Thesis Rough Work

Justin Furlotte (Student #: 64238736)

May 17, 2022

1 Introduction

2 Nonnteracting Setting

Consider the lattice $\mathbb{Z}^2$, on which we define a bulk Hamiltonian H_B, whose matrix elements follow a short-range assumption:

$$\sup_{y\in\mathbb{Z}^2}\sum_{x\in\mathbb{Z}^2}|H_B(x,y)|(e^{\mu|x-y|}-1)<\infty$$

for some $\mu>0$. We define the bulk conductivity

$$\sigma_B(\lambda)=-i\mathrm{Tr}(P_\lambda[[P_\lambda,\Lambda_1],[P_\lambda,\Lambda_2]])$$

where P_λ is the projection onto the eigenstates of H_B with energy lies in $(-\infty,\lambda)$, and where

$$\Lambda_i(x)=\begin{cases}1 & x_i<0\\ 0 & x_i\geq 0\end{cases}$$

are characteristic functions. We construct an edge Hamiltonian on the lattice $\mathbb{Z}^2_a=\{x\in\mathbb{Z}^2 : x_2>-a\}$. We denote the edge Hamiltonian by $H_a:\ell^2(\mathbb{Z}^2_a)\to\ell^2(\mathbb{Z}^2_a)$, requiring only that that the edge operator $E_a:\ell^2(\mathbb{Z}^2_a)\to\ell^2(\mathbb{Z}^2_a)$ define by

$$E_a=J_aH_a-H_BJ_a$$

satisfies the edge assumption

$$\sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2_a}E_a(x,y)|e^{\mu(|x_2+a|-|x_1-y_1|)}<\infty$$

for some $\mu>0$, where $|x|:=|x_1|+|x_2|$. The interpretation

Each site $x\in\mathbb{Z}^2$ get an associated Hilbert space $\mathcal{H}_x$. The dimension of these Hilbert spaces is bounded uniformly in x. We consider the Hilbert space $\ell^2(\mathbb{Z}^2,\mathbb{C}^n)=\{(x_1,x_2,\dots)\subset\mathbb{C}^n:\sum_{i\in\mathbb{Z}^2}\|x_i\|^2<\infty\}$. For example, one might consider a system of spins at the lattice sites, in which case the Hilbert space $\mathcal{H}_x$ at each site would be $\mathbb{C}^2$, and the total Hilbert space $\mathcal{H}=\otimes_x\mathcal{H}_x$ would then be the space of summable wavefunctions $\psi=\otimes_x\psi_x\in\ell^2(\mathbb{Z}^2,\mathbb{C}^2)$.

The Hilbert space $\ell^2(\mathbb{Z}^2)$ is the "bulk" setting, i.e. the setting in which we consider an infinite two-dimensional medium with no edges, and we consider a "bulk Hamiltonian" H_B on this Hilbert space. We also define the "edge" Hilbert space $\ell^2(\mathbb{Z}^2_a)$ and an associated "edge Hamiltonian" H_a, where $\mathbb{Z}^2_a:=$

$\{(n,m) \in \mathbb{Z}^2 : n \geq -a\}$. The bulk and edge Hamiltonians are related by the edge operator $E_a : \ell^2(\mathbb{Z}^2_a) \to \ell(\mathbb{Z}^2)$ defined by

$$E_a := J_a H_a - H_B J_a,$$

where $J_a : \ell^2(\mathbb{Z}^2_a) \to \ell(\mathbb{Z}^2)$ denotes extension by zeroes. We assume that

Assumption 1. *The edge operator satisfies*

$$\sup_{z \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}^2_a} |E_a(x,y)| e^{\mu(|x_2+a|+|x_1-y_1|)} < \infty.$$

The interpretation is that $E_a = J_a H_a - H_B J_a$ is the difference between first applying H_a on $\ell^2(\mathbb{Z}^2_a)$, and then making everything below $-a$ into zeroes, versus first making all $x \in \mathbb{Z}^2$ such that $x_2 < -a$ zeroes, and the applying H_B. The assumption ensures that the effects from introducing the edge at $-a$ die exponentially as we move upward away from the edge (due to the $|x_2 - (-a)|$ term in the exponent), and also terms do not interact too much as their x_1 distance increases (due to the $|x_1 - y_1|$ term in the exponent).

We also make the following assumption about both the bulk and edge Hamiltonians:

Assumption 2. *The Hamiltonians have a spectral gap. There exists an interval Δ such that $\Delta \cap \sigma(H) = \varnothing$.*

Remark: The spectral gap assumption can be relaxed to a "mobility gap" assumption,

$$\sup_{f \in B_c(\Delta)} |f(H_B)(x,y)|(1+|x|)^{-\nu} e^{\mu|x-y|} < \infty$$

for some $\nu > 0$, where $B_c(\Delta)$ is the set of Borel functions f which are constant on $(-\infty, \inf \Delta)$ and on $(\sup \Delta, \infty)$ such that $|f(x)| \leq 1$ for all x. See ? for details.

An example of an edge Hamiltonian satisfying the assumption on E_a is $H_a = J_a^* H_B J_a$, where $J_a : \ell^2(\mathbb{Z}^2_a) \to \ell^2(\mathbb{Z}^2)$ denotes extension by zeros. The idea is that for a state $\psi \in \ell^2(\mathbb{Z}^2_a)$, we have $\langle \psi, H_a \psi \rangle = \langle (J_a \psi), H_B (J_a \psi) \rangle$, which we interpret as the edge Hamiltonian having the same expectation as the bulk Hamiltonian if we just turned all the states ψ_x with $x_2 < -a$ into zeroes. The edge operator is

$$E_a = J_a J_a^* H_B J_a - H_b J_a = (J_a J_a^* - \mathbb{1}) H_B J_a = \begin{cases} -H_B(x,y) & \text{if } x_2 < -a \\ 0 & \text{if } x_2 \geq -a \end{cases}$$

Intuitively, there is no difference between H_B and H_a on $\mathbb{Z}_a^2$. The bound in assumption ? is satisfied by the short range assumption ?.

We define the *bulk conductivity* at Fermi energy μ as follows. Suppose we subject the system to an external electric potential difference V in the x_2 direction. We write this as $-V_0\Lambda_2$, where Λ_i are multiplication operators $\Lambda_i|\psi(x_1,x_2)\rangle = \Lambda(x_i)|\psi(x_1,x_2)\rangle$ which are *switch functions*,

$$\Lambda : \mathbb{R} \to \mathbb{R} \qquad\qquad \Lambda(x_i) = \begin{cases} 1 & \text{if } x_i \leq 0 \\ 0 & \text{if } x_i \geq 1 \end{cases}$$

and are smooth and monotonically decreasing on $(0,1)$. Note that the ensuing physics (in particular, our definition of the Hall conductivity) is independent of the particular choice of switch function Λ_i, since any two switch functions are exactly equal on the lattice.

This gives $\vec{E} = -\nabla V = V_0 \frac{\partial \Lambda_2}{\partial x_2}$, so that $\vec{E}$ is has compact support $\mathrm{supp}(\Lambda_2')$. We introduce a function which grows slowly in time as t grows from $-\infty$ to 0, so as to invoke the adiabatic principle. Here, we choose $e^{\varepsilon t}$, and we will let $\varepsilon \to 0$ at the end. The Hamiltonian therefore experiences a perturbation,

$$\widetilde{H}_B(t) = H_B - V_0\Lambda_2 e^{\varepsilon t}.$$

We define the Hall current operator $J_H = i[\widetilde{H}_B(t), \Lambda_1] = i[H_B, \Lambda_1]$, which is related to the Hall conductivity by $J_H = \sigma_H V$.

Lemma 1. *The ground state expectation $Tr(P_\mu J_H)$ of the Hall current is zero.*

Proof. Notice that since J_H is trace-class and P_μ is bounded, and since $[H_B, P_\mu] = 0$, we have

$$\mathrm{Tr}(P_\mu J_H) = i\mathrm{Tr}(P_\mu[H_B, \Lambda_1]) = i\mathrm{Tr}(P_\mu[H_B, P_\mu\Lambda_1 P_\mu])$$

$\square$

Proposition 1. *The Hall conductivity σ_H in the bulk system is equal to*

$$\sigma_B = -i\mathrm{Tr}\left(P_\mu\left[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\right]\right),$$

where $P_\mu := P((-\infty, \mu))$ is the projection-valued measure associated with H_B onto states with energy less than the Fermi energy μ.

Proof. We begin with the Heisenberg equation of motion for the density matrix, $\dot{\rho}(t) = -i[\widetilde{H}_B(t), \rho(t)]$, with initial condition $\lim_{t\to-\infty} \|\rho(t) - e^{-itH_B}P_\mu e^{itH_B}\| = 0$, which also implies $\lim_{t\to-\infty} \|e^{itH_B}\rho(t)e^{-itH_B} - P_\mu\| = 0$.

We work in the interaction picture, and define $\rho_I(t) = e^{itH_B}\rho(t)e^{-itH_B}$, and $\Delta H_B(t) = -e^{itH_B}V_0\Lambda_2 e^{\varepsilon t}e^{-itH_B}$. Thus

$$\dot{\rho}_I(t) = -i[\Delta H_B(t), \rho_I(t)]$$

The solution to this differential equation is readily verified to be

$$\rho(t) = i\int_{-\infty}^{t}[\Delta H_B(s), P_\mu]ds + P_\mu$$

Indeed, taking the derivative of the right hand side gives $i[\Delta H_B(t), P_\mu] = i[\Delta H_B(t), \rho_I(t)] + \mathcal{O}(V_0^2)$, but P_μ and $\rho_I(t)$ are equal up to zeroth order in V_0. The initial condition is also satisfied.

Using $J_H = i[H_B, \Lambda_1] = \sigma_H V = -\sigma_H V_0 \Lambda_2$, we obtain $\sigma_H = \frac{1}{V_0}\lim_{\varepsilon\to 0}\mathrm{Tr}(\rho(0)i[H_B, \Lambda_1])$. Since the expectation of the ground state current is zero, $\mathrm{Tr}(P_\mu J_H) = 0$, we have

$$\begin{aligned}
\sigma_H &= \frac{i}{V_0}\lim_{\varepsilon\to 0}\mathrm{Tr}\left(i\int_{-\infty}^{0}[\Delta H_B(t), P_\mu][H_B, \Lambda_1]ds\right) \\
&= -\frac{1}{V_0}\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}[-e^{isH_B}V_0\Lambda_2 e^{\varepsilon s}e^{-isH_B}, P\mu][H_B, \Lambda_1]ds\right) \\
&= -\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}e^{isH_B}[\Lambda_2, P_\mu]e^{-isH_B}[H_B, \Lambda_1]e^{\varepsilon s}ds\right) \\
&= -\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}(e^{-isH_B}[H_B, \Lambda_1]e^{isH_B})\cdot([\Lambda_2, P_\mu]e^{\varepsilon s})ds\right)
\end{aligned}$$

Where we used the fact that P_μ and H_B commute. Using integration by parts on the two terms in brackets, and noting that $\frac{d}{ds}(e^{isH_B}[H_B, \Lambda_1]e^{-isH_B}) = -(ie^{isH_B}\Lambda_1 e^{-isH_B} - \Lambda_1)$, we obtain

$$\begin{aligned}
\sigma_H &= i\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}(e^{-isH_B}\Lambda_1 e^{isH_B} - \Lambda_1)\frac{d}{ds}([\Lambda_2, P_\mu]e^{\varepsilon s})ds\right) \\
&= i\lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0}\Lambda_1^s[\Lambda_2, P_\mu]e^{\varepsilon s})ds\right)
\end{aligned}$$

where $\Lambda_1^s := e^{-isH_B}\Lambda_1 e^{isH_B} - \Lambda_1$. Using the notation $\overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, it is readily verified that the commutator $[\Lambda_2, P_\mu]$ is an *off-diagonal* operator, in the sense that $[\Lambda_2, P_\mu] = \overline{[\Lambda_2, P_\mu]}$. Furthermore, a simple computation reveals that for any two operators A and B, $\mathrm{Tr}(\overline{A}B) = \mathrm{Tr}(A\overline{B})$. It therefore follows that

$$\sigma_H = i \lim_{\varepsilon\to 0} \varepsilon \mathrm{Tr}\left(\int_{-\infty}^{0} \overline{\Lambda_1^s}[\Lambda_2, P_\mu]e^{\varepsilon s})ds\right)$$

The integrand can be broken into two terms,

$$\overline{\Lambda_1^s}[\Lambda_2, P_\mu]e^{\varepsilon s} = e^{-isH_B}\overline{\Lambda_1}e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s} - \overline{\Lambda_1}[\Lambda_2, P_\mu]e^{\varepsilon s}$$

by commutativity of P_μ and H_B. We show that the integral of the first term vanishes. We begin by breaking the first term down further into

$$e^{-isH_B}P_\mu\Lambda_1 P_\mu^\perp e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s} + e^{-isH_B}P_\mu^\perp\Lambda_1 P_\mu e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s}.$$

We treat the first of these two terms; the other is handled in an identical manner. We use the spectral theorem to write $e^{-isH_B}P_\mu = \int_{-\infty}^{\mu} e^{-is\lambda}dP_\lambda$, and similarly $P_\mu^\perp e^{isH_B} = (\mathrm{Id} - P_\mu)e^{isH_B} = \int_\mu^\infty e^{is\nu}dP_\nu$.

We remark that, since the Fermi energy μ is assumed to lie in a spectral gap, there must exist a neighbourhood $(\mu-\delta, \mu+\delta)$ in which there are no states. We exploit this fact to rewrite the limits of integration, $\int_{-\infty}^{\mu-\delta} e^{-is\lambda}dP_\lambda$ and $\int_{\mu+\delta}^{\infty} e^{is\nu}dP_\nu$. We therefore obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon \int_{-\infty}^{0} e^{-isH_B}P_\mu\Lambda_1 P_\mu^\perp e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s}ds \\
&\qquad = \lim_{\varepsilon\to 0}\varepsilon \mathrm{Tr}\left(\int_{-\infty}^{0}\int_{-\infty}^{\mu-\delta} e^{-is\lambda}dP_\lambda \Lambda_1 \int_{\mu+\delta}^{\infty} e^{is\nu}dP_\nu[\Lambda_2, P_\mu]e^{\varepsilon s}ds\right) \\
&\qquad = \lim_{\varepsilon\to 0}\varepsilon \mathrm{Tr}\left(\int_{-\infty}^{0}\int_{-\infty}^{\mu-\delta}\int_{\mu+\delta}^{\infty} e^{s(\varepsilon - i\lambda + i\nu)}dP_\lambda \Lambda_1 dP_\nu[\Lambda_2, P_\mu]ds\right)
\end{aligned}$$

Performing the integral over s yields

$$\lim_{\varepsilon\to 0}\varepsilon \int_{-\infty}^{0} e^{s(\varepsilon - i\lambda + i\nu)}ds = -\lim_{\varepsilon\to 0}\frac{\varepsilon}{i\varepsilon + \lambda - \nu}$$

This limit is zero, since $\lambda \neq \nu$. Indeed, due to the spectral gap, the integration variables live in $\lambda \in (-\infty, \mu-\delta)$ and $\nu \in (\mu+\delta, \infty)$. The case for the $e^{-isH_B}P_\mu^\perp\Lambda_1 P_\mu e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s}$ term (where the P_μ and $P_\mu^\perp$ swap places) is treated analogously. Hence the first term in the integrand for σ_H vanishes, as claimed.

Finally, we return to our expression for the Hall conductivity, which now reads

$$\sigma_H = i \lim_{\varepsilon \to 0} \varepsilon \mathrm{Tr} \left(\int_{-\infty}^{0} \overline{\Lambda_1} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right).$$

It is a basic algebraic calculation to show that $\overline{\Lambda_1} = [[\Lambda_1, P_\mu], P_\mu]$. Evaluating the integral over s is now trivial; $\int_{-\infty}^{0} e^{\varepsilon s} ds = \varepsilon^{-1}$. Thus

$$\sigma_H = -i \mathrm{Tr}([[\Lambda_1, P_\mu], P_\mu][\Lambda_2, P_\mu]).$$

Shifting the commutator completes the proof:

$$\begin{aligned} \sigma_H &= -i \mathrm{Tr}(P_\mu [[\Lambda_2, P_\mu], [\Lambda_1, P_\mu]]) \\ &= i \mathrm{Tr}(P_\mu [[\Lambda_1, P_\mu], [\Lambda_2, P_\mu]]) \\ &= i \mathrm{Tr}(P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]). \end{aligned}$$

□

Remark: This is reminiscent of the well-known adiabatic curvature formula,

$$\kappa = \mathrm{Tr}(P[\partial_1 P, \partial_2 P]) = \mathrm{Tr}(P\,[[P, K_1], [P, K_2]]) = \mathrm{Tr}(P[K_1, K_2]),$$

where K_i are called *generators of parallel transport*. We will see the adiabatic curvature formula again later in the interacting setting.

For the *edge conductivity*, we need the current operator across the line $x_1 = 0$, which is given by $-i[H_a, \Lambda_1]$. We define

$$\sigma_E = -i \lim_{a \to \infty} \mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]),$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(r) = \begin{cases} 1 & \text{if } r < \inf \Delta \\ 0 & \text{if } r > \sup \Delta \end{cases}$$

and decreases smoothly and monotonically in Δ. The definition of σ_E is reminiscent of another formula we will see later in the interacting setting, $\mathrm{Tr}(\dot{P} J)$, where J is the current operator. The interpretation of σ_E is that if we apply a small potential difference V across $x_2 = -a$ to $x_2 = \infty$, there will be a net current

$$\begin{aligned} I &= -i \mathrm{Tr}(\rho(H_a + V)[H_a + V, \Lambda_1] - \rho(H_a)[H_a, \Lambda_1]) \\ &= -i \mathrm{Tr}((\rho(H_a + V) - \rho(H_a))[H_a, \Lambda_1]) \end{aligned}$$

Thus we obtain the conductivity

$$\sigma_E = \frac{I}{V} = -i\mathrm{Tr}\left(\frac{(\rho(H_a+V)-\rho(H_a)}{V}[H_a,\Lambda_1]\right) \to -i\mathrm{Tr}(\rho'(H_a)[H_a,\Lambda_1])$$

in the limit as $V \to 0$. As we shall see, it turns out that σ_E is independent of the choice of ρ, and σ_B is independent of λ.

The main result of this section is

Theorem 1. $\sigma_E = \sigma_B$.

2.1 Outline of the Proof

First, let

$$\tilde{\sigma}_E(a,t) = -i\mathrm{Tr}(\rho'(H_a)[H_a,\Lambda_1]\Lambda_{2,a}(t))$$

where $\Lambda_{2,a}(t) = e^{itH_a}\Lambda_2 e^{-itH_a}$ is the time evolution of Λ_2. One can show that, while

$$\sigma_E = \lim_{T\to\infty}\lim_{a\to\infty}\frac{1}{T}\int_0^T \mathrm{Re}(\tilde{\sigma}_E(a,t))dt,$$

it is unfortunately the case that $\lim_{a\to\infty}\|\rho'(H_a)[H_a,\Lambda_1]\Lambda_{2,a}(t)\|_1 = \infty$. However, even though the trace norm diverges, it turns out that the trace itself does not, so we will instead subtract a clever choice of zero-trace operator $Z(a,t)$ to define

$$\sigma_E(a,t) = -i\mathrm{Tr}(\rho'(H_a)[H_a,\Lambda_1]\Lambda_{2,a}(t) - Z(a,t))$$

so that the equation $\sigma_E = \lim_{T\to\infty}\lim_{a\to\infty}\frac{1}{T}\int_0^T \mathrm{Re}(\sigma_E(a,t))dt$ still holds, but we also have $\lim_{a\to\infty}\|\rho'(H_a)[H_a,\Lambda_1]\Lambda_{2,a}(t) - Z(a,t)\|_1 < \infty$. The correct choice of Z will become apparent after writing $\rho(H_a)$ and $\rho'(H_a)$ in terms of their Hellfer-Sjostrand representations,

$$\rho(H_a) = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)$$

$$\rho'(H_a) = -\frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)^2$$

where $R(z) = (H_a - z)^{-1}$ is the resolvent. Using $[R(z),\Lambda_i] = R(z)[H_a,\Lambda_i]R(z)$, we obtain the representations of the following useful operators:

$$[\rho(H_a), \Lambda_1] = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)[H_a, \Lambda_1]R(z)$$

$$\rho'(H_a)[H_a, \Lambda_1] = -\frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)^2[H_a, \Lambda_1]$$

From here, we define the zero-trace operator

$$Z(a,t) = [\rho(H_a), \Lambda_1]\Lambda_2 - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)(R(z)[H_a, \Lambda_1]\Lambda_{2,a}(t) - [H_a, \Lambda_1]\Lambda_{2,a}(t)R(z))$$

from which we obtain

$$\begin{aligned}
\sigma_E(a,t) &= \tilde{\sigma}_E(a,t) - Z(a,t) \\
&= \mathrm{Tr}\left(-[\rho(H_a), \Lambda_1]\Lambda_2 - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)[H_a, \Lambda_1]\Lambda_{2,a}(t)R(z)\right) \\
&= \mathrm{Tr}\left([\rho(H_a), \Lambda_1](\Lambda_{2,a}(t) - \Lambda_2) - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)[H_a, \Lambda_1]R(z)[H_a, \Lambda_{2,a}(t)]R(z)\right)
\end{aligned}$$

All of the statements so far can be verified by calculations. The difficult part of the theorem (aside from proving that the relevant operators are trace-class) is proving that

$$\|J_a\Sigma'_aJ_a^* - \Sigma'_B\|_1, \|J_a\Sigma''_aJ_a^* - \Sigma''_B\|_1 \to 0$$

as $a \to \infty$, where Σ'_B and Σ''_B are the same as with the subscript a, except using the bulk Hamiltonian H_B in their definition rather than H_a. It follows that

$$\sigma_E(a,t) = \mathrm{Tr}(J_a\Sigma'_aJ_a^* + J_a\Sigma''_aJ_a^*) = \mathrm{Tr}(\Sigma'_a + \Sigma''_a) \to \mathrm{Tr}(\Sigma'_B + \Sigma''_B)$$

From there, a calculation shows that $\lim_{T\to\infty}\frac{1}{T}\int_0^T \mathrm{Tr}(\Sigma'_B + \Sigma''_B)dt = \sigma_B$, concluding the proof.

2.2 Gap Simplifications

We now assume that H_B has a spectral gap. In this case, the edge condition guarantees that $\sigma_E(a) := \rho'(H_a)[H_a, \Lambda_1]$ is trace-class (need to add).

Need to add section on why $\sigma_E(a) := -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1])$ is equal to

$$-\frac{i}{2}\mathrm{Tr}(\rho'(H_a)\{[H_a, \Lambda_1], \Lambda_2\}).$$

Theorem 2. $\sigma_E = \sigma_B$

Outline of the Proof

Before giving the proof in its entirety, we outline the basic steps. The key ingredient is the use of the funcional calculus given by the Helffer-Sjostrand representation of self-adjoint operators on a Hilbert space (need to add reference to appendix here).

The Proof

Proof. We posit that the edge conductivity can be rewritten as $\sigma_E = \lim_{a\to\infty} \sigma_E(a)$, where

$$\sigma_E(a) = -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2),$$

since we have assumed that there is a spectral gap (as opposed to a mobility gap), so that there are extended states near the edge, and no bound states or resonances far from the edge. Thus, intuitively, the cutoff introduced by Λ_2 is irrelevant as we take $a \to \infty$. We provide a more concrete justification for this later.

"Concrete justificiation": Since σ_B is translation invariant (need to add), we only need to prove to case $-i\mathrm{Tr}(\rho'(H_{a=0})[H_{a=0}, \Lambda_1] = \sigma_B$. We drop the subscript, $H := H_{a=0}$. Since the multiplication operator $\Lambda_2(n)|\psi\rangle := \Lambda(x_2 - n)|\psi\rangle$ converges strongly to the identity as $n \to \infty$, we can write

$$\sigma_E(a) = -i\mathrm{Tr}(\rho'(H)[H, \Lambda_1]) = -i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H)[H, \Lambda_1]\Lambda_2(n)).$$

Instead of completing the shift $(0, -n)$ with the operator $\Lambda_2(n)$, we can consider a shifted Hamiltonian. Indeed, rather than restrict H_B at $x_2 = -n$ to obtain $H_{a=n}$, consider the shifted bulk Hamiltonian $H_B \mapsto H_B(n)$ obtained by the shift $(0, -n)$, and then restricting this at $x_2 = 0$ to obtain $H(n)$. In other words, $H(n)$ is the edge Hamiltonian associated with the shifted bulk Hamiltonian. Comparing this with the expression above, this is exactly equivalent to

$$-i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H)[H, \Lambda_1]\Lambda_2(n)) = -i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H(n))[H(n), \Lambda_1]\Lambda_2).$$

In other words, the difference between whether we cut off everything above $x_2 = n$ and then apply $H_{a=0}$, or instead cut off everything above $x_2 = 0$ and then apply the Hamiltonian $H(n)$ shifted down by $(0, -n)$ is immaterial.

Thus, our goal is to show that $-i\mathrm{Tr}(\rho'(H(n))[H(n),\Lambda_1]\Lambda_2) \to \sigma_B$ as $n \to \infty$.

End of concrete justification.

Define

$$Z(a) = [\rho(H_a),\Lambda_1]\Lambda_2 - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[R_a(z),[H_a,\Lambda_1]\Lambda_2]dz^2$$

This operator has zero trace. Indeed, the first term has vanishing trace in the position basis, while the second term's integrand involves the trace of $[R,R]=0$. So $\sigma_E(a) = \mathrm{Tr}(\Sigma(a))$, where

$$\begin{aligned}\Sigma(a) &= -i\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 + iZ(a)\\ &= -i\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 + i[\rho(H_a),\Lambda_1]\Lambda_2 - \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[R_a(z),[H_a,\Lambda_1]\Lambda_2]dz^2\end{aligned}$$

Using the Hellfer-Sjostrand representations for the first two terms on the right hand side, we obtain

$$\begin{aligned}\Sigma(a) = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)^2[H_a,\Lambda_1]\Lambda_2 dz^2 &+ \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1]R_a(z)\Lambda_2 dz^2\\ &- \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} (R_a(z)^2[H_z,\Lambda_1]\Lambda_2 - R_a(z)[H_a,\Lambda_1]\Lambda_2 R_a(z))dz^2\\ = -\frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1][R_a(z),\Lambda_2]dz^2&\\ = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1]R_a(z)[H_a,\Lambda_2]R_a(z)dz^2,&\end{aligned}$$

where we used

$$[R_a(z),\Lambda_i] = -R_a(z)[H_a,\Lambda_i]R_a(z)$$

in the final equality. Next, we must prove that the operator above converges to the corresponding bulk operator in trace-norm,

$$\|\Sigma(a) - \Sigma_B\|_1 \to 0,$$

as $a \to \infty$, which in turn proves that $\mathrm{Tr}(\Sigma(a)) \to \mathrm{Tr}(\Sigma_B)$ because of the bound $|\mathrm{Tr}(A)| \le \|A\|_1$. Here, Σ_B is the same operator as before, but using the bulk operators H_B and $R_B(z)$. Once this limit is established, we shall prove that $\sigma_B = \mathrm{Tr}(\Sigma_B)$ to conclude the proof.

To show that the limit is zero as claimed, we bound the trace norm of the integrand of $\Sigma(a)$ by breaking it into three parts,

$$R[H_a,\Lambda_1]R[H_a,\Lambda_2]R = J_a[R,\Lambda_1]e^{\delta|x_1|}J_a^* \cdot e^{-\delta|x_1|}e^{-\delta|x_2|} \cdot J_a e^{\delta|x_2|}[H_a,\Lambda_2]RJ_a^*,$$

and bounding the norm of each, making use of the fact that $\|AB\|_1 \leq \|A\|\|B\|_1$.

1. For the first term, $J_a[R,\Lambda_1]e^{\delta|x_1|}J_a^*$, we bound its operator norm by breaking it down further into

$$\begin{aligned}
\|J_a[R,\Lambda_1]e^{\delta|x_1|}J_a^*\| &= \|[R,\Lambda_1]e^{\delta|x_1|}\| \\
&= \|-R[H_a,\Lambda_1]Re^{\delta|x_1|}\| \\
&= \|-R\cdot[H_a,\Lambda_1]e^{\delta|x_1|}\cdot e^{-\delta|x_1|}Re^{\delta|x_1|}\| \\
&\leq \|R\|\cdot\|[H_a,\Lambda_1]e^{\delta|x_1|}\|\cdot\|e^{-\delta|x_1|}Re^{\delta|x_1|}\|
\end{aligned}$$

The norm of R is bounded by

$$\|R_a(z)\| \leq \frac{1}{|\mathrm{Im}(z)|},$$

since H_a is self-adjoint. The norm of the second operator can be bounded by inspecting its matrix elements.

$$\begin{aligned}
\langle x,[H_a,\Lambda_1]e^{\delta|x_1|}y\rangle &= \langle x,H_a\Lambda_1 y\rangle e^{\delta|y_1|} - \langle x,\Lambda_1 H_a y\rangle e^{\delta|y_1|} \\
&= H_a(x,y)e^{\delta|y_1|}(\Lambda(y_1)-\Lambda(x_1)).
\end{aligned}$$

This is zero if $|x_1-y_1| \leq |y_1|$, since this would imply that x_1 and y_1 have the same sign, yielding $\Lambda(x_1)=\Lambda(y_1)$. So either the matrix element is zero, or $|y_1| \leq |x_1-y_1|$, which implies

$$\begin{aligned}
|H_a(x,y)e^{\delta|y_1|}(\Lambda(y_1)-\Lambda(x_1))| &\leq 2|H_a(x,y)|e^{\delta|x_1-y_1|} \\
&\leq 2|H_a(x,y)|e^{\delta|x-y|} \\
&\leq C|H_a(x,y)|(e^{\delta|x-y|}-1)
\end{aligned}$$

where the final inequality comes from the fact that the diagonal matrix elements are zero. Hence the assumption

$$\sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}|H(x,y)|(e^{\mu|x-y|}-1)<\infty,$$

combined with Holmgren's bound

$$\|A\|\leq\max\left\{\sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}|A(x,y)|,\ \sup_{y\in\mathbb{Z}^2}\sum_{x\in\mathbb{Z}^2}|A(x,y)|\right\},$$

implies that the second term is bounded. Finally, for the third term $e^{-\delta|x_1|}Re^{\delta|x_1|}$, we apply the Combes-Thomas bound,

$$\|e^{-\varepsilon f(x)}R_a(z)e^{\varepsilon f(x)}\|\leq\frac{C}{|\mathrm{Im}(z)|}$$

where $f:\mathbb{Z}^2\to\mathbb{R}$ is any Lipschitz function, and ε can be chosen as $\varepsilon=\frac{1}{C(1+|\mathrm{Im}(z)|)}$.

Altogether, the bound of the first term takes the form

$$\frac{C}{\mathrm{Im}(z)^2}.$$

2. For $e^{-\delta|x_1|}e^{-\delta|x_2|}$, we bound the trace norm by noticing that this is a positive operator satisfying

$$\langle(n,m),e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m)\rangle=\langle e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m),(n,m)\rangle,$$

so that its trace norm is equal to its trace. In the position basis, we see that its trace is given by a geometric series

$$\begin{aligned}\mathrm{Tr}(e^{-\delta|x_1|}e^{-\delta|x_2|})&=\sum_{(n,m)\in\mathbb{Z}^2}\langle(n,m),e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m)\rangle\\&\leq 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}e^{-\delta m}e^{-\delta n}\\&=2\left(\frac{1}{1-e^{-\delta}}\right)^2.\end{aligned}$$

3. For $J_a e^{\delta|x_2|}[H_a, \Lambda_2]R_a(z)J_a^*$, note that analogously to 1. above where we bounded $[H_a, \Lambda_1]e^{\delta|x_1|}$, we also have that $e^{\delta|x_2|}[H_a, \Lambda_2]$ is bounded. Again, the resolvent $R_a(z)$ is also bounded.

   The bound of the third term takes the same form as the bound of the first term,

$$\frac{C}{\mathrm{Im}(z)^2}.$$

Altogether, we see that the integrand is bounded by the product of the three bounds from 1., 2., and 3., and is of the form $\frac{C}{\mathrm{Im}(z)^4}$.

For any $n \in \mathbb{N}$, the quasi-analytic extension $\tilde{\rho}$ of ρ in the Helffer-Sjostrand representation can be chosen so that

$$\int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} \frac{1}{|\mathrm{Im}(z)|^{p+1}} dz^2 \leq C_0 \sum_{k=0}^{n+2} \|\rho^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_m = \int_{-\infty}^{\infty} |f(x)|(x^2+1)^{m/2} dx.$$

Since $|\rho(x)| \leq 1$ and ρ' is compactly supported, these norms are all clearly finite. This fact, combined with the bound

$$\|R_a(z)[H_a, \Lambda_1]R_a(z)[H_a, \Lambda_2]R_a(z)\|_1 \leq \frac{C}{\mathrm{Im}(z)^4}$$

for the trace norm of the integrand of $\Sigma(a)$ provides the necessary bound for Lebesgue dominated convergence. Thus, it suffices to show pointwise convergence in z of the integrand to the associated bulk operator.

In other words, we wish to show

$$J_a[R_a(z), \Lambda_1]e^{\delta|x_1|}J_a^* \xrightarrow{s} [R_B(z), \Lambda_1]e^{\delta|x_1|}$$

and

$$J_a e^{\delta|x_2|}[H_a, \Lambda_2]J_a^* \xrightarrow{s} e^{\delta|x_2|}[H_B, \Lambda_2]$$

for each fixed $z \in \mathbb{C}$. Inspecting the bounds we found for the left hand sides of these limits, it is clear that they are uniformly bounded in a. It therefore suffices to show convergence on a dense subspace of $\ell^2(\mathbb{Z}^2)$; in particular, we may choose the dense subspace of compactly supported states, which allows us to ignore the $e^{\delta|x_i|}$ terms. Thus, we need to prove

$$J_a[R_a(z), \Lambda_1]J_a^* \xrightarrow{s} [R_B(z), \Lambda_1]$$

and

$$J_a[H_a, \Lambda_2]J_a^* \xrightarrow{s} [H_B, \Lambda_2].$$

We appeal to the general fact of functional analysis that strong convergence of the resolvent implies that $J_a f(H_a) J_a^* \xrightarrow{s} f(H_B)$ for any bounded and continuous function f. In particular, the functions $[(\cdot - z)^{-1}, \Lambda_1]$ and $[\cdot, \Lambda_2]$ above are bounded and continuous, so we will have proven the desired limits if we can prove convergence of the resolvent, $J_a R_a(z) J_a^* \xrightarrow{s} R_B(z)$.

To prove this, we use the edge assumption. Recall the edge operator, $E_a = J_a H_a - H_B J_a$. Adding and subtracting zJ_a gives

$$E_a = J_a(H_a - z) - (H_B - z)J_a.$$

Applying R_B from the left and R_a from the right on both sides, we obtain

$$J_a R_a(z) - R_B(z) J_a = -R_B(z) E_a R_a(z).$$

Taking the adjoint, and then multiplying from the left by J_a, we see that

$$\begin{aligned} J_a R_a(z) J_a^* - R_B(z) &= -J_a R_a(z) E_a^* R_B(z) \\ &= -(J_a R_a(z) E_a^* + 1 - J_a J_a^*) R_B(z) \\ &\xrightarrow{s} 0, \end{aligned}$$

since $E_a^* \xrightarrow{s} 0$ by the edge assumption, and $1 - J_a J_a^* \xrightarrow{s} 0$. This proves that the limits above converge to the desired associated bulk operators, and thus $\Sigma(a) \xrightarrow{s} \Sigma_B$.

Finally, it remains to show that

$$\mathrm{Tr}(\Sigma_B) = \sigma_B.$$

First, we manipulate

$$\begin{aligned} \Sigma_B &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1] R_B(z) [H_B, \Lambda_2] R_B(z) dz^2 \\ &= -\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1][R_B(z), \Lambda_2] dz^2 \\ &= i[\rho(H_B), \Lambda_1]\Lambda_2 - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1]\Lambda_2 R_B(z) dz^2. \end{aligned}$$

Define $P_+ := P((\sup\Delta, \infty))$ and $P_- := P((-\infty, \inf\Delta))$, the projections onto states above and below the gap, respectively. Since H_B is assumed to have a gap, we have

$$\mathrm{Tr}(\Sigma_B) = \mathrm{Tr}(P_+\Sigma_B P_+) + \mathrm{Tr}(P_-\Sigma_B P_-).$$

Since $P_\pm R_B(z)$ and $R_B(z)P_\pm$ are analytic on $\mathrm{supp}(\rho(z))$ and $\mathrm{supp}(1-\rho(z))$, respectively, the integral in $P_\pm\Sigma_B P_\pm$ vanishes by integration by parts. Thus

$$\Sigma_B = iP_+[\rho(H_B), \Lambda_1]\Lambda_2 P_+ + iP_-[\rho(H_B), \Lambda_1]\Lambda_2 P_-.$$

By the spectral theorem for projection-valued measures, if the Fermi energy lies in the gap, $\lambda \in \Delta$, we have

$$\rho(H_B) = \int_{-\infty}^{\infty} \rho(\lambda) dP_\nu = \int_{-\infty}^{\lambda} \rho(\lambda) dP_\nu = \int_{-\infty}^{\lambda} dP_\nu = P_\lambda.$$

We may therefore replace $\rho(H_B)$ by P_λ, by which we obtain

$$\mathrm{Tr}(\Sigma_B) = i\mathrm{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+) + i\mathrm{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-).$$

Now, the bulk conductivity is given by

$$\begin{aligned}
\sigma_B &= i\mathrm{Tr}(P_\lambda[[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]]) \\
&= i\mathrm{Tr}(P_\lambda((P_\lambda\Lambda_1 - \Lambda_1 P_\lambda)(P_\lambda\Lambda_2 - \Lambda_2 P_\lambda) - (P_\lambda\Lambda_2 - \Lambda_2 P_\lambda)(P_\lambda\Lambda_1 - \Lambda_1 P_\lambda))) \\
&= i\mathrm{Tr}(P_\lambda(P_\lambda\Lambda_1 P_\lambda\Lambda_2 - P_\lambda\Lambda_1\Lambda_2 P_\lambda - \Lambda_1 P_\lambda\Lambda_2 + \Lambda_1 P_\lambda\Lambda_2 P_\lambda \\
&\qquad - P_\lambda\Lambda_2 P_\lambda\Lambda_1 + P_\lambda\Lambda_2\Lambda_1 P_\lambda + \Lambda_2 P_\lambda\Lambda_1 - \Lambda_2 P_\lambda\Lambda_1 P_\lambda)) \\
&= i\mathrm{Tr}(-P_\lambda\Lambda_1\Lambda_2 P_\lambda + \Lambda_1 P_\lambda\Lambda_2 P_\lambda + P_\lambda\Lambda_2\Lambda_1 P_\lambda - \Lambda_2 P_\lambda\Lambda_1 P_\lambda) \\
&= i\mathrm{Tr}(-P_\lambda\Lambda_1\Lambda_2 P_\lambda + P_\lambda\Lambda_1 P_\lambda\Lambda_2 P_\lambda + P_\lambda\Lambda_2\Lambda_1 P_\lambda - P_\lambda\Lambda_2 P_\lambda\Lambda_1 P_\lambda) \\
&= i\mathrm{Tr}(P_\lambda\Lambda_1 P_\lambda^\perp\Lambda_2 P_\lambda - P_\lambda\Lambda_2 P_\lambda^\perp\Lambda_1 P_\lambda) \\
&= i\mathrm{Tr}(P_\lambda\Lambda_1 P_\lambda^\perp\Lambda_2 P_\lambda - P_\lambda^\perp\Lambda_1 P_\lambda\Lambda_2) \\
&= i\mathrm{Tr}(P_\lambda\Lambda_1 P_\lambda^\perp\Lambda_2 P_\lambda - P_\lambda^\perp\Lambda_1 P_\lambda\Lambda_2 P_\lambda^\perp).
\end{aligned}$$

We define $T_\lambda := P_\lambda\Lambda_1 P_\lambda^\perp\Lambda_2 P_\lambda - P_\lambda^\perp\Lambda_1 P_\lambda\Lambda_2 P_\lambda^\perp$, so that

$$\sigma_B = i\mathrm{Tr}(T_\lambda),$$

and show that $P_\pm T_\lambda P_\pm = P_\pm[P_\lambda, \Lambda_1]\Lambda_2 P_\pm$.

First, notice that because of the gap, we have $P_\lambda^\perp P_- = 0$, and thus also $P_\lambda P_- = P_-$. Thus

$$\begin{aligned}P_-T_\lambda P_- &= P_-P_\lambda\Lambda_1P_\lambda^\perp\Lambda_2P_\lambda P_-\\ &= P_-(P_\lambda\Lambda_1\Lambda_2 - \Lambda_1P_\lambda\Lambda_2)P_-\\ &= P_-[P_\lambda,\Lambda_1]\Lambda_2P_-,\end{aligned}$$

and similarly, for P_+, we have $P_\lambda^\perp P_+ = P_+$, and $P_\lambda P_- = 0$, which implies

$$\begin{aligned}P_+T_\lambda P_+ &= -P_+P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2P_\lambda^\perp P_+\\ &= -P_+P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2P_+\\ &= -P_+P_\lambda^\perp\Lambda_1\Lambda_2P_+ + P_+P_\lambda^\perp\Lambda_1P_\lambda^\perp\Lambda_2P_+\\ &= -P_+P_\lambda^\perp\Lambda_1\Lambda_2P_+ + P_+\Lambda_1P_\lambda^\perp\Lambda_2P_+\\ &= -P_+[P_\lambda^\perp,\Lambda_1]\Lambda_2P_+\\ &= -P_+[(1-P_\lambda),\Lambda_1]\Lambda_2P_+\\ &= P_+[P_\lambda,\Lambda_1]\Lambda_2P_+.\end{aligned}$$

Finally, we obtain

$$\begin{aligned}\sigma_B &= i\mathrm{Tr}(T_\lambda)\\ &= i\mathrm{Tr}(P_-T_\lambda P_-) + i\mathrm{Tr}(P_+T_\lambda P_+)\\ &= i\mathrm{Tr}(P_-[P_\lambda,\Lambda_1]\Lambda_2P_-) + i\mathrm{Tr}(P_+[P_\lambda,\Lambda_1]\Lambda_2P_+)\\ &= \mathrm{Tr}(\Sigma_B),\end{aligned}$$

concluding the proof. □

3 Interacting Setting

Let $L \in \mathbb{N}$, and let $\Gamma_L = \mathbb{Z}_L \times [0, L]$ be the discrete cylinder, equipped with a metric d. To each site $x \in \Gamma_L$, we associate a Hilbert space $\mathcal{H}_x$ whose dimension is bounded uniformly in L. We denote $N = \sup_L \mathcal{H}_L$. For a subset $X \subseteq \Gamma_L$, we define the Hilbert space $\mathcal{H}_X = \otimes_{x\in X}\mathcal{H}_x$, and we set

$$\mathcal{H}_L := \mathcal{H}_{\Gamma_L} = \bigotimes_{x\in\Gamma_L}\mathcal{H}_x.$$

The algebra $\mathcal{U}_X \subset \mathcal{B}(\mathcal{H}_X)$ of observables on $\mathcal{H}_X$ is the set of bounded self-adjoint operators supported in X. For an operator $A_X \in \mathcal{U}_X$, we identify its extension to an operator on $\mathcal{H}_L$ by taking its tensor product with copies of the identity, $(\otimes_{x\in X^c}\mathbb{I}_x)\otimes A_X$. Conversely, we say that an operator $A \in \mathcal{U}_L$ has support X if $A_X := (\otimes_{x\in X^c}\mathbb{I}_x)\otimes(A|_X)$ is equal to A, and write $A_X \in \mathcal{U}_X$. For ease of notation, we omit the subscript L wherever there is no risk of confusion.

A *local interaction* is a map $\Phi : \mathcal{P}(\Gamma_L) \to \mathcal{U}_L$ such that

1. $\Phi(X) = 0$ whenever $\mathrm{diam}(X) > R$ for some $R > 0$.
2. $\Phi(X)$ is supported in X.
3. $\|\Phi(X)\| \leq C$ for all $X \subset \Gamma_L$, for all L.

We consider a region as depicted in Figure 1, with the left and right edges joined together to form a cylinder. In the left white region $[0, L/2] \times [0, L]$, H_0 is a trivial Hamiltonian which we take to be empty space (we take $H_0 = 0$), and in the right blue region $[L/2, L] \times [0, L]$, H_1 is a *local Hamiltonian*, in the sense that $H_1 = \sum_{X \subseteq \Gamma_L} \Phi(X)$, is a sum of local interactions. We define the Hamiltonian of the full system to be

$$H_\mu = H_1 + \mu Q_h,$$

where $Q_h = \sum_{x \in \Gamma_h} a_x^* a_x$ is the number operator for the region $\Gamma_h = [L/4, 3L/4] \times [0, L]$ shown in red. This introduces a driving strength; the μQ_h term can be viewed as a potential difference $V(x)$.

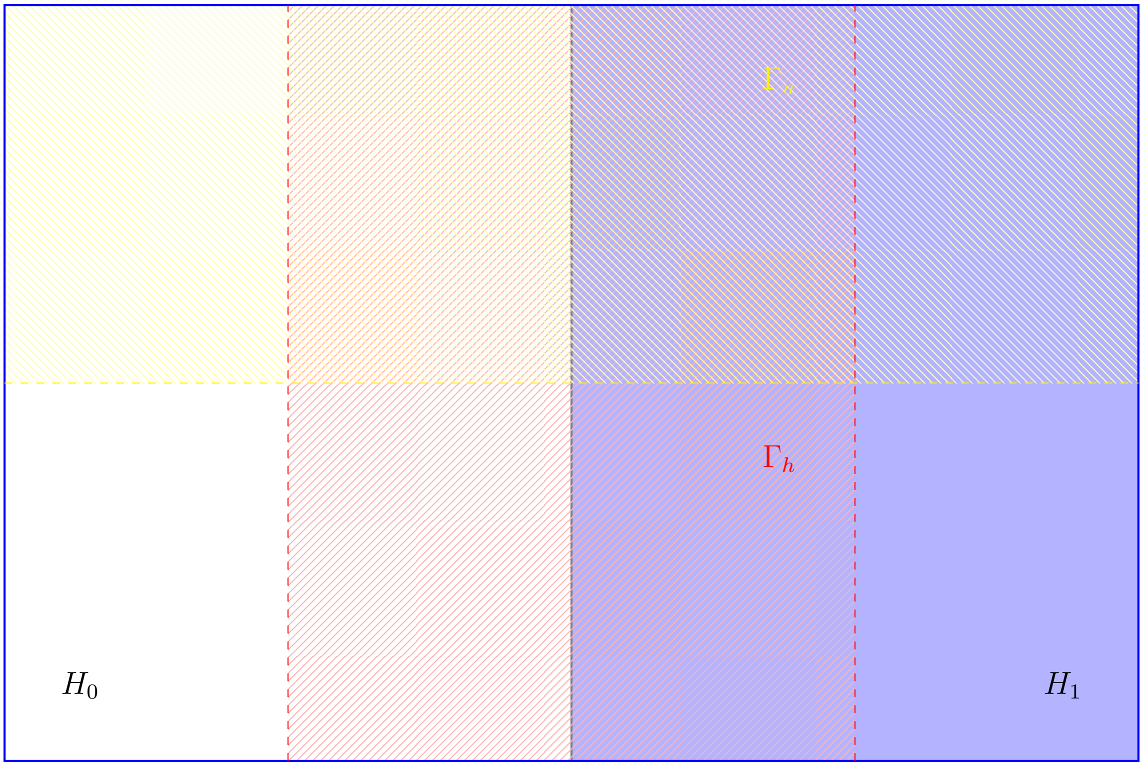

Figure 1: The cylinder Γ_L.

We also consider the plane $\mathbb{Z}^2$. In this setting, there are no edge states, and so the associated "bulk" Hamiltonian H_B is assumed to have a *gapped* spectrum, in the sense that

Assumption 3.

$$\mathrm{Spec}(H_B) = \mathcal{S}_- \cup \mathcal{S}_+,$$

where $\inf \mathcal{S}_+ - \sup \mathcal{S}_- \geq \gamma$ *uniformly in* L *and* μ *for some* $\gamma > 0$.

In the case of the cylinder, this effect does not necessarily occur due to the presence of the edge. We also assume that the Hamiltonian is *charge-conserving*.

Assumption 4. $[H_\mu, Q] = 0$, *where* Q *is the total charge in* Γ_L.

Let P_B be the ground state projection of H_B (the system without an edge), and let P be the ground state projection of H (the system with an edge). We assume that states far from the edge are essentially bulk states, up to tails that vanish quickly in L.

Assumption 5. *Define the bulk region* $\Gamma_B = [3L/4 - k, 3L/4 + k] \times [0, L]$ *for some* $k > 0$. *For any operator* $A \in \mathcal{U}_{\Gamma_B}$,

$$\mathrm{Tr}(PA) = \mathrm{Tr}(P_B A) + \mathcal{O}(L^{-\infty}).$$

The idea is that observables localized far away from the edge are not affected by the edge of the system.

3.1 Equality of Bulk and Edge Currents

3.1.1 Cylinder Geometry

Let P_μ be the (possibly degenerate) ground state projection of H_μ. Let $Q_u = \sum_{x \in \Gamma_u} a_x^* a_x$ be the charge in the upper half of the cylinder $\Gamma_u = [0, L] \times [L/2, L]$ (the yellow region in Figure 1), and define current operator

$$J = i[H_\mu, Q_u],$$

which measures the current across the fiducial line $y = L/2$. Charge conservation 4 implies that this current operator is supported along a strip of width $2R$ centred on the fiducial line $y = L/2$. Indeed, if we inspect a local interaction $\Phi(X)$ of range R with support $(\Gamma_u)_R$, where $(X)_\alpha$ is the α-shrinking of the set X, then clearly $\Phi(X)$ commutes with the charge outside Γ_u, so that $[\Phi(X), Q_u] = [\Phi(X), Q]$, which vanishes by the charge conservation assumption 4. Similarly, if $\Phi(X)$ is supported in $((\Gamma_u)^{\mathsf{c}})_R$, then $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$. It follows that for an interaction $\Phi(X)$ with range R and arbitrary support, $[\Phi(X), Q_u]$ must be supported on a set which is contained in (or equal to) the strip $[L/2, L] \times [L/2 - R, L/2 + R]$. There $[H_\mu, Q_u]$ must be supported there as well, since H_μ is a sum of such local interactions.

Lemma 2. *The ground state expectation of the current* J *is zero.*

Proof. Assuming linearity and cyclicity of the trace hold, the proof is trivial,

$$\mathrm{Tr}(P_\mu J) = i\mathrm{Tr}(P_\mu [H_\mu, Q_u]) = i\mathrm{Tr}([P_\mu, H_\mu] Q_u) = 0.$$

In order for this calculation to hold, we need to prove that

1. $P_\mu H_\mu Q_u$ and $P_\mu Q_u H_\mu$ are separately trace-class to apply linearity of the trace, and

2. $\|H_\mu\| < \infty$ and $P_\mu Q_u \in \mathcal{J}_1$ to apply cyclicity of the trace.

The latter implies the former by the bound $\|AB\|_1 \leq \|A\|_1\|B\|$. To prove (2), fix a finite L. The Hamiltonian is bounded since it is a finite sum of at most $\mathcal{P}(\Gamma_L)$ local interactions $\Phi(X)$, each of which is uniformly bounded by assumption, along with the μQ_h term. But the number operator for the entire space is bounded by $\|Q\| \leq NL^2$, where N is the uniform bound on the dimension of each Hilbert space. This shows that both Q_u and Q_h are bounded in operator norm. Finally, $\|P_\mu\|_1 \leq CL^2$ because the projection is finite-rank, since the dimension of each site is bounded. Therefore $P_\mu Q_u \in \mathcal{J}_1$. $\square$

Next, we define a family of operators indexed by μ called *Hastings operators*,

$$K_\mu = \mathcal{I}_\mu(\dot{H}_\mu),$$

where

$$\mathcal{I}_\mu(A) = \int_{\mathbb{R}} W(t)e^{itH_\mu}Ae^{-itH_\mu}dt.$$

Here, $W : \mathbb{R} \to \mathbb{R}$ is a function satisfying (need to add). More explicitly, in our setting we see that

$$K_\mu = \mathcal{I}_\mu(Q_h).$$

We present two important properties of the map $\mathcal{I}_\mu : \mathcal{U}_L \to \mathcal{U}_L$ in the following lemmas, and leave their proofs to the appendix (need to add).

We also recall a definition from the non-interacting setting: an *off-diagonal* operator is an operator A such that $A = \overline{A} := P_\mu AP_\mu^\perp + P_\mu^\perp AP_\mu$, where $P_\mu^\perp = \mathbb{I} - P_\mu$ is the projection onto the excited states above the gap.

Lemma 3. *For any off-diagonal operator $A = \overline{A}$, $\mathcal{I}_\mu(\cdot)$ and $[H_\mu, \cdot]$ act as inverses of each other, up to a factor of i:*

$$\mathcal{I}_\mu\left([H_\mu, A]\right) = [H_\mu, \mathcal{I}_\mu(A)] = iA.$$

Furthermore, for any (not necessarily off-diagonal) operator A,

$$[\mathcal{I}_\mu([H_\mu, A]), P_\mu] = i[A, P_\mu].$$

Another important property of the map $\mathcal{I}_\mu$ is that it preserves locality.

Lemma 4. *$\mathcal{I}_\mu$ is local in the sense that for any $A \in \mathcal{U}_X$,*

$$\|\mathcal{I}(A)_{(X^r)^c}\| \leq \|A\||X|\mathcal{O}(r^{-\infty})$$

where X^r is the r-fattening of X.

From this point, we drop the subscript μ wherever it is not needed for context.

Proposition 2. *The operator K_μ is the* generator of parallel transport*, satisfying*

$$\dot{P}_\mu = i[K_\mu, P_\mu]$$

for all μ.

Proof. First, we show that $\dot{P}$ is off-diagonal. Taking the derivative on both sides of $P^2 = P$, we see that $\dot{P}P + P\dot{P} = \dot{P}$. Acting on the left and right with P on both sides of this equation gives

$$P\dot{P}P + P\dot{P}P = P\dot{P}P,$$

which implies that $P\dot{P}P = 0$. Thus

$$\begin{aligned}
\overline{\partial_\mu P} &= P\dot{P}(1-P) + (1-P)\dot{P}P \\
&= P\dot{P} - P\dot{P}P + \dot{P}P - P\dot{P}P \\
&= P\dot{P} + \dot{P}P \\
&= \partial_\mu(P^2) \\
&= \partial_\mu P,
\end{aligned}$$

as claimed. By the product rule and the fact that H and P commute,

$$[\dot{H}, P] = -[H, \dot{P}].$$

It therefore follows from Lemma 3 that

$$\dot{P} = -i\mathcal{I}_\mu([H, \dot{P}]) = i\mathcal{I}([\dot{H}, P]) = i[\mathcal{I}(\dot{H}), P] = i[K, P].$$

□

Increasing the electric potential by a small amount $d\mu Q_h$ and expanding to linear order, the change in ground state current is given by

$$\mathrm{Tr}(P_{\mu+d\mu}J) - \mathrm{Tr}(P_\mu J) = \kappa d\mu + \mathcal{O}(d\mu^2).$$

Dividing by $d\mu$ and taking a limit, we see that the linear response coefficient is given by

$$\sigma(\mu) = \mathrm{Tr}\left(\dot{P}_\mu J\right).$$

The *Hall conductivity* of the system on a subset $V \subseteq \Gamma_L$ is defined to be $\sigma_V := \mathrm{Tr}\left(\dot{P} J_V\right)$, where J_V is the restriction of J to V.

Proposition 3. *The Hall conductivity is independent of the driving strength μ.*

Proof. Since $\dot{H}_\mu = Q_h$, we see that for any μ_1 and μ_2

$$\begin{aligned}
\sigma(\mu_1) - \sigma(\mu_2) &= \mathrm{Tr}\left(\dot{P}_{\mu_1} i[H_{\mu_1}, Q_u] - \dot{P}_{\mu_2} i[H_{\mu_2}, Q_u]\right) \\
&= i\mathrm{Tr}\left(\left([\dot{P}_{\mu_1}, H_{\mu_1}] - [\dot{P}_{\mu_2}, H_{\mu_2}]\right) Q_u\right) \\
&= -i\mathrm{Tr}\left(\left([\dot{H}_{\mu_1}, P_{\mu_1}] - [\dot{H}_{\mu_2}, P_{\mu_2}]\right) Q_u\right) \\
&= i\mathrm{Tr}\left([Q_h, P_{\mu_1} - P_{\mu_2}]Q_u\right) \\
&= i\mathrm{Tr}\left([Q_u, Q_h](P_{\mu_1} - P_{\mu_2})\right) \\
&= 0,
\end{aligned}$$

since Q_h and Q_u commute. The proof of Lemma 2 provides the necessary bounds to invoke linearity and cyclicity of the trace to shift the commutator in the second line and the second-last line. (need to add a bound for $\|\dot{P}\|_1$). $\square$

This indicates that the Hall conductivity is independent of μ as one would expect physically. We simply write $\sigma = \sigma(\mu)$ from this point, in accordance with proposition 3.

The following is the main result:

Theorem 3. *The ground state current in the strip $[L/2 + k, 3L/4 - k] \times [0, L]$ between the edge and the bulk current vanishes, in the sense that $\kappa_V = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty})$ for any $V \subseteq [L/2 + R, 3L/4 - R] \times [0, L]$ "in between" the bulk and edge strips, where*

$$r = \mathrm{dist}(V, [L/2 - R, 3L/4 + R] \times [0, L] \cup [3L/4 - R, 3L/4 + R] \times [0, L])$$

is the distance from V to one of the edge or bulk strips.

Proof. By Proposition 2, the Hall conductivity can also be written by the formula $\kappa_V^B = \mathrm{Tr}\left(i[K(\mu), P_B(\mu)]J_V^B\right) = \mathrm{Tr}\left(i[\mathcal{I}_\mu(Q_h), P_B(\mu)]J_V^B\right)$, where $J_V^B = (i[H_B, Q_u])_V$ is the current arising from the bulk Hamiltonian. From commutativity of P_B and H_B along with cyclicity of the trace, we compute

$$\begin{aligned}
\kappa_V^B &= \mathrm{Tr}\left(i[\mathcal{I}_\mu(Q_h), P_B(\mu)]J_V^B\right) \\
&= \mathrm{Tr}\left(i\int_{\mathbb{R}} W(t)e^{itH_B(\mu)}[Q_h, P_B(\mu)]e^{-itH_B(\mu)}dt J_V^B\right) \\
&= \int_{\mathbb{R}} W(t)\mathrm{Tr}\left(i[Q_h, P_B(\mu)]e^{-itH_B(\mu)}J_V^B e^{itH_B(\mu)}\right)dt \\
&= -\int_{\mathbb{R}} W(t)\mathrm{Tr}\left(i[Q_h, P_B(\mu)]e^{itH_B(\mu)}J_V^B e^{-itH_B(\mu)}\right)dt \\
&= -\mathrm{Tr}\left(i[Q_h, P_B(\mu)]\mathcal{I}_\mu(J_V^B)\right),
\end{aligned}$$

since $W(t)$ is odd. Again by cyclicity of trace combined with the fact that $\mathcal{I}_\mu(\cdot)$ is an inverse of $[H_B(\mu), \cdot]$ for commutators with $P_B(\mu)$ (by the remark after lemma 3), we obtain

$$\begin{aligned}
\kappa_V^B &= -\mathrm{Tr}([\mathcal{I}_\mu([H_B(\mu), Q_h]), P_B(\mu)]\mathcal{I}_\mu(J_V^B)) \\
&= -\mathrm{Tr}(\mathcal{I}_\mu([H_B(\mu), Q_h])P_B(\mu)\mathcal{I}_\mu(J_V^B) - P_B(\mu)\mathcal{I}_\mu([H_B(\mu), Q_h])\mathcal{I}_\mu(J_V^B))) \\
&= -\mathrm{Tr}(P_B(\mu)\mathcal{I}_\mu(J_V^B)\mathcal{I}_\mu([H_B(\mu), Q_h]) - P_B(\mu)\mathcal{I}_\mu([H_B(\mu), Q_h])\mathcal{I}_\mu(J_V^B))) \\
&= \mathrm{Tr}\left(P_B(\mu)[\mathcal{I}_\mu([H_B(\mu), Q_h]), \mathcal{I}_\mu(J_V^B)]\right).
\end{aligned}$$

Now, $[H_B(\mu), Q_h]$ is a local operator, supported on the "bulk line" $[3L/4 - R, 3L/4 + R] \times [0, L]$, while J_V^B is a local operator supported on $V \subseteq [L/2 + k, 3L/4 - k] \times [0, L]$. Since $\mathcal{I}_\mu$ preserves locality up to tails, in the sense that $\|\mathcal{I}_\mu(A)_{(S^r)^c}\| \leq \|A\||S|\mathcal{O}(r^{-\infty})$ (Lemma 4), it follows that the commutator $[\mathcal{I}_\mu([H_B(\mu), Q_h]), \mathcal{I}_\mu(J_V^B)] = C\mathcal{O}(r^{-\infty})$ whenever $V \cap ([3L/4 - R, 3L/4 + R] \times [L/2, L]) = \varnothing$.

The previous fact applies to the bulk setting with H_B and P_B. To extend this to the setting with an edge, it is enough to use Assumption 5 to conclude the same result, except with equality up to $\mathcal{O}(L^{-\infty})$, i.e.

$$\kappa_V = \mathrm{Tr}\left(\dot{P}J_V\right) = \mathrm{Tr}\left(\dot{P}(J_V^B + \mathcal{O}(L^{-\infty}))\right) = \kappa_V^B + \mathcal{O}(L^{-\infty}) = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

□

The intuitive picture from the previous result is that, in the bulk region, the Hall conductivity is essentially only nonzero along the bulk line $[3L/4 -$

$R, 3L/4 + R] \times [L/2, L]$. Since the ground state expectation of the current is zero (by lemma 2), it must be that there is an equal current flowing along the edge, but in the opposite direction, see figure (need to add).

3.1.2 Torus Geometry

Our goal is to show the same result on the discrete torus $\mathbb{T}_L := \mathbb{Z}_L \times \mathbb{Z}_L$. We define the same regions Γ_u and Γ_h, and the same current operator $J_u = i[H(\mu), Q_u]$. This time, however, Lemma 2 does not apply. Intuitively, it does not apply because electrons can now flow through both the bottom and the top of the region Γ_u, rather than just the bottom. Mathematically, the lemma fails because our definition of the current is slightly changed.

We use charge conservation and the fact that H is finite range to split the current J_u into two components, $J_u = i[H_-, Q_u] + i[H_+, Q_u] = J_- - J_+$, supported on strips of width $2R$ at $y = L/2$ and $y = L$, respectively. We then define the current operator to be $J = J_-$, which is the current on the lower strip. This is the mathematical reason that the proof in Lemma 2 fails on the torus; we have replaced H by H_-, which may no longer commute with P. We instead proceed by a different approach. We will need a few auxiliary results first.

Lemma 5. *$K_\pm$ is supported on $\partial_\pm$ up to tails.*

Proof. □

Proposition 4. *The operator $Q_h - K$ leaves the ground state space invariant, i.e. $[Q_h - K, P] = 0$.*

Proof. □

Lemma 6. *Show that $\mathrm{Tr}(A, [Q_h, P]) = 0$ for all $A \in \mathcal{U}_{edge}$. This shows that Q_h commutes with P "along the edge".*

Proof. Let $A \in \mathcal{U}_{\text{edge}}$. Since H is charge conserving, we may choose a simultaneous eigenbasis of H and the total charge Q, in which case P and Q commute. It follows that

$$\mathrm{Tr}(A[Q_h, P]) = \mathrm{Tr}([A, Q_h]P) = \mathrm{Tr}([A, Q]P) = \mathrm{Tr}(A[Q, P]) = 0.$$

□

Finally, we will prove that in the bulk system with Hamiltonian $H_B(\mu)$, the ground state expectation of the current vanishes faster than any power as $L \to \infty$.

Lemma 7. *The ground state expectation of the current* $J_B := i[(H_B)_-, Q_h]$ *(of the system without an edge) is* $\mathrm{Tr}(P_B J_B) = \mathcal{O}(L^{-\infty})$.

Proof. First, $K = \mathcal{I}(i[H_B, Q])$ splits into $K = K_- - K_+$, with the support of $K_\pm$ contained in $\partial_\pm$ up to tails:

$$[K_\pm, A_X] = \mathcal{O}(p^{-\infty}),$$

for every $A_X \in \mathcal{U}_X$ such that $\|A_X\| = 1$, and where $p = \mathrm{dist}(X, \partial_\pm)$ (need to add). Using the fact that $K_\pm$ is supported in $\partial_\pm$ up to tails (Lemma 5), we see that

$$i[H_B, K_-] = i[(H_B)_-, K_-] + \mathcal{O}(L^{-\infty}),$$

and similarly $i[(H_B)_-, K_+] = \mathcal{O}(L^{-\infty})$. Putting these facts together, it follows that the current can be rewritten as

$$\begin{aligned} J_B &= i[H_B, Q_h + K_- - K_- + K_+] + \mathcal{O}(L^{-\infty}) \\ &= i[H_B, K_-] + i[(H_B)_-, Q_h - K_- + K_+)] + \mathcal{O}(L^{-\infty}). \end{aligned}$$

From here, we use the fact that H_B and $Q_h - K_- + K_+$ both commute with P_B to write

$$P_B J_B P_B = i[H_B, P_B K_- P_B] + i[P_B (H_B)_- P_B, Q_h - K_- + K_+)] + P_B \mathcal{O}(L^{-\infty}) P_B.$$

Since the trace of any commutator is zero,

$$\mathrm{Tr}(P_B J_B) = \mathrm{Tr}(P_B J_B P_B) = \mathcal{O}(L^{-\infty}).$$

□

Using this, we can show a simple proof of the analogue of Lemma 2 on the torus, in the case of non-interacting systems.

Proposition 5. *Let* $H = \sum_{x \in \mathbb{T}} h_x$ *be a non-interacting Hamiltonian, i.e. a sum of single site Hamiltonians* h_x. *The ground state expectation of the current* $J = i[H_-, Q_h]$ *(of the system with an edge) is* $\mathrm{Tr}(PJ) = \mathcal{O}(L^{-\infty})$.

Proof. Since H is a sum of single site Hamiltonians, we can split H_- into the restrictions $H_- = (H_-)_{\text{edge}} + (H_-)_{\text{bulk}}$, with no fear of any terms which are in both the edge region and the bulk region. By Assumption 5,

$$\begin{aligned}
\mathrm{Tr}(PJ) &= \mathrm{Tr}(Pi[H_-, Q_h]) \\
&= i\mathrm{Tr}([H_-, Q_h]P) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\text{bulk}}[Q_h, P]) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\text{bulk}}[Q_h, (P)_{\text{bulk}}]) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_B)_-[Q_h, P_B]) + \mathcal{O}(L^{-\infty}) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + \mathrm{Tr}(i[(H_B)_-, Q_h]P_B) + \mathcal{O}(L^{-\infty}).
\end{aligned}$$

By Lemma 6, the first term is zero. By Lemma 7, the second term is $\mathcal{O}(L^{-\infty})$. □

A Properties of $\mathcal{I}_\mu$

Proof. (Of Lemma 3). Let $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} W(t) e^{-2\pi i t \xi} dt$ be the Fourier transform of W. One can show that for $|\xi| \geq \gamma$, $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi} i \xi}$ (need to add). Let A be an observable. First, we show that $\mathcal{I}([H, PAP^\perp]) = iPAP^\perp$.

Decomposing

$$\begin{aligned} e^{itH} P &= \sum_{j=0}^{\infty} \frac{(itH)^j}{j!} P \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left(\sum_n E_n^j P_n \right) P \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \sum_{n:E_n=0} E_n^j P_n \\ &= \sum_{n:E_n=0} e^{itE_n} P_n, \end{aligned}$$

and similarly

$$P^\perp e^{-itH} = \sum_{m:E_m \geq \gamma} P_m e^{-itE_m},$$

we see that

$$
\begin{aligned}
\mathcal{I}([H, PAP^\perp]) &= \mathcal{I}(P[H,A]P^\perp) \\
&= \int_{\mathbb{R}} W(t) e^{itH} P[H,A] P^\perp e^{-itH} dt \\
&= \int_{\mathbb{R}} W(t) \sum_{n:E_n=0} e^{itE_n} P_n [H,A] \sum_{m:E_m \geq \gamma} P_m e^{-itE_m} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} \int_{\mathbb{R}} W(t) e^{itE_n} P_n A (E_n - E_m) P_m e^{-itE_m} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \int_{\mathbb{R}} W(t) e^{-it(E_m - E_n)} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \sqrt{2\pi} \widehat{W}(E_m - E_n) \\
&= i \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m \\
&= iPAP^\perp.
\end{aligned}
$$

(need to check the 2π factor)

By the same argument, $\mathcal{I}([H, P^\perp AP]) = iP^\perp AP$ as well, and so $\mathcal{I}([H, \overline{A}]) = i\overline{A}$. □

Proof. (Of Lemma 4). We break the integral into two parts,

$$
\|\mathcal{I}(A)\| \leq \left\| \int_{-T}^{T} W(t) e^{itH} A e^{-itH} dt \right\| + \left\| \int_{\mathbb{R} \setminus [-T,T]} W(t) e^{itH} A e^{-itH} dt \right\|.
$$

The first term can be estimated using the Lieb-Robinson bound found in Appendix B.

□

B Lieb-Robinson Bound

Let N be a uniform upper bound for the dimensions of the Hilbert spaces at each site, i.e. $\dim(\mathcal{H}_x) \leq N$ for all sites x.

The following is a version of the Lieb-Robinson. For any operators $A \in \mathcal{U}_X$ and $B \in \mathcal{U}_Y$ having disjoint supports $X \cap Y = \varnothing$,

$$
\|[e^{itH} A e^{-itH}, B]\| \leq C \|A\| \|B\| |X| |Y| N^{2|X|} e^{2t\|\Phi\|_\lambda - \lambda d(X,Y)}.
$$

C Grönwall's Inequality and Uniqueness

Theorem 4. *(Grönwall's Inequality). Let $\alpha : I \to (0, \infty)$ be positive and continuous on I^o for some interval of the form $[a, b)$, $[a, b]$, or $[a, \infty)$. Suppose $u : \mathbb{R} \to \mathcal{U}$ is a Banach-valued, differentiable function. If $\|u'(t)\| \leq \alpha(t)\|u(t)\|$ for all $t \in I$, then*

$$\|u(t)\| \leq \|u(a)\| e^{\int_a^t \alpha(s) ds} \quad \forall t \in I$$

Proof. Let $f(t) = e^{\int_a^t \alpha(s) ds}$, which is nonzero, has initial value $f(a) = 1$, and has derivative $f'(t) = \alpha(t) f(t)$. Then by the quotient rule,

$$\left(\frac{\|u(t)\|}{f(t)} \right)' = \frac{\|u'(t)\| f(t) - \|u(t)\| \alpha(t) f(t)}{f(t)^2} \leq 0,$$

where the inequality follows from the assumption $\|u'(t)\| \leq \|\alpha(t) u(t)\|$. Thus $\frac{\|u(t)\|}{f(t)}$ is decreasing, so that

$$\frac{\|u(t)\|}{f(t)} \leq \frac{\|u(a)\|}{f(a)} = \|u(a)\|,$$

which is the desired inequality. □

Theorem 5. *(ODE Uniqueness). Let $F : \mathcal{U} \to \mathcal{U}$ be Lipschitz and consider the differential equaton $u'(t) = F(u(t))$ with initial condition $u(a) = u_a$ for some function $u : I \to \mathcal{U}$, where $I = [a, b]$, or $[a, b)$, or $[a, \infty)$. Solutions to this equation are unique.*

Proof. Suppose there are two solutions $u(t)$ and $v(t)$, and let $g(t) = \|u(t) - v(t)\|^2$. By assumption, there exists a constant L_F such that $\|F(u(t)) - F(v(t))\| \leq L_F \|u(t) - v(t)\|$, so that

$$\begin{aligned}
g'(t) &= 2\|u(t) - v(t)\| \|u'(t) - v'(t)\| \\
&= 2\|u(t) - v(t)\| \|F(u(t)) - F(v(t))\| \\
&\leq 2L_F \|u(t) - v(t)\|^2 \\
&= 2L_F g(t).
\end{aligned}$$

Notice that $\alpha := 2L_F$ is a positive continuous function, so we may apply Grönwall's inequality to $g(t)$ to conclude

$$g(t) \leq g(a) e^{2L_f (t-a)} = 0,$$

since $g(a) = 0$. □

D Note on Generators of Parallel Transport

Consider the differential equation $\dot{\rho}(\mu) = i[K_B, \rho(\mu)]$ with initial condition $\rho(0) = P_B(0)$. Here $K_B = \int_{\mathbb{R}} W_\gamma(t) e^{-itH_B} \dot{H}_B e^{itH_B} dt$, and recall that in our setting, $\dot{H}_B = Q_h$. We know that the solution is $\rho(\mu) = P_B(\mu)$ (proposition 2). Notice that the map $F : \mathcal{U} \to \mathcal{U}$ defined by $F(A) = i[K_B, A]$ is Lipschitz, since

$$\|F(A) - F(B)\| = \|[K_B, A - B]\| \leq 2\|K_B\| \|A - B\|.$$

The Lipschitz constant is $2\|K_B\|$, which is finite since K_B is a bounded operator:

$$\|K_B\| \leq \int_{\mathbb{R}} |W_\gamma(t)| \|e^{-itH_B} Q_h e^{itH_B}\| dt \leq \int_{\mathbb{R}} |W_\gamma(t)| dt \|Q_h\|.$$

Indeed, since Q_h is the number operator on a finite volume, by charge conservation and the fact that the dimension of the Hilbert space is bounded uniformly by d, there can only be a finite number of charges in the region Γ_h.

Thus, by Grönwall's uniqueness theorem (appendix C), we see that the solution to the equation $\dot{\rho}(\mu) = F(\rho(\mu)) = i[K_B, \rho(\mu)]$ is unique.

Now define

$$K_E := \int_{\mathbb{R}} W_\gamma(t) e^{-itH_E} Q_h e^{itH_E} dt,$$

which is using the gap γ of H_B to define W_γ, but also using the edge Hamiltonian in the time evolution operators. Consider $\sigma : [0, \infty) \to \mathcal{U}$ defined by

$$\dot{\sigma}(\mu) = i[K_E, \sigma(\mu)] \qquad \sigma(0) = P_E(0).$$

We now show that, similar to how ρ is an approximation of P_B, σ is also a good approximation of P_E (up to $\mathcal{O}(L^{-\infty})$) "in the bulk". Let $A \in \Gamma_B$ be an operator localized in the bulk of the edge system. Then

$$
\begin{aligned}
\mathrm{Tr}(\dot{\sigma}A) &= \mathrm{Tr}(i[K_E,\sigma]A) \\
&= \mathrm{Tr}(i[A,K_E]\sigma) \\
&= \int_{\mathbb{R}} W_\gamma(t)\mathrm{Tr}([e^{-itH_E}Q_h e^{itH_E},A]\sigma)dt \\
&= \int_{\mathbb{R}} W_\gamma(t)\mathrm{Tr}(e^{-itH_E}[Q_h,e^{itH_E}Ae^{-itH_E}]e^{itH_E}\sigma)dt \\
&= \int_{\mathbb{R}} W_\gamma(t)\mathrm{Tr}(e^{-itH_E}[Q_h,e^{itH_B}Ae^{-itH_B}]e^{itH_E}+\mathcal{O}(L^{-\infty})\sigma)dt \\
&= \int_{\mathbb{R}} W_\gamma(t)\mathrm{Tr}(e^{-itH_B}[Q_h,e^{itH_B}Ae^{-itH_B}]e^{itH_B}+\mathcal{O}(L^{-\infty})\sigma)dt \\
&= \int_{\mathbb{R}} W_\gamma(t)\mathrm{Tr}([e^{-itH_B}Q_h e^{itH_B},A]\sigma)dt+\mathcal{O}(L^{-\infty}) \\
&= \mathrm{Tr}(i[A,K_B]\sigma]+\mathcal{O}(L^{-\infty}) \\
&= \mathrm{Tr}(i[K_B,\sigma]A)+\mathcal{O}(L^{-\infty}),
\end{aligned}
$$

since σ is trace-class (?) and $W_\gamma \in L^1$. By linearity of the trace, we see that $\mathrm{Tr}((\dot{\sigma}-i[K_B,\sigma])A)=\mathcal{O}(L^{-\infty})$ for any operator $A\in\Gamma_B$ (does this mean $\dot{\sigma}-i[K_E,\sigma]=0$?). But the solution of $\dot{\sigma}-i[K_B,\sigma]=0$ (with initial condition $\sigma(0)=P_B(0)$) is unique; it is $\rho(\mu)$, or $P_B(\mu)$. Hence

$$
\mathrm{Tr}(P_EA)=\mathrm{Tr}(P_BA)+\mathcal{O}(L^{-\infty})=\mathrm{Tr}(\rho A)+\mathcal{O}(L^{-\infty})=Tr(\sigma A)+\mathcal{O}(L^{-\infty})
$$

for any operator $A\in\Gamma_B$. In particular, this gives another local formula for the Hall conductivity in the bulk of an edge system, by taking $A=J_V$, where J is the current operator and $V\subset\Gamma_B$ is a set localized in the bulk. The Hall conductivity is given by $\mathrm{Tr}(\dot{P}_E J_V)$, and this can be approximated by

$$
\mathrm{Tr}(\dot{P}_EJ_V)=\mathrm{Tr}(\dot{P}_BJ_V)+\mathcal{O}(L^{-\infty})=\mathrm{Tr}(\dot{\rho}J_V)+\mathcal{O}(L^{-\infty})=\mathrm{Tr}(\dot{\sigma}J_V)+\mathcal{O}(L^{-\infty}).
$$

Want to pick a norm s.t. Gronwall gives $\|\rho(\mu)-\sigma(\mu)\|_G\le\|P_B(0)-P_E(0)\|_G e^{2L_F\mu}$. Need $\|P_B(0)-P_E(0)\|_G$ to be small enough to kill the exponential which depends on $L_F=2\|K_B\|_G\le\|W_\gamma\|_{L^1}\|Q_h\|_G$. If we use the operator norm for $\|\cdot\|_G$, we would get $\|Q_h\|_G=d|\Gamma_h|$ in the exponent. Need $\|\cdot\|_G$ to be an actual norm so that $\|\rho-\sigma\|_G=0 \implies \rho=\sigma$.

From Dec 13 Meeting

Let $r(t) = \rho(t) - \sigma(t)$. Notice that

$$\frac{d}{dt} e^{itK_B} \sigma_0 e^{-itK_B} = e^{itK_B} i[K_B, \sigma_0] e^{-itK_B} + e^{-itK_B} \dot{\sigma_0} e^{itK_B}.$$