The Quantum Hall Effect and Bulk-Edge Correspondence on Lattice Systems

by

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Abstract

In this thesis, we investigate the *integer quantum Hall effect* (IQHE). Discovered in 1980 by Nobel Laureate Klaus Von Klitzing, the IQHE is a phenomenon in which the Hall resistivity of a 2-dimensional electron gas is precisely quantized to integer multiples of h/q^2 when subjected to a strong magnetic field at very low temperatures.

In particular, we provide a rigorous theoretical description of a phenomenon known as *bulk-edge correspondence*, wherein the value of the Hall conductivity is unaffected by whether or not one considers a system with an edge or an infinite plane. This is accomplished in two settings. First, we ignore interactions between electrons, and then proceed to the more challenging interacting setting.

Preface

Chapter 2 is based on work conducted in [1]. This thesis is the work of the author, Justin Furlotte, under the supervision of Sven Bachmann.

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Acknowledgements

I would like to thank my supervisor, Sven Bachmann, for his patience, his kindness, and his inspiring mastery in the field of mathematical physics and quantum lattice systems.

Dedication

To Abigail Sanderson, for her unwavering belief in me, and to my parents, for their endless support.

Chapter 1

Introduction

Since its unexpected discovery in 1980, the Integer Quantum Hall Effect (IQHE) has captured significant interest in the mathematical physics community. This effect occurs when a 2 dimensional electron gas at near 0 Kelvin is pierced by a strong magnetic field. As the strength of the field increases, the Hall resistivity - a macroscopic quantity - experiences quantization, as in Figure 1.1.

In this thesis, we begin by proving a simple classical calculation for the Hall conductivity, and emphasize that the result does not agree with experiment. The IQHE, despite being a macro-scale phenomenon, is fundamentally reliant on quantum mechanics. We give another heuristic argument due to Laughlin [15] for why this quantization occurs, before going into the main body of the thesis in Chapters 2, 3.

There we provide a proof of *bulk-edge correspondence*; this is the interesting mathematical fact that the Hall conductivity does not depend on whether or not the system is assumed to have an edge. In the system with an edge, the Hall conductivity is defined by assuming that current is transported along the edge, while in the bulk (i.e. edgeless) system, the Hall current is assumed to be carried throughout the entire bulk of the material. This bulk-edge correspondence between the Hall conductivities in the IQHE is a special case of a more general bulk-edge correspondence between other invariants of *topological insulators*.

1.1 Heuristic Arguments

We provide a brief introduction to the quantum Hall effect. We emphasize that the results of this section are not rigorous.

1.1.1 The Classical Hall Effect

Using classical electromagnetism is not enough to predict the plateaux seen experimentally. Suppose we have a 2-dimensional electron gas, and let

$\vec{B} = B\hat{x_3}$ be a magnetic field piercing the plane of the electrons. They are subjected to a Lorentz force

$$m\dot{\vec{v}} = -q\vec{v} \times \vec{B}.$$

The solution to this differential equation is given by the *cyclotron orbits*,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} a + r\sin(\omega_B t + \phi) \\ b + r\cos(\omega_B t + \phi) \\ 0 \end{pmatrix}$$

where $\omega_B = qB/m$ is the *cylontron frequency*. An electric field $\vec{E} = E\hat{x_1}$ is introduced, and the electrons move in the x_1-direction. We now employ the *Drude model*,

$$m\dot{\vec{v}} = -q(\vec{E} + \vec{v} \times \vec{B}) + \frac{m}{\tau}\vec{v},$$

where the final term is a linear friction term, and τ is the scattering time. At equilibrium, the equation reads

$$\vec{J} + \frac{q\tau}{m}\vec{J} \times \vec{B} = -\frac{q^2 n\tau}{m}\vec{E},$$

where $\vec{J} = -nq\vec{v}$ is the current density, related to the velocity by the density of electrons per unit area n. In matrix notation, this reads

$$\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & q \end{pmatrix} \vec{J} = -\frac{q^2 n\tau}{m}\vec{E}.$$

Since the matrix on the left is invertible, we may write $\vec{J} = \sigma\vec{E}$, which is Ohm's Law. The *conductivity tensor* is given by

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{pmatrix} = -\frac{q^2 n\tau}{m(1 + \omega_B^2\tau^2)} \begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix}.$$

The off-diagonal components, $\pm\frac{q^2 n\tau}{m(1+\omega_B^2\tau^2)}\omega_B\tau$, are responsible for the Hall effect; the magnetic field induces a component of the current in the x_2-direction, in addition to the one in the x_1-direction from the electric field.

When making a measurement, physicists actually measure the resistivity. In particular, the *Hall resistivity*, given by the off-diagonal components of the resistivity tensor $\rho = \sigma^{-1}$, is simply

$$\rho_{xy} = \frac{B}{nq}.$$

The key prediction of the classical theory is that the Hall resistivity increases linearly in response to the strength of the magnetic field.

1.1.2 The Quantum Hall Effect

Von Klitzing's experimental observation in 1980 made it clear that classical electromagnetism is not sufficient to describe the Hall effect. At a temperature of about 8mK, the Hall resistivity looked like this as a function of the magnetic field strength.

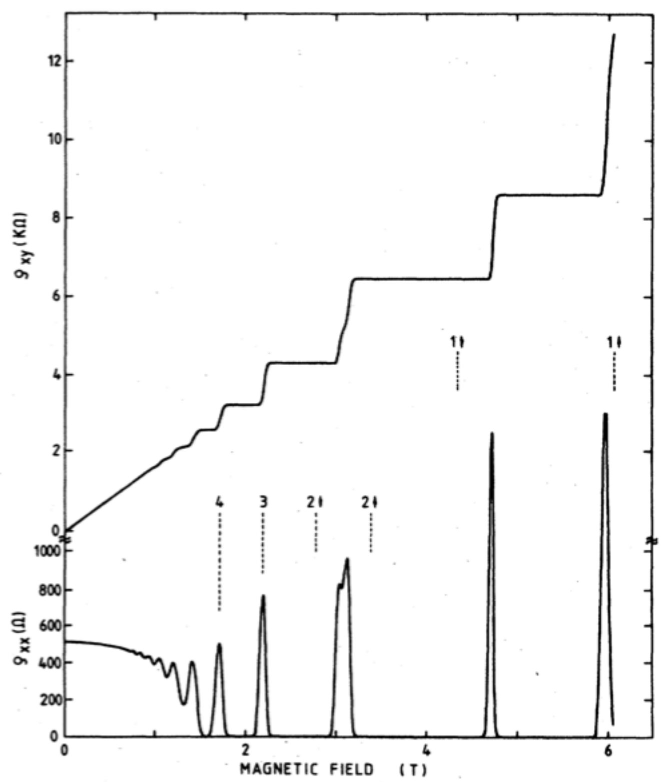

Figure 1.1: Von Klitzing's discovery - the Hall resistivity as a function of the magnetic field strength jumps in distinct plateaux. [23]

Even more surprising, the plateaux occur at the values

$$\rho_{xy} = \frac{h}{q^2}\frac{1}{n},$$

where n is an integer. This phenomenon is known as the *integer quantum Hall effect* (IQHE). In fact, this integer can be measured to such extraordinary precision (about one part in 3×10^{-10}) that as of 2020, the SI definition of the Ohm itself has been redefined in terms of the quantum Hall resistivity. We remark that, although the resistivity is what is measured in a lab, throughout this thesis we work with the quantum Hall *conductivity*,

$$\sigma_H = \frac{q^2}{h} n.$$

The first quantum mechanical explanation of the integer-valued Hall conductivity is due to Laughlin in his short but seminal 1981 paper [15], for which he also won the Nobel prize. Laughlin explained the plateaux using a charge transport argument.

The Laughlin Argument

Rather than a flat sheet, consider gluing the edges to form a cylinder, as depicted in Figure 1.2. The cylinder is pierced by a uniform magnetic field $\vec{B}$ normal to its surface. In addition to this background magnetic field, suppose that a magnetic flux ϕ is also threaded through the cylinder.

As $\phi(t)$ increases slowly from $\phi(0) = \phi_0$ to $\phi(T) = \phi_0 + \Delta\phi$, the charge increases by $\Delta Q = -\sigma_H \Delta\phi$. This can be seen by Faraday's law,

$$-\frac{d\phi}{dt} = \oint_C \vec{E} \cdot d\vec{l}$$

combined with the formula for the Hall current, $\vec{J_H} = -\sigma_H \hat{\rho} \times \vec{E}$. The current pumped across the fiducial line C is

$$\begin{aligned}
\frac{dQ}{dt} &= \oint_C \vec{J_H} \cdot \hat{z} dl \\
&= -\sigma_H \oint (\hat{\rho} \times \vec{E}) \cdot \hat{z} dl \\
&= -\sigma_H \oint_C (\hat{\rho} \times \hat{z}) \cdot \vec{E} dl \\
&= \sigma_H \oint_C \hat{\theta} \cdot \vec{E} dl \\
&= \sigma_H \oint_C \vec{E} \cdot d\vec{l} \\
&= -\sigma_H \frac{d\phi}{dt}.
\end{aligned}$$

Thus $\Delta Q = -\sigma_H \Delta\phi$. We now argue that the Hall conductivity must be an integer multiple of q^2/h. Suppose that we now unroll the cylinder to

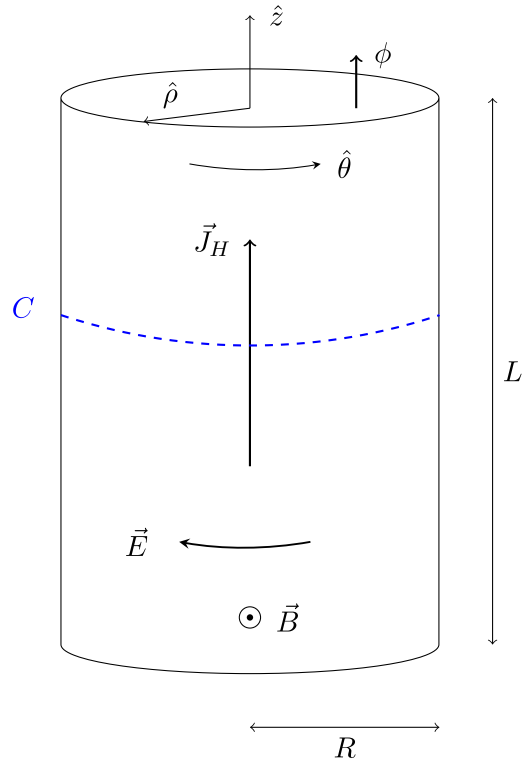

Figure 1.2: Laughlin's cylinder.

work in Cartesian coordinates, but still identify the edges $x = 0$ and $x = L$ with periodic boundary conditions. We take the positive x-direction to be the opposite direction as J_H in figure 1.2, and the positive y-direction to be the same as $\vec{E}$ in the figure. We choose the *Landau gauge* for the magnetic field,

$$\vec{A}_B = \begin{pmatrix} 0 \\ Bx \\ 0 \end{pmatrix},$$

and

$$\vec{A}_\phi = \begin{pmatrix} 0 \\ \frac{\phi}{2\pi R} \\ 0 \end{pmatrix}$$

for the flux potential, which has vanishing curl (as desired since there's no magnetic field from ϕ on the cylinder). The total magnetic vector potential $\vec{A} = \vec{A}_B + \vec{A}_\phi$ appears in the Hamiltonian

$$\begin{aligned} H &= \frac{1}{2m}(\vec{p} - q\vec{A})^2 \\ &= \frac{1}{2m}\left(p_x^2 + \left(p_y - q\left(Bx + \frac{\phi}{2\pi R}\right)\right)^2\right). \end{aligned}$$

The fact that the Hamiltonian commutes with the y-momentum is evident from the expression above, and allows this us to replace p_y with its eigenvalue, $\hbar k$. The wavenumber is quantized by the periodic boundary condition

$$e^{i0k} = e^{i2\pi Rk},$$

which implies $p_y = \hbar k = \hbar j/R$ with $j \in \mathbb{N}$, giving

$$\begin{aligned} H_j &= \frac{1}{2m}\left(p_x^2 + \left(\frac{\hbar j}{R} - q\left(Bx + \frac{\phi}{2\pi R}\right)\right)^2\right) \\ &= \frac{1}{2m}\left(p_x^2 + q^2B^2\left(x - \frac{1}{2\pi RB}\left(\frac{h}{q}j - \phi\right)\right)^2\right). \end{aligned}$$

Define the frequency $\omega_B = \frac{qB}{m}$, and the *shift factor*

$$s_j(\phi) := \frac{1}{2\pi RB}\left(\frac{h}{q}j - \phi\right).$$

In this notation, the Hamiltonian takes the form

$$H_j = \frac{1}{2m}p_x^2 + \frac{1}{2}m\omega_B^2(x - s_j(\phi))^2.$$

This is a shifted quantum harmonic oscillator in x, with frequency ω_B. The spectrum is as usual,

$$E_n = \hbar\omega_B\left(n + \frac{1}{2}\right).$$

The eigenstates are called *Landau levels*, and they are exponentially localized at $x = -s_j(\phi)$. They are of the form

$$\psi_{n,j}(x,y) = C_{n,j}e^{i\frac{yj}{R}}H_n(x - s_j)e^{-\frac{(x-s_j(\phi))^2}{2\ell_B^2}}$$

where $\ell_B^2 = \frac{\hbar}{qB}$, $C_{n,j}$ are normalization constants, and H_n are the Hermite polynomials.

The crucial argument now comes from inspecting the shift $s_j(\phi)$. Suppose we adiabatically increase ϕ by one *flux quantum*, $\phi_0 \mapsto \phi_0 + \frac{h}{q}$, as time evolves from $t = 0$ to $t = T$. The adiabatic principle tells us that the eigenstates of the Hamiltonian at $t = 0$ must also be an eigenstate of the Hamiltonian at $t = T$. But we can find exactly what the new eigenstate is; the only thing that changes is the shift, which by a simple calculation becomes

$$s_j\left(\phi_0 + \frac{h}{q}\right) = \frac{1}{2\pi RB}\left(\frac{h}{q}j - \phi_0 - \frac{h}{q}\right) = s_{j-1}(\phi_0).$$

The (exponentially localized) wavefunctions are each transported upward in the $-x$ direction by an increment of $s_1(0) = \frac{\hbar}{qB}\frac{1}{R}$ as ϕ increases by one flux quantum. We remark that the wavefunctions $\psi_{n,j}$ are Gaussian-like, with standard deviation $\ell_B^2 = \frac{\hbar}{qB}$. If $R \ll 1$, the shift increment $s_1(0)$ is large enough that we may treat the wavefunctions as being localized in a similar manner as depicted in the diagram of Figure 1.3.

Heuristically, this diagram explains why charge transport occurs. The landau levels take their usual discrete spectrum, and bend up sharply at the edges. Inspect one filled Landau level. Each of the states is transported to the left.In particular, on the bottom edge of the cylinder (at $x = L$), the single empty state just above the Fermi energy μ is brought below the Fermi energy. On the top edge, the single filled state just below the Fermi energy is brought above the Fermi energy. Everywhere in between, states are merely transported up, each one filling the other's place. Thus, exactly one charge q is transported from the bottom edge to the top edge. This occurs once per filled Landau level, and thus the total charge transported is

$$\Delta Q = -qN_L$$

where N_L is the (integer) number of filled Landau levels. Our result $\Delta Q = -\sigma_H \Delta\phi$ now gives us exactly the Hall conductivity which agrees with experiment,

$$\sigma_H = \frac{q^2}{h}N_L \qquad\qquad N_L \in \mathbb{N}.$$

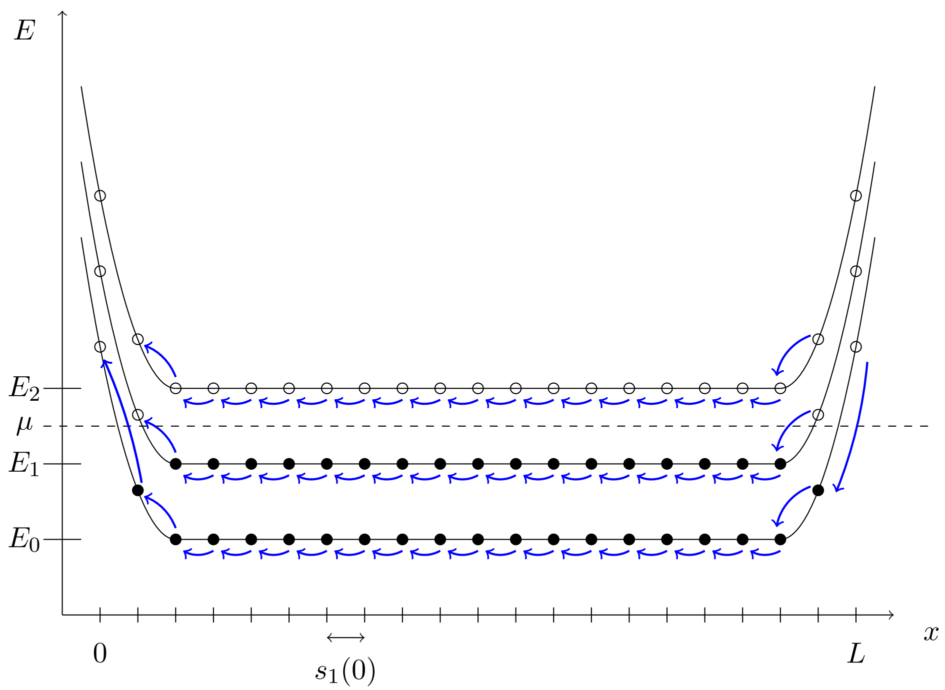

Figure 1.3: Quantum Hall conductance explained using charge transport. The states are exponentially localized and separated by a distance of $s_1(0)$.

Degeneracy, Disorder, and Plateaux

The reasoning above only guarantees the correct value of σ_H if the Landau levels are exactly filled. Furthermore, the argument relies on a fictitious magnetic flux ϕ. As $\phi \mapsto \phi + h/q$, one charge per Landau level is transported from bottom edge to top. Isn't σ_H supposed to increase as B decreases, not as ϕ decreases?

The idea is that this magnetic flux is purely an explanatory tool used to derive the formula $\sigma_H = N_L \frac{q^2}{h}$. The role of the magnetic field strength B becomes apparent from inspecting the degeneracy of the Landau levels. The number of states $\psi_{n,j}$ per fixed Landau level n must be finite, since $0 \leq x \leq L$ implies (roughly) that $-L \leq s_j(\phi) \leq 0$. We ignore ϕ since adding h/q only corresponds to $j = 1$ extra state, which is insignificant to the calculation. Substituting our expression for the shift $s_j(0)$, the inequality becomes

$$-L\frac{qBR}{\hbar} \leq j \leq 0.$$

The number of states N_s per Landau level is therefore

$$N_s = L\frac{qBR}{\hbar} = \frac{qBA}{h},$$

where $A = 2\pi RL$ is the total area of the cylinder. From here we see that as B decreases, so too does N_s; the electrons must therefore begin to populate higher Landau levels. Once the next highest Landau level is filled, the Hall conductivity is increased by exactly q^2/h.

What about the plateaux? Disorder, i.e. the presence of impurities in a real world material, is required to explain why the Hall conductivity does not change even when Landau levels are only partially filled. We model disorder using a random potential. Under the addition of a suitably small random potential to the Hamiltonian, the degeneracy of the Landau levels is lifted; the spectrum broadens into bands as in Figure 1.4. Furthermore, while the center of the bands is continuous spectrum (red), the edges of the bands are pure-point (blue) [2, 12]. This phenomenon is called Anderson localization.

The pure-point states, called *localized states*, contribute nothing to the current, as they have compact support which does not wrap around the cylinder. The effect of ϕ on these states can therefore be gauge transformed away, since the domain of their support is simply connected, ensuring that $\vec{A}_\phi$ is a (locally) conservative vector field. This is not the case on the cylinder as a whole, and in particular it is not the case for *extended states* (red), whose domain is not simply connected. Thus only extended states contribute to charge transport.

As the magnetic field strength B decreases, the degeneracy of each Landau level, $N_s = \frac{q}{h}BA$, also decreases. As electrons fill states of higher energy within the band, the Fermi energy increases. However, the states at the edges of the bands (blue) are localized and contribute nothing to the current, and thus leave the Hall conductivity unchanged. This gives rise to the plateaux.

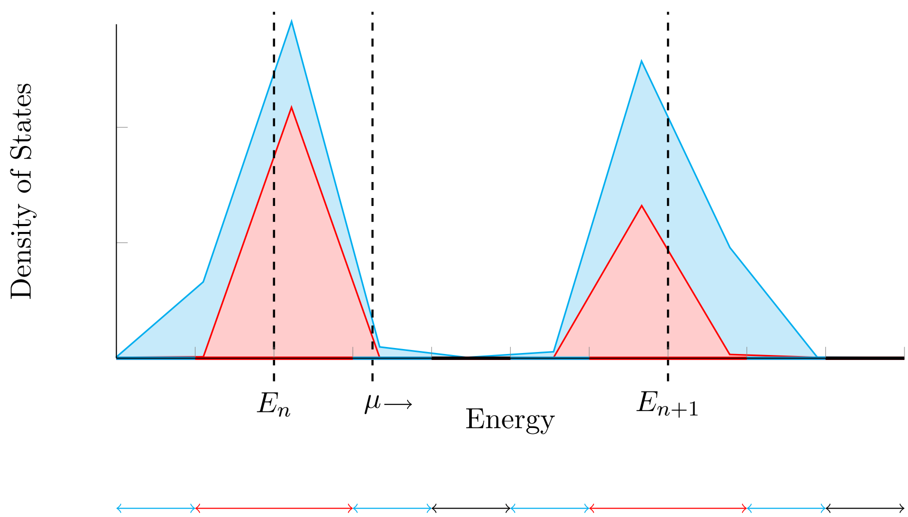

Figure 1.4: Disorder, modelled by a small random potential, breaks the degeneracy of the discrete Landau levels E_n. The blue states are *localized* and do not contribute to charge transport, while the red states are *extended*. As B decreases, the Fermi energy μ increases. When μ moves through the blue (localized) and black (spectral gap) regions, the current (and therefore the Hall conductivity) are not affected; this gives rise to the plateaux observed experimentally. Only once μ begins moving through the red (extended) region will the Hall conductivity begin to increase to its next plateau.

Chapter 2

Noninteracting Bulk-Edge Correspondence

An intriguing mathematical fact of the quantum Hall effect is the equality of bulk and edge conductivity, in the sense that whether the system is assumed to have an edge or not is immaterial. Proving this fact, both in the interacting and noninteracting setting, is the main focus of this thesis.

We begin by generalizing from Landau Hamiltonians to a much more general class of Hamiltonians, and derive appropriate formulae for the "bulk" and "edge" Hall conductivities in this scenario.

2.1 General Setting

Consider the lattice $\mathbb{Z}^2$, and the associated Hilbert space of square-summable sequences of vectors in $\mathbb{C}^n$,

$$\ell^2(\mathbb{Z}^2, \mathbb{C}^n) = \left\{ (x_i)_{i\in\mathbb{Z}^2} \subset \mathbb{C}^n \ : \ \sum_{i\in\mathbb{Z}^2} \|x_i\|^2 < \infty \right\},$$

with inner product $\langle x, y\rangle = \sum_{i\in\mathbb{Z}^2} x_i \overline{y_i}$. We denote this as $\ell^2(\mathbb{Z}^2)$ for short. On this Hilbert space we define a *bulk Hamiltonian* $H_B : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)$, whose matrix elements follow a short-range assumption:

Assumption 1. *There exists some $\alpha > 0$ such that*

$$\sup_{y\in\mathbb{Z}^2} \sum_{x\in\mathbb{Z}^2} |H_B(x, y)|(e^{\alpha|x-y|} - 1) \le C < \infty,$$

where $|x| = |x_1| + |x_2|$ is the taxicab metric.

We also construct an *edge Hamiltonian* on the lattice $\mathbb{Z}^2_a := \{x \in \mathbb{Z}^2 : x_2 > -a\}$, denoted by $H_a : \ell^2(\mathbb{Z}^2_a) \to \ell^2(\mathbb{Z}^2_a)$. The bulk and edge Hamiltonians are related by the *edge operator* $E_a : \ell^2(\mathbb{Z}^2_a) \to \ell(\mathbb{Z}^2)$,

$$E_a := J_a H_a - H_B J_a,$$

where $J_a : \ell^2(\mathbb{Z}_a^2) \to \ell(\mathbb{Z}^2)$ denotes extension by zeroes. We require only that that the edge operator satisfies the edge assumption

Assumption 2. *The edge operator satisfies*

$$\sup_{z\in\mathbb{Z}^2} \sum_{y\in\mathbb{Z}_a^2} |E_a(x,y)| e^{\alpha(|x_2+a|+|x_1-y_1|)} \leq C < \infty$$

for some $\alpha > 0$, where $|x| = |x_1| + |x_2|$ is the taxicab metric.

The interpretation is that $E_a = J_a H_a - H_B J_a$ is the difference between first applying H_a on $\ell^2(\mathbb{Z}_a^2)$, and then making everything below $-a$ into zeroes, versus first making all $x \in \mathbb{Z}^2$ such that $x_2 < -a$ zeroes, and the applying H_B. The assumption ensures that the effects from introducing the edge at $-a$ die exponentially as we move upward away from the edge (due to the $|x_2 - (-a)|$ term in the exponent), and also terms do not interact too much as their x_1 distance increases (due to the $|x_1 - y_1|$ term in the exponent).

A simple example of an edge Hamiltonian satisfying the edge condition is $H_a = J_a^* H_B J_a$, which gives $E_a = (\mathcal{J}_a \mathcal{J}_a^* - 1) H_B$. The idea is that for a state $\psi \in \ell^2(\mathbb{Z}_a^2)$, we have $\langle \psi, H_a \psi \rangle = \langle (J_a \psi), H_B(J_a \psi) \rangle$, which we interpret as the edge Hamiltonian having the same expectation as the bulk Hamiltonian if we just transformed all the states with support below the line $x_2 = -a$ into zeroes below $x_2 = -a$. The edge operator is

$$E_a = (J_a J_a^* - \mathbb{1}) H_B J_a = \begin{cases} -H_B(x,y) & \text{if } x_2 < -a \\ 0 & \text{if } x_2 \geq -a \end{cases}$$

Intuitively, there is no difference between H_B and H_a on $\mathbb{Z}_a^2$. The edge assumption 2 then follows from the bound $|E_a(x,y)| \leq |H_B(x,y)|$ and the short range assumption 1.

We also make the following assumption about the bulk Hamiltonian.

Assumption 3. *The bulk Hamiltonian has a spectral gap. That is, there exists an interval $\Delta \subset \mathbb{R}$ such that for all L,*

$$\Delta \cap \operatorname{Spec}(H_B) = \varnothing.$$

Remark: The spectral gap assumption can be relaxed to a "mobility gap" assumption,

$$\sup_{f \in B_c(\Delta)} |f(H_B)(x,y)|(1+|x|)^{-\alpha_1} e^{\alpha_2 |x-y|} < \infty$$

for some $\alpha_1 > 0$, where $B_c(\Delta)$ is the set of Borel functions f which are constant on $(-\infty, \inf \Delta)$ and on $(\sup \Delta, \infty)$ such that $|f(x)| \leq 1$ for all x [1].

We define the *bulk conductivity* at Fermi energy μ as follows. Suppose we subject the system to an external electric potential difference V in the x_2 direction. We write this as $V_0 \Lambda_2$, where Λ_i are multiplication operators $\Lambda_i |\psi(x_1, x_2)\rangle = \Lambda(x_i) |\psi(x_1, x_2)\rangle$ which are *switch functions*,

$$\Lambda : \mathbb{R} \to \mathbb{R} \qquad \Lambda(x_i) = \begin{cases} 1 & \text{if } x_i \leq 0 \\ 0 & \text{if } x_i \geq 1 \end{cases}$$

and are smooth and monotonically decreasing on $(0, 1)$. Note that the ensuing physics (in particular, our definition of the Hall conductivity) is independent of the particular choice of switch function Λ_i, since any two switch functions are exactly equal on the lattice.

This gives $\vec{E} = -\nabla V = -V_0 \frac{\partial \Lambda_2}{\partial x_2}$, so that $\vec{E}$ is has compact support $\mathrm{supp}(\Lambda_2')$. We introduce a function which grows slowly in time as t grows from $-\infty$ to 0, so as to invoke the adiabatic principle. Here, we choose $e^{\varepsilon t}$, and we will let $\varepsilon \to 0$ at the end. The Hamiltonian therefore experiences a perturbation,

$$\widetilde{H}_B(t) = H_B + V_0 \Lambda_2 e^{\varepsilon t}.$$

We define the Hall current operator $J_H = i[\widetilde{H}_B(t), \Lambda_1] = i[H_B, \Lambda_1]$, which is related to the Hall conductivity by $J_H = \sigma_H V$. We also denote by $P_\mu := P((-\infty, \mu])$ the projection-valued measure associated with H_B onto states with energy below the Fermi energy μ (see Appendix A.1).

Proposition 1. *The Hall conductivity σ_H in the bulk system is equal to*

$$\sigma_B = -i \mathrm{Tr}\left(P_\mu \left[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]\right]\right).$$

Proof. We begin with the Heisenberg equation of motion for the density matrix, $\dot{\rho}(t) = -i[\widetilde{H}_B(t), \rho(t)]$, with initial condition $\lim_{t \to -\infty} \|\rho(t) - e^{-itH_B} P_\mu e^{itH_B}\| = 0$, which also implies $\lim_{t \to -\infty} \|e^{itH_B} \rho(t) e^{-itH_B} - P_\mu\| = 0$.

We work in the interaction picture by defining $\rho_I(t) = e^{itH_B} \rho(t) e^{-itH_B}$ and $\Delta H_B(t) = e^{itH_B} V_0 \Lambda_2 e^{\varepsilon t} e^{-itH_B}$. Thus

$$\dot{\rho}_I(t) = -i[\Delta H_B(t), \rho_I(t)].$$

The exact solution to this differential equation is readily verified to be

$$\rho_I(t) = i\int_{-\infty}^{t} [\Delta H_B(s), \rho_I(t)]ds + P_\mu.$$

Indeed, taking the derivative of the right hand side gives $i[\Delta H_B(t), P_\mu] = i[\Delta H_B(t), \rho_I(t)]$, and the initial condition is also satisfied. This also shows that

$$\|\rho_I(t) - P_\mu\| \leq 2\int_{-\infty}^{t} \|V_0 \Lambda_2 e^{\varepsilon s} P_\mu\| ds = \mathcal{O}(V_0),$$

which implies $[\Delta H_B(s), \rho_I(t)] = [\Delta H_B(s), P_\mu] + \mathcal{O}(V_0^2)$. Therefore

$$\rho_I(t) = i\int_{-\infty}^{t} [\Delta H_B(s), P_\mu]ds + P_\mu + \mathcal{O}(V_0^2).$$

The Hall conductivity is (by definition) the linear response coefficient, which is

$$\begin{aligned}
\sigma_H &= \lim_{V_0\to 0}\lim_{\varepsilon\to 0} \frac{\mathrm{Tr}(\rho_I(0)J_H) - \mathrm{Tr}(P_\mu J_H)}{V_0} \\
&= \lim_{V_0\to 0}\lim_{\varepsilon\to 0} \frac{i}{V_0}\mathrm{Tr}\left(i\int_{-\infty}^{0} [\Delta H_B(s), P_\mu][H_B, \Lambda_1]ds\right),
\end{aligned}$$

where we used the fact that the error $\mathcal{O}(V_0^2)$ introduced by replacing ρ_I with P_μ vanishes in the limit $V_0 \to 0$. Some simplifications can be made,

$$\begin{aligned}
\sigma_H &= -\lim_{V_0\to 0}\lim_{\varepsilon\to 0} \frac{1}{V_0}\mathrm{Tr}\left(\int_{-\infty}^{0} [e^{isH_B} V_0 \Lambda_2 e^{\varepsilon s} e^{-isH_B}, P_\mu][H_B, \Lambda_1]ds\right) \\
&= -\lim_{V_0\to 0}\lim_{\varepsilon\to 0} \mathrm{Tr}\left(\int_{-\infty}^{0} e^{isH_B}[\Lambda_2, P_\mu]e^{-isH_B}[H_B, \Lambda_1]e^{\varepsilon s}ds\right) \\
&= -\lim_{\varepsilon\to 0} \mathrm{Tr}\left(\int_{-\infty}^{0} (e^{isH_B}[H_B, \Lambda_1]e^{-isH_B}) \cdot ([\Lambda_2, P_\mu]e^{\varepsilon s})ds\right),
\end{aligned}$$

where we used the fact that P_μ and H_B commute. We also dropped the limit $V_0 \to 0$ in the final line because, even though the limits may not commute in general, the expression is independent of V_0. Using integration by parts on the two terms in brackets, and noting that $\frac{d}{ds}(e^{isH_B}\Lambda_1 e^{-isH_B} - \Lambda_1) = -ie^{isH_B}[H_B, \Lambda_1]e^{-isH_B}$, we obtain

$$\begin{aligned}\sigma_H &= -i \lim_{\varepsilon\to 0} \operatorname{Tr}\left(\int_{-\infty}^{0} (e^{isH_B}\Lambda_1 e^{-isH_B} - \Lambda_1)\frac{d}{ds}([\Lambda_2, P_\mu]e^{\varepsilon s})ds\right)\\ &= -i \lim_{\varepsilon\to 0} \varepsilon \operatorname{Tr}\left(\int_{-\infty}^{0} \Lambda_1^s[\Lambda_2, P_\mu]e^{\varepsilon s})ds\right)\end{aligned}$$

where $\Lambda_1^s := e^{isH_B}\Lambda_1 e^{-isH_B} - \Lambda_1$. Using the notation $\overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, it is readily verified that the commutator $[\Lambda_2, P_\mu]$ is an *off-diagonal* operator, in the sense that $[\Lambda_2, P_\mu] = \overline{[\Lambda_2, P_\mu]}$. Furthermore, a simple computation reveals that for any two trace-class operators A and B, $\operatorname{Tr}(\overline{A}B) = \operatorname{Tr}(A\overline{B})$. It therefore follows that

$$\sigma_H = -i \lim_{\varepsilon\to 0} \varepsilon \operatorname{Tr}\left(\int_{-\infty}^{0} \overline{\Lambda_1^s}[\Lambda_2, P_\mu]e^{\varepsilon s})ds\right).$$

The integrand can be broken into two terms,

$$\overline{\Lambda_1^s}[\Lambda_2, P_\mu]e^{\varepsilon s} = e^{-isH_B}\overline{\Lambda_1}e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s} - \overline{\Lambda_1}[\Lambda_2, P_\mu]e^{\varepsilon s}$$

by commutativity of P_μ and H_B. We show that the integral of the first term vanishes. We begin by breaking the first term down further into

$$e^{-isH_B}P_\mu\Lambda_1 P_\mu^\perp e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s} + e^{-isH_B}P_\mu^\perp\Lambda_1 P_\mu e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s}.$$

We treat the first of these two terms; the other is handled in an identical manner. We invoke the spectral theorem (Appendix A.1) to write $e^{-isH_B}P_\mu = \int_{-\infty}^{\mu} e^{-is\lambda}dP_\lambda$, and similarly $P_\mu^\perp e^{isH_B} = (\mathbb{1} - P_\mu)e^{isH_B} = \int_{\mu}^{\infty} e^{is\nu}dP_\nu$.

We remark that, since the Fermi energy μ is assumed to lie in a spectral gap, there must exist a neighbourhood $(\mu - \delta, \mu + \delta)$ in which there are no states. We exploit this fact to rewrite the limits of integration, $\int_{-\infty}^{\mu-\delta} e^{-is\lambda}dP_\lambda$ and $\int_{\mu+\delta}^{\infty} e^{is\nu}dP_\nu$. We therefore obtain

$$
\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon \int_{-\infty}^{0} e^{-isH_B} P_\mu \Lambda_1 P_\mu^\perp e^{isH_B}[\Lambda_2, P_\mu] e^{\varepsilon s} ds \\
&\qquad = \lim_{\varepsilon\to 0} \varepsilon \mathrm{Tr}\left(\int_{-\infty}^{0} \int_{-\infty}^{\mu-\delta} e^{-is\lambda} dP_\lambda \Lambda_1 \int_{\mu+\delta}^{\infty} e^{is\nu} dP_\nu [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right) \\
&\qquad = \lim_{\varepsilon\to 0} \varepsilon \mathrm{Tr}\left(\int_{-\infty}^{0} \int_{-\infty}^{\mu-\delta} \int_{\mu+\delta}^{\infty} e^{s(\varepsilon - i\lambda + i\nu)} dP_\lambda \Lambda_1 dP_\nu [\Lambda_2, P_\mu] ds \right)
\end{aligned}
$$

Performing the integral over s yields

$$
\lim_{\varepsilon\to 0} \varepsilon \int_{-\infty}^{0} e^{s(\varepsilon - i\lambda + i\nu)} ds = -\lim_{\varepsilon\to 0} \frac{\varepsilon}{i\varepsilon + \lambda - \nu}
$$

This limit is zero, since $\lambda \neq \nu$. Indeed, due to the spectral gap, the integration variables live in $\lambda \in (-\infty, \mu - \delta)$ and $\nu \in (\mu + \delta, \infty)$. The case for the $e^{-isH_B} P_\mu^\perp \Lambda_1 P_\mu e^{isH_B}[\Lambda_2, P_\mu] e^{\varepsilon s}$ term (where the P_μ and $P_\mu^\perp$ swap places) is treated analogously. Hence the first term in the integrand for σ_H vanishes, as claimed.

Finally, we return to our expression for the Hall conductivity, which now reads

$$
\sigma_H = i \lim_{\varepsilon\to 0} \varepsilon \mathrm{Tr}\left(\int_{-\infty}^{0} \overline{\Lambda_1} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right).
$$

It is a basic algebraic calculation to show that $\overline{\Lambda_1} = [[\Lambda_1, P_\mu], P_\mu]$. Evaluating the integral over s is now trivial; $\int_{-\infty}^{0} e^{\varepsilon s} ds = \varepsilon^{-1}$. Thus

$$
\sigma_H = i \mathrm{Tr}([[\Lambda_1, P_\mu], P_\mu][\Lambda_2, P_\mu]).
$$

Shifting the commutator completes the proof:

$$
\begin{aligned}
\sigma_H &= i \mathrm{Tr}(P_\mu [[\Lambda_2, P_\mu], [\Lambda_1, P_\mu]]) \\
&= -i \mathrm{Tr}(P_\mu [[\Lambda_1, P_\mu], [\Lambda_2, P_\mu]]) \\
&= -i \mathrm{Tr}(P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]).
\end{aligned}
$$

The justification for shifting the commutator is that $[\Lambda_1, P_\mu][\Lambda_2, P_\mu]$ is trace class by the proof of Lemma 4, noting that

$$
\|[\Lambda_1, P_\mu][\Lambda_2, P_\mu]\|_1 \leq \|[\Lambda_1, P_\mu] e^{3\delta|x_1|} e^{-\delta|x|}\| \cdot \|e^{-\delta|x|}\|_1 \cdot \|e^{3\delta|x_2|} e^{-\delta|x|}[\Lambda_2, P_\mu]\|
$$

where $|x| = |x_1| + |x_2|$ and the $e^{-\delta|x|}$ term is trace-class by lemma 3. □

Remark: This is reminiscent of the well-known adiabatic curvature formula,

$$\kappa = \mathrm{Tr}(P[\partial_1 P, \partial_2 P]) = \mathrm{Tr}(P\,[[P, K_1], [P, K_2]]) = \mathrm{Tr}(P[K_1, K_2]),$$

where K_i are called *generators of parallel transport*. We will see the adiabatic curvature formula again later in the interacting setting.

For the *edge conductivity*, we again need the current operator across the line $x_1 = 0$, which is this time given by $-i[H_a, \Lambda_1]$. We define

$$\sigma_E = -i \lim_{a\to\infty} \mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]),$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(r) = \begin{cases} 1 & \text{if } r \leq \inf \Delta \\ 0 & \text{if } r \geq \sup \Delta \end{cases}$$

and decreases smoothly and monotonically in Δ. The definition of σ_E is reminiscent of another formula we will see later in the interacting setting, $\mathrm{Tr}(\dot{P}J)$, where J is the current operator. The interpretation of σ_E is that if we apply a small potential difference V across $x_2 = -a$ to $x_2 = \infty$, there will be a net current

$$\begin{aligned} I &= -i\mathrm{Tr}(\rho(H_a + V)[H_a + V, \Lambda_1] - \rho(H_a)[H_a, \Lambda_1]) \\ &= -i\mathrm{Tr}((\rho(H_a + V) - \rho(H_a))[H_a, \Lambda_1]) \end{aligned}$$

Thus we obtain the conductivity

$$\sigma_E = \frac{I}{V} = -i\mathrm{Tr}\left(\frac{(\rho(H_a + V) - \rho(H_a)}{V}[H_a, \Lambda_1]\right) \to -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1])$$

in the limit as $V \to 0$. As we shall see, it turns out that σ_E is independent of the choice of ρ, and σ_B is independent of λ.

2.2 Equality of Bulk and Edge Conductivities

The main result of this section is

Theorem 1. *$\sigma_E = \sigma_B$.*

2.2.1 Outline of the Proof

Before giving the proof in its entirety, we outline the basic steps. Lemma 1 shows that $\sigma_E := \rho'(H_a)[H_a, \Lambda_1]$ is trace-class. We posit that the edge conductivity can be rewritten as $\sigma_E = \lim_{a\to\infty} \sigma_E(a)$, where

$$\sigma_E(a) = -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2),$$

since we have assumed that there is a spectral gap (as opposed to a mobility gap), so that there are extended states near the edge, and no bound states or resonances far from the edge. Thus, intuitively, the cutoff introduced by Λ_2 is irrelevant as we take $a \to \infty$. We provide a more concrete justification for this later in Lemma 1.

The key ingredient of the proof is the use of the functional calculus given by the Helffer-Sjöstrand representation of self-adjoint operators on a Hilbert space (Appendix F). The two crucial operators written in their Helffer-Sjöstrand representations are

$$\rho(H) = -\frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R(z) dz \wedge d\bar{z}$$

$$\rho'(H) = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R(z)^2 dz \wedge d\bar{z}$$

where $R(z) = (H - z)^{-1}$ is the resolvent of H. For ease of notation, we drop the $dz \wedge d\bar{z}$ from this point.

Another key observation is that it turns out that the bulk conductivity can actually be written

$$\sigma_B = i\mathrm{Tr}([\rho(H_B), \Lambda_1]\Lambda_2).$$

By employing the Helffer-Sjostrand representations above, one can add an operator of zero trace to the edge conductivity, and show that this operator converges in trace to $[\rho(H_B), \Lambda_1]\Lambda_2$ in the limit $a \to \infty$.

2.2.2 The Proof

Lemma 1. *The edge conductivity is equal to* $\lim_{a\to\infty} \sigma_E(a)$*, where*

$$\sigma_E(a) = -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2).$$

Proof. Since σ_B is translation invariant [1], we only need to prove the case $-i\mathrm{Tr}(\rho'(H_{a=0})[H_{a=0},\Lambda_1] = \sigma_B$. We drop the subscript, $H := H_{a=0}$.

A general fact of functional analysis is that if $A_n \xrightarrow{s} 0$ and X is trace-class, then $\|A_n X\|_1 \to 0$. Since the multiplication operator $\Lambda_{2,n}|\psi\rangle := \Lambda(x_2 - n)|\psi\rangle$ converges strongly to the identity as $n \to \infty$, it follows that

$$\|\rho'(H)[H,\Lambda_1](\mathbb{1} - \Lambda_{2,n})\|_1 \to 0,$$

and thus from the inequality $|\mathrm{Tr}(A)| \leq \|A\|_1$ we deduce

$$\sigma_E(a) = -i\mathrm{Tr}(\rho'(H)[H,\Lambda_1]) = -i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H)[H,\Lambda_1]\Lambda_{2,n}).$$

Consider the edge Hamiltonian with cutoff at $a = 0$ associated with the bulk Hamiltonian shifted down by n. We denote this modified edge Hamiltonian by H^n.

Whether we first cut off everything above $x_2 = n$ using $\Lambda_{2,n}$ and then apply $H_{a=0}$, or instead cut off everything above $x_2 = 0$ using Λ_2 and then apply the Hamiltonian H^n is immaterial. In other words,

$$-i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H)[H,\Lambda_1]\Lambda_{2,n}) = -i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H^n)[H^n,\Lambda_1]\Lambda_2).$$

Thus, our goal is to show that

$$\lim_{n\to\infty} \mathrm{Tr}(\rho'(H^n)[H^n,\Lambda_1]\Lambda_2) = \lim_{a\to\infty} \mathrm{Tr}(\rho'(H_a)[H_a,\Lambda_1]\Lambda_2).$$

□

Define

$$Z(a) = [\rho(H_a),\Lambda_1]\Lambda_2 - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[R_a(z),[H_a,\Lambda_1]\Lambda_2]dz^2$$

This operator has zero trace. Indeed, the first term has vanishing trace in the position basis, while the second term's integrand involves the trace of $[R,R] = 0$. The bounds necessary for shifting the commutator like this are on

$$\|[H_a,\Lambda_1]e^{\delta|x_1|}\|, \quad \|e^{-\delta|x_1|}\Lambda_2\|_1, \quad \|R\|,$$

the first two of which are given below in Lemmas 2, 3, and the third is the fact that the resolvent is bounded. So $\sigma_E(a) = \mathrm{Tr}(\Sigma(a))$, where

$$\begin{aligned}\Sigma(a) &= -i\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 + iZ(a)\\ &= -i\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 + i[\rho(H_a),\Lambda_1]\Lambda_2 - \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\rho}}{\partial\bar{z}}R_a(z)[R_a(z),[H_a,\Lambda_1]\Lambda_2]dz^2.\end{aligned}$$

Using the Hellfer-Sjöstrand representations for the first two terms on the right hand side, we obtain

$$\begin{aligned}\Sigma(a) = \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\rho}}{\partial\bar{z}}R_a(z)^2[H_a,\Lambda_1]\Lambda_2 dz^2 + \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\rho}}{\partial\bar{z}}R_a(z)[H_a,\Lambda_1]R_a(z)\Lambda_2 dz^2\\ - \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\rho}}{\partial\bar{z}}(R_a(z)^2[H_z,\Lambda_1]\Lambda_2 - R_a(z)[H_a,\Lambda_1]\Lambda_2 R_a(z))dz^2\\ = -\frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\rho}}{\partial\bar{z}}R_a(z)[H_a,\Lambda_1][R_a(z),\Lambda_2]dz^2\\ = \frac{i}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\rho}}{\partial\bar{z}}R_a(z)[H_a,\Lambda_1]R_a(z)[H_a,\Lambda_2]R_a(z)dz^2,\end{aligned}$$

where we used

$$[R_a(z),\Lambda_i] = -R_a(z)[H_a,\Lambda_i]R_a(z)$$

in the final equality. Next, we must prove that the operator above converges to the corresponding bulk operator in trace-norm,

$$\|\Sigma(a) - \Sigma_B\|_1 \to 0,$$

as $a \to \infty$, which in turn proves that $\mathrm{Tr}(\Sigma(a)) \to \mathrm{Tr}(\Sigma_B)$ because of the bound $|\mathrm{Tr}(A)| \le \|A\|_1$. Here, Σ_B is the same operator as before, but using the bulk operators H_B and $R_B(z)$. Once this limit is established, we shall prove that $\sigma_B = \mathrm{Tr}(\Sigma_B)$ to conclude the proof.

To show that the limit is zero as claimed, we bound the trace norm of the integrand of $\Sigma(a)$ with the ultimate goal of applying dominated convergence. We accomplish this bound by breaking it into three parts,

$$R[H_a,\Lambda_1]R[H_a,\Lambda_2]R = J_a[R,\Lambda_1]e^{\delta|x_1|}J_a^* \cdot e^{-\delta|x_1|}e^{-\delta|x_2|} \cdot J_a e^{\delta|x_2|}[H_a,\Lambda_2]RJ_a^*,$$

and bounding the norm of each with the following two lemmas. We remark that the extension J_a and its adjoint have norm 1.

Lemma 2.
$$\|[H_a, \Lambda_i]e^{\delta|x_i|}\| \leq C.$$

Proof. The operator can be bounded by inspecting its matrix elements

$$\begin{aligned}\langle x, [H_a, \Lambda_i]e^{\delta|x_i|}y\rangle &= \langle x, H_a\Lambda_i y\rangle e^{\delta|y_i|} - \langle x, \Lambda_i H_a y\rangle e^{\delta|y_i|} \\ &= H_a(x,y)e^{\delta|y_i|}(\Lambda(y_i) - \Lambda(x_i)).\end{aligned}$$

This is zero if $|x_i - y_i| \leq |y_i|$, since this would imply that x_i and y_i have the same sign, yielding $\Lambda(x_i) = \Lambda(y_i)$. So either the matrix element is zero, or $|y_i| \leq |x_i - y_i|$, which implies

$$\begin{aligned}|H_a(x,y)e^{\delta|y_i|}(\Lambda(y_i) - \Lambda(x_i))| &\leq 2|H_a(x,y)|e^{\delta|x_i - y_i|} \\ &\leq 2|H_a(x,y)|e^{\delta|x-y|} \\ &\leq C|H_a(x,y)|(e^{\delta|x-y|} - 1),\end{aligned}$$

where the final inequality comes from the fact that the diagonal matrix elements are zero. Hence the short range assumption

$$\sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}|H(x,y)|(e^{\mu|x-y|} - 1) < \infty$$

combined with Holmgren's bound

$$\|A\| \leq \max\left\{\sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}|A(x,y)|,\ \sup_{y\in\mathbb{Z}^2}\sum_{x\in\mathbb{Z}^2}|A(x,y)|\right\}$$

completes the proof. □

Lemma 3. $e^{-\delta|x_1|}e^{-\delta|x_2|}$ *is trace-class.*

Proof. We bound the trace norm by noticing that this is a positive operator satisfying

$$\langle (n,m), e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m)\rangle = \langle e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m), (n,m)\rangle,$$

so that its trace norm is equal to its trace. In the position basis, we see that its trace is given by a geometric series

$$\begin{aligned}
\mathrm{Tr}(e^{-\delta|x_1|}e^{-\delta|x_2|}) &= \sum_{(n,m)\in\mathbb{Z}^2} \langle (n,m), e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m)\rangle \\
&\leq 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-\delta m}e^{-\delta n} \\
&= 2\left(\frac{1}{1-e^{-\delta}}\right)^2 .
\end{aligned}$$

□

Now we return to the integrand which was broken into three parts. For the first term, $J_a[R_a(z),\Lambda_1]e^{\delta|x_1|}J_a^*$, we bound its operator norm by breaking it down further into

$$\begin{aligned}
\|J_a[R_a(z),\Lambda_1]e^{\delta|x_1|}J_a^*\| &= \|[R_a(z),\Lambda_1]e^{\delta|x_1|}\| \\
&= \|-R_a(z)[H_a,\Lambda_1]R_a(z)e^{\delta|x_1|}\| \\
&\leq \|R_a(z)\|\cdot\|[H_a,\Lambda_1]e^{\delta|x_1|}\|\cdot\|e^{-\delta|x_1|}R_a(z)e^{\delta|x_1|}\|.
\end{aligned}$$

The norm of $R_a(z)$ is bounded by

$$\|R_a(z)\| \leq \frac{1}{|\mathrm{Im}(z)|}$$

for any $z \notin \mathbb{R}$ since H_a is self-adjoint. The second term is bounded by Lemma 2. Finally, for the third term $e^{-\delta|x_1|}R_a(z)e^{\delta|x_1|}$, we apply the Combes-Thomas bound,

$$\|e^{-\varepsilon f(x)}R_a(z)e^{\varepsilon f(x)}\| \leq \frac{C}{|\mathrm{Im}(z)|}$$

where $f:\mathbb{Z}^2\to\mathbb{R}$ is any Lipschitz function, and ε can be chosen as $\varepsilon = \frac{1}{C(1+|\mathrm{Im}(z)|)}$. Altogether, the bound of the first term of the integrand takes the form

$$\frac{C}{\mathrm{Im}(z)^2}.$$

The second term of the integrand is bounded by Lemma 3. Finally, the bound for the third term of the integrand, $e^{\delta|x_2|}[H_a,\Lambda_2]R_a(z)$, follows from the bound on R and Lemma 2, and is again of the form $\frac{C}{\mathrm{Im}(z)^2}$.

Altogether, it follows from $\|AB\|_1 \leq \|A\|\|B\|_1$ that the trace norm of the integrand is bounded by $\frac{C}{\mathrm{Im}(z)^4}$.

We now appeal to a general fact of the Helffer-Sjöstrand functional calculus to provide domination. For any $n \in \mathbb{N}$, the quasi-analytic extension $\tilde{\rho}$ of ρ in the Helffer-Sjöstrand representation can be chosen so that

$$\int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} \frac{1}{|\mathrm{Im}(z)|^{p+1}} dz^2 \leq C_0 \sum_{k=0}^{n+2} \|\rho^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_m = \int_{-\infty}^{\infty} |f(x)|(x^2+1)^{m/2} dx.$$

Since $|\rho(x)| \leq 1$ and ρ' is compactly supported, these norms are all clearly finite. This fact, combined with the bound

$$\|R_a(z)[H_a, \Lambda_1] R_a(z) [H_a, \Lambda_2] R_a(z)\|_1 \leq \frac{C}{\mathrm{Im}(z)^4}$$

for the trace norm of the integrand of $\Sigma(a)$ provides the necessary bound for Lebesgue dominated convergence. Thus, it suffices to show pointwise convergence in z of the integrand to the associated bulk operator.

In other words, we wish to show

$$J_a[R_a(z), \Lambda_1] e^{\delta|x_1|} J_a^* \xrightarrow{s} [R_B(z), \Lambda_1] e^{\delta|x_1|},$$

$$J_a e^{\delta|x_2|} [H_a, \Lambda_2] J_a^* \xrightarrow{s} e^{\delta|x_2|} [H_B, \Lambda_2],$$

and

$$J_a R_a(z) J_a^* \xrightarrow{s} R_B(z)$$

for each fixed $z \in \mathbb{C}$. Inspecting the bounds we found for the left hand sides of these limits, it is clear that they are uniformly bounded in a. It therefore suffices to show convergence on a dense subspace of $\ell^2(\mathbb{Z}^2)$ (see Lemma 12); in particular, we may choose the dense subspace of compactly supported states, which allows us to ignore the $e^{\delta|x_i|}$ terms. Thus, we need to prove

$$J_a[R_a(z), \Lambda_1] J_a^* \xrightarrow{s} [R_B(z), \Lambda_1],$$

$$J_a[H_a, \Lambda_2] J_a^* \xrightarrow{s} [H_B, \Lambda_2],$$

and

$$J_a R_a(z) J_a^* \xrightarrow{s} R_B(z).$$

In fact, the final statement implies the first two; we appeal to the general fact of functional analysis that strong convergence of the resolvent of a self-adjoint operator implies that $J_a f(H_a) J_a^* \xrightarrow{s} f(H_B)$ for any bounded and continuous function f. In particular, it follows from Lemma 2 that the functions $[(\cdot - z)^{-1}, \Lambda_1]$ and $[\cdot, \Lambda_2]$ above are bounded and continuous, so we will have proven the desired limits if we can prove the strong convergence of the resolvent, $J_a R_a(z) J_a^* \xrightarrow{s} R_B(z)$.

To prove this, we use the edge assumption. Recall the edge operator, $E_a = J_a H_a - H_B J_a$. Adding and subtracting zJ_a gives

$$E_a = J_a(H_a - z) - (H_B - z)J_a.$$

Applying R_B from the left and R_a from the right on both sides, we obtain

$$R_B(z) E_a R_a(z) = R_B(z) J_a - J_a R_a(z).$$

Taking the adjoint, and then multiplying from the left by J_a, we see that

$$J_a R_a(z) E_a^* R_B(z) = J_a J_a^* R_B(z) - J_a R_a(z) J_a^*.$$

Thus

$$R_B(z) - J_a R_a(z) J_a^* = (J_a R_a(z) E_a^* - J_a J_a^* + 1) R_B(z) \xrightarrow{s} 0,$$

since $E_a^* \xrightarrow{s} 0$ by Lemma 5, and $-J_a J_a^* + 1 \xrightarrow{s} 0$. This proves that the limits above converge to the desired associated bulk operators, and hence $\|\Sigma(a) - \Sigma_B\|_1 \to 0$.

Finally, it remains to show that

$$\mathrm{Tr}(\Sigma_B) = \sigma_B.$$

First, we manipulate

$$\begin{aligned}
\Sigma_B &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1] R_B(z) [H_B, \Lambda_2] R_B(z) dz^2 \\
&= -\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1][R_B(z), \Lambda_2] dz^2 \\
&= i[\rho(H_B), \Lambda_1]\Lambda_2 - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1]\Lambda_2 R_B(z) dz^2.
\end{aligned}$$

Define $P_+ := P((\sup\Delta, \infty))$ and $P_- := P((-\infty, \inf\Delta))$, the projections onto states above and below the gap, respectively. Since H_B is assumed to have a gap, it follows that $P_- + P_+ = \mathbb{1}$, and thus

$$\mathrm{Tr}(\Sigma_B) = \mathrm{Tr}((P_- + P_+)\Sigma_B(P_- + P_+)) = \mathrm{Tr}(P_+\Sigma_B P_+) + \mathrm{Tr}(P_-\Sigma_B P_-).$$

We now argue that the integral term in $P_\pm\Sigma_B P_\pm$ vanishes. Consider P_+R_B; this is analytic on $\mathbb{C}\setminus(-\infty, \sup\Delta)$. However, the integrand is zero on $(-\infty, \sup\Delta)$, since $\tilde{\rho} = 0$ there. Thus $\tilde{\rho}P_+R_B$ is analytic on the entire complex plane. The integral can therefore be written

$$\int_{\mathbb{C}\setminus(-\infty,\sup\Delta)} \frac{\partial\tilde{\rho}}{\partial\bar{z}} P_\pm R_B(z)[H_B, \Lambda_1]\Lambda_2 R_B(z)P_\pm dz^2.$$

Since the operator in the integrand is analytic, the derivative $\frac{\partial}{\partial\bar{z}}$ can be made to act on the entire integrand rather than only on $\tilde{\rho}$. Furthermore, since $R_B(z)$ is bounded in norm by $|\mathrm{Im}(z)|^{-1}$ and $|\partial_{\bar{z}}\tilde{\rho}|$ can be made to decay with any power law $\mathcal{O}(\mathrm{Im}(z)^k)$ as $\mathrm{Im}(z) \to 0$ (see Appendix F), we may write the integral as a limit of disks,

$$\lim_{r\to\infty}\int_{B_r(0)\setminus(-\infty,\sup\Delta)} \frac{\partial}{\partial\bar{z}}(\tilde{\rho}(z)P_+R_B(z)[H_B, \Lambda_1]\Lambda_2 R_B(z)P_+)dz^2,$$

where domination is provided by Lemma 15. Stokes' theorem then says that the above integral is bounded in norm by

$$\lim_{r\to\infty}\int_{\partial B_r(0)} \tilde{\rho}(z)P_+R_B(z)[H_B, \Lambda_1]\Lambda_2 R_B(z)P_+ dz.$$

The decay of $\|R_B(z)\| \leq \mathrm{dist}(z, \mathrm{Spec}(H_B))^{-1}$ and boundedness of the extension $\tilde{\rho}(z)$ (since $|\rho(x)| \leq 1$) ensure that the limit of this integral as $r \to \infty$ is zero. Indeed, the norm of the integral is bounded by

$$2\pi r \sup_{\partial B_r(0)} \frac{|\tilde{\rho}(r)|\|[H_B, \Lambda_1]\Lambda_2\|}{\mathrm{dist}(r, \mathrm{Spec}(H_B))^2}$$

which vanishes as $r \to \infty$ since the spectrum of the bulk Hamiltonian is bounded, and Lemma 2 bounds the norm of the operator in the numerator.

As for the integral appearing in $P_-\Sigma_B P_-$, we note that analogous to the argument above, R_BP_- is analytic on $\mathbb{C}\setminus(\inf\Delta, \infty)$, and thus $R_BP_-(1-\tilde{\rho})$ is analytic on all of $\mathbb{C}$. Replacing $\partial_{\bar{z}}\tilde{\rho} = \partial_{\bar{z}}(1-\tilde{\rho})$ leaves the integral unchanged, and the rest of the argument follows as before. Thus the integral term in Σ_B is zero, and

$$\mathrm{Tr}(\Sigma_B) = i\mathrm{Tr}(P_+[\rho(H_B), \Lambda_1]\Lambda_2 P_+) + i\mathrm{Tr}(P_-[\rho(H_B), \Lambda_1]\Lambda_2 P_-),$$

since $[\rho(H_B), \Lambda_1]\Lambda_2$ is trace-class (need to add) which allows us to employ cyclicity of the trace to cancel the terms with both P_- and P_+. By the spectral theorem for projection-valued measures 3, if the Fermi energy lies in the gap, $\lambda \in \Delta$, we have

$$\rho(H_B) = \int_{-\infty}^{\infty} \rho(\lambda) dP_\nu = \int_{-\infty}^{\lambda} \rho(\lambda) dP_\nu = \int_{-\infty}^{\lambda} dP_\nu = P_\lambda.$$

We may therefore replace $\rho(H_B)$ by P_λ, by which we obtain

$$\mathrm{Tr}(\Sigma_B) = i\mathrm{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+) + i\mathrm{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-).$$

Now we must relate this expression to the bulk conductivity. It is an algebraic check that

$$P_\lambda[[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]] = P_\lambda \Lambda_2 P_\lambda^\perp \Lambda_1 P_\lambda - P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda$$

are equal as operators. The two terms on the right hand side are separately trace-class by Lemma 4, so that the bulk conductivity is given by

$$\begin{aligned} \sigma_B &= i\mathrm{Tr}(P_\lambda \Lambda_2 P_\lambda^\perp \Lambda_1 P_\lambda - P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda) \\ &= i\mathrm{Tr}(P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_\lambda^\perp - P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda). \\ &= \mathrm{Tr}(T_\lambda), \end{aligned}$$

Where we've defined $T_\lambda := P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_\lambda^\perp - P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda$. We must now show that

$$P_\pm T_\lambda P_\pm = P_\pm [P_\lambda, \Lambda_1]\Lambda_2 P_\pm.$$

First, notice that because of the gap, we have $P_\lambda^\perp P_- = 0$, and thus also $P_\lambda P_- = P_-$. Thus

$$\begin{aligned} P_- T_\lambda P_- &= P_- P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda P_- \\ &= P_-(P_\lambda \Lambda_1 \Lambda_2 - \Lambda_1 P_\lambda \Lambda_2) P_- \\ &= P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-, \end{aligned}$$

and similarly, for P_+, we have $P_\lambda^\perp P_+ = P_+$, and $P_\lambda P_- = 0$, which implies

$$\begin{aligned}
P_+T_\lambda P_+ &= -P_+P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2P_\lambda^\perp P_+ \\
&= -P_+P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2P_+ \\
&= -P_+P_\lambda^\perp\Lambda_1\Lambda_2P_+ + P_+P_\lambda^\perp\Lambda_1P_\lambda^\perp\Lambda_2P_+ \\
&= -P_+P_\lambda^\perp\Lambda_1\Lambda_2P_+ + P_+\Lambda_1P_\lambda^\perp\Lambda_2P_+ \\
&= -P_+[P_\lambda^\perp, \Lambda_1]\Lambda_2P_+ \\
&= -P_+[(1-P_\lambda), \Lambda_1]\Lambda_2P_+ \\
&= P_+[P_\lambda, \Lambda_1]\Lambda_2P_+.
\end{aligned}$$

Altogether, we obtain

$$\begin{aligned}
\sigma_B &= i\mathrm{Tr}(P_-T_\lambda P_-) + i\mathrm{Tr}(P_+T_\lambda P_+) \\
&= i\mathrm{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2P_-) + i\mathrm{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2P_+) \\
&= \mathrm{Tr}(\Sigma_B),
\end{aligned}$$

concluding the proof.

Lemma 4. *Let $W \cap V = \varnothing$. Then*

$$P_W\Lambda_1P_V\Lambda_2P_W \in \mathcal{J}_1.$$

In particular, the two terms appearing in T_λ are separately trace-class.

Proof. We break down

$$P_W\Lambda_1P_V\Lambda_2P_W = P_W\Lambda_1P_Ve^{3\delta|x_1|}e^{-\delta|x|} \cdot e^{-\delta|x|} \cdot e^{-\delta|x|}e^{3\delta|x_2|}P_V\Lambda_2P_W.$$

Since the middle term is trace-class by Lemma 3, it suffices to prove boundedness of the first and last terms.

To that end, notice that $P_W\Lambda_iP_V = P_W[\Lambda_i, P_V]$. Since P_W is obviously bounded, we only need to bound the product of the commutator with the appropriate exponentials. This can be done by decomposing

$$[\Lambda_i, P_V]e^{3\delta x_i}e^{-\delta|x|} = \Lambda_iP_V(1-\Lambda_i)e^{3\delta x_i}e^{-\delta|x|} + (1-\Lambda_i)P_V\Lambda_ie^{3\delta x_i}e^{-\delta|x|}.$$

Note the lack of absolute values on the $e^{3\delta x_i}$ terms; we will later also bound the same operator but with $e^{-3\delta x_i}$ to account for this, since $e^{3\delta|x_i|} = e^{\pm 3\delta x_i}$. We begin by bounding the second term. Since multiplication operators commute, the norm of the second term can be broken into

$$\|(1-\Lambda_i)P_V e^{-\delta|x|}\|\|\Lambda_i e^{3\delta x_i}\|$$

both of which are bounded, since for $\Lambda_i e^{3\delta x_i}$ only the negative x_i values remain due to the switch function. Returning to the first term, we insert more exponentials

$$\Lambda_i P_V(1-\Lambda_i)e^{3\delta x_i}e^{-\delta|x|} = \Lambda_i e^{3\delta x_i}\cdot e^{-3\delta x_i}P_V(1-\Lambda_i)e^{3\delta x_i}e^{-\delta|x|}.$$

Again $\Lambda_i e^{3\delta x_i}$ is bounded because of the switch function, and the other part is seen to be bounded by taking the adjoint of the operator in Lemma 14.

An exactly analogous argument works for bounding $[\Lambda_i, P_V]e^{-3\delta x_i}e^{-\delta|x|}$, i.e. the same operator but with the $e^{-3\delta x_i}$ terms.

□

Lemma 5. *E_a and E_a^* converge strongly to zero in the limit $a\to\infty$.*

Proof. Let $\psi\in\ell^2(\mathbb{Z}^2)$. Since E_a has real entries,

$$\begin{aligned}\|E_a^*\psi\|^2 &= \langle E_a^*\psi, E_a^*\psi\rangle\\ &= \sum_z\left(\sum_y \overline{E_a^*(z,y)\psi(y)}\right)\left(\sum_x E_a^*(z,x)\psi(x)\right)\\ &= \sum_z\left(\sum_y E_a(y,z)\overline{\psi(y)}\right)\left(\sum_x E_a(x,z)\psi(x)\right)\end{aligned}$$

Consider the $\sum_x E_a(x,z)\psi(x)$ term. To take advantage of the edge assumption, we insert the exponentials

$$\sum_x E_a(x,z)e^{\alpha(|z_2-a|-|z_1-x_1|)}e^{-\alpha(|z_2-a|-|z_1-x_1|)}\psi(x).$$

By assumption 2, $g(z) := \sup_x E_a(x,z)e^{\alpha(|z_2-a|-|z_1-x_1|)}$ is finite and summable. Precisely the same argument can be used on the sum over y. Altogether,

$$\|E_a^*\psi\|^2 \le e^{-\alpha|a|}\sum_z g(z)^2 e^{-2\alpha(|z_1|+|z_2|)}\sum_x e^{-\alpha|x_1|}\psi(x)\sum_y e^{-\alpha|y_1|}\overline{\psi(y)},$$

where we bounded the exponential by $e^{-\alpha(|z_1|+|z_2|)}e^{-\alpha|x_1|}e^{-\alpha|a|}$. Since g is summable, g^2 is also summable, and so too is the summand over z. The sums over x and y are clearly finite, as they are bounded by the summable state $|\psi|$. Thus E_a^* converges strongly to zero. An exactly analogous argument applies for E_a.

$\square$

Chapter 3

Interacting Bulk-Edge Correspondence

3.1 General Setting

Let $L \in \mathbb{N}$, and let $\Gamma_L = \mathbb{Z}_L \times [0, L]$ be the discrete cylinder, equipped with a metric d. To each site $x \in \Gamma_L$, we associate a Hilbert space $\mathcal{H}_x$ whose dimension is bounded uniformly in L. We denote $N = \sup_L \mathcal{H}_L$. For a subset $X \subseteq \Gamma_L$, we define the Hilbert space $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$, and we set

$$\mathcal{H}_L := \mathcal{H}_{\Gamma_L} = \bigotimes_{x \in \Gamma_L} \mathcal{H}_x.$$

The algebra $\mathcal{U}_X \subset \mathcal{B}(\mathcal{H}_X)$ of observables on $\mathcal{H}_X$ is the set of bounded self-adjoint operators supported in X. For an operator $A_X \in \mathcal{U}_X$, we identify its extension to an operator on $\mathcal{H}_L$ by taking its tensor product with copies of the identity, $(\otimes_{x \in X^c} \mathbb{I}_x) \otimes A_X$. Conversely, we say that an operator $A \in \mathcal{U}_L$ has support X if $A_X := (\otimes_{x \in X^c} \mathbb{I}_x) \otimes (A|_X)$ is equal to A, and write $A_X \in \mathcal{U}_X$. For ease of notation, we omit the subscript L wherever there is no risk of confusion.

A *local interaction* is a map $\Phi : \mathcal{P}(\Gamma_L) \to \mathcal{U}_L$ such that

1. $\Phi(X) = 0$ whenever $\mathrm{diam}(X) > R$ for some $R > 0$.
2. $\Phi(X)$ is supported in X.
3. $\|\Phi(X)\| \leq C$ for all $X \subset \Gamma_L$, for all L.

We consider a region as depicted in Figure 3.1, with the left and right edges joined together to form a cylinder. In the left white region $[0, L/2] \times [0, L]$, H_0 is a trivial Hamiltonian which we take to be empty space (we take $H_0 = 0$), and in the right blue region $[L/2, L] \times [0, L]$, H_1 is a *local Hamiltonian*, in the sense that $H_1 = \sum_{X \subseteq \Gamma_L} \Phi(X)$, is a sum of local interactions. We define the Hamiltonian of the full system to be

$$H_\mu = H_1 + \mu Q_h,$$

where $Q_h = \sum_{x \in \Gamma_h} a_x^* a_x$ is the number operator for the region $\Gamma_h = [L/4, 3L/4] \times [0, L]$ shown in red. This introduces a driving strength; the μQ_h term can be viewed as a potential difference $V(x)$.

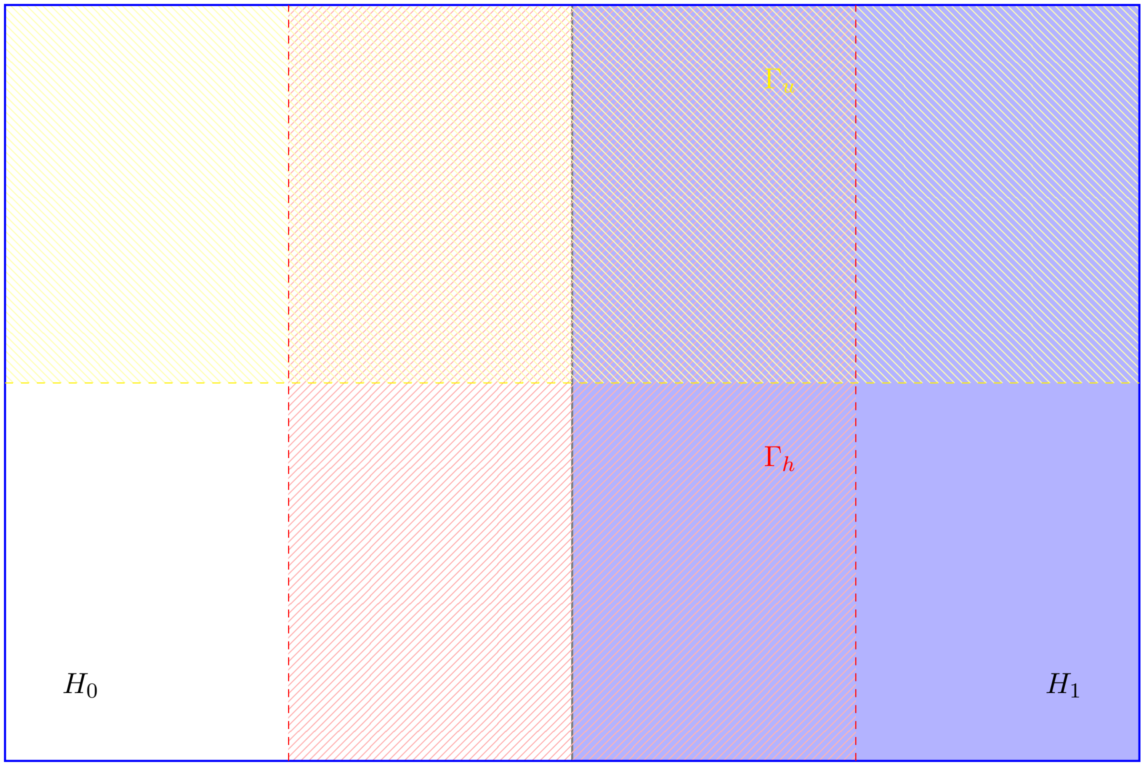

Figure 3.1: The cylinder Γ_L.

We also consider the plane $\mathbb{Z}^2$. In this setting, there are no edge states, and so the associated "bulk" Hamiltonian H_B is assumed to have a *gapped* spectrum, in the sense that

Assumption 4.

$$\mathrm{Spec}(H_B) = \mathcal{S}_- \cup \mathcal{S}_+,$$

where $\inf \mathcal{S}_+ - \sup \mathcal{S}_- \geq \gamma$ *uniformly in* L *and* μ *for some* $\gamma > 0$.

In the case of the cylinder, this effect does not necessarily occur due to the presence of the edge. We also assume that the Hamiltonian is *locally charge-conserving.*

Assumption 5. $[\Phi(X), Q] = 0$, *where* Q *is the total charge in* Γ_L.

Let P_B be the ground state projection of H_B (the system without an edge), and let P be the ground state projection of H (the system with an edge). We assume that states far from the edge are essentially bulk states, up to tails that vanish quickly in L.

Assumption 6. *Define the* edge region

$$\Gamma_E = [L/2 - k, L/2 + k] \times [0, L] \cup [L - k, k] \times [0, L].$$

for some $k > 0$*. For any operator* A *supported on* Γ_E^{c}*,*

$$\mathrm{Tr}(PA) = \mathrm{Tr}(P_B A) + \mathcal{O}(L^{-\infty}).$$

The A *on the right hand side is understood to be the extension by zeroes of* A *to the plane* $\mathbb{Z}^2$*.*

The idea is that observables localized far away from the edge are not affected by the edge of the system. We similarly define the *bulk region*

$$\Gamma_B = [3L/4 - k, 3L/4 + k] \times [0, L],$$

and the *middle region*

$$\Gamma_m = [L/2, L] \cup [0, L] \setminus (\Gamma_E \cup \Gamma_B).$$

The three regions are depicted in figure 3.2.

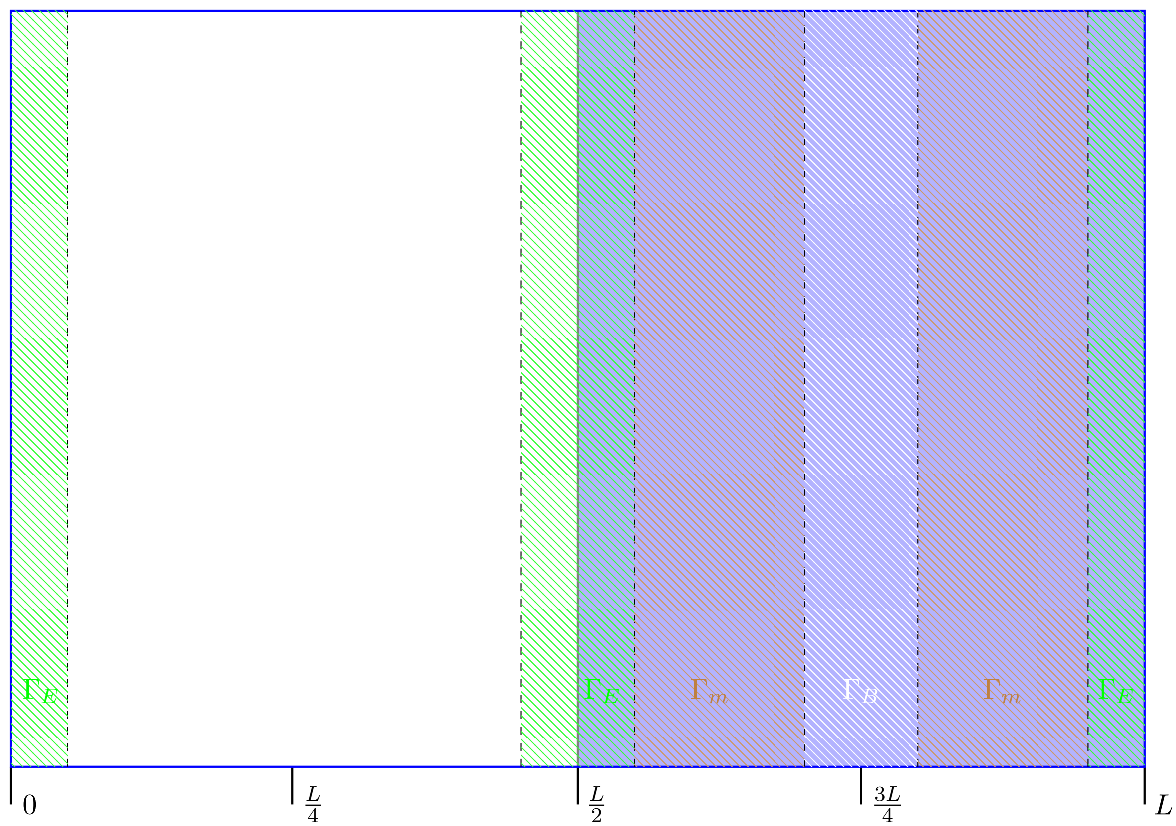

Figure 3.2: The regions Γ_E, Γ_B, and Γ_m.

3.2 Equality of Bulk and Edge Currents

3.2.1 Cylinder Geometry

Let P_μ be the (possibly degenerate) ground state projection of H_μ. Let $Q_u = \sum_{x\in\Gamma_u} a_x^* a_x$ be the charge in the upper half of the cylinder $\Gamma_u = [0, L] \times [L/2, L]$ (the yellow region in Figure 3.1), and define current operator

$$J = i[H_\mu, Q_u],$$

which measures the current across the fiducial line $y = L/2$. Charge conservation 5 implies that this current operator is supported along a strip of width $2R$ centred on the fiducial line $y = L/2$. Indeed, if we inspect a local interaction $\Phi(X)$ of range R with support $(\Gamma_u)_R$, where $(X)_\alpha$ is the α-shrinking of the set X, then clearly $\Phi(X)$ commutes with the charge outside Γ_u, so that $[\Phi(X), Q_u] = [\Phi(X), Q]$, which vanishes by the charge conservation assumption 5. Similarly, if $\Phi(X)$ is supported in $((\Gamma_u)^c)_R$, then $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$. It follows that for an interaction $\Phi(X)$ with range R and arbitrary support, $[\Phi(X), Q_u]$ must be supported on a set which is contained in (or equal to) the strip $[L/2, L] \times [L/2 - R, L/2 + R]$. There $[H_\mu, Q_u]$ must be supported there as well, since H_μ is a sum of such local interactions.

From this point, we drop the subscript μ wherever it is not needed for context.

Lemma 6. *The ground state expectation of the current J is zero.*

Proof. Assuming linearity and cyclicity of the trace hold, the proof is trivial,

$$\mathrm{Tr}(PJ) = i\mathrm{Tr}(P[H, Q_u]) = i\mathrm{Tr}([P, H]Q_u) = 0.$$

In order for this calculation to hold, we need to prove that

1. PHQ_u and PQ_uH are separately trace-class to apply linearity of the trace, and
2. $\|H\| < \infty$ and $PQ_u \in \mathcal{J}_1$ to apply cyclicity of the trace.

The latter implies the former by the bound $\|AB\|_1 \leq \|A\|_1\|B\|$. To prove (2), fix a finite L. The Hamiltonian is bounded since it is a finite sum of at most $\mathcal{P}(\Gamma_L)$ local interactions $\Phi(X)$, each of which is uniformly bounded by assumption, along with the μQ_h term. But the number operator for the entire space is bounded by $\|Q\| \leq NL^2$, where N is the uniform bound on the dimension of each Hilbert space. This shows that both Q_u and Q_h are bounded in operator norm. Finally, $\|P\|_1 \leq CL^2$ because the projection is finite-rank, since the dimension of each site is bounded. Therefore $PQ_u \in \mathcal{J}_1$.

□

Next, we define a family of operators indexed by μ called *Hastings operators*,

$$K_\mu = \mathcal{I}_\mu(\dot{H}_\mu),$$

where

$$\mathcal{I}_\mu(A) = \int_{\mathbb{R}} W(t)e^{itH_\mu}Ae^{-itH_\mu}dt.$$

Here, $W : \mathbb{R} \to \mathbb{R}$ is a bounded, $L^1(\mathbb{R})$ function satisfying

1. $|W(t)| = \mathcal{O}(|t|^{-\infty})$,
2. $\widehat{W}(\xi) = \frac{i}{\xi}$ for all $|\xi| \geq \mathrm{Len}(\Delta)$,

where $\widehat{W}$ is the Fourier transform, taken here to be $\hat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-it\xi}dt$. Such a function can be constructed explicitly [10]. In our setting, we see that

$$K_\mu = \mathcal{I}_\mu(Q_h).$$

We present two important properties of the map $\mathcal{I}_\mu : \mathcal{U}_L \to \mathcal{U}_L$ in the following lemmas, and leave their proofs to the appendix (need to add).

First, recall a definition from the non-interacting setting: an *off-diagonal* operator is an operator A such that $A = \overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, where $P_\mu^\perp = \mathbb{I} - P_\mu$ is the projection onto the excited states above the gap.

Lemma 7. *1. For any off-diagonal operator $A = \overline{A}$, $\mathcal{I}_\mu(\cdot)$ and $[H_\mu, \cdot]$ act as inverses of each other, up to a factor of i:*

$$\mathcal{I}_\mu\left([H_\mu, A]\right) = [H_\mu, \mathcal{I}_\mu(A)] = iA.$$

2. For any (not necessarily off-diagonal) operator A,

$$[\mathcal{I}_\mu([H_\mu, A]), P_\mu] = i[A, P_\mu].$$

Another important property of the map $\mathcal{I}_\mu$ is that it preserves locality.

Lemma 8. *$\mathcal{I}_\mu$ is local in the sense that for any $A \in \mathcal{U}_X$,*

$$\|\mathcal{I}(A)_{(X^r)^c}\| \leq \|A\||X|\mathcal{O}(r^{-\infty})$$

where $X^r = X \cup \{x : d(x, X) \leq r\}$ is the r-fattening of X.

Proposition 2. *The operator K_μ is the* generator of parallel transport*, satisfying*

$$\dot{P}_\mu = i[K_\mu, P_\mu]$$

for all μ.

Proof. First, we show that $\dot{P}$ is off-diagonal. Taking the derivative on both sides of $P^2 = P$, we see that $\dot{P}P + P\dot{P} = \dot{P}$. Acting on the left and right with P on both sides of this equation gives

$$P\dot{P}P + P\dot{P}P = P\dot{P}P,$$

which implies that $P\dot{P}P = 0$. Thus

$$\begin{aligned}
\overline{\partial_\mu P} &= P\dot{P}(1-P) + (1-P)\dot{P}P \\
&= P\dot{P} - P\dot{P}P + \dot{P}P - P\dot{P}P \\
&= P\dot{P} + \dot{P}P \\
&= \partial_\mu(P^2) \\
&= \partial_\mu P,
\end{aligned}$$

as claimed. By the product rule and the fact that H and P commute,

$$[\dot{H}, P] = -[H, \dot{P}].$$

It therefore follows from Lemma 7 that

$$\dot{P} = -i\mathcal{I}_\mu([H, \dot{P}]) = i\mathcal{I}([\dot{H}, P]) = i[\mathcal{I}(\dot{H}), P] = i[K, P].$$

□

Increasing the electric potential by a small amount $d\mu Q_h$ and expanding to linear order, the change in ground state current is given by

$$\mathrm{Tr}(P_{\mu+d\mu}J) - \mathrm{Tr}(P_\mu J) = \kappa d\mu + \mathcal{O}(d\mu^2).$$

Dividing by $d\mu$ and taking a limit, we see that the linear response coefficient is given by

$$\sigma(\mu) = \mathrm{Tr}\left(\dot{P}_\mu J\right).$$

The *Hall conductivity* of the system on a subset $V \subseteq \Gamma_L$ is defined to be $\sigma_V := \mathrm{Tr}\left(\dot{P}J_V\right)$, where J_V is the restriction of J to V.

In particular, we define the *edge conductivity* in the interacting setting as the conductivity on the edge strip, σ_{Γ_E}.

Proposition 3. *The Hall conductivity is independent of the driving strength* μ.

Proof. For any μ_1 and μ_2,

$$\begin{aligned}
\sigma(\mu_1) - \sigma(\mu_2) &= \mathrm{Tr}\left(\dot{P}_{\mu_1} i[H_{\mu_1}, Q_u] - \dot{P}_{\mu_2} i[H_{\mu_2}, Q_u]\right) \\
&= i\mathrm{Tr}\left(\left([\dot{P}_{\mu_1}, H_{\mu_1}] - [\dot{P}_{\mu_2}, H_{\mu_2}]\right) Q_u\right) \\
&= -i\mathrm{Tr}\left(\left([\dot{H}_{\mu_1}, P_{\mu_1}] - [\dot{H}_{\mu_2}, P_{\mu_2}]\right) Q_u\right) \\
&= i\mathrm{Tr}\left([Q_h, P_{\mu_1} - P_{\mu_2}]Q_u\right) \\
&= i\mathrm{Tr}\left([Q_u, Q_h](P_{\mu_1} - P_{\mu_2})\right) \\
&= 0,
\end{aligned}$$

since H and P commute. Note that $\|\dot{P}\|_1 < \infty$ since we are working in a finite-dimensional space. The proof of Lemma 6 provides the other necessary bounds to invoke linearity and cyclicity of the trace to shift the commutator in the second line and second-last line. □

This indicates that the Hall conductivity is independent of μ as one would expect physically. We simply write $\sigma = \sigma(\mu)$ from this point, in accordance with proposition 3.

The following is the main result:

Theorem 2. *Let $V \subseteq \Gamma_m$ be a set contained within the strip in between the edge region Γ_E and the bulk region Γ_B (see Figure 3.2), and define the distance*

$$r = \mathrm{dist}(V, \Gamma_E \cup \Gamma_B)$$

from V to the bulk and edge regions. The Hall conductivity in this regions vanishes in the sense that

$$\sigma_V = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

Proof. By Proposition 2, the bulk Hall conductivity can also be written by the formula

$$\sigma_V^B = \mathrm{Tr}\left(i[K, P_B]J_V^B\right) = \mathrm{Tr}\left(i[\mathcal{I}(Q_h), P_B]J_V^B\right),$$

where $J_V^B = i[H_B, Q_u]|_V$ is the current in the region V arising from the bulk Hamiltonian. From commutativity of P_B and H_B along with cyclicity of the trace, we compute

$$\begin{aligned}\sigma_V^B &= \mathrm{Tr}\left(i\int_{\mathbb{R}} W(t)e^{itH_B}[Q_h,P_B]e^{-itH_B}dtJ_V^B\right)\\ &= \int_{\mathbb{R}} W(t)\mathrm{Tr}\left(i[Q_h,P_B]e^{-itH_B}J_V^Be^{itH_B}\right)dt\\ &= -\int_{\mathbb{R}} W(t)\mathrm{Tr}\left(i[Q_h,P_B]e^{itH_B}J_V^Be^{-itH_B}\right)dt\\ &= -\mathrm{Tr}\left(i[Q_h,P_B]\mathcal{I}(J_V^B)\right),\end{aligned}$$

since $W(t)$ is odd. By part (2) of Lemma 7, we have $i[Q_h,P_B] = [\mathcal{I}([H_B,Q_h]),P_B]$. Therefore

$$\begin{aligned}\sigma_V^B &= -\mathrm{Tr}([\mathcal{I}([H_B,Q_h]),P_B]\mathcal{I}(J_V^B))\\ &= \mathrm{Tr}\left(P_B[\mathcal{I}([H_B,Q_h]),\mathcal{I}(J_V^B)]\right).\end{aligned}$$

Now, $[H_B,Q_h]$ is a local operator supported on Γ_B, while J_V^B is a local operator supported on $V\cap\Gamma_B=\varnothing$. Since $\mathcal{I}$ preserves locality up to tails, in the sense that $\|\mathcal{I}(A)_{(S^r)^{\mathsf{c}}}\| \le \|A\||S|\mathcal{O}(r^{-\infty})$ for any operator A supported in S (Lemma 8), it follows that the commutator can be written

$$[\mathcal{I}([H_B,Q_h])|_{\Gamma_B}+\mathcal{O}(r^{-\infty})A_1,\mathcal{I}(J_V^B)|_V+\mathcal{O}(r^{-\infty})A_2] = C\mathcal{O}(r^{-\infty}),$$

for some operators A_1 and A_2 supported on Γ_B^{c} and V^{c}, respectively. This fact applies to the bulk setting with H_B and P_B. To extend this to the setting with an edge, it is enough to use Assumption 6 to conclude the same result, except with equality up to $\mathcal{O}(L^{-\infty})$, i.e.

$$\sigma_V = \mathrm{Tr}\left(\dot{P}J_V\right) = \mathrm{Tr}\left(\dot{P}(J_V^B+\mathcal{O}(L^{-\infty}))\right) = \sigma_V^B+\mathcal{O}(L^{-\infty}) = \mathcal{O}(r^{-\infty})+\mathcal{O}(L^{-\infty}).$$

$\square$

The intuitive picture from the previous result is that, outside the edge region Γ_E, the Hall conductivity is essentially only nonzero along the bulk strip Γ_B. Since the ground state expectation of the current is zero (by lemma 6), it must be that there is an equal current flowing along the edge strip Γ_E, but in the opposite direction.

Furthermore, consider the regions Γ_E and Γ_B where the Hall conductivity may be nonzero. We conclude using assumption 6 that in the bulk strip Γ_B, the Hall conductivity of the system with an edge matches that of the bulk system, $\sigma_{\Gamma_B}^B = \sigma_{\Gamma_B}+\mathcal{O}(L^{-\infty})$. But from the previous theorem, this is also equal (up to tails) to the edge conductivity σ_{Γ_E}. We therefore have bulk-edge correspondence, $\sigma_{\Gamma_E} = \sigma^B_{\Gamma_B} + \mathcal{O}(L^{-\infty})$. (Need to add).

3.2.2 Torus Geometry

Our goal is to show the same result on the discrete torus $\mathbb{T}_L := \mathbb{Z}_L \times \mathbb{Z}_L$. We define the same regions Γ_u and Γ_h, and the same current operator $J_u = i[H(\mu), Q_u]$. This time, however, Lemma 6 does not apply. Intuitively, it does not apply because electrons can now flow through both the bottom and the top of the region Γ_u, rather than just the bottom. Mathematically, the lemma fails because our definition of the current is slightly changed.

We use charge conservation and the fact that H is finite range to split the current J_u into two components, $J_u = i[H_-, Q_u] + i[H_+, Q_u] = J_- - J_+$, supported on strips of width $2R$ at $y = L/2$ and $y = L$, respectively. We then define the current operator to be $J = J_-$, which is the current on the lower strip. This is the mathematical reason that the proof in Lemma 6 fails on the torus; we have replaced H by H_-, which may no longer commute with P. We instead proceed by a different approach. We will need a few auxiliary results first.

Lemma 9. *$K_\pm$ is supported on $\partial_\pm$ up to tails.*

Proof. □

Proposition 4. *The operator $Q_h - K$ leaves the ground state space invariant, i.e. $[Q_h - K, P] = 0$.*

Proof. □

Lemma 10. *Show that $\mathrm{Tr}(A, [Q_h, P]) = 0$ for all $A \in \mathcal{U}_{edge}$. This shows that Q_h commutes with P "along the edge".*

Proof. Let $A \in \mathcal{U}_{\text{edge}}$. Since H is charge conserving, we may choose a simultaneous eigenbasis of H and the total charge Q, in which case P and Q commute. It follows that

$$\mathrm{Tr}(A[Q_h, P]) = \mathrm{Tr}([A, Q_h]P) = \mathrm{Tr}([A, Q]P) = \mathrm{Tr}(A[Q, P]) = 0.$$

□

Finally, we will prove that in the bulk system with Hamiltonian $H_B(\mu)$, the ground state expectation of the current vanishes faster than any power as $L \to \infty$.

Lemma 11. *The ground state expectation of the current* $J_B := i[(H_B)_-, Q_h]$ *(of the system without an edge) is* $\mathrm{Tr}(P_B J_B) = \mathcal{O}(L^{-\infty})$.

Proof. First, $K = \mathcal{I}(i[H_B, Q])$ splits into $K = K_- - K_+$, with the support of $K_\pm$ contained in $\partial_\pm$ up to tails:

$$[K_\pm, A_X] = \mathcal{O}(p^{-\infty}),$$

for every $A_X \in \mathcal{U}_X$ such that $\|A_X\| = 1$, and where $p = \mathrm{dist}(X, \partial_\pm)$ (need to add). Using the fact that $K_\pm$ is supported in $\partial_\pm$ up to tails (Lemma 9), we see that

$$i[H_B, K_-] = i[(H_B)_-, K_-] + \mathcal{O}(L^{-\infty}),$$

and similarly $i[(H_B)_-, K_+] = \mathcal{O}(L^{-\infty})$. Putting these facts together, it follows that the current can be rewritten as

$$\begin{aligned} J_B &= i[H_B, Q_h + K_- - K_- + K_+] + \mathcal{O}(L^{-\infty}) \\ &= i[H_B, K_-] + i[(H_B)_-, Q_h - K_- + K_+)] + \mathcal{O}(L^{-\infty}). \end{aligned}$$

From here, we use the fact that H_B and $Q_h - K_- + K_+$ both commute with P_B to write

$$P_B J_B P_B = i[H_B, P_B K_- P_B] + i[P_B (H_B)_- P_B, Q_h - K_- + K_+)] + P_B \mathcal{O}(L^{-\infty}) P_B.$$

Since the trace of any commutator is zero,

$$\mathrm{Tr}(P_B J_B) = \mathrm{Tr}(P_B J_B P_B) = \mathcal{O}(L^{-\infty}).$$

□

Using this, we can show a simple proof of the analogue of Lemma 6 on the torus, in the case of non-interacting systems.

Proposition 5. *Let* $H = \sum_{x \in \mathbb{T}} h_x$ *be a non-interacting Hamiltonian, i.e. a sum of single site Hamiltonians* h_x. *The ground state expectation of the current* $J = i[H_-, Q_h]$ *(of the system with an edge) is* $\mathrm{Tr}(PJ) = \mathcal{O}(L^{-\infty})$.

Proof. Since H is a sum of single site Hamiltonians, we can split H_- into the restrictions $H_- = (H_-)_{\mathrm{edge}} + (H_-)_{\mathrm{bulk}}$, with no fear of any terms which are in both the edge region and the bulk region. By Assumption 6,

$$\begin{aligned}
\mathrm{Tr}(PJ) &= \mathrm{Tr}(Pi[H_-, Q_h]) \\
&= i\mathrm{Tr}([H_-, Q_h]P) \\
&= i\mathrm{Tr}((H_-)_{\mathrm{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\mathrm{bulk}}[Q_h, P]) \\
&= i\mathrm{Tr}((H_-)_{\mathrm{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\mathrm{bulk}}[Q_h, (P)_{\mathrm{bulk}}]) \\
&= i\mathrm{Tr}((H_-)_{\mathrm{edge}}[Q_h, P]) + i\mathrm{Tr}((H_B)_-[Q_h, P_B]) + \mathcal{O}(L^{-\infty}) \\
&= i\mathrm{Tr}((H_-)_{\mathrm{edge}}[Q_h, P]) + \mathrm{Tr}(i[(H_B)_-, Q_h]P_B) + \mathcal{O}(L^{-\infty}).
\end{aligned}$$

By Lemma 10, the first term is zero. By Lemma 11, the second term is $\mathcal{O}(L^{-\infty})$. $\square$

Chapter 4

Summary of Results

Need to add. Here is a citation [9]

Bibliography

[1] J.H. Schenker A. Elgart, G.M. Graf. Equality of the bulk and edge hall conductances in a mobility gap. *Commun. Math. Phys.*, 259:185–221, 2005.

[2] P. W. Anderson. Absence of diffusion in certain random lattices. *Phys. Rev.*, 109:1492–1505, Mar 1958.

[3] S. Bachmann, W. De Roeck, and M. Fraas. Adiabatic theorem for quantum spin systems. *Physical Review Letters*, 119(6), aug 2017.

[4] Sven Bachmann, Alex Bols, Wojciech De Roeck, and Martin Fraas. Quantization of conductance in gapped interacting systems, 2017.

[5] Sven Bachmann, Alex Bols, Wojciech De Roeck, and Martin Fraas. Note on linear response for interacting hall insulators, 2018.

[6] Sven Bachmann, Alex Bols, Wojciech De Roeck, and Martin Fraas. A many-body index for quantum charge transport. *Communications in Mathematical Physics*, 375(2):1249–1272, jul 2019.

[7] Sven Bachmann, Alex Bols, Wojciech De Roeck, and Martin Fraas. Rational indices for quantum ground state sectors. *Journal of Mathematical Physics*, 62(1):011901, jan 2021.

[8] Sven Bachmann, Wojciech De Roeck, and Martin Fraas. The adiabatic theorem in a quantum many-body setting, 2018.

[9] Sven Bachmann and Martin Fraas. On the absence of stationary currents. *Reviews in Mathematical Physics*, 33(01):2060011, jul 2020.

[10] Sven Bachmann, Spyridon Michalakis, Bruno Nachtergaele, and Robert Sims. Automorphic equivalence within gapped phases of quantum lattice systems. *Communications in Mathematical Physics*, 309(3):835–871, nov 2011.

[11] S. Bravyi, M. B. Hastings, and F. Verstraete. Lieb-robinson bounds and the generation of correlations and topological quantum order. *Physical Review Letters*, 97(5), jul 2006.

[12] J. M. Combes and P. D. Hislop. Landau hamiltonians with random potentials: Localization and the density of states. *Communications in Mathematical Physics*, 177(3):603–629, 1996.

[13] E. Brian Davies. *Spectral Theory and Differential Operators*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1995.

[14] Gian Graf. Aspects of the integer quantum hall effect. 01 2007.

[15] R. B. Laughlin. Quantized hall conductivity in two dimensions. *Phys. Rev. B*, 23:5632–5633, May 1981.

[16] Pieter Naaijkens. *Quantum Spin Systems on Infinite Lattices*, volume 933 of *Lecture Notes in Physics*. Springer, Oxford; New York, first edition, 2017.

[17] Bruno Nachtergaele, Yoshiko Ogata, and Robert Sims. Propagation of correlations in quantum lattice systems. *Journal of Statistical Physics*, 124(1):1–13, jul 2006.

[18] Bruno Nachtergaele and Robert Sims. Lieb-robinson bounds and the exponential clustering theorem. *Communications in Mathematical Physics*, 265(1):119–130, mar 2006.

[19] Martin Fraas S. Bachmann, Wojciech De Roeck and Markus Lange. Exactness of linear response in the quantum hall effect. *Annales Henri Poincare*, 22:1113–1132, 2021.

[20] Hermann Schulz-Baldes, Johannes Kellendonk, and Thomas Richter. Simultaneous quantization of edge and bulk hall conductivity. *Journal of Physics A: Mathematical and General*, 33(2):L27–L32, dec 1999.

[21] Jacob Shapiro. Notes on topological aspects of condensed matter physics, 2016.

[22] David Tong. *The Quantum Hall effect*. TIFR Infosys Lectures. Cambridge, first edition, 2016.

[23] Klaus von Klitzing. The quantized hall effect. *Rev. Mod. Phys.*, 58:519–531, Jul 1986.

Appendix A

General Functional Analysis

Lemma 12. *Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$. Suppose $A_n \xrightarrow{s} A$ on a dense subspace $\mathcal{D} \subset \mathcal{H}$. If A_n are bounded uniformly in n, then $A_n \xrightarrow{s} A$ on all of $\mathcal{H}$.*

Proof. Let $\psi_n \in \mathcal{D}$ be a sequence converging in norm to $\psi \in \mathcal{H}$. The result follows from a standard $\frac{\varepsilon}{3}$ argument. Let C be a bound for both $\sup_n \|A_n\|$ and $\|A\|$. Then

$$\begin{aligned}\|A_n\psi - A\psi\| &\leq \|A_n(\psi - \psi_m)\| + \|(A_n - A)\psi_m\| + \|A(\psi - \psi_m)\| \\ &\leq C\|(\psi - \psi_m)\| + \|(A_n - A)\psi_m\| + C\|\psi - \psi_m\|.\end{aligned}$$

There exists an M such that the first and third terms are less than $\frac{\epsilon}{3}$ for all $m > M$. For the middle term, observe that for each m, there exists by hypothesis an N_m such that $\|(A_n - A)\psi_m\| < \frac{\epsilon}{3}$ for all $n > N_m$. Thus, by picking some $m > M$ and fixing a suitably large n, the inequality above is less than ε. □

A.1 Spectral Measures and Projection-Values Measures

Projection-valued measures are maps $P : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ from measurable subsets of $\mathbb{R}$ to the space of bounded linear operators on $\mathcal{H}$ satisfying the usual properties of both projections and measures.

1. $P(M) = P(M)^* = P(M)^2$ is an orthogonal projection for all $M \in \mathcal{M}$. Note that this implies that $P(M)$ is a positive operator.

2. $P(\varnothing) = 0$ and $P(\mathbb{R}) = \mathbb{1}_{\mathcal{H}}$.

3. If $\{M_i\}_{i\in\mathbb{N}}$ are pairwise disjoint, then $\sum_{i=1}^n P(M_i) \xrightarrow{s} P(\cup_{i\in\mathbb{N}} M_i)$ as $n \to \infty$ (σ-additivity).

4. $P(M_1 \cap M_2) = P(M_1)P(M_2)$ for any $M_1, M_2 \in \mathcal{M}$.

The heuristic motivation is that $P(M)$ projects onto the subspace of $\mathcal{H}$ spanned by states whose energies lie in M. Using these operator-valued measures, one can construct an operator-valued integral with respect to P in the usual fashion (beginning on nonnegative simple functions, extending to nonnegative measurable functions, and finally to real-valued measurable functions).

Theorem 3 (Spectral Theorem for Projection-Valued Measures)**.** *There exists a one-to-one correspondence between self adjoint operators H and projection-valued measures P given by the formula*

$$H = \int_{\mathbb{R}} \lambda dP_\lambda,$$

where $P_\lambda := P((-\infty, \lambda])$. Moreover, if $g : \mathbb{R} \to \mathbb{R}$ is any bounded Borel function, then $g(H)$ defined via the Borel function calculus coincides with the formula

$$g(H) = \int_{\mathbb{R}} g(\lambda) dP_\lambda,$$

and $g(H) = g(H)^*$.

We remark that it follows from the second part of this theorem that if $\mathbb{1}_M$ denotes the characteristic function of a Borel set $M \subseteq \mathbb{R}$, then

$$\mathbb{1}_M(H) = \int_M dP_\lambda = P(M).$$

We also note that $\operatorname{Spec}(H) = \operatorname{supp}(P)$.

A.2 Greens' Functions and the Combes-Thomas Bound

The short-range assumption 1 is vital for the following non-trivial estimate, the proof of which is omitted.

Theorem 4 (Combes-Thomas Bound)**.** *Let H be a self-adjoint operator on $\ell^2(\mathbb{Z}^2)$ satisfying*

$$S_\alpha := \sup_x \sum_y |H(x,y)|(e^{\alpha|x-y|} - 1) < \infty$$

for some $\alpha > 0$. Suppose z lies outside the spectrum of H, and let $d_z := \operatorname{dist}(z, \operatorname{Spec}(H))$. Then the Greens function of H is exponentially bounded,

$$|G(x,y;z)| \leq \frac{2}{d_z} e^{-\xi_\alpha |x-y|},$$

where $\xi := \frac{\alpha d_z}{2S_\alpha}$.

This theorem gives the crucial decay properties of the spectral projectors.

Lemma 13. *Let H be a self-adjoint operator satisying 1 and with bounded spectrum. Let $S \subseteq \operatorname{Spec}(H)$, and let P_S be the associated spectral projection. Then there exists some $\varepsilon, \nu > 0$ such that the matrix elements of P_S satisfy*

$$\sum_{x,y\in\mathbb{Z}^2} |P_S(x,y)| e^{-\varepsilon|x|} e^{\nu|x-y|} < \infty.$$

Proof. We use the fact that the spectral projection is given by the Riesz integral formula

$$P_S = \frac{1}{2\pi i} \oint_\gamma R(z) dz,$$

where $R(z)$ is the resolvent of H and γ is any smooth closed curve containing S. Since the resolvent is the Greens function of H, it satisfies the Combes-Thomas bound 4. Since the spectrum of H is bounded, it may be enclosed in a curve of finite length, from which we determine that

$$|P_S(x,y)| \leq C e^{-\xi_\alpha |x-y|},$$

since $\inf_{z\in\gamma} d_z = \inf_{z\in\gamma} \operatorname{dist}(z, \operatorname{Spec}(H)) > 0$. Hence

$$\sum_{x,y\in\mathbb{Z}^2} |P_S(x,y)| e^{-\varepsilon|x|} e^{\nu|x-y|} < \infty$$

holds for $\nu = \xi_\alpha$ and for any $\varepsilon > 0$, since $e^{-\varepsilon|x|} = e^{-\varepsilon|x_1|} e^{-\varepsilon|x_2|}$ is summable on $\mathbb{Z}^2$ by Lemma 3.

□

This statement about decay of the matrix elements can be turned into a statement about the norm of the operator. Let Lip^1 be the set of all Lipschtiz functions whose Lipschitz constant is not greater than 1, that is the set of functions ℓ satisfying

$$|\ell(x) - \ell(y)| \leq |x - y|$$

for all $x, y \in \mathbb{Z}^2$.

Lemma 14. *Let P_S be the spectral projection onto $S \subseteq \mathrm{Spec}(H)$. For all $\varepsilon > 0$,*

$$\sup_{\ell \in \mathrm{Lip}^1} \| e^{\nu \ell(x)} e^{-\varepsilon |x|} P_S e^{-\nu \ell(x)} \| < \infty,$$

where ν is the same as in 13.

Proof. The matrix elements of the operator are $P_S(x,y)e^{\nu(\ell(x)-\ell(y))}e^{-\varepsilon|x|}$. By Holmgren's bound, the operator's norm is therefore bounded by

$$\max\{\sup_x \sum_y |P_S(x,y)| e^{\nu(\ell(x)-\ell(y))} e^{-\varepsilon|x|}, \ \sup_y \sum_x |P_S(x,y)| e^{\nu(\ell(x)-\ell(y))} e^{-\varepsilon|x|}\}.$$

Replacing the supremum with a sum yields the bound

$$\|P_S(x,y)e^{\nu(\ell(x)-\ell(y))}e^{-\varepsilon|x|}\| \leq \sum_{x,y} |P_S(x,y)| e^{\nu|\ell(x)-\ell(y)|} e^{-\varepsilon|x|},$$

and taking a supremum over $\ell \in \mathrm{Lip}^1$ completes the proof by Lemma 13. $\square$

Appendix B

Properties of $\mathcal{I}_\mu$

Proof. (Of Lemma 7). Let $\widehat{W}(\xi) = \int_{\mathbb{R}} W(t)e^{-it\xi}dt$ be the Fourier transform of W and let A be any observable. First, we show that $\mathcal{I}([H, PAP^\perp]) = iPAP^\perp$.

Decomposing

$$\begin{aligned}
e^{itH}P &= \sum_{j=0}^{\infty} \frac{(itH)^j}{j!} P \\
&= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left(\sum_n E_n^j P_n \right) P \\
&= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \sum_{n:E_n=0} E_n^j P_n \\
&= \sum_{n:E_n=0} e^{itE_n} P_n,
\end{aligned}$$

and similarly

$$P^\perp e^{-itH} = \sum_{m:E_m \geq \gamma} P_m e^{-itE_m},$$

we see that

$$\begin{aligned}
\mathcal{I}([H, PAP^\perp]) &= \mathcal{I}(P[H,A]P^\perp) \\
&= \int_{\mathbb{R}} W(t) e^{itH} P[H,A]P^\perp e^{-itH} dt \\
&= \int_{\mathbb{R}} W(t) \sum_{n:E_n=0} e^{itE_n} P_n [H,A] \sum_{m:E_m \geq \gamma} P_m e^{-itE_m} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m\geq\gamma} \int_{\mathbb{R}} W(t) e^{itE_n} P_n A (E_n - E_m) P_m e^{-itE_m} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m\geq\gamma} P_n A P_m (E_n - E_m) \int_{\mathbb{R}} W(t) e^{-it(E_m - E_n)} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m\geq\gamma} P_n A P_m (E_n - E_m) \widehat{W}(E_m - E_n) \\
&= i \sum_{n:E_n=0} \sum_{m:E_m\geq\gamma} P_n A P_m \\
&= i PAP^\perp,
\end{aligned}$$

since $\widehat{W}(\xi) = i\xi^{-1}$ for all $|\xi| \geq \mathrm{Len}(\Delta)$. If the spectrum of H is continuous, the sums can be replaced by integrals with respect to the spectral measure P_λ, as in 3. Either way, the use of domination to interchange the integral over t and the other sums/integrals is provided by the fact that $W, \widehat{W} \in L^1$. By the same argument, $\mathcal{I}([H, P^\perp AP]) = iP^\perp AP$ as well, and so $\mathcal{I}([H, \overline{A}]) = i\overline{A}$. □

Proof. (Of Lemma 8). Let A be an operator with support X, and let $A(t) = e^{itH} A e^{-itH}$ be its time evolution. Let $S = (X^r)^{\mathsf{c}}$ be the set of sites of distance at least r from X. We see that $\|\mathcal{I}(A) - \mathrm{tr}_S(\mathcal{I}(A))\|$ is bounded by

$$\int_{-T}^{T} |W(t)| \|A(t) - \mathrm{tr}_S(A(t))\| dt + \int_{\mathbb{R}\setminus[-T,T]} |W(t)| \|A(t) - \mathrm{tr}_S(A(t))\| dt.$$

We treat each of the terms separately. For the first, by Proposition 7 we have

$$\begin{aligned}
\int_{-T}^{T} |W(t)| \|A(t) - \mathrm{tr}_S(A(t))\| dt &\leq |X| \|A\| \|W\|_\infty \int_{-T}^{T} e^{-\frac{r - v|t|}{\xi}} dt \\
&= \frac{2\xi}{v} |X| \|A\| \|W\|_\infty e^{-\frac{r}{\xi}} (e^{\frac{vT}{\xi}} - 1).
\end{aligned}$$

For the second term, we again employ Proposition 7 along with the decay property $|W(t)| \le \mathcal{O}(|t|^{-\infty})$ to obtain

$$\int_{\mathbb{R}\setminus[-T,T]} |W(t)| \|A(t) - \mathrm{tr}_S(A(t))\| dt \le |X| \|A\| e^{-\frac{r}{\xi}} \int_{\mathbb{R}\setminus[-T,T]} |W(t)| e^{\frac{v|t|}{\xi}} dt$$

(need to add)

□

Appendix C

The Lieb-Robinson Bound

Let the dimensions of the Hilbert spaces at each site, i.e. $\dim(\mathcal{H}_x)$, be uniformly bounded. Let $A \in \mathcal{U}_X$ and $B \in \mathcal{U}_Y$ be any operators having disjoint supports $X \cap Y = \varnothing$, and denote by $r = \mathrm{dist}(X, Y)$ the distance between them. Let $A(t) = e^{itH} A e^{-itH}$ be the time evolution of A. Then

The following is a version of the Lieb-Robinson bound [11].

Proposition 6.

$$\|[A(t), B]\| \leq C\|A\|\|B\| \min\{|X|, |Y|\} e^{-\frac{r - v|t|}{\xi}},$$

where C, v, and ξ are positive constants.

A consequence of the Lieb-Robinson bound is the following estimate on the growth of the support of A in time.

Proposition 7. *Let A be any operator with support in X, and let $S = (X^r)^{\mathsf{c}}$ be the set of sites whose distance from X is at least r. Then*

$$\|A(t) - tr_S(A(t))\| \leq C|X|\|A\| e^{-\frac{r - v|t|}{\xi}},$$

where C, v, and ξ are as in 6.

Proof. The partial trace is given by

$$\mathrm{tr}_S(A(t)) = \int_{\mathcal{G}_S} U A(t) U^* d\mu(U),$$

where $\mathcal{G}_S$ is the group of unitaries supported on S and μ denotes the Haar measure. From this we see that

$$\begin{aligned} \|A(t) - \mathrm{tr}_S(A(t))\| &\leq \int_{\mathcal{G}_S} \|A(t) - U A(t) U^*\| d\mu(U) \\ &= \int_{\mathcal{G}_S} \|[A(t), U] U^*\| d\mu(U) \\ &\leq \|[A(t), U]\| \end{aligned}$$

whence the Lieb-Robinson bound 6 completes the proof. ∎

Appendix D

Grönwall's Inequality and Uniqueness

Theorem 5. *(Grönwall's Inequality). Let* $\alpha : I \to (0, \infty)$ *be positive and continuous on* I^o *for some interval of the form* $[a, b)$, $[a, b]$, *or* $[a, \infty)$. *Suppose* $u : \mathbb{R} \to \mathcal{U}$ *is a Banach-valued, differentiable function. If* $\|u'(t)\| \leq \alpha(t)\|u(t)\|$ *for all* $t \in I$, *then*

$$\|u(t)\| \leq \|u(a)\| e^{\int_a^t \alpha(s) ds} \quad \forall t \in I$$

Proof. Let $f(t) = e^{\int_a^t \alpha(s) ds}$, which is nonzero, has initial value $f(a) = 1$, and has derivative $f'(t) = \alpha(t) f(t)$. Then by the quotient rule,

$$\left(\frac{\|u(t)\|}{f(t)} \right)' = \frac{\|u'(t)\| f(t) - \|u(t)\| \alpha(t) f(t)}{f(t)^2} \leq 0,$$

where the inequality follows from the assumption $\|u'(t)\| \leq \|\alpha(t) u(t)\|$. Thus $\frac{\|u(t)\|}{f(t)}$ is decreasing, so that

$$\frac{\|u(t)\|}{f(t)} \leq \frac{\|u(a)\|}{f(a)} = \|u(a)\|,$$

which is the desired inequality. $\square$

Theorem 6. *(ODE Uniqueness). Let* $F : \mathcal{U} \to \mathcal{U}$ *be Lipschitz and consider the differential equation*

$$u'(t) = F(u(t))$$

with initial condition $u(a) = u_a$ *for some function* $u : I \to \mathcal{U}$, *where* $I = [a, b]$, *or* $[a, b)$, *or* $[a, \infty)$. *Solutions to this equation are unique.*

Proof. Suppose there are two solutions $u(t)$ and $v(t)$, and let $g(t) = \|u(t) - v(t)\|^2$. By assumption, there exists a constant L_F such that $\|F(u(t)) - F(v(t))\| \leq L_F\|u(t) - v(t)\|$, so that

$$\begin{aligned} g'(t) &= 2\|u(t) - v(t)\|\|u'(t) - v'(t)\| \\ &= 2\|u(t) - v(t)\|\|F(u(t)) - F(v(t))\| \\ &\leq 2L_F\|u(t) - v(t)\|^2 \\ &= 2L_F g(t). \end{aligned}$$

Notice that $\alpha := 2L_F$ is a positive continuous function, so we may apply Grönwall's inequality to $g(t)$ to conclude

$$g(t) \leq g(a)e^{2L_f(t-a)} = 0,$$

since $g(a) = 0$. □

Appendix E

Note on Generators of Parallel Transport

Consider the differential equation $\dot{\rho}(\mu) = i[K_B, \rho(\mu)]$ with initial condition $\rho(0) = P_B(0)$. Here $K_B = \int_{\mathbb{R}} W_\gamma(t) e^{-itH_B} \dot{H}_B e^{itH_B} dt$, and recall that in our setting, $\dot{H}_B = Q_h$. We know that the solution is $\rho(\mu) = P_B(\mu)$ (proposition 2). Notice that the map $F : \mathcal{U} \to \mathcal{U}$ defined by $F(A) = i[K_B, A]$ is Lipschitz, since

$$\|F(A) - F(B)\| = \|[K_B, A - B]\| \leq 2\|K_B\| \|A - B\|.$$

The Lipschitz constant is $2\|K_B\|$, which is finite since K_B is a bounded operator:

$$\|K_B\| \leq \int_{\mathbb{R}} |W_\gamma(t)| \|e^{-itH_B} Q_h e^{itH_B}\| dt \leq \int_{\mathbb{R}} |W_\gamma(t)| dt \|Q_h\|.$$

Indeed, since Q_h is the number operator on a finite volume, by charge conservation and the fact that the dimension of the Hilbert space is bounded uniformly by d, there can only be a finite number of charges in the region Γ_h.

Thus, by Grönwall's uniqueness theorem (appendix D), we see that the solution to the equation $\dot{\rho}(\mu) = F(\rho(\mu)) = i[K_B, \rho(\mu)]$ is unique.

Now define

$$K_E := \int_{\mathbb{R}} W_\gamma(t) e^{-itH_E} Q_h e^{itH_E} dt,$$

which is using the gap γ of H_B to define W_γ, but also using the edge Hamiltonian in the time evolution operators. Consider $\sigma : [0, \infty) \to \mathcal{U}$ defined by

$$\dot{\sigma}(\mu) = i[K_E, \sigma(\mu)] \qquad \sigma(0) = P_E(0).$$

We now show that, similar to how ρ is an approximation of P_B, σ is also a good approximation of P_E (up to $\mathcal{O}(L^{-\infty})$) "in the bulk". Let $A \in \mathcal{U}_{\Gamma_B}$ be an operator localized in the bulk of the edge system. Then

$$
\begin{aligned}
\mathrm{Tr}(\dot{\sigma}A) &= \mathrm{Tr}(i[K_E,\sigma]A) \\
&= \mathrm{Tr}(i[A,K_E]\sigma) \\
&= \int_{\mathbb{R}} W_\gamma(t)\mathrm{Tr}([e^{-itH_E}Q_h e^{itH_E},A]\sigma)dt \\
&= \int_{\mathbb{R}} W_\gamma(t)\mathrm{Tr}(e^{-itH_E}[Q_h, e^{itH_E}Ae^{-itH_E}]e^{itH_E}\sigma)dt \\
&= \int_{\mathbb{R}} W_\gamma(t)\mathrm{Tr}(e^{-itH_E}[Q_h, e^{itH_B}Ae^{-itH_B}]e^{itH_E}\sigma + \mathcal{O}(L^{-\infty}))dt \\
&= \int_{\mathbb{R}} W_\gamma(t)\mathrm{Tr}(e^{-itH_B}[Q_h, e^{itH_B}Ae^{-itH_B}]e^{itH_B}\sigma + \mathcal{O}(L^{-\infty}))dt \\
&= \int_{\mathbb{R}} W_\gamma(t)\mathrm{Tr}([e^{-itH_B}Q_h e^{itH_B},A]\sigma)dt + \mathcal{O}(L^{-\infty}) \\
&= \mathrm{Tr}(i[A,K_B]\sigma] + \mathcal{O}(L^{-\infty}) \\
&= \mathrm{Tr}(i[K_B,\sigma]A) + \mathcal{O}(L^{-\infty}),
\end{aligned}
$$

since σ is trace-class (?) and $W_\gamma \in L^1$. By linearity of the trace, we see that $\mathrm{Tr}((\dot{\sigma} - i[K_B,\sigma])A) = \mathcal{O}(L^{-\infty})$ for any operator $A \in \Gamma_B$ (does this mean $\dot{\sigma} - i[K_E,\sigma] = 0$?). But the solution of $\dot{\sigma} - i[K_B,\sigma] = 0$ (with initial condition $\sigma(0) = P_B(0)$) is unique; it is $\rho(\mu)$, or $P_B(\mu)$. Hence

$$
\mathrm{Tr}(P_E A) = \mathrm{Tr}(P_B A) + \mathcal{O}(L^{-\infty}) = \mathrm{Tr}(\rho A) + \mathcal{O}(L^{-\infty}) = Tr(\sigma A) + \mathcal{O}(L^{-\infty})
$$

for any operator $A \in \Gamma_B$. In particular, this gives another local formula for the Hall conductivity in the bulk of an edge system, by taking $A = J_V$, where J is the current operator and $V \subset \Gamma_B$ is a set localized in the bulk. The Hall conductivity is given by $\mathrm{Tr}(\dot{P}_E J_V)$, and this can be approximated by

$$
\mathrm{Tr}(\dot{P}_E J_V) = \mathrm{Tr}(\dot{P}_B J_V) + \mathcal{O}(L^{-\infty}) = \mathrm{Tr}(\dot{\rho} J_V) + \mathcal{O}(L^{-\infty}) = \mathrm{Tr}(\dot{\sigma} J_V) + \mathcal{O}(L^{-\infty}).
$$

Want to pick a norm s.t. Gronwall gives $\|\rho(\mu) - \sigma(\mu)\|_G \leq \|P_B(0) - P_E(0)\|_G e^{2L_F\mu}$. Need $\|P_B(0) - P_E(0)\|_G$ to be small enough to kill the exponential which depends on $L_F = 2\|K_B\|_G \leq \|W_\gamma\|_{L^1}\|Q_h\|_G$. If we use the operator norm for $\|\cdot\|_G$, we would get $\|Q_h\|_G = d|\Gamma_h|$ in the exponent. Need $\|\cdot\|_G$ to be an actual norm so that $\|\rho - \sigma\|_G = 0 \implies \rho = \sigma$.

Appendix F

The Helffer-Sjöstrand Representation

The Helffer-Sjöstrand representation is a functional calculus $f \mapsto f(H)$ for arbitrary (possibly unbounded) operators H on the set of functions

$$\mathcal{A} = \bigcup_{\beta<0} \{f : \mathbb{R} \to \mathbb{C} \ : \ f \in C^\infty(\mathbb{R}), \ |f^{(n)}(x)| \leq c_n(1+x^2)^{\frac{\beta-n}{2}}\}.$$

It has the following properties.

Theorem 7. *For any $f \in \mathcal{A}$,*

1. *$f \mapsto f(H)$ is an algebraic homomorphism (linear and multiplicative).*
2. *$\overline{f}(H) = f(H^*)$.*
3. *$\|f(H)\| \leq \|f\|_\infty$.*
4. *For all $w \notin \mathbb{R}$, if $r_w(s) = \frac{1}{s-w}$ then $r_w(H) = (H-z)^{-1}$.*
5. *For all $f \in C_c^\infty(\mathbb{R})$ with supp$(f) \cap$ Spec$(H) = \varnothing$, we have $f(H) = 0$.*

For $f \in \mathcal{A}$ which are also Borel, $f(H)$ agrees with the operator given by the Borel functional calculus. There is an explicit formula for $f(H)$, which is given by

$$f(H) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (H-z)^{-1} dz \wedge d\bar{z},$$

where $\tilde{f} : \mathbb{C} \to \mathbb{C}$ is a *quasi-analytic extension* of $f : \mathbb{R} \to \mathbb{R}$. It is defined as follows. For any smooth f, we set

$$\tilde{f}(z) = \sum_{r=0}^{n} \tau \left(\frac{y}{(1+x^2)^{1/2}} \right) \frac{(iy)^r}{r!} f^{(r)}(x)$$

where $\tau : \mathbb{R} \to \mathbb{R}$ is any smooth function satisfying

$$\tau(s) = \begin{cases} 1 & |s| < 1 \\ 0 & |s| > 2 \end{cases}.$$

The extension turns out to be independent of the choice of n and τ. Furthermore, as $|\mathrm{Im}(z)| \to 0$, the Wirtinger derivative of the extension obeys the bound

$$\left|\frac{\partial \tilde{f}}{\partial \bar{z}}\right| = \mathcal{O}(|y|^n).$$

Thus $\frac{\partial \tilde{f}}{\partial \bar{z}} = 0$ for all real z, which is why $\tilde{f}$ is called a "quasi"-analytic extension (the Wirtinger derivative would be zero everywhere were $\tilde{f}$ analytic).

A crucial property of the Helffer-Sjöstrand functional calculus is the following bound.

Lemma 15. *For any $n \in \mathbb{N}$, the quasi-analytic extension $\tilde{f}$ can be chosen so that*

$$\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \frac{1}{|Im(z)|^{p+1}} dz \wedge d\bar{z} \leq C_0 \sum_{k=0}^{n+2} \|f^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_m = \int_{-\infty}^{\infty} |f(x)|(1+x^2)^{m/2} dx.$$

This is often useful because the resolvent obeys the bound $\|(H-z)^{-1}\| \leq |\mathrm{Im}(z)|^{-1}$.