2018 Fall Data Compression Homework #1

EE 248583
Advisor: 電子所 高立人 副教授

106368002 張昌祺 Justin, Chang-Qi Zhang
justin840727@gmail.com
Due Date: November 12 2018

Problem 1 Entropy

Let X be a random variable with an alphabet $H = \{1, 2, 3, 4, 5\}$. Please determine $H(X)$ for the following three cases of probability mass function $p(i) = prob[X = i]$. (15%)

(a) $P(1) = P(2) = 1/2$:

Ans

$$\begin{aligned} H(X) &= -(P(1)\log_2 P(1) + P(2)\log_2 P(2)) \\ &= -(0.5\log_2(0.5) + 0.5\log_2(0.5)) \\ &= -(-0.5 - 0.5) \\ &= 1\ bits/symbol \end{aligned}$$

(b) $P(i) = 1/4, for\ i = 1, 2, 3, and\ p(4) = p(5) = 1/8$:

Ans

$$\begin{aligned} H(X) &= -(3 \times P(1)\log_2 P(1) + P(4)\log_2 P(4) + P(5)\log_2 P(5)) \\ &= -(3 \times 0.25\log_2(0.25) + 2 \times 0.125\log_2(0.125)) \\ &= -(-1.5 - 0.75) \\ &= 2.25\ bits/symbol \end{aligned}$$

(c) $P(i) = 2^{-i}, for\ i = 1, 2, 3, 4, and\ p(5) = 1/16$:

Ans

$$\begin{aligned} H(X) &= -(\sum_{i=1}^{4} 2^{-i}\log_2 2^{-i} + \frac{1}{16}\log_2\frac{1}{16}) \\ &= -(0.5 \times (-1) + 0.25 \times (-2) + 0.125 \times (-3) + 0.0625 \times (-4) + 0.0625 \times (-4)) \\ &= 1.875\ bits/symbol \end{aligned}$$

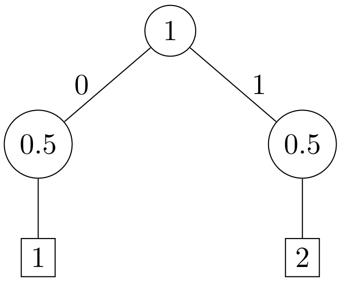
Problem 2 Huffman Code

Design a Huffman code C for the source in Problem 1. (15%)

(a) Specify your codewords for individual pmf model in Problem 1.

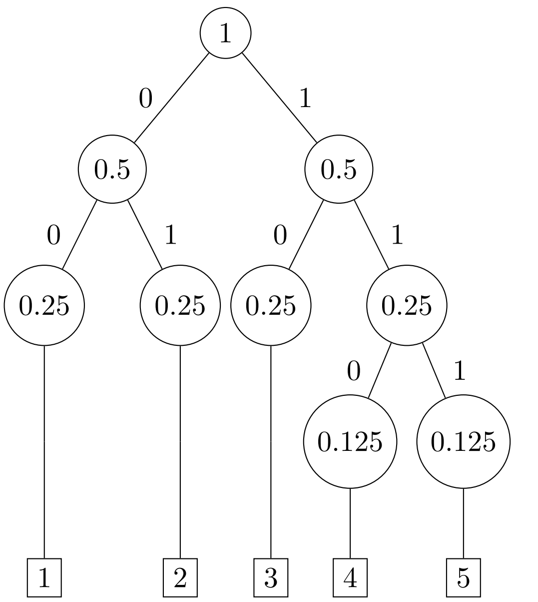
Ans

1.(a)



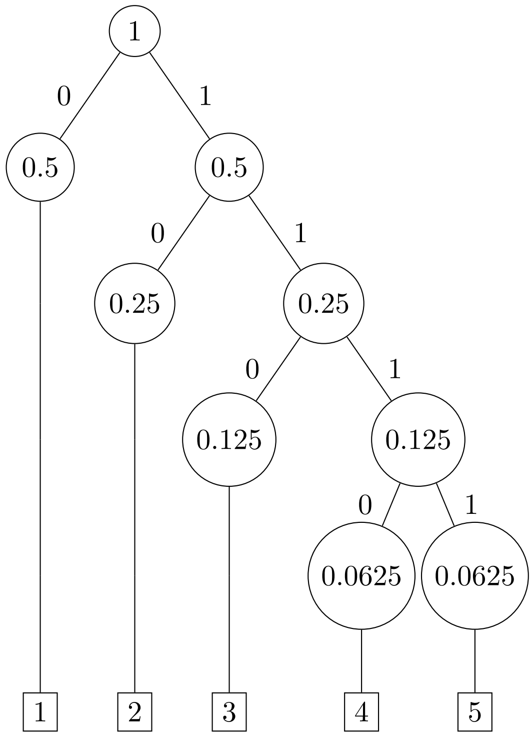
Alphabet	P	Codeword
1	0.5	0
2	0.5	1

1.(b)



Alphabet	P	Codeword
1	0.25	00
2	0.25	01
3	0.25	10
4	0.125	110
5	0.125	111

1.(c)



Alphabet	P	Codeword
1	0.5	0
2	0.25	10
3	0.125	110
4	0.0625	1110
5	0.0625	1111

(b) Compute the expected codeword length and compare with the entropy for your codes in (a).

Ans

1.(b)

$$\begin{aligned} Expected\ codeword\ length &= 0.5 \times 1 + 0.5 \times 1 \\ &= 1\ bits/symbol\ \textbf{(Equal Entropy)} \end{aligned}$$

1.(b)

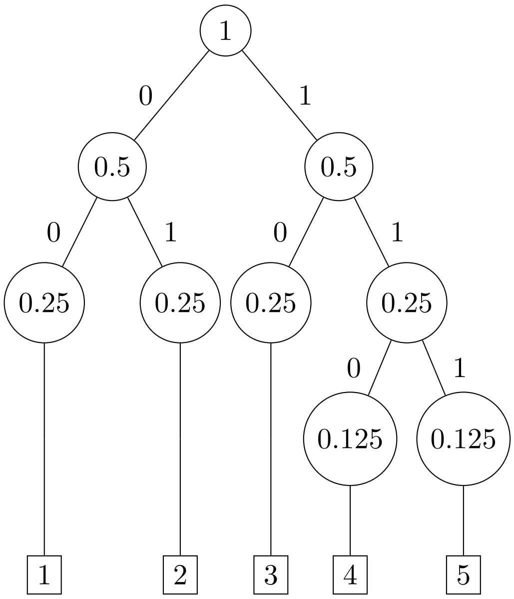
$$\begin{aligned} Expected\ codeword\ length &= 0.25 \times 2 + 0.25 \times 2 + 0.25 \times 2 + 0.125 \times 3 + 0.125 \times 3 \\ &= 2.25\ bits/symbol\ \textbf{(Equal Entropy)} \end{aligned}$$

1.(c)

$$\begin{aligned} Expected\ codeword\ length &= 0.5 \times 1 + 0.25 \times 2 + 0.125 \times 3 + 0.0626 \times 4 + 0.0625 \times 4 \\ &= 4.125\ bits/symbol\ \textbf{(NOT Equal Entropy)} \end{aligned}$$

(c) Design a code with minimum codeword length variance for the pmf model in Problem 1.(b)

Ans



Alphabet	P	Codeword
1	0.25	00
2	0.25	01
3	0.25	10
4	0.125	110
5	0.125	111

Problem 3 Empirical Distribution C++

Empirical distribution. In the case a probability model is not known, it can be estimated from empirical data. Let's say the alphabet is $H = \{1, 2, 3, \ldots, m\}$. Given a set of observations of length N , the empirical distribution is given by $p = total\ number\ of\ symbol\ 1/N,\ for\ i = 1, 2, 3, ..., m$. Please determine the empirical distribution for **santaclaus.txt**, which is an ASCII file with only lower-cased English letters (i.e., $a \sim z$), space and CR (carriage return), totally 28 symbols. The file can be found on the class web site. Compute the entropy. (14%)

Ans

The source code for this problem are available at `https://github.com/justin-changqi/2018_fall_data_compression.git`. Please check README.md to know how to execute the code. After I executed the program the entropy is 4.12 bits/symbol. Empirical distribution shows in Figure 2

```
> ./huffman_code
      Alphaba      SP       o       t       e       n       a       s       i       u      CR       c       y
 Total Number      83      45      41      36      34      29      24      23      20      18      16      15
          PMF   0.171  0.0926  0.0844  0.0741    0.07  0.0597  0.0494  0.0473  0.0412   0.037  0.0329  0.0309
          CDF   0.171   0.263   0.348   0.422   0.492   0.551   0.601   0.648   0.689   0.726   0.759    0.79

      Alphaba       h       w       g       r       l       b       m       d       k       p       f       v
 Total Number      15      15      14      12      10       9       7       7       6       3       3       1
          PMF  0.0309  0.0309  0.0288  0.0247  0.0206  0.0185  0.0144  0.0144  0.0123 0.00617 0.00617 0.00206
          CDF   0.821   0.852   0.881   0.905   0.926   0.944   0.959   0.973   0.986   0.992   0.998       1

Entropy: 4.12 bits/symbol
```

Figure 1: Statistics result for **santaclaus.txt**

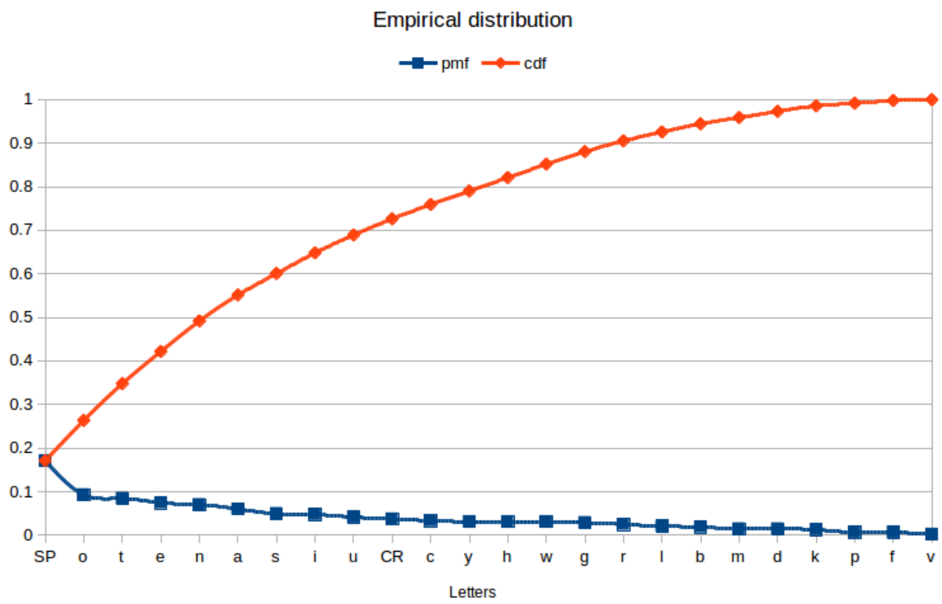


Figure 2: Empirical distribution for **santaclaus.txt**

Problem 4 Huffman Code Encode C++

Write a program that designs a Huffman code for the given distribution in Problem 3. (14

Ans

The program for this problem was wrote together with Problem 3. The execute print the Huffman encode result as Figure 3

```
=================== Codeword Table ================

        SP:         111        t:        1101
         l:      110011        b:      110010
        CR:       11000        e:        1011
         n:        1010        c:       10011
         y:       10010        h:       10001
         w:       10000        a:        0111
         g:       01101        m:      011001
         d:      011000        f:    01011111
         v:    01011110        p:     0101110
         k:      010110        r:       01010
         s:        0100        o:         001
         i:        0001        u:        0000
```

Figure 3: Huffman encode result for **santaclaus.txt**

Problem 5 Adaptive Huffman Tree

Let X be a random variable with an alphabet H , i.e., the 26 lower-case letters. Use adaptive Huffman tree to find the binary code for the sequence **a a b b a**. (24%)
You are asked to use the following 5 bits fixed-length binary code as the initial codewords for the 26 letters. That is

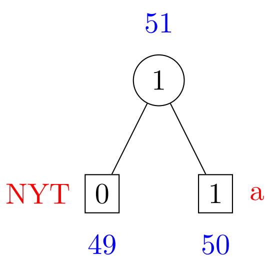
a: 00000
b: 00001
⋮
z: 11001

Note: Show the Huffman tree during your coding process.

Ans

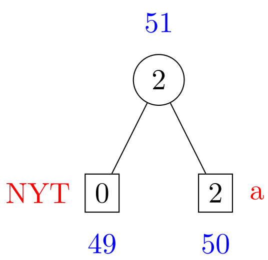
1. Initial step:
   $Total\ nodes = 2m - 1 = 26 \times 2 - 1 = 51$

2. **a** encoded:



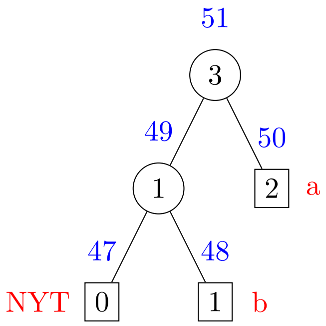
00000
a

3. **a a** encoded:



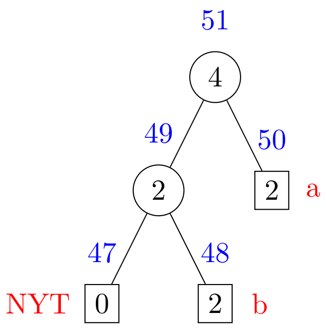
00000 1
a a

4. **a a b** encoded:



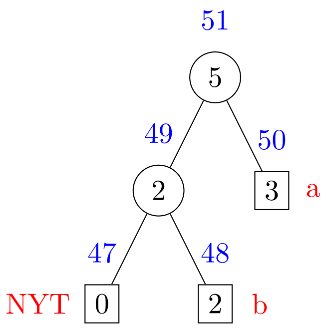
00000	1	0	00001
a	a	NYT	b

5. **a a b b** encoded:



00000	1	0	00001	01
a	a	NYT	b	b

6. **a a b b a** encoded:



00000	1	0	00001	01	1
a	a	NYT	b	b	a

Problem 6 Golomb Encoding and Decoding.

(a) Find the Golomb code of n=21 when m=4.

Ans

$$\begin{aligned} &2^{\lceil \log_2^m \rceil} - m = 2^2 - 4 = 0 \\ &\textit{encoded } 21 = 21/4 = 5 \ldots 1 = 111110\ 01 \end{aligned}$$

(b) Find the Golomb code of n=14 when m=4.

Ans

$$\begin{aligned} &2^{\lceil \log_2^m \rceil} - m = 2^2 - 4 = 0 \\ &\textit{encoded } 14 = 14/4 = 3 \ldots 2 = 1110\ 10 \end{aligned}$$

(c) Find the Golomb code of n=21 when m=5.

Ans

$$\begin{aligned} &2^{\lceil \log_2^m \rceil} - m = 2^3 - 5 = 3 \\ &\textit{encoded } 21 = 21/5 = 2 \ldots 1 = 110\ 01 \end{aligned}$$

(d) Find the Golomb code of n=14 when m=5.

Ans

$$\begin{aligned} &2^{\lceil \log_2^m \rceil} - m = 2^3 - 5 = 3 \\ &\textit{encoded } 14 = 14/5 = 2 \ldots 4 = 110\ 111 \end{aligned}$$

(e) A two-integer sequence is encoded by Golomb code with m=4 to get the bitstream 11101111000. What's the decoded two-integer sequence?

Ans

$$2^{\lceil \log_2^m \rceil} - m = 2^2 - 4 = 0$$

$$\begin{array}{cccc} \underline{1110} & \underline{11} & \underline{110} & \underline{00} \\ 3 & 3 & 2 & 0 \\ \multicolumn{2}{c}{15} & \multicolumn{2}{c}{8} \end{array}$$

(f) A two-integer sequence is encoded by Golomb code with m=5 to get the bitstream 11101111000 (the same bitstream as that in (e)). What's the decoded two-integer sequence?
Hint: The unary code for a positive integer q is simply q 1s followed by a 0.

Ans

$$2^{\lceil \log_2^m \rceil} - m = 2^3 - 4 = 4$$

$$\begin{array}{cccc} \underline{1110} & \underline{11} & \underline{110} & \underline{00} \\ 3 & 3 & 2 & 0 \\ \multicolumn{2}{c}{18} & \multicolumn{2}{c}{10} \end{array}$$