

The Quantum Hall Effect and Bulk-Edge Correspondence on Lattice Systems

by

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Abstract

In this thesis, we investigate the *integer quantum Hall effect* (IQHE). Discovered in 1980 by Nobel Laureate Klaus Von Klitzing, the IQHE is a phenomenon in which the Hall resistivity of a 2-dimensional electron gas is precisely quantized to integer multiples of h/q^2 when subjected to a strong magnetic field at very low temperatures.

In particular, we provide a rigorous theoretical description of a phenomenon known as *bulk-edge correspondence*, wherein the value of the Hall conductivity is unaffected by whether or not one considers a system with an edge or an infinite plane. This is accomplished in two settings. First, we ignore interactions between electrons, and then proceed to the more challenging interacting setting.

Preface

Chapter 2 is based on work conducted in [1]. This thesis is the work of the author, Justin Furlotte, under the supervision of Sven Bachmann.

Table of Contents

Abstract	ii
Preface	iii
Table of Contents	iv
List of Figures	vi
Acknowledgements	vii
Dedication	viii
1 Introduction	1
1.1 Heuristic Arguments	1
1.1.1 The Classical Hall Effect	1
1.1.2 The Quantum Hall Effect	3
2 Noninteracting Bulk-Edge Correspondence	11
2.1 General Setting	11
2.2 Equality of Bulk and Edge Conductivities	17
2.2.1 Outline of the Proof	17
2.2.2 The Proof	18
3 Interacting Bulk-Edge Correspondence	30
3.1 General Setting	30
3.2 Equality of Bulk and Edge Currents	33
3.2.1 Cylinder Geometry	33
4 Summary of Results	42
4.1 Direction for Future Work	42
Bibliography	45

Appendices

A General Functional Analysis	48
A.1 Spectral Measures and Projection-Values Measures	48
A.2 Grönwall's Inequality and Uniqueness	49
A.3 Note on Generators of Parallel Transport	50
A.4 Greens' Functions and the Combes-Thomas Bound	52
B The Lieb-Robinson Bound	55
C The Helffer-Sjöstrand Representation	56

List of Figures

1.1	Hall resistivity as a function of the magnetic field strength jumps in distinct plateaux. [24]	3
1.2	Laughlin's cylinder.	5
1.3	Quantum Hall conductance explained using charge transport. The states are exponentially localized and separated by a distance of $s_1(0)$. As $\phi_0 \mapsto \phi_0 + h/q$, the states all shift to the left by $s_1(0)$.	8
1.4	Disorder, modelled by a small random potential, breaks the degeneracy of the discrete Landau levels E_n into bands. The blue states are <i>localized</i> and do not contribute to charge transport, while the red states are <i>extended</i> . As B decreases and μ moves through the blue (localized) and black (spectral gap) regions, the current (and therefore the Hall conductivity) are not affected; this gives rise to the plateaux observed experimentally. Only once μ begins moving through the red (extended) region will the Hall conductivity begin to increase to its next plateau.	10
3.1	The left and right edges are identified to form the cylinder Γ_L . The charge operator Q_h introduces a driving strength. The Hall current is measured across the dashed yellow line $y = L/2$.	31
3.2	The regions Γ_E , Γ_B , and Γ_m .	33

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I would like to thank my supervisor, Sven Bachmann, for his patience, his kindness, and his inspiring mastery in the field of mathematical physics and quantum lattice systems.

Dedication

To Abigail Sanderson, for her unwavering belief in me, and to my parents,
for their endless support.

Chapter 1

Introduction

Since its discovery in 1980, the Integer Quantum Hall Effect (IQHE) has captured significant interest in the mathematical physics community. The IQHE occurs when a 2 dimensional electron gas at near 0 Kelvin is pierced by a strong magnetic field. As the strength of the field increases, the Hall resistivity - a macroscopic quantity - undergoes quantization in nearly perfect integer multiples of h/q^2 , as in Figure 1.1.

In this thesis, we begin by providing a simple classical calculation for the Hall conductivity, and emphasize that the result does not agree with experiment. The IQHE, as its name suggests, is fundamentally reliant on quantum mechanics. We give another heuristic argument due to Laughlin [15] for why the quantization of Hall resistivity occurs, before going into the main body of the thesis in Chapters 2 and 3.

There, we provide a proof of *bulk-edge correspondence*. This is the interesting mathematical fact that the Hall conductivity does not depend on whether or not the system is assumed to have an edge. In the system with an edge, the Hall conductivity is defined by assuming that current is transported along the edge, while in the bulk (i.e. edgeless) system, the Hall current is assumed to be carried throughout the entire bulk of the material. This correspondence between the bulk and edge Hall conductivities in the IQHE is a special case of a more general bulk-edge correspondence which can occur between other invariants of topological insulators.

1.1 Heuristic Arguments

We provide a brief introduction to the quantum Hall effect. We emphasize that the results of this section are not rigorous.

1.1.1 The Classical Hall Effect

Using classical electromagnetism is not enough to predict the plateaux seen experimentally. Suppose we have a 2-dimensional electron gas, and let

1.1. Heuristic Arguments

$\vec{B} = B\hat{z}$ be a magnetic field piercing the plane of the electrons. They are subjected to a Lorentz force

$$m\dot{\vec{v}} = -q\vec{v} \times \vec{B}.$$

The solution to this differential equation is given by the *cyclotron orbits*,

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} a + r \sin(\omega_B t + \phi) \\ b + r \cos(\omega_B t + \phi) \\ 0 \end{pmatrix}$$

where $\omega_B = qB/m$ is the *cyclotron frequency*. When an electric field $\vec{E} = E\hat{x}$ is introduced, the electrons move in the x -direction, and we employ the *Drude model*,

$$m\dot{\vec{v}} = -q(\vec{E} + \vec{v} \times \vec{B}) + \frac{m}{\tau}\vec{v},$$

where the final term is a linear friction term, and τ is the scattering time. At equilibrium, the equation reads

$$\vec{J} + \frac{q\tau}{m}\vec{J} \times \vec{B} = -\frac{q^2 n \tau}{m}\vec{E},$$

where $\vec{J} = -nq\vec{v}$ is the current density, related to the velocity by the density n of electrons per unit area. In matrix notation, this reads

$$\begin{pmatrix} 1 & \omega_B \tau \\ -\omega_B \tau & q \end{pmatrix} \vec{J} = -\frac{q^2 n \tau}{m}\vec{E}.$$

Since the matrix on the left is invertible, we may write $\vec{J} = \sigma \vec{E}$, which is Ohm's Law. The *conductivity tensor* is given by

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{pmatrix} = -\frac{q^2 n \tau}{m(1 + \omega_B^2 \tau^2)} \begin{pmatrix} 1 & -\omega_B \tau \\ \omega_B \tau & 1 \end{pmatrix}.$$

The off-diagonal components, $\pm \frac{q^2 n \tau}{m(1 + \omega_B^2 \tau^2)} \omega_B \tau$, are responsible for the Hall effect; the magnetic field induces a component of the current in the y -direction, in addition to the one in the x -direction from the electric field.

When making a measurement, physicists actually measure the resistivity. In particular, the *Hall resistivity*, given by the off-diagonal components of the resistivity tensor $\rho = \sigma^{-1}$, is simply

$$\rho_{xy} = \frac{B}{nq}.$$

1.1. Heuristic Arguments

The key prediction of the classical theory is that the Hall resistivity increases linearly in response to the strength of the magnetic field.

1.1.2 The Quantum Hall Effect

Von Klitzing's experimental observation in 1980 made it clear that classical electromagnetism is not sufficient to describe the Hall effect. At a temperature of about 8mK, the Hall resistivity looked like this as a function of the magnetic field strength.

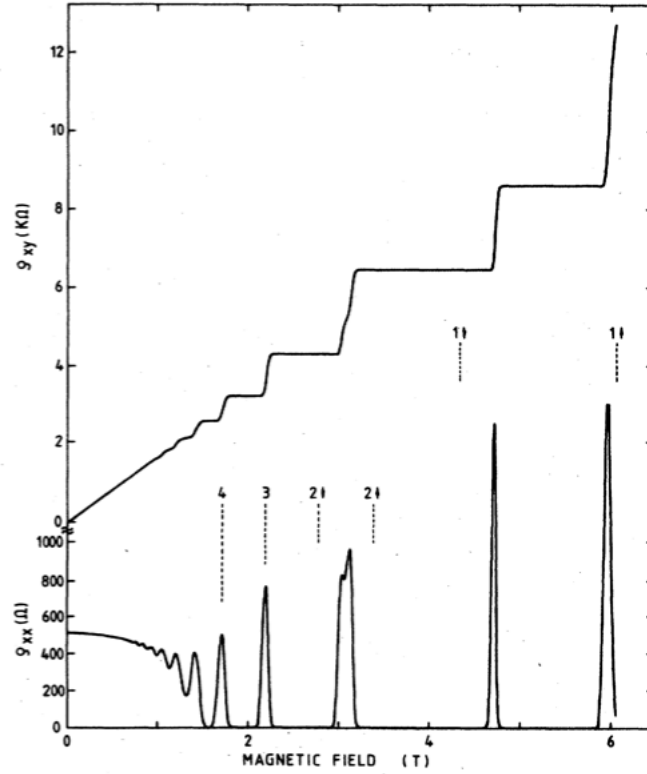


Figure 1.1: Hall resistivity as a function of the magnetic field strength jumps in distinct plateaux. [24]

Even more surprising, the plateaux occur at the values

$$\rho_{xy} = \frac{h}{q^2} \frac{1}{n},$$

1.1. Heuristic Arguments

where n is an integer, hence the name *integer quantum Hall effect*. In fact, even in impure samples, this integer can be measured to such extraordinary precision (about one part in 3×10^{-10}) that as of 2020, the SI definition of the Ohm itself has been redefined in terms of the quantum Hall resistivity. Clearly deeper physics are at play.

The first quantum mechanical explanation of the integer-valued Hall conductivity is due to Laughlin in his short but seminal 1981 paper [15], for which he also won the Nobel prize. In this paper, the plateaux are explained using a charge transport argument.

We remark that, although the resistivity is what is measured in a lab, throughout this thesis we work with the quantum Hall *conductivity*,

$$\sigma_H = \frac{q^2}{h} n.$$

The Laughlin Argument

Rather than a flat sheet, consider gluing together the $x_2 = 0$ and $x_2 = L$ edges to form a cylinder, as depicted in Figure 1.2. The surface of the cylinder is pierced throughout by a uniform magnetic field \vec{B} normal to its surface. In addition to this background magnetic field, suppose that a time-dependent magnetic flux $\phi(t)$ is also threaded through the cylinder.

As $\phi(t)$ increases slowly from $\phi(0) = \phi_0$ to $\phi(T) = \phi_0 + \Delta\phi$, the charge increases by $\Delta Q = -\sigma_H \Delta\phi$. This can be seen by Faraday's law,

$$-\frac{d\phi}{dt} = \oint_C \vec{E} \cdot d\vec{l},$$

combined with the formula for the Hall current, $\vec{J}_H = -\sigma_H \hat{\rho} \times \vec{E}$. The current pumped across the fiducial line C in Figure 1.2 is

1.1. Heuristic Arguments

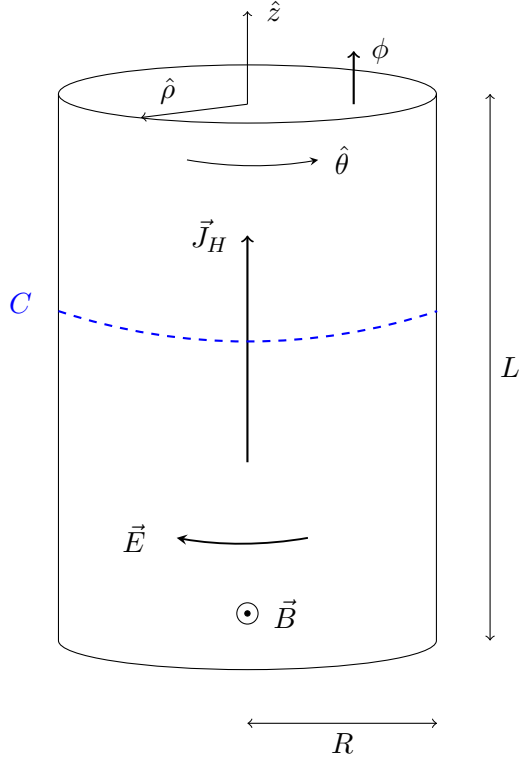


Figure 1.2: Laughlin's cylinder.

$$\begin{aligned}
 \frac{dQ}{dt} &= \oint_C \vec{J}_H \cdot \hat{z} dl \\
 &= -\sigma_H \oint_C (\hat{\rho} \times \vec{E}) \cdot \hat{z} dl \\
 &= -\sigma_H \oint_C (\hat{\rho} \times \hat{z}) \cdot \vec{E} dl \\
 &= \sigma_H \oint_C \hat{\theta} \cdot \vec{E} dl \\
 &= \sigma_H \oint_C \vec{E} \cdot d\vec{l} \\
 &= -\sigma_H \frac{d\phi}{dt}.
 \end{aligned}$$

Thus $\Delta Q = -\sigma_H \Delta\phi$. We now use this fact to argue that the Hall conductivity must be an integer multiple of q^2/h . For convenience, we “unroll”

1.1. Heuristic Arguments

the cylinder to work in Cartesian coordinates, but still identify the edges $x = 0$ and $x = L$. We take the positive x -direction to be the opposite direction as J_H in figure 1.2, and the positive y -direction to be the same as $\hat{\theta}$ in the figure. We choose the *Landau gauge* for the magnetic field $\vec{B} = B\hat{z}$,

$$\vec{A}_B = \begin{pmatrix} 0 \\ Bx \\ 0 \end{pmatrix},$$

and

$$\vec{A}_\phi = \begin{pmatrix} 0 \\ \frac{\phi}{2\pi R} \\ 0 \end{pmatrix}$$

for the flux potential, which has vanishing curl (as desired since there's no magnetic field from ϕ on the cylinder). The total magnetic vector potential $\vec{A} = \vec{A}_B + \vec{A}_\phi$ appears in the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2m} (\vec{p} - q\vec{A})^2 \\ &= \frac{1}{2m} \left(p_x^2 + \left(p_y - q \left(Bx + \frac{\phi}{2\pi R} \right) \right)^2 \right). \end{aligned}$$

The fact that the Hamiltonian commutes with the y -momentum is evident from the expression above, since the only thing that wouldn't commute with p_y is the position operator y , which does not appear. This allows us to replace p_y with its eigenvalue, $\hbar k$. The wavenumber k is quantized by the periodic boundary condition

$$e^{i0k} = e^{i2\pi Rk},$$

which implies $p_y = \hbar k = \hbar j/R$ with $j \in \mathbb{N}$, giving

$$\begin{aligned} H_j &= \frac{1}{2m} \left(p_x^2 + \left(\frac{\hbar j}{R} - q \left(Bx + \frac{\phi}{2\pi R} \right) \right)^2 \right) \\ &= \frac{1}{2m} \left(p_x^2 + q^2 B^2 \left(x - \frac{1}{2\pi RB} \left(\frac{\hbar j}{q} - \phi \right) \right)^2 \right). \end{aligned}$$

Using again the cyclotron frequency $\omega_B = \frac{qB}{m}$, and defining the *shift factor*

1.1. Heuristic Arguments

$$s_j(\phi) := \frac{1}{2\pi RB} \left(\frac{h}{q} j - \phi \right),$$

the Hamiltonian in this notation takes the form

$$H_j = \frac{1}{2m} p_x^2 + \frac{1}{2} m \omega_B^2 (x - s_j(\phi))^2.$$

This is a shifted quantum harmonic oscillator in x , with frequency ω_B . The spectrum is as usual,

$$E_n = \hbar \omega_B \left(n + \frac{1}{2} \right).$$

The eigenstates are called *Landau levels*, and they are exponentially localized at $x = -s_j(\phi)$. They are of the form

$$\psi_{n,j}(x, y) = C_{n,j} e^{i \frac{y j}{R}} H_n(x - s_j) e^{-\frac{(x - s_j(\phi))^2}{2\ell_B^2}}$$

where $\ell_B^2 = \frac{\hbar}{qB}$, $C_{n,j}$ are normalization constants, and H_n are the Hermite polynomials.

The crucial argument now comes from inspecting the shift $s_j(\phi)$. Suppose we adiabatically increase ϕ by one *flux quantum*, $\phi_0 \mapsto \phi_0 + \frac{h}{q}$, as time evolves from $t = 0$ to $t = T$. The adiabatic principle tells us that the eigenstates of the Hamiltonian at $t = 0$ must also be eigenstates of the Hamiltonian at $t = T$. But we can find exactly what the new eigenstates are after the flux quantum is added; the only thing that changes is the shift, which by a simple calculation becomes

$$s_j \left(\phi_0 + \frac{h}{q} \right) = \frac{1}{2\pi RB} \left(\frac{h}{q} j - \phi_0 - \frac{h}{q} \right) = s_{j-1}(\phi_0).$$

The (exponentially localized) wavefunctions are each transported upward in the $-x$ direction by an increment of $s_1(0) = \frac{\hbar}{qB} \frac{1}{R}$ as ϕ increases by one flux quantum. We remark that the wavefunctions $\psi_{n,j}$ are Gaussian-like, with standard deviation $\ell_B^2 = \frac{\hbar}{qB}$. If $R \ll 1$, the shift increment $s_1(0)$ is large enough that we may treat the wavefunctions as being localized in a similar manner as depicted in the diagram of Figure 1.3.

Heuristically, this diagram explains why charge transport occurs. The Landau levels take their usual discrete spectrum and bend up sharply at the edges where the material ends. Consider one filled Landau level. Each of the states in this level is transported to the left. In particular, on the bottom

1.1. Heuristic Arguments

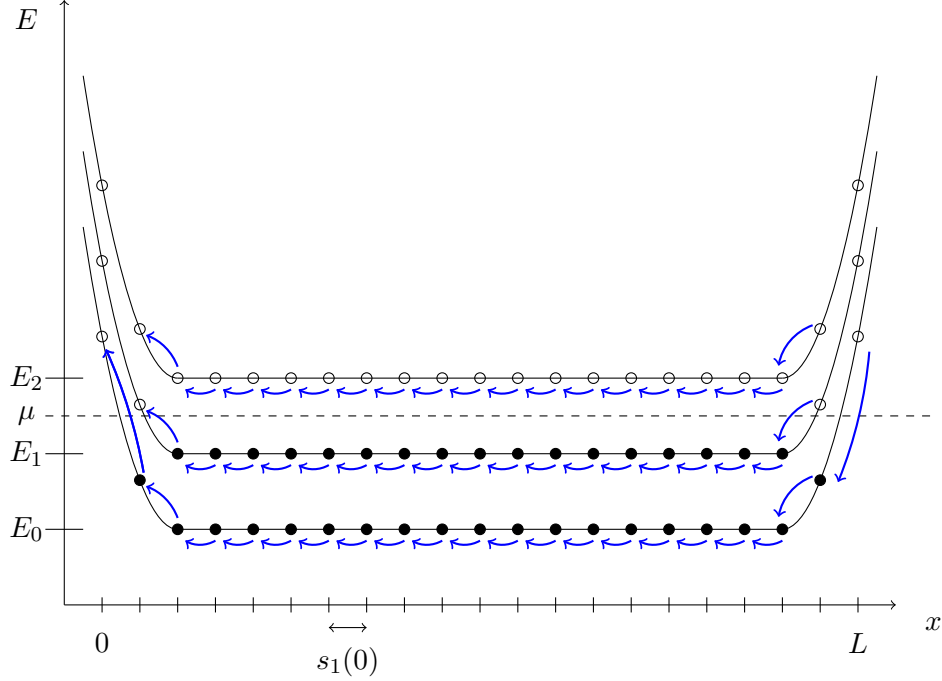


Figure 1.3: Quantum Hall conductance explained using charge transport. The states are exponentially localized and separated by a distance of $s_1(0)$. As $\phi_0 \mapsto \phi_0 + h/q$, the states all shift to the left by $s_1(0)$.

edge of the cylinder at $x = L$, the single empty state just above the Fermi energy μ is brought below the Fermi energy (see Figure 1.3). On the top edge, the single filled state just below the Fermi energy is brought above the Fermi energy. Everywhere in between, states are merely shifted up the cylinder, each one filling the other's place. Thus, exactly one charge q is transported from the bottom edge to the top edge. This occurs once per filled Landau level, so the total charge transported by the system is

$$\Delta Q = -qN_L$$

whenever the magnetic flux increases by $\Delta\phi = \frac{h}{q}$, where N_L is the (integer) number of filled Landau levels. Our result $\Delta Q = -\sigma_H \Delta\phi$ now gives us exactly the Hall conductivity which agrees with experiment,

$$\sigma_H = \frac{q^2}{h} N_L \quad N_L \in \mathbb{N}.$$

Degeneracy, Disorder, and Plateaux

The reasoning above only guarantees the correct value of σ_H if the Landau levels are exactly filled. Furthermore, the argument relies on a fictitious magnetic flux ϕ . As $\phi_0 \mapsto \phi_0 + h/q$, one charge per Landau level is transported from bottom edge to top. But isn't σ_H supposed to increase as B decreases, not as ϕ decreases?

The magnetic flux is purely an explanatory tool used to derive the formula $\sigma_H = \frac{q^2}{h} N_L$. The role of the magnetic field strength B becomes apparent from inspecting the degeneracy of the Landau levels. The number of states $\psi_{n,j}$ per fixed Landau level n must be finite, since $0 \leq x \leq L$ implies (roughly) that $-L \leq s_j(\phi) \leq 0$. We simply take $\phi = 0$, since adding h/q only corresponds to $j = 1$ extra state, which is insignificant to the degeneracy of the Landau levels. Substituting our expression for the shift $s_j(0)$, the inequality becomes

$$-L \frac{qBR}{\hbar} \leq j \leq 0.$$

The number of states N_s per Landau level is therefore

$$N_s = L \frac{qBR}{\hbar} = \frac{qBA}{h},$$

where $A = 2\pi RL$ is the total area of the cylinder. From here we see that as B decreases, so too does the degeneracy N_s ; the electrons must therefore begin to populate higher Landau levels. We know that once the next highest Landau level is completely filled, the Hall conductivity will have increased by exactly q^2/h .

What about the plateaux? Ironically, disorder, i.e. the presence of impurities in a real world material, is required to explain why the Hall conductivity does not change even when a Landau level is only partially filled. Disorder is modelled using a random potential. Under the addition of a suitably small random potential to the Hamiltonian, the degeneracy of the Landau levels is lifted and the spectrum broadens into bands as in Figure 1.4. Furthermore, while the center of the bands is continuous spectrum (red), the edges of the bands are pure-point (blue) [2, 12]. This phenomenon is called Anderson localization.

The pure-point states, called *localized states*, contribute nothing to the current, as they have compact support which does not wrap around the cylinder. The effect of ϕ on these states can therefore be gauge transformed away, since the domain of their support is simply connected, ensuring that

1.1. Heuristic Arguments

\vec{A}_ϕ is a (locally) conservative vector field. This is not the case on the cylinder as a whole, and in particular it is not the case for *extended states* (red), whose domain is not simply connected. Thus only extended states contribute to charge transport.

As B decreases and electrons fill states of higher energy within the band, the Fermi energy μ increases. However, the states at the edges of the bands (blue) are localized and contribute nothing to the current, and thus leave the Hall conductivity unchanged. This gives rise to the plateaux.

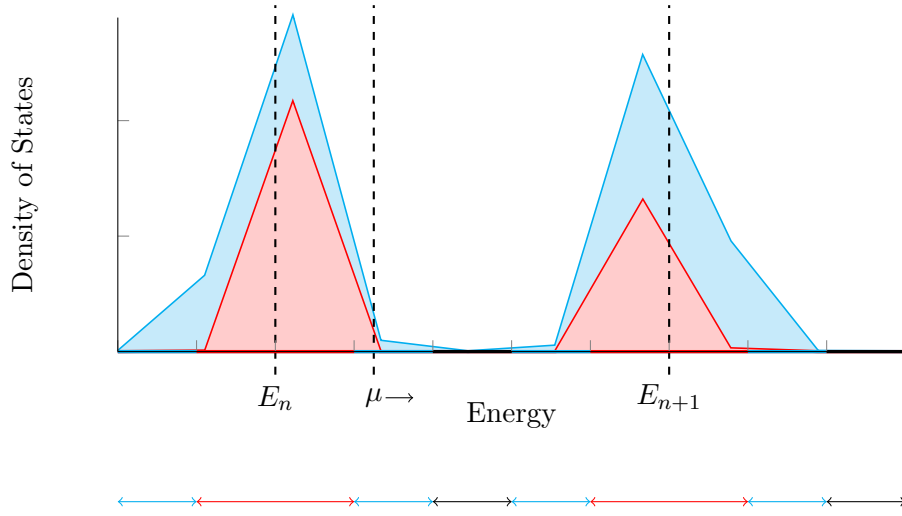


Figure 1.4: Disorder, modelled by a small random potential, breaks the degeneracy of the discrete Landau levels E_n into bands. The blue states are *localized* and do not contribute to charge transport, while the red states are *extended*. As B decreases and μ moves through the blue (localized) and black (spectral gap) regions, the current (and therefore the Hall conductivity) are not affected; this gives rise to the plateaux observed experimentally. Only once μ begins moving through the red (extended) region will the Hall conductivity begin to increase to its next plateau.

Chapter 2

Noninteracting Bulk-Edge Correspondence

An intriguing mathematical fact of the quantum Hall effect is the equality of bulk and edge conductivity, in the sense that whether the system is assumed to have an edge or not is immaterial. Proving this fact, both in the interacting and noninteracting setting, is the main focus of this thesis. In particular, we are not concerned with proving the existence of plateaux, or even the integrality of the Hall conductivity, but rather the equivalence of the bulk and edge Hall conductivities.

We begin by generalizing from Landau Hamiltonians to a much more general class of Hamiltonians, and derive appropriate formulae for the “bulk” and “edge” Hall conductivities in this scenario.

2.1 General Setting

Consider the lattice \mathbb{Z}^2 , and the associated Hilbert space of square-summable sequences of vectors in \mathbb{C}^n ,

$$\ell^2(\mathbb{Z}^2, \mathbb{C}^n) = \left\{ (x_i)_{i \in \mathbb{Z}^2} \subset \mathbb{C}^n : \sum_{i \in \mathbb{Z}^2} \|x_i\|^2 < \infty \right\},$$

with inner product $\langle x, y \rangle = \sum_{i \in \mathbb{Z}^2} x_i \overline{y_i}$. We denote this as $\ell^2(\mathbb{Z}^2)$ for short. On this Hilbert space we define a *bulk Hamiltonian* $H_B : \ell^2(\mathbb{Z}^2) \rightarrow \ell^2(\mathbb{Z}^2)$, whose matrix elements follow a short-range assumption:

Assumption 1. *There exists some $\alpha > 0$ such that*

$$\sup_{y \in \mathbb{Z}^2} \sum_{x \in \mathbb{Z}^2} |H_B(x, y)| (e^{\alpha|x-y|} - 1) \leq C < \infty,$$

where $|x| = |x_1| + |x_2|$ induces the taxicab metric.

2.1. General Setting

We also construct an *edge Hamiltonian* on the lattice $\mathbb{Z}_a^2 := \{x \in \mathbb{Z}^2 : x_2 > -a\}$, denoted by $H_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}_a^2)$. The bulk and edge Hamiltonians are related by the *edge operator* $E_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell(\mathbb{Z}^2)$,

$$E_a := \mathcal{J}_a H_a - H_B \mathcal{J}_a,$$

where $\mathcal{J}_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell(\mathbb{Z}^2)$ denotes extension by zeroes. We require only that the edge operator satisfies the edge assumption

Assumption 2. *The edge operator satisfies*

$$\sup_{z \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}_a^2} |E_a(x, y)| e^{\alpha(|x_2 + a| + |x_1 - y_1|)} \leq C < \infty$$

for some $\alpha > 0$, where $|x| = |x_1| + |x_2|$ is the taxicab metric.

The interpretation is that $E_a = \mathcal{J}_a H_a - H_B \mathcal{J}_a$ is the difference between first applying H_a on $\ell^2(\mathbb{Z}_a^2)$ and then extending by zeroes, versus first making everything below $x_2 = -a$ into zeroes and then applying H_B . The assumption ensures that the effects from introducing the edge at $-a$ die exponentially as we move upward away from the edge (due to the $|x_2 - (-a)|$ term in the exponent), and also that terms do not interact too much as their x_1 distance increases (due to the $|x_1 - y_1|$ term in the exponent).

A simple example of an edge Hamiltonian satisfying the edge condition is $H_a = \mathcal{J}_a^* H_B \mathcal{J}_a$, which gives $E_a = (\mathcal{J}_a \mathcal{J}_a^* - 1) H_B \mathcal{J}_a$. The matrix elements are

$$E_a(x, y) = \begin{cases} -H_B(x, y) & \text{if } x_2 < -a \\ 0 & \text{if } x_2 \geq -a \end{cases}$$

Intuitively, there is no difference between H_B and H_a on \mathbb{Z}_a^2 . The idea is that for a state $\psi \in \ell^2(\mathbb{Z}_a^2)$, we have $\langle \psi, H_a \psi \rangle = \langle (\mathcal{J}_a \psi), H_B (\mathcal{J}_a \psi) \rangle$, which we interpret as the edge Hamiltonian having the same expectation as the bulk Hamiltonian for states supported above $x_2 = -a$. The edge assumption 2 is satisfied since $|E_a(x, y)| \leq |H_B(x, y)|$, and H_B obeys the short range assumption 1.

We also make the following assumption about the bulk Hamiltonian.

Assumption 3. *The bulk Hamiltonian has a spectral gap. That is, there exists an interval $\Delta \subset \mathbb{R}$ such that*

$$\Delta \cap \text{Spec}(H_B) = \emptyset.$$

2.1. General Setting

Remark: The spectral gap assumption can be relaxed to a “mobility gap” assumption,

$$\sup_{f \in B_c(\Delta)} |f(H_B)(x, y)| (1 + |x|)^{-\alpha_1} e^{\alpha_2 |x-y|} < \infty$$

for some $\alpha_1 > 0$, where $B_c(\Delta)$ is the set of Borel functions f which are constant on $(-\infty, \inf \Delta)$ and on $(\sup \Delta, \infty)$ such that $|f(x)| \leq 1$ for all x [1].

We define the *bulk conductivity* at Fermi energy μ as follows. Suppose we subject the system to an external electric potential $V(x_2)$ in the x_2 direction. We write this as $V_0 \Lambda_2$, where Λ_i are multiplication operators

$$\Lambda_i |\psi(x_1, x_2)\rangle = \Lambda(x_i) |\psi(x_1, x_2)\rangle$$

where

$$\Lambda : \mathbb{R} \rightarrow \mathbb{R} \quad \Lambda(x_i) = \begin{cases} 1 & \text{if } x_i \leq 0 \\ 0 & \text{if } x_i \geq 1 \end{cases}$$

is called a *switch functions*, which is smooth and monotonically decreasing on $(0, 1)$. Note that the ensuing physics (in particular, our definition of the Hall conductivity) is independent of the particular choice of switch function Λ_i , since any two switch functions are exactly equal on the lattice.

We introduce a function which grows slowly in time as t grows from $-\infty$ to 0, so as to invoke the adiabatic principle later. We choose $e^{\varepsilon t}$, and we will let $\varepsilon \rightarrow 0$ at the end. The perturbation is $\Delta H_B(t) = V_0 \Lambda_2 e^{\varepsilon t}$, and the perturbed Hamiltonian is

$$\tilde{H}_B(t) = H_B + V_0 \Lambda_2 e^{\varepsilon t}.$$

We define the Hall current operator $J_H = -i[\tilde{H}_B(t), \Lambda_1] = -i[H_B, \Lambda_1]$, which measures the current across the fiducial line $x_1 = 0$.

We also denote by $P_\mu := P((-\infty, \mu])$ the projection-valued measure associated with H_B onto states with energy below the Fermi energy μ , which is assumed to lie in the spectral gap Δ (see Appendix A.1).

Proposition 1. *The Hall conductivity σ_H in the bulk system is equal to*

$$\sigma_B = -i \text{Tr} (P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]) .$$

2.1. General Setting

Proof. We begin with the Heisenberg equation of motion for the density matrix, $\dot{\rho}(t) = -i[\tilde{H}_B(t), \rho(t)]$, with initial condition $\lim_{t \rightarrow -\infty} \|\rho(t) - e^{itH_B} P_\mu e^{-itH_B}\| = 0$, which is equivalent to $\lim_{t \rightarrow -\infty} \|e^{-itH_B} \rho(t) e^{itH_B} - P_\mu\| = 0$.

We work in the interaction picture by defining $\rho_I(t) = e^{-itH_B} \rho(t) e^{itH_B}$ and $\Delta H_B(t) = e^{-itH_B} V_0 \Lambda_2 e^{itH_B}$. Thus

$$\dot{\rho}_I(t) = -i[\Delta H_B(t), \rho_I(t)].$$

The exact solution to this differential equation is readily verified to be

$$\rho_I(t) = i \int_{-\infty}^t [\Delta H_B(s), \rho_I(s)] ds + P_\mu.$$

Indeed, taking the derivative of the right hand side gives $i[\Delta H_B(t), \rho_I(t)]$, and the initial condition $\|\rho_I(t) - P_\mu\| \rightarrow 0$ is also satisfied. This also shows that

$$\|\rho_I(t) - P_\mu\| \leq 2 \int_{-\infty}^t \|V_0 \Lambda_2 e^{\varepsilon s} \rho_I(t)\| ds = \mathcal{O}(V_0),$$

which implies $[\Delta H_B(s), \rho_I(t)] = [\Delta H_B(s), P_\mu] + \mathcal{O}(V_0^2)$. Therefore

$$\rho_I(t) = i \int_{-\infty}^t [\Delta H_B(s), P_\mu] ds + P_\mu + \mathcal{O}(V_0^2).$$

The Hall conductivity is (by definition) the linear response coefficient, which is

$$\begin{aligned} \sigma_H &= \lim_{V_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{\text{Tr}(\rho_I(0) J_H) - \text{Tr}(P_\mu J_H)}{V_0} \\ &= - \lim_{V_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{i}{V_0} \text{Tr} \left(i \int_{-\infty}^0 [\Delta H_B(s), P_\mu] [H_B, \Lambda_1] ds \right), \end{aligned}$$

where we used the fact that the error $\mathcal{O}(V_0^2)$ introduced by replacing ρ_I with P_μ vanishes in the limit $V_0 \rightarrow 0$. Some simplifications can be made,

$$\begin{aligned} \sigma_H &= \lim_{V_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \frac{1}{V_0} \text{Tr} \left(\int_{-\infty}^0 [e^{-isH_B} V_0 \Lambda_2 e^{\varepsilon s} e^{isH_B}, P_\mu] [H_B, \Lambda_1] ds \right) \\ &= \lim_{V_0 \rightarrow 0} \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 e^{-isH_B} [\Lambda_2, P_\mu] e^{isH_B} [H_B, \Lambda_1] e^{\varepsilon s} ds \right) \\ &= \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 (e^{isH_B} [H_B, \Lambda_1] e^{-isH_B}) \cdot ([\Lambda_2, P_\mu] e^{\varepsilon s}) ds \right), \end{aligned}$$

2.1. General Setting

where we used cyclicity of the trace and the fact that P_μ and H_B commute. We also dropped the limit $V_0 \rightarrow 0$ in the final line because, even though the limits may not commute in general, the expression is independent of V_0 . Using integration by parts on the two terms in brackets, and noting that $\frac{d}{ds}(e^{isH_B}\Lambda_1 e^{-isH_B} - \Lambda_1) = ie^{isH_B}[H_B, \Lambda_1]e^{-isH_B}$, we obtain

$$\begin{aligned}\sigma_H &= -i \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 (e^{isH_B}\Lambda_1 e^{-isH_B} - \Lambda_1) \frac{d}{ds} ([\Lambda_2, P_\mu] e^{\varepsilon s}) ds \right) \\ &= -i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \Lambda_1^s [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right)\end{aligned}$$

where $\Lambda_1^s := e^{isH_B}\Lambda_1 e^{-isH_B} - \Lambda_1$. Using the notation $\overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, it is readily verified that the commutator $[\Lambda_2, P_\mu]$ is an *off-diagonal* operator, in the sense that $[\Lambda_2, P_\mu] = \overline{[\Lambda_2, P_\mu]}$. Furthermore, a simple computation reveals that for any two trace-class operators A and B , $\text{Tr}(\overline{AB}) = \text{Tr}(A\overline{B})$. It therefore follows that

$$\sigma_H = -i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \overline{\Lambda_1^s} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right).$$

The integrand can be broken into two terms,

$$\overline{\Lambda_1^s} [\Lambda_2, P_\mu] e^{\varepsilon s} = e^{isH_B} \overline{\Lambda_1} e^{-isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} - \overline{\Lambda_1} [\Lambda_2, P_\mu] e^{\varepsilon s}$$

by commutativity of P_μ and H_B . We show that the integral of the first term vanishes. We begin by breaking the first term down further into

$$e^{isH_B} P_\mu \Lambda_1 P_\mu^\perp e^{-isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} + e^{isH_B} P_\mu^\perp \Lambda_1 P_\mu e^{-isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s}.$$

We treat the first of these two terms; the other is handled in an identical manner. We invoke the spectral theorem (Appendix A.1) to write $e^{isH_B} P_\mu = \int_{-\infty}^\mu e^{is\lambda} dP_\lambda$, and similarly $P_\mu^\perp e^{-isH_B} = (\mathbb{1} - P_\mu) e^{-isH_B} = \int_\mu^\infty e^{-is\nu} dP_\nu$.

Since the Fermi energy μ is assumed to lie in a spectral gap, there must exist a neighbourhood $(\mu - \delta, \mu + \delta)$ in which there are no states. We exploit this fact to rewrite the limits of integration, $\int_{-\infty}^{\mu - \delta} e^{is\lambda} dP_\lambda$ and $\int_{\mu + \delta}^\infty e^{-is\nu} dP_\nu$. We therefore obtain

2.1. General Setting

$$\begin{aligned}
& \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{-\infty}^0 e^{isH_B} P_\mu \Lambda_1 P_\mu^\perp e^{-isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \\
&= \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \int_{-\infty}^{\mu-\delta} e^{is\lambda} dP_\lambda \Lambda_1 \int_{\mu+\delta}^\infty e^{-is\nu} dP_\nu [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right) \\
&= \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \int_{-\infty}^{\mu-\delta} \int_{\mu+\delta}^\infty e^{s(\varepsilon-i\nu+i\lambda)} dP_\lambda \Lambda_1 dP_\nu [\Lambda_2, P_\mu] ds \right)
\end{aligned}$$

Performing the integral over s yields

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{-\infty}^0 e^{s(\varepsilon-i\nu+i\lambda)} ds = - \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{i\varepsilon + \nu - \lambda}$$

This limit is zero, since $\lambda \neq \nu$. Indeed, due to the spectral gap, the integration variables live in $\lambda \in (-\infty, \mu - \delta)$ and $\nu \in (\mu + \delta, \infty)$. The case for the $e^{isH_B} P_\mu^\perp \Lambda_1 P_\mu e^{-isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s}$ term (where the P_μ and P_μ^\perp swap places) is treated analogously. Hence the first term in the integrand for σ_H vanishes, as claimed.

Finally, we return to our expression for the Hall conductivity, which now reads

$$\sigma_H = i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \overline{\Lambda_1} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right).$$

It is a basic algebraic calculation to show that $\overline{\Lambda_1} = [[\Lambda_1, P_\mu], P_\mu]$. Evaluating the integral over s is now trivial; $\int_{-\infty}^0 e^{\varepsilon s} ds = \varepsilon^{-1}$. Thus

$$\sigma_H = i \text{Tr}([[\Lambda_1, P_\mu], P_\mu] [\Lambda_2, P_\mu]).$$

Shifting the commutator completes the proof:

$$\begin{aligned}
\sigma_H &= i \text{Tr}(P_\mu [[\Lambda_2, P_\mu], [\Lambda_1, P_\mu]]) \\
&= -i \text{Tr}(P_\mu [[\Lambda_1, P_\mu], [\Lambda_2, P_\mu]]) \\
&= -i \text{Tr}(P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]).
\end{aligned}$$

The justification for shifting the commutator is that $[\Lambda_1, P_\mu] [\Lambda_2, P_\mu]$ is trace class by the proof of Lemma 5, noting that

$$\|[\Lambda_1, P_\mu] [\Lambda_2, P_\mu]\|_1 \leq \|[\Lambda_1, P_\mu] e^{3\delta|x_1|} e^{-\delta|x|}\| \cdot \|e^{-\delta|x|}\|_1 \cdot \|e^{3\delta|x_2|} e^{-\delta|x|} [\Lambda_2, P_\mu]\|$$

where $|x| = |x_1| + |x_2|$, and the $e^{-\delta|x|}$ term is trace-class by lemma 4. \square

2.2. Equality of Bulk and Edge Conductivities

For the *edge conductivity*, we again need the current operator across the line $x_1 = 0$, which is this time given by $-i[H_a, \Lambda_1]$. We define

$$\sigma_E = -i \lim_{a \rightarrow \infty} \text{Tr}(\rho'(H_a)[H_a, \Lambda_1]),$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(r) = \begin{cases} 1 & \text{if } r \leq \inf \Delta \\ 0 & \text{if } r \geq \sup \Delta \end{cases}$$

and decreases smoothly and monotonically in Δ . The definition of σ_E is reminiscent of another formula we will see later in the interacting setting, $\text{Tr}(\dot{P}J)$, where J is the current operator. The interpretation of σ_E is that if we apply a small potential difference V across $x_2 = -a$ to $x_2 = \infty$, there will be a net Hall current

$$\begin{aligned} I &= -i \text{Tr}(\rho(H_a + V)[H_a + V, \Lambda_1] - \rho(H_a)[H_a, \Lambda_1]) \\ &= -i \text{Tr}((\rho(H_a + V) - \rho(H_a))[H_a, \Lambda_1]) \end{aligned}$$

Thus we obtain the conductivity

$$\sigma_E = \frac{I}{V} = -i \text{Tr} \left(\frac{(\rho(H_a + V) - \rho(H_a))}{V} [H_a, \Lambda_1] \right) \rightarrow -i \text{Tr}(\rho'(H_a)[H_a, \Lambda_1])$$

in the limit as $V \rightarrow 0$. As we shall see, it turns out that σ_E is independent of the choice of ρ , and σ_B is independent of λ .

2.2 Equality of Bulk and Edge Conductivities

The main result of this section is

Theorem 1. $\sigma_E = \sigma_B$.

2.2.1 Outline of the Proof

Before giving the proof in its entirety, we outline the basic steps. Lemma 1 shows that $\sigma_E := \rho'(H_a)[H_a, \Lambda_1]$ is trace-class. We posit that the edge conductivity can be rewritten as $\sigma_E = \lim_{a \rightarrow \infty} \sigma_E(a)$, where

$$\sigma_E(a) = -i \text{Tr}(\rho'(H_a)[H_a, \Lambda_1] \Lambda_2),$$

2.2. Equality of Bulk and Edge Conductivities

since we have assumed that there is a spectral gap (as opposed to a mobility gap), so that there are extended states near the edge, and no bound states or resonances far from the edge. Thus, intuitively, the cutoff introduced by Λ_2 is irrelevant as we take the edge $a \rightarrow \infty$. We provide a more concrete justification for this later in Lemma 1.

The key ingredient of the proof is the use of the functional calculus given by the Helffer-Sjöstrand representation of self-adjoint operators on a Hilbert space (Appendix C). The two crucial operators for the proof written in their Helffer-Sjöstrand representations are

$$\begin{aligned}\rho(H) &= -\frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R(z) dz \wedge d\bar{z} \\ \rho'(H) &= \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R(z)^2 dz \wedge d\bar{z}\end{aligned}$$

where $R(z) = (H - z)^{-1}$ is the resolvent of H . For ease of notation, we write $dz^2 = dz \wedge d\bar{z}$ from this point.

Another key observation is that it turns out that the bulk conductivity can actually be written

$$\sigma_B = i\text{Tr}([\rho(H_B), \Lambda_1] \Lambda_2).$$

By employing the Helffer-Sjöstrand representations above, one can add an operator of zero trace to the edge conductivity, and show that this operator converges in trace to $[\rho(H_B), \Lambda_1] \Lambda_2$ in the limit $a \rightarrow \infty$.

2.2.2 The Proof

Lemma 1. *The edge conductivity is equal to $\lim_{a \rightarrow \infty} \sigma_E(a)$, where*

$$\sigma_E(a) = -i\text{Tr}(\rho'(H_a)[H_a, \Lambda_1] \Lambda_2).$$

Proof. Since σ_B is translation invariant [1], we only need to prove the case $-i\text{Tr}(\rho'(H_{a=0})[H_{a=0}, \Lambda_1]) = \sigma_B$. We drop the subscript, $H := H_{a=0}$.

A general fact of functional analysis is that if $A_n \xrightarrow{s} 0$ and X is trace-class, then $\|A_n X\|_1 \rightarrow 0$. Since the multiplication operator $\Lambda_{2,n}|\psi\rangle := \Lambda(x_2 - n)|\psi\rangle$ converges strongly to the identity as $n \rightarrow \infty$, it follows that

$$\|\rho'(H)[H, \Lambda_1](\mathbb{1} - \Lambda_{2,n})\|_1 \rightarrow 0,$$

and thus from the inequality $|\text{Tr}(A)| \leq \|A\|_1$ we deduce

2.2. Equality of Bulk and Edge Conductivities

$$\sigma_E(a) = -i \text{Tr}(\rho'(H)[H, \Lambda_1]) = -i \lim_{n \rightarrow \infty} \text{Tr}(\rho'(H)[H, \Lambda_1] \Lambda_{2,n}).$$

Consider the edge Hamiltonian with cutoff at $a = 0$ associated with the bulk Hamiltonian shifted down by n . We denote this modified edge Hamiltonian by H^n .

Whether we first cut off everything above $x_2 = n$ using $\Lambda_{2,n}$ and then apply $H_{a=0}$, or instead cut off everything above $x_2 = 0$ using Λ_2 and then apply the Hamiltonian H^n is immaterial. In other words,

$$-i \lim_{n \rightarrow \infty} \text{Tr}(\rho'(H)[H, \Lambda_1] \Lambda_{2,n}) = -i \lim_{n \rightarrow \infty} \text{Tr}(\rho'(H^n)[H^n, \Lambda_1] \Lambda_2).$$

Thus, our goal is to show that

$$\lim_{n \rightarrow \infty} \text{Tr}(\rho'(H^n)[H^n, \Lambda_1] \Lambda_2) = \lim_{a \rightarrow \infty} \text{Tr}(\rho'(H_a)[H_a, \Lambda_1] \Lambda_2).$$

□

Define

$$Z(a) = [\rho(H_a), \Lambda_1] \Lambda_2 - \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [R_a(z), [H_a, \Lambda_1] \Lambda_2] dz^2.$$

Lemma 2. *The two terms in $Z(a)$ are separately trace-class, and the trace of each is zero.*

Proof. In the Helffer-Sjöstrand representation, the first term is

$$[\rho(H_a), \Lambda_1] \Lambda_2 = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} [R_a(z), \Lambda_1] dz^2.$$

The operator in the integrand can be broken into four factors,

$$[R_a(z), \Lambda_1] = -R_a(z) [H_a, \Lambda_1] e^{\delta|x_1|} \cdot e^{-\delta|x|} \cdot e^{\delta|x_2|} R_a(z) e^{-\delta|x_2|} \cdot e^{\delta|x_2|} \Lambda_2.$$

Lemma 3 bounds the first factor, Lemma 4 ensures the second is trace-class, the Combes-Thomas bound $\|e^{\varepsilon f(x)} R(z) e^{-\varepsilon f(x)}\| \leq C |\text{Im}(z)|^{-1}$ takes care of the third, and the fourth is bounded by 1. Finally, Lemma 14 guarantees integrability. Thus $[\rho(H_a), \Lambda_1] \Lambda_2$ is trace-class, and it clearly has vanishing trace since shifting the commutator gives $[\Lambda_1, \Lambda_2] = 0$.

2.2. Equality of Bulk and Edge Conductivities

The second term of $Z(a)$ is also trace-class because $[H_a, \Lambda_1]\Lambda_2$ is, which can be seen by breaking it into

$$[H_a, \Lambda_1]\Lambda_2 = [H_a, \Lambda_1]e^{\delta|x_1|} \cdot e^{-\delta|x|} \cdot e^{\delta|x_2|}\Lambda_2.$$

Again, $\|e^{\delta|x_2|}\Lambda_2\| \leq 1$ and Lemmas 3 and 4 are enough to conclude the operator is trace-class. The commutator in the integrand can therefore be shifted to give $[R_a(z), R_a(z)] = 0$. \square

So $\sigma_E(a) = \text{Tr}(\Sigma(a))$, where

$$\begin{aligned} \Sigma(a) &= -i\rho'(H_a)[H_a, \Lambda_1]\Lambda_2 + iZ(a) \\ &= -i\rho'(H_a)[H_a, \Lambda_1]\Lambda_2 + i[\rho(H_a), \Lambda_1]\Lambda_2 \\ &\quad - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z)[R_a(z), [H_a, \Lambda_1]\Lambda_2] dz^2. \end{aligned}$$

Using the Helffer-Sjöstrand representations for the first two terms on the right hand side, we obtain

$$\begin{aligned} \Sigma(a) &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z)^2 [H_a, \Lambda_1]\Lambda_2 dz^2 + \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [H_a, \Lambda_1] R_a(z) \Lambda_2 dz^2 \\ &\quad - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} (R_a(z)^2 [H_a, \Lambda_1]\Lambda_2 - R_a(z) [H_a, \Lambda_1] \Lambda_2 R_a(z)) dz^2 \\ &= -\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [H_a, \Lambda_1] [R_a(z), \Lambda_2] dz^2 \\ &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [H_a, \Lambda_1] R_a(z) [H_a, \Lambda_2] R_a(z) dz^2, \end{aligned}$$

where we used

$$[R_a(z), \Lambda_i] = -R_a(z)[H_a, \Lambda_i]R_a(z)$$

in the final equality. The goal is to prove that the operator $\Sigma(a)$ converges to the corresponding bulk operator in trace-norm,

$$\|\Sigma(a) - \Sigma_B\|_1 \rightarrow 0$$

as $a \rightarrow \infty$, which in turn proves that $\text{Tr}(\Sigma(a)) \rightarrow \text{Tr}(\Sigma_B)$ because of the bound $|\text{Tr}(A)| \leq \|A\|_1$. Here, Σ_B is the same operator as before, but defined using the bulk operators H_B and $R_B(z)$. Once this limit is established, we shall prove that $\sigma_B = \text{Tr}(\Sigma_B)$ to conclude the proof.

2.2. Equality of Bulk and Edge Conductivities

To show that the limit is zero as claimed, we bound the trace norm of the integrand of $\Sigma(a)$ with the ultimate goal of applying dominated convergence. We accomplish this bound by breaking it into three parts,

$$R_a[H_a, \Lambda_1]R_a[H_a, \Lambda_2]R_a = -\mathcal{J}_a[R_a, \Lambda_1]e^{\delta|x_1|}\mathcal{J}_a^* \cdot e^{-\delta|x|} \cdot \mathcal{J}_a e^{\delta|x_2|}[H_a, \Lambda_2]R_a\mathcal{J}_a^*,$$

and bounding the norm of each with the following two lemmas. We remark that the extension \mathcal{J}_a and its adjoint have norm 1.

Lemma 3.

$$\|[H_a, \Lambda_i]e^{\delta|x_i|}\| \leq C.$$

Proof. The operator can be bounded by inspecting its matrix elements

$$\begin{aligned} \langle x, [H_a, \Lambda_i]e^{\delta|x_i|}y \rangle &= \langle x, H_a\Lambda_i y \rangle e^{\delta|y_i|} - \langle x, \Lambda_i H_a y \rangle e^{\delta|y_i|} \\ &= H_a(x, y)e^{\delta|y_i|}(\Lambda(y_i) - \Lambda(x_i)). \end{aligned}$$

This is zero if $|x_i - y_i| \leq |y_i|$, since this would imply that x_i and y_i have the same sign, yielding $\Lambda(x_i) = \Lambda(y_i)$. So either the matrix element is zero, or $|y_i| \leq |x_i - y_i|$, which implies

$$\begin{aligned} |H_a(x, y)e^{\delta|y_i|}(\Lambda(y_i) - \Lambda(x_i))| &\leq 2|H_a(x, y)|e^{\delta|x_i - y_i|} \\ &\leq 2|H_a(x, y)|e^{\delta|x - y|} \\ &\leq C|H_a(x, y)|(e^{\delta|x - y|} - 1), \end{aligned}$$

where the final inequality comes from the fact that the diagonal matrix elements are zero. Hence the short range assumption

$$\sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} |H(x, y)|(e^{\mu|x - y|} - 1) < \infty$$

combined with Holmgren's bound

$$\|A\| \leq \max \left\{ \sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} |A(x, y)|, \sup_{y \in \mathbb{Z}^2} \sum_{x \in \mathbb{Z}^2} |A(x, y)| \right\}$$

completes the proof. □

Lemma 4. $e^{-\delta|x_1|}e^{-\delta|x_2|}$ is trace-class.

2.2. Equality of Bulk and Edge Conductivities

Proof. We bound the trace norm by noticing that this is a positive operator satisfying

$$\langle (n, m), e^{-\delta|x_1|} e^{-\delta|x_2|} (n, m) \rangle = \langle e^{-\delta|x_1|} e^{-\delta|x_2|} (n, m), (n, m) \rangle,$$

so that its trace norm is equal to its trace. In the position basis, we see that its trace is given by a geometric series

$$\begin{aligned} \text{Tr}(e^{-\delta|x_1|} e^{-\delta|x_2|}) &= \sum_{(n,m) \in \mathbb{Z}^2} \langle (n, m), e^{-\delta|x_1|} e^{-\delta|x_2|} (n, m) \rangle \\ &\leq 2 \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} e^{-\delta m} e^{-\delta n} \\ &= 2 \left(\frac{1}{1 - e^{-\delta}} \right)^2. \end{aligned}$$

□

Now we return to the integrand which was broken into three parts. For the first term, $\mathcal{J}_a[R_a(z), \Lambda_1] e^{\delta|x_1|} \mathcal{J}_a^*$, we bound its operator norm by breaking it down further into

$$\begin{aligned} \|\mathcal{J}_a[R_a(z), \Lambda_1] e^{\delta|x_1|} \mathcal{J}_a^*\| &= \|[R_a(z), \Lambda_1] e^{\delta|x_1|}\| \\ &= \|-R_a(z)[H_a, \Lambda_1] R_a(z) e^{\delta|x_1|}\| \\ &\leq \|R_a(z)\| \cdot \|[H_a, \Lambda_1] e^{\delta|x_1|}\| \cdot \|e^{-\delta|x_1|} R_a(z) e^{\delta|x_1|}\|. \end{aligned}$$

The norm of $R_a(z)$ is bounded by

$$\|R_a(z)\| \leq \frac{1}{|\text{Im}(z)|}$$

for any $z \notin \mathbb{R}$ since H_a is self-adjoint. The second term is bounded by Lemma 3. Finally, for the third term $e^{-\delta|x_1|} R_a(z) e^{\delta|x_1|}$, we apply the Combes-Thomas bound,

$$\|e^{-\varepsilon f(x)} R_a(z) e^{\varepsilon f(x)}\| \leq \frac{C}{|\text{Im}(z)|}$$

where $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$ is any Lipschitz function, and ε can be chosen as $\varepsilon = \frac{1}{C(1+|\text{Im}(z)|)}$. Altogether, the bound of the first term of the integrand takes the form

$$\frac{C}{\operatorname{Im}(z)^2}.$$

The second term of the integrand is trace-class by Lemma 4. Finally, the bound for the third term of the integrand, $e^{\delta|x_2|}[H_a, \Lambda_2]R_a(z)$, follows from the bound on R and Lemma 3, and is again of the form $\frac{C}{\operatorname{Im}(z)^2}$.

Altogether, the trace norm of the integrand is bounded by

$$\|R_a(z)[H_a, \Lambda_1]R_a(z)[H_a, \Lambda_2]R_a(z)\|_1 \leq \frac{C}{\operatorname{Im}(z)^4},$$

so Lemma 14 provides the necessary bound for dominated convergence of $\Sigma(a)$ in a . Thus, it suffices to show pointwise convergence in z of the integrand of $\Sigma(a)$ to the integrand of Σ_B , i.e.

$$\mathcal{J}_a[R_a(z), \Lambda_1]e^{\delta|x_1|}\mathcal{J}_a^* \xrightarrow{s} [R_B(z), \Lambda_1]e^{\delta|x_1|},$$

$$\mathcal{J}_a e^{\delta|x_2|}[H_a, \Lambda_2]\mathcal{J}_a^* \xrightarrow{s} e^{\delta|x_2|}[H_B, \Lambda_2],$$

and

$$\mathcal{J}_a R_a(z)\mathcal{J}_a^* \xrightarrow{s} R_B(z)$$

for each fixed $z \in \mathbb{C}$. Inspecting the bounds we found for the left hand sides of these limits, it is clear that they are in fact uniformly bounded in a . By Lemma 11, it therefore suffices to show convergence on a dense subspace of $\ell^2(\mathbb{Z}^2)$; in particular, we may choose the dense subspace of compactly supported states, which allows us to ignore the $e^{\delta|x_i|}$ terms. Thus, we need to prove

$$\mathcal{J}_a[R_a(z), \Lambda_1]\mathcal{J}_a^* \xrightarrow{s} [R_B(z), \Lambda_1],$$

$$\mathcal{J}_a[H_a, \Lambda_2]\mathcal{J}_a^* \xrightarrow{s} [H_B, \Lambda_2],$$

and

$$\mathcal{J}_a R_a(z)\mathcal{J}_a^* \xrightarrow{s} R_B(z).$$

In fact, the final statement implies the first two; we appeal to the general fact of functional analysis that strong convergence of the resolvent of a self-adjoint operator implies that $\mathcal{J}_a f(H_a)\mathcal{J}_a^* \xrightarrow{s} f(H_B)$ for any bounded and continuous function f ([16] Theorem VIII.20). In particular, it follows from

2.2. Equality of Bulk and Edge Conductivities

$[R, \Lambda_i] = -R[H, \Lambda_i]R$ and Lemma 3 that the functions $[(\cdot - z)^{-1}, \Lambda_1]$ and $[\cdot, \Lambda_2]$ above are bounded and continuous, so we will have proven the desired limits if we can prove strong convergence of the resolvent, $\mathcal{J}_a R_a(z) \mathcal{J}_a^* \xrightarrow{s} R_B(z)$.

To prove this, we use the edge assumption. Recall the edge operator, $E_a = \mathcal{J}_a H_a - H_B \mathcal{J}_a$. Adding and subtracting $z \mathcal{J}_a$ gives

$$E_a = \mathcal{J}_a(H_a - z) - (H_B - z)\mathcal{J}_a.$$

Applying R_B from the left and R_a from the right on both sides, we obtain

$$R_B(z)E_a R_a(z) = R_B(z)\mathcal{J}_a - \mathcal{J}_a R_a(z).$$

Taking the adjoint, and then multiplying from the left by \mathcal{J}_a , we see that

$$\mathcal{J}_a R_a(z) E_a^* R_B(z) = \mathcal{J}_a \mathcal{J}_a^* R_B(z) - \mathcal{J}_a R_a(z) \mathcal{J}_a^*.$$

Thus

$$R_B(z) - \mathcal{J}_a R_a(z) \mathcal{J}_a^* = (\mathcal{J}_a R_a(z) E_a^* - \mathcal{J}_a \mathcal{J}_a^* + \mathbb{1}) R_B(z) \xrightarrow{s} 0,$$

since $E_a^* \xrightarrow{s} 0$ by Lemma 6, and $-\mathcal{J}_a \mathcal{J}_a^* + \mathbb{1} \xrightarrow{s} 0$. This proves that the limits above converge to the desired associated bulk operators, and hence $\|\Sigma(a) - \Sigma_B\|_1 \rightarrow 0$. This also shows that $\text{Tr}(\Sigma(a)) \rightarrow \text{Tr}(\Sigma_B)$, and thus Σ_B is trace-class.

Finally, it remains to show that

$$\text{Tr}(\Sigma_B) = \sigma_B.$$

First, we manipulate

$$\begin{aligned} \Sigma_B &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z) [H_B, \Lambda_1] R_B(z) [H_B, \Lambda_2] R_B(z) dz^2 \\ &= -\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z) [H_B, \Lambda_1] [R_B(z), \Lambda_2] dz^2 \\ &= i[\rho(H_B), \Lambda_1] \Lambda_2 - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z) [H_B, \Lambda_1] \Lambda_2 R_B(z) dz^2. \end{aligned}$$

Define $P_+ := P((\sup \Delta, \infty))$ and $P_- := P((-\infty, \inf \Delta))$, the projections onto states above and below the gap, respectively. Since H_B is assumed to have a gap, it follows that $P_- + P_+ = \mathbb{1}$, and thus

2.2. Equality of Bulk and Edge Conductivities

$$\mathrm{Tr}(\Sigma_B) = \mathrm{Tr}((P_- + P_+)\Sigma_B(P_- + P_+)) = \mathrm{Tr}(P_+\Sigma_BP_+) + \mathrm{Tr}(P_-\Sigma_BP_-),$$

where we've used the fact that Σ_B is trace-class to cycle the projections and get rid of the $P_-P_+ = 0$ terms. We now argue that the integral term in $P_\pm\Sigma_BP_\pm$ vanishes.

Consider P_-R_B and R_BP_- . The only place R_B is non-analytic is on the spectrum of H_B ; in particular, R_B is analytic on the gap Δ . Furthermore, P_- is equal to 0 on $(\inf \Delta, \infty)$. From these two facts, it follows that P_-R_B and R_BP_- are analytic on $\mathbb{C} \setminus (\sup \Delta, \infty)$. However, the integrand is zero on $(\sup \Delta, \infty)$, since $\tilde{\rho} = 0$ there. Thus $\tilde{\rho}P_-R_B$ and $\tilde{\rho}R_BP_-$ are analytic on the entire complex plane. The integral can therefore be written

$$\int_{\mathbb{C}} \frac{\partial}{\partial \bar{z}} (\tilde{\rho}(z)P_-R_B(z)[H_B, \Lambda_1]\Lambda_2R_B(z)P_-) dz^2,$$

with the derivative $\frac{\partial}{\partial \bar{z}}$ acting on the entire integrand rather than only on $\tilde{\rho}(z)$. Furthermore, since $R_B(z)$ is bounded in norm by $|\mathrm{Im}(z)|^{-1}$ and $|\partial_{\bar{z}}\tilde{\rho}|$ can be made to decay with any power law $\mathcal{O}(\mathrm{Im}(z)^k)$ as $\mathrm{Im}(z) \rightarrow 0$ (see Appendix C), we may write the integral as a limit of disks,

$$\lim_{r \rightarrow \infty} \int_{B_r(0)} \frac{\partial}{\partial \bar{z}} (\tilde{\rho}(z)P_-R_B(z)[H_B, \Lambda_1]\Lambda_2R_B(z)P_-) dz^2,$$

where domination is provided by Lemma 14. Stokes' theorem then says that the above integral is bounded in norm by

$$\lim_{r \rightarrow \infty} \int_{\partial B_r(0)} \tilde{\rho}(z) \|P_-R_B(z)[H_B, \Lambda_1]\Lambda_2R_B(z)P_-\| dz.$$

The decay of $\|R_B(z)\| \leq \mathrm{dist}(z, \mathrm{Spec}(H_B))^{-1}$ and boundedness of the extension $\tilde{\rho}(z)$ (since $|\rho(x)| \leq 1$) ensure that the limit of this integral as $r \rightarrow \infty$ is zero. Indeed, the norm of the integral is bounded by

$$2\pi r \sup_{\partial B_r(0)} \frac{|\tilde{\rho}(r)| \| [H_B, \Lambda_1]\Lambda_2 \|}{\mathrm{dist}(r, \mathrm{Spec}(H_B))^2}$$

which vanishes as $r \rightarrow \infty$ since the spectrum of the bulk Hamiltonian is bounded, and Lemma 3 bounds the norm of the operator in the numerator.

As for the integral appearing in $P_+\Sigma_BP_+$, we note that analogous to the argument above, R_BP_+ and P_+R_B are analytic on $\mathbb{C} \setminus (-\infty, \inf \Delta)$, and thus $(1 - \tilde{\rho})R_BP_+$ and $(1 - \tilde{\rho})P_+R_B$ are analytic on all of \mathbb{C} , since $1 - \tilde{\rho}$ is

2.2. Equality of Bulk and Edge Conductivities

zero on $(-\infty, \inf \Delta)$. Replacing $\frac{\partial \tilde{\rho}}{\partial \bar{z}}$ with $\frac{\partial(1-\tilde{\rho})}{\partial \bar{z}}$ of course leaves the integral unchanged, and the rest of the argument follows as before. Thus the integral term in Σ_B is zero, and

$$\text{Tr}(\Sigma_B) = i\text{Tr}(P_+[\rho(H_B), \Lambda_1]\Lambda_2 P_+) + i\text{Tr}(P_-[\rho(H_B), \Lambda_1]\Lambda_2 P_-).$$

By the spectral theorem for projection-valued measures 3, if the Fermi energy lies in the gap, $\lambda \in \Delta$, we have

$$\rho(H_B) = \int_{-\infty}^{\lambda} dP_\nu = P_\lambda.$$

We may therefore replace $\rho(H_B)$ by P_λ , by which we obtain $\text{Tr}(\Sigma_B) = i\text{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+) + i\text{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-)$. We must relate this expression to the bulk conductivity. An algebraic calculation shows that that

$$P_\lambda[[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]] = P_\lambda \Lambda_2 P_\lambda^\perp \Lambda_1 P_\lambda - P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda.$$

The two terms on the right hand side are separately trace-class by Lemma 5, so that by cyclicity of the trace, the bulk conductivity is given by

$$\begin{aligned} \sigma_B &= i\text{Tr}(P_\lambda \Lambda_2 P_\lambda^\perp \Lambda_1 P_\lambda - P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda) \\ &= i\text{Tr}(P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_\lambda^\perp - P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda) \\ &= \text{Tr}(T_\lambda), \end{aligned}$$

Where we've defined $T_\lambda := P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_\lambda^\perp - P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda$. To finish the proof, we must show that

$$P_\pm T_\lambda P_\pm = P_\pm [P_\lambda, \Lambda_1] \Lambda_2 P_\pm.$$

First, notice that because of the gap, we have $P_\lambda^\perp P_- = 0$ and $P_\lambda P_- = P_-$. Thus

$$\begin{aligned} P_- T_\lambda P_- &= P_- P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda P_- \\ &= P_- (P_\lambda \Lambda_1 \Lambda_2 - \Lambda_1 P_\lambda \Lambda_2) P_- \\ &= P_- [P_\lambda, \Lambda_1] \Lambda_2 P_-, \end{aligned}$$

as desired. By a similar argument, for P_+ we have $P_\lambda^\perp P_+ = P_+$ and $P_\lambda P_+ = 0$, which implies

$$\begin{aligned}
P_+ T_\lambda P_+ &= -P_+ P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_\lambda^\perp P_+ \\
&= -P_+ P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_+ \\
&= -P_+ P_\lambda^\perp \Lambda_1 \Lambda_2 P_+ + P_+ P_\lambda^\perp \Lambda_1 P_\lambda^\perp \Lambda_2 P_+ \\
&= -P_+ P_\lambda^\perp \Lambda_1 \Lambda_2 P_+ + P_+ \Lambda_1 P_\lambda^\perp \Lambda_2 P_+ \\
&= -P_+ [P_\lambda^\perp, \Lambda_1] \Lambda_2 P_+ \\
&= -P_+ [(1 - P_\lambda), \Lambda_1] \Lambda_2 P_+ \\
&= P_+ [P_\lambda, \Lambda_1] \Lambda_2 P_+.
\end{aligned}$$

Altogether, we obtain

$$\begin{aligned}
\sigma_B &= i\text{Tr}(P_- T_\lambda P_-) + i\text{Tr}(P_+ T_\lambda P_+) \\
&= i\text{Tr}(P_- [P_\lambda, \Lambda_1] \Lambda_2 P_-) + i\text{Tr}(P_+ [P_\lambda, \Lambda_1] \Lambda_2 P_+) \\
&= \text{Tr}(\Sigma_B),
\end{aligned}$$

concluding the proof.

Lemma 5. *Let $W \cap V = \emptyset$. Then*

$$P_W \Lambda_1 P_V \Lambda_2 P_W \in \mathcal{J}_1.$$

In particular, the two terms appearing in T_λ are separately trace-class.

Proof. We break down

$$P_W \Lambda_1 P_V \Lambda_2 P_W = P_W \Lambda_1 P_V e^{3\delta|x_1|} e^{-\delta|x|} \cdot e^{-\delta|x|} \cdot e^{-\delta|x|} e^{3\delta|x_2|} P_V \Lambda_2 P_W.$$

Since the middle term is trace-class by Lemma 4, it suffices to prove boundedness of the first and last terms.

To that end, notice that $P_W \Lambda_i P_V = P_W [\Lambda_i, P_V]$. Since P_W is obviously bounded, we only need to bound the product of the commutator with the appropriate exponentials. This can be done by decomposing

$$[\Lambda_i, P_V] e^{3\delta x_i} e^{-\delta|x|} = \Lambda_i P_V (1 - \Lambda_i) e^{3\delta x_i} e^{-\delta|x|} + (1 - \Lambda_i) P_V \Lambda_i e^{3\delta x_i} e^{-\delta|x|}.$$

Note the lack of absolute values on the $e^{3\delta x_i}$ terms; we will later also bound the same operator but with $e^{-3\delta x_i}$ to account for this, since $e^{3\delta|x_i|} = e^{\pm 3\delta x_i}$. We begin by bounding the second term. Since multiplication operators commute, the norm of the second term can be broken into

2.2. Equality of Bulk and Edge Conductivities

$$\|(1 - \Lambda_i)P_V e^{-\delta|x|}\| \|\Lambda_i e^{3\delta x_i}\|$$

both of which are bounded, since for $\Lambda_i e^{3\delta x_i}$ only the negative x_i values remain due to the switch function. Returning to the first term, we insert more exponentials

$$\Lambda_i P_V (1 - \Lambda_i) e^{3\delta x_i} e^{-\delta|x|} = \Lambda_i e^{3\delta x_i} \cdot e^{-3\delta x_i} P_V (1 - \Lambda_i) e^{3\delta x_i} e^{-\delta|x|}.$$

Again $\Lambda_i e^{3\delta x_i}$ is bounded because of the switch function, and the other part is seen to be bounded by taking the adjoint of the operator in Lemma 13.

An exactly analogous argument works for bounding $[\Lambda_i, P_V] e^{-3\delta x_i} e^{-\delta|x|}$, i.e. the same operator but with the $e^{-3\delta x_i}$ terms. \square

Lemma 6. E_a and E_a^* converge strongly to zero in the limit $a \rightarrow \infty$.

Proof. Let $\psi \in \ell^2(\mathbb{Z}^2)$. Since E_a has real entries,

$$\begin{aligned} \|E_a^* \psi\|^2 &= \langle E_a^* \psi, E_a^* \psi \rangle \\ &= \sum_z \left(\sum_y \overline{E_a^*(z, y) \psi(y)} \right) \left(\sum_x E_a^*(z, x) \psi(x) \right) \\ &= \sum_z \left(\sum_y E_a(y, z) \overline{\psi(y)} \right) \left(\sum_x E_a(x, z) \psi(x) \right) \end{aligned}$$

Consider the $\sum_x E_a(x, z) \psi(x)$ term. To take advantage of the edge assumption, we insert the exponentials

$$\sum_x E_a(x, z) e^{\alpha(|z_2 - a| - |z_1 - x_1|)} e^{-\alpha(|z_2 - a| - |z_1 - x_1|)} \psi(x).$$

By assumption 2, $g(z) := \sup_x E_a(x, z) e^{\alpha(|z_2 - a| - |z_1 - x_1|)}$ is finite and summable. Precisely the same argument can be used on the sum over y . Altogether,

$$\|E_a^* \psi\|^2 \leq e^{-\alpha|a|} \sum_z g(z)^2 e^{-2\alpha(|z_1| + |z_2|)} \sum_x e^{-\alpha|x_1|} \psi(x) \sum_y e^{-\alpha|y_1|} \overline{\psi(y)},$$

where we bounded the exponential by $e^{-\alpha(|z_1| + |z_2|)} e^{-\alpha|x_1|} e^{-\alpha|a|}$. Since g is summable, g^2 is also summable, and so too is the summand over z . The

2.2. Equality of Bulk and Edge Conductivities

sums over x and y are clearly finite, as they are bounded by the summable state $|\psi|$. Thus E_a^* converges strongly to zero. An exactly analogous argument applies for E_a .

□

Chapter 3

Interacting Bulk-Edge Correspondence

3.1 General Setting

Let $L \in \mathbb{N}$, and let $\Gamma_L = \mathbb{Z}_L \times [0, L]$ be the discrete cylinder, equipped with a metric d . To each site $x \in \Gamma_L$, we associate a Hilbert space \mathcal{H}_x whose dimension is bounded uniformly in L . We denote $N = \sup_L \dim \mathcal{H}_L$. For a subset $X \subseteq \Gamma_L$, we define the Hilbert space $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$, and we set

$$\mathcal{H}_L := \mathcal{H}_{\Gamma_L} = \bigotimes_{x \in \Gamma_L} \mathcal{H}_x.$$

The algebra $\mathcal{U}_X \subset \mathcal{B}(\mathcal{H}_X)$ of observables on \mathcal{H}_X is the set of bounded self-adjoint operators supported in X . For an operator $A_X \in \mathcal{U}_X$, we identify its extension to an operator on \mathcal{H}_L by taking its tensor product with copies of the identity, $(\otimes_{x \in X^c} \mathbb{I}_x) \otimes A_X$. Conversely, we say that an operator $A \in \mathcal{U}_L$ has support X if $A_X := (\otimes_{x \in X^c} \mathbb{I}_x) \otimes (A|_X)$ is equal to A , and write $A_X \in \mathcal{U}_X$. For ease of notation, we omit the subscript L wherever there is no risk of confusion.

A *local interaction* is a map $\Phi : \mathcal{P}(\Gamma_L) \rightarrow \mathcal{U}_L$ such that

1. $\Phi(X) = 0$ whenever $\text{diam}(X) > R$ for some $R > 0$.
2. $\Phi(X)$ is supported in X .
3. $\|\Phi(X)\| \leq C$ for all $X \subset \Gamma_L$, for all L .

We consider a region as depicted in Figure 3.1, with the left and right edges joined together to form a cylinder. In the left white region $[0, L/2] \times [0, L]$, H_0 is a trivial Hamiltonian which we take to be empty space (we take $H_0 = 0$), and in the right blue region $[L/2, L] \times [0, L]$, H_1 is a *local Hamiltonian*, in the sense that $H_1 = \sum_{X \subseteq \Gamma_L} \Phi(X)$, is a sum of local interactions. This replaces assumption 1. We define the Hamiltonian of the full system to be

3.1. General Setting

$$H_\mu = H_1 + \mu Q_h,$$

where $Q_h = \sum_{x \in \Gamma_h} a_x^* a_x$ is the number operator for the region $\Gamma_h = [L/4, 3L/4] \times [0, L]$ shown in red. This introduces a driving strength; the μQ_h term can be viewed as a potential difference $V(x)$.

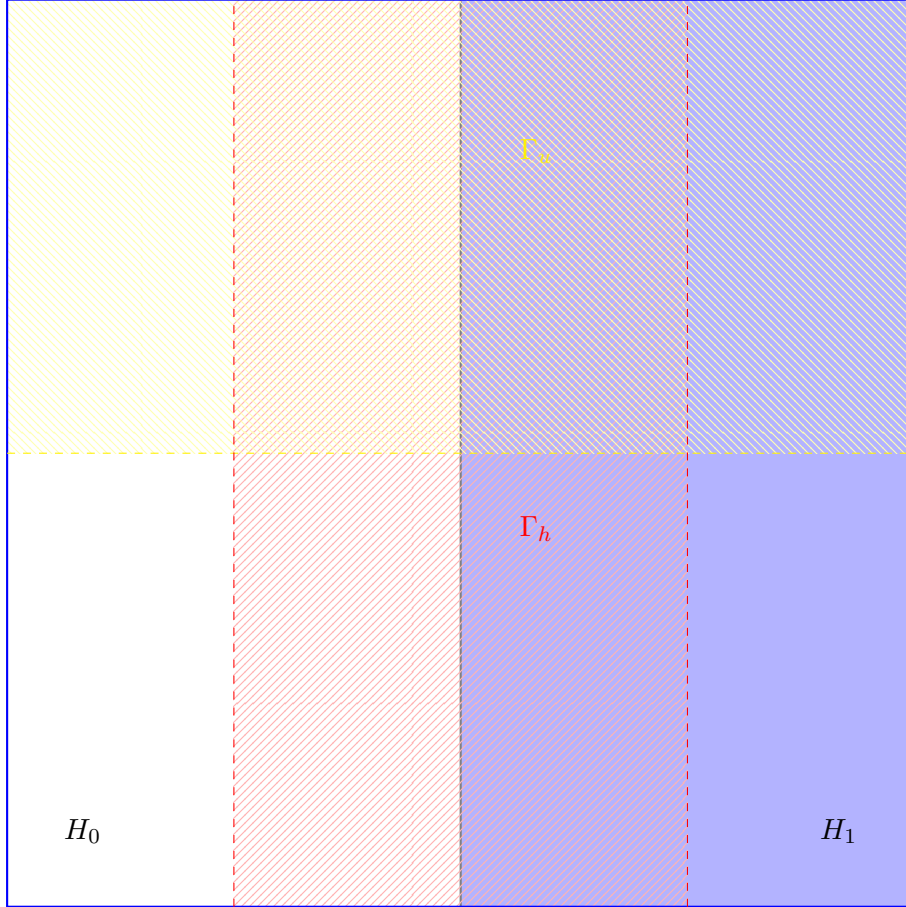


Figure 3.1: The left and right edges are identified to form the cylinder Γ_L . The charge operator Q_h introduces a driving strength. The Hall current is measured across the dashed yellow line $y = L/2$.

We also consider a bulk system, which is modelled in the plane \mathbb{Z}^2 . In this setting, there are no edge states, and so the associated “bulk” Hamiltonian H_B is assumed to have a *gapped* spectrum, as in assumption 3.

3.1. General Setting

In the case of the cylinder, this effect does not necessarily occur due to the presence of the edge. We also assume that the Hamiltonian is *locally charge-conserving*.

Assumption 4. $[\Phi(X), Q] = 0$, where Q is the total charge in Γ_L .

Let P_B be the ground state projection of H_B (the system without an edge), and let P be the ground state projection of H (the system with an edge). We assume that states far from the edge are essentially bulk states, up to tails that vanish quickly in L . In fact, they vanish faster than any power of L . We say that such a function $g(x)$ decays *superpolynomially* or *almost-exponentially* in x if for all n , there exists a C_n such that $g(x) \leq C_n x^{-n}$ for all $x > x_0$ for some x_0 . We denote superpolynomial decay by $|g(x)| = \mathcal{O}(x^{-\infty})$. In our setting, we often write $A = \mathcal{O}(L^{-\infty})$, which is a shorthand to say that the norm of A decays superpolynomially in L .

Assumption 5. Define the edge region

$$\Gamma_E = [L/2 - k, L/2 + k] \times [0, L] \cup [L - k, k] \times [0, L].$$

for some $k > 0$. For any operator A supported on Γ_E^c ,

$$\text{Tr}(PA) = \text{Tr}(P_B A) + \mathcal{O}(L^{-\infty}).$$

The A on the right hand side is understood to be the extension by zeroes of A to the plane \mathbb{Z}^2 .

This replaces the edge assumption 2. The idea is that observables localized far away from the edge are not affected by the edge of the system. We similarly define the *bulk region*

$$\Gamma_B = [3L/4 - k, 3L/4 + k] \times [0, L],$$

and the *middle region*

$$\Gamma_m = [L/2, L] \cup [0, L] \setminus (\Gamma_E \cup \Gamma_B).$$

The three regions are depicted in figure 3.2.

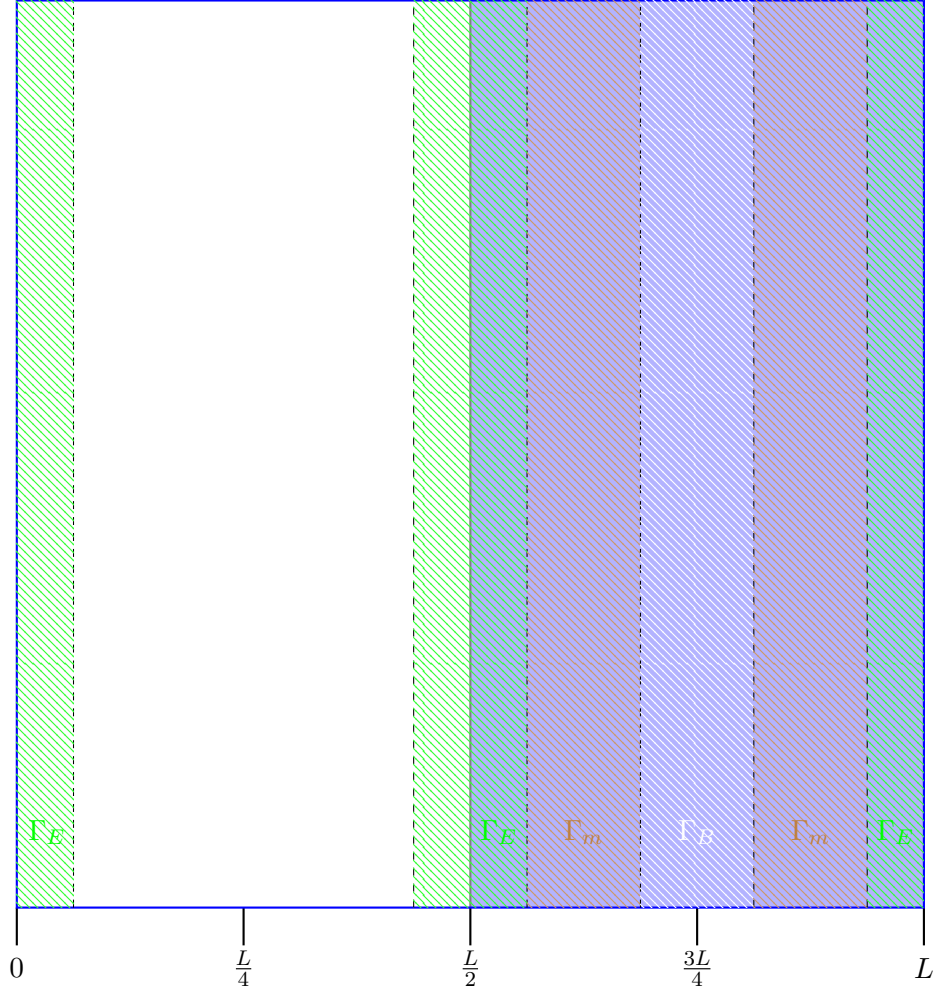


Figure 3.2: The regions Γ_E , Γ_B , and Γ_m .

To summarize, the interacting Hamiltonian of both the edge system and the bulk system is finite-range, and follows assumptions 3, 4, and 5, but not 1 and 2.

3.2 Equality of Bulk and Edge Currents

3.2.1 Cylinder Geometry

Let P_μ be the (possibly degenerate) ground state projection of H_μ . Let $Q_u = \sum_{x \in \Gamma_u} a_x^* a_x$ be the charge in the upper half of the cylinder $\Gamma_u =$

3.2. Equality of Bulk and Edge Currents

$[0, L] \times [L/2, L]$ (the yellow region in Figure 3.1), and define current operator

$$J = i[H_\mu, Q_u],$$

which measures the current across the fiducial line $y = L/2$. Charge conservation 4 implies that this current operator is supported along a strip of width $2R$ centred on the fiducial line $y = L/2$. Indeed, if we inspect a local interaction $\Phi(X)$ of range R with support $(\Gamma_u)_R$, where $(X)_\alpha$ is the α -shrinking of the set X , then clearly $\Phi(X)$ commutes with the charge outside Γ_u , so that $[\Phi(X), Q_u] = [\Phi(X), Q]$, which vanishes by the charge conservation assumption 4. Similarly, if $\Phi(X)$ is supported in $((\Gamma_u)^c)_R$, then $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$. It follows that for an interaction $\Phi(X)$ with range R and arbitrary support, $[\Phi(X), Q_u]$ must be supported on a set which is contained in (or equal to) the strip $[L/2, L] \times [L/2 - R, L/2 + R]$. There $[H_\mu, Q_u]$ must be supported there as well, since H_μ is a sum of such local interactions.

From this point, we drop the subscript μ wherever it is not needed for context.

Lemma 7. *The ground state expectation of the current J is zero.*

Proof. Assuming linearity and cyclicity of the trace hold, the proof is trivial,

$$\text{Tr}(PJ) = i\text{Tr}(P[H, Q_u]) = i\text{Tr}([P, H]Q_u) = 0.$$

In order for this calculation to hold, we need to prove that

1. PHQ_u and PQ_uH are separately trace-class to apply linearity of the trace, and
2. $\|H\| < \infty$ and $PQ_u \in \mathcal{J}_1$ to apply cyclicity of the trace.

The latter implies the former by the bound $\|AB\|_1 \leq \|A\|_1\|B\|$. To prove (2), fix a finite L . The Hamiltonian is bounded since it is a finite sum of at most $\mathcal{P}(\Gamma_L)$ local interactions $\Phi(X)$, each of which is uniformly bounded by assumption, along with the μQ_h term. But the number operator for the entire space is bounded by $\|Q\| \leq NL^2$, where N is the uniform bound on the dimension of each Hilbert space. This shows that both Q_u and Q_h are bounded in operator norm. Finally, $\|P\|_1 \leq CL^2$ because the projection is finite-rank, since the dimension of each site is bounded. Therefore $PQ_u \in \mathcal{J}_1$.

□

3.2. Equality of Bulk and Edge Currents

Next, we define a family of operators indexed by μ called *Hastings operators*,

$$K_\mu = \mathcal{I}_\mu(\dot{H}_\mu),$$

where

$$\mathcal{I}_\mu(A) = \int_{\mathbb{R}} W(t) e^{itH_\mu} A e^{-itH_\mu} dt.$$

Here, $W : \mathbb{R} \rightarrow \mathbb{R}$ is an odd, bounded, $L^1(\mathbb{R})$ function satisfying

1. $|W(t)| = \mathcal{O}(|t|^{-\infty})$,
2. $\widehat{W}(\xi) = \frac{i}{\xi}$ for all $|\xi| \geq \text{Len}(\Delta)$,

where \widehat{W} is the Fourier transform, taken here to be $\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-it\xi} dt$. Such a function can be constructed explicitly [10]. We emphasize the necessity of the spectral gap Δ in the construction of W . In our setting, we see that

$$K_\mu = \mathcal{I}_\mu(Q_h).$$

We present two important properties of the map $\mathcal{I}_\mu : \mathcal{U}_L \rightarrow \mathcal{U}_L$ in the following lemmas.

First, recall a definition from the non-interacting setting: an *off-diagonal* operator is an operator A such that $A = \overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, where $P_\mu^\perp = \mathbb{I} - P_\mu$ is the projection onto the excited states above the gap.

Lemma 8. 1. For any off-diagonal operator $A = \overline{A}$, $\mathcal{I}_\mu(\cdot)$ and $[H_\mu, \cdot]$ act as inverses of each other, up to a factor of i :

$$\mathcal{I}_\mu([H_\mu, A]) = [H_\mu, \mathcal{I}_\mu(A)] = iA.$$

2. For any (not necessarily off-diagonal) operator A ,

$$[\mathcal{I}_\mu([H_\mu, A]), P_\mu] = i[A, P_\mu].$$

Proof. Let $\widehat{W}(\xi) = \int_{\mathbb{R}} W(t) e^{-it\xi} dt$ be the Fourier transform of W and let A be any observable. First, we show that $\mathcal{I}([H, PAP^\perp]) = iPAP^\perp$.

Decomposing

3.2. Equality of Bulk and Edge Currents

$$\begin{aligned}
e^{itH}P &= \sum_{j=0}^{\infty} \frac{(itH)^j}{j!} P \\
&= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left(\sum_n E_n^j P_n \right) P \\
&= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \sum_{n:E_n=0} E_n^j P_n \\
&= \sum_{n:E_n=0} e^{itE_n} P_n,
\end{aligned}$$

and similarly

$$P^\perp e^{-itH} = \sum_{m:E_m \geq \gamma} P_m e^{-itE_m},$$

we see that

$$\begin{aligned}
\mathcal{I}([H, PAP^\perp]) &= \mathcal{I}(P[H, A]P^\perp) \\
&= \int_{\mathbb{R}} W(t) e^{itH} P[H, A] P^\perp e^{-itH} dt \\
&= \int_{\mathbb{R}} W(t) \sum_{n:E_n=0} e^{itE_n} P_n [H, A] \sum_{m:E_m \geq \gamma} P_m e^{-itE_m} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} \int_{\mathbb{R}} W(t) e^{itE_n} P_n A(E_n - E_m) P_m e^{-itE_m} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \int_{\mathbb{R}} W(t) e^{-it(E_m - E_n)} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \widehat{W}(E_m - E_n) \\
&= i \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m \\
&= iPAP^\perp,
\end{aligned}$$

since $\widehat{W}(\xi) = i\xi^{-1}$ for all $|\xi| \geq \text{Len}(\Delta)$. If the spectrum of H is continuous, the sums can be replaced by integrals with respect to the spectral measure P_λ , as in 3. Either way, the use of domination to interchange the integral over t and the other sums/integrals is provided by the fact that

3.2. Equality of Bulk and Edge Currents

$W, \widehat{W} \in L^1$. By the same argument, $\mathcal{I}([H, P^\perp AP]) = iP^\perp AP$ as well, and so $\mathcal{I}([H, \bar{A}]) = i\bar{A}$, which concludes the proof of part (1).

Part (2) of the lemma follows from the observation that $[A, P]$ is off-diagonal, and since P and H commute, we have by part (1)

$$[\mathcal{I}([H, A]), P] = \mathcal{I}([H, A], P) = \mathcal{I}([H, [A, P]]) = i[A, P].$$

□

Another important property of the map \mathcal{I}_μ is that it preserves locality.

Lemma 9. *Let A be a bounded, trace-class operator supported in X , and let $S = (X^r)^c$ be the set of all sites whose distance from X is at least r . \mathcal{I}_μ is quasi-locality preserving in the sense that*

$$\|\mathcal{I}(A) - \text{tr}_S(\mathcal{I}(A))\| = \mathcal{O}(r^{-\infty}).$$

Proof. Let $A(t) = e^{itH} A e^{-itH}$ be the time evolution of A . We see that $\|\mathcal{I}(A) - \text{tr}_S(\mathcal{I}(A))\|$ is bounded by

$$\int_{-T}^T |W(t)| \|A(t) - \text{tr}_S(A(t))\| dt + \int_{\mathbb{R} \setminus [-T, T]} |W(t)| \|A(t) - \text{tr}_S(A(t))\| dt.$$

We treat each of the integrals separately. For the first, by the Lieb-Robinson bound (Proposition 6) we have

$$\begin{aligned} \int_{-T}^T |W(t)| \|A(t) - \text{tr}_S(A(t))\| dt &\leq |X| \|A\| \|W\|_\infty \int_{-T}^T e^{-\frac{r-v|t|}{\xi}} dt \\ &= \frac{2\xi}{v} |X| \|A\| \|W\|_\infty e^{-\frac{r}{\xi}} (e^{\frac{vT}{\xi}} - 1). \end{aligned}$$

For the second term, notice that the bound

$$\|A(t) - \text{tr}_S(A(t))\| \leq \|A\| + |\text{Tr}(A)| =: C$$

holds uniformly in t , and C is finite by assumption. We therefore obtain

$$\int_{\mathbb{R} \setminus [-T, T]} |W(t)| \|A(t) - \text{tr}_S(A(t))\| dt \leq 2C \int_T^\infty |W(t)| dt.$$

Let $M(T) := \int_T^\infty |W(t)| dt$ and fix $n \in \mathbb{N}$. Since $|W|$ is bounded and obeys $|W(t)| = \mathcal{O}(|t|^{-\infty})$, it follows that $|W(t)| \leq C_n |t|^{-n}$ for some C_n , and thus

3.2. Equality of Bulk and Edge Currents

$$M(T) \leq \frac{C_n}{n+1} \frac{1}{T^{n+1}}.$$

Since this holds for all n , we have $M(T) = \mathcal{O}(T^{-\infty})$. Finally, putting the two integrals together, we have

$$\|\mathcal{I}(A) - \text{tr}_S(\mathcal{I}(A))\| \leq \frac{2\xi}{v} \|X\| \|A\| \|W\|_{\infty} e^{-\frac{r}{\xi}} (e^{\frac{vT}{\xi}} - 1) + \mathcal{O}(T^{-\infty}).$$

We conclude by optimizing with the choice $T = \frac{r}{2v}$ so that the first term decays exponentially and the superpolynomial term dominates, leaving $\|\mathcal{I}(A) - \text{tr}_S(\mathcal{I}(A))\| = \mathcal{O}(r^{-\infty})$. \square

The idea of the previous lemma is that for an operator A supported in X , $\mathcal{I}(A)$ can be well-approximated by an operator supported in X^r .

Proposition 2. *The operator K_{μ} is the generator of parallel transport, satisfying*

$$\dot{P}_{\mu} = i[K_{\mu}, P_{\mu}]$$

for all μ .

Proof. First, we show that \dot{P} is off-diagonal. Taking the derivative on both sides of $P^2 = P$, we see that $\dot{P}P + P\dot{P} = \dot{P}$. Acting on the left and right with P on both sides of this equation gives

$$P\dot{P}P + P\dot{P}P = P\dot{P}P,$$

which implies that $P\dot{P}P = 0$. Thus

$$\begin{aligned} \overline{\partial_{\mu}P} &= P\dot{P}(1-P) + (1-P)\dot{P}P \\ &= P\dot{P} - P\dot{P}P + \dot{P}P - P\dot{P}P \\ &= P\dot{P} + \dot{P}P \\ &= \partial_{\mu}(P^2) \\ &= \partial_{\mu}P, \end{aligned}$$

as claimed. By the product rule and the fact that H and P commute,

$$[\dot{H}, P] = -[H, \dot{P}].$$

It therefore follows from Lemma 8 that

3.2. Equality of Bulk and Edge Currents

$$\dot{P} = -i\mathcal{I}_\mu([H, \dot{P}]) = i\mathcal{I}([\dot{H}, P]) = i[\mathcal{I}(\dot{H}), P] = i[K, P].$$

□

Increasing the electric potential by a small amount $d\mu Q_h$ and expanding to linear order, the change in ground state current is given by

$$\text{Tr}(P_{\mu+d\mu}J) - \text{Tr}(P_\mu J) = \kappa d\mu + \mathcal{O}(d\mu^2).$$

Dividing by $d\mu$ and taking a limit, we see that the linear response coefficient is given by

$$\sigma(\mu) = \text{Tr}(\dot{P}_\mu J).$$

The *Hall conductivity* of the system on a subset $V \subseteq \Gamma_L$ is defined to be $\sigma_V := \text{Tr}(\dot{P}J_V)$, where J_V is the restriction of J to V .

In particular, we define the *edge conductivity* in the interacting setting as the conductivity on the edge strip, σ_{Γ_E} .

Proposition 3. *The Hall conductivity is independent of the driving strength μ .*

Proof. For any μ_1 and μ_2 ,

$$\begin{aligned} \sigma(\mu_1) - \sigma(\mu_2) &= \text{Tr}(\dot{P}_{\mu_1} i[H_{\mu_1}, Q_u] - \dot{P}_{\mu_2} i[H_{\mu_2}, Q_u]) \\ &= i\text{Tr}\left(\left([\dot{P}_{\mu_1}, H_{\mu_1}] - [\dot{P}_{\mu_2}, H_{\mu_2}]\right) Q_u\right) \\ &= -i\text{Tr}\left(\left([\dot{H}_{\mu_1}, P_{\mu_1}] - [\dot{H}_{\mu_2}, P_{\mu_2}]\right) Q_u\right) \\ &= i\text{Tr}([Q_h, P_{\mu_1} - P_{\mu_2}]Q_u) \\ &= i\text{Tr}([Q_u, Q_h](P_{\mu_1} - P_{\mu_2})) \\ &= 0, \end{aligned}$$

since H and P commute. Note that $\|\dot{P}\|_1 < \infty$ since we are working in a finite-dimensional space. The proof of Lemma 7 provides the other necessary bounds to invoke linearity and cyclicity of the trace to shift the commutator in the second line and second-last line. □

This indicates that the Hall conductivity is independent of μ as one would expect physically. We simply write $\sigma = \sigma(\mu)$ from this point, in accordance with proposition 3.

The following is the main result:

3.2. Equality of Bulk and Edge Currents

Theorem 2. *Let $V \subseteq \Gamma_m$ be a set contained within the strip in between the edge region Γ_E and the bulk region Γ_B (see Figure 3.2), and define the distance*

$$r = \text{dist}(V, \Gamma_E \cup \Gamma_B)$$

from V to the bulk and edge regions. The Hall conductivity in this regions vanishes in the sense that

$$\sigma_V = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

Proof. By Proposition 2, the bulk Hall conductivity can also be written by the formula

$$\sigma_V^B = \text{Tr} (i[K, P_B]J_V^B) = \text{Tr} (i[\mathcal{I}(Q_h), P_B]J_V^B),$$

where $J_V^B = i[H_B, Q_u]|_V$ is the current in the region V arising from the bulk Hamiltonian. From commutativity of P_B and H_B along with cyclicity of the trace, we compute

$$\begin{aligned} \sigma_V^B &= \text{Tr} \left(i \int_{\mathbb{R}} W(t) e^{itH_B} [Q_h, P_B] e^{-itH_B} dt J_V^B \right) \\ &= \int_{\mathbb{R}} W(t) \text{Tr} (i[Q_h, P_B] e^{-itH_B} J_V^B e^{itH_B}) dt \\ &= - \int_{\mathbb{R}} W(t) \text{Tr} (i[Q_h, P_B] e^{itH_B} J_V^B e^{-itH_B}) dt \\ &= -\text{Tr} (i[Q_h, P_B] \mathcal{I}(J_V^B)), \end{aligned}$$

since $W(t)$ is odd. By part (2) of Lemma 8, we have $i[Q_h, P_B] = [\mathcal{I}([H_B, Q_h]), P_B]$. Therefore

$$\begin{aligned} \sigma_V^B &= -\text{Tr}([\mathcal{I}([H_B, Q_h]), P_B] \mathcal{I}(J_V^B)) \\ &= \text{Tr} (P_B [\mathcal{I}([H_B, Q_h]), \mathcal{I}(J_V^B)]). \end{aligned}$$

Now, $[H_B, Q_h]$ is a local operator supported on Γ_B , while J_V^B is a local operator supported on V . Notice that By Lemma 9, the commutator $[\mathcal{I}([H_B, Q_h]), \mathcal{I}(J_V^B)]$ can be written

$$[A_{(\Gamma_B)^{r/2}} + \mathcal{O}(r^{-\infty}), A_{V^{r/2}} + \mathcal{O}(r^{-\infty})] = \mathcal{O}(r^{-\infty})$$

for some operators $A_{(\Gamma_B)^{r/2}}$ and $A_{V^{r/2}}$ supported on the $\frac{r}{2}$ -fattenings of Γ_B and V , which are disjoint.

3.2. Equality of Bulk and Edge Currents

This fact applies to the bulk setting with H_B and P_B . To extend this to the setting with an edge, it is enough to use Assumption 5 to conclude the same result with equality up to $\mathcal{O}(L^{-\infty})$, i.e.

$$\sigma_V = \text{Tr}(\dot{P}J_V) = \text{Tr}(\dot{P}(J_V^B + \mathcal{O}(L^{-\infty}))) = \sigma_V^B + \mathcal{O}(L^{-\infty}) = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

□

The intuitive picture from the previous result is that, outside the edge region Γ_E , the Hall conductivity is essentially only nonzero along the bulk strip Γ_B . Since the ground state expectation of the current is zero by Lemma 7, it must be that there is an equal current flowing along the edge strip Γ_E , but in the opposite direction; $\text{Tr}(PJ_{\Gamma_E}) = -\text{Tr}(PJ_{\Gamma_B}) + \mathcal{O}(L^{-\infty})$.

Taking a derivative, we see the Hall conductivity at the edge, $\sigma_{\Gamma_E} = \text{Tr}(\dot{P}J_{\Gamma_E})$, must be equal to the Hall conductivity in the bulk strip, $-\text{Tr}(\dot{P}J_{\Gamma_B})$.

We conclude using assumption 5 that the Hall conductivity $-\text{Tr}(\dot{P}J_{\Gamma_B})$ in the region Γ_B is equal (up to $\mathcal{O}(L^{-\infty})$) to that of the actual bulk system, $\sigma_{\Gamma_B}^B := -\text{Tr}(\dot{P}_B J_{\Gamma_B}^B)$, where P_B is the projection of H_B and $J^B = i[H_B, Q_u]$ is the current in the bulk system.

We therefore have bulk-edge correspondence, in the sense that

$$\sigma_{\Gamma_E} = \sigma_{\Gamma_B}^B + \mathcal{O}(L^{-\infty}).$$

Chapter 4

Summary of Results

In this thesis, we proved bulk-edge correspondence in both noninteracting and interacting quantum lattice systems. In the noninteracting setting, the key development was the use of the Helffer-Sjöstrand representation to formally write the difference between σ_E and σ_B as a zero trace operator. In the interacting setting, the key tool is the Hastings operator, which allows us to define an inverse of $[H, \cdot]$, as well as define the ground state projection as the unique solution to the differential equation $\dot{P} = i[K, P]$.

4.1 Direction for Future Work

One interesting result would be a proof of bulk-edge correspondence in the interacting setting on the discrete torus $\mathbb{T}_L := \mathbb{Z}_L \times \mathbb{Z}_L$. One might define the same regions Γ_u and Γ_h as in the cylinder case. However, on \mathbb{T}_L , the vanishing total ground state current (Lemma 7) does not apply. Intuitively, it does not apply because electrons can now flow through both the bottom and the top of the region Γ_u , rather than just the bottom. Mathematically, the lemma fails because our definition of the current is slightly changed.

Charge conservation and the fact that H is finite range can be used to split the current J_u into two components, $J_u = i[H_-, Q_u] + i[H_+, Q_u] = J_- - J_+$, supported on strips of width $2R$ at $y = L/2$ and $y = L$, respectively. The current operator is then defined to be $J = J_-$, which is the current on the lower strip. This is the mathematical reason that the proof in Lemma 7 fails on the torus; we have replaced H by H_- , which may no longer commute with P .

One might proceed in a manner similar to [9]. We define a slightly modified Hastings operator

$$K := - \int_{\mathbb{R}} W(t) e^{itH_B} i[H_B, Q_h] e^{-itH_B} dt$$

so that by Lemma 8, K leaves the bulk system's ground state space invariant,

$$[Q_h - K, P_B] = [Q_h, P_B] - \frac{1}{i}[\mathcal{I}([H_B, Q_h]), P_B] = 0.$$

By Lemma 9, the Hastings operator K is supported (up to tails) on strips of width $2R$ at the edges of Γ_u . We denote these lower and upper strips located at $y = L/2$ and $y = L$ as ∂_- and ∂_+ , respectively.

Furthermore, in the edge system, $\text{Tr}(A, [Q_h, P]) = 0$ for all trace-class operators A supported in Γ_E . The idea is that Q_h “commutes with P along the edge”. Indeed, since H is charge conserving by Assumption 4, we may choose a simultaneous eigenbasis of H and the total charge Q , in which case P and Q commute. It follows that

$$\text{Tr}(A[Q_h, P]) = \text{Tr}([A, Q_h]P) = \text{Tr}([A, Q]P) = \text{Tr}(A[Q, P]) = 0.$$

Finally, we prove that in the bulk system with Hamiltonian $H_B(\mu)$, the ground state expectation of the current vanishes faster than any power as $L \rightarrow \infty$.

Lemma 10. *The ground state expectation of the current $J_B := i[(H_B)_-, Q_u]$ (of the system without an edge) is $\text{Tr}(P_B J_B) = \mathcal{O}(L^{-\infty})$.*

Proof. $K = \mathcal{I}(i[H_B, Q_u])$ splits into $K = K_- - K_+$, with the support of K_{\pm} contained in ∂_{\pm} up to tails by Lemma 9, in the sense that

$$[K_{\pm}, A_X] = \mathcal{O}(r^{-\infty})$$

for every $A_X \in \mathcal{U}_X$ such that $\|A_X\| = 1$, and where $r = \text{dist}(X, \partial_{\pm})$. In particular,

$$i[H_B, K_-] = i[(H_B)_-, K_-] + \mathcal{O}(L^{-\infty}),$$

and similarly $i[(H_B)_-, K_+] = \mathcal{O}(L^{-\infty})$. Putting these facts together, it follows that the current can be rewritten as

$$\begin{aligned} J_B &= i[H_B, Q_u + K_- - K_- + K_+] + \mathcal{O}(L^{-\infty}) \\ &= i[H_B, K_-] + i[(H_B)_-, Q_u - K_- + K_+] + \mathcal{O}(L^{-\infty}). \end{aligned}$$

From here, we use the fact that H_B and $Q_u - K_- + K_+$ both commute with P_B (up to tails) to write

$$P_B J_B P_B = i[H_B, P_B K_- P_B] + i[P_B (H_B)_- P_B, Q_u - K_- + K_+] + P_B \mathcal{O}(L^{-\infty}) P_B.$$

4.1. Direction for Future Work

By boundedness of H_B , Q , and K , along with the fact that P_B is finite rank, it follows that all of these operators are trace-class. Cyclicity allows us to conclude

$$\mathrm{Tr}(P_B J_B) = \mathrm{Tr}(P_B J_B P_B) = \mathcal{O}(L^{-\infty}).$$

□

Of course, the ultimate goal would be to show that the ground state current is zero in the system with an edge, which is what allowed us to conclude bulk-edge correspondence in 3.2.1. However, using our previous results, we can at least show a simple proof of this in the non-interacting case for the edge system.

Proposition 4. *Let $H = \sum_{x \in \mathbb{T}} h_x$ be a non-interacting edge Hamiltonian, i.e. a sum of single site Hamiltonians h_x . The ground state expectation of the Hall current $J = i[H_-, Q_u]$ is $\mathrm{Tr}(PJ) = \mathcal{O}(L^{-\infty})$.*

Proof. Since H is a sum of single site Hamiltonians, we can split H_- into the restrictions $H_- = (H_-)_{\Gamma_E} + (H_-)_{\Gamma_E^c}$, with no fear of any terms which are supported both in the edge and outside the edge. By Assumption 5,

$$\begin{aligned} \mathrm{Tr}(PJ) &= \mathrm{Tr}(Pi[H_-, Q_h]) \\ &= i\mathrm{Tr}([H_-, Q_h]P) \\ &= i\mathrm{Tr}((H_-)_{\Gamma_E}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\Gamma_E^c}[Q_h, P]) \\ &= i\mathrm{Tr}((H_-)_{\Gamma_E}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\Gamma_E^c}[Q_h, (P)_{\mathrm{bulk}}]) \\ &= i\mathrm{Tr}((H_-)_{\Gamma_E}[Q_h, P]) + i\mathrm{Tr}((H_B)_-[Q_h, P_B]) + \mathcal{O}(L^{-\infty}) \\ &= i\mathrm{Tr}((H_-)_{\Gamma_E}[Q_h, P]) + \mathrm{Tr}(i[(H_B)_-, Q_h]P_B) + \mathcal{O}(L^{-\infty}). \end{aligned}$$

Since Q_h commutes with P along the edge, the first term is zero. By Lemma 10, the second term is $\mathcal{O}(L^{-\infty})$. □

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Appendix A

General Functional Analysis

Lemma 11. *Let A be a bounded linear operator on a Hilbert space \mathcal{H} . Suppose $A_n \xrightarrow{s} A$ on a dense subspace $\mathcal{D} \subset \mathcal{H}$. If A_n are bounded uniformly in n , then $A_n \xrightarrow{s} A$ on all of \mathcal{H} .*

Proof. Let $\psi_n \in \mathcal{D}$ be a sequence converging in norm to $\psi \in \mathcal{H}$. The result follows from a standard $\frac{\varepsilon}{3}$ argument. Let C be a bound for both $\sup_n \|A_n\|$ and $\|A\|$. Then

$$\begin{aligned} \|A_n\psi - A\psi\| &\leq \|A_n(\psi - \psi_m)\| + \|(A_n - A)\psi_m\| + \|A(\psi - \psi_m)\| \\ &\leq C\|\psi - \psi_m\| + \|(A_n - A)\psi_m\| + C\|\psi - \psi_m\|. \end{aligned}$$

There exists an M such that the first and third terms are less than $\frac{\varepsilon}{3}$ for all $m > M$. For the middle term, observe that for each m , there exists by hypothesis an N_m such that $\|(A_n - A)\psi_m\| < \frac{\varepsilon}{3}$ for all $n > N_m$. Thus, by picking some $m > M$ and fixing a suitably large n , the inequality above is less than ε . \square

A.1 Spectral Measures and Projection-Values Measures

Projection-valued measures are maps $P : \mathcal{M} \rightarrow \mathcal{B}(\mathcal{H})$ from measurable subsets of \mathbb{R} to the space of bounded linear operators on \mathcal{H} satisfying the usual properties of both projections and measures.

1. $P(M) = P(M)^* = P(M)^2$ is an orthogonal projection for all $M \in \mathcal{M}$.
Note that this implies that $P(M)$ is a positive operator.
2. $P(\emptyset) = 0$ and $P(\mathbb{R}) = \mathbb{1}_{\mathcal{H}}$.
3. If $\{M_i\}_{i \in \mathbb{N}}$ are pairwise disjoint, then $\sum_{i=1}^n P(M_i) \xrightarrow{s} P(\cup_{i \in \mathbb{N}} M_i)$ as $n \rightarrow \infty$ (σ -additivity).
4. $P(M_1 \cap M_2) = P(M_1)P(M_2)$ for any $M_1, M_2 \in \mathcal{M}$.

The heuristic motivation is that $P(M)$ projects onto the subspace of \mathcal{H} spanned by states whose energies lie in M . Using these operator-valued measures, one can construct an operator-valued integral with respect to P in the usual fashion (beginning on nonnegative simple functions, extending to nonnegative measurable functions, and finally to real-valued measurable functions).

Theorem 3 (Spectral Theorem for Projection-Valued Measures). *There exists a one-to-one correspondence between self adjoint operators H and projection-valued measures P given by the formula*

$$H = \int_{\mathbb{R}} \lambda dP_{\lambda},$$

where $P_{\lambda} := P((-\infty, \lambda])$. Moreover, if $g : \mathbb{R} \rightarrow \mathbb{R}$ is any bounded Borel function, then $g(H)$ defined via the Borel function calculus coincides with the formula

$$g(H) = \int_{\mathbb{R}} g(\lambda) dP_{\lambda},$$

and $g(H) = g(H)^*$.

We remark that it follows from the second part of this theorem that if $\mathbb{1}_M$ denotes the characteristic function of a Borel set $M \subseteq \mathbb{R}$, then

$$\mathbb{1}_M(H) = \int_M dP_{\lambda} = P(M).$$

We also note that $\text{Spec}(H) = \text{supp}(P)$.

A.2 Grönwall's Inequality and Uniqueness

Theorem 4. (Grönwall's Inequality). *Let $\alpha : I \rightarrow (0, \infty)$ be positive and continuous on I for some interval of the form $[a, b)$, $[a, b]$, or $[a, \infty)$. Suppose $u : \mathbb{R} \rightarrow \mathcal{U}$ is a Banach-valued, differentiable function. If $\|u'(t)\| \leq \alpha(t)\|u(t)\|$ for all $t \in I$, then*

$$\|u(t)\| \leq \|u(a)\| e^{\int_a^t \alpha(s) ds} \quad \forall t \in I$$

Proof. Let $f(t) = e^{\int_a^t \alpha(s) ds}$, which is nonzero, has initial value $f(a) = 1$, and has derivative $f'(t) = \alpha(t)f(t)$. Then by the quotient rule,

$$\left(\frac{\|u(t)\|}{f(t)} \right)' = \frac{\|u'(t)\|f(t) - \|u(t)\|\alpha(t)f(t)}{f(t)^2} \leq 0,$$

A.3. Note on Generators of Parallel Transport

where the inequality follows from the assumption $\|u'(t)\| \leq \|\alpha(t)u(t)\|$. Thus $\frac{\|u(t)\|}{f(t)}$ is decreasing, so that

$$\frac{\|u(t)\|}{f(t)} \leq \frac{\|u(a)\|}{f(a)} = \|u(a)\|,$$

which is the desired inequality. \square

Theorem 5. (*ODE Uniqueness*). *Let $F : \mathcal{U} \rightarrow \mathcal{U}$ be Lipschitz and consider the differential equation*

$$u'(t) = F(u(t))$$

with initial condition $u(a) = u_a$ for some function $u : I \rightarrow \mathcal{U}$, where $I = [a, b]$, or $[a, b)$, or $[a, \infty)$. Solutions to this equation are unique.

Proof. Suppose there are two solutions $u(t)$ and $v(t)$, and let $g(t) = \|u(t) - v(t)\|^2$. By assumption, there exists a constant L_F such that $\|F(u(t)) - F(v(t))\| \leq L_F \|u(t) - v(t)\|$, so that

$$\begin{aligned} g'(t) &= 2\|u(t) - v(t)\| \|u'(t) - v'(t)\| \\ &= 2\|u(t) - v(t)\| \|F(u(t)) - F(v(t))\| \\ &\leq 2L_F \|u(t) - v(t)\|^2 \\ &= 2L_F g(t). \end{aligned}$$

Notice that $\alpha := 2L_F$ is a positive continuous function, so we may apply Grönwall's inequality to $g(t)$ to conclude

$$g(t) \leq g(a)e^{2L_F(t-a)} = 0,$$

since $g(a) = 0$. \square

A.3 Note on Generators of Parallel Transport

Consider the differential equation $\dot{\rho}(\mu) = i[K_B, \rho(\mu)]$ with initial condition $\rho(0) = P_B(0)$. Here $K_B = \int_{\mathbb{R}} W_\gamma(t) e^{-itH_B} \dot{H}_B e^{itH_B} dt$, and recall that in our setting, $\dot{H}_B = Q_h$. We know that the solution is $\rho(\mu) = P_B(\mu)$ (proposition 2). Notice that the map $F : \mathcal{U} \rightarrow \mathcal{U}$ defined by $F(A) = i[K_B, A]$ is Lipschitz, since

$$\|F(A) - F(B)\| = \|[K_B, A - B]\| \leq 2\|K_B\| \|A - B\|.$$

The Lipschitz constant is $2\|K_B\|$, which is finite since K_B is a bounded operator:

$$\|K_B\| \leq \int_{\mathbb{R}} |W_\gamma(t)| \|e^{-itH_B} Q_h e^{itH_B}\| dt \leq \int_{\mathbb{R}} |W_\gamma(t)| dt \|Q_h\|.$$

Indeed, since Q_h is the number operator on a finite volume, by charge conservation and the fact that the dimension of the Hilbert space is bounded uniformly by d , there can only be a finite number of charges in the region Γ_h .

Thus, by Grönwall's uniqueness theorem (appendix A.2), we see that the solution to the equation $\dot{\rho}(\mu) = F(\rho(\mu)) = i[K_B, \rho(\mu)]$ is unique.

Now define

$$K_E := \int_{\mathbb{R}} W_\gamma(t) e^{-itH_E} Q_h e^{itH_E} dt,$$

which is using the gap γ of H_B to define W_γ , but also using the edge Hamiltonian in the time evolution operators. Consider $\sigma : [0, \infty) \rightarrow \mathcal{U}$ defined by

$$\dot{\sigma}(\mu) = i[K_E, \sigma(\mu)] \quad \sigma(0) = P_E(0).$$

We now show that, similar to how ρ is an approximation of P_B , σ is also a good approximation of P_E (up to $\mathcal{O}(L^{-\infty})$) “in the bulk”. Let $A \in \mathcal{U}_{\Gamma_B}$ be an operator localized in the bulk of the edge system. Then

$$\begin{aligned} \text{Tr}(\dot{\sigma}A) &= \text{Tr}(i[K_E, \sigma]A) \\ &= \text{Tr}(i[A, K_E]\sigma) \\ &= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}([e^{-itH_E} Q_h e^{itH_E}, A]\sigma) dt \\ &= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}(e^{-itH_E} [Q_h, e^{itH_E} A e^{-itH_E}] e^{itH_E} \sigma) dt \\ &= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}(e^{-itH_E} [Q_h, e^{itH_B} A e^{-itH_B}] e^{itH_E} \sigma + \mathcal{O}(L^{-\infty})) dt \\ &= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}(e^{-itH_B} [Q_h, e^{itH_B} A e^{-itH_B}] e^{itH_B} \sigma + \mathcal{O}(L^{-\infty})) dt \\ &= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}([e^{-itH_B} Q_h e^{itH_B}, A]\sigma) dt + \mathcal{O}(L^{-\infty}) \\ &= \text{Tr}(i[A, K_B]\sigma) + \mathcal{O}(L^{-\infty}) \\ &= \text{Tr}(i[K_B, \sigma]A) + \mathcal{O}(L^{-\infty}), \end{aligned}$$

since σ is trace-class (?) and $W_\gamma \in L^1$. By linearity of the trace, we see that $\text{Tr}((\dot{\sigma} - i[K_B, \sigma])A) = \mathcal{O}(L^{-\infty})$ for any operator $A \in \Gamma_B$ (does this mean $\dot{\sigma} - i[K_E, \sigma] = 0$?). But the solution of $\dot{\sigma} - i[K_B, \sigma] = 0$ (with initial condition $\sigma(0) = P_B(0)$) is unique; it is $\rho(\mu)$, or $P_B(\mu)$. Hence

$$\text{Tr}(P_E A) = \text{Tr}(P_B A) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\rho A) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\sigma A) + \mathcal{O}(L^{-\infty})$$

for any operator $A \in \Gamma_B$. In particular, this gives another local formula for the Hall conductivity in the bulk of an edge system, by taking $A = J_V$, where J is the current operator and $V \subset \Gamma_B$ is a set localized in the bulk. The Hall conductivity is given by $\text{Tr}(\dot{P}_E J_V)$, and this can be approximated by

$$\text{Tr}(\dot{P}_E J_V) = \text{Tr}(\dot{P}_B J_V) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\dot{\rho} J_V) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\dot{\sigma} J_V) + \mathcal{O}(L^{-\infty}).$$

Want to pick a norm s.t. Gronwall gives $\|\rho(\mu) - \sigma(\mu)\|_G \leq \|P_B(0) - P_E(0)\|_G e^{2L_F \mu}$. Need $\|P_B(0) - P_E(0)\|_G$ to be small enough to kill the exponential which depends on $L_F = 2\|K_B\|_G \leq \|W_\gamma\|_{L^1} \|Q_h\|_G$. If we use the operator norm for $\|\cdot\|_G$, we would get $\|Q_h\|_G = d|\Gamma_h|$ in the exponent. Need $\|\cdot\|_G$ to be an actual norm so that $\|\rho - \sigma\|_G = 0 \implies \rho = \sigma$. (Need to add)

A.4 Greens' Functions and the Combes-Thomas Bound

The short-range assumption 1 is vital for the following non-trivial estimate, the proof of which is omitted.

Theorem 6 (Combes-Thomas Bound). *Let H be a self-adjoint operator on $\ell^2(\mathbb{Z}^2)$ satisfying*

$$S_\alpha := \sup_x \sum_y |H(x, y)| (e^{\alpha|x-y|} - 1) < \infty$$

for some $\alpha > 0$. Suppose z lies outside the spectrum of H , and let $d_z := \text{dist}(z, \text{Spec}(H))$. Then the Greens function of H is exponentially bounded,

$$|G(x, y; z)| \leq \frac{2}{d_z} e^{-\xi_\alpha |x-y|},$$

where $\xi := \frac{\alpha d_z}{2S_\alpha}$.

This theorem gives the crucial decay properties of the spectral projectors.

Lemma 12. *Let H be a self-adjoint operator satisfying 1 and with bounded spectrum. Let $S \subseteq \text{Spec}(H)$, and let P_S be the associated spectral projection. Then there exists some $\varepsilon, \nu > 0$ such that the matrix elements of P_S satisfy*

$$\sum_{x,y \in \mathbb{Z}^2} |P_S(x,y)| e^{-\varepsilon|x|} e^{\nu|x-y|} < \infty.$$

Proof. We use the fact that the spectral projection is given by the Riesz integral formula

$$P_S = \frac{1}{2\pi i} \oint_{\gamma} R(z) dz,$$

where $R(z)$ is the resolvent of H and γ is any smooth closed curve containing S . Since the resolvent is the Greens function of H , it satisfies the Combes-Thomas bound 6. Since the spectrum of H is bounded, it may be enclosed in a curve of finite length, from which we determine that

$$|P_S(x,y)| \leq C e^{-\xi_{\alpha}|x-y|},$$

since $\inf_{z \in \gamma} d_z = \inf_{z \in \gamma} \text{dist}(z, \text{Spec}(H)) > 0$. Hence

$$\sum_{x,y \in \mathbb{Z}^2} |P_S(x,y)| e^{-\varepsilon|x|} e^{\nu|x-y|} < \infty$$

holds for $\nu = \xi_{\alpha}$ and for any $\varepsilon > 0$, since $e^{-\varepsilon|x|} = e^{-\varepsilon|x_1|} e^{-\varepsilon|x_2|}$ is summable on \mathbb{Z}^2 by Lemma 4. □

This statement about decay of the matrix elements can be turned into a statement about the norm of the operator. Let Lip^1 be the set of all Lipschitz functions whose Lipschitz constant is not greater than 1, that is the set of functions ℓ satisfying

$$|\ell(x) - \ell(y)| \leq |x - y|$$

for all $x, y \in \mathbb{Z}^2$.

Lemma 13. *Let P_S be the spectral projection onto $S \subseteq \text{Spec}(H)$. For all $\varepsilon > 0$,*

$$\sup_{\ell \in \text{Lip}^1} \|e^{\nu\ell(x)} e^{-\varepsilon|x|} P_S e^{-\nu\ell(x)}\| < \infty,$$

A.4. Greens' Functions and the Combes-Thomas Bound

where ν is the same as in 12.

Proof. The matrix elements of the operator are $P_S(x, y)e^{\nu(\ell(x)-\ell(y))}e^{-\varepsilon|x|}$. By Holmgren's bound, the operator's norm is therefore bounded by

$$\max\left\{\sup_x \sum_y |P_S(x, y)|e^{\nu(\ell(x)-\ell(y))}e^{-\varepsilon|x|}, \sup_y \sum_x |P_S(x, y)|e^{\nu(\ell(x)-\ell(y))}e^{-\varepsilon|x|}\right\}.$$

Replacing the supremum with a sum yields the bound

$$\|P_S(x, y)e^{\nu(\ell(x)-\ell(y))}e^{-\varepsilon|x|}\| \leq \sum_{x, y} |P_S(x, y)|e^{\nu|\ell(x)-\ell(y)|}e^{-\varepsilon|x|},$$

and taking a supremum over $\ell \in \text{Lip}^1$ completes the proof by Lemma 12. \square

Appendix B

The Lieb-Robinson Bound

Let the dimensions of the Hilbert spaces at each site, i.e. $\dim(\mathcal{H}_x)$, be uniformly bounded. Let $A \in \mathcal{U}_X$ and $B \in \mathcal{U}_Y$ be any operators having disjoint supports $X \cap Y = \emptyset$, and denote by $r = \text{dist}(X, Y)$ the distance between them. Let $A(t) = e^{itH} A e^{-itH}$ be the time evolution of A . Then

The following is a version of the Lieb-Robinson bound [11].

Proposition 5.

$$\|[A(t), B]\| \leq C \|A\| \|B\| \min\{|X|, |Y|\} e^{-\frac{r-v|t|}{\xi}},$$

where C , v , and ξ are positive constants.

A consequence of the Lieb-Robinson bound is the following estimate on the growth of the support of A in time.

Proposition 6. *Let A be any operator with support in X , and let $S = (X^r)^c$ be the set of sites whose distance from X is at least r . Then*

$$\|A(t) - \text{tr}_S(A(t))\| \leq C |X| \|A\| e^{-\frac{r-v|t|}{\xi}},$$

where C , v , and ξ are as in 5.

Proof. The partial trace is given by

$$\text{tr}_S(A(t)) = \int_{\mathcal{G}_S} U A(t) U^* d\mu(U),$$

where \mathcal{G}_S is the group of unitaries supported on S and μ denotes the Haar measure. From this we see that

$$\begin{aligned} \|A(t) - \text{tr}_S(A(t))\| &\leq \int_{\mathcal{G}_S} \|A(t) - U A(t) U^*\| d\mu(U) \\ &= \int_{\mathcal{G}_S} \|[A(t), U] U^*\| d\mu(U) \\ &\leq \|[A(t), U]\| \end{aligned}$$

whence the Lieb-Robinson bound 5 completes the proof. \square

Appendix C

The Helffer-Sjöstrand Representation

The Helffer-Sjöstrand representation is a functional calculus $f \mapsto f(H)$ for arbitrary (possibly unbounded) operators H on the set of functions

$$\mathcal{A} = \bigcup_{\beta < 0} \{f : \mathbb{R} \rightarrow \mathbb{C} : f \in C^\infty(\mathbb{R}), |f^{(n)}(x)| \leq c_n(1+x^2)^{\frac{\beta-n}{2}}\}.$$

It has the following properties.

Theorem 7. *For any $f \in \mathcal{A}$,*

1. $f \mapsto f(H)$ is an algebraic homomorphism (linear and multiplicative).
2. $\overline{f}(H) = f(H^*)$.
3. $\|f(H)\| \leq \|f\|_\infty$.
4. For all $w \notin \mathbb{R}$, if $r_w(s) = \frac{1}{s-w}$ then $r_w(H) = (H - w)^{-1}$.
5. For all $f \in C_c^\infty(\mathbb{R})$ with $\text{supp}(f) \cap \text{Spec}(H) = \emptyset$, we have $f(H) = 0$.

For $f \in \mathcal{A}$ which are also Borel, $f(H)$ agrees with the operator given by the Borel functional calculus. There is an explicit formula for $f(H)$, which is given by

$$f(H) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (H - z)^{-1} dz \wedge d\bar{z},$$

where $\tilde{f} : \mathbb{C} \rightarrow \mathbb{C}$ is a *quasi-analytic extension* of $f : \mathbb{R} \rightarrow \mathbb{R}$. It is defined as follows. For any smooth f , we set

$$\tilde{f}(z) = \sum_{r=0}^n \tau \left(\frac{y}{(1+x^2)^{1/2}} \right) \frac{(iy)^r}{r!} f^{(r)}(x)$$

where $\tau : \mathbb{R} \rightarrow \mathbb{R}$ is any smooth function satisfying

$$\tau(s) = \begin{cases} 1 & |s| < 1 \\ 0 & |s| > 2 \end{cases}.$$

The extension turns out to be independent of the choice of n and τ . Furthermore, as $|\operatorname{Im}(z)| \rightarrow 0$, the Wirtinger derivative of the extension obeys the bound

$$\left| \frac{\partial \tilde{f}}{\partial \bar{z}} \right| = \mathcal{O}(|y|^n).$$

Thus $\frac{\partial \tilde{f}}{\partial \bar{z}} = 0$ for all real z , which is why \tilde{f} is called a “quasi”-analytic extension (the Wirtinger derivative would be zero everywhere were \tilde{f} analytic).

A crucial property of the Helffer-Sjöstrand functional calculus is the following bound.

Lemma 14. *For any $n \in \mathbb{N}$, the quasi-analytic extension \tilde{f} can be chosen so that*

$$\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \frac{1}{|\operatorname{Im}(z)|^{p+1}} dz \wedge d\bar{z} \leq C_0 \sum_{k=0}^{n+2} \|f^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_m = \int_{-\infty}^{\infty} |f(x)|(1+x^2)^{m/2} dx.$$

In particular, if $f = \rho$ (as defined in chapter 2) then clearly the norms on the right hand side are all finite since ρ is bounded and the derivatives of ρ have compact support. This lemma is often useful for bounding operators because the resolvent, which frequently shows up in the integrand of Helffer-Sjöstrand representations, obeys the bound $\|(H - z)^{-1}\| \leq |\operatorname{Im}(z)|^{-1}$.