# The Quantum Hall Effect and Bulk-Edge Correspondence on Lattice Systems

by

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# Abstract

In this thesis, we investigate the *integer quantum Hall effect* (IQHE). Discovered in 1980 by Nobel Laureate Klaus Von Klitzing, the IQHE is a phenomenon in which the resistivity of a 2-dimensional electron gas is precisely quantized to integer multiples of $h/q^2$ when subjected to a strong magnetic field at very low temperatures.

In particular, we provide a rigorous theoretical description of a phenomenon known as *bulk-edge correspondence*, wherein the value of the Hall conductivity is unaffected by whether or not one considers a system with an edge or a bulk (i.e. edgeless) system. This is accomplished in two settings. First, we ignore interactions between electrons, and then proceed to the more challenging interacting setting.

# Preface

Chapter 2 is based on work conducted in [1]. This thesis is the work of the author, Justin Furlotte, under the supervision of Sven Bachmann.

# Table of Contents

## Appendices

# List of Figures

# Acknowledgements

I would like to thank my supervisor, Sven Bachmann, for his patience, his kindness, and his inspiring mastery in the field of mathematical physics and quantum lattice systems.

# Dedication

To Abigail Sanderson for her unwavering belief in me, and to my parents for their endless support.

# Chapter 1

# Introduction

Since its discovery in 1980, the Integer Quantum Hall Effect (IQHE) has captured significant interest in the mathematical physics community. The IQHE occurs when a 2 dimensional electron gas at near 0 Kelvin is pierced by a strong magnetic field. As the strength of the field increases, the Hall resistivity - a macroscopic quantity - undergoes quantization in nearly perfect integer multiples of $h/q^2$, as in Figure 1.1.

In this thesis, we begin by providing a simple classical calculation for the Hall conductivity, and emphasize that the result does not agree with experiment. The IQHE, as its name suggests, is fundamentally reliant on quantum mechanics. We give another heuristic argument due to Laughlin [15] for why the quantization of Hall resistivity occurs, before going into the main body of the thesis in Chapters 2 and 3.

There, we provide a proof of *bulk-edge correspondence*. This is the interesting mathematical fact that the Hall conductivity does not depend on whether or not the system is assumed to have an edge. In the system with an edge, the Hall conductivity is defined by assuming that current is transported along the edge, while in the bulk (i.e. edgeless) system, the Hall current is assumed to be carried throughout the entire bulk of the material. This correspondence between the bulk and edge Hall conductivities in the IQHE is a special case of a more general bulk-edge correspondence which can occur between other invariants of topological insulators.

## 1.1 Heuristic Arguments

We provide a brief introduction to the quantum Hall effect. We emphasize that the results of this section are not rigorous.

### 1.1.1 The Classical Hall Effect

Using classical electromagnetism is not enough to predict the plateaux seen experimentally. Suppose we have a 2-dimensional electron gas, and let

$\vec{B} = B\hat{z}$ be a magnetic field piercing the plane of the electrons. They are subjected to a Lorentz force

$$m\dot{\vec{v}} = -q\vec{v} \times \vec{B}.$$

The solution to this differential equation is given by the *cyclotron orbits*,

$$\begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} a + r\sin(\omega_B t + \phi) \\ b + r\cos(\omega_B t + \phi) \\ 0 \end{pmatrix}$$

where $\omega_B = qB/m$ is the *cyclotron frequency.* When an electric field $\vec{E} = E\hat{x}$ is introduced, the electrons move in the $x$-direction, and we employ the *Drude model*,

$$m\dot{\vec{v}} = -q(\vec{E} + \vec{v} \times \vec{B}) + \frac{m}{\tau}\vec{v},$$

where the final term is a linear friction term, and $\tau$ is the scattering time. At equilibrium, the equation reads

$$\vec{J} + \frac{q\tau}{m}\vec{J} \times \vec{B} = -\frac{q^2 n\tau}{m}\vec{E},$$

where $\vec{J} = -nq\vec{v}$ is the current density, related to the velocity by the density $n$ of electrons per unit area. In matrix notation, this reads

$$\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & q \end{pmatrix} \vec{J} = -\frac{q^2 n\tau}{m}\vec{E}.$$

Since the matrix on the left is invertible, we may write $\vec{J} = \sigma\vec{E}$, which is Ohm's Law. The *conductivity tensor* is given by

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{pmatrix} = -\frac{q^2 n\tau}{m(1+\omega_B^2\tau^2)} \begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix}.$$

The off-diagonal components, $\pm\frac{q^2 n\tau}{m(1+\omega_B^2\tau^2)}\omega_B\tau$, are responsible for the Hall effect; the magnetic field induces a component of the current in the $y$-direction, in addition to the one in the $x$-direction from the electric field.

When making a measurement, physicists actually measure the resistivity. In particular, the *Hall resistivity*, given by the off-diagonal components of the resistivity tensor $\rho = \sigma^{-1}$, is simply

$$\rho_{xy} = \frac{B}{nq}.$$

The key prediction of the classical theory is that the Hall resistivity increases linearly in response to the strength of the magnetic field.

### 1.1.2 The Quantum Hall Effect

Von Klitzing's experimental observation in 1980 made it clear that classical electromagnetism is not sufficient to describe the Hall effect. At a temperature of about 8mK, the Hall resistivity looked like this as a function of the magnetic field strength.

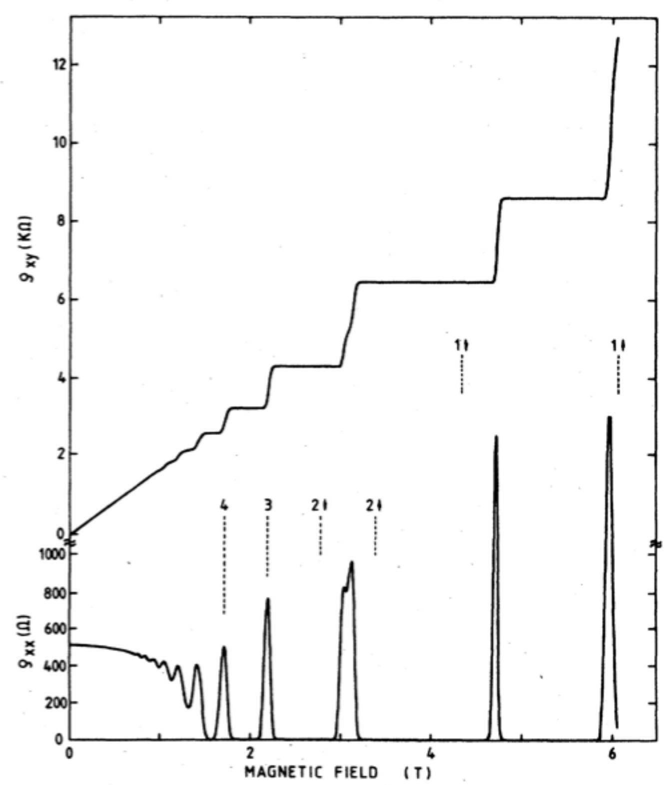


Figure 1.1: Hall resistivity as a function of the magnetic field strength jumps in distinct plateaux. [24]

Even more surprising, the plateaux occur at the values

$$\rho_{xy} = \frac{h}{q^2}\frac{1}{n},$$

where $n$ is an integer, hence the name *integer quantum Hall effect*. In fact, even in impure samples, this integer can be measured to such extraordinary precision (about one part in $3\times10^{-10}$) that as of 2020, the SI definition of the Ohm itself has been redefined in terms of the quantum Hall resistivity. Clearly deeper physics are at play.

The first quantum mechanical explanation of the integer-valued Hall conductivity is due to Laughlin in his short but seminal 1981 paper [15], for which he also won the Nobel prize. In this paper, the plateaux are explained using a charge transport argument.

We remark that, although the resistivity is what is measured in a lab, throughout this thesis we work with the quantum Hall *conductivity*,

$$\sigma_H = \frac{q^2}{h}n.$$

### The Laughlin Argument

Rather than a flat sheet, consider gluing together the $x_2 = 0$ and $x_2 = L$ edges to form a cylinder, as depicted in Figure 1.2. The surface of the cylinder is pierced throughout by a uniform magnetic field $\vec{B}$ normal to its surface. In addition to this background magnetic field, suppose that a time-dependent magnetic flux $\phi(t)$ is also threaded through the cylinder.

As $\phi(t)$ increases slowly from $\phi(0) = \phi_0$ to $\phi(T) = \phi_0 + \Delta\phi$, the charge increases by $\Delta Q = -\sigma_H \Delta\phi$. This can be seen by Faraday's law,

$$-\frac{d\phi}{dt} = \oint_C \vec{E}\cdot\vec{dl},$$

combined with the formula for the Hall current, $\vec{J_H} = -\sigma_H \hat{\rho}\times\vec{E}$. The current pumped across the fiducial line $C$ in Figure 1.2 is

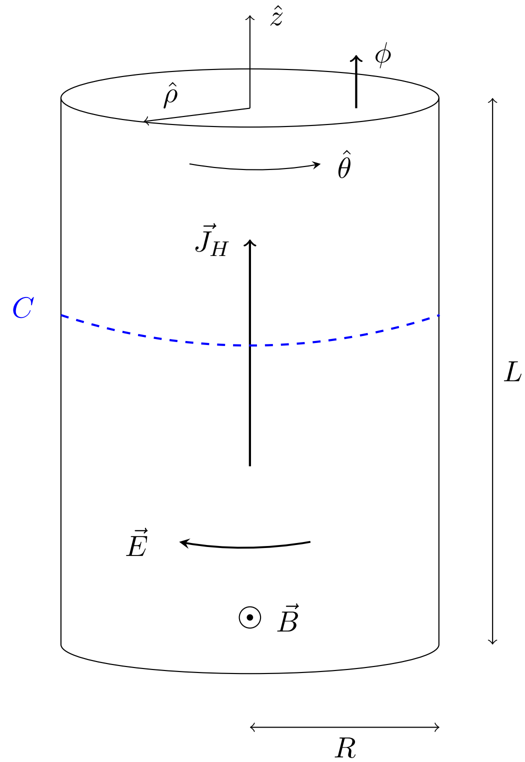


Figure 1.2: Laughlin's cylinder.

$$\begin{aligned}
\frac{dQ}{dt} &= \oint_C \vec{J}_H \cdot \hat{z} dl \\
&= -\sigma_H \oint (\hat{\rho} \times \vec{E}) \cdot \hat{z} dl \\
&= -\sigma_H \oint_C (\hat{\rho} \times \hat{z}) \cdot \vec{E} dl \\
&= \sigma_H \oint_C \hat{\theta} \cdot \vec{E} dl \\
&= \sigma_H \oint_C \vec{E} \cdot \vec{dl} \\
&= -\sigma_H \frac{d\phi}{dt}.
\end{aligned}$$

Thus $\Delta Q = -\sigma_H \Delta\phi$. We now use this fact to argue that the Hall conductivity must be an integer multiple of $q^2/h$. For convenience, we "unroll" the cylinder to work in Cartesian coordinates, but still identify the edges $x = 0$ and $x = L$. We take the positive $x$-direction to be the opposite direction as $J_H$ in figure 1.2, and the positive $y$-direction to be the same as $\hat{\theta}$ in the figure. We choose the *Landau gauge* for the magnetic field $\vec{B} = B\hat{z}$,

$$\vec{A}_B = \begin{pmatrix} 0 \\ Bx \\ 0 \end{pmatrix},$$

and

$$\vec{A}_\phi = \begin{pmatrix} 0 \\ \frac{\phi}{2\pi R} \\ 0 \end{pmatrix}$$

for the flux potential, which has vanishing curl (as desired since there's no magnetic field from $\phi$ on the cylinder). The total magnetic vector potential $\vec{A} = \vec{A}_B + \vec{A}_\phi$ appears in the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2m}(\vec{p} - q\vec{A})^2 \\ &= \frac{1}{2m}\left(p_x^2 + \left(p_y - q\left(Bx + \frac{\phi}{2\pi R}\right)\right)^2\right). \end{aligned}$$

The fact that the Hamiltonian commutes with the $y$-momentum is evident from the expression above, since the only thing that wouldn't commute with $p_y$ is the position operator $y$, which does not appear. This allows us to replace $p_y$ with its eigenvalue, $\hbar k$. The wavenumber $k$ is quantized by the periodic boundary condition

$$e^{i0k} = e^{i2\pi Rk},$$

which implies $p_y = \hbar k = \hbar j/R$ with $j \in \mathbb{N}$, giving

$$\begin{aligned} H_j &= \frac{1}{2m}\left(p_x^2 + \left(\frac{\hbar j}{R} - q\left(Bx + \frac{\phi}{2\pi R}\right)\right)^2\right) \\ &= \frac{1}{2m}\left(p_x^2 + q^2B^2\left(x - \frac{1}{2\pi RB}\left(\frac{h}{q}j - \phi\right)\right)^2\right). \end{aligned}$$

Using again the cyclotron frequency $\omega_B = \frac{qB}{m}$, and defining the *shift factor*

$$s_j(\phi) := \frac{1}{2\pi RB}\left(\frac{h}{q}j - \phi\right),$$

the Hamiltonian in this notation takes the form

$$H_j = \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_B^2(x - s_j(\phi))^2.$$

This is a shifted quantum harmonic oscillator in $x$, with frequency $\omega_B$. The spectrum is as usual,

$$E_n = \hbar\omega_B\left(n + \frac{1}{2}\right).$$

The eigenstates are called *Landau levels*, and they are exponentially localized at $x = -s_j(\phi)$. They are of the form

$$\psi_{n,j}(x,y) = C_{n,j}e^{i\frac{yj}{R}}H_n(x - s_j)e^{-\frac{(x-s_j(\phi))^2}{2\ell_B^2}}$$

where $\ell_B^2 = \frac{\hbar}{qB}$, $C_{n,j}$ are normalization constants, and $H_n$ are the Hermite polynomials.

The crucial argument now comes from inspecting the shift $s_j(\phi)$. Suppose we adiabatically increase $\phi$ by one *flux quantum*, $\phi_0 \mapsto \phi_0 + \frac{h}{q}$, as time evolves from $t = 0$ to $t = T$. The adiabatic principle tells us that the eigenstates of the Hamiltonian at $t = 0$ must also be eigenstates of the Hamiltonian at $t = T$. But we can find exactly what the new eigenstates are after the flux quantum is added; the only thing that changes is the shift, which by a simple calculation becomes

$$s_j\left(\phi_0 + \frac{h}{q}\right) = \frac{1}{2\pi RB}\left(\frac{h}{q}j - \phi_0 - \frac{h}{q}\right) = s_{j-1}(\phi_0).$$

The (exponentially localized) wavefunctions are each transported upward in the $-x$ direction by an increment of $s_1(0) = \frac{\hbar}{qB}\frac{1}{R}$ as $\phi$ increases by one flux quantum. We remark that the wavefunctions $\psi_{n,j}$ are Gaussian-like, with standard deviation $\ell_B^2 = \frac{\hbar}{qB}$. If $R \ll 1$, the shift increment $s_1(0)$ is large enough that we may treat the wavefunctions as being localized in a similar manner as depicted in the diagram of Figure 1.3.

Heuristically, this diagram explains why charge transport occurs. The landau levels take their usual discrete spectrum and bend up sharply at the edges where the material ends. Consider one filled Landau level. Each of the states in this level is transported to the left. In particular, on the bottom

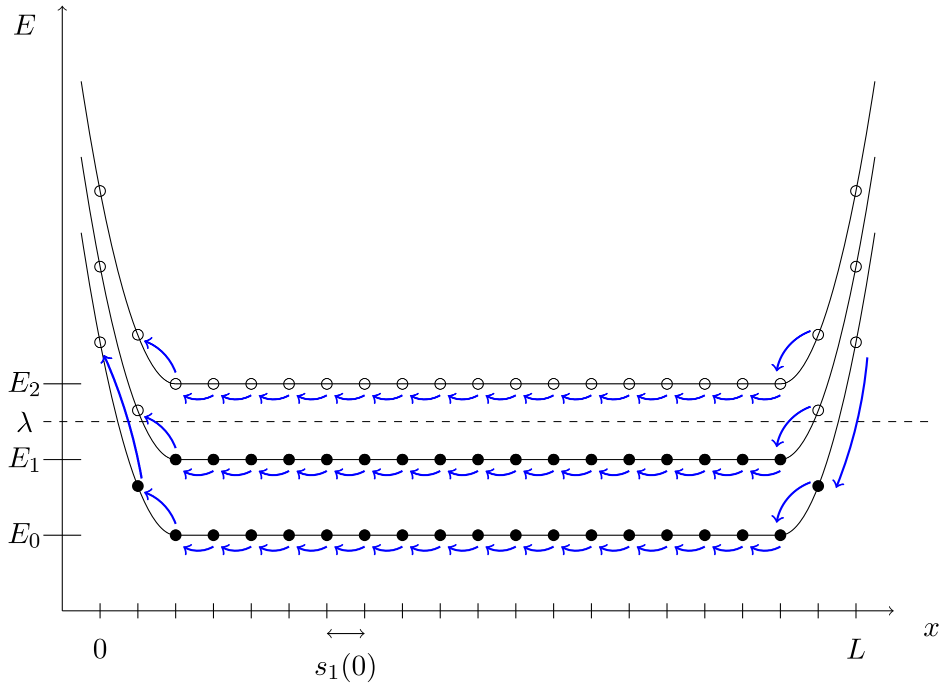


Figure 1.3: Quantum Hall conductance explained using charge transport. The states are exponentially localized and separated by a distance of $s_1(0)$. As $\phi_0 \mapsto \phi_0 + h/q$, the states all shift to the left by $s_1(0)$.

edge of the cylinder at $x = L$, the single empty state just above the Fermi energy $\lambda$ is brought below the Fermi energy (see Figure 1.3). On the top edge, the single filled state just below the Fermi energy is brought above the Fermi energy. Everywhere in between, states are merely shifted up the cylinder, each one filling the other's place. Thus, exactly one charge $q$ is transported from the bottom edge to the top edge. This occurs once per filled Landau level, so the total charge transported by the system is

$$\Delta Q = -qN_L$$

whenever the magnetic flux increases by $\Delta\phi = \frac{h}{q}$, where $N_L$ is the (integer) number of filled Landau levels. Our result $\Delta Q = -\sigma_H \Delta\phi$ now gives us exactly the Hall conductivity which agrees with experiment,

$$\sigma_H = \frac{q^2}{h} N_L \qquad\qquad N_L \in \mathbb{N}.$$

**Degeneracy, Disorder, and Plateaux**

The reasoning above only guarantees the correct value of $\sigma_H$ if the Landau levels are exactly filled. Furthermore, the argument relies on a fictitious magnetic flux $\phi$. As $\phi_0 \mapsto \phi_0 + h/q$, one charge per Landau level is transported from bottom edge to top. But isn't $\sigma_H$ supposed to increase as $B$ decreases, not as $\phi$ decreases?

The magnetic flux is purely an explanatory tool used to derive the formula $\sigma_H = \frac{q^2}{h} N_L$. The role of the magnetic field strength $B$ becomes apparent from inspecting the degeneracy of the Landau levels. The number of states $\psi_{n,j}$ per fixed Landau level $n$ must be finite, since $0 \leq x \leq L$ implies (roughly) that $-L \leq s_j(\phi) \leq 0$. We simply take $\phi = 0$, since adding $h/q$ only corresponds to $j = 1$ extra state, which is insignificant to the degeneracy of the Landau levels. Substituting our expression for the shift $s_j(0)$, the inequality becomes

$$-L\frac{qBR}{\hbar} \leq j \leq 0.$$

The number of states $N_s$ per Landau level is therefore

$$N_s = L\frac{qBR}{\hbar} = \frac{qBA}{h},$$

where $A = 2\pi RL$ is the total area of the cylinder. From here we see that as $B$ decreases, so too does the degeneracy $N_s$; the electrons must therefore begin to populate higher Landau levels. We know that once the next highest Landau level is completely filled, the Hall conductivity will have increased by exactly $q^2/h$.

What about the plateaux? Ironically, disorder, i.e. the presence of impurities in a real world material, is required to explain why the Hall conductivity does not change even when a Landau level is only partially filled. Disorder is modelled using a random potential. Under the addition of a suitably small random potential to the Hamiltonian, the degeneracy of the Landau levels is lifted and the spectrum broadens into bands as in Figure 1.4. Furthermore, while the center of the bands is continuous spectrum (red), the edges of the bands are pure-point (blue) [2, 12]. This phenomenon is called Anderson localization.

The pure-point states, called *localized states*, contribute nothing to the current, as they have compact support which does not wrap around the cylinder. The effect of $\phi$ on these states can therefore be gauge transformed away, since the domain of their support is simply connected, ensuring that

$\vec{A}_\phi$ is a (locally) conservative vector field. This is not the case on the cylinder as a whole, and in particular it is not the case for *extended states* (red), whose domain is not simply connected. Thus only extended states contribute to charge transport.

As $B$ decreases and electrons fill states of higher energy within the band, the Fermi energy $\lambda$ increases. However, the states at the edges of the bands (blue) are localized and contribute nothing to the current, and thus leave the Hall conductivity unchanged. This gives rise to the plateaux.

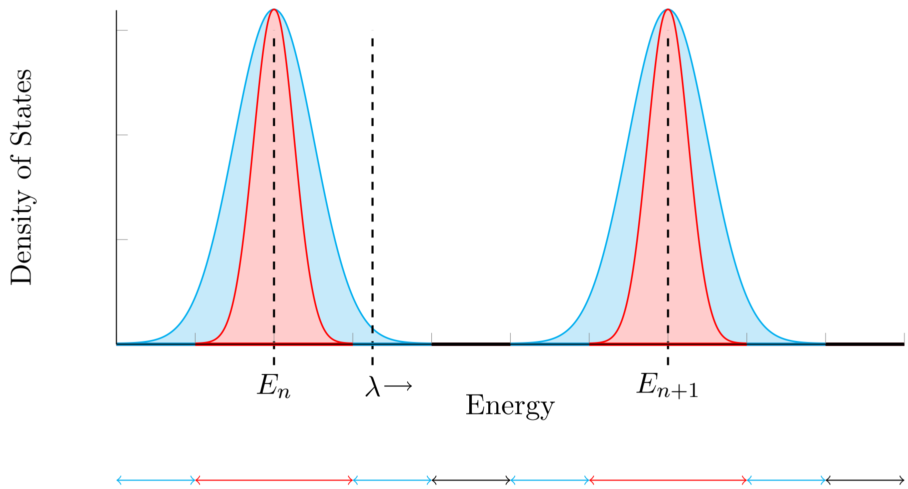


Figure 1.4: Disorder, modelled by a small random potential, breaks the degeneracy of the discrete Landau levels $E_n$ into bands. The blue states are *localized* and do not contribute to charge transport, while the red states are *extended*. As $B$ decreases and $\mu$ moves through the blue (localized) and black (spectral gap) regions, the current (and therefore the Hall conductivity) are not affected; this gives rise to the plateaux observed experimentally. Only once $\mu$ begins moving through the red (extended) region will the Hall conductivity begin to increase to its next plateau.

# Chapter 2

# Noninteracting Bulk-Edge Correspondence

An intriguing mathematical fact of the quantum Hall effect is the equality of bulk and edge conductivity, in the sense that whether the system is assumed to have an edge or not is immaterial. Proving this fact, both in the interacting and noninteracting setting, is the main focus of this thesis. In particular, we are not concerned with proving the existence of plateaux, or even the integrality of the Hall conductivity, but rather the equivalence of the bulk and edge Hall conductivities.

We begin by generalizing from Landau Hamiltonians to a much more general class of Hamiltonians, and derive appropriate formulae for the "bulk" and "edge" Hall conductivities in this scenario.

## 2.1 General Setting

Consider the lattice $\mathbb{Z}^2$, and the associated Hilbert space of square-summable sequences of vectors in $\mathbb{C}^n$,

$$\ell^2(\mathbb{Z}^2, \mathbb{C}^n) = \left\{ (x_i)_{i\in\mathbb{Z}^2} \subset \mathbb{C}^n \;:\; \sum_{i\in\mathbb{Z}^2} \|x_i\|^2 < \infty \right\},$$

with inner product $\langle x, y\rangle = \sum_{i\in\mathbb{Z}^2} x_i \overline{y_i}$. We denote this as $\ell^2(\mathbb{Z}^2)$ for short. On this Hilbert space we define a *bulk Hamiltonian* $H_B : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)$, whose matrix elements follow a short-range assumption:

**Assumption 1.** *There exists some $\alpha > 0$ such that*

$$\sup_{y\in\mathbb{Z}^2} \sum_{x\in\mathbb{Z}^2} |H_B(x,y)|(e^{\alpha|x-y|} - 1) \le C < \infty,$$

*where $|x| = |x_1| + |x_2|$ induces the taxicab metric.*

We also construct an *edge Hamiltonian* on the lattice $\mathbb{Z}^2_a := \{x \in \mathbb{Z}^2 : x_2 > -a\}$, denoted by $H_a : \ell^2(\mathbb{Z}^2_a) \to \ell^2(\mathbb{Z}^2_a)$. The bulk and edge Hamiltonians are related by the *edge operator* $E_a : \ell^2(\mathbb{Z}^2_a) \to \ell(\mathbb{Z}^2)$,

$$E_a := \mathcal{J}_a H_a - H_B \mathcal{J}_a,$$

where $\mathcal{J}_a : \ell^2(\mathbb{Z}^2_a) \to \ell(\mathbb{Z}^2)$ denotes extension by zeroes. We require that $H_a$ also satisfies 1, and that additionally, the edge operator satisfies the edge assumption 2.

**Assumption 2.** *The edge operator satisfies*

$$\sup_{z\in\mathbb{Z}^2} \sum_{y\in\mathbb{Z}^2_a} |E_a(x,y)| e^{\alpha(|x_2+a|+|x_1-y_1|)} \leq C < \infty$$

*for some $\alpha > 0$, where $|x| = |x_1| + |x_2|$ is the taxicab metric.*

We observe some properties of the extension operator. Both $\mathcal{J}_a$ and its adjoint have norm 1. If $A$ is an operator on $\ell^2(\mathbb{Z}^2_a)$, then $\mathcal{J}_a A \mathcal{J}_a^*$ extends $A$ by zeroes to an operator on $\ell^2(\mathbb{Z}^2)$. Furthermore, if $A$ is bounded, then $\|\mathcal{J}_a A \mathcal{J}_a^*\| = \|A\|$, and if $A$ is trace-class, then $\mathrm{Tr}(\mathcal{J}_a A \mathcal{J}_a^*) = \mathrm{Tr}(A)$. Finally, the extensions are related to the identities of the Hilbert spaces by

$$\mathcal{J}_a^* \mathcal{J}_a = \mathbb{1}_{\ell^2(\mathbb{Z}^2_a)},$$

$$\mathcal{J}_a \mathcal{J}_a^* \xrightarrow{s} \mathbb{1}_{\ell^2(\mathbb{Z}^2)}.$$

The interpretation of the edge operator $E_a = \mathcal{J}_a H_a - H_B \mathcal{J}_a$ is that it is the difference between first applying $H_a$ on $\ell^2(\mathbb{Z}^2_a)$ and then extending by zeroes, versus first making everything below $x_2 = -a$ into zeroes and the applying $H_B$. The assumption ensures that the effects from introducing the edge at $-a$ die exponentially as we move upward away from the edge (due to the $|x_2 - (-a)|$ term in the exponent), and also that terms do not interact too much as their $x_1$ distance increases (due to the $|x_1 - y_1|$ term in the exponent).

A simple example of an edge Hamiltonian satisfying the edge condition is $H_a = \mathcal{J}_a^* H_B \mathcal{J}_a$, which gives $E_a = (\mathcal{J}_a \mathcal{J}_a^* - \mathbb{1}) H_B \mathcal{J}_a$. The matrix elements are

$$E_a(x,y) = \begin{cases} -H_B(x,y) & \text{if } x_2 < -a \\ 0 & \text{if } x_2 \geq -a \end{cases}$$

Intuitively, there is no difference between $H_B$ and $H_a$ on $\mathbb{Z}_a^2$. The idea is that for a state $\psi \in \ell^2(\mathbb{Z}_a^2)$, we have $\langle \psi, H_a \psi \rangle = \langle (\mathcal{J}_a \psi), H_B(\mathcal{J}_a \psi) \rangle$, which we interpret as the edge Hamiltonian having the same expectation as the bulk Hamiltonian for states supported above $x_2 = -a$. The edge assumption 2 is satisfied since $|E_a(x, y)| \leq |H_B(x, y)|$, and $H_B$ obeys the short range assumption 1.

We also make the following assumption about the spectrum of the bulk and edge Hamiltonians.

**Assumption 3.** *1. Both $H_a$ and $H_B$ have bounded spectrum.*

*2. The bulk Hamiltonian $H_B$ has a spectral gap. That is, there exists an interval $\Delta \subset \mathbb{R}$ such that*

$$\Delta \cap \operatorname{Spec}(H_B) = \varnothing.$$

*Remark*: The spectral gap assumption can be relaxed to a "mobility gap" assumption,

$$\sup_{f \in B_c(\Delta)} |f(H_B)(x, y)|(1 + |x|)^{-\alpha_1} e^{\alpha_2 |x-y|} < \infty$$

for some $\alpha_1 > 0$, where $B_c(\Delta)$ is the set of Borel functions $f$ which are constant on $(-\infty, \inf \Delta)$ and on $(\sup \Delta, \infty)$ such that $|f(x)| \leq 1$ for all $x$ [1].

We define the *bulk conductivity* at Fermi energy $\lambda$ as follows. Suppose we subject the system to an external electric potential $V(x_2)$ in the $x_2$ direction. We write this as $V_0 \Lambda_2$, where $\Lambda_i$ are multiplication operators

$$\Lambda_i |\psi(x_1, x_2)\rangle = \Lambda(x_i) |\psi(x_1, x_2)\rangle$$

where

$$\Lambda : \mathbb{R} \to \mathbb{R} \qquad\qquad \Lambda(x_i) = \begin{cases} 1 & \text{if } x_i \leq 0 \\ 0 & \text{if } x_i \geq 1 \end{cases}$$

is called a *switch functions*, which is smooth and monotonically decreasing on $(0, 1)$. Note that the ensuing physics (in particular, our definition of the Hall conductivity) is independent of the particular choice of switch function $\Lambda_i$, since any two switch functions are exactly equal on the lattice.

We introduce a function which grows slowly in time as $t$ grows from $-\infty$ to 0, so as to invoke the adiabatic principle later. We choose $e^{\varepsilon t}$, and we will let $\varepsilon \to 0$ at the end. The perturbation is $\Delta H_B(t) = V_0\Lambda_2 e^{\varepsilon t}$, and the perturbed Hamiltonian is

$$\widetilde{H}_B(t) = H_B + V_0\Lambda_2 e^{\varepsilon t}.$$

We define the Hall current operator $J_H = -i[\widetilde{H}_B(t), \Lambda_1] = -i[H_B, \Lambda_1]$, which measures the current across the fiducial line $x_1 = 0$.

We also denote by $P_\lambda := P((-\infty, \lambda])$ the projection-valued measure associated with $H_B$ onto states with energy below the Fermi energy $\lambda$, which is assumed to lie in the spectral gap $\Delta$ (see Appendix A.1).

**Proposition 1.** *The Hall conductivity $\sigma_H$ in the bulk system is equal to*

$$\sigma_B = -i\mathrm{Tr}\left(P_\lambda\left[[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]\right]\right).$$

*Proof.* We begin with the Heisenberg equation of motion for the density matrix, $\dot{\rho}(t) = -i[\widetilde{H}_B(t), \rho(t)]$, with initial condition

$$\lim_{t\to-\infty} \|\rho(t) - e^{itH_B} P_\lambda e^{-itH_B}\| = 0,$$

which is equivalent to $\lim_{t\to-\infty} \|e^{-itH_B}\rho(t)e^{itH_B} - P_\lambda\| = 0$. We work in the interaction picture by defining $\rho_I(t) = e^{-itH_B}\rho(t)e^{itH_B}$ and $\Delta H_B(t) = e^{-itH_B}V_0\Lambda_2 e^{\varepsilon t}e^{itH_B}$. Thus

$$\dot{\rho}_I(t) = -i[\Delta H_B(t), \rho_I(t)].$$

Integrating both sides,

$$\rho_I(t) = i\int_{-\infty}^{t} [\Delta H_B(s), \rho_I(s)]ds + P_\lambda.$$

Indeed, taking the derivative of the right hand side gives $i[\Delta H_B(t), \rho_I(t)]$, and the constant $P_\lambda$ is added so that the initial condition $\|\rho_I(t) - P_\lambda\| \to 0$ is also satisfied. This also shows that

$$\|\rho_I(t) - P_\lambda\| \le 2\int_{-\infty}^{t} \|V_0\Lambda_2 e^{\varepsilon s}\rho_I(t)\|ds = e^{\varepsilon t}\mathcal{O}(V_0),$$

which implies $[\Delta H_B(s), \rho_I(t)] = [\Delta H_B(s), P_\lambda] + e^{2\varepsilon t}\mathcal{O}(V_0^2)$. Therefore

$$\rho_I(t) = i\int_{-\infty}^{t} [\Delta H_B(s), P_\lambda]ds + P_\lambda + e^{2\varepsilon t}\mathcal{O}(V_0^2).$$

The Hall conductivity is (by definition) the linear response coefficient, which is

$$\begin{aligned}\sigma_H &= \lim_{V_0\to 0}\lim_{\varepsilon\to 0}\frac{\mathrm{Tr}(\rho_I(0)J_H)-\mathrm{Tr}(P_\lambda J_H)}{V_0}\\ &= -\lim_{V_0\to 0}\lim_{\varepsilon\to 0}\frac{i}{V_0}\mathrm{Tr}\left(i\int_{-\infty}^{0}[\Delta H_B(s),P_\lambda][H_B,\Lambda_1]ds\right),\end{aligned}$$

where we used the fact that the error $e^{2\varepsilon t}\mathcal{O}(V_0^2)$ introduced by replacing $\rho_I$ with $P_\lambda$ vanishes in the limit $V_0 \to 0$ at $t = 0$. Some simplifications can be made,

$$\begin{aligned}\sigma_H &= \lim_{V_0\to 0}\lim_{\varepsilon\to 0}\frac{1}{V_0}\mathrm{Tr}\left(\int_{-\infty}^{0}[e^{-isH_B}V_0\Lambda_2 e^{\varepsilon s}e^{isH_B},P_\lambda][H_B,\Lambda_1]ds\right)\\ &= \lim_{V_0\to 0}\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}e^{-isH_B}[\Lambda_2,P_\lambda]e^{isH_B}[H_B,\Lambda_1]e^{\varepsilon s}ds\right)\\ &= \lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}(e^{isH_B}[H_B,\Lambda_1]e^{-isH_B})\cdot([\Lambda_2,P_\lambda]e^{\varepsilon s})ds\right),\end{aligned}$$

where we used cyclicity of the trace and the fact that $P_\lambda$ and $H_B$ commute. We also dropped the limit $V_0 \to 0$ in the final line because, even though the limits may not commute in general, the expression is independent of $V_0$. Using integration by parts on the two terms in brackets, and noting that $\frac{d}{ds}(e^{isH_B}\Lambda_1 e^{-isH_B}-\Lambda_1) = ie^{isH_B}[H_B,\Lambda_1]e^{-isH_B}$, we obtain

$$\begin{aligned}\sigma_H &= -i\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}(e^{isH_B}\Lambda_1 e^{-isH_B}-\Lambda_1)\frac{d}{ds}([\Lambda_2,P_\lambda]e^{\varepsilon s})ds\right)\\ &= -i\lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0}\Lambda_1^s[\Lambda_2,P_\lambda]e^{\varepsilon s})ds\right)\end{aligned}$$

where $\Lambda_1^s := e^{isH_B}\Lambda_1 e^{-isH_B}-\Lambda_1$. Using the notation $\overline{A} := P_\lambda A P_\lambda^\perp + P_\lambda^\perp A P_\lambda$, it is readily verified that the commutator $[\Lambda_2, P_\lambda]$ is an *off-diagonal* operator, in the sense that $[\Lambda_2,P_\lambda] = \overline{[\Lambda_2,P_\lambda]}$. Furthermore, a simple computation reveals that for any two trace-class operators $A$ and $B$, $\mathrm{Tr}(\overline{A}B) = \mathrm{Tr}(A\overline{B})$. It therefore follows that

$$\sigma_H = -i\lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0}\overline{\Lambda_1^s}[\Lambda_2,P_\lambda]e^{\varepsilon s})ds\right).$$

The integrand can be broken into two terms,

$$\overline{\Lambda_1^s}[\Lambda_2, P_\lambda]e^{\varepsilon s} = e^{isH_B}\overline{\Lambda_1}e^{-isH_B}[\Lambda_2, P_\lambda]e^{\varepsilon s} - \overline{\Lambda_1}[\Lambda_2, P_\lambda]e^{\varepsilon s}$$

by commutativity of $P_\lambda$ and $H_B$. We show that the integral of the first term vanishes. We begin by breaking the first term down further into

$$e^{isH_B}P_\lambda\Lambda_1 P_\lambda^\perp e^{-isH_B}[\Lambda_2, P_\lambda]e^{\varepsilon s} + e^{isH_B}P_\lambda^\perp\Lambda_1 P_\lambda e^{-isH_B}[\Lambda_2, P_\lambda]e^{\varepsilon s}.$$

We treat the first of these two terms; the other is handled in an identical manner. We invoke the spectral theorem (Appendix A.1) to write $e^{isH_B}P_\lambda = \int_{-\infty}^{\lambda} e^{is\xi}dP_\xi$, and similarly $P_\lambda^\perp e^{-isH_B} = (\mathbb{1} - P_\lambda)e^{-isH_B} = \int_\lambda^\infty e^{-is\nu}dP_\nu$.

Since the Fermi energy $\lambda$ is assumed to lie in a spectral gap, there must exist a neighbourhood $(\lambda-\delta, \lambda+\delta)$ in which there are no states. We exploit this fact to rewrite the limits of integration, $\int_{-\infty}^{\lambda-\delta} e^{is\xi}dP_\xi$ and $\int_{\lambda+\delta}^{\infty} e^{-is\nu}dP_\nu$. We therefore obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{0} e^{isH_B}P_\lambda\Lambda_1 P_\lambda^\perp e^{-isH_B}[\Lambda_2, P_\lambda]e^{\varepsilon s}ds \\
&\qquad = \lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0}\int_{-\infty}^{\lambda-\delta} e^{is\xi}dP_\xi\Lambda_1\int_{\mu+\delta}^{\infty} e^{-is\nu}dP_\nu[\Lambda_2, P_\lambda]e^{\varepsilon s}ds\right) \\
&\qquad = \lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0}\int_{-\infty}^{\lambda-\delta}\int_{\lambda+\delta}^{\infty} e^{s(\varepsilon-i\nu+i\xi)}dP_\xi\Lambda_1 dP_\nu[\Lambda_2, P_\lambda]ds\right)
\end{aligned}$$

Performing the integral over $s$ yields

$$\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{0} e^{s(\varepsilon-i\nu+i\xi)}ds = -\lim_{\varepsilon\to 0}\frac{\varepsilon}{i\varepsilon+\nu-\xi}$$

This limit is zero, since $\xi \neq \nu$. Indeed, due to the spectral gap, the integration variables live in $\xi \in (-\infty, \lambda-\delta)$ and $\nu \in (\lambda+\delta, \infty)$. The case for the $e^{isH_B}P_\lambda^\perp\Lambda_1 P_\lambda e^{-isH_B}[\Lambda_2, P_\lambda]e^{\varepsilon s}$ term (where the $P_\lambda$ and $P_\lambda^\perp$ swap places) is treated analogously. Hence the first term in the integrand for $\sigma_H$ vanishes, as claimed.

Finally, we return to our expression for the Hall conductivity, which now reads

$$\sigma_H = i\lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0}\overline{\Lambda_1}[\Lambda_2, P_\lambda]e^{\varepsilon s}ds\right).$$

It is a basic algebraic calculation to show that $\overline{\Lambda_1} = [[\Lambda_1, P_\lambda], P_\lambda]$. Evaluating the integral over $s$ is now trivial; $\int_{-\infty}^{0} e^{\varepsilon s}ds = \varepsilon^{-1}$. Thus

$$\sigma_H = i\mathrm{Tr}([[\Lambda_1, P_\lambda], P_\lambda][\Lambda_2, P_\lambda]).$$

Shifting the commutator completes the proof:

$$\begin{aligned}\sigma_H &= i\mathrm{Tr}(P_\lambda[[\Lambda_2, P_\lambda], [\Lambda_1, P_\lambda]]) \\ &= -i\mathrm{Tr}(P_\lambda[[\Lambda_1, P_\lambda], [\Lambda_2, P_\lambda]]) \\ &= -i\mathrm{Tr}(P_\lambda[[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]]).\end{aligned}$$

The justification for shifting the commutator is that $[\Lambda_1, P_\lambda][\Lambda_2, P_\lambda]$ is trace class by the proof of Lemma 4, noting that

$$\|[\Lambda_1, P_\lambda][\Lambda_2, P_\lambda]\|_1 \le \|[\Lambda_1, P_\lambda]e^{3\delta|x_1|}e^{-\delta|x|}\| \cdot \|e^{-\delta|x|}\|_1 \cdot \|e^{3\delta|x_2|}e^{-\delta|x|}[\Lambda_2, P_\lambda]\|$$

where $|x| = |x_1| + |x_2|$, and the $e^{-\delta|x|}$ term is trace-class by lemma 2. □

For the *edge conductivity*, we again need the current operator across the line $x = 0$, which is this time given by $-i[H_a, \Lambda_1]$. We define

$$\sigma_E(a) = -i \lim_{a\to\infty} \mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2),$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(r) = \begin{cases} 1 & \text{if } r \le \inf \Delta \\ 0 & \text{if } r \ge \sup \Delta \end{cases}$$

and decreases smoothly and monotonically in $\Delta$. The definition of $\sigma_E$ is reminiscent of another formula we will see later in the interacting setting, $\mathrm{Tr}(\dot{P}J)$, where $J$ is the current operator. The interpretation of $\sigma_E$ is that if we apply a small potential difference $V$ across $x_2 = -a$ to $x_2 = \infty$, there will be a net Hall current

$$\begin{aligned}I &= -i\mathrm{Tr}(\rho(H_a + V)[H_a + V, \Lambda_1] - \rho(H_a)[H_a, \Lambda_1]) \\ &= -i\mathrm{Tr}((\rho(H_a + V) - \rho(H_a))[H_a, \Lambda_1]).\end{aligned}$$

Thus we obtain the conductivity

$$\frac{I}{V} = -i\mathrm{Tr}\left(\frac{(\rho(H_a + V) - \rho(H_a)}{V}[H_a, \Lambda_1]\right) \to -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1])$$

in the limit as $V \to 0$. The operator $\Lambda_2$ is inserted into the trace so that we are measuring the current travelling in the $x$-direction along the strip $(-\infty, \infty) \times [-a, 0]$. The *edge Hall conductivity* is taken to be the limit

$$\sigma_E = \lim_{a \to \infty} \sigma_E(a).$$

As we shall see, it turns out that $\sigma_E$ is independent of the choice of $\rho$, and $\sigma_B$ is independent of $\lambda$, so long as $\lambda \in \Delta$.

## 2.2 Noninteracting Bulk-Edge Correspondence

The main result of this section is

**Theorem 1.** $\sigma_E = \sigma_B$.

### 2.2.1 Outline of the Proof

Before giving the proof in its entirety, we outline the basic steps. The key ingredient of the proof is the use of the functional calculus given by the Helffer-Sjöstrand representation of operators on a Hilbert space (Appendix C). The two crucial operators for the proof written in their Helffer-Sjöstrand representations are

$$\rho(H) = -\frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R(z) dz \wedge d\bar{z}$$
$$\rho'(H) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R(z)^2 dz \wedge d\bar{z}$$

where $R(z) = (H - z)^{-1}$ is the resolvent of $H$. For ease of notation, we write $dz^2 = dz \wedge d\bar{z}$ from this point. Another key observation is that it turns out that the bulk conductivity can actually be written

$$\sigma_B = i\mathrm{Tr}([\rho(H_B), \Lambda_1]\Lambda_2).$$

By employing the Helffer-Sjostrand representations above, one can add an operator $Z$ of zero trace to the edge conductivity operator $\rho'(H_a)[H_a, \Lambda_1]\Lambda_2$, and show that their sum converges in trace norm to $[\rho(H_B), \Lambda_1]\Lambda_2$ in the limit $a \to \infty$.

### 2.2.2 The Proof

We begin with two key lemmas which will be used repeatedly throughout the proof.

**Lemma 1.**

$$\|[H_a, \Lambda_i]e^{\delta|x_i|}\| \leq C.$$

*Proof.* The operator can be bounded by inspecting its matrix elements

$$\begin{aligned}\langle x, [H_a, \Lambda_i]e^{\delta|x_i|}y\rangle &= \langle x, H_a\Lambda_i y\rangle e^{\delta|y_i|} - \langle x, \Lambda_i H_a y\rangle e^{\delta|y_i|} \\ &= H_a(x,y)e^{\delta|y_i|}(\Lambda(y_i) - \Lambda(x_i)).\end{aligned}$$

This is zero if $|x_i - y_i| \leq |y_i|$, since this would imply that $x_i$ and $y_i$ have the same sign, yielding $\Lambda(x_i) = \Lambda(y_i)$. So either the matrix element is zero, or $|y_i| \leq |x_i - y_i|$, which implies

$$\begin{aligned}|H_a(x,y)e^{\delta|y_i|}(\Lambda(y_i) - \Lambda(x_i))| &\leq 2|H_a(x,y)|e^{\delta|x_i - y_i|} \\ &\leq 2|H_a(x,y)|e^{\delta|x-y|} \\ &\leq C|H_a(x,y)|(e^{\delta|x-y|} - 1),\end{aligned}$$

where the final inequality comes from the fact that the diagonal matrix elements are zero. Since the Hamiltonian is self-adjoint, its matrix elements satisfy $|H_a(x,y)^*| = |H_a(y,x)|$, and thus the short range assumption 1 combined with Holmgren's bound 11 completes the proof.

□

**Lemma 2.** $e^{-\delta|x|}$ *is trace-class.*

*Proof.* We bound the trace norm by noticing that $e^{-\delta|x|} = e^{-\delta|x_1|}e^{-\delta|x_2|}$ is a positive operator satisfying

$$\langle (n,m), e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m)\rangle = \langle e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m), (n,m)\rangle,$$

so that its trace norm is equal to its trace. In the position basis, we see that its trace is bounded by a geometric series

$$\begin{aligned}\mathrm{Tr}(e^{-\delta|x_1|}e^{-\delta|x_2|}) &= \sum_{(n,m)\in\mathbb{Z}^2} \langle (n,m), e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m)\rangle \\ &\leq 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-\delta m}e^{-\delta n} \\ &= 2\left(\frac{1}{1-e^{-\delta}}\right)^2.\end{aligned}$$

□

Since we will be using the identity often, we also remark that

$$R(z)[H,\Lambda_i]R(z) = R(z)[H-z,\Lambda_i]R(z) = -[R(z),\Lambda_i].$$

We are now ready to begin the main argument. Define

$$Z(a) = [\rho(H_a),\Lambda_1]\Lambda_2 + \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[R_a(z),[H_a,\Lambda_1]\Lambda_2]dz^2.$$

**Lemma 3.** *The two terms in $Z(a)$ are separately trace-class, and the trace of each is zero.*

*Proof.* In the Helffer-Sjöstrand representation, the first term is

$$[\rho(H_a),\Lambda_1]\Lambda_2 = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}}[R_a(z),\Lambda_1]\Lambda_2 dz^2.$$

The operator in the integrand can be broken into five factors,

$$[R_a(z),\Lambda_1]\Lambda_2 = -R_a(z)\cdot[H_a,\Lambda_1]e^{\delta|x_1|}\cdot e^{-\delta|x|}\cdot e^{\delta|x_2|}R_a(z)e^{-\delta|x_2|}\cdot e^{\delta|x_2|}\Lambda_2.$$

The resolvent is bounded by $C|\mathrm{Im}(z)|^{-1}$, Lemma 1 bounds the second factor, Lemma 2 ensures the third is trace-class, Combes-Thomas 6 bounds the fourth by $C|\mathrm{Im}(z)|^{-1}$, and the fifth is bounded by 1 due to the switch function $\Lambda_2$. Finally, Lemma 15 guarantees integrability. Thus $[\rho(H_a),\Lambda_1]\Lambda_2$ is trace-class, and it clearly has vanishing trace since shifting the commutator gives $[\Lambda_1,\Lambda_2]=0$.

The second term of $Z(a)$ is also trace-class because $[H_a,\Lambda_1]\Lambda_2$ is, which can be seen by breaking it into

$$[H_a,\Lambda_1]\Lambda_2 = [H_a,\Lambda_1]e^{\delta|x_1|}\cdot e^{-\delta|x|}\cdot e^{\delta|x_2|}\Lambda_2.$$

Again, $\|e^{\delta|x_2|}\Lambda_2\|\le 1$ and Lemmas 1 and 2 are enough to conclude the operator is trace-class. The commutator in the integrand can therefore be shifted to give $[R_a(z),R_a(z)]=0$. $\square$

From Lemma 3, it follows that $\sigma_E(a) = \mathrm{Tr}(\Sigma(a))$, where

$$\begin{aligned}\Sigma(a) &= -i\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 + iZ(a)\\ &= -i\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 + i[\rho(H_a),\Lambda_1]\Lambda_2\\ &\qquad + \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\rho}}{\partial\bar{z}}R_a(z)[R_a(z),[H_a,\Lambda_1]\Lambda_2]dz^2.\end{aligned}$$

Using the Hellfer-Sjöstrand representations for the first two terms on the right hand side, we obtain

$$\begin{aligned}\Sigma(a) = -\frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)^2[H_a,\Lambda_1]\Lambda_2 dz^2 \\ + \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1]R_a(z)\Lambda_2 dz^2 \\ + \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} (R_a(z)^2[H_a,\Lambda_1]\Lambda_2 - R_a(z)[H_a,\Lambda_1]\Lambda_2 R_a(z)) dz^2 \\ = -\frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1][R_a(z),\Lambda_2] dz^2 \\ = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1]R_a(z)[H_a,\Lambda_2]R_a(z) dz^2,\end{aligned}$$

where we used

$$[R_a(z),\Lambda_i] = -R_a(z)[H_a,\Lambda_i]R_a(z)$$

in the first and final equality. The goal is to prove that the operator $\Sigma(a)$ converges to the corresponding bulk operator in trace-norm,

$$\|\mathcal{J}_a\Sigma(a)\mathcal{J}_a^* - \Sigma_B\|_1 \to 0$$

as $a \to \infty$, which in turn proves that $\mathrm{Tr}(\Sigma(a)) \to \mathrm{Tr}(\Sigma_B)$ because of the bound $|\mathrm{Tr}(A)| \leq \|A\|_1$. Here, $\Sigma_B$ is the same operator as before, but defined using the bulk operators $H_B$ and $R_B(z)$. Once this limit is established, we shall prove that $\sigma_B = \mathrm{Tr}(\Sigma_B)$ to conclude the proof.

To show that the limit is zero as claimed, we bound the integrand of $\mathcal{J}_a\Sigma(a)\mathcal{J}_a^*$ in trace norm with the ultimate goal of applying dominated convergence to bring the limit inside the integral. We accomplish this bound by breaking the integrand into three parts,

$$\begin{aligned}\mathcal{J}_a R_a[H_a,\Lambda_1]R_a[H_a,\Lambda_2]R_a\mathcal{J}_a^* = -\mathcal{J}_a[R_a,\Lambda_1]e^{\delta|x_1|}\mathcal{J}_a^* \cdot e^{-\delta|x|} \\ \cdot \mathcal{J}_a e^{\delta|x_2|}[H_a,\Lambda_2]R_a\mathcal{J}_a^*,\end{aligned}$$

and bounding the norm of each. We remark that the extension $\mathcal{J}_a$ and its adjoint have norm 1. For the first factor, $\mathcal{J}_a[R_a(z),\Lambda_1]e^{\delta|x_1|}\mathcal{J}_a^*$, we bound its operator norm by breaking it down further into

$$\begin{aligned}\|\mathcal{J}_a[R_a(z),\Lambda_1]e^{\delta|x_1|}\mathcal{J}_a^*\| &= \|[R_a(z),\Lambda_1]e^{\delta|x_1|}\| \\ &= \|-R_a(z)[H_a,\Lambda_1]R_a(z)e^{\delta|x_1|}\| \\ &\le \|R_a(z)\|\cdot\|[H_a,\Lambda_1]e^{\delta|x_1|}\|\cdot\|e^{-\delta|x_1|}R_a(z)e^{\delta|x_1|}\|.\end{aligned}$$

The norm of the resolvent is bounded by $\|R_a(z)\| \le C|\mathrm{Im}(z)|^{-1}$. The middle operator is bounded by Lemma 1. For the operator on the right, we apply the Combes-Thomas bound 6. Altogether, the bound of the first term of the integrand takes the form

$$\frac{C}{\mathrm{Im}(z)^2}.$$

The second factor of the integrand, $e^{-\delta|x|}$, is trace-class by Lemma 2. Finally, the bound for the third factor of the integrand, $e^{\delta|x_2|}[H_a,\Lambda_2]R_a(z)$, follows from the bound on the resolvent and Lemma 1, and is again of the form $\frac{C}{\mathrm{Im}(z)^2}$.

Altogether, the trace norm of the integrand is bounded by

$$\|R_a(z)[H_a,\Lambda_1]R_a(z)[H_a,\Lambda_2]R_a(z)\|_1 \le \frac{C}{\mathrm{Im}(z)^4},$$

so Lemma 15 provides the necessary bound for dominated convergence of $\Sigma(a)$. It therefore suffices to show that the integrand of $\Sigma(a)$ converges pointwise in $z$ to the integrand of $\Sigma_B$ as $a \to \infty$. In particular, we will show

$$\mathcal{J}_a[R_a(z),\Lambda_1]e^{\delta|x_1|}\mathcal{J}_a^* \xrightarrow{s} [R_B(z),\Lambda_1]e^{\delta|x_1|},$$

$$\mathcal{J}_a e^{\delta|x_2|}[H_a,\Lambda_2]\mathcal{J}_a^* \xrightarrow{s} e^{\delta|x_2|}[H_B,\Lambda_2],$$

and

$$\mathcal{J}_a R_a(z)\mathcal{J}_a^* \xrightarrow{s} R_B(z)$$

for each fixed $z \in \mathbb{C}$. Inspecting the bounds we found for the left hand sides of these limits, it is clear that they are in fact uniformly bounded in $a$. By Lemma 10, it therefore suffices to show convergence on a dense subspace of $\ell^2(\mathbb{Z}^2)$; in particular, we may choose the dense subspace of compactly supported states, which allows us to ignore the $e^{\delta|x_i|}$ terms. Thus, we need to prove

$$\mathcal{J}_a[R_a(z),\Lambda_1]\mathcal{J}_a^* \xrightarrow{s} [R_B(z),\Lambda_1],$$

$$\mathcal{J}_a[H_a,\Lambda_2]\mathcal{J}_a^* \xrightarrow{s} [H_B,\Lambda_2],$$

and

$$\mathcal{J}_a R_a(z)\mathcal{J}_a^* \xrightarrow{s} R_B(z).$$

In fact, the final statement implies the first two; we appeal to the general fact of functional analysis that strong convergence of the resolvent of the self-adjoint operator $H_a$ implies that $\mathcal{J}_a f(H_a)\mathcal{J}_a^* \xrightarrow{s} f(H_B)$ for any bounded and continuous function $f$ ([16] Theorem VIII.20). In particular, it follows from Lemma 1 that the functions $[(\cdot - z)^{-1},\Lambda_1]$ and $[\cdot,\Lambda_2]$ above are bounded and continuous, so we will have proven the desired limits if we can prove strong convergence of the resolvent, $\mathcal{J}_a R_a(z)\mathcal{J}_a^* \xrightarrow{s} R_B(z)$.

To prove this, we use the edge assumption. Recall the edge operator, $E_a = \mathcal{J}_a H_a - H_B\mathcal{J}_a$. Adding and subtracting $z\mathcal{J}_a$ gives $E_a = \mathcal{J}_a(H_a - z) - (H_B - z)\mathcal{J}_a$, and applying $R_B$ from the left and $R_a$ from the right on both sides yields

$$R_B(z)E_a R_a(z) = R_B(z)\mathcal{J}_a - \mathcal{J}_a R_a(z).$$

Taking the adjoint and then multiplying from the left by $\mathcal{J}_a$, we see that

$$\mathcal{J}_a R_a(z)E_a^* R_B(z) = \mathcal{J}_a\mathcal{J}_a^* R_B(z) - \mathcal{J}_a R_a(z)\mathcal{J}_a^*.$$

Thus

$$R_B(z) - \mathcal{J}_a R_a(z)\mathcal{J}_a^* = (\mathcal{J}_a R_a(z)E_a^* - \mathcal{J}_a\mathcal{J}_a^* + \mathbb{1}_{\ell^2(\mathbb{Z}^2)})R_B(z) \xrightarrow{s} 0,$$

since $E_a^* \xrightarrow{s} 0$ by Lemma 5, and $-\mathcal{J}_a\mathcal{J}_a^* + \mathbb{1}_{\ell^2(\mathbb{Z}^2)} \xrightarrow{s} 0$. This proves that the three limits above converge to the desired associated bulk operators, and hence $\|\mathcal{J}_a\Sigma(a)\mathcal{J}_a^* - \Sigma_B\|_1 \to 0$. This in turn shows that $\mathrm{Tr}(\Sigma(a)) = \mathrm{Tr}(\mathcal{J}_a\Sigma(a)\mathcal{J}_a^*) \to \mathrm{Tr}(\Sigma_B)$, and also that $\Sigma_B$ is trace-class.

Finally, it remains to show

$$\mathrm{Tr}(\Sigma_B) = \sigma_B.$$

First, we manipulate

$$\begin{aligned}
\Sigma_B &= \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_B(z)[H_B,\Lambda_1]R_B(z)[H_B,\Lambda_2]R_B(z)dz^2 \\
&= -\frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_B(z)[H_B,\Lambda_1][R_B(z),\Lambda_2]dz^2 \\
&= i[\rho(H_B),\Lambda_1]\Lambda_2 - \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_B(z)[H_B,\Lambda_1]\Lambda_2 R_B(z)dz^2.
\end{aligned}$$

Define $P_+ := P((\sup\Delta,\infty))$ and $P_- := P((-\infty,\inf\Delta))$, the projections onto states above and below the gap, respectively. Since $H_B$ is assumed to have a gap, it follows that $P_- + P_+ = \mathbb{1}$, and thus

$$\mathrm{Tr}(\Sigma_B) = \mathrm{Tr}((P_- + P_+)\Sigma_B(P_- + P_+)) = \mathrm{Tr}(P_+\Sigma_B P_+) + \mathrm{Tr}(P_-\Sigma_B P_-),$$

where we've used the fact that $\Sigma_B$ is trace-class to cycle the projections and get rid of the $P_-P_+ = 0$ terms. We now argue that the integral term in $P_\pm\Sigma_B P_\pm$ vanishes.

The integrand is analytic on all of $\mathbb{C}$. Indeed, $[H_B,\Lambda_1]\Lambda_2$ is analytic everywhere. Consider $P_+R_B$ and $R_BP_+$. The resolvent is analytic away from the real line, and $P_+$ is equal to 0 on $(-\infty,\sup\Delta)$, so $P_-R_B$ and $R_BP_-$ are analytic on $\mathbb{C}\setminus(\sup\Delta,\infty)$. However, the integrand is zero on $(\sup\Delta,\infty)$, since $\tilde{\rho}=0$ there. The integrand as a whole is therefore analytic on the entire complex plane, and so the integral can be written

$$\int_{\mathbb{C}} \frac{\partial}{\partial\bar{z}}\left(\tilde{\rho}(z)P_+R_B(z)[H_B,\Lambda_1]\Lambda_2R_B(z)P_+\right)dz^2,$$

with the derivative $\frac{\partial}{\partial\bar{z}}$ acting on the entire integrand rather than only on $\tilde{\rho}(z)$. Furthermore, since $R_B(z)$ is bounded in norm by $|\mathrm{Im}(z)|^{-1}$ and $|\partial_{\bar{z}}\tilde{\rho}|$ can be made to decay with any power law $\mathcal{O}(\mathrm{Im}(z)^k)$ as $\mathrm{Im}(z)\to 0$ (see Appendix C), we may write the integral as a limit of disks,

$$\lim_{r\to\infty}\int_{B_r(0)} \frac{\partial}{\partial\bar{z}}(\tilde{\rho}(z)P_+R_B(z)[H_B,\Lambda_1]\Lambda_2R_B(z)P_+)dz^2,$$

where domination is provided by Lemma 15. Stokes' theorem then says that the above integral is bounded in norm by

$$\lim_{r\to\infty}\int_{\partial B_r(0)} \tilde{\rho}(z)\|P_+R_B(z)[H_B,\Lambda_1]\Lambda_2R_B(z)P_+\|dz.$$

The decay of $\|R_B(z)\| \leq \mathrm{dist}(z, \mathrm{Spec}(H_B))^{-1}$ and boundedness of the extension $\tilde{\rho}(z)$ (since $|\rho(x)| \leq 1$) ensure that the limit of this integral as $r \to \infty$ is zero. Indeed, the norm of the integral is bounded by

$$2\pi r \sup_{\partial B_r(0)} \frac{|\tilde{\rho}(r)| \|[H_B, \Lambda_1]\Lambda_2\|}{\mathrm{dist}(r, \mathrm{Spec}(H_B))^2}$$

which vanishes as $r \to \infty$ since the spectrum of the bulk Hamiltonian is bounded, and Lemma 1 bounds the norm of the operator in the numerator.

As for the integral appearing in $P_-\Sigma_B P_-$, we note that analogous to the argument above, $R_B P_-$ and $P_- R_B$ are analytic on $\mathbb{C} \setminus (-\infty, \inf \Delta)$, and thus $(1-\tilde{\rho})R_B P_-$ and $(1-\tilde{\rho})P_- R_B$ are analytic on all of $\mathbb{C}$, since $1-\tilde{\rho}$ is zero on $(-\infty, \inf \Delta)$. Replacing $\frac{\partial \tilde{\rho}}{\partial \bar{z}}$ with $\frac{\partial(1-\tilde{\rho})}{\partial \bar{z}}$ of course leaves the integral unchanged, and the rest of the argument follows as before. Thus the integral term in $P_\pm \Sigma_B P_\pm$ is zero, and

$$\mathrm{Tr}(\Sigma_B) = i\mathrm{Tr}(P_+[\rho(H_B), \Lambda_1]\Lambda_2 P_+) + i\mathrm{Tr}(P_-[\rho(H_B), \Lambda_1]\Lambda_2 P_-).$$

By the spectral theorem for projection-valued measures 3, if the Fermi energy lies in the gap, $\lambda \in \Delta$, we have

$$\rho(H_B) = \int_{-\infty}^{\lambda} dP_\nu = P_\lambda.$$

We may therefore replace $\rho(H_B)$ with $P_\lambda$, by which we obtain $\mathrm{Tr}(\Sigma_B) = i\mathrm{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+) + i\mathrm{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-)$. We must relate this expression to the bulk conductivity. An algebraic calculation shows that that

$$P_\lambda[[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]] = P_\lambda \Lambda_2 P_\lambda^\perp \Lambda_1 P_\lambda - P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda.$$

The two terms on the right hand side are separately trace-class by Lemma 4, so that by cyclicity of the trace, the bulk conductivity is given by

$$\begin{aligned}\sigma_B &= -i\mathrm{Tr}(P_\lambda \Lambda_2 P_\lambda^\perp \Lambda_1 P_\lambda - P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda) \\ &= i\mathrm{Tr}(P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda - P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_\lambda^\perp) \\ &= \mathrm{Tr}(T_\lambda),\end{aligned}$$

Where we've defined $T_\lambda := P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda - P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_\lambda^\perp$. To finish the proof, we must show that

$$P_\pm T_\lambda P_\pm = P_\pm [P_\lambda, \Lambda_1]\Lambda_2 P_\pm.$$

First, notice that because of the gap, we have $P_\lambda^\perp P_- = 0$ and $P_\lambda P_- = P_-$. Thus

$$\begin{aligned} P_- T_\lambda P_- &= P_- P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda P_- \\ &= P_-(P_\lambda \Lambda_1 \Lambda_2 - \Lambda_1 P_\lambda \Lambda_2) P_- \\ &= P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-, \end{aligned}$$

as desired. By a similar argument, for $P_+$ we have $P_\lambda^\perp P_+ = P_+$ and $P_\lambda P_- = 0$, which implies

$$\begin{aligned} P_+ T_\lambda P_+ &= -P_+ P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_+ \\ &= -P_+ P_\lambda^\perp \Lambda_1 \Lambda_2 P_+ + P_+ P_\lambda^\perp \Lambda_1 P_\lambda^\perp \Lambda_2 P_+ \\ &= -P_+ P_\lambda^\perp \Lambda_1 \Lambda_2 P_+ + P_+ \Lambda_1 P_\lambda^\perp \Lambda_2 P_+ \\ &= -P_+[P_\lambda^\perp, \Lambda_1]\Lambda_2 P_+ \\ &= P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+. \end{aligned}$$

Altogether, we obtain

$$\begin{aligned} \sigma_B &= i\mathrm{Tr}(P_- T_\lambda P_-) + i\mathrm{Tr}(P_+ T_\lambda P_+) \\ &= i\mathrm{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-) + i\mathrm{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+) \\ &= \mathrm{Tr}(\Sigma_B), \end{aligned}$$

concluding the proof.

**Lemma 4.** *Let $W \cap V = \varnothing$. Then*

$$P_W \Lambda_1 P_V \Lambda_2 P_W \in \mathcal{J}_1.$$

*In particular, the two terms appearing in $T_\lambda$ are separately trace-class.*

*Proof.* We break down

$$P_W \Lambda_1 P_V \Lambda_2 P_W = P_W \Lambda_1 P_V e^{3\delta|x_1|} e^{-\delta|x|} \cdot e^{-\delta|x|} \cdot e^{-\delta|x|} e^{3\delta|x_2|} P_V \Lambda_2 P_W.$$

Since the middle factor is trace-class by Lemma 2, it suffices to prove boundedness of the first and last factors. We begin with the first; the last is proved in an analogous manner. We use the fact that $P_W \Lambda_i P_V = P_W[\Lambda_i, P_V]$. Since $P_W$ is obviously bounded, we only need to obtain a bound for $[\Lambda_i, P_V] e^{3\delta|x_1|} e^{-\delta|x|}$. To that end, we decompose

$$[\Lambda_i, P_V] e^{3\delta x_i} e^{-\delta|x|} = \Lambda_i P_V (\mathbb{1} - \Lambda_i) e^{3\delta x_i} e^{-\delta|x|} - (\mathbb{1} - \Lambda_i) P_V \Lambda_i e^{3\delta x_i} e^{-\delta|x|}.$$

Note the lack of absolute values on the $e^{3\delta x_i}$ terms. We will later also bound the same operator but with $e^{-3\delta x_i}$ to account for this, since $e^{3\delta|x_i|} = e^{\pm 3\delta x_i}$. Since multiplication operators commute, the second term can be bounded by

$$\|(\mathbb{1} - \Lambda_i)P_V e^{-\delta|x|}\|\|\Lambda_i e^{3\delta x_i}\|,$$

both of which are bounded, since only the negative $x_i$ values remain in $\Lambda_i e^{3\delta x_i}$ due to the switch function. For the first term, we insert more exponentials

$$\Lambda_i P_V(\mathbb{1} - \Lambda_i)e^{3\delta x_i}e^{-\delta|x|} = \Lambda_i e^{3\delta x_i} \cdot e^{-3\delta x_i}P_V(\mathbb{1} - \Lambda_i)e^{3\delta x_i}e^{-\delta|x|}.$$

Again $\Lambda_i e^{3\delta x_i}$ is bounded because of the switch function, and the other part is seen to be bounded by taking the adjoint of the operator in Lemma 14.

A similar technique works for bounding $[\Lambda_i, P_V]e^{-3\delta x_i}e^{-\delta|x|}$, i.e. the same operator but with the $e^{-3\delta x_i}$ terms:

$$\begin{aligned}[\Lambda_i, P_V]e^{-3\delta x_i}e^{-\delta|x|} = {} & \Lambda_i P_V e^{-\delta|x|} \cdot (\mathbb{1} - \Lambda_i)e^{3\delta x_i} \\ & - (\mathbb{1} - \Lambda_i)e^{-3\delta x_i} \cdot e^{3\delta x_i}P_V\Lambda_i e^{-3\delta x_i}e^{-\delta|x|}.\end{aligned}$$

Each term is bounded by the same arguments as above. Finally, the last factor $e^{-\delta|x|}e^{3\delta|x_2|}P_V\Lambda_2 P_W$ is bounded by repeating exactly the same argument, but beginning this time from $P_W\Lambda_i P_V = [P_W, \Lambda_i]P_V$.

□

**Lemma 5.** *$E_a$ and $E_a^*$ converge in norm to zero in the limit $a \to \infty$.*

*Proof.* We use Holmgren's bound 11 to obtain

$$\|E_a\| \leq \max\{\sup_x \sum_y |E_a(x,y)|, \sup_y \sum_x |E_a(x,y)|\}.$$

There are two cases. First, if the maximum is $\sup_x \sum_y |E_a(x,y)|$, we have

$$\begin{aligned}
\|E_a\| &\leq \sup_x \sum_y |E_a(x,y)| \\
&= \sup_x \sum_y |E_a(x,y)| e^{\alpha(|x_2+a|+|x_1-y_1|)} e^{-\alpha(|x_2+a|+|x_1-y_1|)} \\
&\leq e^{-\alpha|a|} \sup_x \sum_y |E_a(x,y)| e^{\alpha(|x_2+a|+|x_1-y_1|)} e^{-\alpha(|x|+|y_1||)} \\
&\leq e^{-\alpha|a|} \sup_x \sum_y |E_a(x,y)| e^{\alpha(|x_2+a|+|x_1-y_1|)} \\
&= Ce^{-\alpha|a|}
\end{aligned}$$

where the constant $C$ is finite by assumption 2. Thus $\|E_a\| \to 0$. If, on the other hand, the maximum is $\sup_y \sum_x |E_a(x,y)|$, we have

$$\|E_a\| = \|E_a^*\| \leq \sup_y \sum_x |E_a^T(x,y)| = \sup_y \sum_x |E_a(y,x)|,$$

and the remainder of the proof is the same as before.

□

# Chapter 3

# Interacting Bulk-Edge Correspondence

We now move to the interacting setting. We now have a more complicated Hilbert space given by a tensor product of Hilbert spaces. We work on a finite system and only at the end allows its size $L \to \infty$. It is also possible to work directly in the infinite interacting setting through the GNS representation [17].

## 3.1 General Setting

Let $L \in \mathbb{N}$, and let $\Gamma_L = \mathbb{Z}/L\mathbb{Z} \times [0, L]$ be the discrete cylinder equipped with a metric. For convenience we take $L$ to be a large power of 2. To each site $x \in \Gamma_L$, we associate a Hilbert space $\mathcal{H}_x$ whose dimension is bounded uniformly in $x$ and in $L$. For a subset $X \subseteq \Gamma_L$, we define the Hilbert space $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$, and we set

$$\mathcal{H}_L := \mathcal{H}_{\Gamma_L} = \bigotimes_{x \in \Gamma_L} \mathcal{H}_x.$$

We denote $N = \sup_L \mathcal{H}_L$. The $C^*$-algebra $\mathcal{U}_X$ of observables on $\mathcal{H}_X$ consists of bounded self-adjoint operators supported in $X$. For an operator $A_X \in \mathcal{U}_X$, we identify its extension to an operator on $\mathcal{H}_L$ by taking its tensor product with copies of the identity, $(\otimes_{x \notin X} \mathbb{I}_x) \otimes A_X$. Conversely, we say that an operator $A \in \mathcal{U}_L$ has support $X$ if $A_X := (\otimes_{x \notin X} \mathbb{I}_x) \otimes (A|_X)$ is equal to $A$, and write $A_X \in \mathcal{U}_X$. For ease of notation, we omit the subscript $L$ wherever there is no risk of confusion.

A *local interaction* of range $R$ is a map $\Phi : \mathcal{P}(\Gamma_L) \to \mathcal{U}_L$ associating observables to subsets of the cylinder such that

1. $\Phi(X) = 0$ whenever $\mathrm{diam}(X) > R$ for some $R > 0$.

2. $\Phi(X)$ is supported in $X$.

3. $\|\Phi(X)\| \leq C$ for all $X \subset \Gamma_L$, for all $L$.

We consider a region as depicted in Figure 3.1, with the left and right edges joined together to form a cylinder.

### 3.1.1 The Edge System

In Figure 3.1, $H_0 = 0$ is a trivial Hamiltonian which we take to be empty space, supported in the left white region $[0, L/2] \times [0, L]$, while in the right blue region $[L/2, L] \times [0, L]$, $H_1$ is a so-called *local* or *finite-range Hamiltonian*, in the sense that $H_1 = \sum_{X \subseteq \Gamma_{\text{right}}} \Phi(X)$, is a sum of local interactions supported on the right half of the cylinder (the blue region). This replaces assumption 1. We define the Hamiltonian of the full edge system to be

$$H_E(\mu) = H_1 + \mu Q_h,$$

where $Q_h = \sum_{x \in \Gamma_h} a_x^* a_x$ is the *number* or *charge* operator for the region $\Gamma_h = [L/4, 3L/4] \times [0, L]$ shown in red. This introduces a driving strength; the $\mu Q_h$ term can be viewed as a potential difference $V(x)$. We emphasize that $H_1$ is supported on $[L/2, L] \times [0, L]$.

### 3.1.2 The Bulk System

The bulk system is again modelled by the cyclinder $\Gamma_L$, except this time, the Hamiltonian $H_B$ is nonzero on the entire cylinder (as opposed to just on the blue region $[L/2, L] \times [0, L]$, as was the case with the edge Hamiltonian). The associated "bulk" Hamiltonian is also assumed to be local, $H_B(\mu) = \sum_{X \subseteq \Gamma_L} \Phi(X) + \mu Q_h$, where this time the sum is over subsets of the entire cylinder. The exact same finite box-shaped regions $\Gamma_h$, $\Gamma_u$, $\Gamma_E$, $\Gamma_B$, and $\Gamma_m$ in Figures 3.1 and 3.2 are also present in the bulk setting on $\mathbb{Z}^2$; in particular, they are not extended to infinity in any direction.

For any $X \in \Gamma_{\text{right}}$, the local interactions $\Phi(X)$ in the bulk Hamiltonian and the edge Hamiltonian are the same; the edge Hamiltonian is merely a restriction of the Bulk Hamiltonian to the right half of the cylinder.

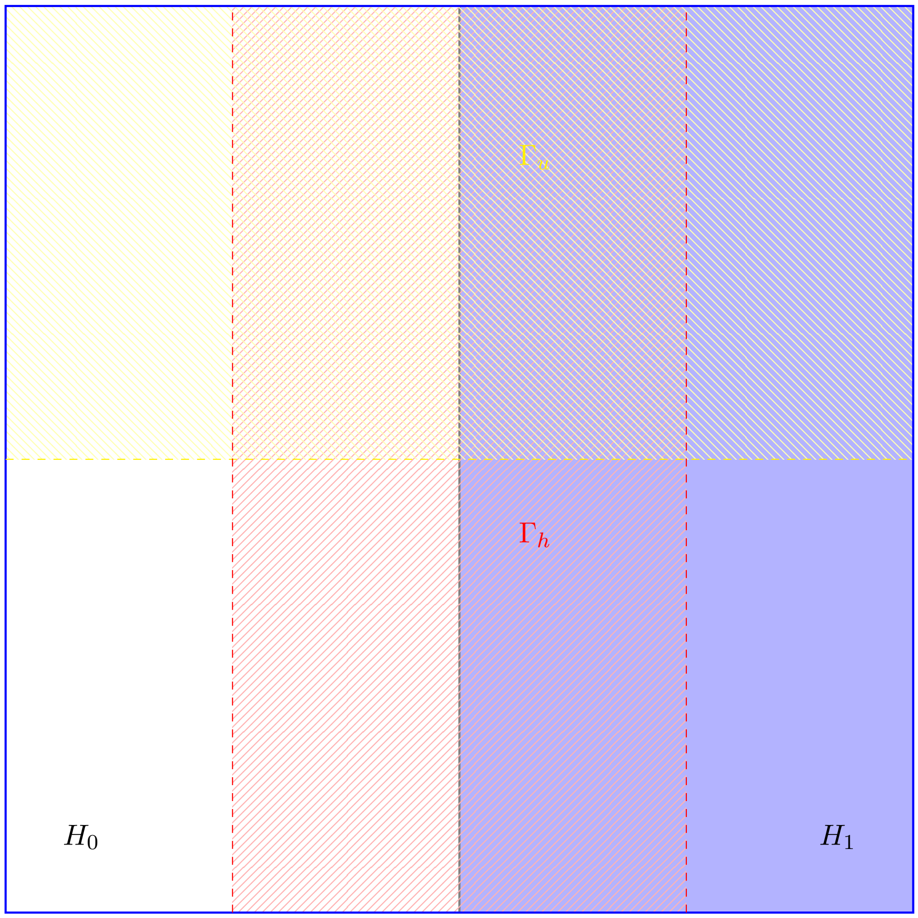


Figure 3.1: The left and right edges are identified to form the cylinder $\Gamma_L$. The charge operator $Q_h$ introduces a driving strength. The Hall current is measured across the dashed yellow line $y = L/2$.

$H_E$ is assumed to have a bounded spectrum, while $H_B$ has a spectrum which is both bounded and gapped, as in assumption 3. We also assume that the bulk and edge Hamiltonians are both *locally charge-conserving*.

**Assumption 4.** $[\Phi(X), Q] = 0$*, where* $Q$ *is the total charge in* $\Gamma_L$*.*

Let $P_B(\mu)$ be the ground state projection of $H_B(\mu)$ (the system without an edge), and let $P_E(\mu)$ be the ground state projection of $H_E(\mu)$ (the system with an edge). The ground state may be degenerate. We assume that states far from the edge are essentially bulk states, up to tails that vanish quickly in $L$. In fact, they vanish faster than any power of $L$. We say that such a function $g(x)$ decays *superpolynomially* or *almost-exponentially* in $x$ if for all $n$, there exists a $C_n$ such that $g(x) \leq C_n x^{-n}$ for all $x > x_0$ for some $x_0$. We denote superpolynomial decay by $g(x) = \mathcal{O}(x^{-\infty})$. For operators, we write $A = \mathcal{O}(L^{-\infty})$, which is a shorthand to say that the norm of $A$ decays superpolynomially in $L$.

**Assumption 5.** *Define the* edge region

$$\Gamma_E = [L/2 - k, L/2 + k] \times [0, L] \cup [L - k, k] \times [0, L].$$

*where $k = L/16$. For any operator $A$ supported on $\Gamma_E^{\mathsf{c}}$,*

$$\mathrm{Tr}(P_E A) = \mathrm{Tr}(P_B A) + \mathcal{O}(L^{-\infty}).$$

This replaces the edge assumption 2. The idea is that observables localized far away from the edge are not affected by the edge of the system. We similarly define the *bulk region*

$$\Gamma_B = [3L/4 - k, 3L/4 + k] \times [0, L],$$

and the *middle region*

$$\Gamma_m = [L/2, L] \cup [0, L] \setminus (\Gamma_E \cup \Gamma_B).$$

The three regions $\Gamma_E$, $\Gamma_B$, and $\Gamma_m$ are depicted in figure 3.2.

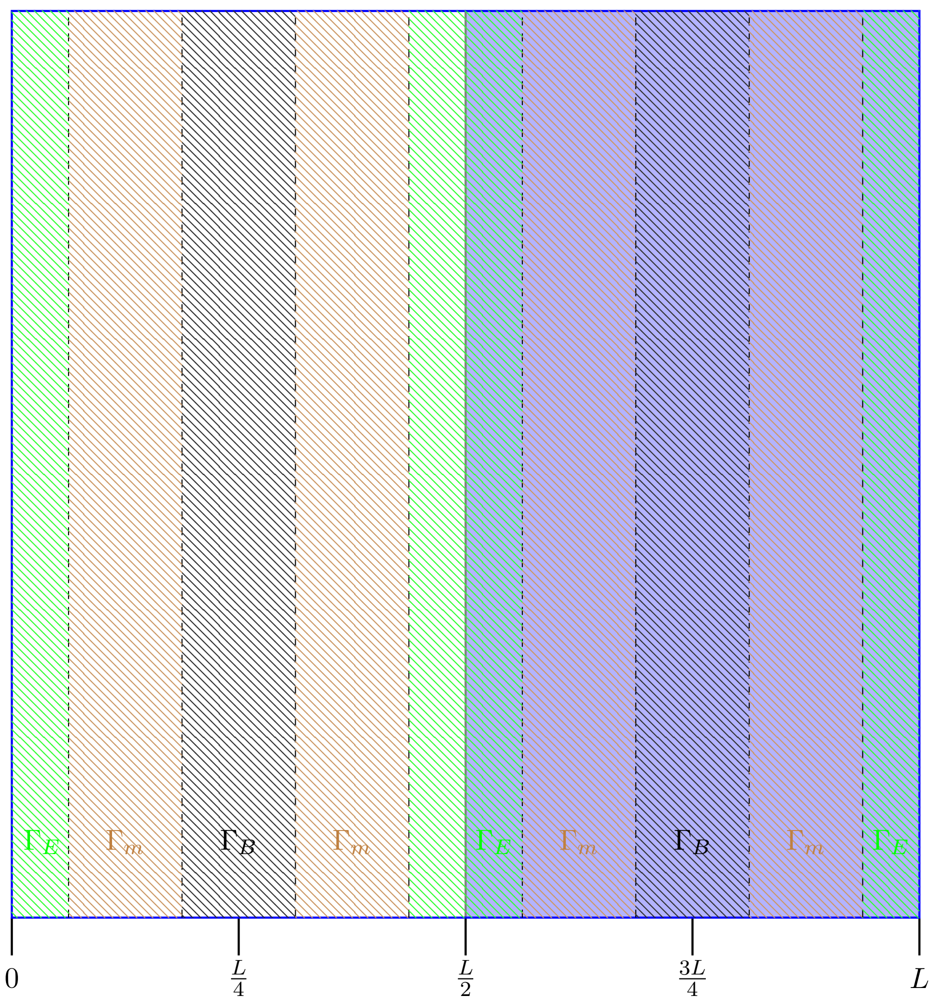


Figure 3.2: The regions $\Gamma_E$, $\Gamma_B$, and $\Gamma_m$.

To summarize, the bulk and edge Hamiltonians are both finite-range, and both follow assumptions 4 and 5, rather than 1 and 2. Additionally, $H_B$ satisfies 3.

## 3.2 Interacting Bulk-Edge Correspondence

### 3.2.1 The Current Operator

Let $Q_u = \sum_{x \in \Gamma_u} a_x^* a_x$ be the charge in the upper half of the cylinder $\Gamma_u = [0, L] \times [L/2, L]$ (the yellow region in Figure 3.1), and define current operator of the edge system,

$$J^E(\mu) = i[H_E(\mu), Q_u],$$

which measures the Hall current across the fiducial line $y = L/2$ in the edge system. We also denote by $J^B(\mu) = i[H_B(\mu), Q_u]$ the bulk Hall current. Charge conservation 4 implies that both current operators are supported along a strip of width $2R$ centred on the fiducial line $y = L/2$. To see why this is the case, let

$$S = [0, L] \times \left[\frac{L}{2} - R, \frac{L}{2} + R\right]^{\mathsf{c}},$$

denote the region outside the strip. Suppose $\Phi(X)$ is a local interaction of range $R$ supported outside the strip, $X \subseteq S$. Then clearly $\Phi(X)$ must commute with the charge in $S$, since $[\Phi(X), Q_S] = [\Phi(X), Q]$ which vanishes by the charge conservation assumption 4. It follows that for an interaction $\Phi(X)$ with range $R$ and arbitrary support $X$, $[\Phi(X), Q_u]$ must be supported in the strip $\left[\frac{L}{2}, L\right] \times \left[\frac{L}{2} - R, \frac{L}{2} + R\right]$. Therefore $[H_{E,B}(\mu), Q_u]$ must be supported there as well, since $H_{E,B}(\mu)$ is a sum of such local interactions.

It is worth noting that by the same argument, in the blue region $\Gamma_{\text{right}}$ in Figure 3.1, the bulk and edge Hamiltonians only differ on a strip of width $2R$ contained in $\Gamma_E$, since the $H_E$ is simply a restriction of $H_B$ to $\Gamma_{\text{right}}$.

From this point, we drop the subscript $\mu$ wherever it is not needed for context.

**Lemma 6.** *The ground state expectation of the both the bulk and edge system current $J^B$ and $J^E$ is zero.*

*Proof.* This is trivial by shifting the commutator under the trace, which we can do because the dimension of $\mathcal{H}_L$ is finite,

$$\mathrm{Tr}(PJ) = i\mathrm{Tr}(P[H, Q_u]) = i\mathrm{Tr}([P, H]Q_u) = 0.$$

□

### 3.2.2 Hastings Operators

Next, we define a family of operators indexed by $\mu$ called *Hastings operators*,

$$K_\mu = \mathcal{I}_\mu(\dot{H}_B(\mu)),$$

where

$$\mathcal{I}_\mu(A) = \int_{\mathbb{R}} W(t) e^{itH_B(\mu)} A e^{-itH_B(\mu)} dt.$$

Here, $W : \mathbb{R} \to \mathbb{R}$ is an odd, bounded, $L^1(\mathbb{R})$ function satisfying

1. $|W(t)| = \mathcal{O}(|t|^{-\infty})$,
2. $\widehat{W}(\xi) = \frac{i}{\xi}$ for all $|\xi| \geq \mathrm{Len}(\Delta)$,

where $\widehat{W}$ is the Fourier transform, taken here to be $\hat{f}(\xi) = \int_{\mathbb{R}} f(t) e^{-it\xi} dt$. Such a function can be constructed explicitly [10]. We emphasize the necessity of the spectral gap $\Delta$ in the construction of $W$. In our setting, we see that

$$K_\mu = \mathcal{I}_\mu(Q_h).$$

We present two important properties of the map $\mathcal{I}_\mu : \mathcal{U}_L \to \mathcal{U}_L$ in the following lemmas. First, recall a definition from the non-interacting setting: an *off-diagonal* operator is an operator $A$ such that $A = \overline{A} := P_B A P_B^\perp + P_B^\perp A P_B$, where $P_B^\perp = \mathbb{I} - P_B$ is the projection onto the excited states above the gap.

**Lemma 7.** *1. For any off-diagonal bounded operator $A = \overline{A}$, $\mathcal{I}_\mu(\cdot)$ and $[H_B(\mu), \cdot]$ act as inverses of each other, up to a factor of $i$:*

$$\mathcal{I}\left([H_B, A]\right) = [H_B, \mathcal{I}(A)] = iA.$$

*2. For any (not necessarily off-diagonal) bounded operator $A$,*

$$[\mathcal{I}([H_B, A]), P_B] = i[A, P_B].$$

*Proof.* We drop the subscript $B$ to clean up the notation. Let $\widehat{W}(\xi) = \int_{\mathbb{R}} W(t) e^{-it\xi} dt$ be the Fourier transform of $W$ and let $A$ be any observable. First, we show that $\mathcal{I}([H, PAP^\perp]) = iPAP^\perp$.

Assume for now the spectrum of the bulk Hamiltonian $H$ is discrete. Decomposing

$$\begin{aligned}
e^{itH}P &= \sum_{j=0}^{\infty} \frac{(itH)^j}{j!} P \\
&= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left( \sum_n E_n^j P_n \right) P \\
&= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \sum_{n:E_n=0} E_n^j P_n \\
&= \sum_{n:E_n=0} e^{itE_n} P_n,
\end{aligned}$$

and similarly

$$P^{\perp} e^{-itH} = \sum_{m:E_m \geq \gamma} P_m e^{-itE_m},$$

we see that

$$\begin{aligned}
\mathcal{I}([H, PAP^{\perp}]) &= \mathcal{I}(P[H,A]P^{\perp}) \\
&= \int_{\mathbb{R}} W(t) e^{itH} P[H,A] P^{\perp} e^{-itH} dt \\
&= \int_{\mathbb{R}} W(t) \sum_{n:E_n=0} e^{itE_n} P_n [H,A] \sum_{m:E_m \geq \gamma} P_m e^{-itE_m} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} \int_{\mathbb{R}} W(t) e^{itE_n} P_n A (E_n - E_m) P_m e^{-itE_m} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \int_{\mathbb{R}} W(t) e^{-it(E_m - E_n)} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \widehat{W}(E_m - E_n) \\
&= i \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m \\
&= iPAP^{\perp},
\end{aligned}$$

since $\widehat{W}(\xi) = i\xi^{-1}$ for all $|\xi| \geq \gamma$, where $\gamma := \mathrm{Len}(\Delta)$. If the spectrum of $H$ is continuous, the sums can be replaced by integrals with respect to the projection-valued measure $P_\lambda$, as in 3. Either way, interchanging the integral over $t$ and the other sums/integrals is allowed by boundedness of $A$ and the fact that $W, \widehat{W} \in L^1$. By the same argument, $\mathcal{I}([H, P^\perp AP]) = iP^\perp AP$ as well, and so $\mathcal{I}([H, \overline{A}]) = i\overline{A}$, which concludes the proof of part (1).

Part (2) of the lemma follows from the observation that $[A, P]$ is always off-diagonal, and since $P$ and $H$ commute, we have by part (1)

$$[\mathcal{I}([H, A]), P] = \mathcal{I}([[H, A], P]) = \mathcal{I}([H, [A, P]]) = i[A, P].$$

□

Another important property of the map $\mathcal{I}$ is that it preserves locality.

**Lemma 8.** *Let $A$ be a bounded, trace-class operator supported in $X \subset \mathbb{Z}^2$, and let $X^r$ denote the $r$-fattening of $X$. Let $S = (X^r)^\mathsf{c}$ be the set of all sites whose distance from $X$ is at least $r$. $\mathcal{I}$ is* quasi-locality preserving *in the sense that*

$$\|\mathcal{I}(A) - tr_S(\mathcal{I}(A))\| = \mathcal{O}(r^{-\infty}),$$

*where $tr_S(\cdot)$ is the partial trace over $S$.*

*Proof.* Let $A(t) = e^{itH_B} A e^{-itH_B}$ be the time evolution of $A$. We see that $\|\mathcal{I}(A) - \mathrm{tr}_S(\mathcal{I}(A))\|$ is bounded by

$$\int_{-T}^{T} |W(t)| \|A(t) - \mathrm{tr}_S(A(t))\| dt + \int_{\mathbb{R}\setminus[-T,T]} |W(t)| \|A(t) - \mathrm{tr}_S(A(t))\| dt.$$

We treat each of the integrals separately. For the first, by the Lieb-Robinson bound (Proposition 4) we have

$$\begin{aligned}\int_{-T}^{T} |W(t)| \|A(t) - \mathrm{tr}_S(A(t))\| dt &\le |X| \|A\| \|W\|_\infty \int_{-T}^{T} e^{-\frac{r - v|t|}{\xi}} dt \\ &= \frac{2\xi}{v} |X| \|A\| \|W\|_\infty e^{-\frac{r}{\xi}} (e^{\frac{vT}{\xi}} - 1).\end{aligned}$$

For the second term, notice that the bound

$$\|A(t) - \mathrm{tr}_S(A(t))\| \le \|A\| + |\mathrm{Tr}(A)| =: C$$

holds uniformly in $t$, and $C$ is finite by assumption. We therefore obtain

$$\int_{\mathbb{R}\setminus[-T,T]} |W(t)| \|A(t) - \mathrm{tr}_S(A(t))\| dt \le 2C \int_{T}^{\infty} |W(t)| dt.$$

Let $M(T) := \int_T^\infty |W(t)|dt$ and fix $n \in \mathbb{N}$. Since $|W|$ is bounded and obeys $|W(t)| = \mathcal{O}(|t|^{-\infty})$, it follows that $|W(t)| \le C_n|t|^{-n}$ for some $C_n$, and thus

$$M(T) \le \frac{C_n}{n+1}\frac{1}{T^{n+1}}.$$

Since this holds for all $n$, we have $M(T) = \mathcal{O}(T^{-\infty})$. Finally, putting the two integrals together, we have

$$\|\mathcal{I}(A) - \mathrm{tr}_S(\mathcal{I}(A))\| \le \frac{2\xi}{v}|X|\|A\|\|W\|_\infty e^{-\frac{r}{\xi}}(e^{\frac{vT}{\xi}} - 1) + \mathcal{O}(T^{-\infty}).$$

We conclude by optimizing with the choice $T = \frac{r}{2v}$ so that the first term decays exponentially and the superpolynomial term dominates, leaving $\|\mathcal{I}(A) - \mathrm{tr}_S(\mathcal{I}(A))\| = \mathcal{O}(r^{-\infty})$.

□

The idea of the previous lemma is that for an operator $A$ supported in $X$, $\mathcal{I}(A)$ can be well-approximated by an operator supported in the $r$-fattening $X^r$.

**Proposition 2.** *The operator $K_\mu$ is the* generator of parallel transport*, meaning that the ground state projection $P_B(\mu)$ is the unique solution to the differential equation*

$$\dot{A}(\mu) = i[K_\mu, A(\mu)]$$

*with initial condition $A(0) = P_B(0)$.*

*Proof.* We drop the $B$ and $\mu$ subscripts to lighten notation. First, we show that $\dot{P}$ is off-diagonal. Taking the derivative on both sides of $P^2 = P$, we see that $\dot{P}P + P\dot{P} = \dot{P}$. Acting on the left and right with $P$ on both sides of this equation gives

$$P\dot{P}P + P\dot{P}P = P\dot{P}P,$$

which implies that $P\dot{P}P = 0$. Thus

$$\begin{aligned}\overline{\partial_\mu P} &= P\dot{P}(1-P) + (1-P)\dot{P}P \\ &= P\dot{P} + \dot{P}P \\ &= \partial_\mu(P^2) \\ &= \partial_\mu P,\end{aligned}$$

as claimed. By the product rule and the fact that $H$ and $P$ commute,

$$[\dot{H}, P] = -[H, \dot{P}].$$

It therefore follows from part (1) of Lemma 7 that

$$\dot{P} = -i\mathcal{I}([H, \dot{P}]) = i\mathcal{I}([\dot{H}, P]) = i[\mathcal{I}(\dot{H}), P] = i[K, P].$$

Thus $P_B$ is a solution. As for uniqueness, notice that the map $F : \mathcal{U} \to \mathcal{U}$ defined by $F(A) = i[K, A]$ is Lipschitz, since

$$\|F(A) - F(B)\| = \|[K, A - B]\| \leq 2\|K\|\|A - B\|.$$

The Lipschitz constant is $2\|K\|$, which is finite since $K$ is a bounded operator:

$$\|K\| \leq \int_{\mathbb{R}} |W_\gamma(t)|\|e^{-itH_B} Q_h e^{itH_B}\|dt \leq \int_{\mathbb{R}} |W_\gamma(t)|dt\|Q_h\|.$$

Indeed, since $Q_h$ is the number operator on a finite volume, by charge conservation and the fact that the dimension of the Hilbert space at every site is uniformly bounded, there can only be a finite number of charges in the region $\Gamma_h$. Thus, by Grönwall's uniqueness theorem for Banach-valued functions A.2, we see that the solution is unique.

□

### 3.2.3 The Main Result

We return to the analogy of $\mu Q_h$ as an electric potential. Let $P = P_E, P_B$ and $J = J^E, J^B$ denote either the edge or bulk system projection and current. Increasing the potential by a small amount $d\mu Q_h$ and expanding to linear order, the change in ground state current is given by

$$\mathrm{Tr}(P(\mu + d\mu)J) - \mathrm{Tr}(P(\mu)J) = \kappa d\mu + \mathcal{O}(d\mu^2).$$

Dividing by $d\mu$ and taking a limit, we see that the linear response coefficient is given by

$$\sigma = \mathrm{Tr}\left(\dot{P}J\right).$$

The Hall conductivity on a subset $V \subseteq \Gamma_L$ is defined to be $\sigma_V := \mathrm{Tr}\left(\dot{P}J_V\right)$, where $J_V$ is the restriction of $J$ to $V$. In particular, we define the *edge conductivity* in the interacting setting as the conductivity measured locally on the edge strip $\Gamma_E$ arising from the edge Hamiltonian,

$$\sigma^E_{\Gamma_E} = \mathrm{Tr}(\dot{P}_E J^E_{\Gamma_E}) = i\mathrm{Tr}(\dot{P}_E [H_E, Q_u]_{\Gamma_E}).$$

We similarly take $\sigma^B_{\Gamma_B} = -\mathrm{Tr}(\dot{P}_B J^B_{\Gamma_B})$ as the definition of the *bulk Hall conductivity*; the reason for the minus sign will become apparent later.

The following is the main result:

**Theorem 2.** *Let $V \subseteq \Gamma_m$ be a set contained within the strip in between the edge region $\Gamma_E$ and the bulk region $\Gamma_B$ (see Figure 3.2), and define the distance*

$$r = \mathrm{dist}(V, \Gamma_E \cup \Gamma_B)$$

*from $V$ to the bulk and edge regions. The edge Hall conductivity in this regions vanishes in the sense that*

$$\sigma^E_V = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

*Proof.* By Proposition 2, the bulk Hall conductivity can also be written by the formula

$$\sigma^B_V = \mathrm{Tr}\left(i[K, P_B] J^B_V\right) = \mathrm{Tr}\left(i[\mathcal{I}(Q_h), P_B] J^B_V\right).$$

From commutativity of $P_B$ and $H_B$ along with cyclicity of the trace (since $\mathcal{H}_L$ is finite), we compute

$$\begin{aligned}
\sigma^B_V &= \mathrm{Tr}\left(i\int_{\mathbb{R}} W(t) e^{itH_B}[Q_h, P_B] e^{-itH_B} dt J^B_V\right) \\
&= \int_{\mathbb{R}} W(t)\mathrm{Tr}\left(i[Q_h, P_B] e^{-itH_B} J^B_V e^{itH_B}\right) dt \\
&= -\int_{\mathbb{R}} W(t)\mathrm{Tr}\left(i[Q_h, P_B] e^{itH_B} J^B_V e^{-itH_B}\right) dt \\
&= -\mathrm{Tr}\left(i[Q_h, P_B]\mathcal{I}(J^B_V)\right),
\end{aligned}$$

since $W(t)$ is odd. By part (2) of Lemma 7, we have $i[Q_h, P_B] = [\mathcal{I}([H_B, Q_h]), P_B]$. Therefore

$$\begin{aligned}
\sigma^B_V &= -\mathrm{Tr}([\mathcal{I}([H_B, Q_h]), P_B]\mathcal{I}(J^B_V)) \\
&= \mathrm{Tr}\left(P_B[\mathcal{I}([H_B, Q_h]), \mathcal{I}(J^B_V)]\right).
\end{aligned}$$

Now, recalling the region $\Gamma_h$ in Figure 3.1, we see that $[H_B, Q_h]$ is supported on vertical strips of width $2R$ centered at $x = L/4$ and $x = 3L/4$, and thus for large enough $L$, it is supported on $\Gamma_B$. On the other hand, $J_V^B$ is supported on $V$. By Lemma 8, the commutator $[\mathcal{I}([H_B, Q_h]), \mathcal{I}(J_V^B)]$ can be written

$$[A_{(\Gamma_B)^{r/2}} + \mathcal{O}(r^{-\infty}), B_{V^{r/2}} + \mathcal{O}(r^{-\infty})] = \mathcal{O}(r^{-\infty})$$

for some operators $A_{(\Gamma_B)^{r/2}}$ and $B_{V^{r/2}}$ supported on the $\frac{r}{2}$-fattenings of $\Gamma_B$ and $V$, and $(\Gamma_B)^{r/2} \cap V^{r/2} = \varnothing$. Thus

$$\sigma_V^B = \mathcal{O}(r^{-\infty}).$$

This fact applies to the bulk setting with $H_B$ and $P_B$. To extend this to the setting with an edge, recall that the bulk and edge Hamiltonians are equivalent in $V$, so $J_V^E = J_V^B$. It is enough to use Assumption 5 to conclude the desired result with equality up to $\mathcal{O}(L^{-\infty})$, i.e.

$$\begin{aligned}
\sigma_V^E &= \operatorname{Tr}\left(\dot{P}_E J_V^E\right) \\
&= \operatorname{Tr}\left(\dot{P}_E J_V^B\right) \\
&= \operatorname{Tr}\left(\dot{P}_B J_V^B\right) + \mathcal{O}(L^{-\infty}) \\
&= \sigma_V^B + \mathcal{O}(L^{-\infty}) \\
&= \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).
\end{aligned}$$

□

The intuitive picture from the previous result is that the Hall conductivity of the edge system is essentially only nonzero along $\Gamma_E$ and $\Gamma_B$; if there is current flowing anywhere, it must be along these strips. In fact, since $H_E$ is trivial outside $\Gamma_{\text{right}}$ in Figure 3.1, the current can only flow along the strips $\Gamma_E$ and the *rightmost* strip $\Gamma_B$.

The ground state expectation of the total edge current $\operatorname{Tr}(P_E J^E)$ is zero by Lemma 6. If there is a Hall current flowing along the strip $\Gamma_B$ (as one would expect physically, since that's where the driving $\mu Q_h$ occurs), then there must therefore be an equal current flowing along the edge strips $\Gamma_E$ in the opposite direction,

$$\operatorname{Tr}(P_E J_{\Gamma_E}^E) = -\operatorname{Tr}(P_E J_{\Gamma_B}^E) + \mathcal{O}(L^{-\infty}).$$

Taking a derivative, and recalling that $J^E_{\Gamma_B} = J^B_{\Gamma_B}$, we see the edge Hall conductivity $\mathrm{Tr}(\dot{P}_E J^E_{\Gamma_E})$ is equal to

$$\sigma^E_{\Gamma_E} = -\mathrm{Tr}(\dot{P}_E J^B_{\Gamma_B}) + \mathcal{O}(L^{-\infty}).$$

We conclude bulk-edge correspondence using assumption 5, in the sense that the edge Hall conductivity is equal (up to tails) to the bulk conductivity,

$$\sigma^E_{\Gamma_E} = \sigma^B_{\Gamma_B} + \mathcal{O}(L^{-\infty}).$$

# Chapter 4

# Summary of Results

In this thesis, we proved bulk-edge correspondence in both noninteracting and interacting quantum lattice systems. In the noninteracting setting, the key development was the use of the Helffer-Sjöstrand representation to formally write the difference between $\sigma_E$ and $\sigma_B$ as a zero trace operator. In the interacting setting, the key tool is the Hastings operator, which allows a certain kind of inverse of $[H, \cdot]$ and defines the ground state projection as the unique solution to the differential equation $\dot{P} = i[K, P]$.

## 4.1 Directions for Future Work

### 4.1.1 Torus Geometry

An interesting result would be a proof of bulk-edge correspondence in the interacting setting on the discrete torus $\mathbb{T}_L := (\mathbb{Z}/L\mathbb{Z})^2$. Indeed, one could easily argue that it would be a "better" bulk edge correspondence than the cylinder geomtry. In the bulk setting, the torus would have no edges at all, whereas in the cylinder, the top and bottom edges are still present in the bulk setting.

One might define the same regions $\Gamma_u$ and $\Gamma_h$ (Figure 3.1) as in the cylinder case. However, on $\mathbb{T}_L$, the vanishing total ground state current (Lemma 6) does not apply. Intuitively, it does not apply because electrons can now flow through both the bottom and the top of the region $\Gamma_u$, rather than just the bottom. Mathematically, the lemma fails because the definition of the current is slightly changed.

Charge conservation and the fact that $H$ is finite range can be used to split the current $J_u$ into two components, $J_u = i[H_-, Q_u] + i[H_+, Q_u] = J_- - J_+$, supported on strips of width $2R$ centered at $y = L/2$ and $y = L$. We denote these lower and upper strips located at $y = L/2$ and $y = L$ as $\partial_-$ and $\partial_+$, respectively. The current operator is then defined to be $J = J_-$, which measures the current passing through the fiducial line $y = L/2$. This is the mathematical reason that the proof in Lemma 6 fails on the torus; we have replaced $H$ by $H_-$, which may no longer commute with $P$.

One might proceed in a manner similar to [9]. We define a slightly modified Hastings operator

$$K := -\mathcal{I}(J) = -\int_{\mathbb{R}} W(t) e^{itH_B} i[H_B, Q_u] e^{-itH_B} dt$$

so that by Lemma 7, $Q_u - K$ has the crucial property of leaving the bulk system's ground state space invariant,

$$[Q_u - K, P_B] = [Q_u, P_B] - \frac{1}{i}[\mathcal{I}([H_B, Q_u]), P_B)] = 0.$$

By Lemma 8, the Hastings operator $K$ is supported (up to tails) on $\partial_\pm$. Finally, we prove that in the bulk system with Hamiltonian $H_B(\mu)$, the ground state expectation of the total current vanishes superpolynomially in $L$.

**Lemma 9.** *The ground state expectation of the bulk system current* $J^B := i[(H_B)_-, Q_u]$ *is* $\mathrm{Tr}(P_B J^B) = \mathcal{O}(L^{-\infty})$.

*Proof.* $K = -\mathcal{I}(i[H_B, Q_u])$ splits into $K = K_- - K_+$, with $K_\pm$ supported in $\partial_\pm$ up to tails by Lemma 8, in the sense that

$$[K_\pm, A_X] = \mathcal{O}(r^{-\infty})$$

for every bounded $A_X \in \mathcal{U}_X$, where $r = \mathrm{dist}(X, \partial_\pm)$. In particular,

$$i[H_B, K_-] = i[(H_B)_-, K_-] + \mathcal{O}(L^{-\infty}),$$

and similarly $i[(H_B)_-, K_+] = \mathcal{O}(L^{-\infty})$. Putting these facts together, it follows that the current can be rewritten as

$$\begin{aligned} J_B &= i[(H_B)_-, Q_u + K_- - K_- + K_+] + \mathcal{O}(L^{-\infty}) \\ &= i[H_B, K_-] + i[(H_B)_-, Q_u - K)] + \mathcal{O}(L^{-\infty}). \end{aligned}$$

From here, we use the fact that both $H_B$ and $Q_u - K$ commute with $P_B$ to write

$$P_B J^B P_B = i[H_B, P_B K_- P_B] + i[P_B (H_B)_- P_B, Q_u - K)] + \mathcal{O}(L^{-\infty}).$$

$H_B$, $Q$, and $K$ are all bounded (see the proof of Proposition 2). Since the Hilbert space is finite dimensional, it follows from cyclicity of the trace that

$$\mathrm{Tr}(P_B J^B) = \mathrm{Tr}(P_B J^B P_B) = \mathcal{O}(L^{-\infty}).$$

□

Of course, the ultimate goal would be to show that the ground state current is zero in the system with an edge, which is what allowed us to conclude bulk-edge correspondence in the cylinder case.

### 4.1.2 Generators of Parallel Transport in Edge Systems

We return to the interacting cylinder picture. One possible extension of this thesis would be to devise a suitable operator $K_E$ which plays the same role as the generator of parallel transport in the edge system as $K$ did in the bulk setting. One might begin with the attempt

$$K_E := \int_{\mathbb{R}} W(t) e^{-itH_E} Q_h e^{itH_E} dt,$$

which is using the gap $\Delta$ of $H_B$ to define $W$, but also using the edge Hamiltonian in the time evolution operators. Consider the differential equation

$$\dot{\zeta}(\mu) = i[K_E, \zeta(\mu)] \qquad \zeta(0) = P_E(0).$$

The idea would be to show that while $P_E$ is not an exact solution, $\zeta$ is still a good approximation of $P_E$ when taking expectations of operators supported outside $\Gamma_E$, i.e. $\mathrm{Tr}(P_E A_X)$ with $X \cap \Gamma_E = \varnothing$. For such an operator $A_X$,

$$
\begin{aligned}
\mathrm{Tr}(\dot{\zeta}A_X) &= \mathrm{Tr}(i[K_E,\zeta]A_X) \\
&= \mathrm{Tr}(i[A_X,K_E]\zeta) \\
&= \int_{\mathbb{R}} W(t)\mathrm{Tr}([e^{-itH_E}Q_h e^{itH_E},A_X]\zeta)dt \\
&= \int_{\mathbb{R}} W(t)\mathrm{Tr}(e^{-itH_E}[Q_h,e^{itH_E}A_X e^{-itH_E}]e^{itH_E}\zeta)dt \\
&= \int_{\mathbb{R}} W(t)\mathrm{Tr}(e^{-itH_E}[Q_h,e^{itH_B}A_X e^{-itH_B}]e^{itH_E}\zeta + \mathcal{O}(L^{-\infty}))dt \\
&= \int_{\mathbb{R}} W(t)\mathrm{Tr}(e^{-itH_B}[Q_h,e^{itH_B}A_X e^{-itH_B}]e^{itH_B}\zeta + \mathcal{O}(L^{-\infty}))dt \\
&= \int_{\mathbb{R}} W(t)\mathrm{Tr}([e^{-itH_B}Q_h e^{itH_B},A_X]\zeta)dt + \mathcal{O}(L^{-\infty}) \\
&= \mathrm{Tr}(i[A_X,K_B]\zeta] + \mathcal{O}(L^{-\infty}) \\
&= \mathrm{Tr}(i[K_B,\zeta]A_X) + \mathcal{O}(L^{-\infty}).
\end{aligned}
$$

We see that

$$
\mathrm{Tr}((\dot{\zeta} - i[K_B,\zeta])A_X) = \mathcal{O}(L^{-\infty})
$$

for any operator $A_X \in \mathcal{U}_X$. By Grönwall A.2, the solution of $\dot{\rho} - i[K_B,\rho] = 0$ with initial condition $\zeta(0) = P_B(0)$ is unique; it is $P_B(\mu)$. From here, one might try to use the properties of $K_B$ and assumption 5 to argue that the solution $\zeta(\mu)$ approximates $P_E(\mu)$. For instance, we know that

$$
\mathrm{Tr}((\dot{\zeta} - i[K_B,\zeta])A_X) = \mathrm{Tr}((\dot{\zeta} - i[K_E,\zeta])A_X) + \mathcal{O}(L^{-\infty})
$$

for any operator $A_X \in \mathcal{U}_X$. One could also attempt to generalize Grönwall's uniqueness theorem A.2 for the setting $u'(t) = F(u(t)) + \mathcal{O}(L^{-\infty})$.

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# Appendix A

# General Functional Analysis

**Lemma 10.** *Let $A$ be a bounded linear operator on a Hilbert space $\mathcal{H}$. Suppose $A_n \xrightarrow{s} A$ on a dense subspace $\mathcal{D} \subset \mathcal{H}$. If $A_n$ are bounded uniformly in $n$, then $A_n \xrightarrow{s} A$ on all of $\mathcal{H}$.*

*Proof.* Let $\psi_m \in \mathcal{D}$ be a sequence converging in norm to $\psi \in \mathcal{H}$. The result follows from a standard $\frac{\varepsilon}{3}$ argument. Let $C$ be a bound for both $\sup_n \|A_n\|$ and $\|A\|$. Then

$$\begin{aligned}\|A_n\psi - A\psi\| &\leq \|A_n(\psi - \psi_m)\| + \|(A_n - A)\psi_m\| + \|A(\psi - \psi_m)\| \\ &\leq C\|(\psi - \psi_m)\| + \|(A_n - A)\psi_m\| + C\|\psi - \psi_m\|.\end{aligned}$$

There exists an $M$ such that the first and third terms are less than $\frac{\epsilon}{3C}$ for all $m > M$. For the middle term, observe that for each $m$, there exists by hypothesis an $N_m$ such that $\|(A_n - A)\psi_m\| < \frac{\epsilon}{3}$ for all $n > N_m$. Thus, by fixing some $m > M$, the inequality above is less than $\varepsilon$ for all $n > N_m$. □

**Lemma 11** (Holmgren's Bound)**.** *If $A$ is any operator on $\ell^2(\mathbb{Z}^2)$, then*

$$\|A\| \leq \max\{\sup_x \sum_y |A(x,y)|,\ \sup_y \sum_x |A(x,y)|\}.$$

## A.1 Projection-Values Measures

*Projection-valued measures* are maps $P : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ from measurable subsets of $\mathbb{R}$ to the space of bounded linear operators on $\mathcal{H}$ satisfying the usual properties of both projections and measures.

1. $P(M) = P(M)^* = P(M)^2$ is an orthogonal projection for all $M \in \mathcal{M}$. Note that this implies that $P(M)$ is a positive operator.

2. $P(\varnothing) = 0$ and $P(\mathbb{R}) = \mathbb{1}_{\mathcal{H}}$.

3. If $\{M_i\}_{i\in\mathbb{N}}$ are pairwise disjoint, then $\sum_{i=1}^{n} P(M_i) \xrightarrow{s} P(\cup_{i\in\mathbb{N}} M_i)$ as $n\to\infty$ ($\sigma$-additivity).

4. $P(M_1 \cap M_2) = P(M_1)P(M_2)$ for any $M_1, M_2 \in \mathcal{M}$.

The heuristic motivation is that $P(M)$ projects onto the subspace of $\mathcal{H}$ spanned by states whose energies lie in $M$. Using these operator-valued measures, one can construct an operator-valued integral with respect to $P$ in the usual fashion (beginning on nonnegative simple functions, extending to nonnegative measurable functions, and finally to real-valued measurable functions).

**Theorem 3** (Spectral Theorem for Projection-Valued Measures)**.** *There exists a one-to-one correspondence between self adjoint operators $H$ and projection-valued measures $P$ given by the formula*

$$H = \int_{\mathbb{R}} \lambda dP_\lambda,$$

*where $P_\lambda := P((-\infty,\lambda])$. Moreover, if $g:\mathbb{R}\to\mathbb{R}$ is any bounded Borel function, then $g(H)$ defined via the Borel function calculus coincides with the formula*

$$g(H) = \int_{\mathbb{R}} g(\lambda) dP_\lambda,$$

*and $g(H) = g(H)^*$.*

We remark that it follows from the second part of this theorem that if $\chi_M$ denotes the characteristic function of a Borel set $M\subseteq\mathbb{R}$, then

$$\chi_M(H) = \int_M dP_\lambda = P(M).$$

We also note that $\mathrm{Spec}(H) = \mathrm{supp}(P)$.

## A.2 Grönwall's Inequality and Uniqueness

**Lemma 12.** *(Grönwall's Inequality). Let $\alpha : I \to (0,\infty)$ be positive and continuous on $I^o$ for some interval of the form $[a,b)$, $[a,b]$, or $[a,\infty)$. Suppose $u:\mathbb{R}\to\mathcal{U}$ is a Banach-valued, differentiable function. If $\|u'(t)\| \leq \alpha(t)\|u(t)\|$ for all $t\in I$, then*

$$\|u(t)\| \leq \|u(a)\| e^{\int_a^t \alpha(s)ds} \quad \forall t \in I$$

*Proof.* Let $f(t) = e^{\int_a^t \alpha(s)ds}$, which is nonzero, has initial value $f(a) = 1$, and has derivative $f'(t) = \alpha(t)f(t)$. Then by the quotient rule,

$$\left(\frac{\|u(t)\|}{f(t)}\right)' = \frac{\|u'(t)\|f(t) - \|u(t)\|\alpha(t)f(t)}{f(t)^2} \leq 0,$$

where the inequality follows from the assumption $\|u'(t)\| \leq \|\alpha(t)u(t)\|$. Thus $\frac{\|u(t)\|}{f(t)}$ is decreasing, so that

$$\frac{\|u(t)\|}{f(t)} \leq \frac{\|u(a)\|}{f(a)} = \|u(a)\|,$$

which is the desired inequality. □

**Theorem 4.** *(ODE Uniqueness). Let $\mathcal{U}$ be a Banach space, and let $F : \mathcal{U} \to \mathcal{U}$ be Lipschitz. Consider the differential equation*

$$u'(t) = F(u(t))$$

*with initial condition $u(a) = u_a$ for some function $u : I \to \mathcal{U}$, where $I = [a, b]$, or $[a, b)$, or $[a, \infty)$. Solutions to this equation are unique.*

*Proof.* Suppose there are two solutions $u(t)$ and $v(t)$, and let $g(t) = \|u(t) - v(t)\|^2$. By assumption, there exists a constant $L_F$ such that $\|F(u(t)) - F(v(t))\| \leq L_F\|u(t) - v(t)\|$, so that

$$\begin{aligned} g'(t) &= 2\|u(t) - v(t)\|\|u'(t) - v'(t)\| \\ &= 2\|u(t) - v(t)\|\|F(u(t)) - F(v(t))\| \\ &\leq 2L_F\|u(t) - v(t)\|^2 \\ &= 2L_F g(t). \end{aligned}$$

Notice that $\alpha := 2L_F$ is a positive continuous function, so we may apply Grönwall's inequality to $g(t)$ to conclude

$$g(t) \leq g(a)e^{2L_f(t-a)} = 0,$$

since $g(a) = 0$. □

## A.3 Greens' Functions and the Combes-Thomas Bound

The short-range assumption 1 is vital for the following non-trivial estimate, the proof of which is omitted.

**Theorem 5** (Combes-Thomas Bound (Version 1))**.** *Let $H$ be a self-adjoint operator on $\ell^2(\mathbb{Z}^2)$ satisfying*

$$S_\alpha := \sup_x \sum_y |H(x,y)|(e^{\alpha|x-y|} - 1) < \infty$$

*for some $\alpha > 0$. Suppose $z$ lies outside the spectrum of $H$, and let $d_z := \mathrm{dist}(z, \mathrm{Spec}(H))$. Then the Greens function of $H$ is exponentially bounded,*

$$|G(x,y;z)| \leq \frac{2}{d_z} e^{-\xi_\alpha |x-y|},$$

*where $\xi := \frac{\alpha d_z}{2S_\alpha}$.*

Let $\mathrm{Lip}^1$ be the set of all Lipschtiz functions whose Lipschitz constant is not greater than 1, that is, the set of functions $\ell$ satisfying

$$|\ell(x) - \ell(y)| \leq |x-y|$$

for all $x, y \in \mathbb{Z}^2$.

**Theorem 6** (Combes-Thomas Bound (Version 2))**.**

$$\|e^{-\varepsilon f(x)} R_a(z) e^{\varepsilon f(x)}\| \leq \frac{C}{|\mathrm{Im}(z)|}$$

*for any* $\mathrm{Lip}^1$ *function $f : \mathbb{Z}^2 \to \mathbb{R}$, and $\varepsilon$ can be chosen as $\varepsilon = \frac{1}{C(1+|\mathrm{Im}(z)|^{-1})}$.*

This theorem gives the crucial decay properties of the spectral projectors.

**Lemma 13.** *Let $H$ be a self-adjoint operator satisying 1 and with bounded spectrum. Let $S \subseteq \mathrm{Spec}(H)$, and let $P_S$ be the associated spectral projection. Then there exists some $\varepsilon, \nu > 0$ such that the matrix elements of $P_S$ satisfy*

$$\sum_{x,y \in \mathbb{Z}^2} |P_S(x,y)| e^{-\varepsilon|x|} e^{\nu|x-y|} < \infty.$$

*Proof.* We use the fact that the spectral projection is given by the Riesz integral formula

$$P_S = \frac{1}{2\pi i}\oint_\gamma R(z)dz,$$

where $R(z)$ is the resolvent of $H$ and $\gamma$ is any smooth closed curve containing $S$. Since the resolvent is the Greens function of $H$, it satisfies the Combes-Thomas bound 5. Since the spectrum of $H$ is bounded, it may be enclosed in a curve of finite length, from which we determine that

$$|P_S(x,y)| \le Ce^{-\xi_\alpha|x-y|},$$

since $\inf_{z\in\gamma} d_z = \inf_{z\in\gamma}\operatorname{dist}(z,\operatorname{Spec}(H)) > 0$. Hence

$$\sum_{x,y\in\mathbb{Z}^2} |P_S(x,y)|e^{-\varepsilon|x|}e^{\nu|x-y|} < \infty$$

holds for $\nu = \xi_\alpha$ and for any $\varepsilon > 0$, since $e^{-\varepsilon|x|} = e^{-\varepsilon|x_1|}e^{-\varepsilon|x_2|}$ is summable on $\mathbb{Z}^2$ by Lemma 2.

□

This statement about decay of the matrix elements can be turned into a statement about the norm of the operator.

**Lemma 14.** *Let $P_S$ be the spectral projection onto $S \subseteq \operatorname{Spec}(H)$. For all $\varepsilon > 0$,*

$$\sup_{\ell\in\mathrm{Lip}^1} \|e^{\nu\ell(x)}e^{-\varepsilon|x|}P_S e^{-\nu\ell(x)}\| < \infty,$$

*where $\nu$ is the same as in 13.*

*Proof.* The matrix elements of the operator are $P_S(x,y)e^{\nu(\ell(x)-\ell(y))}e^{-\varepsilon|x|}$. By Holmgren's bound, the operator's norm is therefore bounded by

$$\max\{\sup_x \sum_y |P_S(x,y)|e^{\nu(\ell(x)-\ell(y))}e^{-\varepsilon|x|},\ \sup_y \sum_x |P_S(x,y)|e^{\nu(\ell(x)-\ell(y))}e^{-\varepsilon|x|}\}.$$

Replacing the supremum with a sum yields the bound

$$\|P_S(x,y)e^{\nu(\ell(x)-\ell(y))}e^{-\varepsilon|x|}\| \le \sum_{x,y} |P_S(x,y)|e^{\nu|\ell(x)-\ell(y)|}e^{-\varepsilon|x|},$$

and taking a supremum over $\ell \in \mathrm{Lip}^1$ completes the proof by Lemma 13. □

# Appendix B

# The Lieb-Robinson Bound

Let the dimensions of the Hilbert spaces at each lattice site $\dim(\mathcal{H}_x)$ be uniformly bounded. Let $A \in \mathcal{U}_X$ and $B \in \mathcal{U}_Y$ be any operators having disjoint supports $X \cap Y = \varnothing$, and denote by $r = \mathrm{dist}(X, Y)$ the distance between them. Let $A(t) = e^{itH} A e^{-itH}$ be the time evolution of $A$. Then

The following is a version of the Lieb-Robinson bound [11].

**Proposition 3.** *Let $A(t) = e^{itH} A e^{-itH}$ be the time evolution of an operator $A$ under a gapped Hamiltonian. Then*

$$\|[A(t), B]\| \leq C\|A\|\|B\| \min\{|X|, |Y|\} e^{-\frac{r - v|t|}{\xi}},$$

*where $C$, $v$, and $\xi$ are positive constants.*

A consequence of the Lieb-Robinson bound is the following estimate on the growth of the support of $A$ in time.

**Proposition 4.** *Let $A$ be any operator with support in a finite set $X$, and let $S = (X^r)^{\mathsf{c}}$ be the set of sites whose distance from $X$ is at least $r$. Let $A(t) = e^{itH} A e^{-itH}$ be its time evolution under a gapped Hamiltonian. Then*

$$\|A(t) - tr_S(A(t))\| \leq C|X|\|A\| e^{-\frac{r - v|t|}{\xi}},$$

*where $C$, $v$, and $\xi$ are as in 3.*

*Proof.* The partial trace is given by

$$\mathrm{tr}_S(A(t)) = \int_{\mathcal{G}_S} U A(t) U^* d\mu(U),$$

where $\mathcal{G}_S$ is the group of unitaries supported on $S$ and $\mu$ denotes the Haar measure. From this we see that

$$\begin{aligned}
\|A(t) - \mathrm{tr}_S(A(t))\| &\leq \int_{\mathcal{G}_S} \|A(t) - U A(t) U^*\| d\mu(U) \\
&= \int_{\mathcal{G}_S} \|[A(t), U] U^*\| d\mu(U) \\
&\leq \|[A(t), U]\|
\end{aligned}$$

whence the Lieb-Robinson bound 3 completes the proof. □

# Appendix C

# The Helffer-Sjöstrand Representation

The Helffer-Sjöstrand representation is a functional calculus $f \mapsto f(H)$ for arbitrary (possibly unbounded) operators $H$ on the set of functions

$$\mathcal{A} = \bigcup_{\beta<0} \{f : \mathbb{R} \to \mathbb{C} \ : \ f \in C^\infty(\mathbb{R}),\ |f^{(n)}(x)| \le c_n (1+x^2)^{\frac{\beta-n}{2}}\}.$$

It has the following properties. For proofs and further reading, see [13].

**Theorem 7.** *For any $f \in \mathcal{A}$,*

1. *$f \mapsto f(H)$ is an algebraic homomorphism (linear and multiplicative).*
2. *$\overline{f}(H) = f(H^*)$.*
3. *$\|f(H)\| \le \|f\|_\infty$.*
4. *For all $w \notin \mathbb{R}$, if $r_w(s) = \frac{1}{s-w}$ then $r_w(H) = (H-z)^{-1}$.*
5. *For all $f \in C_c^\infty(\mathbb{R})$ with supp$(f) \cap$ Spec$(H) = \varnothing$, we have $f(H) = 0$.*

For $f \in \mathcal{A}$ which are also Borel, $f(H)$ agrees with the operator given by the Borel functional calculus. There is an explicit formula for $f(H)$, which is given by

$$f(H) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (H-z)^{-1} dz \wedge d\bar{z},$$

where $\tilde{f} : \mathbb{C} \to \mathbb{C}$ is a *quasi-analytic extension* of $f : \mathbb{R} \to \mathbb{R}$. It is defined as follows. For any smooth $f$, we set

$$\tilde{f}(z) = \sum_{r=0}^{n} \tau\left(\frac{y}{(1+x^2)^{1/2}}\right) \frac{(iy)^r}{r!} f^{(r)}(x)$$

where $\tau : \mathbb{R} \to \mathbb{R}$ is any smooth function satisfying

$$\tau(s) = \begin{cases} 1 & |s| < 1 \\ 0 & |s| > 2 \end{cases}.$$

The extension turns out to be independent of the choice of $n$ and $\tau$. Furthermore, as $|\mathrm{Im}(z)| \to 0$, the Wirtinger derivative of the extension obeys the bound

$$\left| \frac{\partial \tilde{f}}{\partial \bar{z}} \right| = \mathcal{O}(|\mathrm{Im}(z)|^n).$$

Thus $\frac{\partial \tilde{f}}{\partial \bar{z}} = 0$ for all real $z$, which is why $\tilde{f}$ is called a "quasi"-analytic extension (the Wirtinger derivative would be zero everywhere if $\tilde{f}$ were truly analytic).

A crucial property of the Helffer-Sjöstrand functional calculus is the following bound.

**Lemma 15.** *For any $n \in \mathbb{N}$, the quasi-analytic extension $\tilde{f}$ can be chosen so that*

$$\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \frac{1}{|\mathrm{Im}(z)|^{p+1}} dz \wedge d\bar{z} \leq C_0 \sum_{k=0}^{n+2} \|f^{(k)}\|_{k-p-1},$$

*where the norms on the right hand side are defined by*

$$\|f\|_m = \int_{-\infty}^{\infty} |f(x)|(1+x^2)^{m/2} dx.$$

In particular, if $f = \rho$ (as defined in chapter 2) then clearly the norms on the right hand side are all finite since $\rho$ is bounded and the derivatives of $\rho$ have compact support. This lemma is often useful for bounding operators because the resolvent, which frequently shows up in the integrand of Helffer-Sjöstrand representations, obeys the bound $\|(H-z)^{-1}\| \leq |\mathrm{Im}(z)|^{-1}$.