Homework Assignment 1 – 601.455/655 Fall 2025

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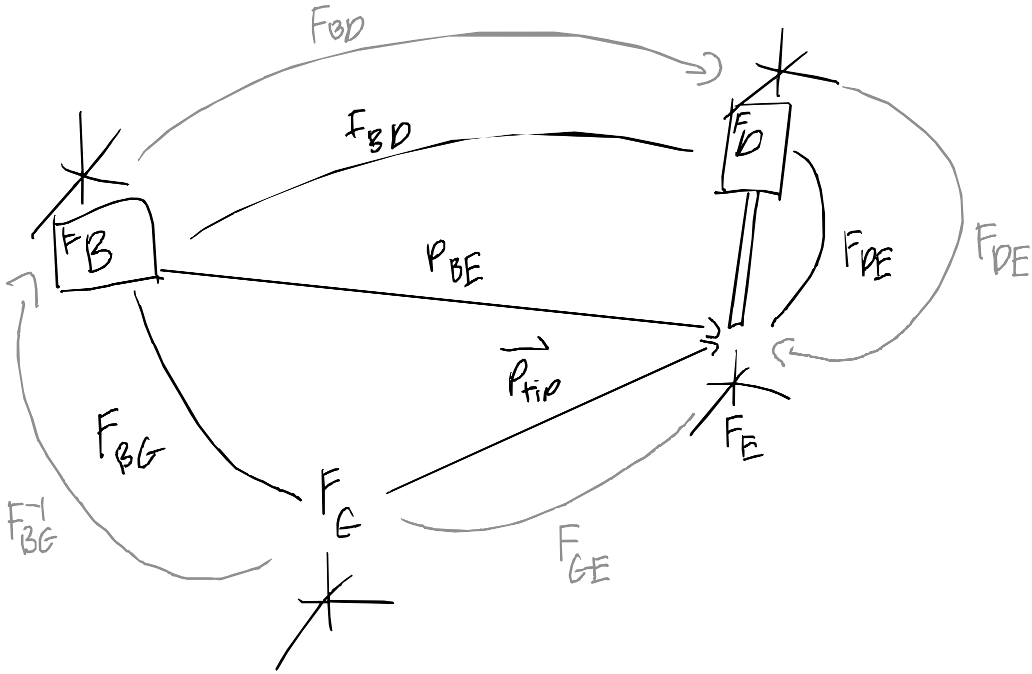
Signature (required) I/We have followed the rules in completing this assignment Alan You, 22SEP2025	Signature (required) I/We have followed the rules in completing this assignment Justin Wang, 23SEP2025

1. Remember that this is a graded homework assignment. It is the functional equivalent of a take-home exam.
2. You are to work **alone** or in **teams of two** and are not to discuss the problems with anyone other than the TAs or the instructor.
3. **IMPORTANT NOTE:** If you work in teams of two, you are **not** to split up the questions and each answer a subset individually. You are to work **together**. I encourage teaming on these problems because I believe that it encourages learning, not as a way to reduce the required work for students taking the course. By signing this sheet you are asserting that each of you has contributed equally to each answer and can individually explain the answer as well as if you had answered the question alone. I view this as a question of trust and ethics.
4. It is otherwise open book, notes, and web. But you should cite any references you consult.
5. Please refer to the course organizational notes for a fuller listing of all the rules. I am not reciting them all here, but they are still in effect.
6. Unless I say otherwise in class, it is due before the start of class on the due date posted on the web.
7. Submit the assignment on GradeScope as a neat and legible PDF file. We will not insist on typesetting your answers, but we must be able to read them. We will not go to extraordinary lengths to decipher what you write. If the graders cannot make out an answer, the score will be 0.
8. Sign and hand in this page as the first sheet of your assignment. If you work with a partner, then you both should sign the sheet, but you should only submit one PDF file for both of you, using the GradeScope teaming feature. Indicate clearly who it is from.
9. This assignment has more than 100 points, but the most that will be applied to your grade is 100.

1A. Pointer Pose

$\vec{p}_{tip} = \vec{p}_{GE}$: tip of pointer relative to pelvis basis



$$[R_{GE}, \vec{p}_{tip}] = F_{BG}^{-1} F_{BD} F_{DE}$$

$$F_{BG}^{-1} = [R_{BG}^{-1}, -R_{BG}^{-1} \cdot \vec{p}_{BG}]$$

$$F_{BG}^{-1} F_{BD} = [R_{BG}^{-1} \cdot R_{BD}, \; R_{BG}^{-1} \vec{p}_{BD} + (-R_{BG}^{-1} \cdot \vec{p}_{BG})]$$

$$F_{BG}^{-1} F_{BD} F_{DE} =$$

$$[(R_{BG}^{-1} \cdot R_{BD}) \cdot R_{DE}, \; (R_{BG}^{-1} \cdot R_{BD}) \vec{p}_{DE} + R_{BG}^{-1} \vec{p}_{BD} + (-R_{BG}^{-1} \cdot \vec{p}_{BG})]$$

$$\underline{\vec{p}_{tip} = R_{BG}^{-1} (R_{BD} \vec{p}_{DE} + \vec{p}_{BD} - \vec{p}_{BG})}$$

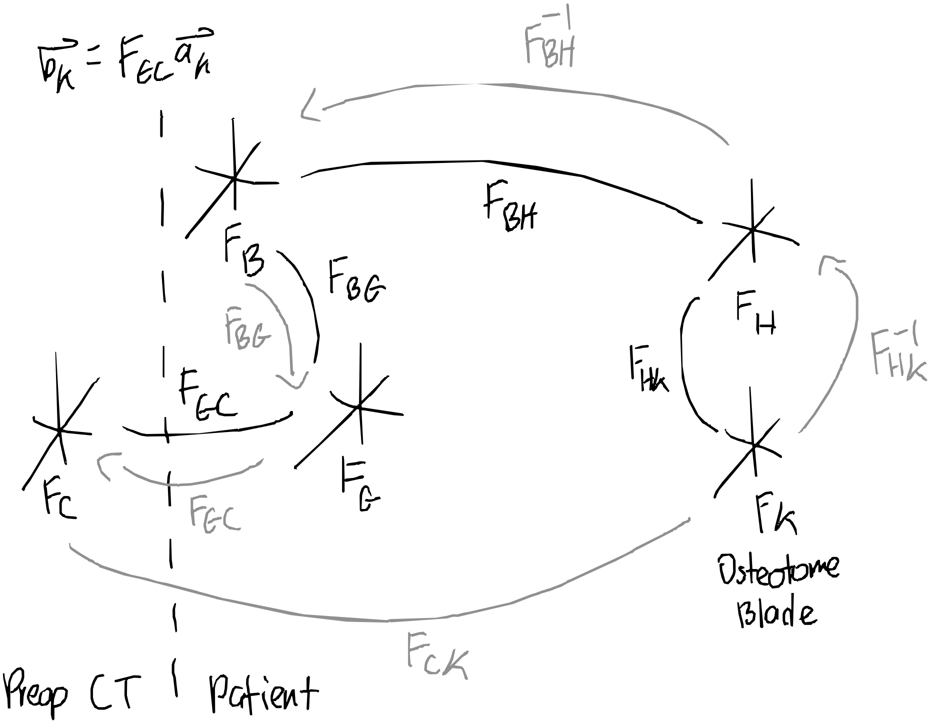
1B. Osteotome Registration

F_{GC}: Frame transformation from patient to CT

$\vec{a}_k$: fiducial relative to CT

$\vec{b}_k$: fiducial relative to patient

$\vec{b}_k = F_{GC}\vec{a}_k$



$$F_{CK} = F_{HK}^{-1} F_{BH}^{-1} F_{BG} F_{GC}$$

$$F_{HK}^{-1} = [R_{HK}^{-1}, -R_{HK}^{-1}\vec{p}_{HK}]$$

$$F_{HK}^{-1}F_{BH}^{-1} = [R_{HK}^{-1}R_{BH}^{-1}, -R_{HK}^{-1}R_{BH}^{-1}\vec{p}_{BH} - R_{HK}^{-1}\vec{p}_{HK}]$$

$$F_{HK}^{-1}F_{BH}^{-1}F_{BG} = [R_{HK}^{-1}R_{BH}^{-1}R_{BG}, R_{HK}^{-1}R_{BH}^{-1}\vec{p}_{BG} - R_{HK}^{-1}R_{BH}^{-1}\vec{p}_{BH} - R_{HK}^{-1}\vec{p}_{HK}]$$

$$F_{HK}^{-1}F_{BH}^{-1}F_{BG}F_{GC} =$$

$$\underline{[R_{HK}^{-1}R_{BH}^{-1}R_{BG}R_{GC}, R_{HK}^{-1}(R_{BH}^{-1}R_{BG}\vec{p}_{GC} + R_{BH}^{-1}\vec{p}_{BG} - R_{BH}^{-1}\vec{p}_{BH} - \vec{p}_{HK})]}$$

1C. Pointer Tip Tracking Error

$$F_{GE} = F_{GD} F_{DE} = [R_{GE}, \vec{p}_{tip}] \longrightarrow F^*_{GE} = [R^*_{GE}, \vec{p}^*_{tip}]$$

$$F_{GD} = F_{BG}^{-1} F_{BD}$$

$$F^*_{GD} = F^{*-1}_{GB} F^*_{BD} = (\cancel{\Delta F_B} F_{BG} \Delta F_{BG})^{-1} (\cancel{\Delta F_B} F_{BD} \Delta F_{BD})$$

$$= \Delta F_{BG}^{-1} F_{GD} \Delta F_{BD}$$

$$= [\Delta R_{BG}^{-1} R_{GD} \Delta R_{BD}, \Delta R_{BG}^{-1}(R_{GD} \vec{p}_{BD} + \vec{p}_{GD} - \Delta \vec{p}_{BG})]$$

$$F^*_{GE} = F^*_{GD} F_{DE}$$

$$= [\Delta R_{BG}^{-1} R_{GD} \Delta R_{BD} R_{DE}, \Delta R_{BG}^{-1}(R_{GD} \Delta R_{BD} \vec{p}_{DE} + R_{GD} \Delta \vec{p}_{BD} + \vec{p}_{GD} - \Delta \vec{p}_{BG})]$$

$$\vec{p}^*_{tip} = \Delta R_{BG}^{-1}(R_{GD} \Delta R_{BD} \vec{p}_{DE} + R_{GD} \Delta \vec{p}_{BD} + \vec{p}_{GD} - \Delta \vec{p}_{BG})$$

1D. Pointer Tip Tracking Error Linear Approximation

$$\vec{p}^{\,*}_{tip} = \Delta R_{BG}^{-1}\left(R_{GD}\Delta R_{BD}\vec{p}_{DE} + R_{GD}\Delta \vec{p}_{BD} + \vec{p}_{GD} - \Delta \vec{p}_{BG}\right)$$

Small Angle Linear Approx:

$$\vec{p}^{\,*}_{tip} \approx \Delta R_{BG}^{-1}\left(R_{GD}\Delta R_{BD}\vec{p}_{DE} + R_{GD}\varepsilon_{BD} + \vec{p}_{GD} - \varepsilon_{BG}\right)$$

$$= (I + sk\,\alpha_{BG})\left[R_{GD}(I + sk\,\alpha_{BD})\vec{p}_{DE} + R_{GD}\varepsilon_{BD} + \vec{p}_{GD} - \varepsilon_{BG}\right]$$

$$= R_{GD}(I + sk\,\alpha_{BD})\vec{p}_{DE} + R_{GD}\varepsilon_{BD} + \vec{p}_{GD} - \varepsilon_{BG}$$

$$+\, sk(\alpha_{BG})R_{GD}\vec{p}_{DE} + \cancelto{0}{sk(\alpha_{BG})\,sk(\alpha_{BD})}\,R_{GD}\vec{p}_{DE}$$

$$+\, \cancelto{0}{sk(\alpha_{BG})R_{GD}}\,\varepsilon_{BD} + sk(\alpha_{BG})\vec{p}_{GD} - \cancelto{0}{sk(\alpha_{BG})}\,\varepsilon_{BG}$$

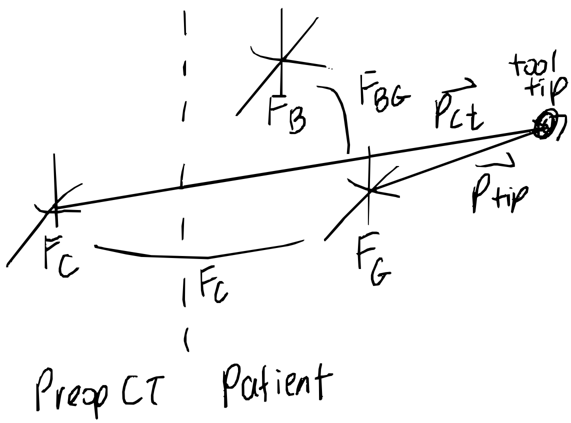
$$= R_{GD}\vec{p}_{DE} + R_{GD}\,sk(\alpha_{BD})\vec{p}_{DE} + R_{GD}\varepsilon_{BD} + \vec{p}_{GD} - \varepsilon_{BG} + sk(\alpha_{BG})R_{GD}\vec{p}_{DE}$$
$$+\, sk(\alpha_{BG})\vec{p}_{GD}$$

$$= R_{GD}\left[\vec{p}_{DE} - sk(\vec{p}_{DE})\alpha_{BD} + \varepsilon_{BD}\right] + \vec{p}_{GD} + \left[sk(R_{GD}\vec{p}_{DE}) + sk(\vec{p}_{GD})\right]\alpha_{BG}$$

1E. Registration Error

$$F_{GC}^{*} = F_{GC}\Delta F_{GC} = [R_{GC}\Delta R_{GC},\ \vec{p}_{GC} + R_{GC}\Delta \vec{p}_{GC}]$$

$\vec{p}_{ct}^{\,*}$: $\vec{p}_{tip}$ in G mapped to F_C



$$\vec{p}_{ct}^{\,*} = F_{GC}^{*}\vec{p}_{tip}^{\,*} = R_{GC}^{*}\vec{p}_{tip}^{\,*} + \vec{p}_{GC}^{\,*}$$

$$= \underline{R_{GC}\Delta R_{GC}(\vec{p}_{tip} + \Delta\vec{p}_{tip}) + \vec{p}_{GC} + R_{GC}\Delta\vec{p}_{GC}}$$

1F. Registration Error Linear Approximation

$$\Delta\vec{p}_{ct} = \vec{p}^{*}_{ct} - \vec{p}_{ct}$$

$$= R_{GC}\Delta R_{GC}(\vec{p}_{tip} + \Delta\vec{p}_{tip}) + \cancel{\vec{p}_{GC}} + R_{GC}\Delta\vec{p}_{GC} - R_{GC}\vec{p}_{tip} - \cancel{\vec{p}_{GC}}$$

$$= R_{GC}\left[\Delta R_{GC}(\vec{p}_{tip} + \Delta\vec{p}_{tip}) + \Delta\vec{p}_{GC} - \vec{p}_{tip}\right]$$

$$\approx R_{GC}\left[(I + sk(\alpha_{GC}))(\vec{p}_{tip} + \varepsilon_{tip}) + \varepsilon_{GC} - \vec{p}_{tip}\right]$$

$$= R_{GC}\left[\cancel{\vec{p}_{tip}} + \Delta\vec{p}_{tip} + sk(\alpha_{GC})\vec{p}_{tip} + \cancel{sk(\alpha_{GC})\vec{\varepsilon}_{tip}}^{\,0} + \varepsilon_{GC} - \cancel{\vec{p}_{tip}}\right]$$

$$= R_{GC}\left[sk(\alpha_{GC})\vec{p}_{tip} + \Delta\vec{p}_{tip} + \varepsilon_{GC}\right]$$

$$= \underline{R_{GC}(\Delta\vec{p}_{tip} + \varepsilon_{GC} - sk(\vec{p}_{tip})\alpha_{GC})}$$

1G. Fiducial Pin Errors w/ Constraint

$$\vec{b}_k^* = F_{EC}^* \vec{a}_k$$

$$\Rightarrow \vec{b}_k + \Delta\vec{b}_k = F_{EC}\,\Delta F_{EC}\,\vec{a}_k$$

$$\approx F_{EC}\left(\left[I + sk(\alpha_{EC})\right]\vec{a}_k + \varepsilon_{EC}\right)$$

$$= R_{EC}\left(\vec{a}_k + sk(\alpha_{EC})\vec{a}_k + \varepsilon_{EC}\right) + \vec{p}_{EC}$$

$$= \underbrace{R_{EC}\vec{a}_k + \vec{p}_{EC}}_{\vec{b}_k} + R_{EC}\,sk(\alpha_{EC})\vec{a}_k + R_{EC}\varepsilon_{EC}$$

$$\Rightarrow \underline{\Delta\vec{b}_k = R_{EC}\,sk(\alpha_{EC})\vec{a}_k + R_{EC}\varepsilon_{EC}}$$

$$f(\Delta\vec{b}_k) \le \sigma_k$$

$$\Rightarrow f\left(R_{EC}\,sk(\alpha_{EC})\vec{a}_k + R_{EC}\varepsilon_{EC}\right) \le \sigma_k \quad \forall k$$

$$\Rightarrow \underline{f\left(R_{EC}\left(-sk(\vec{a}_k)\alpha_{EC} + \varepsilon_{EC}\right)\right) \le \sigma_k \quad \forall k}$$

1H. Euclidian Norm

$R_{EC} \in SO(3) \rightarrow R_{EC}^T R_{EC} = I, \; \det(R_{EC}) = +1, \; \|R_{EC}\|_2 = 1$

Norm properties: ☆ source: RDKDC MLS Textbook [2]

1. Positive definitness: $\|f\| \geq 0, \; \forall f$

$\therefore \; 0 \leq f(A\vec{b_k}) \leq \sigma_k$

$\Rightarrow \; 0 \leq \|\cancel{R_{EC}}^{\,1}(-sk(\vec{a_k})\alpha_{GC} + \varepsilon_{GC})\| \leq \sigma_k$

$\Rightarrow \; \underline{0 \leq \|-sk(\vec{a_k})\alpha_{GC} + \varepsilon_{GC}\| \leq \sigma_k}$

II. Matrix Simplification

$$\vec{\eta}_{EC} = \begin{bmatrix} \alpha_{EC} \\ \epsilon_{EC} \end{bmatrix}$$

$$f(\Delta \vec{p}_k) = \left\| \Delta \vec{b}_k \right\| = \left\| -sk(\vec{a}_k)\alpha_{EC} + \epsilon_{EC} \right\| = \left\| \begin{bmatrix} I & -sk(a_k) \end{bmatrix} \vec{\eta}_{EC} \right\|$$

$$= \left[\left(\begin{bmatrix} I & -sk(\vec{a}_k) \end{bmatrix} \vec{\eta}_{EC} \right)^T \left(\begin{bmatrix} I & -sk(\vec{a}_k) \end{bmatrix} \vec{\eta}_{EC} \right) \right]^{\frac{1}{2}} \leq \sigma_k$$

$$\Rightarrow \left(\begin{bmatrix} I & -sk(\vec{a}_k) \end{bmatrix} \vec{\eta}_{EC} \right)^T \left(\begin{bmatrix} I & -sk(\vec{a}_k) \end{bmatrix} \vec{\eta}_{EC} \right) \leq \sigma_k^2$$

$$\Rightarrow \vec{\eta}_{EC}^{\,T} \underbrace{\begin{bmatrix} I & -sk(\vec{a}_k) \end{bmatrix}^T \begin{bmatrix} I & -sk(a_k) \end{bmatrix}}_{\begin{bmatrix} sk(\vec{a}_k)^T sk(a_k)_{3\times3} & -sk(a_k)^T_{3\times3} \\ sk(a_k)_{3\times3} & I_3 \end{bmatrix}_{6\times6}} \vec{\eta}_{EC} \leq \sigma_k^2$$

$$\boxed{\begin{aligned} A_k &= sk(\vec{a}_k)^T sk(a_k) \\ B_k &= -sk(a_k)^T \\ C_k &= sk(a_k) \\ D_k &= I_3 \end{aligned}}$$

1J. Segmentation Error

$$\vec{b}_k^* = F_{EC}^* \vec{a}_k^*$$

$$\Rightarrow \vec{b}_k + \Delta\vec{b}_k = F_{EC} \Delta F_{EC}(\vec{a}_k + \Delta\vec{a}_k)$$

$$\approx F_{EC}\Big(\big(I + sk(\alpha_{EC})\big)(\vec{a}_k + \Delta\vec{a}_k) + \vec{\varepsilon}_{EC}\Big)$$

$$= R_{EC}\big(\vec{a}_k + \Delta\vec{a}_k + sk(\alpha_{EC})\vec{a}_k + \underbrace{sk(\cancel{\alpha_{EC}})\Delta\vec{a}_k}_{0}$$

$$+ \vec{\varepsilon}_{EC}\big) + \vec{p}_{EC}$$

$$= \underbrace{R_{EC}\vec{a}_k + \vec{p}_{EC}}_{\vec{b}_k} + R_{EC}(\Delta\vec{a}_k + sk(\alpha_{EC})\vec{a}_k + \varepsilon_{EC})$$

$$\Rightarrow \Delta\vec{b}_k = R_{EC}(\Delta\vec{a}_k - sk(\vec{a}_k)\alpha_{EC} + \varepsilon_{EC})$$

$$\boxed{\begin{cases} f(R_{EC}(\Delta\vec{a}_k - sk(\vec{a}_k)\alpha_{EC} + \varepsilon_{EC}) \le \sigma_k \\ 0 \le f(\Delta\vec{a}_k) \le \xi_k \end{cases}}$$

1K. Gaussian Error Propagation

$$\vec{\Delta p_{ct}} = R_{GC}\left(\vec{\Delta p_{tip}} + \varepsilon_{GC} - sk(\vec{p_{tip}})\,\alpha_{GC}\right)$$

in terms of η_{GC} and $\vec{\Delta p_{tip}} = \varepsilon_{tip}$

$$\vec{\Delta p_{ct}} = \begin{bmatrix} -R_{GC}\,sk(\vec{p_{tip}}) & R_{GC} \end{bmatrix} \overbrace{\begin{bmatrix} \alpha_{GC} \\ \varepsilon_{GC} \end{bmatrix}}^{\eta_{GC}} + R_{GC}\,\varepsilon_{tip}$$

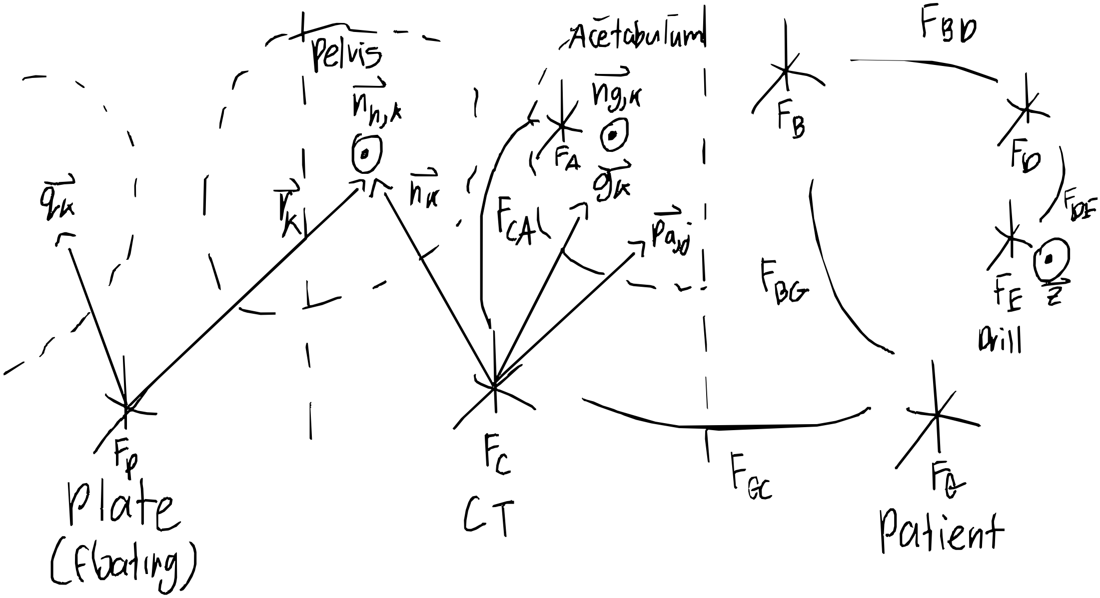
We know $\eta_{GC} \sim N(0, C_{GC})$ and $\varepsilon_{tip} \sim N(0, C_{tip})$

AND η_{GC}, ε_{tip} are indep

$$\boxed{\begin{aligned} C_{ct} = \mathrm{Cov}(\vec{\Delta p_{ct}}) &= A C_{GC} A^T + B C_{tip} B^T \\ \text{w/ } A &= \begin{bmatrix} -R_{GC}\,sk(\vec{p_{tip}}) & R_{GC} \end{bmatrix} \\ B &= R_{GC} \end{aligned}}$$

☆ Internet Prob Stats Textbook

2A. Plate Hole Alignment

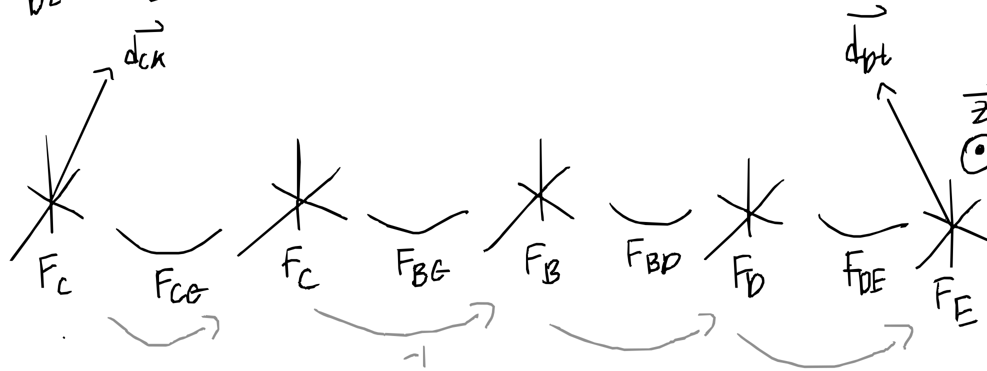


$\vec{r_k} = \vec{n_k}$

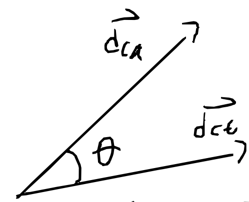
$\vec{q_k} = F_{CA}\,\vec{g_k}$

2B. Drill Axis Alignment

$\vec{d}_{DE} = R_{DE}\vec{z}$



$$\vec{d}_{CT} = R_{GC} R_{BG}^{-1} R_{BD} \underbrace{R_{DE}\vec{z}}_{\vec{d}_{DE}}$$



$$\vec{d}_{CA} \cdot \vec{d}_{CT} = \|\vec{d}_{CA}\| \, \|\vec{d}_{CT}\| \cos\theta$$

$$\Rightarrow \theta = \arccos(\vec{d}_{CA} \, \vec{d}_{CT})$$

both $\|\vec{d}_{CA}\| = \|\vec{d}_{CT}\| = 1$

$$= \underline{\arccos(\vec{d}_{CA} R_{GC} R_{BG}^{-1} R_{BD} R_{DE} \vec{z})}$$

2C. Registration Error

$$\Delta F_{CA} = (F_{GC} F_{CA})^{-1} F^{*}_{GC} F_{CA}$$

$$= F_{CA}^{-1} \cancel{F_{GC}^{-1} F_{GC}} \Delta F_{GC} F_{CA}$$

$$= F_{CA}^{-1} \Delta F_{GC} F_{CA}$$

$$\approx [R_{CA}^{-1}, -R_{CA}^{-1}\vec{p_{CA}}] \cdot [I + sk(\alpha_{GC}), \varepsilon_{GC}] \cdot [R_{CA}, \vec{p_{CA}}]$$

$$= [\Delta R_{CA}, \Delta\vec{p_{CA}}] \approx [I + sk(\alpha_{CA}), \varepsilon_{CA}]$$

Rotation:

$$I + sk(\alpha_{CA}) = R_{CA}^{-1}(I + sk(\alpha_{GC}))R_{CA}$$

$$= I + R_{CA}^{-1} sk(\alpha_{GC}) R_{CA}$$

☆ Triple Rotation Product from Sides

$$= I + sk(R_{CA}^{-1}\alpha_{GC})$$

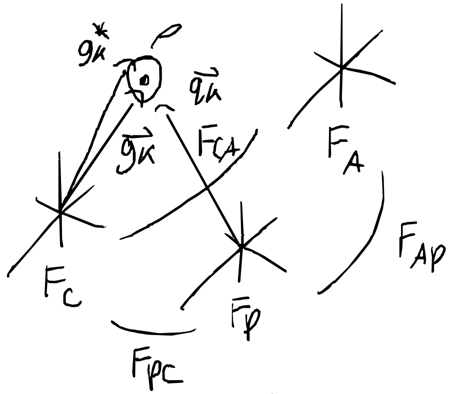
$$\Rightarrow \underline{\alpha_{CA} = R_{CA}^{-1}\alpha_{GC}}$$

Translation:

$$\varepsilon_{CA} = -R_{CA}^{-1}\vec{p_{CA}} + R_{CA}^{-1}\varepsilon_{GC} + R_{CA}^{-1}(I + sk(\alpha_{GC}))\vec{p_{CA}}$$

$$= \underline{R_{CA}^{-1}\varepsilon_{GC} - R_{CA}^{-1} sk(\vec{p_{CA}})\alpha_{GC}}$$

2D. Hole Tolerancing



For any hole has ρ tolerance

$$\begin{cases} \|\vec{g_k^*} - (F_{AP}\vec{q_k})\| \le \rho \\ \|\vec{h_k^*} - (F_{PC}\vec{r_k})\| \le \rho \end{cases}$$

$\|\vec{g_k^*} - \vec{g_k}\| \le \rho$

$\vec{g_k^*} = F_{AP}^* \vec{q_k}$

$\vec{h_k^*} = F_{PC}^* \vec{r_k}$

Perfect Registration:

$F_{CA}^* = F_{PC}^* F_{AP}^{*-1}$

$F_{AP}^* = F_{AP} \Delta F_{AP}$

$F_{PC}^* = F_{PC} \Delta F_{PC}$

$\vec{h_k} = F_{PC} \Delta F_{PC} F_{AP}^{-1} \Delta F_{AP}^{-1} \vec{g_k}$

Works Cited

[1] Taylor, R. H. (2025). *Frames* [PowerPoint slides]. Johns Hopkins University. https://ciis.lcsr.jhu.edu/lib/exe/fetch.php?media=courses:455-655:lectures:frames.pdf

[2] Bretscher, O. (2019). Linear algebra with applications (5th ed.). Pearson.

[3] Ross, S. M. (2020). Probability and statistics for engineers and scientists (6th ed.). Academic Press.

[4] Murray, R. M., Li, Z., & Sastry, S. S. (1994). A mathematical introduction to robotic manipulation. CRC Press.