$, that is,

$$x(t) = e^{\alpha t}\sin(\omega t)\, u + e^{\alpha t}\cos(\omega t)\, w$$

We can now make the following conclusion: If the initial state is of the form

$$x_0 = \xi_{10}u + \xi_{20}w$$

where ξ_{10}, ξ_{20} are any two real scalars (that is, the initial state is in the real two-dimensional plane spanned by u, w) then, the resulting solution is given by

$$x(t) = \xi_1(t)u + \xi_2(t)w$$

where

$$\begin{aligned} \xi_1(t) &= e^{\alpha t}\cos(\omega t)\xi_{10} + e^{\alpha t}\sin(\omega t)\xi_{20} \\ \xi_2(t) &= -e^{\alpha t}\sin(\omega t)\xi_{10} + e^{\alpha t}\cos(\omega t)\xi_{20} \end{aligned}$$

In component form, this implies that the behavior of each state variable can be described

$$\boxed{x_i(t) = X_i e^{\alpha t}\cos(\omega t - \phi_i) \qquad \text{for} \quad i = 1, 2, \cdots, n\,.}$$

where $X_1, \ldots, X_n$ and $\phi_1, \ldots, \phi_n$ are constants which depend on initial conditions. So, we can make the following conclusion for real systems and genuinely complex eigenvectors.

If the initial state is a linear combination of the real and imaginary parts of a complex eigenvector, then so is the resulting solution.

Example 58 Recall example 55. Considering

$$v = v^1 = \begin{pmatrix} 1 \\ \jmath \end{pmatrix}$$

we have

$$u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad w = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

So,

$$x = x_1 u + x_2 w$$

Hence, all solutions are given by

$$\begin{aligned} x_1(t) &= e^{\alpha t}\cos(\omega t)x_{10} + e^{\alpha t}\sin(\omega t)x_{20} \\ x_2(t) &= -e^{\alpha t}\sin(\omega t)x_{10} + e^{\alpha t}\cos(\omega t)x_{20} \end{aligned} \quad = \quad \begin{pmatrix} e^{\alpha t}\cos(\omega t) & e^{\alpha t}\sin(\omega t) \\ -e^{\alpha t}\sin(\omega t) & e^{\alpha t}\cos(\omega t) \end{pmatrix} x_0$$

Here the $u - w$ plane is the whole space.

Example 59 Beavis and Butthead

$$\begin{aligned} m\ddot{q}_1 &= k(q_2 - q_1) \\ m\ddot{q}_2 &= -k(q_2 - q_1) \end{aligned}$$

Introducing state variables

$$x_1 := q_1 \,, \quad x_2 := q_2 \,, \quad x_3 := \dot{q}_1 \,, \quad x_4 := \dot{q}_2 \,,$$

this system is described by $\dot{x} = Ax$ where

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{k}{m} & \frac{k}{m} & 0 & 0 \\ \frac{k}{m} & -\frac{k}{m} & 0 & 0 \end{pmatrix}$$

The characteristic polynomial of A is given by

$$p(s) = s^2(s^2 + \omega^2) \,, \qquad \omega = \sqrt{2k/m}$$

Hence, A has three eigenvalues

$$\lambda_1 = \jmath\omega \,, \qquad \lambda_2 = -\jmath\omega \,, \qquad \lambda_3 = 0 \,.$$

Corresponding to each eigenvalue λ_i, there is at most one linearly independent eigenvector v^i, e.g.,

$$v^1 = \begin{pmatrix} 1 \\ -1 \\ \jmath\omega \\ -\jmath\omega \end{pmatrix} \,, \qquad v^2 = \overline{v^1} \,, \qquad v^3 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \,.$$

Consider now the plane spanned by the real and imaginary parts of v^1, that is. the plane spanned by

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ \omega \\ -\omega \end{pmatrix}$$

This is the same as the plane spanned by the vectors:

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \end{pmatrix} , \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}$$

This plane is characterized by

$$q_2 = -q_1 \qquad \dot{q}_2 = -\dot{q}_1$$

Hence every state in this plane has the form

$$\begin{pmatrix} q_1 \\ -q_1 \\ \dot{q}_1 \\ -\dot{q}_1 \end{pmatrix},$$

Whenever a motion originates in this plane it remains there; also all motions in this plane are purely oscillatory and consist of terms of the form $A\cos(\omega t - \phi)$.

Consider now the line spanned by the eigenvector v^3. Every state in this line has the form

$$\begin{pmatrix} q_1 \\ q_1 \\ 0 \\ 0 \end{pmatrix},$$

that is, q_1 is arbitrary, $q_2 = q_1$, and $\dot{q}_1 = \dot{q}_2 = 0$. Since the eigenvalue corresponding to this eigenspace is zero, every state in this line is an equilibrium state.

To examine how the size of a modal solution changes with time, we need to define what we mean by the size or magnitude of a vector.

9.6 A characterization of all solutions

We first make the following observation: Suppose that $v^1, \dots, v^m$ are any eigenvectors of A with corresponding eigenvalues, $\lambda_1, \dots, \lambda_m$ and consider any initial state of the form

$$x_0 = \xi_{10} v^1 + \dots + \xi_{m0} v^m \tag{9.16}$$

where $\xi_{10}, \dots, \xi_{m0}$ are scalars. Then the corresponding solution of (9.15) is given by

$$x(t) = \xi_{10} e^{\lambda_1 t} v^1 + \dots + \xi_{m0} e^{\lambda_m t} v^m \tag{9.17}$$

The big question now is whether or not every initial state x_0 can be expressed as in (9.16), that is, as a linear combination of eigenvectors of A. The answer to this question is yes, except in special cases. If an $n \times n$ matrix A has n distinct eigenvalues or if A is symmetric, then the answer is always yes. If A has an eigenvalue which is a repeated root of its characteristic polynomial, the answer may be no. This is illustrated in Example 60. Note that, if v is an eigenvalue of a matrix A then so is ξv for any scalar ξ. So we have the following conclusion:

Except in special cases, every solution of a linear time-invariant system $\dot{x} = Ax$ can be expressed as

$$\boxed{x(t) = e^{\lambda_1 t} v^1 + \dots + e^{\lambda_m t} v^m} \tag{9.18}$$

where $v^1, \dots, v^m$ are eigenvectors of A with corresponding eigenvalues, $\lambda_1, \dots, \lambda_m$.

9.6.1 Repeated roots

When an eigenvalue λ is a repeated root of the characteristic polynomial then one may have solutions containing terms of the form

$$t^k e^{\lambda t}$$

where k is a nonzero positive integer, that is, $k = 1, 2, \cdots$. This is illustrated in the following example.

Example 60 Recall the A matrix for the unattached mass:

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$

The characteristic polynomial for this matrix is

$$p(s) = s^2 .$$

Hence, this matrix has a single eigenvalue $\lambda = 0$ which is a repeated root of the characteristic polynomial. All eigenvectors are of the form

$$v = \begin{pmatrix} v_1 \\ 0 \end{pmatrix} .$$

Thus, every state cannot be expressed as a linear combination of eigenvectors, for example, consider $x_0 = (0,\ 1)$. This initial state results in the solution

$$x(t) = \begin{pmatrix} t \\ 1 \end{pmatrix}$$

9.7 Phugoid mode and short period mode

Recall that if we use state variables V, α, θ, q, to describe the longitudinal dynamics of an aircraft, then the linearization of the nonlinear dynamics about any trim condition $V^e, \alpha^e, \theta^e, 0$ can be described by

$$\delta\dot{x} = A\delta x$$

where $\delta x = \begin{pmatrix} \delta V & \delta\alpha & \delta\theta & \delta q \end{pmatrix}^T$ and A is a 4×4 matrix.

For most stable aircraft and trim conditions, the matrix A has four (genuine) complex eigenvalues. Since A is real, this results in two complex conjugate pairs of eigenvalues

$$\lambda_p = \alpha_p + \jmath\omega_p \qquad \bar{\lambda}_p = \alpha_p - \jmath\omega_p$$

$$\lambda_s = \alpha_s + \jmath\omega_s \qquad \bar{\lambda}_s = \alpha_s - \jmath\omega_s$$

with

$$\alpha_s, \alpha_p < 0$$

Usually

$$\omega_s >> \omega_p \qquad \alpha_s << \alpha_p$$

hence the mode corresponding to the eigenvalues $\lambda_s, \bar{\lambda}_s$ is called the short period mode. The mode corresponding to $\lambda_p, \bar{\lambda}_p$ is called the phugoid mode.

The short period eigenvalues give rise to terms of the form

$$e^{\alpha_s t}\cos(\omega_s t)$$

The phugoid eigenvalues give rise to terms of the form

$$e^{\alpha_p t}\cos(\omega_p t)$$

9.8 The magnitude of a vector

What is the size or magnitude of a vector? Meet norm.

- Consider any complex n-vector x.
  The Euclidean norm or 2-norm of x is the nonegative real number given by

$$\boxed{||x|| := (|x_1|^2 + \ldots + |x_n|^2)^{\frac{1}{2}}}$$

Note that

$$\begin{aligned} ||x|| &= (\bar{x}_1 x_1 + \ldots + \bar{x}_n x_n)^{\frac{1}{2}} \\ &= (x^* x)^{\frac{1}{2}} \end{aligned}$$

If x is *real*, these expressions become

$$\begin{aligned} ||x|| &= (x_1^2 + \ldots + x_n^2)^{\frac{1}{2}} \\ &= (x^T x)^{\frac{1}{2}} \end{aligned}$$

```
>>  norm([3; 4])

ans = 5

>>  norm([1; j])

ans = 1.4142
```

Note that in the last example $x^T x = 0$, but $x^* x = 2$.

Properties of $||\cdot||$. The Euclidean norm has the following properties.

(i) For every vector x,

$$||x|| \geq 0$$

and

$$||x|| = 0 \qquad \text{if and only if} \qquad x = 0$$

(ii) (*Triangle Inequality.*) For every pair of fectors x, y,

$$||x + y|| \leq ||x|| + ||y||$$

(iii) For every vector x and every scalar λ.

$$||\lambda x|| = |\lambda| ||x||$$

Any real valued function on a vector space which has the above three properties is defined to be a norm. You may meet other norms later.

Let us now examine how the magnitude of a modal solution $x(t) = e^{\lambda t}v$ changes with time. Let

$$\lambda = \alpha + \jmath\omega\,,$$

that is, α are ω are the real and imaginary parts of λ. Then,

$$|e^{\lambda t}| = |e^{(\alpha+\jmath\omega)t}| = |e^{\alpha t}||e^{\jmath\omega t}| \;=\; e^{\alpha t}\,;$$

hence

$$\|x(t)\| = ||e^{\lambda t}v|| = |e^{\lambda t}|\,\|v\| \;=\; e^{\alpha t}\,\|v\|$$

where $\alpha = \Re(\lambda)$. So,

(a) If $\Re(\lambda) < 0$ you decay with increasing age.

(b) If $\Re(\lambda) > 0$ you grow with increasing age.

(c) If $\Re(\lambda) = 0$ your magnitude remains constant.

(e) If $\lambda = 0$ you stay put; every eigenvector corresponding to $\lambda = 0$ is an equilibrium state.

9.9 Similarity transformations

Motivation. Consider a system described by

$$\begin{aligned} \dot{x}_1 &= 3x_1 - x_2 \\ \dot{x}_2 &= -x_1 + 3x_2 \end{aligned}$$

If we introduce new states,

$$\begin{aligned} \xi_1 &= (x_1 + x_2)/2 \\ \xi_2 &= (x_1 - x_2)/2 \end{aligned}$$

then, the system is simply described by

$$\begin{aligned} \dot{\xi}_1 &= 2\xi_1 \\ \dot{\xi}_2 &= 4\xi_2\,. \end{aligned}$$

This new description consists of two uncoupled scalar systems. Where did the new states come from? First note that

$$\begin{aligned} x_1 &= \xi_1 + \xi_2 \\ x_2 &= \xi_1 - \xi_2 \end{aligned}$$

and we obtain the following geometric interpretation of the new states. Since,

$$x = x_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \xi_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \xi_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

the scalars x_1, x_2 are the coordinates of the vector x with respect to the usual basis for $\mathbb{R}^2$, The scalars ξ_1, ξ_2 are the coordinates of x with respect to the basis

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Where did these new basis vectors come from? Actually, these new basis vectors are eigenvectors of the matrix

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

and the system is described by $\dot{x} = Ax$.

Coordinate transformations. Suppose we have n scalar variables $x_1, x_2, \ldots, x_n$ and we implicitly define n new scalar variables $\xi_1, \xi_2, \ldots, \xi_n$ by the relationship

$$\boxed{x = T\xi}$$

where T is an $n \times n$ invertible matrix. Then, ξ is explicitly given by

$$\xi = T^{-1}x$$

We can obtain a geometric interpretation of this change of variables as follows. First, observe that

$$\underbrace{\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}}_{x} = x_1 \underbrace{\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{e^1} + x_2 \underbrace{\begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}}_{e^2} + \ldots + x_n \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}}_{e^n},$$

that is, the scalars $x_1, x_2, \ldots, x_n$ are the coordinates of the vector x with respect to the standard basis $e^1, e^2, \ldots, e^n$, or, x is the coordinate vector of itself with respect to the standard basis. Suppose

$$T = \begin{pmatrix} v^1 & \ldots & v^j & \ldots & v^n \end{pmatrix}$$

that is, v^j is the j-th column of T. Then, the bunch $v^1, v^2, \ldots, v^n$ form a basis for $\mathbb{C}^n$ and $x = T\xi$ can be written as

$$x = \xi_1 v^1 + \xi_2 v^2 + \ldots + \xi_n v^n$$

From this we see that $\xi_1, \xi_2, \ldots, \xi_n$ are the coordinates of x with respect to to the new basis and the vector ξ is the coordinate vector of the vector x with respect to this new basis. So $x = T\xi$ defines a coordinate transformation.

Similarity transformations. Consider now a system described by

$$\dot{x} = Ax$$

and suppose we implicitly define a new state vector ξ by $x = T\xi$ where T is a nonsingular square matrix; then the behavior of ξ is governed by

$$\dot{\xi} = \Lambda\xi$$

where

$$\boxed{\Lambda = T^{-1}AT}$$

DEFN. A square matrix Λ similar to another square matrix A if there exists a nonsingular matrix T such that $\Lambda = T^{-1}AT$.

Example 61 Recall example 53. Note that the eigenvectors v^1 and v^2 are linearly independent; hence the matrix

$$T := \begin{pmatrix} v^1 & v^2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

is invertible. Computing $T^{-1}AT$, we obtain

$$\Lambda = T^{-1}AT = \begin{pmatrix} 2 & 0 \\ 0 & 4 \end{pmatrix}$$

Example 62 Recalling example 55 we can follow the procedure of the last example and define

$$T := \begin{pmatrix} 1 & 1 \\ \jmath & -\jmath \end{pmatrix}$$

to obtain

$$\Lambda = T^{-1}AT = \begin{pmatrix} 1+\jmath & 0 \\ 0 & 1-\jmath \end{pmatrix}$$

Example 63 B&B

Some properties: If A is similar to Λ then the following hold

(a) $\det A = \det \Lambda$

(b) charpoly A = charpoly Λ and, hence, A and Λ have the same eigenvalues with the same algebraic multiplicities.

9.10 Diagonalizable systems

9.10.1 Diagonalizable matrices

We say that a square matrix A is diagonalizable if it is similar to a diagonal matrix, that is, there is an invertible matrix T so that $T^{-1}AT$ is diagonal. We have the following useful fact.

Fact 2 *Suppose that an $n \times n$ matrix A has n* distinct *eigenvalues $\lambda_1, \cdots, \lambda_n$ and let*

$$T := \begin{pmatrix} v^1 & v^2 & \dots & v^n \end{pmatrix}$$

where v^i is an eigenvector corresponding to λ_i. Then, T is invertible and

$$\begin{aligned} T^{-1}AT &= \textit{diag}\,(\lambda_1, \lambda_2, \dots, \lambda_n) \\ &:= \underbrace{\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}}_{\textit{Complex Jordan form of } A} \end{aligned}$$

The above fact tells us that if A has n distinct eigenvalues, then it is similar to a diagonal matrix.

Example 64

$$A = \begin{pmatrix} 4 & -2 & -2 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

The characteristic polynomial of A is

$$p(s) = \det(sI - A) = (s-4)[(s-1)(s-1) - 1] = s(s-2)(s-4)$$

So A has 3 distinct eigenvalues,

$$\lambda_1 = 0, \qquad \lambda_2 = 2, \qquad \lambda_3 = 4$$

Since the the 3×3 matrix A has 3 distinct eigenvalues, A is diagonalizable and its Jordan form is

$$\Lambda := \begin{pmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{pmatrix}$$

Let us check this by computing the eigenvectors of A.

The equation $(A - \lambda I)v = 0$ yields:

$$\begin{array}{rrrcl} (4-\lambda)v_1 & -2v_2 & -2v_3 & = & 0 \\ & (1-\lambda)v_2 & -v_3 & = & 0 \\ & -v_2 & +(1-\lambda)v_3 & = & 0 \end{array}$$

For $\lambda_1 = 0$, we obtain

$$\begin{array}{rrrcl} 4v_1 & -2v_2 & -2v_3 & = & 0 \\ & v_2 & -v_3 & = & 0 \\ & -v_2 & +v_3 & = & 0 \end{array}$$

Hence, an eigenvector v^1 for λ_1 is given by

$$v^1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

For $\lambda_2 = 2$, we obtain

$$\begin{array}{rrrcl} 2v_1 & -2v_2 & -2v_3 & = & 0 \\ & -v_2 & -v_3 & = & 0 \\ & -v_2 & -v_3 & = & 0 \end{array}$$

Hence, an eigenvector v^2 for λ_2 is given by

$$v^2 = \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$$

For $\lambda_3 = 4$, we obtain

$$\begin{array}{rrcl} -2v_2 & -2v_3 & = & 0 \\ -3v_2 & -v_3 & = & 0 \\ -v_2 & -3v_3 & = & 0 \end{array}$$

Hence, an eigenvector v^3 for λ_3 is given by

$$v^3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

A quick check with MATLAB yields $\Lambda = T^{-1}AT$ where $T = \begin{pmatrix} v^1 & v^2 & v^3 \end{pmatrix}$.

9.10.2 Diagonalizable systems

Consider a system

$$\dot{x} = Ax$$

and suppose A is diagonalizable, that is, there is a nonsingular matrix T such that

$$\Lambda = T^{-1}AT$$

is diagonal; hence the diagonal elements of Λ are the eigenvalues $\lambda_1, \ldots, \lambda_n$ of A.

Introducing the state transformation,

$$x = T\xi$$

we have $\xi = T^{-1}x$ and its behavior is described by:

$$\dot{\xi} = \Lambda \xi$$

Hence,

$$\begin{aligned}
\dot{\xi}_1 &= \lambda_1 \xi_1 \\
\dot{\xi}_2 &= \lambda_2 \xi_2 \\
&\vdots \\
\dot{\xi}_n &= \lambda_n \xi_n
\end{aligned}$$

So,

$$\xi_i(t) = e^{\lambda_i t}\xi_{i0}, \qquad i = 1, 2, \ldots, n$$

or,

$$\xi(t) = \begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix} \xi_0$$

where $\xi_o = \xi(0)$ and ξ_{0i} is the i-th component of ξ_0. If $x(0) = x_0$ then,

$$\xi_0 = T^{-1}x_0$$

Since

$$x = \xi_1 v^1 + \xi_2 v^2 + \ldots + \xi_n v^n$$

where $v^1, v^2, \ldots, v^n$ are the columns of T, the solution for $x(t)$ is given by

$$\boxed{x(t) = e^{\lambda_1 t}\xi_{10}v^1 + e^{\lambda_2 t}\xi_{20}v^2 + \ldots + e^{\lambda_n t}\xi_{n0}v^n}$$

Or,

$$x(t) = \Phi(t)x_0$$

where the state transition matrix $\Phi(\cdot)$ is given by

$$\boxed{\Phi(t) := T \begin{pmatrix} e^{\lambda_1 t} & 0 & \ldots & 0 \\ 0 & e^{\lambda_2 t} & \ldots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \ldots & e^{\lambda_n t} \end{pmatrix} T^{-1}}$$

Example 65 Consider a system with A from example 53, that is,

$$A = \begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix}$$

Here,

$$T = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

and $\lambda_1 = 2, \lambda_2 = 4$.

Hence,

$$\begin{aligned} \Phi(t) &= T \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{4t} \end{pmatrix} T^{-1} \\ &= \frac{1}{2} \begin{pmatrix} e^{2t} + e^{4t} & e^{2t} - e^{4t} \\ e^{2t} - e^{4t} & e^{2t} + e^{4t} \end{pmatrix} \end{aligned}$$

and all solutions are given by

$$\begin{aligned} x_1(t) &= \frac{1}{2}(e^{2t} + e^{4t})x_{10} + \frac{1}{2}(e^{2t} - e^{4t})x_{20} \\ x_2(t) &= \frac{1}{2}(e^{2t} - e^{4t})x_{10} + \frac{1}{2}(e^{2t} + e^{4t})x_{20} \end{aligned}$$

Exercise 58 Compute the state transition matrix for the system corresponding to the A matrix in example 64.

9.11 Real Jordan form for real diagonalizable systems

Consider a *real* $n \times n$ matrix A with n distinct eigenvalues. Suppose

$$\lambda_1, \cdots, \lambda_l$$

are real eigenvalues with real eigenvectors

$$v^1, \cdots, v^l$$

Suppose the remaining $2m = n - l$ eigenvalues are genuine complex and given by

$$\begin{array}{ccc} \alpha_1 + j\omega_1\,, & \ldots\,, & \alpha_m + j\omega_m \\ \alpha_1 - j\omega_1\,, & \ldots\,, & \alpha_m - j\omega_m \end{array}$$

where α_i and $\omega_i \neq 0$ are real. Suppose the corresponding complex eigenvectors are given by

$$\begin{array}{ccc} u^1 + \jmath w^1\,, & \ldots, & u^m + \jmath w^m \\ u^1 - \jmath w^1\,, & \ldots\,, & u^m - \jmath w^m \end{array}$$

where u^i and w^i are real.

Letting

$$T = \begin{pmatrix} v^1 & \cdots & v^l & u^1 & w^1 & \ldots & u^m & w^m \end{pmatrix}$$

it can be shown that T is invertible and $\Lambda = T^{-1}AT$ is given by

$$\begin{aligned} \Lambda &= \text{block-diag}\left(\lambda_1, \ldots, \lambda_l, \begin{pmatrix} \alpha_1 & \omega_1 \\ -\omega_1 & \alpha_1 \end{pmatrix}, \ldots, \begin{pmatrix} \alpha_m & \omega_m \\ -\omega_m & \alpha_m \end{pmatrix}\right) \\ &= \underbrace{\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & & \\ & & \lambda_l & & & \\ & & & \begin{pmatrix} \alpha_1 & \omega_1 \\ -\omega_1 & \alpha_1 \end{pmatrix} & & \\ & & & & \ddots & \\ & & & & & \begin{pmatrix} \alpha_m & \omega_m \\ -\omega_m & \alpha_m \end{pmatrix} \end{pmatrix}}_{\text{Real Jordan form of } A} \end{aligned}$$

Example 66 Recall example 55.

Example 67

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

By inspection (recall companion matrices),

$$\det(sI - A) = s^4 - 1$$

So eigenvalues are:

$$\lambda_1 = 1 \quad \lambda_2 = -1 \quad \lambda_2 = \jmath \quad \lambda_3 = -\jmath$$

Since there are $n = 4$ distinct eigenvalues, the complex Jordan form is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \jmath & 0 \\ 0 & 0 & 0 & -\jmath \end{pmatrix}$$

The real Jordan form is:

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

The solutions of

$$\dot{x} = Ax, \qquad x(0) = x_0$$

are given by

$$x(t) = \Phi(t)x_0$$

where the state transition matrix $\Phi(\cdot)$ is given by

$$\Phi(t) = T\hat{\Phi}(t)T^{-1}$$

with $\hat{\Phi}(t)$ being the block diagonal matrix:

$$\text{block-diag}\left(e^{\lambda_1 t}, \ldots, e^{\lambda_l t}, \begin{pmatrix} e^{\alpha_1 t}\cos(\omega_1 t) & e^{\alpha_1 t}\sin(\omega_1 t) \\ -e^{\alpha_1 t}\sin(\omega_1 t) & e^{\alpha_1 t}\cos(\omega_1 t) \end{pmatrix}, \ldots, \begin{pmatrix} e^{\alpha_m t}\cos(\omega_m t) & e^{\alpha_m t}\sin(\omega_m t) \\ -e^{\alpha_m t}\sin(\omega_m t) & e^{\alpha_m t}\cos(\omega_m t) \end{pmatrix}\right)$$

9.12 Exercises

Exercise 59 For each of the following second order systems, determine whether they are stable, unstable, or neutrally stable about the zero solution.

(a)

$$\ddot{y} - \dot{y} + y = 0$$

(b)

$$-\ddot{y} - \dot{y} - y = 0$$

(c)

$$\ddot{y} + \dot{y} = 0$$

(d)

$$\ddot{y} - \dot{y} = 0$$

(e)

$$\ddot{y} + \dot{y} - \sin y = 0$$

(f)

$$\ddot{y} + \dot{y}^3 + y = 0$$

Chapter 10

e^{At}

10.1 State transition matrix

Consider a linear time-invariant system described by

$$\begin{aligned} \dot{x} &= Ax & (10.1a) \\ x(t_0) &= x_0 & (10.1b) \end{aligned}$$

where the n-vector $x(t)$ is the system state at time t. For each initial time t_0 and each initial state x_0, there is an unique solution $x(\cdot)$ to (10.1) and this solution is defined for all time t. Due to linearity and time-invariance, we must have

$$\boxed{x(t) = \Phi(t-t_0)x_0}$$

for some matrix $\Phi(t - t_0)$; we call this guy the state transition matrix associated with the system $\dot{x} = Ax$. Note that this matrix is independent of x_0 and depends only on $t-t_0$. Since, the solution only depends on $t - t_0$, we can consider $t_0 = 0$ wlog (without loss of generality). Also, Φ must satisfy the following *matrix differential equation* and initial condition:

$$\boxed{\begin{aligned} \dot{\Phi} &= A\Phi \\ \Phi(0) &= I \end{aligned}}$$

Note that the above two conditions on Φ uniquely specify Φ (Why?) The purpose of this section is to examine Φ and look at ways of evaluating it.

10.2 Polynomials of a square matrix

We define *polynomials of a matrix* as follows. Suppose p is a polynomial given by

$$p(s) = a_0 + a_1 s + a_2 s^2 + \ldots + a_m s^m \,,$$

where s and $a_0, \ldots, a_m$ are scalars, and A is a square matrix. Then, we define $p(A)$ by

$$p(A) := a_0 I + a_1 A + a_2 A^2 + \ldots + a_m A^m$$

- The following properties can be readily deduced from the above definition.

(a) The matrices A and $p(A)$ commute, that is,

$$Ap(A) = p(A)A$$

(b) Suppose A is similar to another matrix B, that is, for some nonsingular T,

$$A = TBT^{-1} \,.$$

Then

$$p(A) = Tp(B)T^{-1} \,.$$

(c) Suppose A is diagonal, that is,

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} .$$

Then

$$p(A) = \begin{bmatrix} p(\lambda_1) & 0 & \cdots & 0 \\ 0 & p(\lambda_2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & p(\lambda_n) \end{bmatrix} .$$

(d) Suppose A is diagonalizable, that is, for some nonsingular T,

$$A = T \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} T^{-1} .$$

Then

$$p(A) = T \begin{bmatrix} p(\lambda_1) & 0 & \cdots & 0 \\ 0 & p(\lambda_2) & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & p(\lambda_n) \end{bmatrix} T^{-1} .$$

(e) If λ is an eigenvalue of A with eigenvector v, then $p(\lambda)$ is an eigenvalue of $p(A)$ with eigenvector v.

Properties (b)-(d) are useful for computing $p(A)$.

Example 68 For the matrix

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}.$$

we found that

$$T = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

results in

$$\Lambda = T^{-1}AT = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$$

Hence, $A = T\Lambda T^{-1}$ and

$$\begin{aligned} p(A) &= Tp(\Lambda)T^{-1} \\ &= T \begin{bmatrix} p(2) & 0 \\ 0 & p(4) \end{bmatrix} T^{-1} \\ &= \frac{1}{2} \begin{bmatrix} p(2)+p(4) & p(2)-p(4) \\ p(2)-p(4) & p(2)+p(4) \end{bmatrix}. \end{aligned}$$

Suppose p is the characteristic polynomial of A. Since $2, 4$ are the eigenvalues of A, we have $p(2) = p(4) = 0$; hence $p(A) = 0$.

10.3 The matrix exponential: e^A

Recall the exponential function:

$$f(\lambda) = e^\lambda$$

For any complex number λ, e^λ is given by the sum of a convergent series:

$$e^\lambda = \sum_{k=0}^{\infty} \frac{1}{k!}\lambda^k = 1 + \lambda + \frac{1}{2}\lambda^2 + \frac{1}{3!}\lambda^3 + \dots$$

The exponential of any matrix A is defined by:

$$\boxed{e^A := \sum_{k=0}^{\infty} \frac{1}{k!}A^k = I + A + \frac{1}{2!}A^2 + \dots}$$

The following properties can be readily deduced from the above definition.

(a) The matrices A and e^A commute, that is,

$$Ae^A = e^A A$$

(b) Suppose A is similar to another matrix B, that is, for some nonsingular T,

$$A = TBT^{-1}.$$

Then

$$e^A = Te^B T^{-1}.$$

(c) Suppose A is diagonal, that is,

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Then

$$e^A = \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix}.$$

(d) Suppose A is diagonalizable, that is, for some nonsingular T,

$$A = T\begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}T^{-1}.$$

Then

$$e^A = T \begin{bmatrix} e^{\lambda_1} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n} \end{bmatrix} T^{-1}$$

(e) If λ is an eigenvalue of A with eigenvector v, then e^λ is an eigenvalue of e^A with eigenvector v.

10.4 The state transition matrix: e^{At}

Using the above definitions and abusing notation (e^{At} instead of e^{tA}),

$$\boxed{e^{At} \;:=\; \sum_{k=0}^{\infty} \frac{1}{k!}(tA)^k \quad = \quad I + tA + \frac{1}{2!}(tA)^2 + \frac{1}{3!}(tA)^3 + \ldots}$$

Letting

$$\Phi(t) := e^{At}$$

we have

$$\begin{aligned} \Phi(0) &= I \\ \dot{\Phi} &= A\Phi \end{aligned}$$

Hence, the solution of

$$\dot{x} = Ax \qquad x(0) = x_0$$

is given by

$$\boxed{x(t) = e^{At}x_0}$$

Some further properties of e^{At}. From the previous section, we obtain the following properties:

(a) The matrices A and e^{At} commute, that is,

$$Ae^{At} = e^{At}A$$

(b) Suppose A is similar to another matrix B, that is, for some nonsingular T,

$$A = TBT^{-1}\,.$$

Then

$$e^{At} = Te^{Bt}T^{-1}\,.$$

(c) Suppose A is diagonal, that is,

$$A = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}.$$

Then

$$e^{At} = \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix}.$$

(d) Suppose A is diagonalizable, that is, for some nonsingular T,

$$A = T \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix} T^{-1}.$$

Then

$$e^{At} = T \begin{bmatrix} e^{\lambda_1 t} & 0 & \cdots & 0 \\ 0 & e^{\lambda_2 t} & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & 0 & \cdots & e^{\lambda_n t} \end{bmatrix} T^{-1}$$

(e) If λ is an eigenvalue of A with eigenvector v, then $e^{\lambda t}$ is an eigenvalue of e^{At} with eigenvector v.

Properties (b)-(d) are useful for computing e^{At}.

10.5 Computation of e^{At}

10.5.1 MATLAB

```
>> help expm

 EXPM   Matrix exponential.
        EXPM(X) is the matrix exponential of X.  EXPM is computed using
        a scaling and squaring algorithm with a Pade approximation.
        Although it is not computed this way, if X has a full set
        of eigenvectors V with corresponding eigenvalues D, then
        [V,D] = EIG(X) and EXPM(X) = V*diag(exp(diag(D)))/V.
        See EXPM1, EXPM2 and EXPM3 for alternative methods.

        EXP(X) (that's without the M) does it element-by-element.
```

10.5.2 Numerical simulation

Numerically solve

$$\dot{\Phi} = A\Phi, \quad \Phi(0) = I\,.$$

This can be carried out in in MATLAB as follows. For $i = 1, 2, \cdots, n$ let e_i be the n-vector corresponding to the i-th column of the $n \times n$ identiryt matrix, that is, the i-th element of e_i is one while all other elements are zero. Then use a differential solver to solve

$$\begin{aligned}
\dot{\phi}_1 &= A\phi_1 \qquad \phi_1(0) = e_1\\
\dot{\phi}_2 &= A\phi_2 \qquad \phi_2(0) = e_2\\
&\vdots\\
\dot{\phi}_n &= A\phi_n \qquad \phi_n(0) = e_n
\end{aligned}$$

Funally, let

$$\Phi(t) = \begin{bmatrix} \phi_1(t) & \phi_2(t) & \cdots & \phi_n(t) \end{bmatrix}$$

10.5.3 Jordan form

$$\boxed{e^{At} = Te^{\Lambda t}T^{-1}}$$

where $\Lambda = T^{-1}AT$ and Λ is the Jordan form of A. We have already seen this for diagonalizable systems.

10.5.4 Laplace style

$$\dot{x} = Ax \qquad\qquad x(0) = x_0$$

Suppose

$$X(s) = \mathcal{L}(x)(s)$$

is the *Laplace transform* of $x(\cdot)$ evaluated at $s \in \mathbb{C}$. Taking the Laplace transform of $\dot{x} = Ax$ yields:

$$sX(s) - x_0 = AX(s)$$

Hence, except when s is an eigenvalue of A, $sI - A$ is invertible and

$$\mathcal{L}(x)(s) = X(s) = (sI - A)^{-1}x_0$$

Since $x(t) = e^{At}x_0$ for all x_0, we must have

$$\boxed{\begin{array}{rcl} \mathcal{L}(e^{At}) & = & (sI - A)^{-1} \\ \\ \mathcal{L}^{-1}\left((sI - A)^{-1}\right) & = & e^{At} \end{array}}$$

Example 69 Recall

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

So,

$$(sI - A)^{-1} = \begin{bmatrix} \frac{s-3}{(s-2)(s-4)} & \frac{-1}{(s-2)(s-4)} \\ \\ \frac{-1}{(s-2)(s-4)} & \frac{s-3}{(s-2)(s-4)} \end{bmatrix}$$

Since

$$\begin{array}{rcl} \dfrac{s-3}{(s-2)(s-4)} & = & \dfrac{1}{2}\left(\dfrac{1}{s-2} + \dfrac{1}{s-4}\right) \\ \dfrac{-1}{(s-2)(s-4)} & = & \dfrac{1}{2}\left(\dfrac{1}{s-2} - \dfrac{1}{s-4}\right) \end{array}$$

and

$$\mathcal{L}^{-1}(\frac{1}{s-2}) = e^{2t} \qquad \mathcal{L}^{-1}(\frac{1}{s-4}) = e^{4t}$$

we have,

$$\begin{array}{rcl} e^{At} & = & \mathcal{L}^{-1}\left((sI - A)^{-1}\right) \\ \\ & = & \dfrac{1}{2}\begin{bmatrix} e^{2t} + e^{4t} & e^{2t} - e^{4t} \\ e^{2t} - e^{4t} & e^{2t} + e^{4t} \end{bmatrix} \end{array}$$

Exercise 60 Compute e^{At} at $t = \ln(2)$ for

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

using *all* methods of this section.

Chapter 11

Stability

11.1 Introduction

Consider a general system (linear or nonlinear) described by

$$\dot{x} = f(x) \tag{11.1}$$

where the state $x(t)$ is an n-vector and the real scalar variable t represents time. By a solution of (11.1) we mean a continuous function $x(\cdot)$ which satisfies $\dot{x}(t) = f(x(t))$ for all $t \geq 0$.

In the following sections, we assume that x^e is an equilibrium state of the above system So, if the system starts at x^e, it remains at x^e for all time. What happens if the system does not start at x^e? If x^e is a desirable equilibrium state, we would like the system to have the property that if it is perturbed from x^e, it returns to x^e.

11.2 Stability

Global stability. *An equilibrium state x^e is* globally stable (GS) *if every solution converges to it, that is, every solution satisfies*

$$\lim_{t\to\infty} x(t) = x^e\,.$$

If a system has a globally stable equilibrium state x^e then, it cannot have any other equilibrium states. In this case we say that the system is globally stable. Also, if a system has a globally stable equilibrium state then, all solutions are bounded; the system cannot have any unbounded solutions.

Example 70

$$\dot{x} = -x$$

The origin is GS.

Example 71

$$\dot{x} = -x^3$$

The origin is GS. Why?

For nonlinear systems, we also need the following definition of stability.

Stability. *An equilibrium state x^e is* stable (S) *if it has a neighborhood with the property that whenever the initial state is in this neighborhood then, the resulting solution converges to x^e.*

The above definition of stability does not require that every solution converge to x^e but, only those solutions which originate in some neighborhood of x^e. Only nonlinear systems can have equilibrium states which are stable but not globally stable. For linear systems, stability and global stability are equivalent concepts. So, if the origin of a linear system is stable then the origin is globally stable and all solutions are bounded. However, a nonlinear system can have a stable equilibrium state but also have some unbounded solutions. This is illustrated in the next example.

Example 72

$$\dot{x} = -x(1-x^2)$$

The zero state is S but not GS. This system has some unbounded solutions.

Example 73 Damped simple pendulum

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - x_2\end{aligned}$$

The origin is S but not GS.

Region of attraction. When an equilibrium state is stable, we refer to its region of attraction as the set of states with the following property: if the system starts at one of these states, the resulting solution converges to the equilibrium state. Hence, an equilibrium state is globally stable if it is asymptotically stable with the whole state space as its region of attraction.

Figure 11.1: Region of attraction

Example 74

$$\dot{x} = -x(\epsilon^2 - x^2)$$

Linear systems. Consider now a linear system described by

$$\dot{x} = Ax \tag{11.2}$$

where A is a constant $n \times n$ matrix. If A has n distinct eigenvalues then, every solution of the above system can be expressed as

$$x(t) = \sum_{i=1}^{m} e^{\lambda_i t} v^i$$

where $\lambda_1, \ldots, \lambda_m$ are the eigenvalues of A and v^i is an eigenvector for λ_i. From this general form of the solution we have the following conclusion.

The zero state of $\dot{x} = Ax$ is globally stable if and only if all the eigenvalues of A have negative real parts.

It can be shown that the above statement also holds when A does not have n distinct eigenvalues; hence it holds for any square matrix A.

Suppose A is a 2×2 matrix and

$$p(s) = s^2 + a_1 s + a_0$$

is the characteristic polynomial of A. We have shown before that both roots of such a second order polynomial have negative real part if and only if both a_0 and a_1 are positive. Hence, if A is a 2×2 matrix then, $\dot{x} = Ax$ is stable if and only if the coefficients of the characteristic polynomial of A are positive.

For a general linear system $\dot{x} = Ax$, one can determine whether or not all the eigenvalues of matrix A have negative real part by applying the Routh test to the characteristic polynomial of A.

Example 75 The system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= -x_2\end{aligned}$$

is GS.

Example 76 The system

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= -x_1 - x_2\end{aligned}$$

is GS.

Example 77 The system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1\end{aligned}$$

is not GS.

Example 78 The system

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_1\end{aligned}$$

is not GS.

11.3 Neutral stability and instability

Consider a linear undamped spring mass system described by $\ddot{q}+q = 0$. All nonzero solutions oscillate with a constant amplitude which depends on the initial state. Also, the amplitude is small if the magnitude of the initial state is small. The origin is not stable in the sense just defined. However it is neutrally stable.

We say that an equilibrium state x^e of a system is neutrally stable (NS) or marginally stable if it is not stable as defined previously, but has the following property. *If the perturbation $x(0)-x^e$ of the initial state from the equilibrium state is sufficiently small, then the perturbation $x(t)-x^e$ of the resulting solution from the equilibrium state is small.*

Instability. *An equilibrium state is* unstable (US) *if it is neither stable nor neutrally stable.*

For a linear system, it can be shown that neutral stability of the origin implies all solutions are bounded; also, instability of the origin implies the system has some unbounded solutions. However, for a nonlinear system, instability of an equilibrium state does not imply that the system has some unbounded solutions. Also, the existence of a stable or neutrally stable equilibrium state does not imply all solutions are bounded.

Figure 11.2: Neutral stability

Example 79

$$\dot{x} = 0$$

Every state is a neutrally stable equilibrium state. All solutions are bounded.

Example 80

$$\dot{x} = x$$

The origin is unstable. Also the system has unbounded solutions.

Example 81 Undamped oscillator

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -x_1\end{aligned}$$

The origin is neutrally stable. All solutions are bounded.

Example 82 Simple pendulum

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= -\sin x_1\end{aligned}$$

$(0,0)$ is neutrally stable; $(\pi,0)$ is unstable.

Example 83

$$\dot{x} = x(1-x^2)$$

The origin here is unstable. However, all solutions of this system are bounded. This cannot occur in a linear system.

Example 84 Van der Pol oscillator This nonlinear system is interesting in that is has a single equilibrium at the origin and this equilibrium is unstable. However all solutions of the system are bounded,

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= (1-x_1^2)x_2 - x_1\end{aligned}$$

Figure 11.3: Van der Pol oscillator

Linear systems. Consider now a linear system described by $\dot{x} = Ax$ where A is a constant $n \times n$ matrix. If A has n distinct eigenvalues, we have seen that every solution of the above system has the form

$$x(t) = \sum_{i=1}^{m} e^{\lambda_i t} v^i$$

where $\lambda_1, \ldots, \lambda_m$ are eigenvalues of A. From the general form of the solution we have the following conclusion:

If the matrix A has at least one eigenvalue with a positive real part then, the system $\dot{x} = Ax$ is unstable *about zero and has some unbounded solutions.*

The above conclusion also holds when A does not have n distinct eigenvalues; so, it holds for any square A.

Example 85 The system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 \end{aligned}$$

is US about zero.

We also have the following conclusion from the above general form of the solution:

Suppose that none of the eigenvalues of the matrix A have positive real part and there is at least one eigenvalue with zero real part. Then, the system $\dot{x} = Ax$ is neutrally stable *about zero and all solutions are bounded.*

Example 86 The system

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 \end{aligned}$$

is NS about 0.

One has to be a little careful in applying the last conclusion above. It may not hold if A has an eigenvalue λ with a zero real part and this eigenvalue is a repeated root of the characteristic polynomial of A. This is illustrated in the next example.

Example 87 Unattached mass

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= 0\end{aligned}$$

Here

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the characteristic polynomial of A is $p(s) = s^2$. Hence, A has 0 a single eigenvalue 0 and this eigenvalue is a repeated root of the characteristic polynomial of A. This system is unstable about zero and has unbounded solutions. To see this consider the solution

$$x(t) = \begin{bmatrix} t \\ 1 \end{bmatrix}$$

11.3.1 Exercises

Exercise 61 Determine the stability properties of the zero equilibrium state of the following systems.
(a)

$$\dot{x} = 2x$$

(b)

$$\begin{aligned}\dot{x}_1 &= 2x_2\\ \dot{x}_2 &= -2x_1\end{aligned}$$

(c)

$$\begin{aligned}\dot{x}_1 &= 2x_2\\ \dot{x}_2 &= 2x_1\end{aligned}$$

Exercise 62 Determine the stability properties of the following systems.

(♠)

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2\\ \dot{x}_2 &= x_1 - x_2\end{aligned}$$

(♣)

$$\begin{aligned}\dot{x}_1 &= -2x_1 + x_2\\ \dot{x}_2 &= x_1 + x_2\end{aligned}$$

(♢)

$$\begin{aligned}\dot{x}_1 &= -x_1 + x_2\\ \dot{x}_2 &= -4x_1 - x_2\end{aligned}$$

(♡)

$$\begin{aligned}\dot{x}_1 &= 2x_2\\ \dot{x}_2 &= -4x_1\end{aligned}$$

Exercise 63 Determine whether or not the following system is stable, neutrally stable, or unstable about the origin.

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= x_4\\ \dot{x}_4 &= x_1\end{aligned}$$

Exercise 64 Determine the **stability** properties of the following systems about the zero solution.

(a)

$$\begin{aligned}\dot{x}_1 &= x_3\\ \dot{x}_2 &= x_2 + x_3\\ \dot{x}_3 &= -x_2 + x_3\end{aligned}$$

(b)

$$\begin{aligned}\dot{x}_1 &= 2x_2\\ \dot{x}_2 &= 2x_1\end{aligned}$$

(c)

$$\begin{aligned}\dot{x}_1 &= -x_1 + 2x_2\\ \dot{x}_2 &= -2x_1 - x_2\end{aligned}$$

11.4 Linearization and stability

We know that a linear system is stable is stable if all its associated eigenvalues have negative real part. How do we determine the stability properties of an equilibrium state of a nonlinear system? On approach is to linearize the nonlinear system about the equilibrium state and look at the stability properties of the linearized system, The big question then is: what do the stability properties of the linearized system tell us about the nonlinear system? We will see that, except in a special case, the local stability behavior of the nonlinear system about the equilibrium state is predicted by the stability property of the linearized system.

Consider a nonlinear time-invariant system described by

$$\dot{x} = f(x)$$

where $x(t)$ is an n-vector at each time t. Suppose x^e is an equilibrium state for this system and let

$$\delta\dot{x} = A\delta x$$

be the linearization of this system about x^e. Then we have the following fundamental results.

If all the eigenvalues of the matrix A have negative real part, then the nonlinear system is stable *about x^e.*

If the matrix A matrix has at least one eigenvalue with positive real part, then the nonlinear system is unstable *about x^e.*

If none of the eigenvalues of the matrix A have positive real part and there is at least one eigenvalue with zero real part, then, one cannot make any conclusions on the stability of the equilibrium state based on linearization.

Note that we cannot conclude global stability of a nonlinear system based on linearization. We can only conclude local results.

Example 88

$$\dot{x} = -x(1-x^2)$$

Example 89 Undamped pendulum.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 \end{aligned}$$

Example 90 Pendulum with linear damping.

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_2 - \sin x_1 \end{aligned}$$

Example 91 Van der Pol oscillator

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= (1-x_1^2)x_2 - x_1 \end{aligned}$$

Example 92

$$\dot{x} = ax^3$$

11.4.1 Exercises

Exercise 65 For each of the following systems, determine (if possible) the stability properties of the zero equilibrium state. If not possible, state the reason. Justify your results.

(a)

$$\dot{x} = -x^3$$

(b)

$$\dot{x} = x - 10x^3$$

(c)

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\sin x_1 - (1 - x_1^2)x_2\end{aligned}$$

Exercise 66 For each of following set(s) of differential equations,

(a)

$$(1 + y^2)\ddot{y} + (|\dot{y}| - 1)\dot{y} + \cos(\dot{y})\sin(y) = 0$$

(b)

$$\begin{aligned}\dot{x}_1 &= x_1^3 + x_2 \\ \dot{x}_2 &= -x_1 + x_2^3\end{aligned}$$

answer the following questions.

(i) Linearize the differential equation(s) about the zero solution.

(ii) What are the stability properties of the linearized differential equation(s)?

(iii) Based on part (ii), what can you say about the stability properties of the original set of nonlinear equation(s)?

Exercise 67 Consider the system described by

$$\begin{aligned}\ddot{y}_1 + (\sin y_2)\ddot{y}_2 + (\cos y_1)\dot{y}_2^2 - \sin y_2 &= 0 \\ (1-e^{y_1})\ddot{y}_1 + \ddot{y}_2 + y_2\dot{y}_1^2 + e^{-y_1} - 1 &= 0\end{aligned}$$

where y_1 and y_2 are real scalar variables. Determine whether or not this system is stable about the zero solution ($y_1(t) \equiv y_2(t) \equiv 0$.)

Exercise 68 For each of the following second order systems, determine whether they are stable, unstable, or neutrally stable about the zero solution.

(a)

$$\ddot{y} - \dot{y} + y = 0$$

(b)

$$-\ddot{y} - \dot{y} - y = 0$$

(c)

$$\ddot{y} + \dot{y} = 0$$

(d)

$$\ddot{y} - \dot{y} = 0$$

(e)

$$\ddot{y} + \dot{y} - \sin y = 0$$

(f)

$$\ddot{y} + \dot{y}^3 + y = 0$$

Exercise 69 If possible, use linearization to determine whether or not the following system is stable, unstable, or marginally stable about the equilibrium solution $\omega_1^e = 1$, $\omega_2^e = \omega_3^e = 0$.

$$\begin{aligned}
I_1\dot{\omega}_1 + (I_3 - I_2)\omega_2\omega_3 &= 0 \\
I_2\dot{\omega}_2 + (I_1 - I_3)\omega_1\omega_3 &= 0 \\
I_3\dot{\omega}_3 + (I_2 - I_1)\omega_1\omega_2 &= 0
\end{aligned}$$

If not possible, state why. Consider three different cases:
(a)

$$I_1 = 1, \quad I_2 = 2, \quad I_3 = 3$$

(b)

$$I_1 = 2, \quad I_2 = 3, \quad I_3 = 1$$

(c)

$$I_1 = 3, \quad I_2 = 1, \quad I_3 = 2$$

Exercise 70 (Simple pendulum in drag)
Recall the simple pendulum in drag whose motion is described by

$$ml\ddot{\theta} + \kappa V(l\dot{\theta} - w\sin\theta) + mg\sin\theta = 0$$

where

$$V = \sqrt{l^2\dot{\theta}^2 + w^2 - 2lw\sin(\theta)\dot{\theta}}$$

(a) Obtain (by hand) the equilibrium values θ^e of the angle θ.
(a) For each equilibrium condition, determine the range of values of the wind speed w for which that condition is stable.

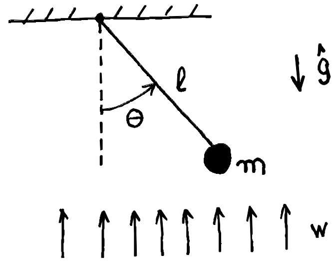


Figure 11.4: Pendulum in drag

11.5 Root loci

Here, we look at how the roots of a polynomial or the eigenvalues of a matrix vary with a scalar parameter.

11.5.1 Polynomials

Suppose n and d are two polynomials and k is a real scalar parameter. Consider now the parameter dependent polynomial given by

$$p(s;k) = d(s) + kn(s)\,.$$

This new polynomial depends on k in an affine linear manner. A root locus consists of a plot in the complex plane of the roots of p for values of k between 0 and ∞. This can be accomplished with the MATLAB command

`rlocus`

We will not go into the usual root locus rules here. It is assumed that they are covered elsewhere.

Example 93 Consider the polynomials

$$d(s) = s^2 - 2s + 2 \qquad \text{and} \qquad n(s) = s^2 + 2s + 2\,.$$

Then

$$d(s) + kn(s) = (1+k)s^2 + 2(k-1)s + 2(1+k)\,.$$

We now illustrate the use of the MATLAB command `rlocus`. The output of that command is shown in Figure 11.5.

```
 d=[1 -2 2];
 n=[1 2 2];
 roots(d)
ans =
   1.0000 + 1.0000i
   1.0000 - 1.0000i
 roots(n)
```

```
ans =
  -1.0000 + 1.0000i
  -1.0000 - 1.0000i
 rlocus(n,d)
```

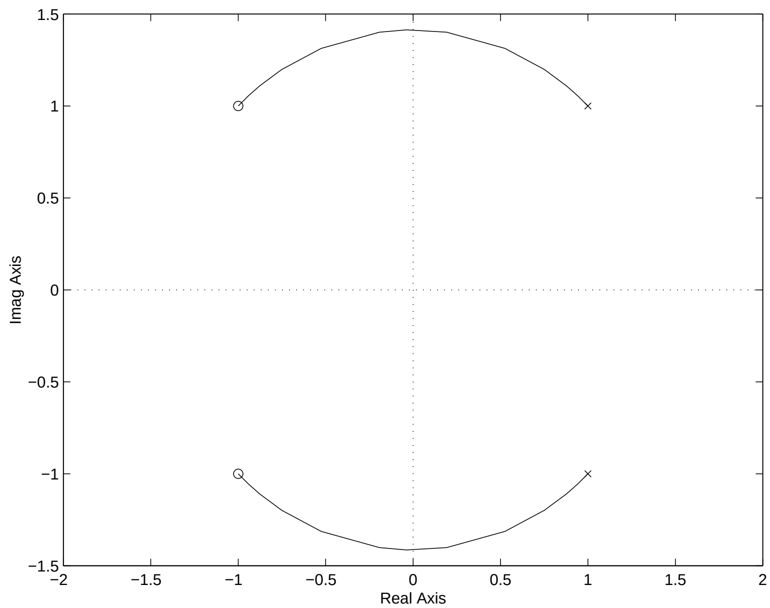


Figure 11.5: A root locus

Example 94 The MATLAB output for `rlocus(d,n)` corresponding to

$$d(s) = (s-1)(s+2)(s+4) = s^3 + 5s^2 + 2s - 8 \qquad \text{and} \qquad n(s) = s+3$$

is shown in Figure 11.6.

11.5.2 Matrices

Here we look at the manner in which the eigenvalues of a matrix vary with a scalar parameter. To do this we need the following result on determinants.

Fact 3 *Suppose M is an $n \times m$ matrix and N is $m \times n$ matrix. Then*

$$\det(I_n + MN) = \det(I_m + NM)\,. \tag{11.3}$$

In particular, if $m = 1$, then

$$\det(I_n + MN) = 1 + NM\,. \tag{11.4}$$

Consider now a parameter dependent matrix described by

$$A - bkc$$

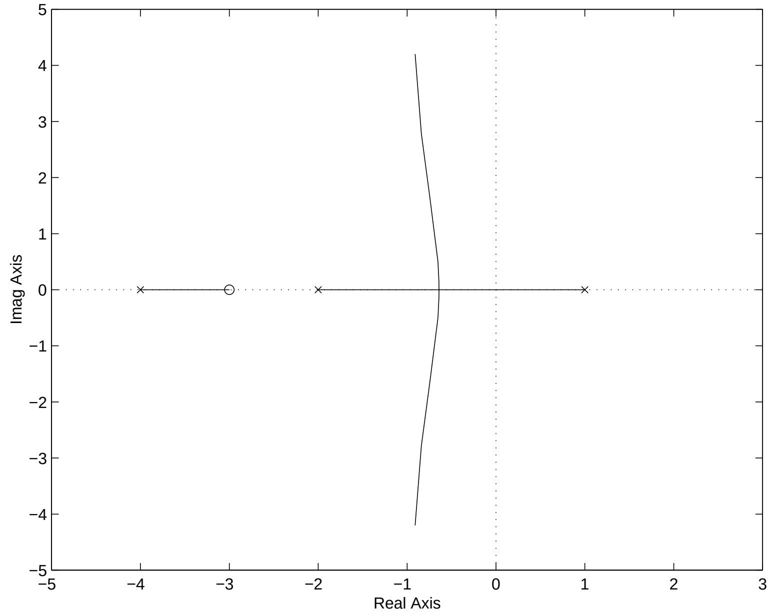


Figure 11.6: Another root locus

where k is a scalar parameter and the matrices A, b and c are $n \times n$, $n \times 1$, and $1 \times n$, respectively. Here a root locus consists of a plot in the complex plane of the eigenvalues of $A - bkc$ for values of k between 0 and ∞.

Let

$$d(s) = \det(sI - A) \qquad \text{and} \qquad g(s) = c(sI - A)^{-1}b\,. \tag{11.5}$$

Then, it can be shown that g can be uniquely expressed as

$$g(s) = \frac{n(s)}{d(s)} \tag{11.6}$$

where n is a polynomial of degree strictly less than d. We now show that

$$\boxed{\det(sI - A + bkc) = d(s) + kn(s)} \tag{11.7}$$

This yields the following conclusion: *The eigenvalues of the matrix $A - bkc$ are the roots of the polynomial $d(s) + kn(s)$. Also, the coefficients of the characteristic polynomial of $A - bkc$ depend on k in an affine linear fashion.*

To demonstrate the above result, consider any complex number s which is not an eigenvalue of A. Then $sI - A$ is an invertible matrix and

$$sI - A + bkc = (sI - A)(I + (sI - A)^{-1}bkc)\,.$$

Using Fact 3 and other properties of determinants, we obtain that

$$\begin{aligned}
\det(sI - A + bkc) &= \det\Big((sI - A)(I + (sI - A^{-1})bkc)\Big) \\
&= \det(sI - A)\det(I + (sI - A)^{-1}bkc) \\
&= \det(sI - A)(1 + kc(sI - A)^{-1}b) \\
&= d(s)(1 + kg(s)) \\
&= d(s) + kn(s)\,.
\end{aligned}$$

That is, $\det(sI - A + bkc) = d(s) + kn(s)$. Since we have shown that this relationship holds for every s except the eigenvalues of A, and both sides of this equation have no poles at the eigenvalues of A, the relationship also holds at the eigenvalues of A.

To use the MATLAB command `rlocus`, we do not actually have to compute n and d. We can simply type

```
rlocus(A,b,c,0)
```

Notice the "0".

Example 95 With

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 8 & -2 & -5 \end{bmatrix} \qquad b = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad c = \begin{bmatrix} 3 & 1 & 0 \end{bmatrix},$$

Figure 11.7 illustrates the output of the MATLAB command `rlocus(A,b,c,0)`. Why are

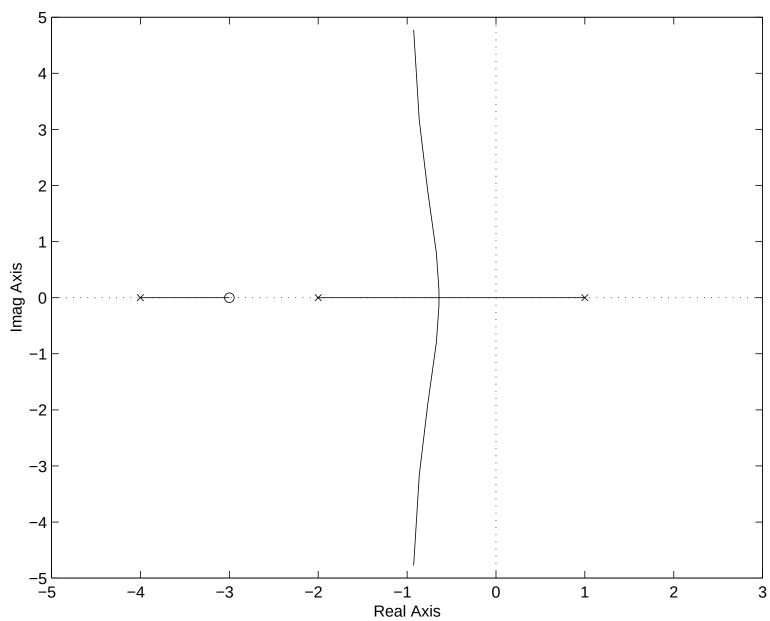


Figure 11.7: Yet another root locus

Figures 11.6 and 11.7 the same?

Chapter 12

Equations of motion for general flight

In this chapter we derive the equations of motion for the general three-dimensional motion of an aircraft. We can regard an aircraft as a dynamical system with the following control inputs:

th: throttle setting
el: elevator deflection
δ_a: deflection of the ailerons
δ_r: deflection of the rudder

If we model an aircraft as a single rigid body, we can completely describe its motion relative to the earth by specifying the position of its mass center relative to some earth fixed point and by specifying the orientation of the aircraft relative to the earth. Hence the complete state of an aircraft can be described by the position and velocity of aircraft mass center, and the orientation and angular velocity of the aircraft.

We have already looked at the longitudinal dynamics of an aircraft. Before proceeding with the general aircraft dynamics, we look at a simpler scenario to obtain some familiarity with some of the variables involved in the lateral dynamics of an aircraft.

12.1 Ice-racer

Our Ice-racer is basically an aircraft moving on a lake of ice; we assume that the vehicle has a plane of symmetry which is vertical and we ignore friction between the vehicle and the ice; see Figure 12.1. The inputs to the vehicle are the throttle setting th and the rudder deflection δ_r. The throttle setting affects the thrust T; we assume that the thrust vector lies in the plane of symmetry of the vehicle. The rudder deflection produces a yawing moment on the vehicle. The purpose of studying this system is to gain an initial appreciation of the lateral dynamics of an aircraft.

Kinematics. Introducing reference frame e fixed in the ice, the location of the mass center of the vehicle can be described by the range coordinates p_1 and p_2 which are the cartesian coordinates of the mass center of the vehicle relative to e; see Figure 12.2. The orientation of the vehicle can be described by the yaw angle ψ which is the angle between a horizontal

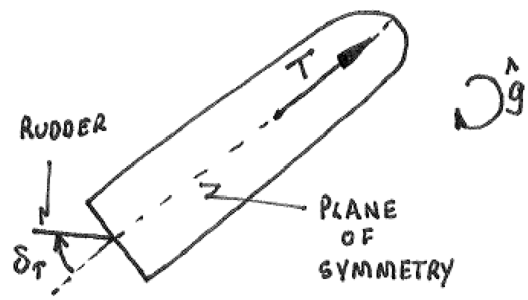


Figure 12.1: Ice-racer

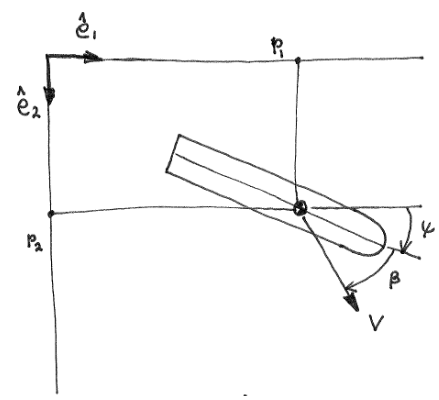


Figure 12.2: Ice-racer kinematics

reference line fixed in the plane of symmetry of the vehicle and the $\hat{e}_1$ axis. It is considered positive as shown, that is, when it corresponds to a positive rotation about the $\hat{e}_3$ axis. We let r be the yaw rate, that is,

$$r = \dot{\psi}\,. \tag{12.1}$$

We will describe the velocity $\bar{V}$ of the vehicle mass center by its magnitude V (speed) and the sideslip angle β; this is the angle between $\bar{V}$ and the reference line fixed in the vehicle; it is considered positive as shown. Since

$$\bar{V} = \dot{p}_1\hat{e}_1 + \dot{p}_2\hat{e}_2$$

and

$$\bar{V} = V\cos(\psi+\beta)\hat{e}_1 + V\sin(\psi+\beta)\hat{e}_2$$

we obtain the following navigation equations:

$$\dot{p}_1 = V\cos(\psi+\beta) \tag{12.2a}$$
$$\dot{p}_2 = V\sin(\psi+\beta) \tag{12.2b}$$

We introduce the wind reference frame which is determined by $\bar{V}$ and shown in Figure 12.3. Using this reference frame, we have

$$\bar{V} = V\hat{w}_1\,.$$

Note that ${}^e\bar{\omega}^w = (r + \dot{\beta})\hat{w}_3$. To obtain an expression for $\bar{a}$, the inertial acceleration of the vehicle mass center, we use the BKE to obtain

$$\bar{a} = \frac{{}^e d\bar{V}}{dt} = \frac{{}^w d\bar{V}}{dt} + {}^e\bar{\omega}^w \times \bar{V} = \dot{V}\hat{w}_1 + (r + \dot{\beta})V\hat{w}_2 \,.$$

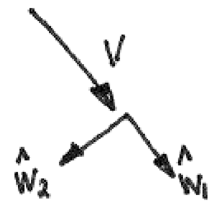


Figure 12.3: w-frame

Forces and moments. Since the vehicle moves in a horizontal plane, we only consider the forces in that plane and the moments perpendicular to that plane. The corresponding free-body-diagram of the vehicle is contained in Figure 12.4.

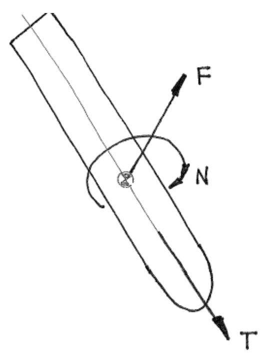


Figure 12.4: Relevant ice-racer force and moments

Here $\bar{F}$ is the total aerodynamic force on the vehicle and N is the yawing moment N. We can decompose $\bar{F}$ in two ways:

Drag D and Crosswind C. The drag force is always in the direction opposite the velocity vector $\bar{V}$. The crosswind force is perpendicular to the drag and considered positive as shown; thus

$$\bar{F} = -D\hat{w}_1 - C\hat{w}_2$$

Force component X and sideforce Y. These are the components of $\bar{F}$ relative to a reference frame b fixed in the vehicle.

$$\bar{F} = X\hat{b}_1 + Y\hat{b}_2$$

The sideforce force $\bar{Y}$ is perpendicular to the vehicle and Y is considered positive as shown, that is to the right of the vehicle.

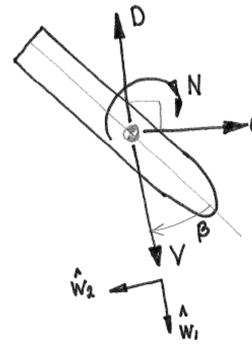


Figure 12.5: Drag and crosswind

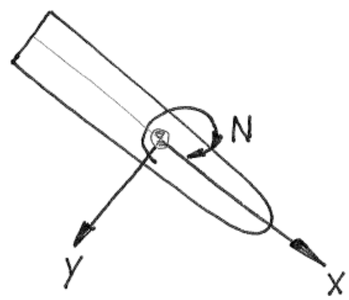


Figure 12.6: X and sideforce Y

From Figure 12.7 we see that $Y = -C\cos\beta - D\sin\beta$; hence the crosswind C can be expressed in terms of D and Y:

$$C\cos\beta = -Y - D\sin\beta$$

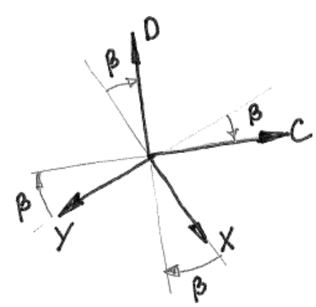


Figure 12.7: Resolving forces

Dimensionless coefficients. As usual, we let $\bar{q} = \frac{1}{2}\rho V^2$ be the dynamic pressure where ρ is the air density.

We express the drag D as

$$D = \bar{q}SC_D$$

where C_D is the drag coefficient and S is a reference area associated with the vehicle. We will consider C_D to be constant here.

We can express the sideforce Y as

$$Y = \bar{q} S C_Y$$

where C_Y is called the sideforce coefficient. The static value of C_Y depends mainly on β and δ_r. These dependencies can be described by the derivatives

$$C_{Y_\beta} = \frac{\partial Y}{\partial \beta} \qquad \text{and} \qquad C_{Y_{\delta r}} = \frac{\partial Y}{\partial \delta_r}$$

We will consider C_{Y_β} and $C_{Y_{\delta r}}$ constant. Including rate dependent terms, we approximate C_Y by

$$\boxed{C_Y = C_{Y_\beta}\beta + \tfrac{b}{2V} C_{Y_r} r + C_{Y_{\delta_r}} \delta_r} \tag{12.3}$$

where b is a reference length like the wing span. Note that a lot of the sideforce is due to the vertical tail and rudder. Usually

$$C_{Y_\beta} < 0\,, \qquad C_{Y_r} > 0\,, \qquad C_{Y_{\delta r}} > 0$$

We can express the yawing moment as

$$N = \bar{q} S b C_n$$

where C_n is the yawing moment coefficient and is usually expressed as

$$\boxed{C_n = C_{n_\beta}\beta + \tfrac{b}{2V} C_{n_r} r + C_{n_{\delta_r}} \delta_r} \tag{12.4}$$

Usually,

$$C_{n_\beta} > 0\,, \qquad C_{n_r} < 0\,, \qquad C_{n_{\delta r}} < 0\,,$$

Equations of motion. Looking at the free-body-diagram in Figure 12.8 and recalling the expression for the CM acceleration, the motion of the ice-racer is described by

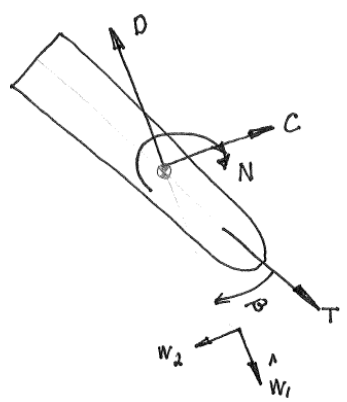


Figure 12.8: Free body diagram

$$\boxed{\begin{array}{rcl} m\dot{V} & = & -D + T\cos\beta \\ mV\dot{\beta} & = & -C - T\sin\beta - mVr \\ J_{33}\dot{r} & = & N \end{array}} \tag{12.5}$$

where the crosswind C is given by

$$C\cos\beta = -Y - D\sin\beta$$

and J_{33} is the moment of inertia of the vehicle about the yaw axis at the CM. This is a state-space description with two inputs T and δ_r and three state variables: V, β and r.

Linearization. Consider a trim condition corresponding to $V = V^e$ and $\beta = r = 0$. The linearized ice-racer is described by

$$\delta\dot{V} = -(D_V - T_V)\delta V \tag{12.6}$$

$$V^e\delta\dot{\beta} = Y_\beta\delta\beta + (Y_r - V^e)\delta r + Y_{\delta_r}\delta_r \tag{12.7}$$

$$\delta\dot{r} = N_\beta\delta\beta + N_r\delta r + N_{\delta_r}\delta_r \tag{12.8}$$

where

$$\begin{aligned}
Y_\beta &= \frac{1}{m}\frac{\partial Y}{\partial\beta} = \frac{\bar{q}SC_{Y_\beta}}{m} < 0 \\
Y_r &= \frac{1}{m}\frac{\partial Y}{\partial r} = \frac{\bar{q}SbC_{Y_r}}{2mV^e} > 0 \\
N_\beta &= \frac{1}{J_{33}}\frac{\partial N}{\partial\beta} = \frac{\bar{q}SbC_{n_\beta}}{J_{33}} > 0 \\
N_r &= \frac{1}{J_{33}}\frac{\partial N}{\partial r} = \frac{\bar{q}Sb^2C_{n_r}}{2J_{33}V^e} < 0
\end{aligned}$$

and

$$\begin{aligned}
Y_{\delta_r} &= \frac{1}{m}\frac{\partial Y}{\partial\delta_r} = \frac{\bar{q}SC_{Y_{\delta_r}}}{m} > 0 \\
N_{\delta_r} &= \frac{1}{J_{33}}\frac{\partial Y}{\partial\delta_r} = \frac{\bar{q}SbC_{n_{\delta_r}}}{J_{33}} < 0
\end{aligned}$$

Note that the dynamics of δV are decoupled from the dynamics of $\delta\beta, \delta r$. The A matrix for the latter dynamics is given by

$$\begin{pmatrix} Y_\beta/V^e & Y_r/V^e - 1 \\ N_\beta & N_r \end{pmatrix}$$

Example 96 (Cessna 182) Some data:

$$\begin{aligned}
\text{W} &= \text{2650 lbs} \\
J_{33} &= \text{1967 slug ft}^2 \\
b &= \text{36 ft} \\
S &= \text{174 ft}^2 \\
C_{Y_\beta} &= \text{-0.3930} \\
C_{Y_r} &= 0.2140 \\
C_{n_\beta} &= 0.0587 \\
C_{n_r} &= \text{-0.0937} \\
C_{Y_{\delta r}} &= 0.1870 \\
C_{n_{\delta r}} &= \text{-0.0645} \\
V^e &= \text{150 ft sec}^{-2}
\end{aligned}$$

Consider the A matrix corresponding to the $\delta\beta, \delta r$ dynamics, we have

$$A = \begin{pmatrix} -0.1872 & -0.9917 \\ 9.2719 & -1.2104 \end{pmatrix}$$

whose eigenvalues are

$$-0.6988 \pm 2.9888\jmath$$

12.2 Translational kinematics

12.2.1 Review of vectors and components

Suppose $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$ is a orthonormal set of basis vectors. Consider any vector $\bar{V}$ and let V be the magnitude of $\bar{V}$, that is, $V = |\bar{V}|$. If we let $\bar{V}_I$ be the projection of $\bar{V}$ onto the $\hat{b}_1 - \hat{b}_3$ plane then,

$$\bar{V} = \bar{V}_I + V_2 \hat{b}_2 \,;$$

see Figure 12.9.

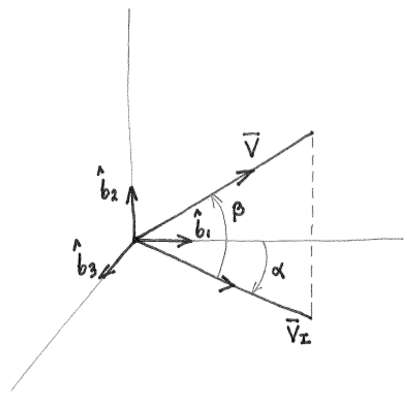


Figure 12.9: Components

Let β be the angle between $\bar{V}$ and its projection $\bar{V}_I$ where β is considered positive if V_2 is positive. Note that β lies between -90^o and 90^o and can be regarded as the angle between $\bar{V}$ and the $\hat{b}_1 - \hat{b}_3$ plane. Also,

$$V_2 = V \sin \beta \qquad \text{and} \qquad V_I = V \cos \beta$$

where V_I denotes the magnitude of $\bar{V}_I$. Let α be the angle between the vector $\bar{V}_I$ and $\hat{b}_1$ where α is considered positive if it is clockwise as seen by $\hat{b}_2$; see Figure 12.10. Resolve $\bar{V}_I$

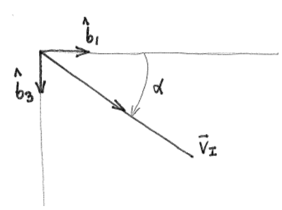


Figure 12.10: Components

into components along $\hat{b}_1$ and $\hat{b}_3$ to obtain

$$\bar{V}_I = V_I \cos \alpha \; \hat{b}_1 + V_I \sin \alpha \; \hat{b}_3$$

It now follows that

$$\bar{V} = V_1 \hat{b}_1 + V_2 \hat{b}_2 + V_3 \hat{b}_3$$

where

$$\begin{aligned} V_1 &= V\cos\beta\cos\alpha \\ V_2 &= V\sin\beta \\ V_3 &= V\cos\beta\sin\alpha \end{aligned}$$

Inverting these relationships, we obtain

$$\begin{aligned} V &= \sqrt{V_1^2+V_2^2+V_3^2} \\ \alpha &= \arctan(V_3/V_1) \\ \beta &= \arcsin(V_2/V) \end{aligned}$$

The three scalars, V_1, V_2, V_3, are called the cartesian components of $\bar{V}$ relative to $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$.

The three scalars V, α, β, are called the spherical components of $\bar{V}$ relative to $(\hat{b}_1, \hat{b}_2, \hat{b}_3)$.

12.2.2 Aircraft translational kinematics

The configuration of a rigid aircraft relative to the earth can be completely specified by specifying the position of the mass center of the aircraft and the orientation of the aircraft. The translational kinematics of the aircraft describe the motion of the aircraft mass center.

Earth-fixed reference frame. As an inertial reference frame, we will choose a reference frame

$$\mathbf{e} = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$$

fixed in the earth where $\hat{e}_3$ points vertically downward and $\hat{e}_1$ and $\hat{e}_3$ lie in the surface of the (assumed) flat earth; see Figure 12.11.

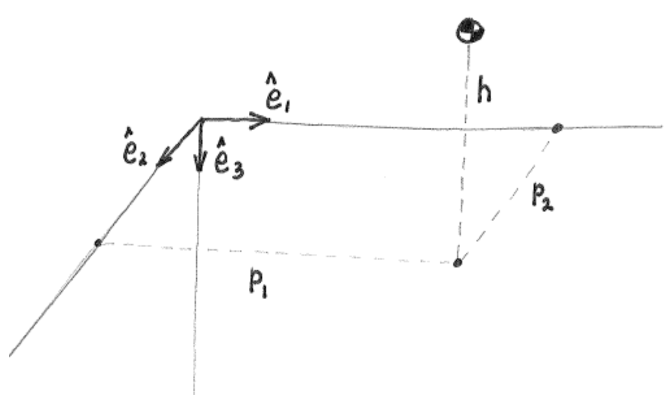


Figure 12.11: Position coordinates

The position vector $\bar{r}$. Choosing an arbitrary point fixed in the earth as the origin of the inertial reference frame, the position of the aircraft mass center relative to this point can be expressed as

$$\boxed{\bar{r} = p_1\hat{e}_1 + p_2\hat{e}_2 - h\hat{e}_3} \tag{12.9}$$

where h is the altitude or height of the aircraft. So, the scalars $p_1, p_2, -h$ are the cartesian components of $\bar{r}$ relative to $\mathbf{e}$; see Figure 12.11.

The velocity vector $\bar{V}$. If we let $\bar{V}$ be the velocity of the mass center of the aircraft relative to earth, then

$$\bar{V} = \dot{p}_1\hat{e}_1 + \dot{p}_2\hat{e}_2 - \dot{h}\hat{e}_3 \,. \tag{12.10}$$

So, we can regard the scalars $\dot{p}_1, \dot{p}_2, -\dot{h}$ as the cartesian components of $\bar{V}$ relative to $\mathbf{e}$.

Aircraft fixed reference frame. We choose a reference frame

$$\mathbf{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$$

fixed in the aircraft with

$\hat{b}_1$ (aircraft X-axis)	in the plane of symmetry of the aircraft and pointing forward
$\hat{b}_2$ (aircraft Y-axis)	perpendicular to the plane of symmetry of the aircraft and pointing starboard
$\hat{b}_3$ (aircraft Z-axis)	in the plane of symmetry of the aircraft and pointing downward

Thus, the $\hat{b}_1 - \hat{b}_3$ plane is the aircraft plane of symmetry; see Figure 12.12.

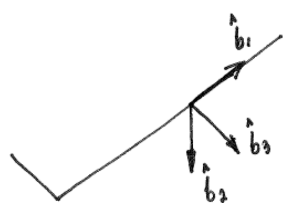


Figure 12.12: Aircraft-fixed frame

It is common to describe the vector $\bar{V}$ in components taken relative to the aircraft frame. Relative to the aircraft frame, the cartesian components of $\bar{V}$ are denoted by u, v, w; hence,

$$\boxed{\bar{V} = u\hat{b}_1 + v\hat{b}_2 + w\hat{b}_3}$$

The spherical components of $\bar{V}$ relative to the aircraft frame are denoted by V, α, β and described as follows; see Figure 12.13

airspeed: $V = |\bar{V}|$.

sideslip angle, β is the angle between $\bar{V}$ and the the aircraft-fixed $\hat{b}_1 - \hat{b}_3$ plane (aircraft X-Z plane). It is considered positive if v, the $\hat{b}_2$ (aircraft Y) component of $\bar{V}$, is positive.

angle of attack, α is the angle between the projection of $\bar{V}$ onto the $\hat{b}_1 - \hat{b}_3$ plane and the $\hat{b}_1$ (aircraft X-axis). It is considered positive if it is clockwise as seen by $\hat{b}_2$ (aircraft Y-axis.)

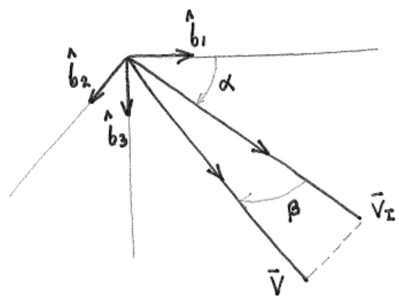


Figure 12.13: Velocity description

Using the results of the previous section, we obtain the following relationships between the cartesian components and the spherical components.

$$\boxed{\begin{aligned} u &= V \cos\alpha \cos\beta \\ v &= V \sin\beta \\ w &= V \sin\alpha \cos\beta \end{aligned}}$$

and

$$\boxed{\begin{aligned} V &= \sqrt{u^2 + v^2 + w^2} \\ \alpha &= \arctan(w/u) \\ \beta &= \arcsin(v/V) \end{aligned}}$$

12.3 Rotational kinematics

To discuss the rotational kinematics of a rigid body, we need only discuss the rotational kinematics of a reference frame fixed in the body. So, to discuss the rotational kinematics of a rigid body, we will consider the rotation of one reference frame relative to another.

12.3.1 Simple rotations

Consider any two reference frames

$$\mathbf{e} = (\hat{e}_1, \hat{e}_2, \hat{e}_3) \qquad \text{and} \qquad \mathbf{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)\,.$$

Suppose reference frame $\mathbf{b}$ is obtained from reference frame $\mathbf{e}$ by a simple counter-clockwise rotation through angle θ about $\hat{e}_2$; see Figure 12.14. Then

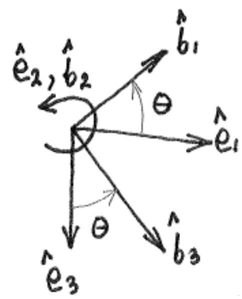


Figure 12.14: A simple rotation

$$\begin{aligned}
\hat{b}_1 &= \cos\theta\,\hat{e}_1 - \sin\theta\,\hat{e}_3 && (12.11a)\\
\hat{b}_2 &= \hat{e}_2 && (12.11b)\\
\hat{b}_3 &= \sin\theta\,\hat{e}_1 + \cos\theta\,\hat{e}_3\,. && (12.11c)
\end{aligned}$$

Rotation matrix. The rotation matrix R associated with the above rotation of $\mathbf{b}$ relative to $\mathbf{e}$ is defined by

$$R = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}. \tag{12.12}$$

Note that this matrix is obtained by letting its first, second, and third columns consist of the cartesian components of $\hat{b}_1$, $\hat{b}_2$, $\hat{b}_3$, respectively, relative to $\mathbf{e}$. Note also that

$$R^T R = I \tag{12.13}$$

where I is the 3×3 identity matrix. From this it follows that R is invertible and its inverse is its transpose, that is, $R^{-1} = R^T$. Also, $RR^T = I$.

Exercise 71 Verify (12.13).

Coordinate transformations. Consider *any* vector $\bar{V}$ (it does not have to be a velocity vector) and suppose we express it in components relative to $\mathbf{e}$ and $\mathbf{b}$, that is,

$$\bar{V} = {}^eV_1\hat{e}_1 + {}^eV_2\hat{e}_2 + {}^eV_3\hat{e}_3 \qquad \text{and} \qquad \bar{V} = {}^bV_1\hat{b}_1 + {}^bV_2\hat{b}_2 + {}^bV_3\hat{b}_3\,,$$

respectively. Introducing the coordinate vectors,

$${}^eV = \begin{pmatrix} {}^eV_1 \\ {}^eV_2 \\ {}^eV_3 \end{pmatrix} \qquad \text{and} \qquad {}^bV = \begin{pmatrix} {}^bV_1 \\ {}^bV_2 \\ {}^bV_3 \end{pmatrix}, \tag{12.14}$$

one may readily obtain the following transformation relationship from (12.11):

$$\boxed{{}^eV = R\,{}^bV} \tag{12.15}$$

Premultiplying both sides of the last equation by R^T and using $R^TR = I$, we also have

$$\boxed{{}^bV = R^T\,{}^eV} \tag{12.16}$$

For example, if $\bar{V} = \hat{b}_1$, then

$${}^bV = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad {}^eV = \begin{pmatrix} \cos\theta \\ 0 \\ -\sin\theta \end{pmatrix}.$$

Clearly ${}^eV = R\,{}^bV$ holds for this vector.

Exercise 72 Verify (12.15).

Angular velocity vector. Recalling the above rotation of $\mathbf{b}$ relative to $\mathbf{e}$, we define the angular velocity of $\mathbf{b}$ relative to $\mathbf{e}$ as follows:

$${}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} = \dot{\theta}\hat{e}_2\,. \tag{12.17}$$

We can obtain the angular velocity by differentiating the rotation matrix R with respect to time. To see this, define

$$\Omega := R^T\dot{R} \tag{12.18}$$

then, one may readily obtain that

$$\Omega = \begin{pmatrix} 0 & 0 & \dot{\theta} \\ 0 & 0 & 0 \\ -\dot{\theta} & 0 & 0 \end{pmatrix}. \tag{12.19}$$

Thus, the matrix Ω is skew symmetric ($\Omega^T = -\Omega$) and its 13 element equals $\dot{\theta}$.

Exercise 73 Using (12.18) as the definition of Ω where R is given by (12.12), prove that Ω is given by (12.19).

Basic Kinematic Equation (BKE). Consider any vector $\bar{V}$ and let

$$\frac{^{\mathbf{e}}d\bar{V}}{dt} \qquad \text{and} \qquad \frac{^{\mathbf{b}}d\bar{V}}{dt}$$

denote the time rate of change of $\bar{V}$ relative to reference frames $\mathbf{e}$ and $\mathbf{b}$, respectively, that is,

$$\begin{aligned} \frac{^{\mathbf{e}}d\bar{V}}{dt} &= {}^e\dot{V}_1\hat{e}_1 + {}^e\dot{V}_2\hat{e}_2 + {}^e\dot{V}_3\hat{e}_3 \\ \frac{^{\mathbf{b}}d\bar{V}}{dt} &= {}^b\dot{V}_1\hat{b}_1 + {}^b\dot{V}_2\hat{b}_2 + {}^b\dot{V}_3\hat{b}_3 \,. \end{aligned}$$

Since the cartesian components of the vector $\frac{^{\mathbf{e}}d\bar{V}}{dt}$ relative to $\mathbf{e}$ are given by

$${}^e\dot{V} = \begin{pmatrix} {}^e\dot{V}_1 \\ {}^e\dot{V}_2 \\ {}^e\dot{V}_3 \end{pmatrix},$$

it follows from relationship (12.16), that the cartesian components of $\frac{^{\mathbf{e}}d\bar{V}}{dt}$ relative to $\mathbf{b}$ are given by $R^T\,{}^e\dot{V}$. The cartesian components of the vector $\frac{^{\mathbf{b}}d\bar{V}}{dt}$ relative to $\mathbf{b}$ are given by

$${}^b\dot{V} = \begin{pmatrix} {}^b\dot{V}_1 \\ {}^b\dot{V}_2 \\ {}^b\dot{V}_3 \end{pmatrix}.$$

If we time-differentiate the relationship ${}^eV = R\,{}^bV$, we obtain

$${}^e\dot{V} = R\,{}^b\dot{V} + \dot{R}\,{}^bV\,.$$

Pre-multiplying both sides of this relationship by R^T yields

$$R^T\,{}^e\dot{V} = {}^b\dot{V} + \Omega\,{}^bV \tag{12.20}$$

where $\Omega = R^T\dot{R}$. Recalling the expression (12.19) for Ω, we see that

$$\Omega\,{}^bV = \begin{pmatrix} \dot{\theta}\ \ {}^bV_3 \\ 0 \\ -\dot{\theta}\ \ {}^bV_1 \end{pmatrix}.$$

Noting that

$$^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} \times \bar{V} = (\dot{\theta}\hat{b}_2) \times ({}^bV_1\hat{b}_1 + {}^bV_2\hat{b}_2 + {}^bV_3\hat{b}_3) = (\dot{\theta}\ \ {}^bV_3)\,\hat{b}_1 - (\dot{\theta}\ \ {}^bV_1)\,\hat{b}_3\,,$$

we see that $\Omega\,{}^bV$ is the component vector of $^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} \times \bar{V}$ relative to frame $\mathbf{b}$. Hence, relationship (12.20) yields the basic kinematic equation (BKE):

$$\boxed{\frac{^{\mathbf{e}}d\bar{V}}{dt} = \frac{^{\mathbf{b}}d\bar{V}}{dt} + {}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} \times \bar{V}} \tag{12.21}$$

12.3.2 General rotations

Here we look at various ways of representing the rotation of any reference frame $\mathbf{b} = (\hat{b}_1, \hat{b}_2, \hat{b}_3)$ relative to any other reference frame $\mathbf{e} = (\hat{e}_1, \hat{e}_2, \hat{e}_3)$.

Rotation matrix (R): Suppose we express each of the b-vectors in terms of the e-vectors, that is,

$$\begin{aligned}\hat{b}_1 &= R_{11}\hat{e}_1 + R_{21}\hat{e}_2 + R_{31}\hat{e}_3 \\ \hat{b}_2 &= R_{12}\hat{e}_1 + R_{22}\hat{e}_2 + R_{32}\hat{e}_3 \\ \hat{b}_3 &= R_{13}\hat{e}_1 + R_{23}\hat{e}_2 + R_{33}\hat{e}_3 \,.\end{aligned} \tag{12.22}$$

We define the rotation matrix R associated with the rotation of reference frame $\mathbf{b}$ relative to reference frame $\mathbf{e}$ by

$$R = \begin{pmatrix} R_{11} & R_{12} & R_{13} \\ R_{21} & R_{22} & R_{23} \\ R_{31} & R_{32} & R_{33} \end{pmatrix} .$$

Note that this matrix is obtained by letting its first, second, and third columns consist of the scalar components of $\hat{b}_1$, $\hat{b}_2$ $\hat{b}_3$, respectively, relative to $\mathbf{e}$. Note also that

$$R^T R = I \tag{12.23}$$

where I is the 3×3 identity matrix. From this it follows that R is invertible and its inverse is its transpose, that is, $R^{-1} = R^T$. Also, $RR^T = I$.

Exercise 74 Verify (12.23).

Coordinate transformations. Consider any vector $\bar{V}$ and suppose we express it in components relative to $\mathbf{e}$ and $\mathbf{b}$, that is,

$$\bar{V} = {}^eV_1\hat{e}_1 + {}^eV_2\hat{e}_2 + {}^eV_3\hat{e}_3 \qquad \text{and} \qquad \bar{V} = {}^bV_1\hat{b}_1 + {}^bV_2\hat{b}_2 + {}^bV_3\hat{b}_3,$$

respectively. Now introduce the coordinate vectors,

$${}^eV = \begin{pmatrix} {}^eV_1 \\ {}^eV_2 \\ {}^eV_3 \end{pmatrix} \qquad \text{and} \qquad {}^bV = \begin{pmatrix} {}^bV_1 \\ {}^bV_2 \\ {}^bV_3 \end{pmatrix} .$$

Using relationships (12.22) one may readily obtain the following transformation relationship:

$$\boxed{{}^eV = R\,{}^bV} \tag{12.24}$$

Premultiplying both sides by R^T and using $R^T R = I$, we also have

$$\boxed{{}^bV = R^T\,{}^eV} \tag{12.25}$$

For example, if $\bar{V} = \hat{b}_1$, then

$${}^bV = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \qquad \text{and} \qquad {}^eV = \begin{pmatrix} R_{11} \\ R_{21} \\ R_{31} \end{pmatrix} .$$

Clearly ${}^eV = R\,{}^bV$ holds here.

Exercise 75 Verify (12.24).

Composition of rotations. If **f** and **g** are any two reference frames, we let ${}^fR^g$ denote the rotation matrix associated with the rotation of frame g relative to frame f. We now demonstrate the following fact: *If* **a**, **b** *and* **c** *are any three reference frames, then*

$$ {}^aR^c = {}^aR^b\, {}^bR^c \,. \tag{12.26} $$

To see this, consider any vector $\bar{V}$ and let ${}^aV$, ${}^bV$ and ${}^cV$ be the coordinate vectors of $\bar{V}$ relative to reference frames **a**, **b** and **c**, respectively. Utilizing relationship (12.24) between **a** and **b** and between **b** and **c**, we obtain ${}^aV = {}^aR^b\, {}^bV$ and ${}^bV = {}^bR^c\, {}^cV$, respectively. It now follows that

$$ {}^aV = {}^aR^b\, {}^bR^c\, {}^cV \,. $$

Now utilizing relationship (12.24) between **a** and **c**, we also have ${}^aV = {}^aR^c\, {}^cV$; hence

$$ {}^aR^c\, {}^cV = {}^aR^b\, {}^bR^c\, {}^cV \,. $$

Since the above relationship holds for any vector ${}^cV$ we must have the desired relationship: ${}^aR^c = {}^aR^b\, {}^bR^c$.

Angular velocity vector. Consider any two reference frames **e** and **b** and let R be the rotation matrix associated with the rotation of **b** relative to **e**. Define the following matrix

$$ \boxed{\Omega := R^T \dot{R}} \tag{12.27} $$

Using the relationship $R^T R = I$, one can readily show that Ω is skew-symmetric, that is,

$$ \Omega^T = -\Omega \,. $$

Hence, Ω must have the following structure:

$$ \Omega = \begin{pmatrix} 0 & \Omega_{12} & \Omega_{13} \\ -\Omega_{12} & 0 & \Omega_{23} \\ -\Omega_{13} & -\Omega_{23} & 0 \end{pmatrix} . $$

Suppose we let

$$ \omega_1 := -\Omega_{23} \,, \qquad \omega_2 := -\Omega_{31} \,, \qquad \omega_3 := -\Omega_{12} \,. $$

Then

$$ \Omega = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} $$

We define the angular velocity of **b** relative to **e** as :

$$ \boxed{{}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} = \omega_1 \hat{b}_1 + \omega_2 \hat{b}_2 + \omega_3 \hat{b}_3} $$

Composition of angular velocities. One can readily demonstrate the following fact: *If* **a**, **b** *and* **c** *are any three reference frames, then*

$$ {}^{\mathbf{a}}\bar{\omega}^{\mathbf{c}} = {}^{\mathbf{a}}\bar{\omega}^{\mathbf{b}} + {}^{\mathbf{b}}\bar{\omega}^{\mathbf{c}} \,. \tag{12.28} $$

Example 97 Consider the machine illustrated in Figure 12.15.

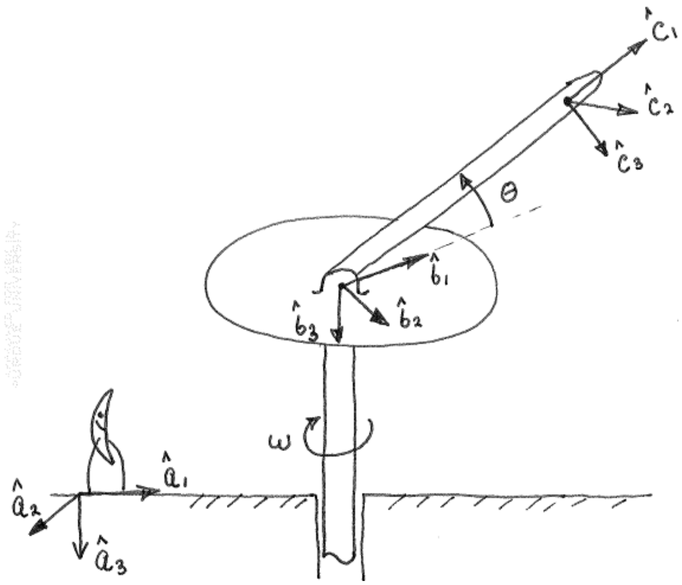


Figure 12.15: Example

Basic Kinematic Equation (BKE). We let

$$\frac{{}^{\mathbf{e}}d\bar{V}}{dt} \qquad \text{and} \qquad \frac{{}^{\mathbf{b}}d\bar{V}}{dt}$$

denote the time rate of change of $\bar{V}$ relative to reference frames $\mathbf{e}$ and $\mathbf{b}$, respectively; thus

$$\begin{aligned}
\frac{{}^{\mathbf{e}}d\bar{V}}{dt} &= {}^{e}\dot{V}_1\hat{e}_1 + {}^{e}\dot{V}_2\hat{e}_2 + {}^{e}\dot{V}_3\hat{e}_3 \\
\frac{{}^{\mathbf{b}}d\bar{V}}{dt} &= {}^{b}\dot{V}_1\hat{b}_1 + {}^{b}\dot{V}_2\hat{b}_2 + {}^{b}\dot{V}_3\hat{b}_3
\end{aligned}$$

The scalar components of the vector $\frac{{}^{\mathbf{e}}d\bar{V}}{dt}$ relative to $\mathbf{e}$ are given by ${}^{e}\dot{V}$. Recalling relationship (12.25), the scalar components of this vector relative to $\mathbf{b}$ are given by $R^T\ {}^{e}\dot{V}$. The scalar components of the vector $\frac{{}^{\mathbf{b}}d\bar{V}}{dt}$ relative relative to $\mathbf{b}$ are given by ${}^{b}\dot{V}$. If we time-differentiate relationship (12.24) we obtain

$$\dot{V}^e = R\ {}^{b}\dot{V} + \dot{R}\ {}^{b}V$$

Pre-multiplying both sides of this relationship by R^T and using $R^TR = I$ yields

$$R^T\ {}^{e}\dot{V} = {}^{b}\dot{V} + \Omega\ {}^{b}V$$

where $\Omega = R^T\dot{R}$. Recalling the above expression for Ω, we see that

$$\Omega\ {}^{b}V = \begin{pmatrix} \omega_2\ {}^{b}V_3 - \omega_3\ {}^{b}V_2 \\ \omega_3\ {}^{b}V_1 - \omega_1\ {}^{b}V_3 \\ \omega_1\ {}^{b}V_2 - \omega_2\ {}^{b}V_1 \end{pmatrix}$$

Noting that

$${}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} \times \bar{V} = (\omega_2\ {}^{b}V_3 - \omega_3\ {}^{b}V_2)\,\hat{e}_1 + (\omega_3\ {}^{b}V_1 - \omega_1\ {}^{b}V_3)\,\hat{e}_2 + (\omega_1\ {}^{b}V_2 - \omega_2\ {}^{b}V_1)\,\hat{e}_3$$

we see that $\Omega\ {}^{b}V$ is the component vector of ${}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} \times \bar{V}$ relative to frame $\mathbf{b}$. Hence, we obtain the basic kinematic equation (BKE):

$$\boxed{\frac{{}^{\mathbf{e}}d\bar{V}}{dt} = \frac{{}^{\mathbf{b}}d\bar{V}}{dt} + {}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} \times \bar{V}}$$

12.3.3 Euler angles

We have seen how to represent rotations with rotations with rotation matrices. Here we look at a more intuitive way of representing rotations. Consider a rotation of a reference frame $\mathbf{b}$ relative to another reference frame $\mathbf{e}$ obtained by the following sequence of three simple rotations:

Rotation of $\mathbf{b}$ about $\hat{b}_3$ through angle ψ.
Rotation of $\mathbf{b}$ about (new) $\hat{b}_2$ through angle θ.
Rotation of $\mathbf{b}$ about (new) $\hat{b}_1$ through angle ϕ.

This is called a 3-2-1 Euler angle sequence.

Letting R_ψ denote the rotation matrix for the first rotation, we have

$$R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Letting R_θ denote the rotation matrix for the second rotation, we have

$$R_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

and the rotation matrix for the composition of the first two rotations is

$$R_\psi R_\theta = \begin{pmatrix} \cos\psi\cos\theta & -\sin\psi & \cos\psi\sin\theta \\ \sin\psi\cos\theta & \cos\psi & \sin\psi\sin\theta \\ -\sin\theta & 0 & \cos\theta \end{pmatrix}$$

Letting R_ϕ denote the rotation matrix for the third rotation, we have

$$R_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}.$$

The rotation matrix R for the overall rotation which is a composition of the three simple rotations is given by

$$R = R_\psi R_\theta R_\phi\,.$$

Carrying our the matrix multiplications yields

$$R = \begin{pmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix}. \tag{12.29}$$

The angular velocity of $\mathbf{b}$ in $\mathbf{e}$ is now given by

$${}^e\bar{\omega}^b = \omega_1\hat{b}_1 + \omega_2\hat{b}_2 + \omega_3\hat{b}_3\,.$$

where

$$\begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix} = \Omega = R^T\dot{R}\,.$$

Differentiating R and pre-multiplying it by R^T yields

$$\begin{aligned} \omega_1 &= \dot{\phi} - \dot{\psi}\sin\theta \\ \omega_2 &= \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\ \omega_3 &= -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi \end{aligned} \tag{12.30}$$

These relationships may also be written as

$$\begin{pmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \dot\phi + \begin{pmatrix} 0 \\ \cos\phi \\ -\sin\phi \end{pmatrix} \dot\theta + \begin{pmatrix} -\sin\theta \\ \cos\theta\sin\phi \\ \cos\theta\cos\phi \end{pmatrix} \dot\psi$$

Note that these equations do not depend on the angle ψ. Solving for $\dot\phi, \dot\theta, \dot\psi$ in terms of $\omega_1, \omega_2, \omega_3$ yields

$$\dot\phi = \omega_1 + \tan\theta\sin\phi\,\omega_2 + \tan\theta\cos\phi\,\omega_3 \tag{12.31a}$$
$$\dot\theta = \cos\phi\,\omega_2 - \sin\phi\,\omega_3 \tag{12.31b}$$
$$\dot\psi = (\sin\phi/\cos\theta)\omega_2 + (\cos\phi/\cos\theta)\omega_3 \tag{12.31c}$$

12.3.4 Aircraft rotational kinematics

Recall the earth-fixed frame **e** and the aircraft-fixed **b** frame introduced earlier. We will describe the orientation of **b** relative to **e** with a 3-2-1 Euler angle sequence.

In going from earth frame **e** to aircraft frame **b** carry out the following simple rotations in sequence..

Rotate about $\hat b_3$ (aircraft Z-axis)	yaw angle ψ	Positive when nose right
Rotate about new $\hat b_2$ (aircraft Y-axis)	pitch angle θ	Positive when nose up
Rotate about new $\hat b_1$ (aircraft X-axis)	roll angle ϕ	Positive when right wing down

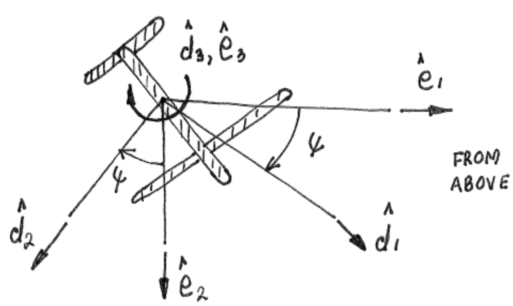


Figure 12.16: Yaw

Rotation matrices.

$$R_\psi = \begin{pmatrix} \cos\psi & -\sin\psi & 0 \\ \sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_\theta = \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \qquad R_\phi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix}$$

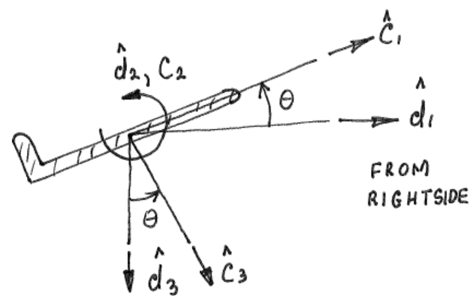


Figure 12.17: Pitch

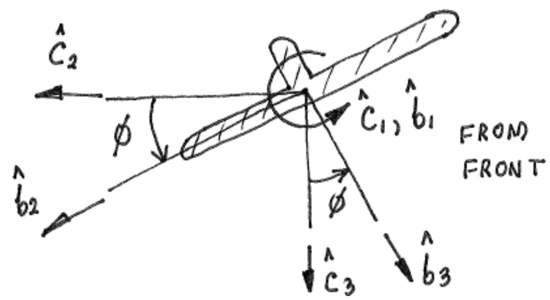


Figure 12.18: Roll

$$R = R_\psi R_\theta R_\phi = \begin{pmatrix} \cos\psi\cos\theta & -\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi & \sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi \\ \sin\psi\cos\theta & \cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi & -\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi \\ -\sin\theta & \cos\theta\sin\phi & \cos\theta\cos\phi \end{pmatrix}$$

Angular velocity. The notation for the angular velocity of the aircraft relative to the earth is

$$\boxed{{}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} = p\hat{b}_1 + q\hat{b}_2 + r\hat{b}_3}$$

Sometimes the following nomenclature is used:

p	roll rate
q	pitch rate
r	yaw rate

From the previous section, we have that

$$\begin{aligned} p &= \dot{\phi} - \dot{\psi}\sin\theta \\ q &= \dot{\theta}\cos\phi + \dot{\psi}\cos\theta\sin\phi \\ r &= -\dot{\theta}\sin\phi + \dot{\psi}\cos\theta\cos\phi \end{aligned} \tag{12.32}$$

Utilizing (12.31) we can obtain the following differential equations for the rotational kinematics:

$$\boxed{\begin{aligned} \dot\phi &= p + \tan\theta\sin\phi\, q + \tan\theta\cos\phi\, r \\ \dot\theta &= \cos\phi\, q - \sin\phi\, r \\ \dot\psi &= (\sin\phi/\cos\theta)\, q + (\cos\phi/\cos\theta)\, r \end{aligned}} \tag{12.33}$$

Also, using

$$^{e}V = R\,^{b}V$$

we obtain the following differential equations for the translational kinematics:

$$\boxed{\begin{array}{rclll} \dot p_1 &=& \cos\psi\cos\theta\, u & +(-\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi)\, v & +(\sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi)\, w \\ \dot p_2 &=& \sin\psi\cos\theta\, u & +(\cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi)\, v & +(-\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi)\, w \\ \dot h &=& \sin\theta\, u & -\cos\theta\sin\phi\, v & -\cos\theta\cos\phi\, w \end{array}} \tag{12.34}$$

Two other reference frames in common use in describing the dynamics of an aircraft are the stability reference frame and the wind reference frame.

Stability reference frame. Obtained via a clockwise rotation of aircraft frame about $\hat b_2$ (aircraft Y-axis) through α. The corresponding rotation matrix is:

$$R_\alpha = \begin{pmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{pmatrix}$$

Wind reference frame. Rotate stability frame counterclockwise about $\hat s_3$ (stability Z-axis) through β. Note that $\hat w_1$ (the wind X-axis) is aligned with $\bar V$. The corresponding rotation matrix is:

$$R_\beta = \begin{pmatrix} \cos\beta & -\sin\beta & 0 \\ \sin\beta & \cos\beta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The rotation matrix corresponding to the rotation of the wind frame relative to the aircraft frame is:

$$R_{\alpha\beta} = R_\alpha R_\beta = \begin{pmatrix} \cos\alpha\cos\beta & -\cos\alpha\sin\beta & -\sin\alpha \\ \sin\beta & \cos\beta & 0 \\ \sin\alpha\cos\beta & -\sin\alpha\sin\beta & \cos\alpha \end{pmatrix}$$

The angular velocity of the wind frame relative to the aircraft frame is

$$^{\mathbf{b}}\bar\omega^{\mathbf{w}} = -\dot\alpha\sin\beta\,\hat w_1 - \dot\alpha\cos\beta\,\hat w_2 + \dot\beta\,\hat w_3$$

12.4 Forces and moments on aircraft

12.4.1 Aerodynamic and thrust

Forces. A common way to model the total aerodynamic force $\bar{F}_A$ is

$$\bar{F}_A = -D\hat{w}_1 - C\hat{w}_2 - L\hat{w}_3 \tag{12.35}$$

where D is drag, C is called the crosswind and L is lift. In terms of the body-fixed frame b we have

$$\bar{F}_A = X_A\hat{b}_1 + Y_A\hat{b}_2 + Z_A\hat{b}_3$$

where

$$\begin{pmatrix} X_A \\ Y_A \\ Z_A \end{pmatrix} = R_{\alpha\beta}\begin{pmatrix} -D \\ -C \\ -L \end{pmatrix}$$

Let $\bar{F}_T$ be the resultant thrust; usually,

$$\bar{F}_T = TC_\epsilon\hat{b}_1 + TS_\epsilon\hat{b}_3\,.$$

We will let the resultant aerodynamic and thrust force be given by

$$\bar{F}_A + \bar{F}_T = X\,\hat{b}_1 + Y\,\hat{b}_2 + Z\,\hat{b}_3 \tag{12.36}$$

Note that we are using Y to mean two different things. The meaning of Y will be clear from the context

Moments. We will let the resultant moment about mass center due to aerodynamic and thrust forces be given by

$$\bar{M}_A + \bar{M}_T = L^\clubsuit\,\hat{b}_1 + M\,\hat{b}_2 + N\,\hat{b}_3 \tag{12.37}$$

12.4.2 Gravity

$$\bar{W} = m\,\bar{g}\,, \qquad \bar{g} = g\hat{e}_3$$

Aircraft-fixed components. Using

$$^b g = R^T\,{}^e g \qquad \text{and} \qquad {}^e g = \begin{pmatrix} 0 \\ 0 \\ g \end{pmatrix}$$

we obtain

$$\bar{g} = -g\sin\theta\,\hat{b}_1 + g\cos\theta\sin\phi\,\hat{b}_2 + g\cos\theta\cos\phi\,\hat{b}_3 \tag{12.38}$$

12.5 Equations of motion

12.5.1 Translational dynamics

Apply

$$\Sigma \bar{F} = m\bar{a}$$

to mass center of aircraft to yield

$$m\frac{{}^{\mathbf{e}}d\bar{V}}{dt} = \bar{W} + \bar{F}_A + \bar{F}_T\,. \tag{12.39}$$

Body fixed components. We shall consider components of (12.39) relative to the aircraft fixed frame. Since

$$\bar{V} = u\,\hat{b}_1 + v\,\hat{b}_2 + w\,\hat{b}_3$$

and

$${}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} = p\,\hat{b}_1 + q\,\hat{b}_2 + r\,\hat{b}_3\,,$$

application of the BKE yields

$$\begin{aligned}
\frac{{}^{\mathbf{e}}d\bar{V}}{dt} &= \frac{{}^{\mathbf{b}}d\bar{V}}{dt} + {}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} \times \bar{V} \\
&= (\dot{u} + qw - rv)\,\hat{b}_1 + (\dot{v} + ru - pw)\,\hat{b}_2 + (\dot{w} + pv - qu)\,\hat{b}_3\,.
\end{aligned}$$

Recalling (12.39), (12.36), and (12.38), we obtain

$$\boxed{\begin{array}{rclccc}
\dot{u} &=& rv - qw & -\,g\sin\theta & +\,X/m \\
\dot{v} &=& -ru + pw & +\,g\cos\theta\sin\phi & +\,Y/m \\
\dot{w} &=& qu - pv & +\,g\cos\theta\cos\phi & +\,Z/m
\end{array}} \tag{12.40}$$

12.5.2 Rotational dynamics

The rotational dynamics of an aircraft are governed by

$$\Sigma \bar{M} = \frac{{}^{\mathbf{e}}d\bar{H}}{dt} \tag{12.41}$$

where $\bar{H}$ is the angular momentum of the aircraft about its mass center and $\Sigma\bar{M}$ is the resultant moment about the mass center of all forces acting on the aircraft. Hence

$$\frac{{}^{\mathbf{e}}d\bar{H}}{dt} = \bar{M}_A + \bar{M}_T\,. \tag{12.42}$$

If we use the aircraft-fixed frame, we have

$$\bar{H} = H_1\hat{b}_1 + H_2\hat{b}_2 + H_3\hat{b}_3$$

where

$$\begin{pmatrix} H_1 \\ H_2 \\ H_3 \end{pmatrix} = J\omega \qquad \text{and} \qquad \omega = \begin{pmatrix} p \\ q \\ r \end{pmatrix}$$

while

$$J = \begin{pmatrix} J_{11} & 0 & -J_{13} \\ 0 & J_{22} & 0 \\ -J_{13} & 0 & J_{33} \end{pmatrix}$$

is the inertia matrix of the aircraft about its mass center and relative to the aircraft-fixed reference frame. Since the aircraft is symmetric wrt to the $\hat{b}_1$–$\hat{b}_3$ plane; J_{21} and J_{23} are both zero. Thus,

$$\begin{aligned} H_1 &= J_{11}p - J_{13}r \\ H_2 &= J_{22}q \\ H_3 &= -J_{13}p + J_{33}r \end{aligned} \tag{12.43}$$

Applying the BKE with ${}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} = p\hat{b}_1 + q\,\hat{b}_2 + r\,\hat{b}_3$ yields

$$\begin{aligned} \frac{{}^{\mathbf{e}}d\bar{H}}{dt} &= \frac{{}^{\mathbf{b}}d\bar{H}}{dt} + {}^{\mathbf{e}}\bar{\omega}^{\mathbf{b}} \times \bar{H} \\ &= (\dot{H}_1 + qH_3 - rH_2)\,\hat{b}_1 + (\dot{H}_2 + rH_1 - pH_3)\,\hat{b}_2 + (\dot{H}_3 + pH_2 - qH_1)\,\hat{b}_3 \end{aligned} \tag{12.44}$$

Substituting (12.44), (12.43) and

$$\bar{M}_A + \bar{M}_T = L^{\clubsuit}\hat{b}_1 + M\hat{b}_2 + N\hat{b}_3 \tag{12.45}$$

into (12.42) we obtain

$$\boxed{\begin{array}{rcll} J_{11}\dot{p} - J_{13}\dot{r} &=& (J_{22} - J_{33})qr + J_{13}pq & +L^{\clubsuit} \\ J_{22}\dot{q} &=& (J_{33} - J_{11})pr + J_{13}(r^2 - p^2) & +M \\ -J_{13}\dot{p} + J_{33}\dot{r} &=& (J_{11} - J_{22})pq - J_{13}qr & +N \end{array}} \tag{12.46}$$

12.6 The whole story

The general three dimensional motion of an aircraft can be completely described by the following 12 differential equations:

$$\begin{array}{rcl}
\dot{p}_1 & = & \cos\psi\cos\theta\, u + (-\sin\psi\cos\phi + \cos\psi\sin\theta\sin\phi)\, v + (\sin\psi\sin\phi + \cos\psi\sin\theta\cos\phi)\, w \\
\dot{p}_2 & = & \sin\psi\cos\theta\, u + (\cos\psi\cos\phi + \sin\psi\sin\theta\sin\phi)\, v + (-\cos\psi\sin\phi + \sin\psi\sin\theta\cos\phi)\, w \\
\dot{h} & = & \sin\theta\, u - \cos\theta\sin\phi\, v - \cos\theta\cos\phi\, w \\
\\
\dot{\phi} & = & p + \tan\theta\sin\phi\, q + \tan\theta\cos\phi\, r \\
\dot{\theta} & = & \cos\phi\, q - \sin\phi\, r \\
\dot{\psi} & = & (\sin\phi/\cos\theta)\, q + (\cos\phi/\cos\theta)\, r \\
\\
\dot{u} & = & rv - qw - g\sin\theta + X/m \\
\dot{v} & = & -ru + pw + g\cos\theta\sin\phi + Y/m \\
\dot{w} & = & qu - pv + g\cos\theta\cos\phi + Z/m \\
\\
J_{11}\dot{p} - J_{13}\dot{r} & = & (J_{22} - J_{33})qr + J_{13}pq + L^{\clubsuit} \\
J_{22}\dot{q} & = & (J_{33} - J_{11})pr + J_{13}(r^2 - p^2) + M \\
-J_{13}\dot{p} + J_{33}\dot{r} & = & (J_{11} - J_{22})pq - J_{13}qr + N
\end{array}$$

This is a state space description with state and control input

$$x = \begin{pmatrix} p_1 \\ p_2 \\ h \\ \phi \\ \theta \\ \psi \\ u \\ v \\ w \\ p \\ q \\ r \end{pmatrix} \qquad \text{and} \qquad u = \begin{pmatrix} th \\ el \\ \delta_a \\ \delta_r \end{pmatrix},$$

respectively.

12.7 Lateral dimenionless coefficients and derivatives

We have

$$\begin{aligned} Y &= \bar{q}SC_Y \\ L^{\clubsuit} &= \bar{q}SbC_l \\ N &= \bar{q}SbC_n \end{aligned}$$

Usually, the above coefficients are modeled as depending only on

$$\beta, \delta_a, \delta_r, p, r,$$

Stability derivatives

Sideforce derivative due to sideslip: C_{Y_β}:

Dihedral derivative C_{l_β}:

Yaw stiffness derivative C_{n_β}:

Damping derivatives

$$C_{Y_p},\, C_{Y_r}\, C_{l_p},\, C_{l_r}\, C_{n_p},\, C_{n_r}$$

Control derivatives

$$C_{Y_{\delta a}},\, C_{Y_{\delta r}}\, C_{l_{\delta a}},\, C_{l_{\delta r}}\, C_{n_{\delta a}},\, C_{n_{\delta r}}$$

Cessna 182.

$$\begin{aligned} C_{Y_\beta} &= -0.393\,\mathrm{rad}^{-1} \\ C_{l_\beta} &= -0.0923\,\mathrm{rad}^{-1} \\ C_{n_\beta} &= 0.0587\,\mathrm{rad}^{-1} \end{aligned}$$

Using linear approximations we get

$$\begin{aligned} C_Y &= C_{Y_\beta}\beta + \frac{b}{2V}C_{Y_p}p + \frac{b}{2V}C_{Y_r}r + C_{Y_{\delta a}}\delta_a + C_{Y_{\delta r}}\delta_r \\ C_l &= C_{l_\beta}\beta + \frac{b}{2V}C_{l_p}p + \frac{b}{2V}C_{l_r}r + C_{l_{\delta a}}\delta_a + C_{l_{\delta r}}\delta_r \\ C_n &= C_{n_\beta}\beta + \frac{b}{2V}C_{n_p}p + \frac{b}{2V}C_{n_r}r + C_{n_{\delta a}}\delta_a + C_{n_{\delta r}}\delta_r \end{aligned}$$

12.8 Linearization

Since the above equations of motion do not depend on the variables p_1, p_2, h and yaw angle ψ we will forget about these variables for now. We will split the remaining variables into two groups

Longitudinal variables:	u, w, θ, q
Lateral variables:	v, ϕ, p, r

The corresponding differential equations are:

$$\begin{aligned}
\dot{u} &= rv - qw - g\sin\theta + X/m \\
\dot{w} &= qu - pv + g\cos\theta\cos\phi + Z/m \\
\dot{\theta} &= \cos\phi\, q - \sin\phi\, r \\
J_{22}\dot{q} &= (J_{33} - J_{11})pr + J_{13}(r^2 - p^2) + M
\end{aligned}$$

and

$$\begin{aligned}
\dot{v} &= -ru + pw + g\cos\theta\sin\phi + Y/m \\
\dot{\phi} &= p + \tan\theta\sin\phi\, q + \tan\theta\cos\phi\, r \\
J_{11}\dot{p} - J_{13}\dot{r} &= (J_{22} - J_{33})qr + J_{13}pq + L^{\clubsuit} \\
-J_{13}\dot{p} + J_{33}\dot{r} &= (J_{11} - J_{22})pq - J_{13}qr + N
\end{aligned}$$

12.8.1 Longitudinal Motion

Here

$$\phi, v, p, r = 0$$

and the motion of the aircraft is described by

$$\begin{aligned}
\dot{u} &= -qw - g\sin\theta + X/m \\
\dot{w} &= qu + g\cos\theta + Z/m \\
\dot{\theta} &= q \\
J_{22}\dot{q} &= M
\end{aligned}$$

We have already examined this.

12.8.2 Linearized lateral dynamics

If we linearize about a steady state condition corresponding to longitudinal motion at constant velocity and constant pitch angle, we shall see that the lateral dynamics decouple from the longitudinal dynamics. So we are linearizing about

$$\begin{aligned}
&u = u^e, \quad w = w^e, \quad \theta = \theta^e \\
&v, \phi, p, q, r = 0
\end{aligned}$$

We let

$$\begin{pmatrix} \delta v \\ \delta\phi \\ \delta p \\ \delta r \end{pmatrix} = \begin{pmatrix} v \\ \phi \\ p \\ r \end{pmatrix}$$

In this linearization, we will assume

$$Y/m \approx Y_v \delta v + Y_p \delta p + Y_r \delta r + Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r$$

and

$$\begin{aligned} L^{\clubsuit}/J_{11} &\approx L_v \delta v + L_p \delta p + L_r \delta r + L_{\delta_a} \delta_a + L_{\delta_r} \delta_r \\ N/J_{33} &\approx N_v \delta v + N_p \delta p + N_r \delta r + N_{\delta_a} \delta_a + N_{\delta_r} \delta_r \end{aligned}$$

Linearizing the lateral dynamics we obtain

$$\begin{aligned} \delta \dot{v} &= Y_v \delta v + (g \cos \theta^e) \delta \phi + (w^e + Y_p) \delta p + (-u^e + Y_r) \delta r + Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r \\ \delta \dot{\phi} &= \delta p + (\tan \theta^e) \delta r \\ J_{11} \delta \dot{p} - J_{13} \delta \dot{r} &= J_{11} [L_v \delta v + L_p \delta p + L_r \delta r + L_{\delta_a} \delta_a + L_{\delta_r} \delta_r] \\ -J_{13} \delta \dot{p} + J_{33} \delta \dot{r} &= J_{33} [N_v \delta v + N p \delta_p + N_r \delta r + N_{\delta_a} \delta_a + N_{\delta_r} \delta_r] \end{aligned}$$

The last two equations can be solved explicitly for $\delta \dot{p}$ and $\delta \dot{r}$ to obtain

$$\boxed{\begin{aligned} \delta \dot{v} &= Y_v \delta v + (g \cos \theta^e) \delta \phi + (w^e + Y_p) \delta p + (-u^e + Y_r) \delta r + Y_{\delta_a} \delta_a + Y_{\delta_r} \delta_r \\ \delta \dot{\phi} &= \delta p + (\tan \theta^e) \delta r \\ \delta \dot{p} &= L'_v \delta v + L'_p \delta p + L'_r \delta r + L'_{\delta_a} \delta_a + L'_{\delta_r} \delta_r \\ \delta \dot{r} &= N'_v \delta v + N'_p \delta p + N'_r \delta r + N'_{\delta_a} \delta_a + N'_{\delta_r} \delta_r \end{aligned}}$$

where

$$L'_{\spadesuit} = \mu L_{\spadesuit} + \sigma_1 N_{\spadesuit} \qquad N'_{\spadesuit} = \mu N_{\spadesuit} + \sigma_2 L_{\spadesuit} \qquad \text{for} \quad \spadesuit = v, p, r, \delta_a, \delta_r$$

and

$$\mu = \frac{J_{11} J_{33}}{\Gamma} \qquad \sigma_1 = \frac{J_{33} J_{13}}{\Gamma} \qquad \sigma_2 = \frac{J_{11} J_{13}}{\Gamma} \qquad \Gamma = J_{11} J_{33} - J_{13}^2$$

Introducing the state and input,

$$x = \begin{pmatrix} \delta v \\ \delta \phi \\ \delta p \\ \delta r \end{pmatrix} \qquad u = \begin{pmatrix} \delta_a \\ \delta_r \end{pmatrix}$$

respectively, the linearization of the lateral dynamics can be described by

$$\dot{x} = Ax + Bu$$

with

$$A = \begin{pmatrix} Y_v & g \cos \theta^e & w^e + Y_p & -u^e + Y_r \\ 0 & 0 & 1 & \tan \theta^e \\ L'_v & 0 & L'_p & L'_r \\ N'_v & 0 & N'_p & N'_r \end{pmatrix} \qquad B = \begin{pmatrix} Y_{\delta_a} & Y_{\delta_r} \\ 0 & 0 \\ L'_{\delta_a} & L'_{\delta_r} \\ N'_{\delta_a} & N'_{\delta_r} \end{pmatrix} \tag{12.47}$$

Lateral modes

Dutch roll
roll subsidence
spiral

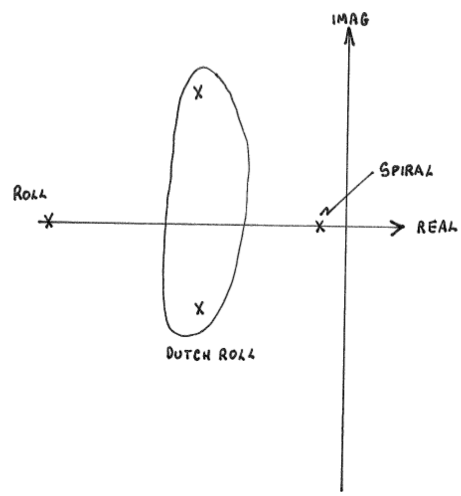


Figure 12.19: Lateral modes

Example 98 Navion, a small general-aviation aircraft; see Bryson, 1994.

$$A = \begin{pmatrix} -0.254 & 0.322 & 0 & -1.76 \\ 0 & 0 & 1 & 0 \\ -9.08 & 0 & -8.40 & 2.19 \\ 2.55 & 0 & -0.35 & -0.76 \end{pmatrix}$$

Application of the MATLAB eig command yields:

mode	eigenvalue(s)
dutch roll	-0.4862 $\pm$ 2.3335$\jmath$
roll	-8.4327
spiral	-0.0088

where the eigenvectors are given by

	dutch roll	roll	spiral
δv	0.4083 $\pm$ 0.2868$\jmath$	-0.0134	-0.0498
$\delta \phi$	0.1674 $\pm$ 0.1615$\jmath$	0.1177	-0.9836
δp	-0.4582 $\pm$ 0.3121$\jmath$	-0.9921	0.0087
δr	-0.2958 $\pm$ 0.5496$\jmath$	-0.0408	-0.1730

References

1. Bryson, A.E., *Control of Spacecraft and Aircraft*, Princeton University Press, 1994.

12.8.3 extra

If $(\hat{w}_1, \hat{w}_2, \hat{w}_3)$ is the wind frame, the total aerodynamic force on the aircraft can be expressed as

$$\bar{F}_A = -D\hat{w}_1 - C\hat{w}_2 - L\hat{w}_3 \tag{12.48}$$

where

D	drag
L	lift
C	crosswind

Hence, the coordinate vector ${}^wF_A$ of $\bar{F}_A$ relative to the wind frame is given by

$$ {}^wF_A = \begin{pmatrix} -D \\ -C \\ -L \end{pmatrix} $$

If we let

$$ {}^bF_A = \begin{pmatrix} X_A \\ Y_A \\ Z_A \end{pmatrix} $$

be the coordinate vector of $\bar{F}_A$ relative to the aircraft frame, then, using ${}^bF_A = {}^bR^w\,{}^wF_A$, we obtain that

$$\begin{aligned} X_A &= -D\cos\alpha\cos\beta + C\cos\alpha\sin\beta + L\sin\alpha \\ Y_A &= -D\sin\beta - C\cos\beta \\ Z_A &= -D\sin\alpha\cos\beta + C\sin\alpha\sin\beta - L\sin\alpha \end{aligned}$$

Chapter 13

Controllability

13.1 Controllability

In this section, we introduce a concept of controllability which is fundamental in the state space approach to control systems. Consider a general linear time-invariant system described by

$$\dot{x} = Ax + Bu \tag{13.1}$$

where the scalar t is time, the n-vector $x(t)$ is the state, and $u(t)$ is an m-vector of control inputs; it is called the control input (vector). We say that this system is controllable if it can be 'driven' from any initial state to any other state by appropriate choice of the control input

Example 99 (Two unattached masses: an uncontrollable system.)

$$\begin{aligned} \dot{x}_1 &= u \\ \dot{x}_2 &= 0 \end{aligned}$$

Clearly uncontrollable.

Example 100 (Simple integrator: a controllable system.) Consider the scalar system described by

$$\dot{x} = u$$

Consider any initial state x^0 and any desired final state x^1. Take any $T > 0$ and consider the constant control:

$$u(t) = (x^1 - x^0)/T \qquad 0 \leq t \leq T$$

Then the state trajectory $x(\cdot)$ of this system with $x(0) = x^0$ satisfies $x(T) = x^1$. We consider this system to be controllable.

13.2 Main controllability result

The controllability matrix associated with the system $\dot{x} = Ax + Bu$ or with the matrix pair $(A,\,B)$ is defined to be the following $n \times nm$ matrix:

$$Q_c := \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix}$$

The main result of this section follows.

Theorem 2 (Main controllability result.) *The linear time-invariant system (13.1) is controllable if and only if its controllability matrix has maximum rank.*

Since the rank of a matrix is less than or equal to the number of rows or the number of columns in the matrix and Q_c has n rows and nm columns, its maximum rank is n, the number of state variables. So, the above controllability condition can be expressed as

$$\boxed{\operatorname{rank} Q_c = n}$$

that is,

$$\boxed{\operatorname{rank} \begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = n}$$

Since controllability only depends on the matrix pair (A, B), we say that this pair is controllable if system (13.1) is controllable.

For single input ($m = 1$) systems, Q_c is a square $n \times n$ matrix; hence it has maximum rank n if and only if it is invertible. This in turn is equivalent to Q_c having nonzero determinant. So, the above theorem has the following corollary.

Corollary 1 *A single input system of the form (13.1) is controllable if and only if the determinant of its controllability matrix is nonzero.*

Example 101 (Two unattached masses.) This is a scalar input system with

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Hence,

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Clearly Q_c has rank one which is less than $n = 2$. Hence this system is not controllable. Note also that Q_c has zero determinant;

Example 102 At first sight the system

$$\begin{aligned} \dot{x}_1 &= x_1 + u \\ \dot{x}_2 &= x_2 + u \end{aligned}$$

might look controllable. However, it is not.

Since

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

we have

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Since rank $(Q_c) = 1 < n$ or $\det(Q_c) = 0$, this system is not controllable.

Example 103 Consider

$$\begin{aligned} \dot{x}_1 &= x_1 + u \\ \dot{x}_2 &= 2x_2 + u \end{aligned}$$

Since

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

we have

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$$

Since $\det(Q_c) = 1 \neq 0$, this system is controllable.

Example 104 Consider

$$\begin{aligned} \dot{x}_1 &= x_2 + u \\ \dot{x}_2 &= x_1 - u \end{aligned}$$

Since

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

we have

$$Q_c = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Since rank $(Q_c) = 1 < n$, this system is not controllable.

Example 105 This example has two input components.

$$\begin{aligned} \dot{x}_1 &= x_3 + u_1 \\ \dot{x}_2 &= x_3 + u_1 \\ \dot{x}_3 &= u_2 \end{aligned}$$

Here $n = 3$ and

$$A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

So,

$$Q_c = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and

$$\text{rank } [Q_c] \quad = \quad \text{rank} \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = 2$$

Since $\text{rank}\, Q_c = 2$, we have $\text{rank}\, Q_c \neq n$; hence this system is not controllable.

MATLAB

```
>> help ctrb

 CTRB   Form controllability matrix.
        CTRB(A,B) returns the controllability matrix
        Co = [B AB A^2B ...]

>> A=[0 0 1; 0 0 1; 0 0 0];
>> B=[1 0; 1 0; 0 1];

>> ctrb(A,B)
ans =
     1     0     0     1     0     0
     1     0     0     1     0     0
     0     1     0     0     0     0
```

• **Useful fact.** Suppose that for some integer n^* we have

$$\text{rank}\left(\begin{bmatrix} B & AB & \dots & A^{n^*}B \end{bmatrix}\right) = \text{rank}\left(\begin{bmatrix} B & AB & \dots & A^{n^*-1}B \end{bmatrix}\right)$$

Then

$$\text{rank}\,(Q_c) = \text{rank}\left(\begin{bmatrix} B & AB & \dots & A^{n^*-1}B \end{bmatrix}\right)$$

See next example.

Example 106 (Beavis and Butthead: self-controlled)

$$
\begin{aligned}
m\ddot{q}_1 &= k(q_2-q_1)-u \\
m\ddot{q}_2 &= -k(q_2-q_1)+u
\end{aligned}
$$

With

$$
x_1 = q_1 \qquad x_2 = q_2 \qquad x_3 = \dot{q}_1 \qquad x_4 = \dot{q}_2
$$

we have

$$
A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m & k/m & 0 & 0 \\ k/m & -k/m & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ -1/m \\ 1/m \end{bmatrix}
$$

Hence,

$$
\begin{aligned}
\begin{bmatrix} B & AB \end{bmatrix} &= \begin{bmatrix} 0 & -1/m \\ 0 & 1/m \\ -1/m & 0 \\ 1/m & 0 \end{bmatrix} \\
\begin{bmatrix} B & AB & A^2B \end{bmatrix} &= \begin{bmatrix} 0 & -1/m & 0 \\ 0 & 1/m & 0 \\ -1/m & 0 & 2k/m^2 \\ 1/m & 0 & -2k/m^2 \end{bmatrix}
\end{aligned}
$$

Since

$$
\operatorname{rank}\left(\begin{bmatrix} B & AB & A^2B \end{bmatrix}\right) = \operatorname{rank}\left(\begin{bmatrix} B & AB \end{bmatrix}\right)
$$

we have

$$
\operatorname{rank}(Q_c) = \operatorname{rank}\left(\begin{bmatrix} B & AB \end{bmatrix}\right) = 2 < 4
$$

Hence, this system is not controllable.

MATLAB time.

```
>> a = [0     0     1     0
     0     0     0     1
    -1     1     0     0
     1    -1     0     0];
>> b = [0; 0; -1; 1];

>> rank(ctrb(a,b))
ans = 2
```

Example 107 (Beavis and Butthead with external control)

$$
\begin{aligned}
m\ddot{q}_1 &= k(q_2-q_1) \\
m\ddot{q}_2 &= -k(q_2-q_1)+u
\end{aligned}
$$

With

$$x_1 = q_1 \qquad x_2 = q_2 \qquad x_3 = \dot{q}_1 \qquad x_4 = \dot{q}_2$$

we have

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m & k/m & 0 & 0 \\ k/m & -k/m & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1/m \end{bmatrix}$$

$$\begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1/m \\ 0 & 0 \\ 1/m & 0 \end{bmatrix}$$

$$\begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/m & 0 \\ 0 & 0 & k/m^2 \\ 1/m & 0 & -k/m^2 \end{bmatrix}$$

$$Q_c = \begin{bmatrix} B & AB & A^2B & A^3B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & k/m^2 \\ 0 & 1/m & 0 & -k/m^2 \\ 0 & 0 & k/m^2 & 0 \\ 1/m & 0 & -k/m^2 & 0 \end{bmatrix}$$

Clearly, $\operatorname{rank} Q_c = 4 = n$; hence this system is controllable.

13.2.1 Invariance of controllability under state transformations*

Suppose one has a system described by

$$\dot{x} = Ax + Bu$$

and introduces a state transformation

$$x = T\xi$$

where T is an invertible matrix. Then the transformed system is decribed by

$$\dot{\xi} = \tilde{A}\xi + \tilde{B}u$$

with $\tilde{A} = T^{-1}AT$, $\tilde{B} = T^{-1}B$. The next result states that controllability is invariant under state transformations.

Fact 4 *Suppose A is $n \times n$, B is $n \times m$, T is $n \times n$ and nonsingular and let*

$$\tilde{A} = T^{-1}AT \qquad \tilde{B} = T^{-1}B$$

Then (A, B) is controllable if and only if $(\tilde{A}, \tilde{B})$ is controllable.

13.3 PBH test

The next result provides another useful characterization of controllability for LTI systems.

Lemma 1 (PBH controllability test) *A pair (A, B) is controllable if and only if*

$$\text{rank}\,[A - \lambda I \quad B] = n$$

for every complex number λ where n is the number of rows in A.

If λ is not an eigenvalue of A, then $A - \lambda I$ is invertible and, hence, the rank of $A - \lambda I$ is n. From this it follows that rank $[A - \lambda I \quad B] = n$. So, in applying the above PBH test, one only has the check the rank of $[A - \lambda I \quad B]$ when λ is an eigenvalue of A. This leads to the next definition.

DEFN. *A complex number λ is an* uncontrollable eigenvalue *of the pair (A, B) if*

$$\text{rank}\,[A - \lambda I \quad B] < n$$

where n is the number of rows in A.

We will see the significance of uncontrollable eigenvalues when we discuss the concept of stabilizability.

Example 108 Two unattached masses. In this example,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Hence,

$$[A - \lambda I \quad B] = \begin{bmatrix} -\lambda & 0 & 0 \\ 0 & -\lambda & 1 \end{bmatrix}$$

The above rank test fails for $\lambda = 0$. So, zero is an uncontrollable eigenvalue of this system.

Example 109 Consider

$$\begin{aligned} \dot{x}_1 &= x_2 - u \\ \dot{x}_2 &= x_1 + u \end{aligned}$$

Here,

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

and

$$[A - \lambda I \quad B] = \begin{bmatrix} -\lambda & 1 & 1 \\ 1 & -\lambda & -1 \end{bmatrix}$$

The above rank test fails for $\lambda = 1$

Example 110

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + u \end{aligned}$$

13.4 Exercises

Exercise 76 Determine (by hand) whether or not the following systems are controllable. Check your answers with MATLAB.
(a)

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= -x_1 + x_3 \\ \dot{x}_3 &= x_2 + u\end{aligned}$$

(b)

$$\begin{aligned}\dot{x}_1 &= x_1 \\ \dot{x}_2 &= -x_1 + x_3 \\ \dot{x}_3 &= x_2 + u\end{aligned}$$

Exercise 77 Determine whether or not the following systems are controllable.
(a)

$$\begin{aligned}\dot{x}_1 &= 2x_1 + u \\ \dot{x}_2 &= 2x_2 + u\end{aligned}$$

(b)

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= x_2 + x_3 \\ \dot{x}_3 &= u\end{aligned}$$

(c)

$$\begin{aligned}\dot{x}_1 &= x_1 + u_1 + u_2 \\ \dot{x}_2 &= x_2 + u_1 + u_2\end{aligned}$$

Exercise 78 (BB in Laundromat: self excited) Obtain a state space representation of the following system:

$$\begin{aligned}m\ddot{q}_1 - m\Omega^2 q_1 + \tfrac{k}{2}(q_1 - q_2) &= -u \\ m\ddot{q}_2 - m\Omega^2 q_2 - \tfrac{k}{2}(q_1 - q_2) &= u \\ y &= q_1\end{aligned}$$

Determine whether or not this state space representation is controllable.

Exercise 79 (BB in Laundromat: external excitation) Obtain a state space representation of the following system:

$$\begin{aligned}m\ddot{q}_1 - m\Omega^2 q_1 + \tfrac{k}{2}(q_1 - q_2) &= 0 \\ m\ddot{q}_2 - m\Omega^2 q_2 - \tfrac{k}{2}(q_1 - q_2) &= u \\ y &= q_1\end{aligned}$$

Determine whether or not this state space representation is controllable.

Chapter 14

Observability

14.1 Observability

Consider a system with input u, output y, and state x. Consider any time interval $[0, T]$ with $T > 0$. The basic observability problem is as follows. Suppose we have knowledge of the input $u(t)$ and the output $y(t)$ over the interval $[0, T]$; can we uniquely determine the initial state x_0? If we can do this for all input histories and initial states x_0 we say the system is observable. So, for observability, we require that for each input history, different initial states produce different output histories, or, equivalently, if two output histories are identical, then the corresponding initial states are the same. Note that knowledge of the initial state and the input history over the interval $[0, T]$ allows one to compute the state $x(t)$ for $0 \leq t \leq T$.

Example 111 Two unattached masses

$$
\begin{aligned}
\dot{x}_1 &= 0 \\
\dot{x}_2 &= 0 \\
y &= x_1
\end{aligned}
$$

Clearly, this system is unobservable.

Example 112

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u
\end{aligned}
$$

If $y = x_2$, the system is unobservable. If $y = x_1$, the system is observable.

Consider a general linear time-invariant system described by

$$
\begin{aligned}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{aligned}
\tag{14.1}
$$

with initial condition

$$x(0) = x_0$$

where the n-vector $x(t)$ is the state, the m-vector $u(t)$ is the input and the p-vector $y(t)$ is a vector of measured outputs; we call y the measured output (vector).

14.2 Main observability result

The observability matrix associated with $(C,\, A)$ is defined to be the following $pn \times n$ matrix

$$Q_o = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$$

The main result of this section follows.

Theorem 3 (Main observability theorem) *For each $T > 0$, system (14.1) is observable over the time interval $[0, T]$ if and only if*

$$rank \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} = n$$

where n is the number of rows of A.

An immediate consequence of the above theorem is that observability does not depend on the interval $[0, T]$. So, from now on we drop reference to the interval. Since observability only depends on the matrix pair $(C,\, A)$, we say that this pair is observable if system (14.1) is observable.

For single input $(m = 1)$ systems, Q_o is a square $n \times n$ matrix; hence it has rank n if and only if it has nonzero determinant. So, the above theorem has the following corollary.

Corollary 2 *A single input system of the form (14.1) is observable if and only if $\det Q_o$ is nonzero.*

Example 113 (Two unattached masses.) Recall example 111. Here $n = 2$,

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Hence,

$$Q_o \;=\; \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Since $\det Q_o = 0$, this system is unobservable.

Example 114 At first sight, the system

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_2 \\ y &= x_1 + x_2 \end{aligned}$$

might look observable. However, it is not. To see this, introduce

$$x_o := x_1 + x_2$$

Then

$$\begin{aligned} \dot{x}_o &= x_o \\ y &= x_o \end{aligned}$$

Hence, $x_o(0) = 0$ implies $x_o(t) = 0$ and, hence, $y(t) = 0$ for all t. In other words, if the initial state $x(0)$ satisfies

$$x_1(0) + x_2(0) = 0\,,$$

then, the resulting output history is zero.

Now lets look at the observability test. Here

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Hence,

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Since $\det(Q_o) = 0$, this system is not observable.

Example 115 (The unattached mass)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \end{aligned}$$

We consider two cases:

(a) (Velocity measurement.) Suppose

$$y = x_2$$

Then

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

So,

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Since $\det Q_0 = 0$, we have unobservability.

(b) (Position measurement.) Suppose

$$y = x_1$$

Then

$$Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Since $\det Q_o = 1 \neq 0$, we have observability.

Exercise 80 (Damped linear oscillator.) Recall

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(k/m)\,x_1 - (d/m)\,x_2 \end{aligned}$$

Show the following:

(a) (Position measurement.) If $y = x_1$, we have observability.

(b) (Velocity measurement.) If $y = x_2$, we have observability if and only if $k \neq 0$.

(c) (Acceleration measurement.) If $y = \dot{x}_2$, we have observability if and only if $k \neq 0$.

A useful fact for computing rank Q_o. In computing rank Q_o, the following fact is useful. Suppose there exists n^* such that

$$\text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n^*-1} \\ CA^{n^*} \end{bmatrix} = \text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n^*-1} \end{bmatrix}$$

Then

$$\text{rank}\,(Q_o) = \text{rank} \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n^*-1} \end{bmatrix}$$

See next example.

Example 116 (Beavis and Butthead: mass center measurement)

$$\begin{aligned} m\ddot{q}_1 &= k(q_2 - q_1) \\ m\ddot{q}_2 &= -k(q_2 - q_1) \\ y &= q_1 + q_2 \end{aligned}$$

With

$$x_1 = q_1 \qquad x_2 = q_2 \qquad x_3 = \dot{q}_1 \qquad x_4 = \dot{q}_2$$

we have

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m & k/m & 0 & 0 \\ k/m & -k/m & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 & 0 & 0 \end{bmatrix}$$

Hence,

$$\begin{aligned} CA &= \begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix} \\ CA^2 &= \begin{bmatrix} 0 & 0 & 0 & 0 \end{bmatrix} \end{aligned}$$

So, using useful fact (b) above,

$$\operatorname{rank}(Q_o) = \operatorname{rank}\begin{bmatrix} C \\ CA \end{bmatrix} = \operatorname{rank}\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} = 2$$

Since $\operatorname{rank}(Q_o) < n$, this system is not observable.

Example 117 (Beavis and Butthead: observable)

$$\begin{aligned} m\ddot{q}_1 &= k(q_2 - q_1) \\ m\ddot{q}_2 &= -k(q_2 - q_1) \\ y &= q_1 \end{aligned}$$

With

$$x_1 = q_1 \qquad x_2 = q_2 \qquad x_3 = \dot{q}_1 \qquad x_4 = \dot{q}_2$$

we have

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -k/m & k/m & 0 & 0 \\ k/m & -k/m & 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ CA^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -k/m & k/m & 0 & 0 \\ 0 & 0 & -k/m & k/m \end{bmatrix}$$

Since $\operatorname{rank} Q_o = 4 = n$, this system is observable.

14.3 Controllability, observability, and duality

Recalling that for any matrix M, $\operatorname{rank} M' = \operatorname{rank} M$, it follows from the main observability theorem that a pair $(C,\, A)$ is observable if and only if

$$\operatorname{rank} \begin{bmatrix} C' & A'C' & \cdots & (A')^{n-1}C' \end{bmatrix} = n$$

where A is $n \times n$. Recalling the main controllability theorem, we obtain the following result:

• $(C,\, A)$ is observable if and only if $(A',\, C')$ is controllable.

The above statement is equivalent to

• $(A,\, B)$ is controllable if and only if $(B',\, A')$ is observable.

This is an example of what is meant by the duality between controllability and observability.

14.4 PBH Test

Defn. Unobservable eigenvalue. A complex number λ is an unobservable eigenvalue of the pair (C, A) if

$$\operatorname{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} < n$$

where n is the number of rows in A.

• If λ is an unobservable eigenvalue then the system

$$\dot{x} = Ax\,, \qquad y = Cx$$

has a nonzero solution x which satisfies

$$\boxed{x(t) \equiv e^{\lambda t} v \qquad \text{and} \qquad y(t) \equiv 0}$$

We call this or $e^{\lambda t}$ an unobservable mode of the system.

Lemma 2 (PBH observability test) *The pair $(C,\, A)$ is observable if and only if , for every complex number λ,*

$$\mathit{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} = n$$

where n is the number of rows in A

Example 118

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_1 + u \\ y &= -x_1 + x_2 \end{aligned}$$

Exercises

Exercise 81 Determine (by hand) whether or not the following systems are observable. Check your answers with MATLAB.

(a)

$$\begin{aligned}\dot{x}_1 &= x_2\\ \dot{x}_2 &= x_3\\ \dot{x}_3 &= 0\\ y &= x_1\end{aligned}$$

(b)

$$\begin{aligned}\dot{x}_1 &= x_1\\ \dot{x}_2 &= x_2\\ \dot{x}_3 &= x_3\\ y &= x_1 + x_2\end{aligned}$$

(c)

$$\begin{aligned}\dot{x}_1 &= x_1 + u\\ \dot{x}_2 &= 2x_1 + x_2\\ \dot{x}_3 &= 3x_3\\ y &= x_1 + x_2\end{aligned}$$

(d)

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 + u\\ \dot{x}_2 &= x_2\\ \dot{x}_3 &= x_3\\ y &= x_1 + x_2\end{aligned}$$

(e)

$$\begin{aligned}\dot{x}_1 &= x_1 + x_2 + 3u\\ \dot{x}_2 &= x_2 + x_3\\ \dot{x}_3 &= x_3\\ y &= x_1\end{aligned}$$

Exercise 82 *BB in laundromat* Obtain a state space representation of the following system.

$$\begin{aligned}m\ddot{\phi}_1 - m\Omega^2\phi_1 + \tfrac{k}{2}(\phi_1 - \phi_2) &= 0\\ m\ddot{\phi}_2 - m\Omega^2\phi_2 - \tfrac{k}{2}(\phi_1 - \phi_2) &= u\\ y &= \phi_2\end{aligned}$$

Determine whether or not this state space representation is observable.

Exercise 83 *BB in laundromat* Obtain a state space representation of the following system.

$$
\begin{aligned}
m\ddot{\phi}_1 - m\Omega^2\phi_1 + \tfrac{k}{2}(\phi_1 - \phi_2) &= 0 \\
m\ddot{\phi}_2 - m\Omega^2\phi_2 - \tfrac{k}{2}(\phi_1 - \phi_2) &= u \\
y &= \phi_2 - \phi_1
\end{aligned}
$$

Determine whether or not this state space representation is observable.

Exercise 84 Compute the unobservable eigenvalues of the following systems.

(a)

$$
\begin{aligned}
\dot{x}_1 &= -x_1 + u \\
\dot{x}_2 &= x_1 + u \\
y &= x_1
\end{aligned}
$$

(b)

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= 4x_1 + u \\
y &= -2x_1 + x_2 + u
\end{aligned}
$$

Chapter 15

State feedback controllers

15.1 State feedback and stabilizability

Consider a system with control input u described by

$$\dot{x} = Ax + Bu \tag{15.1}$$

For now, we consider linear static state feedback controllers: at each instant t of time the current control input $u(t)$ depends linearly on the current state $x(t)$, that is, we consider the control input to be given by

$$\boxed{u(t) = Kx(t)} \tag{15.2}$$

where K is a constant $m \times n$ matrix, sometimes called a gain matrix. When system (15.1) is subject to such a controller, its behavior is governed by

$$\boxed{\dot{x} = (A + BK)x} \tag{15.3}$$

We call this the closed loop system resulting from (15.2).

Figure 15.1: State feedback

Suppose the open loop system, that is, the system with zero feedback, or,

$$\dot{x} = Ax$$

is unstable or at most marginally stable. A natural question is the following. Can we choose again matrix K so that $A + BK$ is stable? If yes, we say that the system $\dot{x} = Ax + Bu$ is stabilizable.

DEFN. *The system $\dot{x} = Ax + Bu$ is* stabilizable *if there exists a matrix K such that $A+BK$ is stable.*

We sometimes say that the pair (A, B) is stabilizable if system (15.1) is stabilizable.

The big question is: under what conditions is a given pair (A, B) stabilizable? We shall see shortly that controllability is a sufficient condition for stabilizability; that is, if a system is controllable it is stabilizable. However, controllability is not necessary for stabilizability, that is, it is possible for a system to be stabilizable but not controllable; see next two examples.

Example 119 *Not stabilizable and not controllable.*

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= u \end{aligned}$$

For any gain matrix K, the matrix

$$A + BK = \begin{bmatrix} 1 & 0 \\ k_1 & k_2 \end{bmatrix}$$

has eigenvalues 1 and k_2. Hence, $A + BK$ is unstable for all K.

Example 120 *Stabilizable but not controllable.*

$$\begin{aligned} \dot{x}_1 &= -x_1 \\ \dot{x}_2 &= u \end{aligned}$$

For any gain matrix K, the matrix

$$A + BK = \begin{bmatrix} -1 & 0 \\ k_1 & k_2 \end{bmatrix}$$

has eigenvalues -1 and k_2. Hence, $A + BK$ is asymptotically stable if $k_2 < 0$.

15.2 Controllable canonical form

Here we consider single input ($m = 1$) systems of the form

$$\dot{x} = Ax + Bu$$

15.2.1 Controllable canonical form

Suppose A and B have the following structure:

$$A = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-2} & -a_{n-1} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Then we say that the pair (A, B) is in controllable canonical form.

Note that $\dot{x} = Ax + Bu$ looks like:

$$\begin{array}{rcl}\dot{x}_1 &=& x_2 \\ &\vdots& \\ \dot{x}_{n-1} &=& x_n \\ \dot{x}_n &=& -a_0x_1 + a_1x_2 + \ldots + a_{n-1}x_n + u\end{array}$$

Also,

$$\det(sI - A) = a_0 + a_1s + \cdots + a_{n-1}s^{n-1} + s^n$$

Example 121 (*Controlled unattached mass*)

$$\begin{array}{rcl}\dot{x}_1 &=& x_2 \\ \dot{x}_2 &=& u\end{array}$$

Here,

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Fact 5 *If (A, B) is in controllable canonical form then, it is controllable.*

15.2.2 Transformation to controllable canonical form

Here we show that any controllable single-input system

$$\dot{x} = Ax + Bu$$

can be transformed via a state transformation to a system which is in controllable canonical form.

State transformations. Suppose T is a nonsingular matrix and consider the state transformation

$$x = T\tilde{x}$$

Then the evolution of $\tilde{x}$ is governed by

$$\dot{\tilde{x}} = \tilde{A}\tilde{x} + \tilde{B}u$$

where

$$\tilde{A} = T^{-1}AT \qquad \tilde{B} = T^{-1}B$$

Transformation to controllable canonical form. Suppose (A, B) is controllable and $m = 1$. Then the following algorithm yields a state transformation which transforms $(A,\ B)$ into controllable canonical form.

Algorithm. Let

$$a_0 + \ldots + a_{n-1}s^{n-1} + s^n = \det(sI - A)$$

Recursively define the following sequence $t^1, t^2, \cdots, t^n$ of vectors:

$$\boxed{\begin{array}{rcl} t^n &=& B \\ t^j &=& At^{j+1} + a_j B\,, \qquad j = n{-}1,\ \ldots,\ 1 \end{array}}$$

and let

$$T = \left[\ t^1 \quad \ldots \quad t^n\ \right] \qquad \blacksquare$$

Note that we start with t^n and work back to t^1.

Fact 6 *If (A, B) is controllable then, the matrix T is generated by the above algorithm is nonsingular and the pair $(T^{-1}AT,\ T^{-1}B)$ is in controllable canonical form.*

We can now state the following result.

Theorem 4 *A single input system (A, B) is controllable if and only if there is a nonsingular matrix T such that $(T^{-1}AT,\ T^{-1}B)$ is in controllable canonical form.*

Example 122 (Beavis and Butthead with external control.) Consider $m = k = 1$. Then,

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Hence,

$$\det(sI - A) = s^4 + 2s^2$$

and

$$\begin{aligned} t^4 &= B \\ t^3 &= At^4 + a_3 B = At^4 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \\ t^2 &= At^3 + a_2 B = At^3 + 2B = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} \\ t^1 &= At^2 + a_1 B = At^2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \end{aligned}$$

Hence,

$$T = \begin{bmatrix} t^1 & t^2 & t^3 & t^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

```
>> inv(t)
ans =
     1     0     0     0
     0     0     1     0
    -1     1     0     0
     0     0    -1     1

>> inv(t)*a*t
ans =
     0     1     0     0
     0     0     1     0
     0     0     0     1
     0     0    -2     0
```

15.3 Eigenvalue placement by state feedback

Recall the stabilizability problem for the system

$$\dot{x} = Ax + Bu$$

and suppose A and B are real matrices. Here we consider a more general question: Can we arbitrarily assign the eigenvalues of $A + BK$ by choice of the gain matrix K? The main result of this section is that the answer is yes if and only if (A, B) is controllable.

We demonstrate this result for single input systems. Consider any bunch of complex numbers $\lambda_1, \ldots, \lambda_n$ with the property that if λ is in the bunch then so is $\bar{\lambda}$ and suppose we wish to choose a gain matrix K such that the eigenvalues of $A + BK$ are precisely these complex numbers.

Let

$$\hat{p}(s) = s^n + \hat{a}_{n-1}s^{n-1} + \ldots + \hat{a}_0 = \prod_{i=1}^{n}(s - \lambda_i)$$

that is, the real numbers, $\hat{a}_0, \ldots, \hat{a}_{n-1}$, are the coefficients of the unique monic polynomial $\hat{p}$ whose roots are $\lambda_1, \ldots, \lambda_n$. If $(A,\ B)$ is controllable and $m = 1$, there exists a nonsingular matrix T such that the pair $(\tilde{A},\ \tilde{B})$, is in controllable canonical form, where

$$\tilde{A} = T^{-1}AT \qquad \tilde{B} = T^{-1}B$$

that is,

$$\tilde{A} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-2} & -a_{n-1} \end{bmatrix} \qquad \tilde{B} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

where

$$s^n + a_{n-1}s^{n-1} + \ldots + a_0 = \det(sI - A)$$

Consider the $1 \times n$ real matrix

$$\tilde{K} = \begin{bmatrix} a_0 - \hat{a}_0 & a_1 - \hat{a}_1 & \ldots & a_{n-1} - \hat{a}_{n-1} \end{bmatrix}$$

Then

$$\tilde{A} + \tilde{B}\tilde{K} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 & 0 \\ 0 & 0 & 1 & \ldots & 0 & 0 \\ \vdots & \vdots & & \ddots & & \vdots \\ \vdots & \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 1 \\ -\hat{a}_0 & -\hat{a}_1 & -\hat{a}_2 & \ldots & -\hat{a}_{n-2} & -\hat{a}_{n-1} \end{bmatrix}$$

hence, its characteristic polynomial is $\hat{p}$.

Let

$$K = \tilde{K}T^{-1}$$

Then

$$\begin{aligned} A + BK &= T\tilde{A}T^{-1} + T\tilde{B}\tilde{K}T^{-1} \\ &= T[\tilde{A} + \tilde{B}\tilde{K}]T^{-1} \end{aligned}$$

and hence

$$\det(sI - A - BK) = \det(sI - \tilde{A} - \tilde{B}\tilde{K}) = \hat{p}(s)$$

in other words, the eigenvalues of $A + BK$ are as desired.

The following theorem states that the above result also holds in the general multi-input case.

Theorem 5 (Pole placement theorem.) *Suppose the real matrix pair (A, B) is controllable and $\lambda_1, \ldots, \lambda_n$ is any bunch of n complex numbers with the property that if λ is in the bunch then so is $\bar{\lambda}$. Then there exists a real matrix K such that*

$$\det(sI - A - BK) = \prod_{i=1}^{n}(s - \lambda_i)$$

• It follows from the above theorem that

$$\boxed{\text{controllability} \quad \Longrightarrow \quad \text{stabilizability}}$$

MATLAB

```
>> help place

 PLACE  K = place(A,B,P)  computes the state feedback matrix K such that
        the eigenvalues of  A-B*K  are those specified in vector P.
        The complex eigenvalues in the vector P must appear in consecu-
        tive complex conjugate pairs. No eigenvalue may be placed with
        multiplicity greater than the number of inputs.

        The  displayed "ndigits" is an  estimate of how well the
        eigenvalues were placed.   The value seems to give an estimate
        of how many decimal digits in the eigenvalues of A-B*K match
        the specified numbers given in the array P.

        A warning message is printed if the nonzero closed loop poles
        are greater than 10% from the desired locations specified in P.

        See also: LQR and RLOCUS.
```

```
>> help acker

 ACKER  Pole placement gain selection using Ackermann's formula.
        K = ACKER(A,B,P)  calculates the feedback gain matrix K such that
        the single input system
                .
                x = Ax + Bu

        with a feedback law of  u = -Kx  has closed loop poles at the
        values specified in vector P, i.e.,  P = eig(A-B*K).

        See also PLACE.

        Note: This algorithm uses Ackermann's formula.  This method
        is NOT numerically reliable and starts to break down rapidly
        for problems of order greater than 10, or for weakly controllable
        systems.  A warning message is printed if the nonzero closed loop
        poles are greater than 10% from the desired locations specified
        in P.
```

Example 123 Consider

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 \\ \dot{x}_2 &= u \end{aligned}$$

Here

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Since A has eigenvalues 1 and 0, the open loop system is unstable. Letting

$$K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$$

we have

$$A + BK = \begin{bmatrix} 1 & 1 \\ k_1 & k_2 \end{bmatrix}$$

Hence, the characteristic polynomial of the closed loop system is given by

$$\det(sI - A - BK) = \begin{vmatrix} s-1 & -1 \\ -k_1 & s-k_2 \end{vmatrix} = s^2 - (1+k_2)s - k_1 + k_2\,.$$

Suppose we desire the closed loop eigenvalues to be $\lambda_1 = -2 + \jmath$ and $\lambda_2 = -2 - \jmath$. Then the characteristic polynomial of the closed loop system is

$$p(s) = (s-\lambda_1)(s-\lambda_2) = (s+2-\jmath)(s+2+\jmath) = s^2 + 4s + 5\,.$$

Equating corresponding coefficients of the two expressions for the characteristic polynomial we obtain that

$$\begin{aligned} -k_1 + k_2 &= 5 \\ -1 - k_2 &= 4 \end{aligned}$$

Solving for k_1 and k_2 yields

$$k_2 = -5 \qquad \text{and} \qquad k_1 = -10\,.$$

Example 124 (Beavis and Butthead with external control.) Here

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix} \qquad\qquad B = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

We have already shown that this system is controllable. Let

$$\hat{p}(s) = s^4 + \hat{a}_3 s^3 + \hat{a}_2 s^2 + \hat{a}_1 s + \hat{a}_0$$

be the desired characteristic polynomial of $A + BK$ and let

$$K = \begin{bmatrix} k_1 & k_2 & k_3 & k_4 \end{bmatrix}$$

Then

$$\begin{aligned} \det(sI - A - BK) &= \det \begin{bmatrix} s & 0 & -1 & 0 \\ 0 & s & 0 & -1 \\ 1 & -1 & s & 0 \\ -1-k_1 & 1-k_2 & -k_3 & s-k_4 \end{bmatrix} \\ &= s^4 - k_4 s^3 + (2-k_2)s^2 - (k_3+k_4)s - (k_1+k_2) \end{aligned}$$

Letting this equal $\hat{p}(s)$ results in

$$\begin{aligned} -k_4 &= \hat{a}_3 \\ 2 - k_2 &= \hat{a}_2 \\ -k_3 - k_4 &= \hat{a}_1 \\ -k_1 - k_2 &= \hat{a}_0 \end{aligned}$$

Solving for $k_1, \ldots k_4$ yields

$$\begin{aligned} k_1 &= -2 - \hat{a}_0 + \hat{a}_2 \\ k_2 &= 2 - \hat{a}_2 \\ k_3 &= -\hat{a}_1 + \hat{a}_3 \\ k_4 &= -\hat{a}_3 \end{aligned}$$

• To illustrate MATLAB, consider desired closed loop eigenvalues: $-1, -2-3, -4$.

```
>> poles=[-1 -2 -3 -4];

>> poly(poles)
ans =
     1    10    35    50    24
```

Hence

$$\hat{a}_0 = 24 \qquad \hat{a}_1 = 50 \qquad \hat{a}_2 = 35 \qquad \hat{a}_3 = 10$$

and the above expressions for the gain matrix yield

$$k_1 = 9 \qquad k_2 = -33 \qquad k_3 = -40 \qquad k_4 = -10$$

Using MATLAB pole placement commands, we obtain:

```
>> k= place(a,b,poles)
place: ndigits= 18

k =
   -9.0000   33.0000   40.0000   10.0000

>> k= acker(a,b,poles)

k =
    -9    33    40    10
```

Remembering that MATLAB yields $-K$, these results agree with our 'hand' calculated results. Lets check to see if we get the desired closed loop eigenvalues.

```
>> eig(a-b*k)

ans =
   -1.0000
   -2.0000
   -3.0000
   -4.0000
```

YEES!

15.4 Uncontrollable eigenvalues (you can't touch them)*

In Examples 119 and 120, the matrix A had an eigenvalue λ with the property that for every gain matrix K, it is an eigenvalue of $A+BK$. We now demonstrate the following result.

If λ is a complex number with the property that

$$\text{rank} \begin{bmatrix} A-\lambda I & B \end{bmatrix} < n \tag{15.4}$$

then, for every gain matrix K, λ is an eigenvalue of $A+BK$.

First note that

$$\text{rank} \begin{bmatrix} (A-\lambda I)^* \\ B^* \end{bmatrix} < n$$

hence, there exists a nonzero n-vector v such that

$$\begin{aligned} (A-\lambda I)^* v &= 0 \\ B^* v &= 0 \end{aligned}$$

So, for any gain matrix K, we have

$$(A+BK)^* v = \bar{\lambda} v$$

that is, $\bar{\lambda}$ is an eigenvalue of $(A+BK)^*$; hence λ is an eigenvalue of $A+BK$. So, regardless of the feedback gain matrix, λ is always an eigenvalue of $A+BK$. We cannot alter this eigenvalue by feedback. So, if $\Re(\lambda) \geq 0$, the pair (A,B) is not stabilizable. Actually, the following result can be demonstrated.

Theorem 6 (PBH stabilizability theorem) *System (15.1) is stabilizable if and only if for every eigenvalue λ of A with nonnegative real part,*

$$\text{rank}\, [A-\lambda I \quad B] = n$$

where n is the number of rows of A.

Example 125 Consider the system described by

$$\begin{aligned} \dot{x}_1 &= -11x_1 + 8x_2 + 2u \\ \dot{x}_2 &= 8x_1 + x_2 - u\,. \end{aligned}$$

Here

$$A = \begin{bmatrix} -11 & 8 \\ 8 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Since

$$AB = \begin{bmatrix} -30 \\ 15 \end{bmatrix} = -15B\,,$$

it follows that the controllability matrix $\begin{bmatrix} B & AB \end{bmatrix}$ for this system only has rank one; hence this system is not controllable.

Since $AB = -15B$, it follows that B is an eigenvector of A corresponding to the eigenvalue -15. Since the eigenvalues of A are -15 and 5, we expect that the eigenvalue 5 is not controllable. Applying the PBH condition with $\lambda = 5$ we have

$$\begin{bmatrix} A - \lambda I & B \end{bmatrix} = \begin{bmatrix} -16 & 8 & 2 \\ 8 & -4 & -1 \end{bmatrix}.$$

Since the rank of the above matrix is only one, it follows that the eigenvalue 5 is not controllable. Since this uncontrollable eigenvalue has positive real part, the system under consideration is not stabilizable.

Let us directly verify that the system is not stabilizable. Consider any state feedback controller of the form

$$u = k_1 x_1 + k_2 x_2 .$$

This is of the form $u = Kx$ where $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$. The system matrix $A + BK$ for the resulting closed loop system is given by

$$A + BK = \begin{bmatrix} -11 + 2k_1 & 8 + 2k_2 \\ 8 - k_1 & 1 - k_2 \end{bmatrix}.$$

The characteristic polynomial for this matrix is

$$\begin{aligned} \det(sI - A - BK) &= s^2 + (10 - 2k_1 + k_2)s - 75 + 10k_1 - 5k_2 \\ &= (s - 5)(s + 15 - 2k_1 + k_2) . \end{aligned}$$

As expected, five is an eigenvalue of $A + BK$, regardless of K. Hence this system is not stabilizable.

15.5 Stabilization of nonlinear systems

Consider a nonlinear system described by

$$\dot{x} = F(x, u) \tag{15.5}$$

where the n-vector $x(t)$ is the system state at time t and the m-vector $u(t)$ is the control input. Let x^e be a controlled equilibrium state for this system, that is, there is a constant input u^e such that

$$F(x^e, u^e) = 0 .$$

Suppose that we wish to construct a state feedback controller which asymptotically stabilizes the above system about x^e. This can be achieved as follows.

Let

$$\delta\dot{x} = A\delta x + B\delta u$$

be the linearization of the above nonlinear system about (x^e, u^e). Thus,

$$A = \frac{\partial F}{\partial x}(x^e, u^e) \qquad \text{and} \qquad B = \frac{\partial F}{\partial u}(x^e, u^e) .$$

If the pair (A, B) is stabilizable, choose any gain matrix K with the property that all the eigenvalues of $A + BK$ have negative real parts. Then a state feedback controller which asymptotically stabilizes the original nonlinear system about x^e is given by

$$u = u^e + K(x - x^e)\,. \tag{15.6}$$

If the pair (A, B) is not stabilizable, the nonlinear system may not be stabilizable about x^e or one may have to use nonlinear control design techniques.

Example 126 Stabilization of inverted pendulum

$$I\ddot{\theta} + Wl\sin\theta = u$$

15.6 Exercises

Exercise 85 For the system

$$\begin{aligned}
\dot{x}_1 &= x_1 + x_2 \\
\dot{x}_2 &= x_3 \\
\dot{x}_3 &= x_1 + x_3 + u
\end{aligned}$$

obtain a linear state feedback controller which results in a closed loop system with eigenvalues

$$-1, -2, -3$$

Use the following methods.

(a) Obtaining an expression for the closed loop characteristic polynomial in terms of the components of the gain matrix.

(b) `acker`

(c) `place`

Numerically simulate the open loop system (u=0) and the closed loop system with the same initial conditions.

Exercise 86 Consider the system

$$\begin{aligned}
\dot{x}_1 &= x_1 + u \\
\dot{x}_2 &= 2x_2 + u \\
\dot{x}_3 &= 3x_3 + u
\end{aligned}$$

(a) Is the open loop system asymptotically stable?

(b) Without using MATLAB, obtain a linear state feedback controller which results in an asymptotically stable closed loop system.

(c) Using MATLAB, obtain a linear state feedback controller which results in an asymptotically stable closed loop system.

Exercise 87 Is the system

$$\begin{aligned} \dot{x}_1 &= x_1 \\ \dot{x}_2 &= x_3 \\ \dot{x}_3 &= x_1 + u \end{aligned}$$

stabilizable? Justify your answer.

Exercise 88 Consider the single link robotic manipulator with flexible joint described by

$$\begin{aligned} I\ddot{\theta} - Wl\sin\theta + k(\theta - \phi) &= 0 \\ J\ddot{\phi} - k(\theta - \phi) &= u\,. \end{aligned}$$

with parameter values

$$I = 10\,, \qquad Wl = 5\,, \qquad k = 2\,, \qquad J = 1\,.$$

Obtain a state feedback controller which stabilizes this system about the upward vertical configuration ($\theta = 0$).

Exercise 89 For the system,

$$\begin{aligned} \dot{x}_1 &= 2x_2 + u \\ \dot{x}_2 &= 2x_1 + u \end{aligned}$$

obtain a linear state feedback controller which results in a stable closed loop system.

Chapter 16

State estimators

16.1 Observers and detectability

Consider the system (plant)

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{16.1}$$

Suppose that at each instant time t we can measure the plant output $y(t)$ (a p-vector) and the plant input $u(t)$ (an m-vector) and we wish to estimate the plant state $x(t)$ (an n-vector). In this section we demonstrate how to obtain an estimate $\hat{x}(t)$ of the plant state $x(t)$ with the property that as $t \to \infty$ the state estimation error $\hat{x}(t) - x(t)$ goes to zero. We do this by constructing an observer or state estimator.

Observer or state estimator

$$\boxed{\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + L(\hat{y} - y) \\ \hat{y} &= C\hat{x} + Du \end{aligned}} \qquad \hat{x}(0) = \hat{x}_0 \tag{16.2}$$

where the n-vector $\hat{x}(t)$ is the estimated state; the $n \times p$ matrix L is called the observer gain matrix and is yet to be determined. The initial state $\hat{x}_0$ for the observer is arbitrary. Note that the above observer is simply a copy of the plant along with a correction term $L(\hat{y} - y)$ which accounts for the difference between the plant output y and the estimate $\hat{y}$ of the plant output based on the state estimate $\hat{x}$.

If we rewrite the observer description as

$$\dot{\hat{x}} = (A + LC)\hat{x} + (B + LD)u - Ly$$

it should be clear that we can regard the observer as a linear system whose inputs are the plant input u and plant output y.

To study the properties of an observer, we introduce the state estimation error

$$\tilde{x} := \hat{x} - x \tag{16.3}$$

Figure 16.1: Observer

Recall that we want $\hat{x}$ to go to zero. Using the description of the plant (16.1) and the observer (16.2), we see that

$$\begin{aligned}
\dot{\tilde{x}} &= \dot{\hat{x}} - \dot{x} \\
&= A\dot{\hat{x}} + Bu + L(C\hat{x} + Du - Cx - Du) - Ax - Bu \\
&= A(\hat{x} - x) + LC(\hat{x} - x) \\
&= (A + LC)\tilde{x}\,.
\end{aligned}$$

Thus the behavior of the state estimation error is governed by the linear system:

$$\boxed{\dot{\tilde{x}} = (A + LC)\tilde{x}} \tag{16.4}$$

If one can choose the observer gain matrix L such that the matrix $A + LC$ is stable, then

$$\boxed{\lim_{t \to \infty} \tilde{x}(t) = 0}$$

In other words the estimation error goes to zero as t goes to infinity. When this happens, we call the observer an asymptotic observer. This leads to the next definition.

DEFN.[Detectability] *The pair* $(C,\, A)$ *is* detectable *if there exists a matrix* L *such that* $A{+}LC$ *is asymptotically stable.*

It should be clear that we have the following duality between stabilizability and detectectability.

- $(C,\, A)$ is detectable if and only if $(A',\, C')$ is stabilizable.
- $(A,\, B)$ is stabilizable. if and only if $(B',\, A')$ is detectable.
- Also, it follows that

$$\boxed{\text{observability} \quad \Longrightarrow \quad \text{detectability}}$$

16.2 Eigenvalue placement for estimation error dynamics

Suppose $(C,\, A)$ is observable and real. Then the pair $(A',\, C')$ is controllable and real. Hence given any bunch of complex numbers

$$\lambda_1, \lambda_2, \cdots, \lambda_n$$

with the property that if λ is in the bunch then so is $\bar{\lambda}$, there is a real gain matrix K so that the eigenvalues of $A' + C'K$ are precisely $\lambda_1, \lambda_2, \cdots, \lambda_n$. Letting $L = K'$, we have that the adjoint of $A' + C'K$ is $A + LC$. Since the eiegenvalues of a real matrix and its adjoint are the same, it follows that the the eigenvalues of $A + LC$ are precisely $\lambda_1, \lambda_2, \cdots, \lambda_n$.

Hence, if (C, A) is observable, one can abitrarily place the eigenvalues of $A + LC$ by appropriate choice of the observer gain matrix L.

Example 127 (The unattached mass with position measurement and an input.)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned}$$

Here

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

Since this system is observable, it is detectable. If

$$L = \begin{bmatrix} l_1 \\ l_2 \end{bmatrix}$$

we have

$$A + LC = \begin{bmatrix} l_1 & 1 \\ l_2 & 0 \end{bmatrix}$$

Hence,

$$\det(sI - A - LC) = s^2 - l_1 s - l_2$$

and $A + LC$ is asymptotically stable if

$$l_1 < 0 \qquad l_2 < 0$$

An asymptotic observer is then given by

$$\begin{aligned} \dot{\hat{x}}_1 &= \hat{x}_2 + l_1(\hat{x}_1 - y) \\ \dot{\hat{x}}_2 &= u + l_2(\hat{x}_1 - y) \end{aligned}$$

and the estimation error $\tilde{x} = \hat{x} - x$ satisfies

$$\begin{aligned} \dot{\tilde{x}}_1 &= l_1\tilde{x}_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= l_2\tilde{x}_1 \end{aligned}$$

16.3 Unobservable eigenvalues

Suppose (C, A) is an unobservable pair and λ is a complex number for which the PBH observability test fails, that is,

$$\text{rank} \begin{bmatrix} A - \lambda I \\ C \end{bmatrix} < n$$

where n is the number of columns of A. Then λ an unobservable eigenvalue of $(C,\ A)$. We now show that if λ an unobservable eigenvalue of $(C,\ A)$, then, it is an eigenvalue of $A+LC$ for every observer gain matrix L. We demonstrate this as follows: when the above λ dependent matrix has rank less than n, there is a nonzero n-vector v such that

$$\left[\begin{array}{c} A-\lambda I \\ C \end{array}\right] v = 0$$

or, equivalently,

$$\begin{aligned} Av &= \lambda v \\ Cv &= 0 \end{aligned}$$

Hence, for any matrix L

$$(A+LC)v = \lambda v$$

Since $v \neq 0$, it follows that λ is an eigenvalue of $A+LC$.

The next result states that a necessary and sufficient condition for detectability is that all unobservable eigenvalues are asymptotically stable.

Theorem 7 (PBH detectability theorem) *A pair (C, A) is detectable if and only if for every eigenvalue λ of A with nonegative real part,*

$$\mathit{rank} \left[\begin{array}{c} A-\lambda I \\ C \end{array}\right] = n$$

where n is the number of columns in A.

Example 128 (The unattached mass with velocity measurement.)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= 0 \\ y &= x_2 \end{aligned}$$

Here

$$A = \left[\begin{array}{cc} 0 & 1 \\ 0 & 0 \end{array}\right] \qquad C = \left[\begin{array}{cc} 0 & 1 \end{array}\right]$$

Hence

$$\left[\begin{array}{c} A-\lambda I \\ C \end{array}\right] = \left[\begin{array}{cc} -\lambda & 1 \\ 0 & -\lambda \\ 0 & 1 \end{array}\right]$$

The above matrix has rank $1 < n$ for $\lambda = 0$; hence $\lambda = 0$ is an unobservable eigenvalue and this system in not detectable.

Note that for any observer gain matrix

$$L = \left[\begin{array}{c} l_1 \\ l_2 \end{array}\right]$$

we have

$$A + LC = \begin{bmatrix} 0 & 1 + l_1 \\ 0 & l_2 \end{bmatrix}$$

Hence,

$$\det(sI - A - LC) = s(s - l_2)$$

So, regardless of choice of L, the matrix $A + LC$ always has an eigenvalue at 0.

Chapter 17

Output feedback controllers

Plant (The object of your control desires.)

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx \end{aligned} \tag{17.1}$$

The n-vector $x(t)$ is the state, the m-vector $u(t)$ is the control input, and the p-vector $y(t)$ is the measured output.

17.1 Memoryless (static) output feedback

A memoryless (or static) linear output feedback controller is of the form

$$\boxed{u = Ky}$$

where K is a real $m \times p$ matrix, sometimes called a gain matrix. In scalar form, this looks like

$$\begin{aligned} u_1 &= k_{11}y_1 + \ldots + k_{1p}y_p \\ &\vdots \\ u_m &= k_{m1}y_1 + \ldots + k_{mp}y_p \end{aligned}$$

This controller results in the following closed loop system:

$$\dot{x} = (A + BKC)x$$

If the open loop system $\dot{x} = Ax$ is not asymptotically stable, a natural question is whether one can choose K so that the closed loop system is asymptotically stable. For full state feedback ($C = I$), we have seen that it is possible to do this if $(A,\, B)$ is controllable or, less restrictively, if (A, B) is stabilizable. If, in addition, $(C,\, A)$ is observable or detectable, we might expect to be able to stabilize the plant with static output feedback. This is not the case as the following example illustrates.

Example 129 (Unattached mass with position measurement)

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= u \\
y &= x_1
\end{aligned}
$$

This system is both controllable and observable. All linear static output feedback controllers are given by

$$u = ky$$

which results in

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= kx_1
\end{aligned}
$$

Such a system is never asymptotically stable.

Note that if the plant has some damping in it, i.e.,

$$
\begin{array}{rcrcl}
\dot{x}_1 &=& x_2 && \\
\dot{x}_2 &=& -dx_2 &+& u \\
y &=& x_1 &&
\end{array}
$$

where $d > 0$, the closed loop system is given by

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= kx_1 - dx_2
\end{aligned}
$$

This is asymptotically stable provided $k < 0$.

For a general system, there are currently are no easilily verifiable conditions which are both necessary and sufficient for stabilizability via static output feedback. It is a topic of current research. So where do we go now?

17.2 Dynamic output feedback

In general, a linear dynamic output feedback controller is described by a linear time-invariant system of some order n_c:

$$
\boxed{
\begin{aligned}
\dot{x}_c &= A_c x_c + B_c y \\
u &= C_c x_c + D_c y
\end{aligned}
}
\tag{17.2}
$$

where the n_c-vector $x_c(t)$ is called the controller state and n_c is the order of the controller. This controller is a linear time-invariant system whose input is the plant measured output

Figure 17.1: Closed loop system

and whose output is the plant control input. We will regard a memoryless controller as a controller of order zero.

Using Laplace, the above controller can be described by

$$\hat{u}(s) = \hat{G}_c(s)\hat{y}(s)$$

where

$$\hat{G}_c(s) = C_c(sI - A_c)^{-1}B_c + D_c$$

The closed loop system that results when a controller of the form (17.2) is applied to plant (17.1) is a linear time invariant system of order $n + n_c$ and is described by

$$\begin{array}{rcllll} \dot{x} & = & (A + BD_cC)x & + & BC_cx_c \\ \dot{x}_c & = & B_cCx & + & A_cx_c \end{array} \tag{17.3}$$

This is an LTI system with state $\begin{bmatrix} x \\ x_c \end{bmatrix}$ and "A-matrix"

$$\mathcal{A} = \begin{bmatrix} A + BD_cC & BC_c \\ B_cC & A_c \end{bmatrix}$$

So the order of this system is $n + n_c$.

Example 130 Consider a SISO system and recall that a PI (proportional integral) controller is described by

$$u(t) = -k_py(t) - k_i\int_{t_0}^{t} y(\tau)\,dt$$

Letting

$$x_c(t) = \int_{t_0}^{t} y(\tau)\,dt$$

it can readily be seen that this controller is a first order dynamic system described by

$$\begin{array}{rcl} \dot{x}_c & = & y \\ u & = & -k_ix_c - k_py \end{array}$$

We now ask the following question: under what conditions does there exist a dynamic output feedback controller which results in a stable closed loop system? Before answering this we have some preliminary facts.

Fact 7

(a) If a complex number λ is an unobservable eigenvalue of (C, A) then λ is an eigenvalue of $\mathcal{A}$.

(b) If a complex number λ is an uncontrollable eigenvalue of (A, B) then λ is an eigenvalue of $\mathcal{A}$.

We can now state the following intuitively appealing result.

Lemma 3 *If either (A, B) is not stabilizable or (C, A) is not detectable, then plant (17.1) is not stabilizable by a linear dynamic output feedback controller of any order.*

The last lemma states that stabilizability of (A, B) and and detectability of (C, A) is necessary for stabilizability via dynamic output feedback control. In the next section, we demonstrate that if these conditions are satisfied then closed loop asymptotic stability can be achieved with a controller of order no more than the plant order.

17.3 Observer based controllers

An observer based controller has the following structure:

$$\boxed{\begin{array}{rcl} \dot{\hat{x}} & = & A\hat{x} + Bu + L(C\hat{x} - y) \\ u & = & K\hat{x} \end{array}} \tag{17.4}$$

Note that this controller is completely specified by specifying the gain matrices K and L; also this controller can be written as

$$\begin{array}{rcl} \dot{\hat{x}} & = & (A + BK + LC)\hat{x} - Ly \\ u & = & K\hat{x} \end{array}$$

This is a dynamic output feedback controller with $x_c = \hat{x}$ (hence $n_c = n$, that is, the controller has same order as the plant) and $A_c = A + BK + LC$, $B_c = -L$, $C_c = K$, and $D_c = 0$.

Closed loop system. Combining the plant description (17.1) with the controller description (17.4), the closed loop system can be described by

$$\begin{array}{rcrcl} \dot{x} & = & Ax & + & BK\hat{x} \\ \dot{\hat{x}} & = & -LCx & + & (A + BK + LC)\hat{x} \end{array}$$

Let $\tilde{x}$ be the estimation error, i.e., $\tilde{x} = \hat{x} - x$. Then the closed loop system is described by

$$\begin{array}{rcl} \dot{x} & = & (A + BK)x + BK\tilde{x} \\ \dot{\tilde{x}} & = & (A + LC)\tilde{x} \end{array} \tag{17.5}$$

This is an LTI system with "A matrix"

$$\mathcal{A} = \begin{bmatrix} A + BK & BK \\ 0 & A + LC \end{bmatrix}$$

Noting that

$$\det(sI - \mathcal{A}) = \det(sI - A - BK)\det(sI - A - LC)$$

it follows that the set of eigenvalues of the closed loop system are simply the union of those of $A+BK$ and those of $A+LC$. So, if both $A+BK$ and $A+LC$ are asymptotically stable, the closed loop system is asymptotically stable. If $(A,\, B)$ is stabilizable, one can choose K so that $A+BK$ is asymptotically stable. If $(C,\, A)$ is detectable, one can choose L so that $A+LC$ is asymptotically stable. Combining these observations with lemma 3 leads to the following result.

Theorem 8 *The following statements are equivalent.*

(a) (A, B) *is stabilizable and* (C, A) *is detectable.*

(b) Plant (17.1) is stabilizable via a linear dynamic output feedback controller.

(c) Plant (17.1) is stabilizable via a linear dynamic output feedback controller whose order is less than or equal to that of the plant.

Example 131 (The unattached mass with position measurement)

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= u \\ y &= x_1 \end{aligned}$$

Observer based controllers are given by

$$\begin{array}{rcllll} \dot{\hat{x}}_1 &=& \hat{x}_2 & + & l_1(\hat{x}_1 - y) \\ \dot{\hat{x}}_2 &=& u & & l_2(\hat{x}_1 - y) \\ u &=& k_1\hat{x}_1 & + & k_2\hat{x}_2 \end{array}$$

which is the same as

$$\begin{array}{rcllllll} \dot{\hat{x}}_1 &=& l_1\hat{x}_1 & + & \hat{x}_2 & - & l_1 y \\ \dot{\hat{x}}_2 &=& (k_1 + l_2)\hat{x}_1 & + & k_2\hat{x}_2 & - & l_2 y \end{array}$$

The closed loop system is described by

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= k_1 x_1 + k_2 x_2 + k_1\tilde{x}_1 + k_2\tilde{x}_2 \\ \dot{\tilde{x}}_1 &= l_1\tilde{x}_1 + \tilde{x}_2 \\ \dot{\tilde{x}}_2 &= l_2\tilde{x}_1 \end{aligned}$$

where $\tilde{x}_i = \hat{x}_i - x_i$, $i = 1, 2$ are the estimation error state variables. The characteristic polynomial of the closed loop system is

$$p(s) = (s^2 - k_2 s - k_1)(s^2 - l_1 s - l_2)$$

Hence, if

$$k_1, k_2, l_1, l_2 < 0$$

the closed loop system is asymptotically stable.