

# Thesis Rough Work

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# 1 Introduction

## 2 Noninteracting Setting

Consider the lattice  $\mathbb{Z}^2$ , on which we define a bulk Hamiltonian  $H_B$ , whose matrix elements follow a short-range assumption:

$$\sup_{y \in \mathbb{Z}^2} \sum_{x \in \mathbb{Z}^2} |H_B(x, y)| (e^{\mu|x-y|} - 1) < \infty$$

for some  $\mu > 0$ . We define the bulk conductivity

$$\sigma_B(\lambda) = -i \text{Tr}(P_\lambda [[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]])$$

where  $P_\lambda$  is the projection onto the eigenstates of  $H_B$  with energy lies in  $(-\infty, \lambda)$ , and where

$$\Lambda_i(x) = \begin{cases} 1 & x_i < 0 \\ 0 & x_i \geq 0 \end{cases}$$

are characteristic functions. We construct an edge Hamiltonian on the lattice  $\mathbb{Z}_a^2 = \{x \in \mathbb{Z}^2 : x_2 > -a\}$ . We denote the edge Hamiltonian by  $H_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}_a^2)$ , requiring only that the edge operator  $E_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}_a^2)$  define by

$$E_a = J_a H_a - H_B J_a$$

satisfies the edge assumption

$$\sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}_a^2} E_a(x, y) |e^{\mu(|x_2+a|-|x_1-y_1|)}| < \infty$$

for some  $\mu > 0$ , where  $|x| := |x_1| + |x_2|$ . The interpretation

Each site  $x \in \mathbb{Z}^2$  get an associated Hilbert space  $\mathcal{H}_x$ . The dimension of these Hilbert spaces is bounded uniformly in  $x$ . We consider the Hilbert space  $\ell^2(\mathbb{Z}^2, \mathbb{C}^n) = \{(x_1, x_2, \dots) \in \mathbb{C}^n : \sum_{i \in \mathbb{Z}^2} \|x_i\|^2 < \infty\}$ . For example, one might consider a system of spins at the lattice sites, in which case the Hilbert space  $\mathcal{H}_x$  at each site would be  $\mathbb{C}^2$ , and the total Hilbert space  $\mathcal{H} = \otimes_x \mathcal{H}_x$  would then be the space of summable wavefunctions  $\psi = \otimes_x \psi_x \in \ell^2(\mathbb{Z}^2, \mathbb{C}^2)$ .

The Hilbert space  $\ell^2(\mathbb{Z}^2)$  is the “bulk” setting, i.e. the setting in which we consider an infinite two-dimensional medium with no edges, and we consider a “bulk Hamiltonian”  $H_B$  on this Hilbert space. We also define the “edge” Hilbert space  $\ell^2(\mathbb{Z}_a^2)$  and an associated “edge Hamiltonian”  $H_a$ , where  $\mathbb{Z}_a^2 :=$

$\{(n, m) \in \mathbb{Z}^2 : n \geq -a\}$ . The bulk and edge Hamiltonians are related by the edge operator  $E_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell(\mathbb{Z}^2)$  defined by

$$E_a := J_a H_a - H_B J_a,$$

where  $J_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell(\mathbb{Z}^2)$  denotes extension by zeroes. We assume that

**Assumption 1.** *The edge operator satisfies*

$$\sup_{z \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}_a^2} |E_a(x, y)| e^{\mu(|x_2+a|+|x_1-y_1|)} < \infty.$$

The interpretation is that  $E_a = J_a H_a - H_B J_a$  is the difference between first applying  $H_a$  on  $\ell^2(\mathbb{Z}_a^2)$ , and then making everything below  $-a$  into zeroes, versus first making all  $x \in \mathbb{Z}^2$  such that  $x_2 < -a$  zeroes, and then applying  $H_B$ . The assumption ensures that the effects from introducing the edge at  $-a$  die exponentially as we move upward away from the edge (due to the  $|x_2 - (-a)|$  term in the exponent), and also terms do not interact too much as their  $x_1$  distance increases (due to the  $|x_1 - y_1|$  term in the exponent).

We also make the following assumption about both the bulk and edge Hamiltonians:

**Assumption 2.** *The Hamiltonians have a spectral gap. There exists an interval  $\Delta$  such that  $\Delta \cap \sigma(H) = \emptyset$ .*

*Remark:* The spectral gap assumption can be relaxed to a “mobility gap” assumption,

$$\sup_{f \in B_c(\Delta)} |f(H_B)(x, y)| (1 + |x|)^{-\nu} e^{\mu|x-y|} < \infty$$

for some  $\nu > 0$ , where  $B_c(\Delta)$  is the set of Borel functions  $f$  which are constant on  $(-\infty, \inf \Delta)$  and on  $(\sup \Delta, \infty)$  such that  $|f(x)| \leq 1$  for all  $x$ . See ? for details.

An example of an edge Hamiltonian satisfying the assumption on  $E_a$  is  $H_a = J_a^* H_B J_a$ , where  $J_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}^2)$  denotes extension by zeros. The idea is that for a state  $\psi \in \ell^2(\mathbb{Z}_a^2)$ , we have  $\langle \psi, H_a \psi \rangle = \langle (J_a \psi), H_B (J_a \psi) \rangle$ , which we interpret as the edge Hamiltonian having the same expectation as the bulk Hamiltonian if we just turned all the states  $\psi_x$  with  $x_2 < -a$  into zeroes. The edge operator is

$$E_a = J_a J_a^* H_B J_a - H_B J_a = (J_a J_a^* - \mathbb{1}) H_B J_a = \begin{cases} -H_B(x, y) & \text{if } x_2 < -a \\ 0 & \text{if } x_2 \geq -a \end{cases}$$

Intuitively, there is no difference between  $H_B$  and  $H_a$  on  $\mathbb{Z}_a^2$ . The bound in assumption ? is satisfied by the short range assumption ?.

We define the *bulk conductivity* at Fermi energy  $\mu$  as follows. Suppose we subject the system to an external electric potential difference  $V$  in the  $x_2$  direction. We write this as  $-V_0\Lambda_2$ , where  $\Lambda_i$  are multiplication operators  $\Lambda_i|\psi(x_1, x_2)\rangle = \Lambda(x_i)|\psi(x_1, x_2)\rangle$  which are *switch functions*,

$$\Lambda : \mathbb{R} \rightarrow \mathbb{R} \quad \Lambda(x_i) = \begin{cases} 1 & \text{if } x_i \leq 0 \\ 0 & \text{if } x_i \geq 1 \end{cases}$$

and are smooth and monotonically decreasing on  $(0, 1)$ . Note that the ensuing physics (in particular, our definition of the Hall conductivity) is independent of the particular choice of switch function  $\Lambda_i$ , since any two switch functions are exactly equal on the lattice.

This gives  $\vec{E} = -\nabla V = V_0 \frac{\partial \Lambda_2}{\partial x_2}$ , so that  $\vec{E}$  has compact support  $\text{supp}(\Lambda_2')$ . We introduce a function which grows slowly in time as  $t$  grows from  $-\infty$  to 0, so as to invoke the adiabatic principle. Here, we choose  $e^{\varepsilon t}$ , and we will let  $\varepsilon \rightarrow 0$  at the end. The Hamiltonian therefore experiences a perturbation,

$$\tilde{H}_B(t) = H_B - V_0\Lambda_2 e^{\varepsilon t}.$$

We define the Hall current operator  $J_H = i[\tilde{H}_B(t), \Lambda_1] = i[H_B, \Lambda_1]$ , which is related to the Hall conductivity by  $J_H = \sigma_H V$ .

**Lemma 1.** *The ground state expectation  $\text{Tr}(P_\mu J_H)$  of the Hall current is zero.*

*Proof.* Notice that since  $J_H$  is trace-class and  $P_\mu$  is bounded, and since  $[H_B, P_\mu] = 0$ , we have

$$\text{Tr}(P_\mu J_H) = i\text{Tr}(P_\mu [H_B, \Lambda_1]) = i\text{Tr}(P_\mu [H_B, P_\mu \Lambda_1 P_\mu])$$

□

**Proposition 1.** *The Hall conductivity  $\sigma_H$  in the bulk system is equal to*

$$\sigma_B = -i\text{Tr}(P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]),$$

where  $P_\mu := P((-\infty, \mu))$  is the projection-valued measure associated with  $H_B$  onto states with energy less than the Fermi energy  $\mu$ .

*Proof.* We begin with the Heisenberg equation of motion for the density matrix,  $\dot{\rho}(t) = -i[\tilde{H}_B(t), \rho(t)]$ , with initial condition  $\lim_{t \rightarrow -\infty} \|\rho(t) - e^{-itH_B} P_\mu e^{itH_B}\| = 0$ , which also implies  $\lim_{t \rightarrow -\infty} \|e^{itH_B} \rho(t) e^{-itH_B} - P_\mu\| = 0$ .

We work in the interaction picture, and define  $\rho_I(t) = e^{itH_B}\rho(t)e^{-itH_B}$ , and  $\Delta H_B(t) = -e^{itH_B}V_0\Lambda_2e^{\varepsilon t}e^{-itH_B}$ . Thus

$$\dot{\rho}_I(t) = -i[\Delta H_B(t), \rho_I(t)]$$

The solution to this differential equation is readily verified to be

$$\rho(t) = i \int_{-\infty}^t [\Delta H_B(s), P_\mu] ds + P_\mu$$

Indeed, taking the derivative of the right hand side gives  $i[\Delta H_B(t), P_\mu] = i[\Delta H_B(t), \rho_I(t)] + \mathcal{O}(V_0^2)$ , but  $P_\mu$  and  $\rho_I(t)$  are equal up to zeroth order in  $V_0$ . The initial condition is also satisfied.

Using  $J_H = i[H_B, \Lambda_1] = \sigma_H V = -\sigma_H V_0 \Lambda_2$ , we obtain  $\sigma_H = \frac{1}{V_0} \lim_{\varepsilon \rightarrow 0} \text{Tr}(\rho(0)i[H_B, \Lambda_1])$ . Since the expectation of the ground state current is zero,  $\text{Tr}(P_\mu J_H) = 0$ , we have

$$\begin{aligned} \sigma_H &= \frac{i}{V_0} \lim_{\varepsilon \rightarrow 0} \text{Tr} \left( i \int_{-\infty}^0 [\Delta H_B(t), P_\mu] [H_B, \Lambda_1] ds \right) \\ &= -\frac{1}{V_0} \lim_{\varepsilon \rightarrow 0} \text{Tr} \left( \int_{-\infty}^0 [-e^{isH_B} V_0 \Lambda_2 e^{\varepsilon s} e^{-isH_B}, P_\mu] [H_B, \Lambda_1] ds \right) \\ &= -\lim_{\varepsilon \rightarrow 0} \text{Tr} \left( \int_{-\infty}^0 e^{isH_B} [\Lambda_2, P_\mu] e^{-isH_B} [H_B, \Lambda_1] e^{\varepsilon s} ds \right) \\ &= -\lim_{\varepsilon \rightarrow 0} \text{Tr} \left( \int_{-\infty}^0 (e^{-isH_B} [H_B, \Lambda_1] e^{isH_B}) \cdot ([\Lambda_2, P_\mu] e^{\varepsilon s}) ds \right) \end{aligned}$$

Where we used the fact that  $P_\mu$  and  $H_B$  commute. Using integration by parts on the two terms in brackets, and noting that  $\frac{d}{ds}(e^{isH_B} [H_B, \Lambda_1] e^{-isH_B}) = -(ie^{isH_B} \Lambda_1 e^{-isH_B} - \Lambda_1)$ , we obtain

$$\begin{aligned} \sigma_H &= i \lim_{\varepsilon \rightarrow 0} \text{Tr} \left( \int_{-\infty}^0 (e^{-isH_B} \Lambda_1 e^{isH_B} - \Lambda_1) \frac{d}{ds} ([\Lambda_2, P_\mu] e^{\varepsilon s}) ds \right) \\ &= i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left( \int_{-\infty}^0 \Lambda_1^s [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right) \end{aligned}$$

where  $\Lambda_1^s := e^{-isH_B} \Lambda_1 e^{isH_B} - \Lambda_1$ . Using the notation  $\overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$ , it is readily verified that the commutator  $[\Lambda_2, P_\mu]$  is an *off-diagonal* operator, in the sense that  $[\Lambda_2, P_\mu] = \overline{[\Lambda_2, P_\mu]}$ . Furthermore, a simple computation reveals that for any two operators  $A$  and  $B$ ,  $\text{Tr}(\overline{AB}) = \text{Tr}(AB)$ . It therefore follows that

$$\sigma_H = i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left( \int_{-\infty}^0 \overline{\Lambda}_1^s [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right)$$

The integrand can be broken into two terms,

$$\overline{\Lambda}_1^s [\Lambda_2, P_\mu] e^{\varepsilon s} = e^{-isH_B} \overline{\Lambda}_1 e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} - \overline{\Lambda}_1 [\Lambda_2, P_\mu] e^{\varepsilon s}$$

by commutativity of  $P_\mu$  and  $H_B$ . We show that the integral of the first term vanishes. We begin by breaking the first term down further into

$$e^{-isH_B} P_\mu \Lambda_1 P_\mu^\perp e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} + e^{-isH_B} P_\mu^\perp \Lambda_1 P_\mu e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s}.$$

We treat the first of these two terms; the other is handled in an identical manner. We use the spectral theorem to write  $e^{-isH_B} P_\mu = \int_{-\infty}^\mu e^{-is\lambda} dP_\lambda$ , and similarly  $P_\mu^\perp e^{isH_B} = (\text{Id} - P_\mu) e^{isH_B} = \int_\mu^\infty e^{is\nu} dP_\nu$ .

We remark that, since the Fermi energy  $\mu$  is assumed to lie in a spectral gap, there must exist a neighbourhood  $(\mu - \delta, \mu + \delta)$  in which there are no states. We exploit this fact to rewrite the limits of integration,  $\int_{-\infty}^{\mu-\delta} e^{-is\lambda} dP_\lambda$  and  $\int_{\mu+\delta}^\infty e^{is\nu} dP_\nu$ . We therefore obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{-\infty}^0 e^{-isH_B} P_\mu \Lambda_1 P_\mu^\perp e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \\ = \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left( \int_{-\infty}^0 \int_{-\infty}^{\mu-\delta} e^{-is\lambda} dP_\lambda \Lambda_1 \int_{\mu+\delta}^\infty e^{is\nu} dP_\nu [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right) \\ = \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left( \int_{-\infty}^0 \int_{-\infty}^{\mu-\delta} \int_{\mu+\delta}^\infty e^{s(\varepsilon - i\lambda + i\nu)} dP_\lambda \Lambda_1 dP_\nu [\Lambda_2, P_\mu] ds \right) \end{aligned}$$

Performing the integral over  $s$  yields

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{-\infty}^0 e^{s(\varepsilon - i\lambda + i\nu)} ds = - \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{i\varepsilon + \lambda - \nu}$$

This limit is zero, since  $\lambda \neq \nu$ . Indeed, due to the spectral gap, the integration variables live in  $\lambda \in (-\infty, \mu - \delta)$  and  $\nu \in (\mu + \delta, \infty)$ . The case for the  $e^{-isH_B} P_\mu^\perp \Lambda_1 P_\mu e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s}$  term (where the  $P_\mu$  and  $P_\mu^\perp$  swap places) is treated analogously. Hence the first term in the integrand for  $\sigma_H$  vanishes, as claimed.

Finally, we return to our expression for the Hall conductivity, which now reads

$$\sigma_H = i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left( \int_{-\infty}^0 \overline{\Lambda_1} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right).$$

It is a basic algebraic calculation to show that  $\overline{\Lambda_1} = [[\Lambda_1, P_\mu], P_\mu]$ . Evaluating the integral over  $s$  is now trivial;  $\int_{-\infty}^0 e^{\varepsilon s} ds = \varepsilon^{-1}$ . Thus

$$\sigma_H = -i \text{Tr}([[\Lambda_1, P_\mu], P_\mu] [\Lambda_2, P_\mu]).$$

Shifting the commutator completes the proof:

$$\begin{aligned} \sigma_H &= -i \text{Tr}(P_\mu [[\Lambda_2, P_\mu], [\Lambda_1, P_\mu]]) \\ &= i \text{Tr}(P_\mu [[\Lambda_1, P_\mu], [\Lambda_2, P_\mu]]) \\ &= i \text{Tr}(P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]). \end{aligned}$$

□

*Remark:* This is reminiscent of the well-known adiabatic curvature formula,

$$\kappa = \text{Tr}(P[\partial_1 P, \partial_2 P]) = \text{Tr}(P[[P, K_1], [P, K_2]]) = \text{Tr}(P[K_1, K_2]),$$

where  $K_i$  are called *generators of parallel transport*. We will see the adiabatic curvature formula again later in the interacting setting.

For the *edge conductivity*, we need the current operator across the line  $x_1 = 0$ , which is given by  $-i[H_a, \Lambda_1]$ . We define

$$\sigma_E = -i \lim_{a \rightarrow \infty} \text{Tr}(\rho'(H_a)[H_a, \Lambda_1]),$$

where  $\rho \in C^\infty(\mathbb{R})$  satisfies

$$\rho(r) = \begin{cases} 1 & \text{if } r < \inf \Delta \\ 0 & \text{if } r > \sup \Delta \end{cases}$$

and decreases smoothly and monotonically in  $\Delta$ . The definition of  $\sigma_E$  is reminiscent of another formula we will see later in the interacting setting,  $\text{Tr}(\dot{P}J)$ , where  $J$  is the current operator. The interpretation of  $\sigma_E$  is that if we apply a small potential difference  $V$  across  $x_2 = -a$  to  $x_2 = \infty$ , there will be a net current

$$\begin{aligned} I &= -i \text{Tr}(\rho(H_a + V)[H_a + V, \Lambda_1] - \rho(H_a)[H_a, \Lambda_1]) \\ &= -i \text{Tr}((\rho(H_a + V) - \rho(H_a))[H_a, \Lambda_1]) \end{aligned}$$

Thus we obtain the conductivity

$$\sigma_E = \frac{I}{V} = -i\text{Tr} \left( \frac{(\rho(H_a + V) - \rho(H_a))}{V} [H_a, \Lambda_1] \right) \rightarrow -i\text{Tr}(\rho'(H_a)[H_a, \Lambda_1])$$

in the limit as  $V \rightarrow 0$ . As we shall see, it turns out that  $\sigma_E$  is independent of the choice of  $\rho$ , and  $\sigma_B$  is independent of  $\lambda$ .

The main result of this section is

**Theorem 1.**  $\sigma_E = \sigma_B$ .

## 2.1 Outline of the Proof

First, let

$$\tilde{\sigma}_E(a, t) = -i\text{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_{2,a}(t))$$

where  $\Lambda_{2,a}(t) = e^{itH_a}\Lambda_2e^{-itH_a}$  is the time evolution of  $\Lambda_2$ . One can show that, while

$$\sigma_E = \lim_{T \rightarrow \infty} \lim_{a \rightarrow \infty} \frac{1}{T} \int_0^T \text{Re}(\tilde{\sigma}_E(a, t)) dt,$$

it is unfortunately the case that  $\lim_{a \rightarrow \infty} \|\rho'(H_a)[H_a, \Lambda_1]\Lambda_{2,a}(t)\|_1 = \infty$ . However, even though the trace norm diverges, it turns out that the trace itself does not, so we will instead subtract a clever choice of zero-trace operator  $Z(a, t)$  to define

$$\sigma_E(a, t) = -i\text{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_{2,a}(t) - Z(a, t))$$

so that the equation  $\sigma_E = \lim_{T \rightarrow \infty} \lim_{a \rightarrow \infty} \frac{1}{T} \int_0^T \text{Re}(\sigma_E(a, t)) dt$  still holds, but we also have  $\lim_{a \rightarrow \infty} \|\rho'(H_a)[H_a, \Lambda_1]\Lambda_{2,a}(t) - Z(a, t)\|_1 < \infty$ . The correct choice of  $Z$  will become apparent after writing  $\rho(H_a)$  and  $\rho'(H_a)$  in terms of their Hellfer-Sjostrand representations,

$$\begin{aligned} \rho(H_a) &= \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z) \\ \rho'(H_a) &= -\frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z)^2 \end{aligned}$$

where  $R(z) = (H_a - z)^{-1}$  is the resolvent. Using  $[R(z), \Lambda_i] = R(z)[H_a, \Lambda_i]R(z)$ , we obtain the representations of the following useful operators:



$$[\rho(H_a), \Lambda_1] = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z) [H_a, \Lambda_1] R(z)$$

$$\rho'(H_a) [H_a, \Lambda_1] = -\frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z)^2 [H_a, \Lambda_1]$$

From here, we define the zero-trace operator

$$Z(a, t) = [\rho(H_a), \Lambda_1] \Lambda_2 - \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z) (R(z) [H_a, \Lambda_1] \Lambda_{2,a}(t) - [H_a, \Lambda_1] \Lambda_{2,a}(t) R(z))$$

from which we obtain

$$\begin{aligned} \sigma_E(a, t) &= \tilde{\sigma}_E(a, t) - Z(a, t) \\ &= \text{Tr} \left( -[\rho(H_a), \Lambda_1] \Lambda_2 - \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z) [H_a, \Lambda_1] \Lambda_{2,a}(t) R(z) \right) \\ &= \text{Tr} \left( [\rho(H_a), \Lambda_1] (\Lambda_{2,a}(t) - \Lambda_2) - \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z) [H_a, \Lambda_1] R(z) [H_a, \Lambda_{2,a}(t)] R(z) \right) \end{aligned}$$

All of the statements so far can be verified by calculations. The difficult part of the theorem (aside from proving that the relevant operators are trace-class) is proving that

$$\|J_a \Sigma'_a J_a^* - \Sigma'_B\|_1, \|J_a \Sigma''_a J_a^* - \Sigma''_B\|_1 \rightarrow 0$$

as  $a \rightarrow \infty$ , where  $\Sigma'_B$  and  $\Sigma''_B$  are the same as with the subscript  $a$ , except using the bulk Hamiltonian  $H_B$  in their definition rather than  $H_a$ . It follows that

$$\sigma_E(a, t) = \text{Tr}(J_a \Sigma'_a J_a^* + J_a \Sigma''_a J_a^*) = \text{Tr}(\Sigma'_a + \Sigma''_a) \rightarrow \text{Tr}(\Sigma'_B + \Sigma''_B)$$

From there, a calculation shows that  $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{Tr}(\Sigma'_B + \Sigma''_B) dt = \sigma_B$ , concluding the proof.

## 2.2 Gap Simplifications

We now assume that  $H_B$  has a spectral gap. In this case, the edge condition guarantees that  $\sigma_E(a) := \rho'(H_a) [H_a, \Lambda_1]$  is trace-class (need to add).

Need to add section on why  $\sigma_E(a) := -i \text{Tr}(\rho'(H_a) [H_a, \Lambda_1])$  is equal to

$$-\frac{i}{2} \text{Tr}(\rho'(H_a) \{[H_a, \Lambda_1], \Lambda_2\}).$$

**Theorem 2.**  $\sigma_E = \sigma_B$

### Outline of the Proof

Before giving the proof in its entirety, we outline the basic steps.

*Proof.* We prove that

$$\sigma_E(a) = -i\text{Tr}(\rho'(H_a)[H_a, \Lambda_1]) = \sigma_B$$

for all  $a \in \mathbb{Z}$ , and thus in particular,  $\sigma_E = \lim_{a \rightarrow \infty} \sigma_E(a) = \sigma_B$ . Since  $\sigma_B$  is translation invariant (need to add), we only need to prove to case  $-i\text{Tr}(\rho'(H_{a=0})[H_{a=0}, \Lambda_1]) = \sigma_B$ . We drop the subscript,  $H := H_{a=0}$ . Since the multiplication operator  $\Lambda_2(n)|\psi\rangle := \Lambda(x_2 - n)|\psi\rangle$  converges strongly to the identity as  $n \rightarrow \infty$ , we can write

$$\sigma_E(a) = -i\text{Tr}(\rho'(H)[H, \Lambda_1]) = -i \lim_{n \rightarrow \infty} \text{Tr}(\rho'(H)[H, \Lambda_1]\Lambda_2(n)).$$

Instead of completing the shift  $(0, -n)$  with the operator  $\Lambda_2(n)$ , we can consider a shifted Hamiltonian. Indeed, rather than restrict  $H_B$  at  $x_2 = -n$  to obtain  $H_{a=n}$ , consider the shifted bulk Hamiltonian  $H_B \mapsto H_B(n)$  obtained by the shift  $(0, -n)$ , and then restricting this at  $x_2 = 0$  to obtain  $H(n)$ . In other words,  $H(n)$  is the edge Hamiltonian associated with the shifted bulk Hamiltonian. Comparing this with the expression above, this is exactly equivalent to

$$-i \lim_{n \rightarrow \infty} \text{Tr}(\rho'(H)[H, \Lambda_1]\Lambda_2(n)) = -i \lim_{n \rightarrow \infty} \text{Tr}(\rho'(H(n))[H(n), \Lambda_1]\Lambda_2).$$

In other words, the difference between whether we cut off everything above  $x_2 = n$  and then apply  $H_{a=0}$ , or instead cut off everything above  $x_2 = 0$  and then apply the Hamiltonian  $H(n)$  shifted down by  $(0, -n)$  is immaterial.

Thus, our goal is to show that  $-i\text{Tr}(\rho'(H(n))[H(n), \Lambda_1]\Lambda_2) \rightarrow \sigma_B$  as  $n \rightarrow \infty$ . Define  $\widetilde{\sigma}_E(a) := -i\text{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2)$ .

$$Z(a) = [\rho(H_a), \Lambda_1]\Lambda_2 - \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [R_a(z), [H_a, \Lambda_1]\Lambda_2] dz^2$$

This operator has zero trace, so  $\widetilde{\sigma}_E(a) = \text{Tr}(\Sigma(a))$ , where

$$\begin{aligned} \Sigma(a) &= -i\rho'(H_a)[H_a, \Lambda_1]\Lambda_2 + iZ(a) \\ &= -i\rho'(H_a)[H_a, \Lambda_1]\Lambda_2 + i[\rho(H_a), \Lambda_1]\Lambda_2 - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [R_a(z), [H_a, \Lambda_1]\Lambda_2] dz^2 \end{aligned}$$

Using the Helffer-Sjostrand representations for the first two terms on the right hand side, we obtain

$$\begin{aligned}
\Sigma(a) &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z)^2 [H_a, \Lambda_1] \Lambda_2 dz^2 + \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [H_a, \Lambda_1] R_a(z) \Lambda_2 dz^2 \\
&\quad - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} (R_a(z)^2 [H_z, \Lambda_1] \Lambda_2 - R_a(z) [H_a, \Lambda_1] \Lambda_2 R_a(z)) dz^2 \\
&= -\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [H_a, \Lambda_1] [R_a(z), \Lambda_2] dz^2 \\
&= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_a(z) [H_a, \Lambda_1] R_a(z) [H_a, \Lambda_2] R_a(z) dz^2,
\end{aligned}$$

where we used

$$[R_a(z), \Lambda_i] = -R_a(z) [H_a, \Lambda_i] R_a(z)$$

in the final equality. Next, we must prove that the operator above converges to the corresponding bulk operator in trace-norm,

$$\|\Sigma(a) - \Sigma_B\|_1 \rightarrow 0,$$

as  $a \rightarrow \infty$ . Here,  $\Sigma_B$  is the same operator but using the bulk operators  $H_B$  and  $R_B(z)$ . Once this limit is established, we shall prove that  $\sigma_B = -i\text{Tr}(\Sigma_B)$  to conclude the proof.

To show that the limit is zero as claimed, we bound the integrand of  $\Sigma(a)$  in trace-norm by breaking it into three parts,

$$R[H_a, \Lambda_1] R[H_a, \Lambda_2] R = J_a[R, \Lambda_1] e^{\delta|x_1|} J_a^* \cdot e^{-\delta|x_1|} e^{-\delta|x_2|} \cdot J_a e^{\delta|x_2|} [H_a, \Lambda_2] R J_a^*,$$

and bounding the norm of each, making use of the fact that  $\|AB\|_1 \leq \|A\| \|B\|_1$ .

1. For the first term,  $J_a[R, \Lambda_1] e^{\delta|x_1|} J_a^*$ , we bound its operator norm by breaking it down further into

$$\begin{aligned}
\|J_a[R, \Lambda_1] e^{\delta|x_1|} J_a^*\| &= \|[R, \Lambda_1] e^{\delta|x_1|}\| \\
&= \|-R[H_a, \Lambda_1] R e^{\delta|x_1|}\| \\
&= \|-R \cdot [H_a, \Lambda_1] e^{\delta|x_1|} \cdot e^{-\delta|x_1|} R e^{\delta|x_1|}\| \\
&\leq \|R\| \cdot \|[H_a, \Lambda_1] e^{\delta|x_1|}\| \cdot \|e^{-\delta|x_1|} R e^{\delta|x_1|}\|
\end{aligned}$$

The norm of  $R$  is bounded by

$$\|R_a(z)\| \leq \frac{1}{\text{dist}(z, \sigma(H_a))}.$$

The norm of the second operator can be bounded by inspecting its matrix elements.

$$\begin{aligned} \langle x, [H_a, \Lambda_1] e^{\delta|x_1|} y \rangle &= \langle x, H_a \Lambda_1 y \rangle e^{\delta|y_1|} - \langle x, \Lambda_1 H_a y \rangle e^{\delta|y_1|} \\ &= H_a(x, y) e^{\delta|y_1|} (\Lambda(y_1) - \Lambda(x_1)). \end{aligned}$$

This is zero if  $|x_1 - y_1| \leq |y_1|$ , since this would imply that  $x_1$  and  $y_1$  have the same sign, and thus  $\Lambda(x_1) = \Lambda(y_1)$ . So either the matrix element is zero, or  $|y_1| \leq |x_1 - y_1|$ , which implies

$$\begin{aligned} |H_a(x, y) e^{\delta|y_1|} (\Lambda(y_1) - \Lambda(x_1))| &\leq 2|H_a(x, y)| e^{\delta|x_1 - y_1|} \\ &\leq 2|H_a(x, y)| e^{\delta|x - y|} \\ &\leq C|H_a(x, y)| (e^{\delta|x - y|} - 1) \end{aligned}$$

where the final inequality comes from the fact that the diagonal matrix elements are zero. Hence, by the assumption

$$\sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} |H(x, y)| (e^{\mu|x - y|} - 1) < \infty,$$

combined with Holmgren's bound,

$$\|A\| \leq \max\left\{\sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2} |A(x, y)|, \sup_{y \in \mathbb{Z}^2} \sum_{x \in \mathbb{Z}^2} |A(x, y)|\right\},$$

the second term is bounded. Finally, for the third term  $e^{-\delta|x_1|} R e^{\delta|x_1|}$ , we apply the Combes-Thomas bound,

$$\|e^{-\varepsilon f(x)} R_a(z) e^{\varepsilon f(x)}\| \leq \frac{C}{|\text{Im}(z)|}$$

where  $f : \mathbb{Z}^2 \rightarrow \mathbb{R}$  is any Lipschitz function, and  $\varepsilon$  can be chosen as  $\varepsilon = \frac{1}{C(1+|\text{Im}(z)|)}$ .

Altogether, the bound of the first term takes the form

$$\frac{C}{|\operatorname{Im}(z)|}.$$

2. For  $e^{-\delta|x_1|}e^{-\delta|x_2|}$ , we bound the trace norm by noticing that this is a summable function on  $\mathbb{Z}^2$ . Indeed, in the position basis, we see that the trace is nothing but the geometric series,

$$\begin{aligned} \operatorname{Tr}(e^{-\delta|x_1|}e^{-\delta|x_2|}) &= \sum_{(m,m) \in \mathbb{Z}^2} \langle (m, m), e^{-\delta|x_1|}e^{-\delta|x_2|}(m, m) \rangle \\ &= \sum_{m \in \mathbb{Z}} e^{-\delta|m|}e^{-\delta|m|} \\ &= \sum_{m \in \mathbb{Z}} e^{-2\delta|m|} \\ &= \frac{e^2 + 1}{e^2 - 1}. \end{aligned}$$

3. For  $J_a e^{\delta|x_2|} [H_a, \Lambda_2] R_a(z) J_a^*$ , note that analogously to 1. above where we bounded  $[H_a, \Lambda_1] e^{\delta|x_1|}$ , we also have that  $e^{\delta|x_2|} [H_a, \Lambda_2]$  is bounded. Again, the resolvent  $R_a(z)$  is also bounded.

The bound of the third term takes the same form as the bound of the first term,

$$\frac{C}{|\operatorname{Im}(z)|}.$$

Altogether, we see that the integrand is bounded by the product of the three bounds from 1., 2., and 3., and is of the form  $\frac{C}{|\operatorname{Im}(z)|^2}$ .

For any  $n \in \mathbb{N}$ , the quasi-analytic extension  $\tilde{\rho}$  of  $\rho$  in the Helffer-Sjostrand representation can be chosen so that

$$\int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} \frac{1}{|\operatorname{Im}(z)|^{p+1}} dz^2 \leq C_0 \sum_{k=0}^n + 2 \|\rho^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_{k-p-1} = \int_{-\infty}^{\infty} \frac{|f(x)|}{(x^2 + 1)^{k/2}} dx.$$

Since  $\rho'$  is compactly supported, this is clearly finite. This fact, combined with the bound

$$\|R_a(z)[H_a, \Lambda_1]R_a(z)[H_a, \Lambda_2]R_a(z)\|_1 \leq \frac{C}{\text{Im}(z)}$$

for the trace norm of the integrand of  $\Sigma(a)$  provides the necessary bound for Lebesgue dominated convergence. Thus, it suffices to show pointwise convergence in  $z$  of the integrand to the associated bulk operator.

In other words, we wish to show

$$J_a[R_a(z), \Lambda_1]e^{\delta|x_1|}J_a^* \xrightarrow{s} [R_B(z), \Lambda_1]e^{\delta|x_1|}$$

and

$$J_a e^{\delta|x_2|}[H_a, \Lambda_2]J_a^* \xrightarrow{s} e^{\delta|x_2|}[H_B, \Lambda_2].$$

But inspecting the bounds we found for the left hand sides, it is clear that they are uniformly bounded in  $a$ . It therefore suffices to show convergence on a dense subspace; in particular, we may choose the dense subspace of compactly supported states, which allows us to eliminate the  $e^{\delta|x_i|}$  terms.

Thus, we need to prove

$$J_a[R_a(z), \Lambda_1]J_a^* \xrightarrow{s} [R_B(z), \Lambda_1]$$

and

$$J_a[H_a, \Lambda_2]J_a^* \xrightarrow{s} [H_B, \Lambda_2].$$

We appeal to the general fact of functional analysis that strong convergence of the resolvent implies that  $J_a f(H_a)J_a^* \xrightarrow{s} f(H_B)$  for any bounded and continuous function  $f$ . In particular, the functions  $[(\cdot - z)^{-1}, \Lambda_1]$  and  $[\cdot, \Lambda_2]$  above are bounded and continuous, so we will have proven the desired limits if we can prove convergence of the resolvent,  $J_a R_a(z)J_a^* \xrightarrow{s} R_B(z)$ .

To prove this, we use the edge assumption. Recall the edge operator,  $E_a = J_a H_a - H_B J_a$ . Adding and subtracting  $zJ_a$  gives

$$E_a = J_a(H_a - z) - (H_B - z)J_a.$$

Applying  $R_B$  from the left and  $R_a$  from the right on both sides, we obtain

$$J_a R_a(z) - R_B(z)J_a = -R_B(z)E_a R_a(z).$$

Taking the adjoint, and then multiplying from the left by  $J_a$ , we see that

$$\begin{aligned}
J_a R_a(z) J_a^* - R_B(z) &= -J_a R_a(z) E_a^* R_B(z) \\
&= -(J_a R_a(z) E_a^* + 1 - J_a J_a^*) R_B(z) \\
&\xrightarrow{s} 0,
\end{aligned}$$

since  $E_a^* \xrightarrow{s} 0$  by the edge assumption, and  $1 - J_a J_a^* \xrightarrow{s} 0$ . This proves that the limits above converge to the desired associated bulk operators, and thus  $\Sigma(a) \xrightarrow{s} \Sigma_B$ .

Finally, it remains to show that

$$\text{Tr}(\Sigma_B) = \sigma_B.$$

First, we manipulate

$$\begin{aligned}
\Sigma_B &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z) [H_B, \Lambda_1] R_B(z) [H_B, \Lambda_2] R_B(z) dz^2 \\
&= -\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z) [H_B, \Lambda_1] [R_B(z), \Lambda_2] dz^2 \\
&= i[\rho(H_B), \Lambda_1] \Lambda_2 - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z) [H_B, \Lambda_1] \Lambda_2 R_B(z) dz^2.
\end{aligned}$$

Define  $P_+ := P((\sup \Delta, \infty))$  and  $P_- := P((-\infty, \inf \Delta))$ . Since  $H_B$  is assumed to have a gap, we have  $\text{Tr}(\Sigma_B) = \text{Tr}(P_+ \Sigma_B P_+) + \text{Tr}(P_- \Sigma_B P_-)$ . Since  $P_{\pm} R_B(z)$  and  $R_B(z) P_{\pm}$  are analytic on  $\text{supp}(\rho(z))$  and  $\text{supp}(1 - \rho(z))$ , respectively, the integral in  $P_{\pm} \Sigma_B P_{\pm}$  vanishes by integration by parts. Thus

$$\Sigma_B = iP_+[\rho(H_B), \Lambda_1] \Lambda_2 P_+ + iP_-[\rho(H_B), \Lambda_1] \Lambda_2 P_-.$$

By the spectral theorem for projection-valued measures, if the Fermi energy lies in the gap,  $\lambda \in \Delta$ , we have

$$\rho(H_B) = \int_{-\infty}^{\infty} \rho(\lambda) dP_{\nu} = \int_{-\infty}^{\lambda} \rho(\lambda) dP_{\nu} = \int_{-\infty}^{\lambda} dP_{\nu} = P_{\lambda}.$$

We may therefore replace  $\rho(H_B)$  by  $P_{\lambda}$ , by which we obtain

$$\text{Tr}(\Sigma_B) = i\text{Tr}(P_+[P_{\lambda}, \Lambda_1] \Lambda_2 P_+) + i\text{Tr}(P_-[P_{\lambda}, \Lambda_1] \Lambda_2 P_-).$$

Now, the bulk conductivity is given by

$$\begin{aligned}
\sigma_B &= i\text{Tr}(P_\lambda[[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]]) \\
&= i\text{Tr}(P_\lambda((P_\lambda\Lambda_1 - \Lambda_1P_\lambda)(P_\lambda\Lambda_2 - \Lambda_2P_\lambda) - (P_\lambda\Lambda_2 - \Lambda_2P_\lambda)(P_\lambda\Lambda_1 - \Lambda_1P_\lambda))) \\
&= i\text{Tr}(P_\lambda(P_\lambda\Lambda_1P_\lambda\Lambda_2 - P_\lambda\Lambda_1\Lambda_2P_\lambda - \Lambda_1P_\lambda\Lambda_2 + \Lambda_1P_\lambda\Lambda_2P_\lambda \\
&\quad - P_\lambda\Lambda_2P_\lambda\Lambda_1 + P_\lambda\Lambda_2\Lambda_1P_\lambda + \Lambda_2P_\lambda\Lambda_1 - \Lambda_2P_\lambda\Lambda_1P_\lambda)) \\
&= i\text{Tr}(-P_\lambda\Lambda_1\Lambda_2P_\lambda + \Lambda_1P_\lambda\Lambda_2P_\lambda + P_\lambda\Lambda_2\Lambda_1P_\lambda - \Lambda_2P_\lambda\Lambda_1P_\lambda) \\
&= i\text{Tr}(-P_\lambda\Lambda_1\Lambda_2P_\lambda + P_\lambda\Lambda_1P_\lambda\Lambda_2P_\lambda + P_\lambda\Lambda_2\Lambda_1P_\lambda - P_\lambda\Lambda_2P_\lambda\Lambda_1P_\lambda) \\
&= i\text{Tr}(P_\lambda\Lambda_1P_\lambda^\perp\Lambda_2P_\lambda - P_\lambda\Lambda_2P_\lambda^\perp\Lambda_1P_\lambda) \\
&= i\text{Tr}(P_\lambda\Lambda_1P_\lambda^\perp\Lambda_2P_\lambda - P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2) \\
&= i\text{Tr}(P_\lambda\Lambda_1P_\lambda^\perp\Lambda_2P_\lambda - P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2P_\lambda^\perp).
\end{aligned}$$

We define  $T_\lambda := P_\lambda\Lambda_1P_\lambda^\perp\Lambda_2P_\lambda - P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2P_\lambda^\perp$ , so that

$$\sigma_B = i\text{Tr}(T_\lambda),$$

and show that  $P_\pm T_\lambda P_\pm = P_\pm[P_\lambda, \Lambda_1]\Lambda_2P_\pm$ .

First, notice that because of the gap, we have  $P_\lambda^\perp P_- = 0$ , and thus also  $P_\lambda P_- = P_-$ . Thus

$$\begin{aligned}
P_- T_\lambda P_- &= P_- P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda P_- \\
&= P_- (P_\lambda \Lambda_1 \Lambda_2 - \Lambda_1 P_\lambda \Lambda_2) P_- \\
&= P_- [P_\lambda, \Lambda_1] \Lambda_2 P_-,
\end{aligned}$$

and similarly, for  $P_+$ , we have  $P_\lambda^\perp P_+ = P_+$ , and  $P_\lambda P_- = 0$ , which implies

$$\begin{aligned}
P_+ T_\lambda P_+ &= -P_+ P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_\lambda^\perp P_+ \\
&= -P_+ P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_+ \\
&= -P_+ P_\lambda^\perp \Lambda_1 \Lambda_2 P_+ + P_+ P_\lambda^\perp \Lambda_1 P_\lambda^\perp \Lambda_2 P_+ \\
&= -P_+ P_\lambda^\perp \Lambda_1 \Lambda_2 P_+ + P_+ \Lambda_1 P_\lambda^\perp \Lambda_2 P_+ \\
&= -P_+ [P_\lambda^\perp, \Lambda_1] \Lambda_2 P_+ \\
&= -P_+ [(1 - P_\lambda), \Lambda_1] \Lambda_2 P_+ \\
&= P_+ [P_\lambda, \Lambda_1] \Lambda_2 P_+.
\end{aligned}$$

Finally, we obtain



$$\begin{aligned}
\sigma_B &= i\text{Tr}(T_\lambda) \\
&= i\text{Tr}(P_- T_\lambda P_-) + i\text{Tr}(P_+ T_\lambda P_+) \\
&= i\text{Tr}(P_- [P_\lambda, \Lambda_1] \Lambda_2 P_-) + i\text{Tr}(P_+ [P_\lambda, \Lambda_1] \Lambda_2 P_+) \\
&= \text{Tr}(\Sigma_B),
\end{aligned}$$

concluding the proof.  $\square$

### 3 Interacting Setting

Let  $L \in \mathbb{N}$ , and let  $\Lambda_L = \mathbb{Z}_L \times [0, L]$  be the discrete cylinder, equipped with a metric  $d$ . To each site  $x \in \Lambda_L$ , we associate a Hilbert space  $\mathcal{H}_x$  whose dimension is bounded uniformly in  $L$ , i.e. there exists some  $N > 0$  such that for all  $L \in \mathbb{N}$ , we have  $\dim(\mathcal{H}_x) \leq N$  for all  $x \in \Lambda_L$ . For a subset  $X \subseteq \Lambda_L$ , we define the Hilbert space  $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$ , and we set  $\mathcal{H}_L := \mathcal{H}_{\Lambda_L} = \otimes_{x \in \Lambda_L} \mathcal{H}_x$ . For simplicity, throughout we take  $L = 2^n$  for some large  $n$ .

The algebra  $\mathcal{U}_L \subset \mathcal{B}(\mathcal{H}_L)$  of observables on  $\Lambda_L$  is the set of bounded operators on  $\mathcal{H}_L$  which are self-adjoint. For an operator  $A_X \in \mathcal{U}_X$ , we identify its extension to  $\mathcal{U}_L$  by taking its tensor product with copies of the identity,  $A_X \otimes_{x \in X^c} \mathbb{I}_x$ . Conversely, we say that an operator  $A_X \in \mathcal{U}_L$  has support  $X$  if  $A_X = A_X|_X \otimes_{x \in X^c} \mathbb{I}_x$ , and write  $A_X \in \mathcal{U}_X$ .

A *local interaction* is a map  $\Phi : \mathcal{P}(\Lambda_L) \rightarrow \mathcal{U}_L$  such that  $\Phi(X) = 0$  whenever  $\text{diam}(X) > R$ ,  $\Phi(X)$  is supported in  $X$ , and  $\|\Phi(X)\| \leq C$  for all  $X \subset \Lambda_L$ , for all  $L$ .

We consider a region as depicted in Figure 1. In the left white region  $[0, L/2] \times [0, L]$ ,  $H_0$  is a trivial Hamiltonian which we take to be the void (for example  $H_0 = 0$ ), and in the right blue region  $[L/2, L] \times [0, L]$ ,  $H_1$  is a nontrivial *local Hamiltonian*, in the sense that  $H_1 = \sum_{X \subseteq \Lambda} \Phi(X)$ , where  $\Phi$  is a local interaction. We define the Hamiltonian of the full system to be

$$H(\mu) = H_1 + \mu Q_h,$$

where  $Q_h = \sum_{x \in \Gamma_h} a_x^* a_x$  is the number operator for the region  $\Gamma_h = [L/4, 3L/4] \times [0, L]$ .

We also consider the plane  $\mathbb{Z}^2$ . In this setting, there are no edge states, and so the associated “bulk” Hamiltonian  $H_B(\mu)$  is assumed to have a *gapped* spectrum, in the sense that

**Assumption 3.**

$$\sigma(H_B) = \mathcal{S}_- \cup \mathcal{S}_+,$$

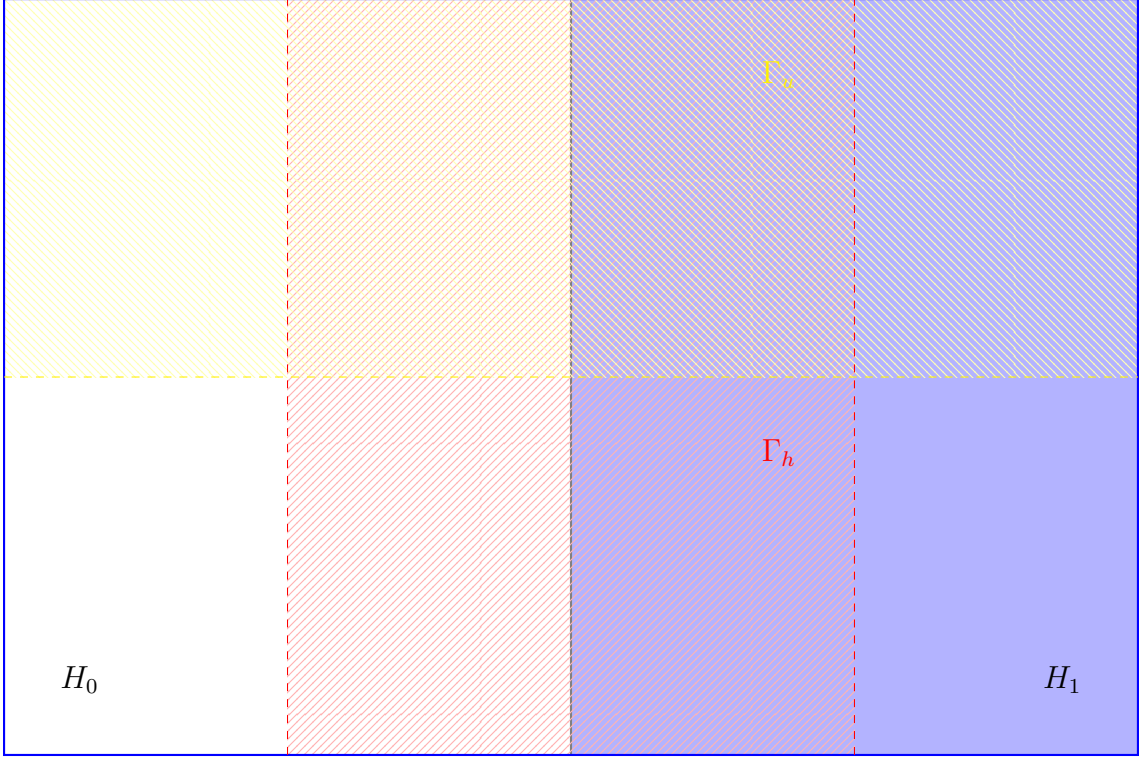


Figure 1: The setup for  $\Lambda_L$ . The left and right edges are identified to form a cylinder.

where  $\inf \mathcal{S}_+ - \sup \mathcal{S}_- \geq \gamma$  uniformly in  $L$  and  $\mu$  for some  $\gamma > 0$ .

In the case of the cylinder, this effect does not necessarily occur due to the presence of the edge. We also assume that the Hamiltonian is *charge-conserving*.

**Assumption 4.**  $[H(\mu), Q] = 0$ , where  $Q$  is the total charge in  $\Lambda_L$ .

Let  $P_B$  be the ground state projection of  $H_B$  (the system without an edge), and let  $P$  be the ground state projection of  $H$  (the system with an edge). We assume that states far from the edge are essentially bulk states, up to tails that vanish quickly in  $L$ .

**Assumption 5.** Define the bulk region  $\Gamma_B = [L/2 + k, L] \times [0, L]$  for some  $k > 2R$ . For any operator  $A \in \mathcal{U}_{\Gamma_B}$ ,

$$\text{Tr}(PA) = \text{Tr}(P_B A) + \mathcal{O}(L^{-\infty}).$$

(need to add justification)

For ease of notation, we omit the subscript  $L$  wherever there is no risk of confusion.

## 4 Equality of Bulk and Edge Currents

### 4.1 Cylinder Geometry

Let  $P(\mu)$  be the (possibly degenerate) ground state projection of  $H(\mu)$ . Let  $Q_u = \sum_{x \in \Gamma_u} a_x^* a_x$  be the charge in the upper half of the cylinder  $\Gamma_u = [0, L] \times [L/2, L]$ , and define current operator

$$J_u = i[H(\mu), Q_u].$$

For simplicity, we drop the subscript  $u$  and simply write  $J = J_u$ . Charge conservation 4 implies that this current operator is supported on the strip  $[L/2, L] \times [L/2 - R, L/2 + R]$ , i.e. along a strip of width  $2R$  centred at the line  $y = L/2$ . Indeed, if we inspect a local interaction  $\Phi(X)$  of range  $R$  with support  $S \subset (\Gamma_u)_R$ , where  $(\Gamma)_\alpha$  is the  $\alpha$ -shrinking of  $\Gamma$ , then clearly  $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$ , by assumption 4. Similarly, if  $\Phi(X)$  has support  $S \subset ((\Gamma_u)^c)_R$ , then  $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$ . It follows that for an interaction  $\Phi(X)$  with range  $R$  and arbitrary support,  $[\Phi(X), Q_u]$  must be supported on a set which is contained in (or equal to) the strip  $[L/2, L] \times [L/2 - R, L/2 + R]$ , and so  $[H, Q_u]$  must be as well, since  $H$  is a sum of such local interactions.

**Lemma 2.** *The ground state expectation of the current  $J$  is zero.*

*Proof.* By cyclicity of the trace and commutativity of  $P(\mu)$  with  $H(\mu)$ ,

$$\begin{aligned} \text{Tr}(P(\mu)J) &= \text{Tr}(P(\mu)i[H(\mu), Q_u]) \\ &= \text{Tr}(iP(\mu)H(\mu)Q_u) - \text{Tr}(iP(\mu)Q_uH(\mu)) \\ &= \text{Tr}(iH(\mu)P(\mu)Q_u) - \text{Tr}(iH(\mu)P(\mu)Q_u) \\ &= 0. \end{aligned}$$

□

Define the operators

$$K(\mu) = \mathcal{I}_\mu(\dot{H}(\mu)),$$

where

$$\mathcal{I}_\mu(A) = \int_{\mathbb{R}} W(t) e^{itH(\mu)} A e^{-itH(\mu)} dt.$$

More explicitly, in this setting we see that  $K(\mu) = \mathcal{I}_\mu(Q_h)$ . As a shorthand, we use the notation  $\dot{A}(\mu_0) = \frac{d}{d\mu} A|_{\mu_0}$ .

We present two important properties of the map  $\mathcal{I}_\mu$  in the following lemmas, and leave their proofs to the appendix (need to add). We will also need a definition: an *off-diagonal* operator is an operator  $A$  such that  $A = \bar{A} := PAP^\perp + P^\perp AP$ , where  $P^\perp = \mathbb{I} - P$  is the projection onto the excited states above the gap.

**Lemma 3.** *For any off-diagonal operator  $A$ ,  $\mathcal{I}_\mu(\cdot)$  and  $[H(\mu), \cdot]$  act as inverses of each other, up to a factor of  $i$ :*

$$\mathcal{I}_\mu([H(\mu), A]) = [H(\mu), \mathcal{I}_\mu(A)] = iA$$

It is easy to verify that any operator  $A$  behaves as an off-diagonal operator when taking a commutator with  $P$ , in the sense that

$$[\bar{A}, P] = [PA(1 - P) + (1 - P)AP, P] = [A, P].$$

It follows that for any (not necessarily off-diagonal) operator  $A$ ,

$$[\mathcal{I}([H, A]), P] = i[A, P]$$

**Lemma 4.**  $\mathcal{I}_\mu$  is local in the sense that for any  $A \in \mathcal{U}_X$ ,

$$\|\mathcal{I}(A)_{(X^r)^c}\| \leq \|A\| \|X\| \mathcal{O}(r^{-\infty}).$$

**Proposition 2.** *The operator  $K(\mu)$  is the generator of parallel transport, satisfying*

$$\dot{P}(\mu) = i[K(\mu), P(\mu)].$$

*Proof.* By the product rule and the fact that  $H(\mu)$  and  $P(\mu)$  commute,

$$[\dot{H}(\mu), P] = -[H, \dot{P}(\mu)].$$

Now we show that  $\dot{P}(\mu)$  is off-diagonal. Taking the derivative on both sides of  $P^2 = P$ , we see that  $\dot{P}P + P\dot{P} = \dot{P}$ . Acting on the left and right with  $P$  on both sides of this equation gives

$$P\dot{P}P + P\dot{P}P = P\dot{P}P,$$

which implies that  $P\dot{P}P = 0$ . Now,

$$\begin{aligned}\overline{\partial_\mu P} &= P(\partial_\mu P)(1 - P) + (1 - P)(\partial_\mu P)P \\ &= P(\partial_\mu P) - P(\partial_\mu P)P + (\partial_\mu P)P - P(\partial_\mu P)P \\ &= P(\partial_\mu P) + (\partial_\mu P)P \\ &= \partial_\mu(P^2) \\ &= \partial_\mu P,\end{aligned}$$

as claimed. It therefore follows from Lemma 3 that

$$\begin{aligned}\dot{P}(\mu) &= -i\mathcal{I}_\mu([H(\mu), \dot{P}(\mu)]) \\ &= i\mathcal{I}_\mu([\dot{H}(\mu), P(\mu)]) \\ &= i[\mathcal{I}_\mu(\dot{H}(\mu)), P(\mu)] \\ &= i[K(\mu), P(\mu)].\end{aligned}$$

□

Increasing the “electric potential” by a small amount  $d\mu Q_h$  and expanding to linear order, the change in ground state current is given by

$$\text{Tr}(P(\mu + d\mu)J) - \text{Tr}(P(\mu)J) = \kappa d\mu + \mathcal{O}(d\mu^2).$$

Dividing by  $d\mu$  and taking a limit, we see that the linear response coefficient is given by

$$\kappa = \text{Tr} \left( \dot{P}(\mu)J \right).$$

The *Hall conductivity* of the system on a subset  $V \subseteq \Lambda$  is defined to be  $\kappa_V := \text{Tr} \left( \dot{P}(\mu)J_V \right)$ , where  $J_V$  is the restriction of the operator  $J$  to  $V$ .

**Proposition 3.** *The Hall conductivity is independent of the driving strength  $\mu$ .*

*Proof.* We see by cyclicity of the trace and the formula  $\dot{H}(\mu) = Q_h$  that for any  $\mu_1$  and  $\mu_2$ ,

$$\begin{aligned}
\kappa(\mu_1) - \kappa(\mu_2) &= \text{Tr} \left( \dot{P}(\mu_1) i[H(\mu_1), Q_u] - \dot{P}(\mu_2) i[H(\mu_1), Q_u] \right) \\
&= i \text{Tr} \left( \left( [\dot{P}(\mu_1), H(\mu_1)] - [\dot{P}(\mu_2), H(\mu_2)] \right) Q_u \right) \\
&= i \text{Tr} \left( \left( [\dot{H}(\mu_1), P(\mu_1)] - [\dot{H}(\mu_2), P(\mu_2)] \right) Q_u \right) \\
&= i \text{Tr} ([Q_h, P(\mu_1) - P(\mu_2)] Q_u) \\
&= i \text{Tr} (Q_h (P(\mu_1) - P(\mu_2)) Q_u - (P(\mu_1) - P(\mu_2)) Q_u Q_h) \\
&= 0,
\end{aligned}$$

since  $Q_h$  and  $Q_u$  commute, indicating that the Hall conductivity is independent of  $\mu$  as one would expect physically.  $\square$

The following is the main result:

**Theorem 3.** *The ground state current in the strip  $[L/2 + k, 3L/4 - k] \times [0, L]$  between the edge and the bulk current vanishes, in the sense that  $\kappa_V = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty})$  for any  $V \subseteq [L/2 + R, 3L/4 - R] \times [0, L]$  “in between” the bulk and edge strips, where*

$$r = \text{dist}(V, [L/2 - R, 3L/4 + R] \times [0, L] \cup [3L/4 - R, 3L/4 + R] \times [0, L])$$

*is the distance from  $V$  to one of the edge or bulk strips.*

*Proof.* By Proposition 2, the Hall conductivity can also be written by the formula  $\kappa_V^B = \text{Tr} (i[K(\mu), P_B(\mu)] J_V^B) = \text{Tr} (i[\mathcal{I}_\mu(Q_h), P_B(\mu)] J_V^B)$ , where  $J_V^B = (i[H_B, Q_u])_V$  is the current arising from the bulk Hamiltonian. From commutativity of  $P_B$  and  $H_B$  along with cyclicity of the trace, we compute

$$\begin{aligned}
\kappa_V^B &= \text{Tr} (i[\mathcal{I}_\mu(Q_h), P_B(\mu)] J_V^B) \\
&= \text{Tr} \left( i \int_{\mathbb{R}} W(t) e^{itH_B(\mu)} [Q_h, P_B(\mu)] e^{-itH_B(\mu)} dt J_V^B \right) \\
&= \int_{\mathbb{R}} W(t) \text{Tr} (i[Q_h, P_B(\mu)] e^{-itH_B(\mu)} J_V^B e^{itH_B(\mu)}) dt \\
&= - \int_{\mathbb{R}} W(t) \text{Tr} (i[Q_h, P_B(\mu)] e^{itH_B(\mu)} J_V^B e^{-itH_B(\mu)}) dt \\
&= - \text{Tr} (i[Q_h, P_B(\mu)] \mathcal{I}_\mu(J_V^B)),
\end{aligned}$$

since  $W(t)$  is odd. Again by cyclicity of trace combined with the fact that  $\mathcal{I}_\mu(\cdot)$  is an inverse of  $[H_B(\mu), \cdot]$  for commutators with  $P_B(\mu)$  (by the remark after lemma 3), we obtain

$$\begin{aligned}
\kappa_V^B &= -\text{Tr}([\mathcal{I}_\mu([H_B(\mu), Q_h]), P_B(\mu)]\mathcal{I}_\mu(J_V^B)) \\
&= -\text{Tr}(\mathcal{I}_\mu([H_B(\mu), Q_h])P_B(\mu)\mathcal{I}_\mu(J_V^B) - P_B(\mu)\mathcal{I}_\mu([H_B(\mu), Q_h])\mathcal{I}_\mu(J_V^B)) \\
&= -\text{Tr}(P_B(\mu)\mathcal{I}_\mu(J_V^B)\mathcal{I}_\mu([H_B(\mu), Q_h]) - P_B(\mu)\mathcal{I}_\mu([H_B(\mu), Q_h])\mathcal{I}_\mu(J_V^B)) \\
&= \text{Tr}(P_B(\mu)[\mathcal{I}_\mu([H_B(\mu), Q_h]), \mathcal{I}_\mu(J_V^B)]).
\end{aligned}$$

Now,  $[H_B(\mu), Q_h]$  is a local operator, supported on the “bulk line”  $[3L/4 - R, 3L/4 + R] \times [0, L]$ , while  $J_V^B$  is a local operator supported on  $V \subseteq [L/2 + k, 3L/4 - k] \times [0, L]$ . Since  $\mathcal{I}_\mu$  preserves locality up to tails, in the sense that  $\|\mathcal{I}_\mu(A)_{(S^r)^c}\| \leq \|A\| |S| \mathcal{O}(r^{-\infty})$  (Lemma 4), it follows that the commutator  $[\mathcal{I}_\mu([H_B(\mu), Q_h]), \mathcal{I}_\mu(J_V^B)] = \mathcal{O}(r^{-\infty})$  whenever  $V \cap ([3L/4 - R, 3L/4 + R] \times [L/2, L]) = \emptyset$ .

The previous fact applies to the bulk setting with  $H_B$  and  $P_B$ . To extend this to the setting with an edge, it is enough to use Assumption 5 to conclude the same result, except with equality up to  $\mathcal{O}(L^{-\infty})$ , i.e.

$$\kappa_V = \text{Tr}(\dot{P} J_V) = \text{Tr}(\dot{P}(J_V^B + \mathcal{O}(L^{-\infty}))) = \kappa_V^B + \mathcal{O}(L^{-\infty}) = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

□

The intuitive picture from the previous result is that, in the bulk region, the Hall conductivity is essentially only nonzero along the bulk line  $[3L/4 - R, 3L/4 + R] \times [L/2, L]$ . Since the ground state expectation of the current is zero (by lemma 2), it must be that there is an equal current flowing along the edge, but in the opposite direction, see figure (need to add).

## 4.2 Torus Geometry

Our goal is to show the same result on the discrete torus  $\mathbb{T}_L := \mathbb{Z}_L \times \mathbb{Z}_L$ . We define the same regions  $\Gamma_u$  and  $\Gamma_h$ , and the same current operator  $J_u = i[H(\mu), Q_u]$ . This time, however, Lemma 2 does not apply. Intuitively, it does not apply because electrons can now flow through both the bottom and the top of the region  $\Gamma_u$ , rather than just the bottom. Mathematically, the lemma fails because our definition of the current is slightly changed.

We use charge conservation and the fact that  $H$  is finite range to split the current  $J_u$  into two components,  $J_u = i[H_-, Q_u] + i[H_+, Q_u] = J_- - J_+$ , supported on strips of width  $2R$  at  $y = L/2$  and  $y = L$ , respectively. We then define the current operator to be  $J = J_-$ , which is the current on the lower

strip. This is the mathematical reason that the proof in Lemma 2 fails on the torus; we have replaced  $H$  by  $H_-$ , which may no longer commute with  $P$ . We instead proceed by a different approach. We will need a few auxiliary results first.

**Lemma 5.**  $K_{\pm}$  is supported on  $\partial_{\pm}$  up to tails.

*Proof.* □

**Proposition 4.** The operator  $Q_h - K$  leaves the ground state space invariant, i.e.  $[Q_h - K, P] = 0$ .

*Proof.* □

**Lemma 6.** Show that  $\text{Tr}(A, [Q_h, P]) = 0$  for all  $A \in \mathcal{U}_{\text{edge}}$ . This shows that  $Q_h$  commutes with  $P$  “along the edge”.

*Proof.* Let  $A \in \mathcal{U}_{\text{edge}}$ . Since  $H$  is charge conserving, we may choose a simultaneous eigenbasis of  $H$  and the total charge  $Q$ , in which case  $P$  and  $Q$  commute. It follows that

$$\text{Tr}(A[Q_h, P]) = \text{Tr}([A, Q_h]P) = \text{Tr}([A, Q]P) = \text{Tr}(A[Q, P]) = 0.$$

□

Finally, we will prove that in the bulk system with Hamiltonian  $H_B(\mu)$ , the ground state expectation of the current vanishes faster than any power as  $L \rightarrow \infty$ .

**Lemma 7.** The ground state expectation of the current  $J_B := i[(H_B)_-, Q_h]$  (of the system without an edge) is  $\text{Tr}(P_B J_B) = \mathcal{O}(L^{-\infty})$ .

*Proof.* First,  $K = \mathcal{I}(i[H_B, Q])$  splits into  $K = K_- - K_+$ , with the support of  $K_{\pm}$  contained in  $\partial_{\pm}$  up to tails:

$$[K_{\pm}, A_X] = \mathcal{O}(p^{-\infty}),$$

for every  $A_X \in \mathcal{U}_X$  such that  $\|A_X\| = 1$ , and where  $p = \text{dist}(X, \partial_{\pm})$  (need to add). Using the fact that  $K_{\pm}$  is supported in  $\partial_{\pm}$  up to tails (Lemma 5), we see that

$$i[H_B, K_-] = i[(H_B)_-, K_-] + \mathcal{O}(L^{-\infty}),$$



and similarly  $i[(H_B)_-, K_+] = \mathcal{O}(L^{-\infty})$ . Putting these facts together, it follows that the current can be rewritten as

$$\begin{aligned} J_B &= i[H_B, Q_h + K_- - K_- + K_+] + \mathcal{O}(L^{-\infty}) \\ &= i[H_B, K_-] + i[(H_B)_-, Q_h - K_- + K_+] + \mathcal{O}(L^{-\infty}). \end{aligned}$$

From here, we use the fact that  $H_B$  and  $Q_h - K_- + K_+$  both commute with  $P_B$  to write

$$P_B J_B P_B = i[H_B, P_B K_- P_B] + i[P_B (H_B)_- P_B, Q_h - K_- + K_+] + P_B \mathcal{O}(L^{-\infty}) P_B.$$

Since the trace of any commutator is zero,

$$\text{Tr}(P_B J_B) = \text{Tr}(P_B J_B P_B) = \mathcal{O}(L^{-\infty}).$$

□

Using this, we can show a simple proof of the analogue of Lemma 2 on the torus, in the case of non-interacting systems.

**Proposition 5.** *Let  $H = \sum_{x \in \mathbb{T}} h_x$  be a non-interacting Hamiltonian, i.e. a sum of single site Hamiltonians  $h_x$ . The ground state expectation of the current  $J = i[H_-, Q_h]$  (of the system with an edge) is  $\text{Tr}(PJ) = \mathcal{O}(L^{-\infty})$ .*

*Proof.* Since  $H$  is a sum of single site Hamiltonians, we can split  $H_-$  into the restrictions  $H_- = (H_-)_{\text{edge}} + (H_-)_{\text{bulk}}$ , with no fear of any terms which are in both the edge region and the bulk region. By Assumption 5,

$$\begin{aligned} \text{Tr}(PJ) &= \text{Tr}(Pi[H_-, Q_h]) \\ &= i\text{Tr}([H_-, Q_h]P) \\ &= i\text{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\text{Tr}((H_-)_{\text{bulk}}[Q_h, P]) \\ &= i\text{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\text{Tr}((H_-)_{\text{bulk}}[Q_h, (P)_{\text{bulk}}]) \\ &= i\text{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\text{Tr}((H_B)_-[Q_h, P_B]) + \mathcal{O}(L^{-\infty}) \\ &= i\text{Tr}((H_-)_{\text{edge}}[Q_h, P]) + \text{Tr}(i[(H_B)_-, Q_h]P_B) + \mathcal{O}(L^{-\infty}). \end{aligned}$$

By Lemma 6, the first term is zero. By Lemma 7, the second term is  $\mathcal{O}(L^{-\infty})$ . □

## A Properties of $\mathcal{I}_\mu$

*Proof.* (Of Lemma 3). Let  $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} W(t) e^{-2\pi i t \xi} dt$  be the Fourier transform of  $W$ . One can show that for  $|\xi| \geq \gamma$ ,  $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi i \xi}}$  (need to add). Let  $A$  be an observable. First, we show that  $\mathcal{I}([H, PAP^\perp]) = i PAP^\perp$ .

Decomposing

$$\begin{aligned} e^{itH} P &= \sum_{j=0}^{\infty} \frac{(itH)^j}{j!} P \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left( \sum_n E_n^j P_n \right) P \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \sum_{n: E_n=0} E_n^j P_n \\ &= \sum_{n: E_n=0} e^{itE_n} P_n, \end{aligned}$$

and similarly

$$P^\perp e^{-itH} = \sum_{m: E_m \geq \gamma} P_m e^{-itE_m},$$

we see that

$$\begin{aligned}
\mathcal{I}([H, PAP^\perp]) &= \mathcal{I}(P[H, A]P^\perp) \\
&= \int_{\mathbb{R}} W(t) e^{itH} P[H, A] P^\perp e^{-itH} dt \\
&= \int_{\mathbb{R}} W(t) \sum_{n: E_n=0} e^{itE_n} P_n[H, A] \sum_{m: E_m \geq \gamma} P_m e^{-itE_m} dt \\
&= \sum_{n: E_n=0} \sum_{m: E_m \geq \gamma} \int_{\mathbb{R}} W(t) e^{itE_n} P_n A(E_n - E_m) P_m e^{-itE_m} dt \\
&= \sum_{n: E_n=0} \sum_{m: E_m \geq \gamma} P_n A P_m (E_n - E_m) \int_{\mathbb{R}} W(t) e^{-it(E_m - E_n)} dt \\
&= \sum_{n: E_n=0} \sum_{m: E_m \geq \gamma} P_n A P_m (E_n - E_m) \sqrt{2\pi} \widehat{W}(E_m - E_n) \\
&= i \sum_{n: E_n=0} \sum_{m: E_m \geq \gamma} P_n A P_m \\
&= iPAP^\perp.
\end{aligned}$$

(need to check the  $2\pi$  factor)

By the same argument,  $\mathcal{I}([H, P^\perp AP]) = iP^\perp AP$  as well, and so  $\mathcal{I}([H, \bar{A}]) = i\bar{A}$ .  $\square$

*Proof.* (Of Lemma 4). We break the integral into two parts,

$$\|\mathcal{I}(A)\| \leq \left\| \int_{-T}^T W(t) e^{itH} A e^{-itH} dt \right\| + \left\| \int_{\mathbb{R} \setminus [-T, T]} W(t) e^{itH} A e^{-itH} dt \right\|.$$

The first term can be estimated using the Lieb-Robinson bound found in Appendix B.  $\square$

## B Lieb-Robinson Bound

Let  $N$  be a uniform upper bound for the dimensions of the Hilbert spaces at each site, i.e.  $\dim(\mathcal{H}_x) \leq N$  for all sites  $x$ .

The following is a version of the Lieb-Robinson. For any operators  $A \in \mathcal{U}_X$  and  $B \in \mathcal{U}_Y$  having disjoint supports  $X \cap Y = \emptyset$ ,

$$\| [e^{itH} A e^{-itH}, B] \| \leq C \|A\| \|B\| |X| |Y| N^{2|X|} e^{2t\|\Phi\|_\lambda - \lambda d(X, Y)}.$$

## C Grönwall's Inequality and Uniqueness

**Theorem 4.** (*Grönwall's Inequality*). Let  $\alpha : I \rightarrow (0, \infty)$  be positive and continuous on  $I^\circ$  for some interval of the form  $[a, b)$ ,  $[a, b]$ , or  $[a, \infty)$ . Suppose  $u : \mathbb{R} \rightarrow \mathcal{U}$  is a Banach-valued, differentiable function. If  $\|u'(t)\| \leq \alpha(t)\|u(t)\|$  for all  $t \in I$ , then

$$\|u(t)\| \leq \|u(a)\| e^{\int_a^t \alpha(s) ds} \quad \forall t \in I$$

*Proof.* Let  $f(t) = e^{\int_a^t \alpha(s) ds}$ , which is nonzero, has initial value  $f(a) = 1$ , and has derivative  $f'(t) = \alpha(t)f(t)$ . Then by the quotient rule,

$$\left( \frac{\|u(t)\|}{f(t)} \right)' = \frac{\|u'(t)\|f(t) - \|u(t)\|\alpha(t)f(t)}{f(t)^2} \leq 0,$$

where the inequality follows from the assumption  $\|u'(t)\| \leq \alpha(t)\|u(t)\|$ . Thus  $\frac{\|u(t)\|}{f(t)}$  is decreasing, so that

$$\frac{\|u(t)\|}{f(t)} \leq \frac{\|u(a)\|}{f(a)} = \|u(a)\|,$$

which is the desired inequality.  $\square$

**Theorem 5.** (*ODE Uniqueness*). Let  $F : \mathcal{U} \rightarrow \mathcal{U}$  be Lipschitz and consider the differential equation  $u'(t) = F(u(t))$  with initial condition  $u(a) = u_a$  for some function  $u : I \rightarrow \mathcal{U}$ , where  $I = [a, b]$ , or  $[a, b)$ , or  $[a, \infty)$ . Solutions to this equation are unique.

*Proof.* Suppose there are two solutions  $u(t)$  and  $v(t)$ , and let  $g(t) = \|u(t) - v(t)\|^2$ . By assumption, there exists a constant  $L_F$  such that  $\|F(u(t)) - F(v(t))\| \leq L_F\|u(t) - v(t)\|$ , so that

$$\begin{aligned} g'(t) &= 2\|u(t) - v(t)\|\|u'(t) - v'(t)\| \\ &= 2\|u(t) - v(t)\|\|F(u(t)) - F(v(t))\| \\ &\leq 2L_F\|u(t) - v(t)\|^2 \\ &= 2L_F g(t). \end{aligned}$$

Notice that  $\alpha := 2L_F$  is a positive continuous function, so we may apply Grönwall's inequality to  $g(t)$  to conclude

$$g(t) \leq g(a)e^{2L_F(t-a)} = 0,$$

since  $g(a) = 0$ .  $\square$

## D Note on Generators of Parallel Transport

Consider the differential equation  $\dot{\rho}(\mu) = i[K_B, \rho(\mu)]$  with initial condition  $\rho(0) = P_B(0)$ . Here  $K_B = \int_{\mathbb{R}} W_{\gamma}(t) e^{-itH_B} \dot{H}_B e^{itH_B} dt$ , and recall that in our setting,  $\dot{H}_B = Q_h$ . We know that the solution is  $\rho(\mu) = P_B(\mu)$  (proposition 2). Notice that the map  $F : \mathcal{U} \rightarrow \mathcal{U}$  defined by  $F(A) = i[K_B, A]$  is Lipschitz, since

$$\|F(A) - F(B)\| = \|[K_B, A - B]\| \leq 2\|K_B\| \|A - B\|.$$

The Lipschitz constant is  $2\|K_B\|$ , which is finite since  $K_B$  is a bounded operator:

$$\|K_B\| \leq \int_{\mathbb{R}} |W_{\gamma}(t)| \|e^{-itH_B} Q_h e^{itH_B}\| dt \leq \int_{\mathbb{R}} |W_{\gamma}(t)| dt \|Q_h\|.$$

Indeed, since  $Q_h$  is the number operator on a finite volume, by charge conservation and the fact that the dimension of the Hilbert space is bounded uniformly by  $d$ , there can only be a finite number of charges in the region  $\Gamma_h$ .

Thus, by Grönwall's uniqueness theorem (appendix C), we see that the solution to the equation  $\dot{\rho}(\mu) = F(\rho(\mu)) = i[K_B, \rho(\mu)]$  is unique.

Now define

$$K_E := \int_{\mathbb{R}} W_{\gamma}(t) e^{-itH_E} Q_h e^{itH_E} dt,$$

which is using the gap  $\gamma$  of  $H_B$  to define  $W_{\gamma}$ , but also using the edge Hamiltonian in the time evolution operators. Consider  $\sigma : [0, \infty) \rightarrow \mathcal{U}$  defined by

$$\dot{\sigma}(\mu) = i[K_E, \sigma(\mu)] \quad \sigma(0) = P_E(0).$$

We now show that, similar to how  $\rho$  is an approximation of  $P_B$ ,  $\sigma$  is also a good approximation of  $P_E$  (up to  $\mathcal{O}(L^{-\infty})$ ) “in the bulk”. Let  $A \in \Gamma_B$  be an operator localized in the bulk of the edge system. Then

$$\begin{aligned}
\text{Tr}(\dot{\sigma}A) &= \text{Tr}(i[K_E, \sigma]A) \\
&= \text{Tr}(i[A, K_E]\sigma) \\
&= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}([e^{-itH_E} Q_h e^{itH_E}, A]\sigma) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}(e^{-itH_E} [Q_h, e^{itH_E} A e^{-itH_E}] e^{itH_E} \sigma) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}(e^{-itH_E} [Q_h, e^{itH_B} A e^{-itH_B}] e^{itH_E} + \mathcal{O}(L^{-\infty}) \sigma) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}(e^{-itH_B} [Q_h, e^{itH_B} A e^{-itH_B}] e^{itH_B} + \mathcal{O}(L^{-\infty}) \sigma) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}([e^{-itH_B} Q_h e^{itH_B}, A]\sigma) dt + \mathcal{O}(L^{-\infty}) \\
&= \text{Tr}(i[A, K_B]\sigma) + \mathcal{O}(L^{-\infty}) \\
&= \text{Tr}(i[K_B, \sigma]A) + \mathcal{O}(L^{-\infty}),
\end{aligned}$$

since  $\sigma$  is trace-class (?) and  $W_\gamma \in L^1$ . By linearity of the trace, we see that  $\text{Tr}((\dot{\sigma} - i[K_B, \sigma])A) = \mathcal{O}(L^{-\infty})$  for any operator  $A \in \Gamma_B$  (does this mean  $\dot{\sigma} - i[K_B, \sigma] = 0$ ?). But the solution of  $\dot{\sigma} - i[K_B, \sigma] = 0$  (with initial condition  $\sigma(0) = P_B(0)$ ) is unique; it is  $\rho(\mu)$ , or  $P_B(\mu)$ . Hence

$$\text{Tr}(P_E A) = \text{Tr}(P_B A) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\rho A) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\sigma A) + \mathcal{O}(L^{-\infty})$$

for any operator  $A \in \Gamma_B$ . In particular, this gives another local formula for the Hall conductivity in the bulk of an edge system, by taking  $A = J_V$ , where  $J$  is the current operator and  $V \subset \Gamma_B$  is a set localized in the bulk. The Hall conductivity is given by  $\text{Tr}(\dot{P}_E J_V)$ , and this can be approximated by

$$\text{Tr}(\dot{P}_E J_V) = \text{Tr}(\dot{P}_B J_V) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\dot{\rho} J_V) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\dot{\sigma} J_V) + \mathcal{O}(L^{-\infty}).$$

Want to pick a norm s.t. Gronwall gives  $\|\rho(\mu) - \sigma(\mu)\|_G \leq \|P_B(0) - P_E(0)\|_G e^{2L_F \mu}$ . Need  $\|P_B(0) - P_E(0)\|_G$  to be small enough to kill the exponential which depends on  $L_F = 2\|K_B\|_G \leq \|W_\gamma\|_{L^1} \|Q_h\|_G$ . If we use the operator norm for  $\|\cdot\|_G$ , we would get  $\|Q_h\|_G = d|\Gamma_h|$  in the exponent. Need  $\|\cdot\|_G$  to be an actual norm so that  $\|\rho - \sigma\|_G = 0 \implies \rho = \sigma$ .

## From Dec 13 Meeting

Let  $r(t) = \rho(t) - \sigma(t)$ . Notice that

$$\frac{d}{dt} e^{itK_B} \sigma_0 e^{-itK_B} = e^{itK_B} i[K_B, \sigma_0] e^{-itK_B} + e^{-itK_B} \dot{\sigma}_0 e^{itK_B}.$$