

The Quantum Hall Effect and Bulk-Edge Correspondence on Lattice Systems

by

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Abstract

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Preface

You must include a preface if any part of your research was partly or wholly published in articles, was part of a collaboration, or required the approval of UBC Research Ethics Boards.

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Examples

Chapter 2 is based on work conducted in UBC's Maple Syrup Laboratory by Dr. A. Apple, Professor B. Boat, and Michael McNeil Forbes. I was responsible for tapping the trees in forests X and Z, conducted and supervised all boiling operations, and performed frequent quality control tests on the product.

A version of chapter 2 has been published [?]. I conducted all the testing and wrote most of the manuscript. The section on "Testing Implements" was originally drafted by Boat, B. Check the first pages of this chapter to see footnotes with similar information.

Note that this preface must come before the table of contents. Note also that this section "Examples" should not be listed in the table of contents, so we have used the starred form: `\section*{Example}`.

Table of Contents

Abstract	ii
Preface	iii
Table of Contents	iv
List of Tables	v
List of Figures	vi
List of Programs	vii
Acknowledgements	viii
Dedication	ix
1 This is a Chapter	1
1.1 Introduction	1
1.1.1 History	1
1.1.2 Heuristic Arguments	1
1.2 Noninteracting Setting	4
1.2.1 Equality of Bulk and Edge Conduvitivities	10
1.3 Interacting Setting	20
1.3.1 Equality of Bulk and Edge Currents	24
1.4 A Section	31
1.4.1 This is a Subsection	32
1.5 Quote	34
1.6 Programs	34
2 Another Chapter...	36
2.1 Another Section	36
2.2 Tables	38

Table of Contents

3 Landscape Mode	41
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Appendices

A General Functional Analysis	43
B Properties of \mathcal{I}_μ	44
C The Lieb-Robinson Bound	47
D Grönwall's Inequality and Uniqueness	48
E Note on Generators of Parallel Transport	50
F From Dec 13 Meeting	52
G The Helffer-Sjöstrand Representation	53
H Spectral Measures and Projection-Values Measures	55
I Greens' Functions and the Combes-Thomas Bound	57

Appendices

A First Appendix	59
B Second Appendix	60

List of Tables

- 1.1 Here is the caption for this wonderful table... 33
- 2.1 Another table. 38
- 2.2 Feasible triples for highly variable Grid 38

List of Figures

1.1	The Hall resistivity as a function of the magnetic field strength. Distinct plateaux are clearly visible.	3
1.2	The cylinder Γ_L	22
1.3	The regions Γ_E , Γ_B , and Γ_m	24
2.1	Happy Face: figure example.	37

List of Programs

1.1	Python program that computes the n^{th} Fibonacci number using memoization.	35
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Acknowledgements

This is the place to thank professional colleagues and people who have given you the most help during the course of your graduate work.

Dedication

The dedication is usually quite short, and is a personal rather than an academic recognition. The *Dedication* does not have to be titled, but it must appear in the table of contents. If you want to skip the chapter title but still enter it into the Table of Contents, use this command `\chapter[Dedication]{}`.

Note that this section is the last of the preliminary pages (with lowercase Roman numeral page numbers). It must be placed *before* the `\mainmatter` command. After that, Arabic numbered pages will begin.

Chapter 1

This is a Chapter

1.1 Introduction

1.1.1 History

Since its unexpected discovery in 1980, the Quantum Hall Effect (QHE) has captured significant interest in the mathematical physics community.

1.1.2 Heuristic Arguments

The Classical Hall Effect

Using classical electromagnetism is not enough to predict the plateaux seen experimentally. Suppose we have a 2-dimensional electron gas, and let $\vec{B} = B\hat{x}_3$ be a magnetic field piercing the plane of the electrons. They are subjected to a Lorentz force

$$m\dot{\vec{v}} = -q\vec{v} \times \vec{B}.$$

The solution to this differential equation is given by the *cyclotron orbits*,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} a + r \sin(\omega_B t + \phi) \\ b + r \cos(\omega_B t + \phi) \\ 0 \end{pmatrix}$$

where $\omega_B = qB/m$ is the *cyclotron frequency*. An electric field $\vec{E} = E\hat{x}_1$ is introduced, and the electrons move in the x_1 -direction. We now employ the *Drude model*,

$$m\dot{\vec{v}} = -q(\vec{E} + \vec{v} \times \vec{B}) + \frac{m}{\tau}\vec{v},$$

where the final term is a linear friction term, and τ is the scattering time. At equilibrium, the equation reads

$$\vec{J} + \frac{q\tau}{m}\vec{J} \times \vec{B} = -\frac{q^2 n \tau}{m}\vec{E},$$

1.1. Introduction

where $\vec{J} = -nq\vec{v}$ is the current density, related to the velocity by the density of electrons per unit area n . In matrix notation, this reads

$$\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & q \end{pmatrix} \vec{J} = -\frac{q^2n\tau}{m} \vec{E}.$$

Since the matrix on the left is invertible, we may write $\vec{J} = \sigma \vec{E}$, which is Ohm's Law. The *conductivity tensor* is given by

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{pmatrix} = -\frac{q^2n\tau}{m(1+\omega_B^2\tau^2)} \begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix}.$$

The off-diagonal components, $\pm \frac{q^2n\tau}{m(1+\omega_B^2\tau^2)}\omega_B\tau$, are responsible for the Hall effect; the magnetic field induces a component of the current in the x_2 -direction, in addition to the one in the x_1 -direction from the electric field.

When making a measurement, physicists actually measure the resistivity $\rho = \sigma^{-1}$. In particular, the *Hall resistivity* is given by

$$\rho_{xy} = \frac{B}{nq}.$$

The key prediction of the classical theory is that the Hall resistivity is increases linearly in response to the strength of the magnetic field.

The Quantum Hall Effect

Von Klitzing's experimental observation in 1980 made it clear that classical electromagnetism is not sufficient to describe the Hall effect. At a temperature of about 8mK, the Hall resistivity looked like this as a function of the magnetic field strength.

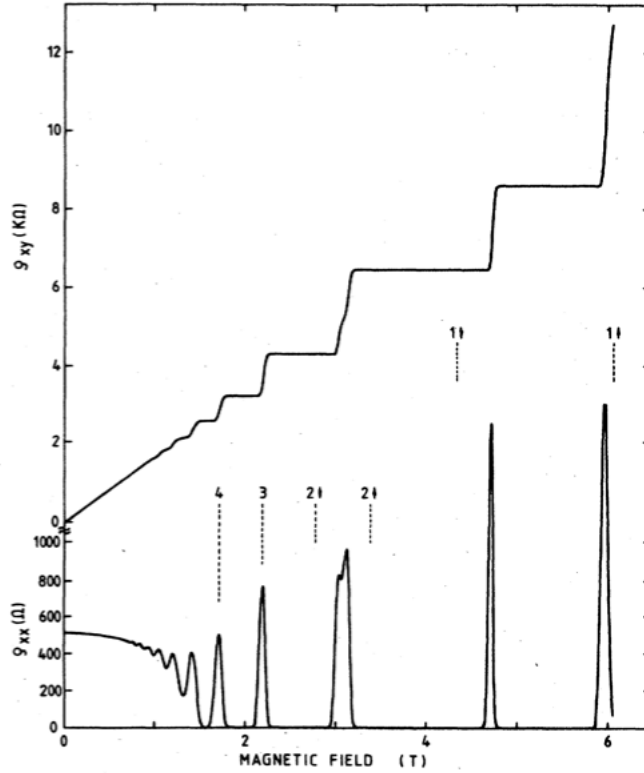


Figure 1.1: The Hall resistivity as a function of the magnetic field strength. Distinct plateaux are clearly visible.

Even more surprising, the plateaux occur at the values

$$\rho_{xy} = \frac{2\pi\hbar}{nq},$$

where n is an integer. In fact, this integer can be measured to such extraordinary precision (to about 3×10^{-10}) that as of 2020, the SI definition of the Ohm itself has been redefined in terms of the quantum Hall resistivity.

The first explanation employing quantum mechanics is due to Laughlin in (need to add).

An intriguing mathematical fact of the quantum Hall effect is the equality of bulk and edge conductivity, in the sense that whether the system is assumed to have an edge or not is immaterial. Proving this fact, both in the interacting and noninteracting setting, is the main focus of this thesis.

1.2 Noninteracting Setting

Consider the lattice \mathbb{Z}^2 , and the associated Hilbert space of square-summable sequences of vectors in \mathbb{C}^n ,

$$\ell^2(\mathbb{Z}^2, \mathbb{C}^n) = \left\{ (x_i)_{i \in \mathbb{Z}^2} \subset \mathbb{C}^n : \sum_{i \in \mathbb{Z}^2} \|x_i\|^2 < \infty \right\},$$

with inner product $\langle x, y \rangle = \sum_{i \in \mathbb{Z}^2} x_i \overline{y_i}$. We denote this as $\ell^2(\mathbb{Z}^2)$ for short. On this Hilbert space we define a *bulk Hamiltonian* $H_B : \ell^2(\mathbb{Z}^2) \rightarrow \ell^2(\mathbb{Z}^2)$, whose matrix elements follow a short-range assumption:

Assumption 1. *There exists some $\alpha > 0$ such that*

$$\sup_{y \in \mathbb{Z}^2} \sum_{x \in \mathbb{Z}^2} |H_B(x, y)| (e^{\alpha|x-y|} - 1) \leq C < \infty,$$

where $|x| = |x_1| + |x_2|$ is the taxicab metric.

We also construct an *edge Hamiltonian* on the lattice $\mathbb{Z}_a^2 := \{x \in \mathbb{Z}^2 : x_2 > -a\}$, denoted by $H_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}_a^2)$. The bulk and edge Hamiltonians are related by the *edge operator*

$$E_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}^2),$$

$$E_a := J_a H_a - H_B J_a,$$

where $J_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}^2)$ denotes extension by zeroes. We require only that the edge operator satisfies the edge assumption

Assumption 2. *The edge operator satisfies*

$$\sup_{z \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}_a^2} |E_a(x, y)| e^{\alpha(|x_2+a|+|x_1-y_1|)} \leq C < \infty$$

for some $\alpha > 0$, where $|x| = |x_1| + |x_2|$ is the taxicab metric.

1.2. Noninteracting Setting

The interpretation is that $E_a = J_a H_a - H_B J_a$ is the difference between first applying H_a on $\ell^2(\mathbb{Z}_a^2)$, and then making everything below $-a$ into zeroes, versus first making all $x \in \mathbb{Z}^2$ such that $x_2 < -a$ zeroes, and then applying H_B . The assumption ensures that the effects from introducing the edge at $-a$ die exponentially as we move upward away from the edge (due to the $|x_2 - (-a)|$ term in the exponent), and also terms do not interact too much as their x_1 distance increases (due to the $|x_1 - y_1|$ term in the exponent).

A simple example of an edge Hamiltonian satisfying the edge condition is $H_a = J_a^* H_B J_a$, which gives $E_a = (J_a J_a^* - 1) H_B$. The idea is that for a state $\psi \in \ell^2(\mathbb{Z}_a^2)$, we have $\langle \psi, H_a \psi \rangle = \langle (J_a \psi), H_B (J_a \psi) \rangle$, which we interpret as the edge Hamiltonian having the same expectation as the bulk Hamiltonian if we just transformed all the states with support below the line $x_2 = -a$ into zeroes below $x_2 = -a$. The edge operator is

$$E_a = (J_a J_a^* - 1) H_B J_a = \begin{cases} -H_B(x, y) & \text{if } x_2 < -a \\ 0 & \text{if } x_2 \geq -a \end{cases}$$

Intuitively, there is no difference between H_B and H_a on \mathbb{Z}_a^2 . The edge assumption 2 then follows from the bound $|E_a(x, y)| \leq |H_B(x, y)|$ and the short range assumption 1.

We also make the following assumption about the bulk Hamiltonian.

Assumption 3. *The bulk Hamiltonian has a spectral gap. That is, there exists an interval $\Delta \subset \mathbb{R}$ such that for all L ,*

$$\Delta \cap \text{Spec}(H_B) = \emptyset.$$

Remark: The spectral gap assumption can be relaxed to a “mobility gap” assumption,

$$\sup_{f \in B_c(\Delta)} |f(H_B)(x, y)| (1 + |x|)^{-\alpha_1} e^{\alpha_2 |x-y|} < \infty$$

for some $\alpha_1 > 0$, where $B_c(\Delta)$ is the set of Borel functions f which are constant on $(-\infty, \inf \Delta)$ and on $(\sup \Delta, \infty)$ such that $|f(x)| \leq 1$ for all x . See (need to add) for details.

We define the *bulk conductivity* at Fermi energy μ as follows. Suppose we subject the system to an external electric potential difference V in the x_2 direction. We write this as $V_0 \Lambda_2$, where Λ_i are multiplication operators $\Lambda_i |\psi(x_1, x_2)\rangle = \Lambda(x_i) |\psi(x_1, x_2)\rangle$ which are *switch functions*,

1.2. Noninteracting Setting

$$\Lambda : \mathbb{R} \rightarrow \mathbb{R} \quad \Lambda(x_i) = \begin{cases} 1 & \text{if } x_i \leq 0 \\ 0 & \text{if } x_i \geq 1 \end{cases}$$

and are smooth and monotonically decreasing on $(0, 1)$. Note that the ensuing physics (in particular, our definition of the Hall conductivity) is independent of the particular choice of switch function Λ_i , since any two switch functions are exactly equal on the lattice.

This gives $\vec{E} = -\nabla V = -V_0 \frac{\partial \Lambda_2}{\partial x_2}$, so that \vec{E} has compact support $\text{supp}(\Lambda_2')$. We introduce a function which grows slowly in time as t grows from $-\infty$ to 0, so as to invoke the adiabatic principle. Here, we choose $e^{\varepsilon t}$, and we will let $\varepsilon \rightarrow 0$ at the end. The Hamiltonian therefore experiences a perturbation,

$$\tilde{H}_B(t) = H_B + V_0 \Lambda_2 e^{\varepsilon t}.$$

We define the Hall current operator $J_H = i[\tilde{H}_B(t), \Lambda_1] = i[H_B, \Lambda_1]$, which is related to the Hall conductivity by $J_H = \sigma_H V$. We also denote by $P_\mu := P((-\infty, \mu])$ the projection-valued measure associated with H_B onto states with energy below the Fermi energy μ (see Appendix H).

Lemma 1. *The ground state expectation $\text{Tr}(P_\mu J_H)$ of the Hall current is zero.*

Proof. Notice that J_H is trace-class, since its trace norm can be broken into $\|J\|_1 \leq \|[H_B, \Lambda_1]e^{\delta|x_1|}\| \|e^{-\delta|x_1|}\|_1$. The first term is bounded by lemma (need to add), while the second is a summable function on \mathbb{Z}^2 . This fact, together with P_μ being bounded, allows us to exploit linearity and cyclicity of the trace. Since the Hamiltonian commutes with its ground state projection, we have

$$\text{Tr}(P_\mu [H_B, \Lambda_1]) = \text{Tr}([P_\mu, H_B] \Lambda_1) = 0.$$

□

Proposition 1. *The Hall conductivity σ_H in the bulk system is equal to*

$$\sigma_B = -i \text{Tr} (P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]).$$

Proof. We begin with the Heisenberg equation of motion for the density matrix, $\dot{\rho}(t) = -i[\tilde{H}_B(t), \rho(t)]$, with initial condition $\lim_{t \rightarrow -\infty} \|\rho(t) - e^{-itH_B} P_\mu e^{itH_B}\| = 0$, which also implies $\lim_{t \rightarrow -\infty} \|e^{itH_B} \rho(t) e^{-itH_B} - P_\mu\| = 0$.

1.2. Noninteracting Setting

We work in the interaction picture by defining $\rho_I(t) = e^{itH_B}\rho(t)e^{-itH_B}$ and $\Delta H_B(t) = e^{itH_B}V_0\Lambda_2e^{\varepsilon t}e^{-itH_B}$. Thus

$$\dot{\rho}_I(t) = -i[\Delta H_B(t), \rho_I(t)].$$

The solution to this differential equation is readily verified to be

$$\rho(t) = i \int_{-\infty}^t [\Delta H_B(s), P_\mu] ds + P_\mu$$

Indeed, taking the derivative of the right hand side gives $i[\Delta H_B(t), P_\mu] = i[\Delta H_B(t), \rho_I(t)] + \mathcal{O}(V_0^2)$, but P_μ and $\rho_I(t)$ are equal up to zeroth order in V_0 (need to add). The initial condition is also satisfied.

Using $J_H = i[H_B, \Lambda_1] = \sigma_H V = \sigma_H V_0 \Lambda_2$, we obtain $\sigma_H = \frac{1}{V_0} \lim_{\varepsilon \rightarrow 0} \text{Tr}(\rho_I(0)i[H_B, \Lambda_1])$ (need to add). By Lemma 1, the expectation of the ground state current is zero. Thus

$$\begin{aligned} \sigma_H &= \frac{i}{V_0} \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(i \int_{-\infty}^0 [\Delta H_B(s), P_\mu] [H_B, \Lambda_1] ds \right) \\ &= -\frac{1}{V_0} \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 [-e^{isH_B} V_0 \Lambda_2 e^{\varepsilon s} e^{-isH_B}, P_\mu] [H_B, \Lambda_1] ds \right) \\ &= \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 e^{isH_B} [\Lambda_2, P_\mu] e^{-isH_B} [H_B, \Lambda_1] e^{\varepsilon s} ds \right) \\ &= \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 (e^{isH_B} [H_B, \Lambda_1] e^{-isH_B}) \cdot ([\Lambda_2, P_\mu] e^{\varepsilon s}) ds \right) \end{aligned}$$

Where we used the fact that P_μ and H_B commute. Using integration by parts on the two terms in brackets, and noting that $\frac{d}{ds}(e^{isH_B} \Lambda_1 e^{-isH_B} - \Lambda_1) = -ie^{isH_B} [H_B, \Lambda_1] e^{-isH_B}$, we obtain

$$\begin{aligned} \sigma_H &= -i \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 (e^{isH_B} \Lambda_1 e^{-isH_B} - \Lambda_1) \frac{d}{ds} ([\Lambda_2, P_\mu] e^{\varepsilon s}) ds \right) \\ &= -i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \Lambda_1^s [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right) \end{aligned}$$

where $\Lambda_1^s := e^{isH_B} \Lambda_1 e^{-isH_B} - \Lambda_1$. Using the notation $\overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, it is readily verified that the commutator $[\Lambda_2, P_\mu]$ is an *off-diagonal* operator, in the sense that $[\Lambda_2, P_\mu] = \overline{[\Lambda_2, P_\mu]}$. Furthermore, a simple computation reveals that for any two operators A and B , $\text{Tr}(\overline{AB}) = \text{Tr}(AB)$. It therefore follows that

1.2. Noninteracting Setting

$$\sigma_H = -i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \overline{\Lambda_1^s} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right)$$

The integrand can be broken into two terms,

$$\overline{\Lambda_1^s} [\Lambda_2, P_\mu] e^{\varepsilon s} = e^{-isH_B} \overline{\Lambda_1} e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} - \overline{\Lambda_1} [\Lambda_2, P_\mu] e^{\varepsilon s}$$

by commutativity of P_μ and H_B . We show that the integral of the first term vanishes. We begin by breaking the first term down further into

$$e^{-isH_B} P_\mu \Lambda_1 P_\mu^\perp e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} + e^{-isH_B} P_\mu^\perp \Lambda_1 P_\mu e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s}.$$

We treat the first of these two terms; the other is handled in an identical manner. We invoke the spectral theorem (Appendix H) to write $e^{-isH_B} P_\mu = \int_{-\infty}^\mu e^{-is\lambda} dP_\lambda$, and similarly $P_\mu^\perp e^{isH_B} = (\mathbb{1} - P_\mu) e^{isH_B} = \int_\mu^\infty e^{is\nu} dP_\nu$.

We remark that, since the Fermi energy μ is assumed to lie in a spectral gap, there must exist a neighbourhood $(\mu - \delta, \mu + \delta)$ in which there are no states. We exploit this fact to rewrite the limits of integration, $\int_{-\infty}^{\mu-\delta} e^{-is\lambda} dP_\lambda$ and $\int_{\mu+\delta}^\infty e^{is\nu} dP_\nu$. We therefore obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{-\infty}^0 e^{-isH_B} P_\mu \Lambda_1 P_\mu^\perp e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \\ = \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \int_{-\infty}^{\mu-\delta} e^{-is\lambda} dP_\lambda \Lambda_1 \int_{\mu+\delta}^\infty e^{is\nu} dP_\nu [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right) \\ = \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \int_{-\infty}^{\mu-\delta} \int_{\mu+\delta}^\infty e^{s(\varepsilon - i\lambda + i\nu)} dP_\lambda \Lambda_1 dP_\nu [\Lambda_2, P_\mu] ds \right) \end{aligned}$$

Performing the integral over s yields

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{-\infty}^0 e^{s(\varepsilon - i\lambda + i\nu)} ds = - \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{i\varepsilon + \lambda - \nu}$$

This limit is zero, since $\lambda \neq \nu$. Indeed, due to the spectral gap, the integration variables live in $\lambda \in (-\infty, \mu - \delta)$ and $\nu \in (\mu + \delta, \infty)$. The case for the $e^{-isH_B} P_\mu^\perp \Lambda_1 P_\mu e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s}$ term (where the P_μ and P_μ^\perp swap places) is treated analogously. Hence the first term in the integrand for σ_H vanishes, as claimed.

Finally, we return to our expression for the Hall conductivity, which now reads

1.2. Noninteracting Setting

$$\sigma_H = i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \overline{\Lambda_1} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right).$$

It is a basic algebraic calculation to show that $\overline{\Lambda_1} = [[\Lambda_1, P_\mu], P_\mu]$. Evaluating the integral over s is now trivial; $\int_{-\infty}^0 e^{\varepsilon s} ds = \varepsilon^{-1}$. Thus

$$\sigma_H = i \text{Tr}([[\Lambda_1, P_\mu], P_\mu] [\Lambda_2, P_\mu]).$$

Shifting the commutator completes the proof:

$$\begin{aligned} \sigma_H &= i \text{Tr}(P_\mu [[\Lambda_2, P_\mu], [\Lambda_1, P_\mu]]) \\ &= -i \text{Tr}(P_\mu [[\Lambda_1, P_\mu], [\Lambda_2, P_\mu]]) \\ &= -i \text{Tr}(P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]). \end{aligned}$$

The justification for shifting the commutator is that $[\Lambda_i, P_\mu]$ is trace class, by the proof of Lemma 5. (Need to add) \square

Remark: This is reminiscent of the well-known adiabatic curvature formula,

$$\kappa = \text{Tr}(P[\partial_1 P, \partial_2 P]) = \text{Tr}(P[[P, K_1], [P, K_2]]) = \text{Tr}(P[K_1, K_2]),$$

where K_i are called *generators of parallel transport*. We will see the adiabatic curvature formula again later in the interacting setting.

For the *edge conductivity*, we again need the current operator across the line $x_1 = 0$, which is this time given by $-i[H_a, \Lambda_1]$. We define

$$\sigma_E = -i \lim_{a \rightarrow \infty} \text{Tr}(\rho'(H_a)[H_a, \Lambda_1]),$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(r) = \begin{cases} 1 & \text{if } r \leq \inf \Delta \\ 0 & \text{if } r \geq \sup \Delta \end{cases}$$

and decreases smoothly and monotonically in Δ . The definition of σ_E is reminiscent of another formula we will see later in the interacting setting, $\text{Tr}(\dot{P}J)$, where J is the current operator. The interpretation of σ_E is that if we apply a small potential difference V across $x_2 = -a$ to $x_2 = \infty$, there will be a net current

$$\begin{aligned} I &= -i\text{Tr}(\rho(H_a + V)[H_a + V, \Lambda_1] - \rho(H_a)[H_a, \Lambda_1]) \\ &= -i\text{Tr}((\rho(H_a + V) - \rho(H_a))[H_a, \Lambda_1]) \end{aligned}$$

Thus we obtain the conductivity

$$\sigma_E = \frac{I}{V} = -i\text{Tr}\left(\frac{(\rho(H_a + V) - \rho(H_a))}{V}[H_a, \Lambda_1]\right) \rightarrow -i\text{Tr}(\rho'(H_a)[H_a, \Lambda_1])$$

in the limit as $V \rightarrow 0$. As we shall see, it turns out that σ_E is independent of the choice of ρ , and σ_B is independent of λ .

1.2.1 Equality of Bulk and Edge Conduvivities

The main result of this section is

Theorem 1. $\sigma_E = \sigma_B$.

Outline of the Proof

Before giving the proof in its entirety, we outline the basic steps. The edge condition guarantees that $\sigma_E(a) := \rho'(H_a)[H_a, \Lambda_1]$ is trace-class (need to add). We posit that the edge conductivity can be rewritten as $\sigma_E = \lim_{a \rightarrow \infty} \sigma_E(a)$, where

$$\sigma_E(a) = -i\text{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2),$$

since we have assumed that there is a spectral gap (as opposed to a mobility gap), so that there are extended states near the edge, and no bound states or resonances far from the edge. Thus, intuitively, the cutoff introduced by Λ_2 is irrelevant as we take $a \rightarrow \infty$. We provide a more concrete justification for this later in Lemma 2.

The key ingredient of the proof is the use of the functional calculus given by the Helffer-Sjöstrand representation of self-adjoint operators on a Hilbert space (Appendix G). The two crucial operators written in their Helffer-Sjöstrand representations are

$$\begin{aligned} \rho(H) &= -\frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R(z) dz \wedge d\bar{z} \\ \rho'(H) &= \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R(z)^2 dz \wedge d\bar{z} \end{aligned}$$

1.2. Noninteracting Setting

where $R(z) = (H - z)^{-1}$ is the resolvent of H . For ease of notation, we drop the $dz \wedge d\bar{z}$ from this point.

Another key observation is that it turns out that the bulk conductivity can actually be written

$$\sigma_B = i\text{Tr}([\rho(H_B), \Lambda_1]\Lambda_2).$$

By employing the Helffer-Sjostrand representations above, one can add an operator of zero trace to the edge conductivity, and show that this operator converges in trace to $[\rho(H_B), \Lambda_1]\Lambda_2$ in the limit $a \rightarrow \infty$.

The Proof

Lemma 2. *The edge conductivity is equal to $\lim_{a \rightarrow \infty} \sigma_E(a)$, where*

$$\sigma_E(a) = -i\text{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2).$$

Proof. Since σ_B is translation invariant (need to add), we only need to prove the case $-i\text{Tr}(\rho'(H_{a=0})[H_{a=0}, \Lambda_1]) = \sigma_B$. We drop the subscript, $H := H_{a=0}$.

A general fact of functional analysis is that if $A_n \xrightarrow{s} 0$ and X is trace-class, then $\|A_n X\|_1 \rightarrow 0$. Since the multiplication operator $\Lambda_{2,n}|\psi\rangle := \Lambda(x_2 - n)|\psi\rangle$ converges strongly to the identity as $n \rightarrow \infty$, it follows that

$$\|\rho'(H)[H, \Lambda_1](\mathbb{1} - \Lambda_{2,n})\|_1 \rightarrow 0,$$

and thus from the inequality $|\text{Tr}(A)| \leq \|A\|_1$ we deduce

$$\sigma_E(a) = -i\text{Tr}(\rho'(H)[H, \Lambda_1]) = -i \lim_{n \rightarrow \infty} \text{Tr}(\rho'(H)[H, \Lambda_1]\Lambda_{2,n}).$$

Consider the edge Hamiltonian with cutoff at $a = 0$ associated with the bulk Hamiltonian shifted down by n . We denote this modified edge Hamiltonian by H^n .

Whether we first cut off everything above $x_2 = n$ using $\Lambda_{2,n}$ and then apply $H_{a=0}$, or instead cut off everything above $x_2 = 0$ using Λ_2 and then apply the Hamiltonian H^n is immaterial. In other words,

$$-i \lim_{n \rightarrow \infty} \text{Tr}(\rho'(H)[H, \Lambda_1]\Lambda_{2,n}) = -i \lim_{n \rightarrow \infty} \text{Tr}(\rho'(H^n)[H^n, \Lambda_1]\Lambda_2).$$

Thus, our goal is to show that

1.2. Noninteracting Setting

$$\lim_{n \rightarrow \infty} \text{Tr}(\rho'(H^n)[H^n, \Lambda_1]\Lambda_2) = \lim_{a \rightarrow \infty} \text{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2).$$

□

Define

$$Z(a) = [\rho(H_a), \Lambda_1]\Lambda_2 - \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [R_a(z), [H_a, \Lambda_1]\Lambda_2] dz^2$$

This operator has zero trace. Indeed, the first term has vanishing trace in the position basis, while the second term's integrand involves the trace of $[R, R] = 0$. The bounds necessary for shifting the commutator like this are on

$$\|[H_a, \Lambda_1]e^{\delta|x_1|}\|, \quad \|e^{-\delta|x_1|}\Lambda_2\|_1, \quad \|R\|,$$

the first two of which are given below in Lemmas 3, 4, and the third is the fact that the resolvent is bounded. So $\sigma_E(a) = \text{Tr}(\Sigma(a))$, where

$$\begin{aligned} \Sigma(a) &= -i\rho'(H_a)[H_a, \Lambda_1]\Lambda_2 + iZ(a) \\ &= -i\rho'(H_a)[H_a, \Lambda_1]\Lambda_2 + i[\rho(H_a), \Lambda_1]\Lambda_2 - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [R_a(z), [H_a, \Lambda_1]\Lambda_2] dz^2. \end{aligned}$$

Using the Hellfer-Sjöstrand representations for the first two terms on the right hand side, we obtain

$$\begin{aligned} \Sigma(a) &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z)^2 [H_a, \Lambda_1]\Lambda_2 dz^2 + \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [H_a, \Lambda_1] R_a(z) \Lambda_2 dz^2 \\ &\quad - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} (R_a(z)^2 [H_z, \Lambda_1]\Lambda_2 - R_a(z) [H_a, \Lambda_1] \Lambda_2 R_a(z)) dz^2 \\ &= -\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [H_a, \Lambda_1] [R_a(z), \Lambda_2] dz^2 \\ &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [H_a, \Lambda_1] R_a(z) [H_a, \Lambda_2] R_a(z) dz^2, \end{aligned}$$

where we used

$$[R_a(z), \Lambda_i] = -R_a(z) [H_a, \Lambda_i] R_a(z)$$

1.2. Noninteracting Setting

in the final equality. Next, we must prove that the operator above converges to the corresponding bulk operator in trace-norm,

$$\|\Sigma(a) - \Sigma_B\|_1 \rightarrow 0,$$

as $a \rightarrow \infty$, which in turn proves that $\text{Tr}(\Sigma(a)) \rightarrow \text{Tr}(\Sigma_B)$ because of the bound $|\text{Tr}(A)| \leq \|A\|_1$. Here, Σ_B is the same operator as before, but using the bulk operators H_B and $R_B(z)$. Once this limit is established, we shall prove that $\sigma_B = \text{Tr}(\Sigma_B)$ to conclude the proof.

To show that the limit is zero as claimed, we bound the trace norm of the integrand of $\Sigma(a)$ with the ultimate goal of applying dominated convergence. We accomplish this bound by breaking it into three parts,

$$R[H_a, \Lambda_1]R[H_a, \Lambda_2]R = J_a[R, \Lambda_1]e^{\delta|x_1|}J_a^* \cdot e^{-\delta|x_1|}e^{-\delta|x_2|} \cdot J_a e^{\delta|x_2|}[H_a, \Lambda_2]R J_a^*,$$

and bounding the norm of each with the following two lemmas. We remark that the extension J_a and its adjoint have norm 1.

Lemma 3.

$$\|[H_a, \Lambda_i]e^{\delta|x_i|}\| \leq C.$$

Proof. The operator can be bounded by inspecting its matrix elements

$$\begin{aligned} \langle x, [H_a, \Lambda_i]e^{\delta|x_i|}y \rangle &= \langle x, H_a \Lambda_i y \rangle e^{\delta|y_i|} - \langle x, \Lambda_i H_a y \rangle e^{\delta|y_i|} \\ &= H_a(x, y) e^{\delta|y_i|} (\Lambda(y_i) - \Lambda(x_i)). \end{aligned}$$

This is zero if $|x_i - y_i| \leq |y_i|$, since this would imply that x_i and y_i have the same sign, yielding $\Lambda(x_i) = \Lambda(y_i)$. So either the matrix element is zero, or $|y_i| \leq |x_i - y_i|$, which implies

$$\begin{aligned} |H_a(x, y) e^{\delta|y_i|} (\Lambda(y_i) - \Lambda(x_i))| &\leq 2|H_a(x, y)| e^{\delta|x_i - y_i|} \\ &\leq 2|H_a(x, y)| e^{\delta|x - y|} \\ &\leq C|H_a(x, y)| (e^{\delta|x - y|} - 1), \end{aligned}$$

where the final inequality comes from the fact that the diagonal matrix elements are zero. Hence the short range assumption

$$\sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} |H(x, y)| (e^{\mu|x - y|} - 1) < \infty$$

1.2. Noninteracting Setting

combined with Holmgren's bound

$$\|A\| \leq \max \left\{ \sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} |A(x, y)|, \sup_{y \in \mathbb{Z}^2} \sum_{x \in \mathbb{Z}^2} |A(x, y)| \right\}$$

completes the proof. \square

Lemma 4. $e^{-\delta|x_1|}e^{-\delta|x_2|}$ is trace-class.

Proof. We bound the trace norm by noticing that this is a positive operator satisfying

$$\langle (n, m), e^{-\delta|x_1|}e^{-\delta|x_2|}(n, m) \rangle = \langle e^{-\delta|x_1|}e^{-\delta|x_2|}(n, m), (n, m) \rangle,$$

so that its trace norm is equal to its trace. In the position basis, we see that its trace is given by a geometric series

$$\begin{aligned} \text{Tr}(e^{-\delta|x_1|}e^{-\delta|x_2|}) &= \sum_{(n, m) \in \mathbb{Z}^2} \langle (n, m), e^{-\delta|x_1|}e^{-\delta|x_2|}(n, m) \rangle \\ &\leq 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-\delta m} e^{-\delta n} \\ &= 2 \left(\frac{1}{1 - e^{-\delta}} \right)^2. \end{aligned}$$

\square

Now we return to the integrand which was broken into three parts. For the first term, $J_a[R_a(z), \Lambda_1]e^{\delta|x_1|}J_a^*$, we bound its operator norm by breaking it down further into

$$\begin{aligned} \|J_a[R_a(z), \Lambda_1]e^{\delta|x_1|}J_a^*\| &= \|[R_a(z), \Lambda_1]e^{\delta|x_1|}\| \\ &= \|-R_a(z)[H_a, \Lambda_1]R_a(z)e^{\delta|x_1|}\| \\ &\leq \|R_a(z)\| \cdot \|[H_a, \Lambda_1]e^{\delta|x_1|}\| \cdot \|e^{-\delta|x_1|}R_a(z)e^{\delta|x_1|}\|. \end{aligned}$$

The norm of $R_a(z)$ is bounded by

$$\|R_a(z)\| \leq \frac{1}{|\text{Im}(z)|}$$

1.2. Noninteracting Setting

for any $z \notin \mathbb{R}$ since H_a is self-adjoint. The second term is bounded by Lemma 3. Finally, for the third term $e^{-\delta|x_1|}R_a(z)e^{\delta|x_1|}$, we apply the Combes-Thomas bound,

$$\|e^{-\varepsilon f(x)}R_a(z)e^{\varepsilon f(x)}\| \leq \frac{C}{|\operatorname{Im}(z)|}$$

where $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is any Lipschitz function, and ε can be chosen as $\varepsilon = \frac{1}{C(1+|\operatorname{Im}(z)|)}$. Altogether, the bound of the first term of the integrand takes the form

$$\frac{C}{\operatorname{Im}(z)^2}.$$

The second term of the integrand is bounded by Lemma 4. Finally, the bound for the third term of the integrand, $e^{\delta|x_2|}[H_a, \Lambda_2]R_a(z)$, follows from the bound on R and Lemma 3, and is again of the form $\frac{C}{\operatorname{Im}(z)^2}$.

Altogether, it follows from $\|AB\|_1 \leq \|A\|\|B\|_1$ that the trace norm of the integrand is bounded by $\frac{C}{\operatorname{Im}(z)^4}$.

We now appeal to a general fact of the Helffer-Sjöstrand functional calculus to provide domination. For any $n \in \mathbb{N}$, the quasi-analytic extension $\tilde{\rho}$ of ρ in the Helffer-Sjöstrand representation can be chosen so that

$$\int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} \frac{1}{|\operatorname{Im}(z)|^{p+1}} dz^2 \leq C_0 \sum_{k=0}^{n+2} \|\rho^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_m = \int_{-\infty}^{\infty} |f(x)|(x^2 + 1)^{m/2} dx.$$

Since $|\rho(x)| \leq 1$ and ρ' is compactly supported, these norms are all clearly finite. This fact, combined with the bound

$$\|R_a(z)[H_a, \Lambda_1]R_a(z)[H_a, \Lambda_2]R_a(z)\|_1 \leq \frac{C}{\operatorname{Im}(z)^4}$$

for the trace norm of the integrand of $\Sigma(a)$ provides the necessary bound for Lebesgue dominated convergence. Thus, it suffices to show pointwise convergence in z of the integrand to the associated bulk operator.

In other words, we wish to show

$$J_a[R_a(z), \Lambda_1]e^{\delta|x_1|}J_a^* \xrightarrow{s} [R_B(z), \Lambda_1]e^{\delta|x_1|},$$

1.2. Noninteracting Setting

$$J_a e^{\delta|x_2|} [H_a, \Lambda_2] J_a^* \xrightarrow{s} e^{\delta|x_2|} [H_B, \Lambda_2],$$

and

$$J_a R_a(z) J_a^* \xrightarrow{s} R_B(z)$$

for each fixed $z \in \mathbb{C}$. Inspecting the bounds we found for the left hand sides of these limits, it is clear that they are uniformly bounded in a . It therefore suffices to show convergence on a dense subspace of $\ell^2(\mathbb{Z}^2)$ (see Lemma 13); in particular, we may choose the dense subspace of compactly supported states, which allows us to ignore the $e^{\delta|x_i|}$ terms. Thus, we need to prove

$$J_a [R_a(z), \Lambda_1] J_a^* \xrightarrow{s} [R_B(z), \Lambda_1],$$

$$J_a [H_a, \Lambda_2] J_a^* \xrightarrow{s} [H_B, \Lambda_2],$$

and

$$J_a R_a(z) J_a^* \xrightarrow{s} R_B(z).$$

In fact, the final statement implies the first two; we appeal to the general fact of functional analysis that strong convergence of the resolvent of a self-adjoint operator implies that $J_a f(H_a) J_a^* \xrightarrow{s} f(H_B)$ for any bounded and continuous function f . In particular, it follows from Lemma 3 that the functions $[(\cdot - z)^{-1}, \Lambda_1]$ and $[\cdot, \Lambda_2]$ above are bounded and continuous, so we will have proven the desired limits if we can prove the strong convergence of the resolvent, $J_a R_a(z) J_a^* \xrightarrow{s} R_B(z)$.

To prove this, we use the edge assumption. Recall the edge operator, $E_a = J_a H_a - H_B J_a$. Adding and subtracting $z J_a$ gives

$$E_a = J_a (H_a - z) - (H_B - z) J_a.$$

Applying R_B from the left and R_a from the right on both sides, we obtain

$$R_B(z) E_a R_a(z) = R_B(z) J_a - J_a R_a(z).$$

Taking the adjoint, and then multiplying from the left by J_a , we see that

$$J_a R_a(z) E_a^* R_B(z) = J_a J_a^* R_B(z) - J_a R_a(z) J_a^*.$$

Thus

$$R_B(z) - J_a R_a(z) J_a^* = (J_a R_a(z) E_a^* - J_a J_a^* + 1) R_B(z) \xrightarrow{s} 0,$$

since $E_a^* \xrightarrow{s} 0$ by Lemma 6, and $-J_a J_a^* + 1 \xrightarrow{s} 0$. This proves that the limits above converge to the desired associated bulk operators, and hence $\|\Sigma(a) - \Sigma_B\|_1 \rightarrow 0$.

Finally, it remains to show that

$$\text{Tr}(\Sigma_B) = \sigma_B.$$

First, we manipulate

$$\begin{aligned} \Sigma_B &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z) [H_B, \Lambda_1] R_B(z) [H_B, \Lambda_2] R_B(z) dz^2 \\ &= -\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z) [H_B, \Lambda_1] [R_B(z), \Lambda_2] dz^2 \\ &= i[\rho(H_B), \Lambda_1] \Lambda_2 - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z) [H_B, \Lambda_1] \Lambda_2 R_B(z) dz^2. \end{aligned}$$

Define $P_+ := P((\sup \Delta, \infty))$ and $P_- := P((-\infty, \inf \Delta))$, the projections onto states above and below the gap, respectively. Since H_B is assumed to have a gap, it follows that $P_- + P_+ = \mathbb{1}$, and thus

$$\text{Tr}(\Sigma_B) = \text{Tr}((P_- + P_+) \Sigma_B (P_- + P_+)).$$

Since $P_{\pm} R_B(z)$ and $R_B(z) P_{\pm}$ are analytic on $\text{supp}(\rho(z))$ and $\text{supp}(1 - \rho(z))$, the integral in $P_{\pm} \Sigma_B P_{\pm}$ vanishes by integration by parts (need to add). Thus

$$\text{Tr}(\Sigma_B) = i\text{Tr}(P_+ [\rho(H_B), \Lambda_1] \Lambda_2 P_+) + i\text{Tr}(P_- [\rho(H_B), \Lambda_1] \Lambda_2 P_-),$$

since $[\rho(H_B), \Lambda_1] \Lambda_2$ is trace-class (need to add) which allows us to cycle and cancel the terms with both P_- and P_+ . By the spectral theorem for projection-valued measures 6, if the Fermi energy lies in the gap, $\lambda \in \Delta$, we have

$$\rho(H_B) = \int_{-\infty}^{\infty} \rho(\lambda) dP_{\nu} = \int_{-\infty}^{\lambda} \rho(\lambda) dP_{\nu} = \int_{-\infty}^{\lambda} dP_{\nu} = P_{\lambda}.$$

1.2. Noninteracting Setting

We may therefore replace $\rho(H_B)$ by P_λ , by which we obtain

$$\text{Tr}(\Sigma_B) = i\text{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+) + i\text{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-).$$

Now we must relate this expression to the bulk conductivity. It is an algebraic check that

$$P_\lambda[[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]] = P_\lambda\Lambda_2 P_\lambda^\perp \Lambda_1 P_\lambda - P_\lambda\Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda$$

are equal as operators. The two terms on the right hand side are separately trace-class by Lemma 5, so that the bulk conductivity is given by

$$\begin{aligned} \sigma_B &= i\text{Tr}(P_\lambda\Lambda_2 P_\lambda^\perp \Lambda_1 P_\lambda - P_\lambda\Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda) \\ &= i\text{Tr}(P_\lambda^\perp \Lambda_1 P_\lambda\Lambda_2 P_\lambda^\perp - P_\lambda\Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda). \\ &= \text{Tr}(T_\lambda), \end{aligned}$$

Where we've defined $T_\lambda := P_\lambda^\perp \Lambda_1 P_\lambda\Lambda_2 P_\lambda^\perp - P_\lambda\Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda$. Since both terms of T_λ are separately trace-class, we obtain

$$\sigma_B = \text{Tr}((P_+ + P_-)T_\lambda(P_+ + P_-)) = \text{Tr}(P_+T_\lambda P_+) + \text{Tr}(P_-T_\lambda P_-)$$

by cyclicity of the trace. We now must show that

$$P_\pm T_\lambda P_\pm = P_\pm[P_\lambda, \Lambda_1]\Lambda_2 P_\pm.$$

First, notice that because of the gap, we have $P_\lambda^\perp P_- = 0$, and thus also $P_\lambda P_- = P_-$. Thus

$$\begin{aligned} P_-T_\lambda P_- &= P_-P_\lambda\Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda P_- \\ &= P_-(P_\lambda\Lambda_1\Lambda_2 - \Lambda_1 P_\lambda\Lambda_2)P_- \\ &= P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-, \end{aligned}$$

and similarly, for P_+ , we have $P_\lambda^\perp P_+ = P_+$, and $P_\lambda P_- = 0$, which implies

$$\begin{aligned} P_+T_\lambda P_+ &= -P_+P_\lambda^\perp \Lambda_1 P_\lambda\Lambda_2 P_\lambda^\perp P_+ \\ &= -P_+P_\lambda^\perp \Lambda_1 P_\lambda\Lambda_2 P_+ \\ &= -P_+P_\lambda^\perp \Lambda_1\Lambda_2 P_+ + P_+P_\lambda^\perp \Lambda_1 P_\lambda^\perp \Lambda_2 P_+ \\ &= -P_+P_\lambda^\perp \Lambda_1\Lambda_2 P_+ + P_+\Lambda_1 P_\lambda^\perp \Lambda_2 P_+ \\ &= -P_+[P_\lambda^\perp, \Lambda_1]\Lambda_2 P_+ \\ &= -P_+[(1 - P_\lambda), \Lambda_1]\Lambda_2 P_+ \\ &= P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+. \end{aligned}$$

1.2. Noninteracting Setting

Finally, we obtain

$$\begin{aligned}\sigma_B &= i\text{Tr}(P_- T_\lambda P_-) + i\text{Tr}(P_+ T_\lambda P_+) \\ &= i\text{Tr}(P_- [P_\lambda, \Lambda_1] \Lambda_2 P_-) + i\text{Tr}(P_+ [P_\lambda, \Lambda_1] \Lambda_2 P_+) \\ &= \text{Tr}(\Sigma_B),\end{aligned}$$

concluding the proof.

Lemma 5. *Let $W, V = \emptyset$. Then*

$$P_W \Lambda_1 P_V \Lambda_2 P_W \in \mathcal{J}_1.$$

In particular, the two terms appearing in T_λ are separately trace-class.

Proof. We break down

$$P_W \Lambda_1 P_V \Lambda_2 P_W = P_W \Lambda_1 P_V e^{3\delta|x_1|} e^{-\delta|x|} \cdot e^{-\delta|x|} \cdot e^{-\delta|x|} e^{3\delta|x_2|} P_V \Lambda_2 P_W.$$

Since the middle term is trace-class by Lemma 4, it suffices to prove boundedness of the first and last terms.

To that end, notice that $P_W \Lambda_i P_V = P_W [\Lambda_i, P_V]$. Since P_W is obviously bounded, we only need to bound the product of the commutator with the appropriate exponentials. This can be done by decomposing

$$[\Lambda_i, P_V] e^{3\delta x_i} e^{-\delta|x|} = \Lambda_i P_V (1 - \Lambda_i) e^{3\delta x_i} e^{-\delta|x|} + (1 - \Lambda_i) P_V \Lambda_i e^{3\delta x_i} e^{-\delta|x|}.$$

Note the lack of absolute values on the $e^{3\delta x_i}$ terms; we will later also bound the same operator but with $e^{-3\delta x_i}$ to account for this, since $e^{3\delta|x_i|} = e^{\pm 3\delta x_i}$. We begin by bounding the second term. Since multiplication operators commute, the norm of the second term can be broken into

$$\|(1 - \Lambda_i) P_V e^{-\delta|x|}\| \|\Lambda_i e^{3\delta x_i}\|.$$

Now, the first term is clearly bounded, and $\Lambda_i e^{3\delta x_i}$ is also bounded; only the negative x_i values remain due to the switch function.

For the first term, notice that (need to add).

□

Lemma 6. *E_a and E_a^* converge strongly to zero in the limit $a \rightarrow \infty$.*

1.3. Interacting Setting

Proof. Let $\psi \in \ell^2(\mathbb{Z}^2)$. Since E_a has real entries,

$$\begin{aligned} \|E_a^* \psi\|^2 &= \langle E_a^* \psi, E_a^* \psi \rangle \\ &= \sum_z \left(\sum_y \overline{E_a^*(z, y) \psi(y)} \right) \left(\sum_x E_a^*(z, x) \psi(x) \right) \\ &= \sum_z \left(\sum_y E_a(y, z) \overline{\psi(y)} \right) \left(\sum_x E_a(x, z) \psi(x) \right) \end{aligned}$$

Consider the $\sum_x E_a(x, z) \psi(x)$ term. To take advantage of the edge assumption, we insert the exponentials

$$\sum_x E_a(x, z) e^{\alpha(|z_2 - a| - |z_1 - x_1|)} e^{-\alpha(|z_2 - a| - |z_1 - x_1|)} \psi(x).$$

By assumption 2, $g(z) := \sup_x E_a(x, z) e^{\alpha(|z_2 - a| - |z_1 - x_1|)}$ is finite and summable. Precisely the same argument can be used on the sum over y . Altogether,

$$\|E_a^* \psi\|^2 \leq e^{-\alpha|a|} \sum_z g(z)^2 e^{-2\alpha(|z_1| + |z_2|)} \sum_x e^{-\alpha|x_1|} \psi(x) \sum_y e^{-\alpha|y_1|} \overline{\psi(y)},$$

where we bounded the exponential by $e^{-\alpha(|z_1| + |z_2|)} e^{-\alpha|x_1|} e^{-\alpha|a|}$. Since g is summable, g^2 is also summable, and so too is the summand over z . The sums over x and y are clearly finite, as they are bounded by the summable state $|\psi|$. Thus E_a^* converges strongly to zero. An exactly analogous argument applies for E_a . □

1.3 Interacting Setting

Let $L \in \mathbb{N}$, and let $\Gamma_L = \mathbb{Z}_L \times [0, L]$ be the discrete cylinder, equipped with a metric d . To each site $x \in \Gamma_L$, we associate a Hilbert space \mathcal{H}_x whose dimension is bounded uniformly in L . We denote $N = \sup_L \mathcal{H}_L$. For a subset $X \subseteq \Gamma_L$, we define the Hilbert space $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$, and we set

$$\mathcal{H}_L := \mathcal{H}_{\Gamma_L} = \bigotimes_{x \in \Gamma_L} \mathcal{H}_x.$$

The algebra $\mathcal{U}_X \subset \mathcal{B}(\mathcal{H}_X)$ of observables on \mathcal{H}_X is the set of bounded self-adjoint operators supported in X . For an operator $A_X \in \mathcal{U}_X$, we identify

1.3. Interacting Setting

its extension to an operator on \mathcal{H}_L by taking its tensor product with copies of the identity, $(\otimes_{x \in X^c} \mathbb{I}_x) \otimes A_X$. Conversely, we say that an operator $A \in \mathcal{U}_L$ has support X if $A_X := (\otimes_{x \in X^c} \mathbb{I}_x) \otimes (A|_X)$ is equal to A , and write $A_X \in \mathcal{U}_X$. For ease of notation, we omit the subscript L wherever there is no risk of confusion.

A *local interaction* is a map $\Phi : \mathcal{P}(\Gamma_L) \rightarrow \mathcal{U}_L$ such that

1. $\Phi(X) = 0$ whenever $\text{diam}(X) > R$ for some $R > 0$.
2. $\Phi(X)$ is supported in X .
3. $\|\Phi(X)\| \leq C$ for all $X \subset \Gamma_L$, for all L .

We consider a region as depicted in Figure 1.2, with the left and right edges joined together to form a cylinder. In the left white region $[0, L/2] \times [0, L]$, H_0 is a trivial Hamiltonian which we take to be empty space (we take $H_0 = 0$), and in the right blue region $[L/2, L] \times [0, L]$, H_1 is a *local Hamiltonian*, in the sense that $H_1 = \sum_{X \subset \Gamma_L} \Phi(X)$, is a sum of local interactions. We define the Hamiltonian of the full system to be

$$H_\mu = H_1 + \mu Q_h,$$

where $Q_h = \sum_{x \in \Gamma_h} a_x^* a_x$ is the number operator for the region $\Gamma_h = [L/4, 3L/4] \times [0, L]$ shown in red. This introduces a driving strength; the μQ_h term can be viewed as a potential difference $V(x)$.

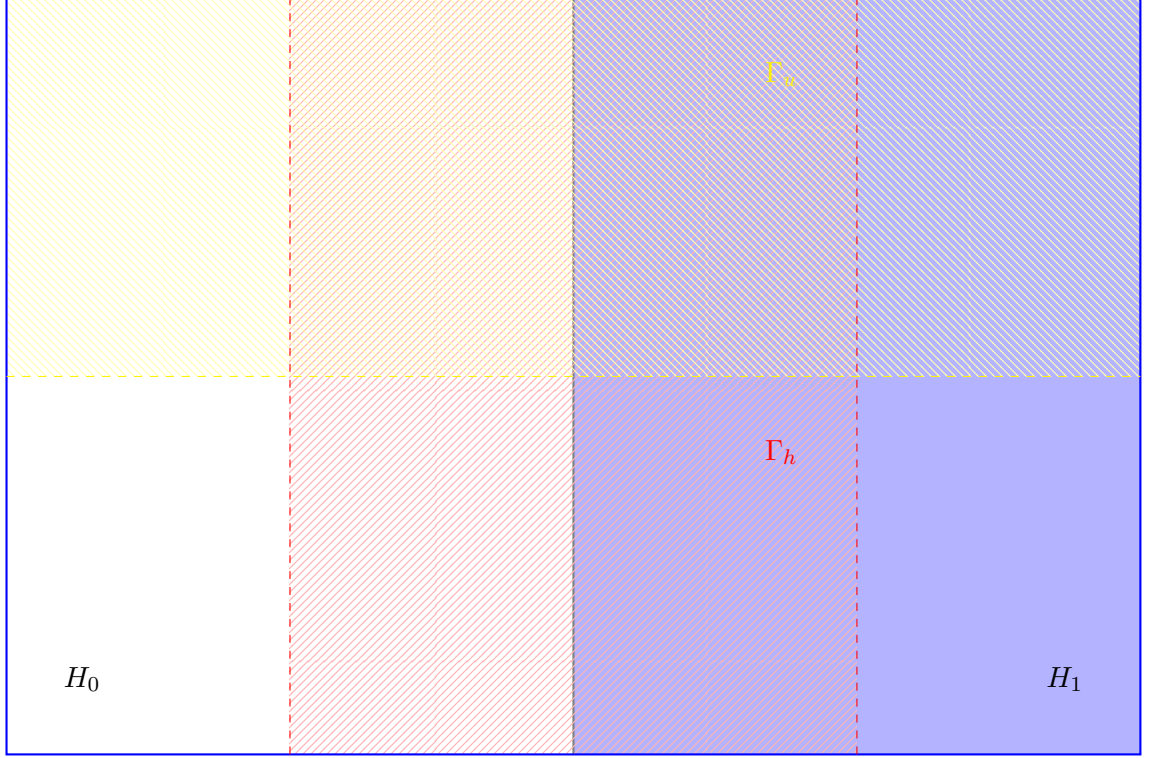


Figure 1.2: The cylinder Γ_L .

We also consider the plane \mathbb{Z}^2 . In this setting, there are no edge states, and so the associated “bulk” Hamiltonian H_B is assumed to have a *gapped* spectrum, in the sense that

Assumption 4.

$$\text{Spec}(H_B) = \mathcal{S}_- \cup \mathcal{S}_+,$$

where $\inf \mathcal{S}_+ - \sup \mathcal{S}_- \geq \gamma$ uniformly in L and μ for some $\gamma > 0$.

In the case of the cylinder, this effect does not necessarily occur due to the presence of the edge. We also assume that the Hamiltonian is *locally charge-conserving*.

Assumption 5. $[\Phi(X), Q] = 0$, where Q is the total charge in Γ_L .

Let P_B be the ground state projection of H_B (the system without an edge), and let P be the ground state projection of H (the system with an edge). We assume that states far from the edge are essentially bulk states, up to tails that vanish quickly in L .

1.3. Interacting Setting

Assumption 6. *Define the edge region*

$$\Gamma_E = [L/2 - k, L/2 + k] \times [0, L] \cup [L - k, k] \times [0, L].$$

for some $k > 0$. For any operator A supported on Γ_E^c ,

$$\text{Tr}(PA) = \text{Tr}(P_B A) + \mathcal{O}(L^{-\infty}).$$

The A on the right hand side is understood to be the extension by zeroes of A to the plane \mathbb{Z}^2 .

The idea is that observables localized far away from the edge are not affected by the edge of the system. We similarly define the *bulk region*

$$\Gamma_B = [3L/4 - k, 3L/4 + k] \times [0, L],$$

and the *middle region*

$$\Gamma_m = [L/2, L] \cup [0, L] \setminus (\Gamma_E \cup \Gamma_B).$$

The three regions are depicted in figure 1.3.

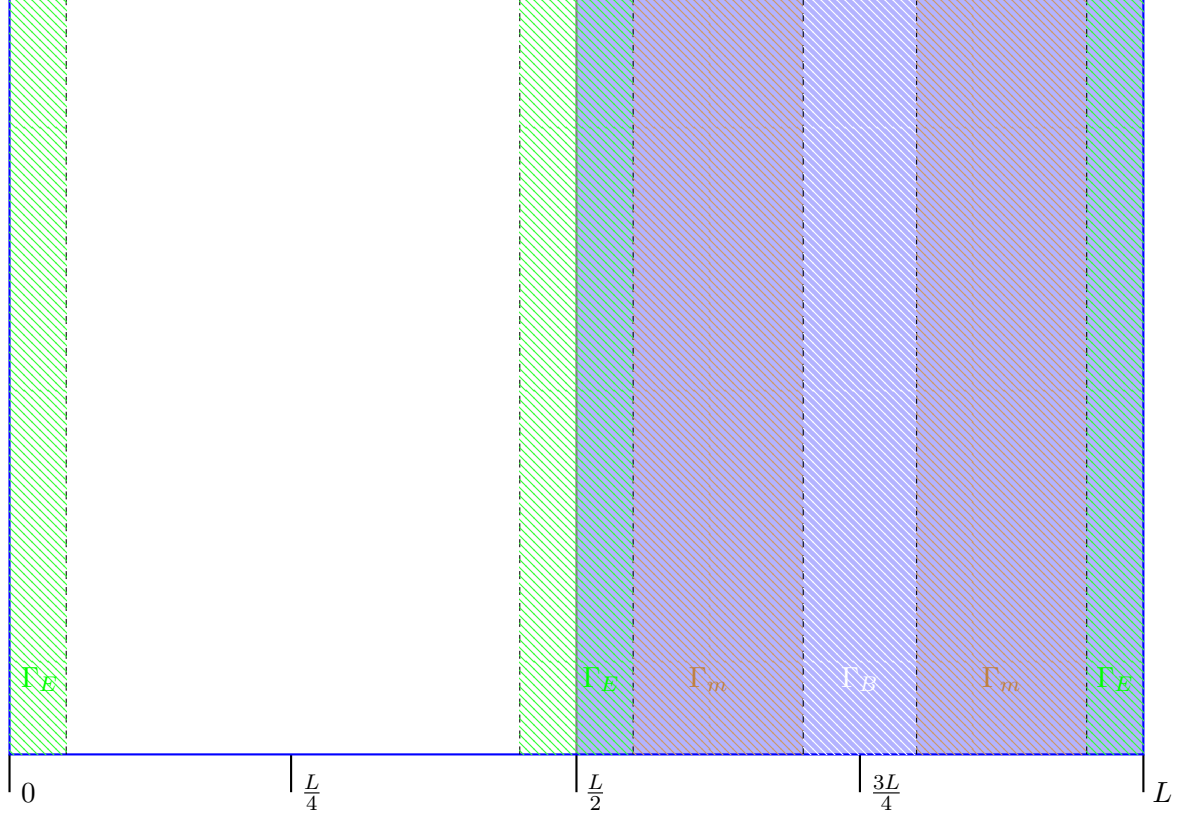


Figure 1.3: The regions Γ_E , Γ_B , and Γ_m .

1.3.1 Equality of Bulk and Edge Currents

Cylinder Geometry

Let P_μ be the (possibly degenerate) ground state projection of H_μ . Let $Q_u = \sum_{x \in \Gamma_u} a_x^* a_x$ be the charge in the upper half of the cylinder $\Gamma_u = [0, L] \times [L/2, L]$ (the yellow region in Figure 1.2), and define current operator

$$J = i[H_\mu, Q_u],$$

which measures the current across the fiducial line $y = L/2$. Charge conservation 5 implies that this current operator is supported along a strip of width $2R$ centred on the fiducial line $y = L/2$. Indeed, if we inspect a local interaction $\Phi(X)$ of range R with support $(\Gamma_u)_R$, where $(X)_\alpha$ is the α -shrinking of the set X , then clearly $\Phi(X)$ commutes with the charge

outside Γ_u , so that $[\Phi(X), Q_u] = [\Phi(X), Q]$, which vanishes by the charge conservation assumption 5. Similarly, if $\Phi(X)$ is supported in $((\Gamma_u)^c)_R$, then $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$. It follows that for an interaction $\Phi(X)$ with range R and arbitrary support, $[\Phi(X), Q_u]$ must be supported on a set which is contained in (or equal to) the strip $[L/2, L] \times [L/2 - R, L/2 + R]$. There $[H_\mu, Q_u]$ must be supported there as well, since H_μ is a sum of such local interactions.

From this point, we drop the subscript μ wherever it is not needed for context.

Lemma 7. *The ground state expectation of the current J is zero.*

Proof. Assuming linearity and cyclicity of the trace hold, the proof is trivial,

$$\text{Tr}(PJ) = i\text{Tr}(P[H, Q_u]) = i\text{Tr}([P, H]Q_u) = 0.$$

In order for this calculation to hold, we need to prove that

1. PHQ_u and PQ_uH are separately trace-class to apply linearity of the trace, and
2. $\|H\| < \infty$ and $PQ_u \in \mathcal{J}_1$ to apply cyclicity of the trace.

The latter implies the former by the bound $\|AB\|_1 \leq \|A\|_1\|B\|$. To prove (2), fix a finite L . The Hamiltonian is bounded since it is a finite sum of at most $\mathcal{P}(\Gamma_L)$ local interactions $\Phi(X)$, each of which is uniformly bounded by assumption, along with the μQ_h term. But the number operator for the entire space is bounded by $\|Q\| \leq NL^2$, where N is the uniform bound on the dimension of each Hilbert space. This shows that both Q_u and Q_h are bounded in operator norm. Finally, $\|P\|_1 \leq CL^2$ because the projection is finite-rank, since the dimension of each site is bounded. Therefore $PQ_u \in \mathcal{J}_1$. □

Next, we define a family of operators indexed by μ called *Hastings operators*,

$$K_\mu = \mathcal{I}_\mu(\dot{H}_\mu),$$

where

$$\mathcal{I}_\mu(A) = \int_{\mathbb{R}} W(t) e^{itH_\mu} A e^{-itH_\mu} dt.$$

1.3. Interacting Setting

Here, $W : \mathbb{R} \rightarrow \mathbb{R}$ is a function satisfying (need to add). More explicitly, in our setting we see that

$$K_\mu = \mathcal{I}_\mu(Q_h).$$

We present two important properties of the map $\mathcal{I}_\mu : \mathcal{U}_L \rightarrow \mathcal{U}_L$ in the following lemmas, and leave their proofs to the appendix (need to add).

First, recall a definition from the non-interacting setting: an *off-diagonal* operator is an operator A such that $A = \bar{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, where $P_\mu^\perp = \mathbb{I} - P_\mu$ is the projection onto the excited states above the gap.

Lemma 8. *1. For any off-diagonal operator $A = \bar{A}$, $\mathcal{I}_\mu(\cdot)$ and $[H_\mu, \cdot]$ act as inverses of each other, up to a factor of i :*

$$\mathcal{I}_\mu([H_\mu, A]) = [H_\mu, \mathcal{I}_\mu(A)] = iA.$$

2. For any (not necessarily off-diagonal) operator A ,

$$[\mathcal{I}_\mu([H_\mu, A]), P_\mu] = i[A, P_\mu].$$

Another important property of the map \mathcal{I}_μ is that it preserves locality.

Lemma 9. *\mathcal{I}_μ is local in the sense that for any $A \in \mathcal{U}_X$,*

$$\|\mathcal{I}(A)_{(X^r)^c}\| \leq \|A\| |X| \mathcal{O}(r^{-\infty})$$

where $X^r = X \cup \{x : d(x, X) \leq r\}$ is the r -fattening of X .

Proposition 2. *The operator K_μ is the generator of parallel transport, satisfying*

$$\dot{P}_\mu = i[K_\mu, P_\mu]$$

for all μ .

Proof. First, we show that \dot{P} is off-diagonal. Taking the derivative on both sides of $P^2 = P$, we see that $\dot{P}P + P\dot{P} = \dot{P}$. Acting on the left and right with P on both sides of this equation gives

$$P\dot{P}P + P\dot{P}P = P\dot{P}P,$$

which implies that $P\dot{P}P = 0$. Thus

$$\begin{aligned}
\overline{\partial_\mu P} &= P\dot{P}(1-P) + (1-P)\dot{P}P \\
&= P\dot{P} - P\dot{P}P + \dot{P}P - P\dot{P}P \\
&= P\dot{P} + \dot{P}P \\
&= \partial_\mu(P^2) \\
&= \partial_\mu P,
\end{aligned}$$

as claimed. By the product rule and the fact that H and P commute,

$$[\dot{H}, P] = -[H, \dot{P}].$$

It therefore follows from Lemma 8 that

$$\dot{P} = -i\mathcal{I}_\mu([H, \dot{P}]) = i\mathcal{I}([\dot{H}, P]) = i[\mathcal{I}(\dot{H}), P] = i[K, P].$$

□

Increasing the electric potential by a small amount $d\mu Q_h$ and expanding to linear order, the change in ground state current is given by

$$\text{Tr}(P_{\mu+d\mu}J) - \text{Tr}(P_\mu J) = \kappa d\mu + \mathcal{O}(d\mu^2).$$

Dividing by $d\mu$ and taking a limit, we see that the linear response coefficient is given by

$$\sigma(\mu) = \text{Tr}(\dot{P}_\mu J).$$

The *Hall conductivity* of the system on a subset $V \subseteq \Gamma_L$ is defined to be $\sigma_V := \text{Tr}(\dot{P}J_V)$, where J_V is the restriction of J to V .

Proposition 3. *The Hall conductivity is independent of the driving strength μ .*

Proof. For any μ_1 and μ_2 ,

$$\begin{aligned}
\sigma(\mu_1) - \sigma(\mu_2) &= \text{Tr}(\dot{P}_{\mu_1}i[H_{\mu_1}, Q_u] - \dot{P}_{\mu_2}i[H_{\mu_2}, Q_u]) \\
&= i\text{Tr}([[\dot{P}_{\mu_1}, H_{\mu_1}] - [\dot{P}_{\mu_2}, H_{\mu_2}]]Q_u) \\
&= -i\text{Tr}([\dot{H}_{\mu_1}, P_{\mu_1}] - [\dot{H}_{\mu_2}, P_{\mu_2}])Q_u \\
&= i\text{Tr}([Q_h, P_{\mu_1} - P_{\mu_2}]Q_u) \\
&= i\text{Tr}([Q_u, Q_h](P_{\mu_1} - P_{\mu_2})) \\
&= 0,
\end{aligned}$$

1.3. Interacting Setting

since H and P commute. Note that $\|\dot{P}\|_1 < \infty$ since we are working in a finite-dimensional space. The proof of Lemma 7 provides the other necessary bounds to invoke linearity and cyclicity of the trace to shift the commutator in the second line and second-last line. \square

This indicates that the Hall conductivity is independent of μ as one would expect physically. We simply write $\sigma = \sigma(\mu)$ from this point, in accordance with proposition 3.

The following is the main result:

Theorem 2. *Let $V \subseteq \Gamma_m$ be a set contained within the strip in between the edge region Γ_E and the bulk region Γ_B (see Figure 1.3), and define the distance*

$$r = \text{dist}(V, \Gamma_E \cup \Gamma_B)$$

from V to the bulk and edge regions. The Hall conductivity in this regions vanishes in the sense that

$$\sigma_V = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

Proof. By Proposition 2, the bulk Hall conductivity can also be written by the formula

$$\sigma_V^B = \text{Tr} (i[K, P_B]J_V^B) = \text{Tr} (i[\mathcal{I}(Q_h), P_B]J_V^B),$$

where $J_V^B = i[H_B, Q_u]|_V$ is the current in the region V arising from the bulk Hamiltonian. From commutativity of P_B and H_B along with cyclicity of the trace, we compute

$$\begin{aligned} \sigma_V^B &= \text{Tr} \left(i \int_{\mathbb{R}} W(t) e^{itH_B} [Q_h, P_B] e^{-itH_B} dt J_V^B \right) \\ &= \int_{\mathbb{R}} W(t) \text{Tr} (i[Q_h, P_B] e^{-itH_B} J_V^B e^{itH_B}) dt \\ &= - \int_{\mathbb{R}} W(t) \text{Tr} (i[Q_h, P_B] e^{itH_B} J_V^B e^{-itH_B}) dt \\ &= -\text{Tr} (i[Q_h, P_B] \mathcal{I}(J_V^B)), \end{aligned}$$

since $W(t)$ is odd. By part (2) of Lemma 8, we have $i[Q_h, P_B] = [\mathcal{I}([H_B, Q_h]), P_B]$. Therefore

$$\begin{aligned} \sigma_V^B &= -\text{Tr}([\mathcal{I}([H_B, Q_h]), P_B] \mathcal{I}(J_V^B)) \\ &= \text{Tr} (P_B [\mathcal{I}([H_B, Q_h]), \mathcal{I}(J_V^B)]) . \end{aligned}$$

1.3. Interacting Setting

Now, $[H_B, Q_h]$ is a local operator supported on Γ_B , while J_V^B is a local operator supported on $V \cap \Gamma_B = \emptyset$. Since \mathcal{I} preserves locality up to tails, in the sense that $\|\mathcal{I}(A)_{(S^r)^c}\| \leq \|A\| \|S\| \mathcal{O}(r^{-\infty})$ for any operator A supported in S (Lemma 9), it follows that the commutator can be written

$$[\mathcal{I}([H_B, Q_h])|_{\Gamma_B} + \mathcal{O}(r^{-\infty})A_1, \mathcal{I}(J_V^B)|_V + \mathcal{O}(r^{-\infty})A_2] = C\mathcal{O}(r^{-\infty}),$$

for some operators A_1 and A_2 supported on Γ_B^c and V^c , respectively. This fact applies to the bulk setting with H_B and P_B . To extend this to the setting with an edge, it is enough to use Assumption 6 to conclude the same result, except with equality up to $\mathcal{O}(L^{-\infty})$, i.e.

$$\sigma_V = \text{Tr} \left(\dot{P} J_V \right) = \text{Tr} \left(\dot{P} (J_V^B + \mathcal{O}(L^{-\infty})) \right) = \sigma_V^B + \mathcal{O}(L^{-\infty}) = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

□

The intuitive picture from the previous result is that, in the bulk region, the Hall conductivity is essentially only nonzero along the bulk strip Γ_B . Since the ground state expectation of the current is zero (by lemma 7), it must be that there is an equal current flowing along the edge strip Γ_E , but in the opposite direction.

Torus Geometry

Our goal is to show the same result on the discrete torus $\mathbb{T}_L := \mathbb{Z}_L \times \mathbb{Z}_L$. We define the same regions Γ_u and Γ_h , and the same current operator $J_u = i[H(\mu), Q_u]$. This time, however, Lemma 7 does not apply. Intuitively, it does not apply because electrons can now flow through both the bottom and the top of the region Γ_u , rather than just the bottom. Mathematically, the lemma fails because our definition of the current is slightly changed.

We use charge conservation and the fact that H is finite range to split the current J_u into two components, $J_u = i[H_-, Q_u] + i[H_+, Q_u] = J_- - J_+$, supported on strips of width $2R$ at $y = L/2$ and $y = L$, respectively. We then define the current operator to be $J = J_-$, which is the current on the lower strip. This is the mathematical reason that the proof in Lemma 7 fails on the torus; we have replaced H by H_- , which may no longer commute with P . We instead proceed by a different approach. We will need a few auxiliary results first.

Lemma 10. K_{\pm} is supported on ∂_{\pm} up to tails.

1.3. Interacting Setting

Proof. □

Proposition 4. *The operator $Q_h - K$ leaves the ground state space invariant, i.e. $[Q_h - K, P] = 0$.*

Proof. □

Lemma 11. *Show that $\text{Tr}(A, [Q_h, P]) = 0$ for all $A \in \mathcal{U}_{\text{edge}}$. This shows that Q_h commutes with P “along the edge”.*

Proof. Let $A \in \mathcal{U}_{\text{edge}}$. Since H is charge conserving, we may choose a simultaneous eigenbasis of H and the total charge Q , in which case P and Q commute. It follows that

$$\text{Tr}(A[Q_h, P]) = \text{Tr}([A, Q_h]P) = \text{Tr}([A, Q]P) = \text{Tr}(A[Q, P]) = 0.$$

□

Finally, we will prove that in the bulk system with Hamiltonian $H_B(\mu)$, the ground state expectation of the current vanishes faster than any power as $L \rightarrow \infty$.

Lemma 12. *The ground state expectation of the current $J_B := i[(H_B)_-, Q_h]$ (of the system without an edge) is $\text{Tr}(P_B J_B) = \mathcal{O}(L^{-\infty})$.*

Proof. First, $K = \mathcal{I}(i[H_B, Q])$ splits into $K = K_- - K_+$, with the support of K_{\pm} contained in ∂_{\pm} up to tails:

$$[K_{\pm}, A_X] = \mathcal{O}(p^{-\infty}),$$

for every $A_X \in \mathcal{U}_X$ such that $\|A_X\| = 1$, and where $p = \text{dist}(X, \partial_{\pm})$ (need to add). Using the fact that K_{\pm} is supported in ∂_{\pm} up to tails (Lemma 10), we see that

$$i[H_B, K_-] = i[(H_B)_-, K_-] + \mathcal{O}(L^{-\infty}),$$

and similarly $i[(H_B)_-, K_+] = \mathcal{O}(L^{-\infty})$. Putting these facts together, it follows that the current can be rewritten as

$$\begin{aligned} J_B &= i[H_B, Q_h + K_- - K_- + K_+] + \mathcal{O}(L^{-\infty}) \\ &= i[H_B, K_-] + i[(H_B)_-, Q_h - K_- + K_+] + \mathcal{O}(L^{-\infty}). \end{aligned}$$

From here, we use the fact that H_B and $Q_h - K_- + K_+$ both commute with P_B to write

$$P_B J_B P_B = i[H_B, P_B K_- P_B] + i[P_B(H_B)_- P_B, Q_h - K_- + K_+] + P_B \mathcal{O}(L^{-\infty}) P_B.$$

Since the trace of any commutator is zero,

$$\mathrm{Tr}(P_B J_B) = \mathrm{Tr}(P_B J_B P_B) = \mathcal{O}(L^{-\infty}).$$

□

Using this, we can show a simple proof of the analogue of Lemma 7 on the torus, in the case of non-interacting systems.

Proposition 5. *Let $H = \sum_{x \in \mathbb{T}} h_x$ be a non-interacting Hamiltonian, i.e. a sum of single site Hamiltonians h_x . The ground state expectation of the current $J = i[H_-, Q_h]$ (of the system with an edge) is $\mathrm{Tr}(PJ) = \mathcal{O}(L^{-\infty})$.*

Proof. Since H is a sum of single site Hamiltonians, we can split H_- into the restrictions $H_- = (H_-)_{\text{edge}} + (H_-)_{\text{bulk}}$, with no fear of any terms which are in both the edge region and the bulk region. By Assumption 6,

$$\begin{aligned} \mathrm{Tr}(PJ) &= \mathrm{Tr}(Pi[H_-, Q_h]) \\ &= i\mathrm{Tr}([H_-, Q_h]P) \\ &= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\text{bulk}}[Q_h, P]) \\ &= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\text{bulk}}[Q_h, (P)_{\text{bulk}}]) \\ &= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_B)_-[Q_h, P_B]) + \mathcal{O}(L^{-\infty}) \\ &= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + \mathrm{Tr}(i[(H_B)_-, Q_h]P_B) + \mathcal{O}(L^{-\infty}). \end{aligned}$$

By Lemma 11, the first term is zero. By Lemma 12, the second term is $\mathcal{O}(L^{-\infty})$. □

1.4 A Section

Here is a section with some text. Equations look like this $y = x$.¹

This is an example of a second paragraph in a section so you can see how much it is indented by.

¹Here is a footnote.

Here is an example of a citation: [?]. The actual form of the citation is governed by the `bibliographystyle`. These citations are maintained in a BibTeX file `sample.bib`. You could type these directly into the file. For an example of the format to use look at the file `ubcsample.bbl` after you compile this file.²

This is a Subsubsection

This is an example of a second paragraph in a subsubsection so you can see how much it is indented by.

This is a Subparagraph This is the last level of organisation. If you need more than this, you should consider reorganizing your work...

$$f(x) = \int_{-\infty}^x e^{-\frac{y^2}{2}} dy \int_{-\infty}^x e^{-z^2} dz \quad (1.1)$$

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32

1.4. A Section

Phoenix	\$960.35
Calgary	\$250.00

Table 1.1: Here is the caption for this wonderful table. It has not been centered and the positioning has been specified to be at the top of the page. Thus it appears above the babble rather than below where it is defined in the source file.

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1.5 Quote

Here is a quote:

This is a small poem,
a little poem, a Haiku,
to show you how to.
—Michael McNeil Forbes.

This small poem shows several features:

- The use of the `quote` and `center` environments.
- The `\newpage` command has been used to force a page break. (Sections do not usually start on a new page.)
- The `pagestyle` has been set to suppress the headers using the command `\thispagestyle{plain}`. Note that using `\pagestyle{plain}` would have affected all of the subsequent pages.

1.6 Programs

Here we give an example of a new float as defined using the `float` package. In the preamble we have used the commands

```
\floatstyle{ruled}  
\newfloat{Program}{htbp}{lop}[chapter]
```

This creates a “Program” environment that may be used for program fragments. A sample `python` program is shown in Program 1.1. (Note that Python places a fairly restrictive limit on recursion so trying to call this with a large n before building up the cache is likely to fail unless you increase the recursion depth.) Instead of using a `verbatim` environment for your program chunks, you might like to `include` them within an `alltt` environment by including the `\usepackage{alltt}` package (see page 187 of the *L^AT_EX* book). Another useful package is the `\usepackage{listings}` which can pretty-print many different types of source code.

Program 1.1 Python program that computes the n^{th} Fibonacci number using memoization.

```
def fib(n, _cache={}):
    if n < 2:
        return 1
    if n in _cache:
        return _cache[n]
    else:
        result = fib(n-1)+fib(n-2)
        _cache[n] = result
    return result
```

Chapter 2

Another Chapter with a Very Long Chapter-name that will Probably Cause Problems

This chapter name is very long and does not display properly in the running headers or in the table of contents. To deal with this, we provide a shorter version of the title as the optional argument to the `\chapter[]{}{}` command.

For example, this chapter's title and associated table of contents heading and running header was created with
`\chapter[Another Chapter\ldots]{Another Chapter with a Very Long Chapter-name that will Probably Cause Problems}`.

Note that, according to the thesis regulations, the heading included in the table of contents must be a truncation of the actual heading.

This Chapter was used as a demonstration in the Preface for how to attribute contribution from collaborators. If there are any such contributions, details must be included in the Preface. If you wish, you may additionally use a footnote such as this.³

2.1 Another Section

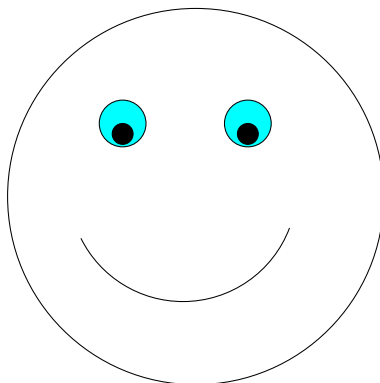
Another bunch of text to demonstrate what this file does. You might want a list for example:⁴

- An item in a list.
- Another item in a list.

³This chapter is based on work conducted in UBC's Maple Syrup Laboratory by Dr. A. Apple, Professor B. Boat, and C. Cat.

⁴Here is a footnote in a different chapter. Footnotes should come after punctuation.

An Unnumbered Section That is Not Included in the Table of Contents



pie makes me happy!

Figure 2.1: This is a figure of a happy face with a `psfrag` replacement. The original figure (drawn in `xfig` and exported to a `.eps` file) has the text “pie makes me happy!”. The `psfrag` package replaces this with “ π makes me happy!”. Note: the Makefile compiles the sample using `pdfLATEX` which cannot use `psfrag` directly. For some options that work with `pdfLATEX`, please see this discussion: <http://tex.stackexchange.com/questions/11839>. For the caption, we have used the optional argument for the caption command so that only a short version of this caption occurs in the list of figures.

Here is an example of a figure environment. Perhaps I should say that the example of a figure can be seen in Figure 2.1. Figure placement can be tricky with `LATEX` because figures and tables are treated as “floats”: text can flow around them, but if there is not enough space, they will appear later. To prevent figures from going too far, the `\afterpage{\clearpage}` command can be used. This makes sure that the figure are typeset at the end of the page (possibly appear on their own on the following pages) and before any subsequent text.

The `\clearpage` forces a page break so that the figure can be placed, but without the the `\afterpage{\clearpage}` command, the page would be broken too early (at the `\clearpage` statement). The `\afterpage{\clearpage}` command tells `LATEX` to issue the command after the present page has been rendered.

2.2 Tables

We have already included one table: 1.1. Another table is plopped right here. Well, actually, as with Figures, tables do not necessarily appear right

	Singular		Plural	
	English	Gaeilge	English	Gaeilge
1st Person	at me	agam	at us	againn
2nd Person	at you	agat	at you	agaibh
3rd Person	at him	aige	at them	acu
	at her	aici		

Table 2.1: Another table.

“here” because tables are also “floats”. \LaTeX puts them where it can. Because of this, one should refer to floats by their labels rather than by their location. This example is demonstrated by Table 2.1. This one is pretty close, however. (Note: you should generally not put tables or figures in the middle of a paragraph. This example is for demonstration purposes only.)

Another useful package is `\usepackage{longtable}` which provides the `longtable` environment. This is nice because it allows tables to span multiple pages. Table 2.2 has been formatted this way.

Table 2.2: Feasible triples for highly variable Grid

Time (s)	Triple chosen	Other feasible triples
0	(1, 11, 13725)	(1, 12, 10980), (1, 13, 8235), (2, 2, 0), (3, 1, 0)
274	(1, 12, 10980)	(1, 13, 8235), (2, 2, 0), (2, 3, 0), (3, 1, 0)
5490	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
8235	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
10980	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
13725	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
16470	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
19215	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
21960	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
24705	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
27450	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
Continued on next page		

Table 2.2 – continued from previous page

Time (s)	Triple chosen	Other feasible triples
30195	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
32940	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
35685	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
38430	(1, 13, 10980)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
41175	(1, 12, 13725)	(1, 13, 10980), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
43920	(1, 13, 10980)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
46665	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
49410	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
52155	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
54900	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
57645	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
60390	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
63135	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
65880	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
68625	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
71370	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
74115	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
76860	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
79605	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
82350	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
85095	(1, 12, 13725)	(1, 13, 10980), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
87840	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
90585	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
93330	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
96075	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
98820	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
101565	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
104310	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
107055	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
109800	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
112545	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
115290	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
118035	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
120780	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
123525	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
126270	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)

Continued on next page

Table 2.2 – continued from previous page

Time (s)	Triple chosen	Other feasible triples
129015	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
131760	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
134505	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
137250	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
139995	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
142740	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
145485	(1, 12, 16470)	(1, 13, 13725), (2, 2, 2745), (2, 3, 0), (3, 1, 0)
148230	(2, 2, 2745)	(2, 3, 0), (3, 1, 0)
150975	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
153720	(1, 12, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
156465	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
159210	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
161955	(1, 13, 16470)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)
164700	(1, 13, 13725)	(2, 2, 2745), (2, 3, 0), (3, 1, 0)

An Unnumbered Subsection

Note that if you use subsections or further divisions under an unnumbered section, then you should make them unnumbered as well otherwise you will end up with zeros in the section numbering.

Chapter 3

Landscape Mode

The landscape mode allows you to rotate a page through 90 degrees. It is generally not a good idea to make the chapter heading landscape, but it can be useful for long tables etc.

This text should appear rotated, allowing for formatting of very wide tables etc. Note that this might only work after you convert the `dvi` file to a postscript (`ps`) or `pdf` file using `dvips` or `dvipdf` etc. This feature is provided by the `lscape` and the `pdflscape` packages. The latter is preferred if it works as it also rotates the pages in the `pdf` file for easier viewing.

Appendix A

General Functional Analysis

Lemma 13. *Let A be a bounded linear operator on a Hilbert space \mathcal{H} . Suppose $A_n \xrightarrow{s} A$ on a dense subspace $\mathcal{D} \subset \mathcal{H}$. If A_n are bounded uniformly in n , then $A_n \xrightarrow{s} A$ on all of \mathcal{H} .*

Proof. Let $\psi_n \in \mathcal{D}$ be a sequence converging in norm to $\psi \in \mathcal{H}$. The result follows from a standard $\frac{\varepsilon}{3}$ argument. Let C be a bound for both $\sup_n \|A_n\|$ and $\|A\|$. Then

$$\begin{aligned} \|A_n\psi - A\psi\| &\leq \|A_n(\psi - \psi_m)\| + \|(A_n - A)\psi_m\| + \|A(\psi - \psi_m)\| \\ &\leq C\|\psi - \psi_m\| + \|(A_n - A)\psi_m\| + C\|\psi - \psi_m\|. \end{aligned}$$

There exists an M such that the first and third terms are less than $\frac{\varepsilon}{3}$ for all $m > M$. For the middle term, observe that for each m , there exists by hypothesis an N_m such that $\|(A_n - A)\psi_m\| < \frac{\varepsilon}{3}$ for all $n > N_m$. Thus, by picking some $m > M$ and fixing a suitably large n , the inequality above is less than ε . \square

Appendix B

Properties of \mathcal{I}_μ

Proof. (Of Lemma 8). Let $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} W(t) e^{-2\pi i t \xi} dt$ be the Fourier transform of W . One can show that for $|\xi| \geq \gamma$, $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi i \xi}}$ (need to add). Let A be an observable. First, we show that $\mathcal{I}([H, PAP^\perp]) = i PAP^\perp$.

Decomposing

$$\begin{aligned} e^{itH} P &= \sum_{j=0}^{\infty} \frac{(itH)^j}{j!} P \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left(\sum_n E_n^j P_n \right) P \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \sum_{n: E_n=0} E_n^j P_n \\ &= \sum_{n: E_n=0} e^{itE_n} P_n, \end{aligned}$$

and similarly

$$P^\perp e^{-itH} = \sum_{m: E_m \geq \gamma} P_m e^{-itE_m},$$

we see that

$$\begin{aligned}
 \mathcal{I}([H, PAP^\perp]) &= \mathcal{I}(P[H, A]P^\perp) \\
 &= \int_{\mathbb{R}} W(t) e^{itH} P[H, A] P^\perp e^{-itH} dt \\
 &= \int_{\mathbb{R}} W(t) \sum_{n: E_n=0} e^{itE_n} P_n[H, A] \sum_{m: E_m \geq \gamma} P_m e^{-itE_m} dt \\
 &= \sum_{n: E_n=0} \sum_{m: E_m \geq \gamma} \int_{\mathbb{R}} W(t) e^{itE_n} P_n A(E_n - E_m) P_m e^{-itE_m} dt \\
 &= \sum_{n: E_n=0} \sum_{m: E_m \geq \gamma} P_n A P_m(E_n - E_m) \int_{\mathbb{R}} W(t) e^{-it(E_m - E_n)} dt \\
 &= \sum_{n: E_n=0} \sum_{m: E_m \geq \gamma} P_n A P_m(E_n - E_m) \sqrt{2\pi} \widehat{W}(E_m - E_n) \\
 &= i \sum_{n: E_n=0} \sum_{m: E_m \geq \gamma} P_n A P_m \\
 &= iPAP^\perp.
 \end{aligned}$$

(need to check the 2π factor)

By the same argument, $\mathcal{I}([H, P^\perp AP]) = iP^\perp AP$ as well, and so $\mathcal{I}([H, \overline{A}]) = i\overline{A}$. \square

Proof. (Of Lemma 9). Note that any operator can be written as the telescoping sum

$$\mathcal{I}(A) = \text{tr}_X(\mathcal{I}(A)) + \sum_{j=1}^{\infty} (\text{tr}_{X^{(j)}}(\mathcal{I}(A)) - \text{tr}_{X^{(j-1)}}(\mathcal{I}(A))),$$

where tr_Y denoted the partial trace over Y , and Y^α denotes the α -fattening of Y .

Our goal is to prove that $\|\text{tr}_{X^{(j)}}(\mathcal{I}(A)) - \mathcal{I}(A)\| \leq \|A\| |X| \mathcal{O}(j^{-\infty})$. To that end, we break the integral $\mathcal{I}(A)$ into two terms,

$$\mathcal{I}(A) = \int_{-T}^T W(t) e^{itH} A e^{-itH} dt + \int_{\mathbb{R} \setminus [-T, T]} W(t) e^{itH} A e^{-itH} dt.$$

We break the integral into two parts,

$$\|\mathcal{I}(A)\| \leq \left\| \int_{-T}^T W(t) e^{itH} A e^{-itH} dt \right\| + \left\| \int_{\mathbb{R} \setminus [-T, T]} W(t) e^{itH} A e^{-itH} dt \right\|.$$

Appendix B. Properties of \mathcal{I}_μ

The first term can be estimated using the Lieb-Robinson bound found in Appendix C.

□

Appendix C

The Lieb-Robinson Bound

Let N be a uniform upper bound for the dimensions of the Hilbert spaces at each site, i.e. $\dim(\mathcal{H}_x) \leq N$ for all sites x .

The following is a version of the Lieb-Robinson. For any operators $A \in \mathcal{U}_X$ and $B \in \mathcal{U}_Y$ having disjoint supports $X \cap Y = \emptyset$,

$$\|[e^{itH} A e^{-itH}, B]\| \leq C \|A\| \|B\| |X| |Y| N^{2|X|} e^{2t\|\Phi\|_\lambda - \lambda d(X,Y)}.$$

Appendix D

Grönwall's Inequality and Uniqueness

Theorem 3. (*Grönwall's Inequality*). Let $\alpha : I \rightarrow (0, \infty)$ be positive and continuous on I° for some interval of the form $[a, b)$, $[a, b]$, or $[a, \infty)$. Suppose $u : \mathbb{R} \rightarrow \mathcal{U}$ is a Banach-valued, differentiable function. If $\|u'(t)\| \leq \alpha(t)\|u(t)\|$ for all $t \in I$, then

$$\|u(t)\| \leq \|u(a)\| e^{\int_a^t \alpha(s) ds} \quad \forall t \in I$$

Proof. Let $f(t) = e^{\int_a^t \alpha(s) ds}$, which is nonzero, has initial value $f(a) = 1$, and has derivative $f'(t) = \alpha(t)f(t)$. Then by the quotient rule,

$$\left(\frac{\|u(t)\|}{f(t)} \right)' = \frac{\|u'(t)\|f(t) - \|u(t)\|\alpha(t)f(t)}{f(t)^2} \leq 0,$$

where the inequality follows from the assumption $\|u'(t)\| \leq \alpha(t)\|u(t)\|$. Thus $\frac{\|u(t)\|}{f(t)}$ is decreasing, so that

$$\frac{\|u(t)\|}{f(t)} \leq \frac{\|u(a)\|}{f(a)} = \|u(a)\|,$$

which is the desired inequality. \square

Theorem 4. (*ODE Uniqueness*). Let $F : \mathcal{U} \rightarrow \mathcal{U}$ be Lipschitz and consider the differential equation $u'(t) = F(u(t))$ with initial condition $u(a) = u_a$ for some function $u : I \rightarrow \mathcal{U}$, where $I = [a, b)$, or $[a, b]$, or $[a, \infty)$. Solutions to this equation are unique.

Proof. Suppose there are two solutions $u(t)$ and $v(t)$, and let $g(t) = \|u(t) - v(t)\|^2$. By assumption, there exists a constant L_F such that $\|F(u(t)) - F(v(t))\| \leq L_F \|u(t) - v(t)\|$.

$\|F(v(t))\| \leq L_F \|u(t) - v(t)\|$, so that

$$\begin{aligned} g'(t) &= 2\|u(t) - v(t)\| \|u'(t) - v'(t)\| \\ &= 2\|u(t) - v(t)\| \|F(u(t)) - F(v(t))\| \\ &\leq 2L_F \|u(t) - v(t)\|^2 \\ &= 2L_F g(t). \end{aligned}$$

Notice that $\alpha := 2L_F$ is a positive continuous function, so we may apply Grönwall's inequality to $g(t)$ to conclude

$$g(t) \leq g(a)e^{2L_F(t-a)} = 0,$$

since $g(a) = 0$. □

Appendix E

Note on Generators of Parallel Transport

Consider the differential equation $\dot{\rho}(\mu) = i[K_B, \rho(\mu)]$ with initial condition $\rho(0) = P_B(0)$. Here $K_B = \int_{\mathbb{R}} W_{\gamma}(t) e^{-itH_B} \dot{H}_B e^{itH_B} dt$, and recall that in our setting, $\dot{H}_B = Q_h$. We know that the solution is $\rho(\mu) = P_B(\mu)$ (proposition 2). Notice that the map $F : \mathcal{U} \rightarrow \mathcal{U}$ defined by $F(A) = i[K_B, A]$ is Lipschitz, since

$$\|F(A) - F(B)\| = \|[K_B, A - B]\| \leq 2\|K_B\| \|A - B\|.$$

The Lipschitz constant is $2\|K_B\|$, which is finite since K_B is a bounded operator:

$$\|K_B\| \leq \int_{\mathbb{R}} |W_{\gamma}(t)| \|e^{-itH_B} Q_h e^{itH_B}\| dt \leq \int_{\mathbb{R}} |W_{\gamma}(t)| dt \|Q_h\|.$$

Indeed, since Q_h is the number operator on a finite volume, by charge conservation and the fact that the dimension of the Hilbert space is bounded uniformly by d , there can only be a finite number of charges in the region Γ_h .

Thus, by Grönwall's uniqueness theorem (appendix D), we see that the solution to the equation $\dot{\rho}(\mu) = F(\rho(\mu)) = i[K_B, \rho(\mu)]$ is unique.

Now define

$$K_E := \int_{\mathbb{R}} W_{\gamma}(t) e^{-itH_E} Q_h e^{itH_E} dt,$$

which is using the gap γ of H_B to define W_{γ} , but also using the edge Hamiltonian in the time evolution operators. Consider $\sigma : [0, \infty) \rightarrow \mathcal{U}$ defined by

$$\dot{\sigma}(\mu) = i[K_E, \sigma(\mu)] \quad \sigma(0) = P_E(0).$$

We now show that, similar to how ρ is an approximation of P_B , σ is also a good approximation of P_E (up to $\mathcal{O}(L^{-\infty})$) “in the bulk”. Let $A \in \mathcal{U}_{\Gamma_B}$ be an operator localized in the bulk of the edge system. Then

$$\begin{aligned}
 \text{Tr}(\dot{\sigma}A) &= \text{Tr}(i[K_E, \sigma]A) \\
 &= \text{Tr}(i[A, K_E]\sigma) \\
 &= \int_{\mathbb{R}} W_{\gamma}(t) \text{Tr}([e^{-itH_E} Q_h e^{itH_E}, A]\sigma) dt \\
 &= \int_{\mathbb{R}} W_{\gamma}(t) \text{Tr}(e^{-itH_E} [Q_h, e^{itH_E} A e^{-itH_E}] e^{itH_E} \sigma) dt \\
 &= \int_{\mathbb{R}} W_{\gamma}(t) \text{Tr}(e^{-itH_E} [Q_h, e^{itH_B} A e^{-itH_B}] e^{itH_E} \sigma + \mathcal{O}(L^{-\infty})) dt \\
 &= \int_{\mathbb{R}} W_{\gamma}(t) \text{Tr}(e^{-itH_B} [Q_h, e^{itH_B} A e^{-itH_B}] e^{itH_B} \sigma + \mathcal{O}(L^{-\infty})) dt \\
 &= \int_{\mathbb{R}} W_{\gamma}(t) \text{Tr}([e^{-itH_B} Q_h e^{itH_B}, A]\sigma) dt + \mathcal{O}(L^{-\infty}) \\
 &= \text{Tr}(i[A, K_B]\sigma) + \mathcal{O}(L^{-\infty}) \\
 &= \text{Tr}(i[K_B, \sigma]A) + \mathcal{O}(L^{-\infty}),
 \end{aligned}$$

since σ is trace-class (?) and $W_{\gamma} \in L^1$. By linearity of the trace, we see that $\text{Tr}((\dot{\sigma} - i[K_B, \sigma])A) = \mathcal{O}(L^{-\infty})$ for any operator $A \in \Gamma_B$ (does this mean $\dot{\sigma} - i[K_E, \sigma] = 0$?). But the solution of $\dot{\sigma} - i[K_B, \sigma] = 0$ (with initial condition $\sigma(0) = P_B(0)$) is unique; it is $\rho(\mu)$, or $P_B(\mu)$. Hence

$$\text{Tr}(P_E A) = \text{Tr}(P_B A) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\rho A) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\sigma A) + \mathcal{O}(L^{-\infty})$$

for any operator $A \in \Gamma_B$. In particular, this gives another local formula for the Hall conductivity in the bulk of an edge system, by taking $A = J_V$, where J is the current operator and $V \subset \Gamma_B$ is a set localized in the bulk. The Hall conductivity is given by $\text{Tr}(\dot{P}_E J_V)$, and this can be approximated by

$$\text{Tr}(\dot{P}_E J_V) = \text{Tr}(\dot{P}_B J_V) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\dot{\rho} J_V) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\dot{\sigma} J_V) + \mathcal{O}(L^{-\infty}).$$

Want to pick a norm s.t. Gronwall gives $\|\rho(\mu) - \sigma(\mu)\|_G \leq \|P_B(0) - P_E(0)\|_G e^{2L_F \mu}$. Need $\|P_B(0) - P_E(0)\|_G$ to be small enough to kill the exponential which depends on $L_F = 2\|K_B\|_G \leq \|W_{\gamma}\|_{L^1} \|Q_h\|_G$. If we use the operator norm for $\|\cdot\|_G$, we would get $\|Q_h\|_G = d|\Gamma_h|$ in the exponent. Need $\|\cdot\|_G$ to be an actual norm so that $\|\rho - \sigma\|_G = 0 \implies \rho = \sigma$.

Appendix F

From Dec 13 Meeting

Let $r(t) = \rho(t) - \sigma(t)$. Notice that

$$\frac{d}{dt} e^{itK_B} \sigma_0 e^{-itK_B} = e^{itK_B} i[K_B, \sigma_0] e^{-itK_B} + e^{-itK_B} \dot{\sigma}_0 e^{itK_B}.$$

Appendix G

The Helffer-Sjöstrand Representation

The Helffer-Sjöstrand representation is a functional calculus $f \mapsto f(H)$ for arbitrary (possibly unbounded) operators H on the set of functions

$$\mathcal{A} = \bigcup_{\beta < 0} \{f : \mathbb{R} \rightarrow \mathbb{C} : f \in C^\infty(\mathbb{R}), |f^{(n)}(x)| \leq c_n(1+x^2)^{\frac{\beta-n}{2}}\}.$$

It has the following properties.

Theorem 5. *For any $f \in \mathcal{A}$,*

1. $f \mapsto f(H)$ is an algebraic homomorphism (linear and multiplicative).
2. $\overline{f}(H) = f(H^*)$.
3. $\|f(H)\| \leq \|f\|_\infty$.
4. For all $w \notin \mathbb{R}$, if $r_w(s) = \frac{1}{s-w}$ then $r_w(H) = (H - w)^{-1}$.
5. For all $f \in C_c^\infty(\mathbb{R})$ with $\text{supp}(f) \cap \text{Spec}(H) = \emptyset$, we have $f(H) = 0$.

There is an explicit formula for $f(H)$, which is given by

$$f(H) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (H - z)^{-1} dz \wedge d\bar{z},$$

where $\tilde{f} : \mathbb{C} \rightarrow \mathbb{C}$ is a *quasi-analytic extension* of $f : \mathbb{R} \rightarrow \mathbb{R}$. It is defined as follows. For any smooth f , we set

$$\tilde{f}(z) = \sum_{r=0}^n \tau \left(\frac{y}{(1+x^2)^{1/2}} \right) \frac{(iy)^r}{r!} f^{(r)}(x)$$

where $\tau : \mathbb{R} \rightarrow \mathbb{R}$ is any smooth function satisfying

$$\tau(s) = \begin{cases} 1 & |s| < 1 \\ 0 & |s| > 2 \end{cases}.$$

The extension turns out to be independent of the choice of n and τ . Furthermore, as $|\operatorname{Im}(z)| \rightarrow 0$, the Wirtinger derivative of the extension obeys the bound

$$\left| \frac{\partial \tilde{f}}{\partial \bar{z}} \right| = \mathcal{O}(|y|^n).$$

Thus $\frac{\partial \tilde{f}}{\partial \bar{z}} = 0$ for all real z , which is why \tilde{f} is called a “quasi”-analytic extension (the Wirtinger derivative would be zero everywhere were \tilde{f} analytic).

A crucial property of the Helffer-Sjöstrand functional calculus is the following bound. For any $n \in \mathbb{N}$, the quasi-analytic extension \tilde{f} can be chosen so that

$$\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \frac{1}{|\operatorname{Im}(z)|^{p+1}} dz \wedge d\bar{z} \leq C_0 \sum_{k=0}^{n+2} \|f^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_m = \int_{-\infty}^{\infty} |f(x)| (1+x^2)^{m/2} dx.$$

This is often useful because the resolvent obeys the bound $\|(H-z)^{-1}\| \leq |\operatorname{Im}(z)|^{-1}$.

Appendix H

Spectral Measures and Projection-Values Measures

Projection-valued measures are maps $P : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H})$ from measurable subsets of \mathbb{R} to the space of bounded linear operators on \mathcal{H} satisfying the usual properties of both projections and measures.

1. $P(M) = P(M)^* = P(M)^2$ is an orthogonal projection for all $M \in \mathcal{M}$.
Note that this implies that $P(M)$ is a positive operator.
2. $P(\emptyset) = 0$ and $P(\mathbb{R}) = \mathbb{1}_{\mathcal{H}}$.
3. If $\{M_i\}_{i \in \mathbb{N}}$ are pairwise disjoint, then $\sum_{i=1}^n P(M_i) \xrightarrow{s} P(\cup_{i \in \mathbb{N}} M_i)$ as $n \rightarrow \infty$ (σ -additivity).
4. $P(M_1 \cap M_2) = P(M_1)P(M_2)$ for any $M_1, M_2 \in \mathcal{M}$.

The heuristic motivation is that $P(M)$ projects onto the subspace of \mathcal{H} spanned by states whose energies lie in M . Using these operator-valued measures, one can construct an operator-valued integral with respect to P in the usual fashion (beginning on nonnegative simple functions, extending to nonnegative measurable functions, and finally to real-valued measurable functions).

Theorem 6 (Spectral Theorem for Projection-Valued Measures). *There exists a one-to-one correspondence between self adjoint operators H and projection-valued measures P given by the formula*

$$H = \int_{\mathbb{R}} \lambda dP_{\lambda},$$

where $P_{\lambda} := P((-\infty, \lambda])$. Moreover, if $g : \mathbb{R} \rightarrow \mathbb{R}$ is any bounded Borel function, then $g(H)$ defined via the Borel function calculus coincides with the formula

$$g(H) = \int_{\mathbb{R}} g(\lambda) dP_{\lambda},$$

and $g(H) = g(H)^*$.

We remark that it follows from the second part of this theorem that if $\mathbb{1}_M$ denotes the characteristic function of a Borel set $M \subseteq \mathbb{R}$, then

$$\mathbb{1}_M(H) = \int_M dP_\lambda = P(M).$$

We also note that $\text{Spec}(H) = \text{supp}(P)$.

Appendix I

Greens' Functions and the Combes-Thomas Bound

The short-range assumption 1 is vital for the following non-trivial estimate, the proof of which is omitted.

Theorem 7 (Combes-Thomas Bound). *Let H be a self-adjoint operator on $\ell^2(\mathbb{Z}^2)$ satisfying*

$$S_\alpha := \sup_x \sum_y |H(x, y)| (e^{\alpha|x-y|} - 1) < \infty$$

for some $\alpha > 0$. Suppose z lies outside the spectrum of H , and let $d_z := \text{dist}(z, \text{Spec}(H))$. Then the Greens function of H is exponentially bounded,

$$|G(x, y; z)| \leq \frac{2}{d_z} e^{-\xi_\alpha |x-y|},$$

where $\xi := \frac{\alpha d_z}{2S_\alpha}$.

This theorem gives the crucial decay properties of the spectral projectors.

Lemma 14. *Let H be a self-adjoint operator satisfying 1 and with bounded spectrum. Let $S \subseteq \text{Spec}(H)$, and let P_S be the associated spectral projection. Then there exists some $\varepsilon, \nu > 0$ such that the matrix elements of P_S satisfy*

$$\sum_{x, y \in \mathbb{Z}^2} |P_S(x, y)| e^{-\varepsilon|x|} e^{\nu|x-y|} < \infty.$$

Proof. We use the fact that the spectral projection is given by the Riesz integral formula

$$P_S = \frac{1}{2\pi i} \oint_\gamma R(z) dz,$$

where $R(z)$ is the resolvent of H and γ is any smooth closed curve containing S . Since the resolvent is the Greens function of H , it satisfies the Combes-Thomas bound 7. Since the spectrum of H is bounded, it may be enclosed in a curve of finite length, from which we determine that

$$|P_S(x, y)| \leq C e^{-\xi_\alpha |x-y|},$$

since $\inf_{z \in \gamma} d_z = \inf_{z \in \gamma} \text{dist}(z, \text{Spec}(H)) > 0$. Hence

$$\sum_{x, y \in \mathbb{Z}^2} |P_S(x, y)| e^{-\varepsilon |x|} e^{\nu |x-y|} < \infty$$

holds for $\nu = \xi_\alpha$ and for any $\varepsilon > 0$, since $e^{-\varepsilon |x|} = e^{-\varepsilon |x_1|} e^{-\varepsilon |x_2|}$ is summable on \mathbb{Z}^2 by Lemma 4. □

This statement about decay of the matrix elements can be turned into a statement about the norm of the operator. Let Lip^1 be the set of all Lipschitz functions whose Lipschitz constant is not greater than 1, that is the set of functions ℓ satisfying

$$|\ell(x) - \ell(y)| \leq |x - y|$$

for all $x, y \in \mathbb{Z}^2$.

Lemma 15. *Let P_S be the spectral projection onto $S \subseteq \text{Spec}(H)$. For all $\varepsilon > 0$,*

$$\sup_{\ell \in \text{Lip}^1} \|e^{\nu \ell(x)} e^{-\varepsilon |x|} P_S e^{-\nu \ell(x)}\| < \infty,$$

where ν is the same as in 14.

Proof. The matrix elements of the operator are $P_S(x, y) e^{\nu(\ell(x) - \ell(y))} e^{-\varepsilon |x|}$. By Holmgren's bound, the operator's norm is therefore bounded by

$$\max\left\{\sup_x \sum_y |P_S(x, y)| e^{\nu(\ell(x) - \ell(y))} e^{-\varepsilon |x|}, \sup_y \sum_x |P_S(x, y)| e^{\nu(\ell(x) - \ell(y))} e^{-\varepsilon |x|}\right\}.$$

Replacing the supremum with a sum yields the bound

$$\|P_S(x, y) e^{\nu(\ell(x) - \ell(y))} e^{-\varepsilon |x|}\| \leq \sum_{x, y} |P_S(x, y)| e^{\nu|\ell(x) - \ell(y)|} e^{-\varepsilon |x|},$$

and taking a supremum over $\ell \in \text{Lip}^1$ completes the proof by Lemma 14. □

Appendix A

First Appendix

Here you can have your appendices. Note that if you only have a single appendix, you should issue

Appendix B

Second Appendix

Here is the second appendix.

Additional Information

This chapter shows you how to include additional information in your thesis, the removal of which will not affect the submission. Such material should be removed before the thesis is actually submitted.

First, the chapter is unnumbered and not included in the Table of Contents. Second, it is the last section of the thesis, so its removal will not alter any of the page numbering etc. for the previous sections. Do not include any floats, however, as these will appear in the initial lists.

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