Bulk-Edge Correspondence in Quantum Hall Lattice Systems

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1 Introduction

1.1 History

Since its unexpected discovery in 1980, the Quantum Hall Effect (QHE) has captured significant interest in the mathematical physics community.

1.2 Heuristic Arguments

1.2.1 The Classical Hall Effect

Using classical electromagnetism is not enough to predict the plateaux seen experimentally. Suppose we have a 2-dimensional electron gas, and let $\vec{B} = B\hat{x_3}$ be a magnetic field piercing the plane of the electrons. They are subjected to a Lorentz force

$$m\dot{\vec{v}} = -q\vec{v} \times \vec{B}.$$

The solution to this differential equation is given by the *cyclotron orbits*,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} a + r\sin(\omega_B t + \phi) \\ b + r\cos(\omega_B t + \phi) \\ 0 \end{pmatrix}$$

where $\omega_B = qB/m$ is the *cylontron frequency.* An electric field $\vec{E} = E\hat{x_1}$ is introduced, and the electrons move in the x_1-direction. We now employ the *Drude model*,

$$m\dot{\vec{v}} = -q(\vec{E} + \vec{v} \times \vec{B}) + \frac{m}{\tau}\vec{v},$$

where the final term is a linear friction term, and τ is the scattering time. At equilibrium, the equation reads

$$\vec{J} + \frac{q\tau}{m}\vec{J} \times \vec{B} = -\frac{q^2 n\tau}{m}\vec{E},$$

where $\vec{J} = -nq\vec{v}$ is the current density, related to the velocity by the density of electrons per unit area n. In matrix notation, this reads

$$\begin{pmatrix} 1 & \omega_B\tau \\ -\omega_B\tau & q \end{pmatrix} \vec{J} = -\frac{q^2 n\tau}{m}\vec{E}.$$

Since the matrix on the left is invertible, we may write $\vec{J} = \sigma\vec{E}$, which is Ohm's Law. The *conductivity tensor* is given by

$$\sigma = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ -\sigma_{xy} & \sigma_{yy} \end{pmatrix} = -\frac{q^2 n\tau}{m(1+\omega_B^2\tau^2)} \begin{pmatrix} 1 & -\omega_B\tau \\ \omega_B\tau & 1 \end{pmatrix}.$$

The off-diagonal components, $\pm\frac{q^2 n\tau}{m(1+\omega_B^2\tau^2)}\omega_B\tau$, are responsible for the Hall effect; the magnetic field induces a component of the current in the x_2-direction, in addition to the one in the x_1-direction from the electric field.

When making a measurement, physicists actually measure the resistivity $\rho = \sigma^{-1}$. In particular, the *Hall resistivity* is given by

$$\rho_{xy} = \frac{B}{nq}.$$

The key prediction of the classical theory is that the Hall resistivity is increases linearly in response to the strength of the magnetic field.

1.2.2 The Quantum Hall Effect

Von Klitzing's experimental observation in 1980 made it clear that classical electromagnetism is not sufficient to describe the Hall effect. At a temperature of about 8mK, the Hall resistivity looked like this as a function of the magnetic field strength.

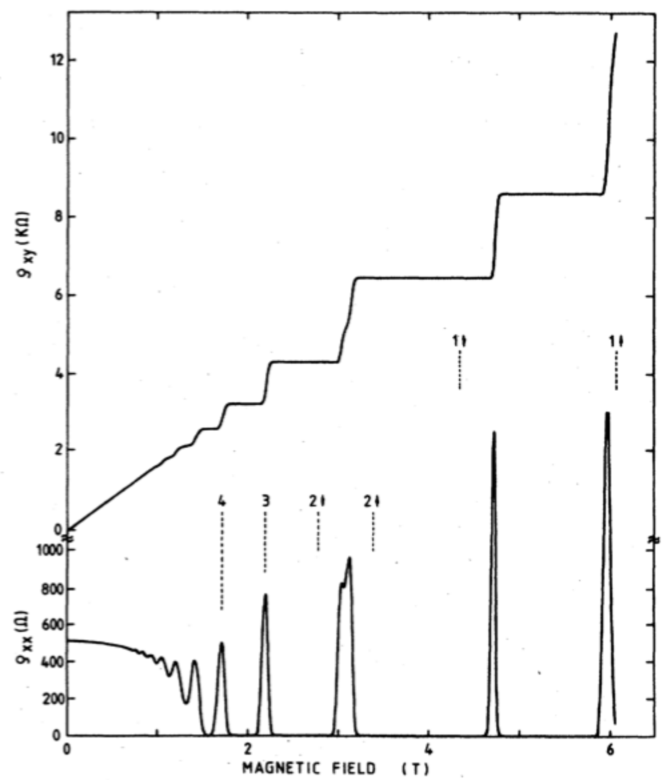

Figure 1: The Hall resistivity as a function of the magnetic field strength. Distinct plateaux are clearly visible.

Even more surprising, the plateaux occur at the values

$$\rho_{xy} = \frac{2\pi\hbar}{nq},$$

where n is an integer. In fact, this integer can be measured to such extraordinary precision (to about 3×10^{-10}) that as of 2020, the SI definition of the Ohm itself has been redefined in terms of the quantum Hall resistivity.

The first explanation employing quantum mechanics is due to Laughlin in (need to add).

An intriguing mathematical fact of the quantum Hall effect is the equality of bulk and edge conductivity, in the sense that whether the system is assumed to have an edge or not is immaterial. Proving this fact, both in the interacting and noninteracting setting, is the main focus of this thesis.

2 Nonnteracting Setting

Consider the lattice $\mathbb{Z}^2$, and the associated Hilbert space of square-summable sequences of vectors in $\mathbb{C}^n$,

$$\ell^2(\mathbb{Z}^2, \mathbb{C}^n) = \left\{ (x_i)_{i\in\mathbb{Z}^2} \subset \mathbb{C}^n \ : \ \sum_{i\in\mathbb{Z}^2} \|x_i\|^2 < \infty \right\},$$

with inner product $\langle x, y\rangle = \sum_{i\in\mathbb{Z}^2} x_i \overline{y_i}$. We denote this as $\ell^2(\mathbb{Z}^2)$ for short. On this Hilbert space we define a *bulk Hamiltonian* $H_B : \ell^2(\mathbb{Z}^2) \to \ell^2(\mathbb{Z}^2)$, whose matrix elements follow a short-range assumption:

Assumption 1. *There exists some $\alpha > 0$ such that*

$$\sup_{y\in\mathbb{Z}^2} \sum_{x\in\mathbb{Z}^2} |H_B(x, y)|(e^{\alpha|x-y|} - 1) \leq C < \infty,$$

where $|x| = |x_1| + |x_2|$ is the taxicab metric.

We also construct an *edge Hamiltonian* on the lattice $\mathbb{Z}^2_a := \{x \in \mathbb{Z}^2 : x_2 > -a\}$, denoted by $H_a : \ell^2(\mathbb{Z}^2_a) \to \ell^2(\mathbb{Z}^2_a)$. The bulk and edge Hamiltonians are related by the *edge operator*

$$E_a : \ell^2(\mathbb{Z}^2_a) \to \ell(\mathbb{Z}^2),$$

$$E_a := J_a H_a - H_B J_a,$$

where $J_a : \ell^2(\mathbb{Z}^2_a) \to \ell(\mathbb{Z}^2)$ denotes extension by zeroes. We require only that that the edge operator satisfies the edge assumption

Assumption 2. *The edge operator satisfies*

$$\sup_{z\in\mathbb{Z}^2} \sum_{y\in\mathbb{Z}^2_a} |E_a(x, y)|e^{\alpha(|x_2+a|+|x_1-y_1|)} \leq C < \infty$$

for some $\alpha > 0$, where $|x| = |x_1| + |x_2|$ is the taxicab metric.

The interpretation is that $E_a = J_a H_a - H_B J_a$ is the difference between first applying H_a on $\ell^2(\mathbb{Z}_a^2)$, and then making everything below $-a$ into zeroes, versus first making all $x \in \mathbb{Z}^2$ such that $x_2 < -a$ zeroes, and the applying H_B. The assumption ensures that the effects from introducing the edge at $-a$ die exponentially as we move upward away from the edge (due to the $|x_2 - (-a)|$ term in the exponent), and also terms do not interact too much as their x_1 distance increases (due to the $|x_1 - y_1|$ term in the exponent).

A simple example of an edge Hamiltonian satisfying the edge condition is $H_a = J_a^* H_B J_a$, which gives $E_a = (\mathcal{J}_a \mathcal{J}_a^* - 1)H_B$. The idea is that for a state $\psi \in \ell^2(\mathbb{Z}_a^2)$, we have $\langle \psi, H_a \psi \rangle = \langle (J_a \psi), H_B(J_a \psi) \rangle$, which we interpret as the edge Hamiltonian having the same expectation as the bulk Hamiltonian if we just transformed all the states with support below the line $x_2 = -a$ into zeroes below $x_2 = -a$. The edge operator is

$$E_a = (J_a J_a^* - \mathbb{1}) H_B J_a = \begin{cases} -H_B(x,y) & \text{if } x_2 < -a \\ 0 & \text{if } x_2 \geq -a \end{cases}$$

Intuitively, there is no difference between H_B and H_a on $\mathbb{Z}_a^2$. The edge assumption 2 then follows from the bound $|E_a(x,y)| \leq |H_B(x,y)|$ and the short range assumption 1.

We also make the following assumption about the bulk Hamiltonian.

Assumption 3. *The bulk Hamiltonian has a spectral gap. That is, there exists an interval $\Delta \subset \mathbb{R}$ such that for all L,*

$$\Delta \cap \operatorname{Spec}(H_B) = \varnothing.$$

Remark: The spectral gap assumption can be relaxed to a "mobility gap" assumption,

$$\sup_{f \in B_c(\Delta)} |f(H_B)(x,y)|(1 + |x|)^{-\alpha_1} e^{\alpha_2 |x-y|} < \infty$$

for some $\alpha_1 > 0$, where $B_c(\Delta)$ is the set of Borel functions f which are constant on $(-\infty, \inf \Delta)$ and on $(\sup \Delta, \infty)$ such that $|f(x)| \leq 1$ for all x. See (need to add) for details.

We define the *bulk conductivity* at Fermi energy μ as follows. Suppose we subject the system to an external electric potential difference V in the x_2 direction. We write this as $-V_0 \Lambda_2$, where Λ_i are multiplication operators $\Lambda_i |\psi(x_1, x_2)\rangle = \Lambda(x_i) |\psi(x_1, x_2)\rangle$ which are *switch functions*,

$$\Lambda : \mathbb{R} \to \mathbb{R} \qquad \Lambda(x_i) = \begin{cases} 1 & \text{if } x_i \leq 0 \\ 0 & \text{if } x_i \geq 1 \end{cases}$$

and are smooth and monotonically decreasing on $(0,1)$. Note that the ensuing physics (in particular, our definition of the Hall conductivity) is independent of the particular choice of switch function Λ_i, since any two switch functions are exactly equal on the lattice.

This gives $\vec{E} = -\nabla V = V_0 \frac{\partial \Lambda_2}{\partial x_2}$, so that $\vec{E}$ is has compact support $\mathrm{supp}(\Lambda_2')$. We introduce a function which grows slowly in time as t grows from $-\infty$ to 0, so as to invoke the adiabatic principle. Here, we choose $e^{\varepsilon t}$, and we will let $\varepsilon \to 0$ at the end. The Hamiltonian therefore experiences a perturbation,

$$\widetilde{H}_B(t) = H_B - V_0 \Lambda_2 e^{\varepsilon t}.$$

We define the Hall current operator $J_H = i[\widetilde{H}_B(t), \Lambda_1] = i[H_B, \Lambda_1]$, which is related to the Hall conductivity by $J_H = \sigma_H V$. We also denote by $P_\mu := P((-\infty, \mu])$ the projection-valued measure associated with H_B onto states with energy below the Fermi energy μ (see Appendix G).

Lemma 1. *The ground state expectation* $\mathrm{Tr}(P_\mu J_H)$ *of the Hall current is zero.*

Proof. Notice that J_H is trace-class, since its trace norm can be broken into $\|J\|_1 \leq \|[H_B, \Lambda_1] e^{\delta |x_1|}\| \|e^{-\delta |x_1|}\|_1$. The first term is bounded by lemma (need to add), while the second is a summable function on $\mathbb{Z}^2$. This fact, together with P_μ being bounded, allows us to exploit linearity and cyclicity of the trace. Since the Hamiltonian commutes with its ground state projection, we have

$$\mathrm{Tr}(P_\mu [H_B, \Lambda_1]) = \mathrm{Tr}([P_\mu, H_B] \Lambda_1) = 0.$$

$\square$

Proposition 1. *The Hall conductivity* σ_H *in the bulk system is equal to*

$$\sigma_B = i \mathrm{Tr}\left(P_\mu \left[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2] \right] \right).$$

Proof. We begin with the Heisenberg equation of motion for the density matrix, $\dot{\rho}(t) = -i[\widetilde{H}_B(t), \rho(t)]$, with initial condition $\lim_{t \to -\infty} \|\rho(t) - e^{-itH_B} P_\mu e^{itH_B}\| = 0$, which also implies $\lim_{t \to -\infty} \|e^{itH_B} \rho(t) e^{-itH_B} - P_\mu\| = 0$.

We work in the interaction picture by defining $\rho_I(t) = e^{itH_B} \rho(t) e^{-itH_B}$ and $\Delta H_B(t) = -e^{itH_B} V_0 \Lambda_2 e^{\varepsilon t} e^{-itH_B}$. Thus

$$\dot{\rho}_I(t) = -i[\Delta H_B(t), \rho_I(t)].$$

The solution to this differential equation is readily verified to be

$$\rho(t) = i\int_{-\infty}^{t} [\Delta H_B(s), P_\mu] ds + P_\mu$$

Indeed, taking the derivative of the right hand side gives $i[\Delta H_B(t), P_\mu] = i[\Delta H_B(t), \rho_I(t)] + \mathcal{O}(V_0^2)$, but P_μ and $\rho_I(t)$ are equal up to zeroth order in V_0 (need to add). The initial condition is also satisfied.

Using $J_H = i[H_B, \Lambda_1] = \sigma_H V = -\sigma_H V_0 \Lambda_2$, we obtain $\sigma_H = \frac{1}{V_0}\lim_{\varepsilon\to 0}\mathrm{Tr}(\rho_I(0) i[H_B, \Lambda_1])$ (need to add). By Lemma 1, the expectation of the ground state current is zero. Thus

$$\begin{aligned}
\sigma_H &= \frac{i}{V_0}\lim_{\varepsilon\to 0}\mathrm{Tr}\left(i\int_{-\infty}^{0}[\Delta H_B(t), P_\mu][H_B, \Lambda_1] ds\right) \\
&= -\frac{1}{V_0}\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}[-e^{isH_B}V_0\Lambda_2 e^{\varepsilon s}e^{-isH_B}, P_\mu][H_B, \Lambda_1] ds\right) \\
&= -\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0} e^{isH_B}[\Lambda_2, P_\mu]e^{-isH_B}[H_B, \Lambda_1]e^{\varepsilon s} ds\right) \\
&= -\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0} (e^{-isH_B}[H_B, \Lambda_1]e^{isH_B})\cdot([\Lambda_2, P_\mu]e^{\varepsilon s}) ds\right)
\end{aligned}$$

Where we used the fact that P_μ and H_B commute. Using integration by parts on the two terms in brackets, and noting that $\frac{d}{ds}(e^{isH_B}[H_B, \Lambda_1]e^{-isH_B}) = -(ie^{isH_B}\Lambda_1 e^{-isH_B} - \Lambda_1)$, we obtain

$$\begin{aligned}
\sigma_H &= i\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0} (e^{-isH_B}\Lambda_1 e^{isH_B} - \Lambda_1)\frac{d}{ds}([\Lambda_2, P_\mu]e^{\varepsilon s}) ds\right) \\
&= i\lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0} \Lambda_1^s[\Lambda_2, P_\mu]e^{\varepsilon s}) ds\right)
\end{aligned}$$

where $\Lambda_1^s := e^{-isH_B}\Lambda_1 e^{isH_B} - \Lambda_1$. Using the notation $\overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, it is readily verified that the commutator $[\Lambda_2, P_\mu]$ is an *off-diagonal* operator, in the sense that $[\Lambda_2, P_\mu] = \overline{[\Lambda_2, P_\mu]}$. Furthermore, a simple computation reveals that for any two operators A and B, $\mathrm{Tr}(\overline{A}B) = \mathrm{Tr}(A\overline{B})$. It therefore follows that

$$\sigma_H = i\lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0} \overline{\Lambda_1^s}[\Lambda_2, P_\mu]e^{\varepsilon s}) ds\right)$$

The integrand can be broken into two terms,

$$\overline{\Lambda_1^s}[\Lambda_2, P_\mu]e^{\varepsilon s} = e^{-isH_B}\overline{\Lambda_1}e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s} - \overline{\Lambda_1}[\Lambda_2, P_\mu]e^{\varepsilon s}$$

by commutativity of P_μ and H_B. We show that the integral of the first term vanishes. We begin by breaking the first term down further into

$$e^{-isH_B}P_\mu\Lambda_1 P_\mu^\perp e^{isH_B}[\Lambda_2,P_\mu]e^{\varepsilon s}+e^{-isH_B}P_\mu^\perp\Lambda_1 P_\mu e^{isH_B}[\Lambda_2,P_\mu]e^{\varepsilon s}.$$

We treat the first of these two terms; the other is handled in an identical manner. We invoke the spectral theorem (Appendix G) to write $e^{-isH_B}P_\mu = \int_{-\infty}^{\mu} e^{-is\lambda}dP_\lambda$, and similarly $P_\mu^\perp e^{isH_B} = (\mathbb{1}-P_\mu)e^{isH_B} = \int_\mu^\infty e^{is\nu}dP_\nu$.

We remark that, since the Fermi energy μ is assumed to lie in a spectral gap, there must exist a neighbourhood $(\mu-\delta,\mu+\delta)$ in which there are no states. We exploit this fact to rewrite the limits of integration, $\int_{-\infty}^{\mu-\delta} e^{-is\lambda}dP_\lambda$ and $\int_{\mu+\delta}^{\infty} e^{is\nu}dP_\nu$. We therefore obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{0} e^{-isH_B}P_\mu\Lambda_1 P_\mu^\perp e^{isH_B}[\Lambda_2,P_\mu]e^{\varepsilon s}ds\\
&\qquad=\lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0}\int_{-\infty}^{\mu-\delta} e^{-is\lambda}dP_\lambda\Lambda_1\int_{\mu+\delta}^{\infty} e^{is\nu}dP_\nu[\Lambda_2,P_\mu]e^{\varepsilon s}ds\right)\\
&\qquad=\lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0}\int_{-\infty}^{\mu-\delta}\int_{\mu+\delta}^{\infty} e^{s(\varepsilon-i\lambda+i\nu)}dP_\lambda\Lambda_1 dP_\nu[\Lambda_2,P_\mu]ds\right)
\end{aligned}$$

Performing the integral over s yields

$$\lim_{\varepsilon\to 0}\varepsilon\int_{-\infty}^{0} e^{s(\varepsilon-i\lambda+i\nu)}ds = -\lim_{\varepsilon\to 0}\frac{\varepsilon}{i\varepsilon+\lambda-\nu}$$

This limit is zero, since $\lambda\neq\nu$. Indeed, due to the spectral gap, the integration variables live in $\lambda\in(-\infty,\mu-\delta)$ and $\nu\in(\mu+\delta,\infty)$. The case for the $e^{-isH_B}P_\mu^\perp\Lambda_1 P_\mu e^{isH_B}[\Lambda_2,P_\mu]e^{\varepsilon s}$ term (where the P_μ and $P_\mu^\perp$ swap places) is treated analogously. Hence the first term in the integrand for σ_H vanishes, as claimed.

Finally, we return to our expression for the Hall conductivity, which now reads

$$\sigma_H = i\lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0}\overline{\Lambda_1}[\Lambda_2,P_\mu]e^{\varepsilon s}ds\right).$$

It is a basic algebraic calculation to show that $\overline{\Lambda_1}=[[\Lambda_1,P_\mu],P_\mu]$. Evaluating the integral over s is now trivial; $\int_{-\infty}^{0} e^{\varepsilon s}ds=\varepsilon^{-1}$. Thus

$$\sigma_H = -i\mathrm{Tr}([[\Lambda_1,P_\mu],P_\mu][\Lambda_2,P_\mu]).$$

Shifting the commutator completes the proof:

$$\begin{aligned}\sigma_H &= -i\mathrm{Tr}(P_\mu[[\Lambda_2, P_\mu], [\Lambda_1, P_\mu]]) \\ &= i\mathrm{Tr}(P_\mu[[\Lambda_1, P_\mu], [\Lambda_2, P_\mu]]) \\ &= i\mathrm{Tr}(P_\mu[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]).\end{aligned}$$

(need to add justification for shifting commutator). □

Remark: This is reminiscent of the well-known adiabatic curvature formula,

$$\kappa = \mathrm{Tr}(P[\partial_1 P, \partial_2 P]) = \mathrm{Tr}(P\,[[P, K_1], [P, K_2]]) = \mathrm{Tr}(P[K_1, K_2]),$$

where K_i are called *generators of parallel transport*. We will see the adiabatic curvature formula again later in the interacting setting.

For the *edge conductivity*, we again need the current operator across the line $x_1 = 0$, which is this time given by $-i[H_a, \Lambda_1]$. We define

$$\sigma_E = -i \lim_{a\to\infty} \mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]),$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(r) = \begin{cases} 1 & \text{if } r \leq \inf \Delta \\ 0 & \text{if } r \geq \sup \Delta \end{cases}$$

and decreases smoothly and monotonically in Δ. The definition of σ_E is reminiscent of another formula we will see later in the interacting setting, $\mathrm{Tr}(\dot{P}J)$, where J is the current operator. The interpretation of σ_E is that if we apply a small potential difference V across $x_2 = -a$ to $x_2 = \infty$, there will be a net current

$$\begin{aligned}I &= -i\mathrm{Tr}(\rho(H_a + V)[H_a + V, \Lambda_1] - \rho(H_a)[H_a, \Lambda_1]) \\ &= -i\mathrm{Tr}((\rho(H_a + V) - \rho(H_a))[H_a, \Lambda_1])\end{aligned}$$

Thus we obtain the conductivity

$$\sigma_E = \frac{I}{V} = -i\mathrm{Tr}\left(\frac{(\rho(H_a + V) - \rho(H_a)}{V}[H_a, \Lambda_1]\right) \to -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1])$$

in the limit as $V \to 0$. As we shall see, it turns out that σ_E is independent of the choice of ρ, and σ_B is independent of λ.

2.1 Equality of Bulk and Edge Conduvitivities

The main result of this section is

Theorem 1. $\sigma_E = \sigma_B$.

Outline of the Proof

Before giving the proof in its entirety, we outline the basic steps. The edge condition guarantees that $\sigma_E(a) := \rho'(H_a)[H_a, \Lambda_1]$ is trace-class (need to add). We posit that the edge conductivity can be rewritten as $\sigma_E = \lim_{a\to\infty} \sigma_E(a)$, where

$$\sigma_E(a) = -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2),$$

since we have assumed that there is a spectral gap (as opposed to a mobility gap), so that there are extended states near the edge, and no bound states or resonances far from the edge. Thus, intuitively, the cutoff introduced by Λ_2 is irrelevant as we take $a \to \infty$. We provide a more concrete justification for this later in Lemma 2.

The key ingredient of the proof is the use of the functional calculus given by the Helffer-Sjöstrand representation of self-adjoint operators on a Hilbert space (Appendix F). The two crucial operators written in their Helffer-Sjöstrand representations are

$$\rho(H) = -\frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R(z)dz \wedge d\bar{z}$$

$$\rho'(H) = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R(z)^2 dz \wedge d\bar{z}$$

where $R(z) = (H - z)^{-1}$ is the resolvent of H. For ease of notation, we drop the $dz \wedge d\bar{z}$ from this point.

Another key observation is that it turns out that the bulk conductivity can actually be written

$$\sigma_B = i\mathrm{Tr}([\rho(H_B), \Lambda_1]\Lambda_2).$$

By employing the Helffer-Sjostrand representations above, one can add an operator of zero trace to the edge conductivity, and show that this operator converges in trace to $[\rho(H_B), \Lambda_1]\Lambda_2$ in the limit $a \to \infty$.

The Proof

Lemma 2. *The edge conductivity is equal to* $\lim_{a\to\infty}\sigma_E(a)$*, where*

$$\sigma_E(a) = -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2).$$

Proof. Since σ_B is translation invariant (need to add), we only need to prove the case $-i\mathrm{Tr}(\rho'(H_{a=0})[H_{a=0}, \Lambda_1] = \sigma_B$. We drop the subscript, $H := H_{a=0}$.

A general fact of functional analysis is that if $A_n \xrightarrow{s} 0$ and X is trace-class, then $\|A_n X\|_1 \to 0$. Since the multiplication operator $\Lambda_{2,n}|\psi\rangle := \Lambda(x_2 - n)|\psi\rangle$ converges strongly to the identity as $n \to \infty$, it follows that

$$\|\rho'(H)[H, \Lambda_1](\mathbb{1} - \Lambda_{2,n})\|_1 \to 0,$$

and thus from the inequality $|\mathrm{Tr}(A)| \leq \|A\|_1$ we deduce

$$\sigma_E(a) = -i\mathrm{Tr}(\rho'(H)[H, \Lambda_1]) = -i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H)[H, \Lambda_1]\Lambda_{2,n}).$$

Consider the edge Hamiltonian with cutoff at $a = 0$ associated with the bulk Hamiltonian shifted down by n. We denote this modified edge Hamiltonian by H^n.

Whether we first cut off everything above $x_2 = n$ using $\Lambda_{2,n}$ and then apply $H_{a=0}$, or instead cut off everything above $x_2 = 0$ using Λ_2 and then apply the Hamiltonian H^n is immaterial. In other words,

$$-i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H)[H, \Lambda_1]\Lambda_{2,n}) = -i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H^n)[H^n, \Lambda_1]\Lambda_2).$$

Thus, our goal is to show that

$$\lim_{n\to\infty} \mathrm{Tr}(\rho'(H^n)[H^n, \Lambda_1]\Lambda_2) = \lim_{a\to\infty} \mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2).$$

□

Define

$$Z(a) = [\rho(H_a), \Lambda_1]\Lambda_2 - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[R_a(z), [H_a, \Lambda_1]\Lambda_2] dz^2$$

This operator has zero trace. Indeed, the first term has vanishing trace in the position basis, while the second term's integrand involves the trace of $[R, R] = 0$. The bounds necessary for shifting the commutator like this are on

$$\|[H_a,\Lambda_1]e^{\delta|x_1|}\|, \quad \|e^{-\delta|x_1|}\Lambda_2\|_1, \quad \|R\|,$$

the first two of which are given below in Lemmas 3, 4, and the third is the fact that the resolvent is bounded. So $\sigma_E(a) = \mathrm{Tr}(\Sigma(a))$, where

$$\begin{aligned}\Sigma(a) &= -i\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 + iZ(a)\\ &= -i\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 + i[\rho(H_a),\Lambda_1]\Lambda_2 - \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[R_a(z),[H_a,\Lambda_1]\Lambda_2]dz^2.\end{aligned}$$

Using the Hellfer-Sjöstrand representations for the first two terms on the right hand side, we obtain

$$\begin{aligned}\Sigma(a) = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)^2[H_a,\Lambda_1]\Lambda_2 dz^2 &+ \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1]R_a(z)\Lambda_2 dz^2\\ &- \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} (R_a(z)^2[H_z,\Lambda_1]\Lambda_2 - R_a(z)[H_a,\Lambda_1]\Lambda_2 R_a(z))dz^2\\ = -\frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1][R_a(z),\Lambda_2]dz^2&\\ = \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1]R_a(z)[H_a,\Lambda_2]R_a(z)dz^2,&\end{aligned}$$

where we used

$$[R_a(z),\Lambda_i] = -R_a(z)[H_a,\Lambda_i]R_a(z)$$

in the final equality. Next, we must prove that the operator above converges to the corresponding bulk operator in trace-norm,

$$\|\Sigma(a) - \Sigma_B\|_1 \to 0,$$

as $a \to \infty$, which in turn proves that $\mathrm{Tr}(\Sigma(a)) \to \mathrm{Tr}(\Sigma_B)$ because of the bound $|\mathrm{Tr}(A)| \le \|A\|_1$. Here, Σ_B is the same operator as before, but using the bulk operators H_B and $R_B(z)$. Once this limit is established, we shall prove that $\sigma_B = \mathrm{Tr}(\Sigma_B)$ to conclude the proof.

To show that the limit is zero as claimed, we bound the trace norm of the integrand of $\Sigma(a)$ with the ultimate goal of applying dominated convergence. We accomplish this bound by breaking it into three parts,

$$R[H_a,\Lambda_1]R[H_a,\Lambda_2]R = J_a[R,\Lambda_1]e^{\delta|x_1|}J_a^* \cdot e^{-\delta|x_1|}e^{-\delta|x_2|} \cdot J_a e^{\delta|x_2|}[H_a,\Lambda_2]RJ_a^*,$$

and bounding the norm of each with the following two lemmas. We remark that the extension J_a and its adjoint have norm 1.

Lemma 3.

$$\|[H_a, \Lambda_i]e^{\delta|x_i|}\| \leq C.$$

Proof. The operator can be bounded by inspecting its matrix elements

$$\begin{aligned}\langle x, [H_a, \Lambda_i]e^{\delta|x_i|}y\rangle &= \langle x, H_a\Lambda_i y\rangle e^{\delta|y_i|} - \langle x, \Lambda_i H_a y\rangle e^{\delta|y_i|} \\ &= H_a(x,y)e^{\delta|y_i|}(\Lambda(y_i) - \Lambda(x_i)).\end{aligned}$$

This is zero if $|x_i - y_i| \leq |y_i|$, since this would imply that x_i and y_i have the same sign, yielding $\Lambda(x_i) = \Lambda(y_i)$. So either the matrix element is zero, or $|y_i| \leq |x_i - y_i|$, which implies

$$\begin{aligned}|H_a(x,y)e^{\delta|y_i|}(\Lambda(y_i) - \Lambda(x_i))| &\leq 2|H_a(x,y)|e^{\delta|x_i - y_i|} \\ &\leq 2|H_a(x,y)|e^{\delta|x-y|} \\ &\leq C|H_a(x,y)|(e^{\delta|x-y|} - 1),\end{aligned}$$

where the final inequality comes from the fact that the diagonal matrix elements are zero. Hence the short range assumption

$$\sup_{x\in\mathbb{Z}^2} \sum_{y\in\mathbb{Z}^2} |H(x,y)|(e^{\mu|x-y|} - 1) < \infty$$

combined with Holmgren's bound

$$\|A\| \leq \max\left\{\sup_{x\in\mathbb{Z}^2} \sum_{y\in\mathbb{Z}^2} |A(x,y)|,\ \sup_{y\in\mathbb{Z}^2} \sum_{x\in\mathbb{Z}^2} |A(x,y)|\right\}$$

completes the proof. □

Lemma 4. $e^{-\delta|x_1|}e^{-\delta|x_2|}$ *is trace-class.*

Proof. We bound the trace norm by noticing that this is a positive operator satisfying

$$\langle (n,m), e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m)\rangle = \langle e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m), (n,m)\rangle,$$

so that its trace norm is equal to its trace. In the position basis, we see that its trace is given by a geometric series

$$\begin{aligned}\mathrm{Tr}(e^{-\delta|x_1|}e^{-\delta|x_2|}) &= \sum_{(n,m)\in\mathbb{Z}^2} \langle (n,m), e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m)\rangle \\ &\leq 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-\delta m}e^{-\delta n} \\ &= 2\left(\frac{1}{1-e^{-\delta}}\right)^2.\end{aligned}$$

□

Now we return to the integrand which was broken into three parts. For the first term, $J_a[R_a(z), \Lambda_1]e^{\delta|x_1|}J_a^*$, we bound its operator norm by breaking it down further into

$$\begin{aligned}\|J_a[R_a(z),\Lambda_1]e^{\delta|x_1|}J_a^*\| &= \|[R_a(z),\Lambda_1]e^{\delta|x_1|}\| \\ &= \|-R_a(z)[H_a,\Lambda_1]R_a(z)e^{\delta|x_1|}\| \\ &\leq \|R_a(z)\|\cdot\|[H_a,\Lambda_1]e^{\delta|x_1|}\|\cdot\|e^{-\delta|x_1|}R_a(z)e^{\delta|x_1|}\|.\end{aligned}$$

The norm of $R_a(z)$ is bounded by

$$\|R_a(z)\| \leq \frac{1}{|\mathrm{Im}(z)|}$$

for any $z \notin \mathbb{R}$ since H_a is self-adjoint. The second term is bounded by Lemma 3. Finally, for the third term $e^{-\delta|x_1|}R_a(z)e^{\delta|x_1|}$, we apply the Combes-Thomas bound,

$$\|e^{-\varepsilon f(x)}R_a(z)e^{\varepsilon f(x)}\| \leq \frac{C}{|\mathrm{Im}(z)|}$$

where $f : \mathbb{Z}^2 \to \mathbb{R}$ is any Lipschitz function, and ε can be chosen as $\varepsilon = \frac{1}{C(1+|\mathrm{Im}(z)|)}$. Altogether, the bound of the first term of the integrand takes the form

$$\frac{C}{\mathrm{Im}(z)^2}.$$

The second term of the integrand is bounded by Lemma 4. Finally, the bound for the third term of the integrand, $e^{\delta|x_2|}[H_a,\Lambda_2]R_a(z)$, follows from the bound on R and Lemma 3, and is again of the form $\frac{C}{\mathrm{Im}(z)^2}$.

Altogether, it follows from $\|AB\|_1 \leq \|A\|\|B\|_1$ that the trace norm of the integrand is bounded by $\frac{C}{\mathrm{Im}(z)^4}$.

We now appeal to a general fact of the Helffer-Sjöstrand functional calculus to provide domination. For any $n \in \mathbb{N}$, the quasi-analytic extension $\tilde{\rho}$ of ρ in the Helffer-Sjöstrand representation can be chosen so that

$$\int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} \frac{1}{|\mathrm{Im}(z)|^{p+1}} dz^2 \leq C_0 \sum_{k=0}^{n+2} \|\rho^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_m = \int_{-\infty}^{\infty} |f(x)|(x^2+1)^{m/2} dx.$$

Since $|\rho(x)| \leq 1$ and ρ' is compactly supported, these norms are all clearly finite. This fact, combined with the bound

$$\|R_a(z)[H_a, \Lambda_1]R_a(z)[H_a, \Lambda_2]R_a(z)\|_1 \leq \frac{C}{\mathrm{Im}(z)^4}$$

for the trace norm of the integrand of $\Sigma(a)$ provides the necessary bound for Lebesgue dominated convergence. Thus, it suffices to show pointwise convergence in z of the integrand to the associated bulk operator.

In other words, we wish to show

$$J_a[R_a(z), \Lambda_1]e^{\delta|x_1|}J_a^* \xrightarrow{s} [R_B(z), \Lambda_1]e^{\delta|x_1|},$$

$$J_a e^{\delta|x_2|}[H_a, \Lambda_2]J_a^* \xrightarrow{s} e^{\delta|x_2|}[H_B, \Lambda_2],$$

and

$$J_a R_a(z) J_a^* \xrightarrow{s} R_B(z)$$

for each fixed $z \in \mathbb{C}$. Inspecting the bounds we found for the left hand sides of these limits, it is clear that they are uniformly bounded in a. It therefore suffices to show convergence on a dense subspace of $\ell^2(\mathbb{Z}^2)$ (see Lemma 12); in particular, we may choose the dense subspace of compactly supported states, which allows us to ignore the $e^{\delta|x_i|}$ terms. Thus, we need to prove

$$J_a[R_a(z), \Lambda_1]J_a^* \xrightarrow{s} [R_B(z), \Lambda_1],$$

$$J_a[H_a, \Lambda_2]J_a^* \xrightarrow{s} [H_B, \Lambda_2],$$

and

$$J_a R_a(z) J_a^* \xrightarrow{s} R_B(z).$$

In fact, the final statement implies the first two; we appeal to the general fact of functional analysis that strong convergence of the resolvent of a self-adjoint operator implies that $J_a f(H_a) J_a^* \xrightarrow{s} f(H_B)$ for any bounded and continuous function f. In particular, it follows from Lemma 3 that the functions $[(\cdot - z)^{-1}, \Lambda_1]$ and $[\cdot, \Lambda_2]$ above are bounded and continuous, so we will have proven the desired limits if we can prove the strong convergence of the resolvent, $J_a R_a(z) J_a^* \xrightarrow{s} R_B(z)$.

To prove this, we use the edge assumption. Recall the edge operator, $E_a = J_a H_a - H_B J_a$. Adding and subtracting zJ_a gives

$$E_a = J_a(H_a - z) - (H_B - z)J_a.$$

Applying R_B from the left and R_a from the right on both sides, we obtain

$$R_B(z) E_a R_a(z) = R_B(z) J_a - J_a R_a(z).$$

Taking the adjoint, and then multiplying from the left by J_a, we see that

$$J_a R_a(z) E_a^* R_B(z) = J_a J_a^* R_B(z) - J_a R_a(z) J_a^*.$$

Thus

$$R_B(z) - J_a R_a(z) J_a^* = (J_a R_a(z) E_a^* - J_a J_a^* + 1) R_B(z) \xrightarrow{s} 0,$$

since $E_a^* \xrightarrow{s} 0$ by Lemma 5, and $-J_a J_a^* + 1 \xrightarrow{s} 0$. This proves that the limits above converge to the desired associated bulk operators, and hence $\|\Sigma(a) - \Sigma_B\|_1 \to 0$.

Finally, it remains to show that

$$\mathrm{Tr}(\Sigma_B) = \sigma_B.$$

First, we manipulate

$$\begin{aligned}
\Sigma_B &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1] R_B(z) [H_B, \Lambda_2] R_B(z) dz^2 \\
&= -\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1][R_B(z), \Lambda_2] dz^2 \\
&= i[\rho(H_B), \Lambda_1]\Lambda_2 - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1]\Lambda_2 R_B(z) dz^2.
\end{aligned}$$

Define $P_+ := P((\sup\Delta, \infty))$ and $P_- := P((-\infty, \inf\Delta))$, the projections onto states above and below the gap, respectively. Since H_B is assumed to have a gap, it follows that $P_- + P_+ = \mathbb{1}$, and thus

$$\mathrm{Tr}(\Sigma_B) = \mathrm{Tr}((P_- + P_+)\Sigma_B(P_- + P_+)).$$

Since $P_\pm R_B(z)$ and $R_B(z)P_\pm$ are analytic on $\mathrm{supp}(\rho(z))$ and $\mathrm{supp}(1-\rho(z))$, the integral in $P_\pm\Sigma_B P_\pm$ vanishes by integration by parts (need to add). Thus

$$\Sigma_B = iP_+[\rho(H_B), \Lambda_1]\Lambda_2 P_+ + iP_-[\rho(H_B), \Lambda_1]\Lambda_2 P_-.$$

(Need to add reasoning for why the operators are trace-class, because really we have $(P_- + P_+)A(P_- + P_+) = A$, and we need to cylce $P_\pm$ to rid the terms $P_- P_+$). By the spectral theorem for projection-valued measures, if the Fermi energy lies in the gap, $\lambda \in \Delta$, we have

$$\rho(H_B) = \int_{-\infty}^{\infty} \rho(\lambda) dP_\nu = \int_{-\infty}^{\lambda} \rho(\lambda) dP_\nu = \int_{-\infty}^{\lambda} dP_\nu = P_\lambda.$$

We may therefore replace $\rho(H_B)$ by P_λ, by which we obtain

$$\mathrm{Tr}(\Sigma_B) = i\mathrm{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+) + i\mathrm{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-).$$

Now, the bulk conductivity is given by

$$\begin{aligned}
\sigma_B &= i\mathrm{Tr}(P_\lambda[[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]]) \\
&= i\mathrm{Tr}(P_\lambda((P_\lambda\Lambda_1 - \Lambda_1 P_\lambda)(P_\lambda\Lambda_2 - \Lambda_2 P_\lambda) - (P_\lambda\Lambda_2 - \Lambda_2 P_\lambda)(P_\lambda\Lambda_1 - \Lambda_1 P_\lambda))) \\
&= i\mathrm{Tr}(P_\lambda(P_\lambda\Lambda_1 P_\lambda\Lambda_2 - P_\lambda\Lambda_1\Lambda_2 P_\lambda - \Lambda_1 P_\lambda\Lambda_2 + \Lambda_1 P_\lambda\Lambda_2 P_\lambda \\
&\qquad\qquad - P_\lambda\Lambda_2 P_\lambda\Lambda_1 + P_\lambda\Lambda_2\Lambda_1 P_\lambda + \Lambda_2 P_\lambda\Lambda_1 - \Lambda_2 P_\lambda\Lambda_1 P_\lambda)) \\
&= i\mathrm{Tr}(-P_\lambda\Lambda_1\Lambda_2 P_\lambda + \Lambda_1 P_\lambda\Lambda_2 P_\lambda + P_\lambda\Lambda_2\Lambda_1 P_\lambda - \Lambda_2 P_\lambda\Lambda_1 P_\lambda) \\
&= i\mathrm{Tr}(-P_\lambda\Lambda_1\Lambda_2 P_\lambda + P_\lambda\Lambda_1 P_\lambda\Lambda_2 P_\lambda + P_\lambda\Lambda_2\Lambda_1 P_\lambda - P_\lambda\Lambda_2 P_\lambda\Lambda_1 P_\lambda) \\
&= i\mathrm{Tr}(P_\lambda\Lambda_1 P_\lambda^\perp\Lambda_2 P_\lambda - P_\lambda\Lambda_2 P_\lambda^\perp\Lambda_1 P_\lambda) \\
&= i\mathrm{Tr}(P_\lambda\Lambda_1 P_\lambda^\perp\Lambda_2 P_\lambda - P_\lambda^\perp\Lambda_1 P_\lambda\Lambda_2) \\
&= i\mathrm{Tr}(P_\lambda\Lambda_1 P_\lambda^\perp\Lambda_2 P_\lambda - P_\lambda^\perp\Lambda_1 P_\lambda\Lambda_2 P_\lambda^\perp).
\end{aligned}$$

We define $T_\lambda := P_\lambda\Lambda_1 P_\lambda^\perp\Lambda_2 P_\lambda - P_\lambda^\perp\Lambda_1 P_\lambda\Lambda_2 P_\lambda^\perp$, so that

$$\sigma_B = i\mathrm{Tr}(T_\lambda),$$

and show that $P_\pm T_\lambda P_\pm = P_\pm[P_\lambda, \Lambda_1]\Lambda_2 P_\pm$.

First, notice that because of the gap, we have $P_\lambda^\perp P_- = 0$, and thus also $P_\lambda P_- = P_-$. Thus

$$\begin{aligned} P_- T_\lambda P_- &= P_- P_\lambda \Lambda_1 P_\lambda^\perp \Lambda_2 P_\lambda P_- \\ &= P_-(P_\lambda \Lambda_1 \Lambda_2 - \Lambda_1 P_\lambda \Lambda_2) P_- \\ &= P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-, \end{aligned}$$

and similarly, for P_+, we have $P_\lambda^\perp P_+ = P_+$, and $P_\lambda P_- = 0$, which implies

$$\begin{aligned} P_+ T_\lambda P_+ &= -P_+ P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_\lambda^\perp P_+ \\ &= -P_+ P_\lambda^\perp \Lambda_1 P_\lambda \Lambda_2 P_+ \\ &= -P_+ P_\lambda^\perp \Lambda_1 \Lambda_2 P_+ + P_+ P_\lambda^\perp \Lambda_1 P_\lambda^\perp \Lambda_2 P_+ \\ &= -P_+ P_\lambda^\perp \Lambda_1 \Lambda_2 P_+ + P_+ \Lambda_1 P_\lambda^\perp \Lambda_2 P_+ \\ &= -P_+[P_\lambda^\perp, \Lambda_1]\Lambda_2 P_+ \\ &= -P_+[(1 - P_\lambda), \Lambda_1]\Lambda_2 P_+ \\ &= P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+. \end{aligned}$$

Finally, we obtain

$$\begin{aligned} \sigma_B &= i\mathrm{Tr}(T_\lambda) \\ &= i\mathrm{Tr}(P_- T_\lambda P_-) + i\mathrm{Tr}(P_+ T_\lambda P_+) \\ &= i\mathrm{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-) + i\mathrm{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+) \\ &= \mathrm{Tr}(\Sigma_B), \end{aligned}$$

concluding the proof.

Lemma 5. *E_a and E_a^* converge strongly to zero in the limit $a \to \infty$.*

Proof. Let $\psi \in \ell^2(\mathbb{Z}^2)$. Since E_a has real entries,

$$\begin{aligned} \|E_a^* \psi\|^2 &= \langle E_a^* \psi, E_a^* \psi \rangle \\ &= \sum_z \left(\sum_y \overline{E_a^*(z, y)\psi(y)} \right) \left(\sum_x E_a^*(z, x)\psi(x) \right) \\ &= \sum_z \left(\sum_y E_a(y, z)\overline{\psi(y)} \right) \left(\sum_x E_a(x, z)\psi(x) \right) \end{aligned}$$

Consider the $\sum_x E_a(x, z)\psi(x)$ term. To take advantage of the edge assumption, we insert the exponentials

$$\sum_x E_a(x,z) e^{\alpha(|z_2-a|-|z_1-x_1|)} e^{-\alpha(|z_2-a|-|z_1-x_1|)} \psi(x).$$

By assumption 2, $g(z) := \sup_x E_a(x,z) e^{\alpha(|z_2-a|-|z_1-x_1|)}$ is finite and summable. Precisely the same argument can be used on the sum over y. Altogether,

$$\|E_a^* \psi\|^2 \leq e^{-\alpha|a|} \sum_z g(z)^2 e^{-2\alpha(|z_1|+|z_2|)} \sum_x e^{-\alpha|x_1|} \psi(x) \sum_y e^{-\alpha|y_1|} \overline{\psi(y)},$$

where we bounded the exponential by $e^{-\alpha(|z_1|+|z_2|)} e^{-\alpha|x_1|} e^{-\alpha|a|}$. Since g is summable, g^2 is also summable, and so too is the summand over z. The sums over x and y are clearly finite, as they are bounded by the summable state $|\psi|$. Thus E_a^* converges strongly to zero. An exactly analogous argument applies for E_a.

□

3 Interacting Setting

Let $L \in \mathbb{N}$, and let $\Gamma_L = \mathbb{Z}_L \times [0, L]$ be the discrete cylinder, equipped with a metric d. To each site $x \in \Gamma_L$, we associate a Hilbert space $\mathcal{H}_x$ whose dimension is bounded uniformly in L. We denote $N = \sup_L \mathcal{H}_L$. For a subset $X \subseteq \Gamma_L$, we define the Hilbert space $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$, and we set

$$\mathcal{H}_L := \mathcal{H}_{\Gamma_L} = \bigotimes_{x \in \Gamma_L} \mathcal{H}_x.$$

The algebra $\mathcal{U}_X \subset \mathcal{B}(\mathcal{H}_X)$ of observables on $\mathcal{H}_X$ is the set of bounded self-adjoint operators supported in X. For an operator $A_X \in \mathcal{U}_X$, we identify its extension to an operator on $\mathcal{H}_L$ by taking its tensor product with copies of the identity, $(\otimes_{x \in X^c} \mathbb{I}_x) \otimes A_X$. Conversely, we say that an operator $A \in \mathcal{U}_L$ has support X if $A_X := (\otimes_{x \in X^c} \mathbb{I}_x) \otimes (A|_X)$ is equal to A, and write $A_X \in \mathcal{U}_X$. For ease of notation, we omit the subscript L wherever there is no risk of confusion.

A *local interaction* is a map $\Phi : \mathcal{P}(\Gamma_L) \to \mathcal{U}_L$ such that

1. $\Phi(X) = 0$ whenever $\text{diam}(X) > R$ for some $R > 0$.

2. $\Phi(X)$ is supported in X.

3. $\|\Phi(X)\| \leq C$ for all $X \subset \Gamma_L$, for all L.

We consider a region as depicted in Figure 2, with the left and right edges joined together to form a cylinder. In the left white region $[0, L/2] \times [0, L]$, H_0 is a trivial Hamiltonian which we take to be empty space (we take $H_0 = 0$), and in the right blue region $[L/2, L] \times [0, L]$, H_1 is a *local Hamiltonian*, in the sense that $H_1 = \sum_{X \subseteq \Gamma_L} \Phi(X)$, is a sum of local interactions. We define the Hamiltonian of the full system to be

$$H_\mu = H_1 + \mu Q_h,$$

where $Q_h = \sum_{x \in \Gamma_h} a_x^* a_x$ is the number operator for the region $\Gamma_h = [L/4, 3L/4] \times [0, L]$ shown in red. This introduces a driving strength; the μQ_h term can be viewed as a potential difference $V(x)$.

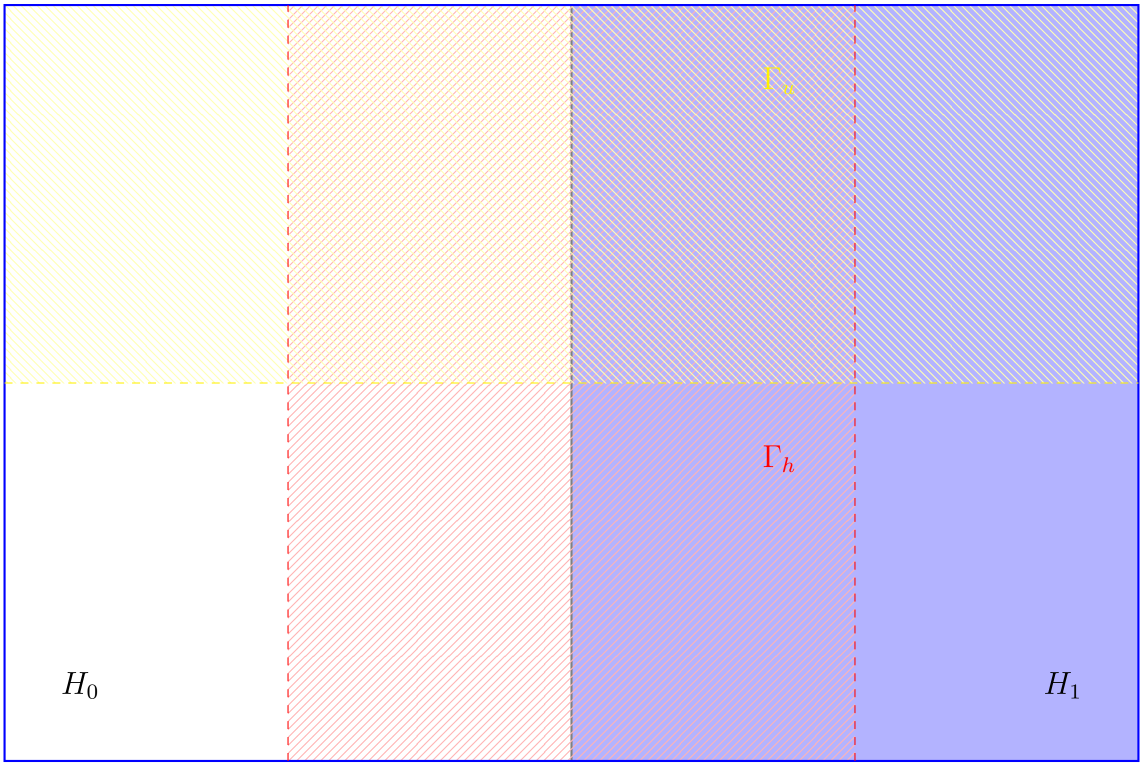

Figure 2: The cylinder Γ_L.

We also consider the plane $\mathbb{Z}^2$. In this setting, there are no edge states, and so the associated "bulk" Hamiltonian H_B is assumed to have a *gapped* spectrum, in the sense that

Assumption 4.

$$\mathrm{Spec}(H_B) = \mathcal{S}_- \cup \mathcal{S}_+,$$

where $\inf \mathcal{S}_+ - \sup \mathcal{S}_- \geq \gamma$ *uniformly in* L *and* μ *for some* $\gamma > 0$.

In the case of the cylinder, this effect does not necessarily occur due to the presence of the edge. We also assume that the Hamiltonian is *locally charge-conserving.*

Assumption 5. $[\Phi(X), Q] = 0$*, where* Q *is the total charge in* Γ_L*.*

Let P_B be the ground state projection of H_B (the system without an edge), and let P be the ground state projection of H (the system with an edge). We assume that states far from the edge are essentially bulk states, up to tails that vanish quickly in L.

Assumption 6. *Define the* edge region

$$\Gamma_E = [L/2 - k, L/2 + k] \times [0, L] \cup [L - k, k] \times [0, L].$$

for some $k > 0$. For any operator A supported on Γ_E^{c},

$$\mathrm{Tr}(PA) = \mathrm{Tr}(P_B A) + \mathcal{O}(L^{-\infty}).$$

The A on the right hand side is understood to be the extension by zeroes of A to the plane $\mathbb{Z}^2$.

The idea is that observables localized far away from the edge are not affected by the edge of the system. We similarly define the *bulk region*

$$\Gamma_B = [3L/4 - k, 3L/4 + k] \times [0, L],$$

and the *middle region*

$$\Gamma_m = [L/2, L] \cup [0, L] \setminus (\Gamma_E \cup \Gamma_B).$$

The three regions are depicted in figure 3.

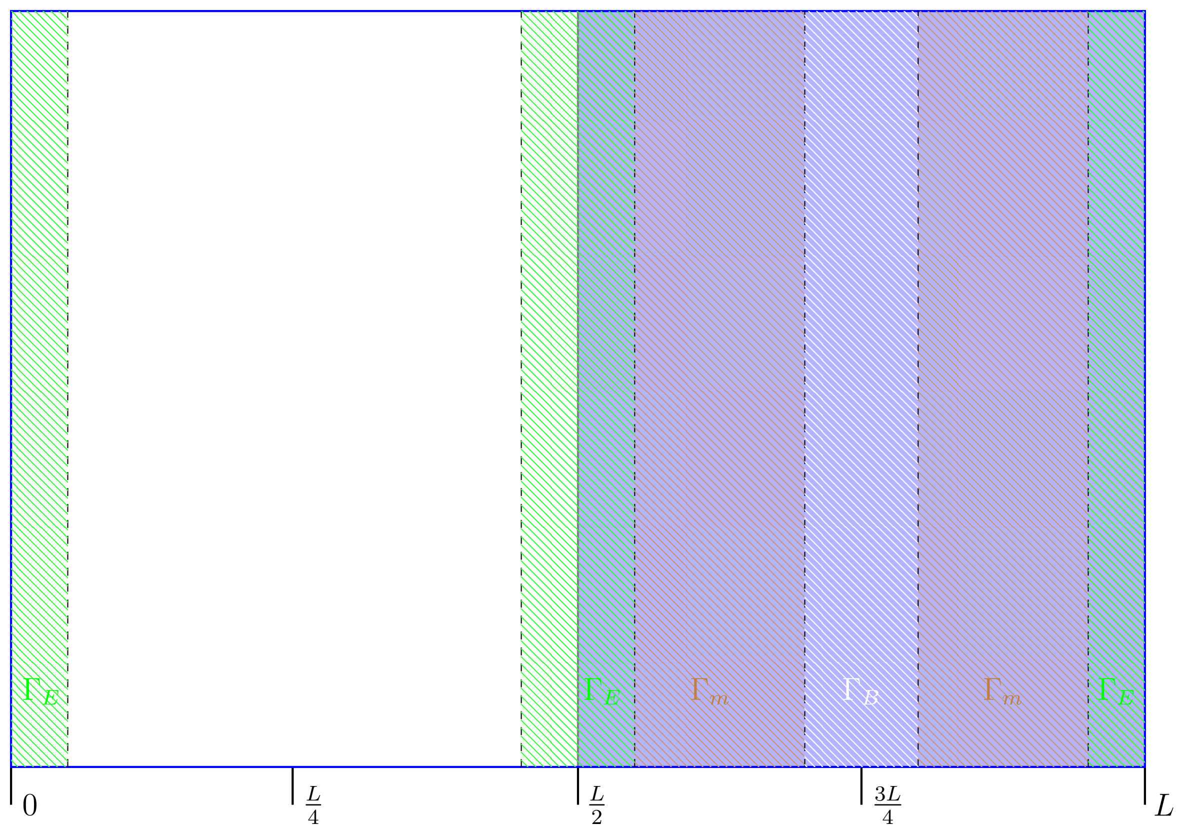

Figure 3: The regions Γ_E, Γ_B, and Γ_m.

3.1 Equality of Bulk and Edge Currents

3.1.1 Cylinder Geometry

Let P_μ be the (possibly degenerate) ground state projection of H_μ. Let $Q_u = \sum_{x\in\Gamma_u} a_x^* a_x$ be the charge in the upper half of the cylinder $\Gamma_u = [0, L]\times[L/2, L]$ (the yellow region in Figure 2), and define current operator

$$J = i[H_\mu, Q_u],$$

which measures the current across the fiducial line $y = L/2$. Charge conservation 5 implies that this current operator is supported along a strip of width $2R$ centred on the fiducial line $y = L/2$. Indeed, if we inspect a local interaction $\Phi(X)$ of range R with support $(\Gamma_u)_R$, where $(X)_\alpha$ is the α-shrinking of the set X, then clearly $\Phi(X)$ commutes with the charge outside Γ_u, so that $[\Phi(X), Q_u] = [\Phi(X), Q]$, which vanishes by the charge conservation assumption 5. Similarly, if $\Phi(X)$ is supported in $((\Gamma_u)^{\mathsf{c}})_R$, then $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$. It follows that for an interaction $\Phi(X)$ with range R and arbitrary support, $[\Phi(X), Q_u]$ must be supported on a set which is contained in (or equal to) the strip $[L/2, L] \times [L/2 - R, L/2 + R]$. There $[H_\mu, Q_u]$ must be supported there as well, since H_μ is a sum of such local interactions.

From this point, we drop the subscript μ wherever it is not needed for context.

Lemma 6. *The ground state expectation of the current J is zero.*

Proof. Assuming linearity and cyclicity of the trace hold, the proof is trivial,

$$\mathrm{Tr}(PJ) = i\mathrm{Tr}(P[H, Q_u]) = i\mathrm{Tr}([P, H]Q_u) = 0.$$

In order for this calculation to hold, we need to prove that

1. PHQ_u and PQ_uH are separately trace-class to apply linearity of the trace, and
2. $\|H\| < \infty$ and $PQ_u \in \mathcal{J}_1$ to apply cyclicity of the trace.

The latter implies the former by the bound $\|AB\|_1 \leq \|A\|_1\|B\|$. To prove (2), fix a finite L. The Hamiltonian is bounded since it is a finite sum of at most $\mathcal{P}(\Gamma_L)$ local interactions $\Phi(X)$, each of which is uniformly bounded by assumption, along with the μQ_h term. But the number operator for the entire space is bounded by $\|Q\| \leq NL^2$, where N is the uniform bound on the dimension of each Hilbert space. This shows that both Q_u and Q_h are bounded in operator norm. Finally, $\|P\|_1 \leq CL^2$ because the projection is finite-rank, since the dimension of each site is bounded. Therefore $PQ_u \in \mathcal{J}_1$. $\square$

Next, we define a family of operators indexed by μ called *Hastings operators*,

$$K_\mu = \mathcal{I}_\mu(\dot{H}_\mu),$$

where

$$\mathcal{I}_\mu(A) = \int_{\mathbb{R}} W(t) e^{itH_\mu} A e^{-itH_\mu} dt.$$

Here, $W : \mathbb{R} \to \mathbb{R}$ is a function satisfying (need to add). More explicitly, in our setting we see that

$$K_\mu = \mathcal{I}_\mu(Q_h).$$

We present two important properties of the map $\mathcal{I}_\mu : \mathcal{U}_L \to \mathcal{U}_L$ in the following lemmas, and leave their proofs to the appendix (need to add).

First, recall a definition from the non-interacting setting: an *off-diagonal* operator is an operator A such that $A = \overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, where $P_\mu^\perp = \mathbb{I} - P_\mu$ is the projection onto the excited states above the gap.

Lemma 7. *1. For any off-diagonal operator $A = \overline{A}$, $\mathcal{I}_\mu(\cdot)$ and $[H_\mu, \cdot]$ act as inverses of each other, up to a factor of i:*

$$\mathcal{I}_\mu\left([H_\mu, A]\right) = [H_\mu, \mathcal{I}_\mu(A)] = iA.$$

2. For any (not necessarily off-diagonal) operator A,

$$[\mathcal{I}_\mu([H_\mu, A]), P_\mu] = i[A, P_\mu].$$

Another important property of the map $\mathcal{I}_\mu$ is that it preserves locality.

Lemma 8. *$\mathcal{I}_\mu$ is local in the sense that for any $A \in \mathcal{U}_X$,*

$$\|\mathcal{I}(A)_{(X^r)^c}\| \le \|A\| |X| \mathcal{O}(r^{-\infty})$$

where $X^r = X \cup \{x : d(x, X) \le r\}$ is the r-fattening of X.

Proposition 2. *The operator K_μ is the* generator of parallel transport*, satisfying*

$$\dot{P}_\mu = i[K_\mu, P_\mu]$$

for all μ.

Proof. First, we show that $\dot{P}$ is off-diagonal. Taking the derivative on both sides of $P^2 = P$, we see that $\dot{P}P + P\dot{P} = \dot{P}$. Acting on the left and right with P on both sides of this equation gives

$$P\dot{P}P + P\dot{P}P = P\dot{P}P,$$

which implies that $P\dot{P}P = 0$. Thus

$$\begin{aligned}\overline{\partial_\mu P} &= P\dot{P}(1-P) + (1-P)\dot{P}P \\ &= P\dot{P} - P\dot{P}P + \dot{P}P - P\dot{P}P \\ &= P\dot{P} + \dot{P}P \\ &= \partial_\mu(P^2) \\ &= \partial_\mu P,\end{aligned}$$

as claimed. By the product rule and the fact that H and P commute,

$$[\dot{H}, P] = -[H, \dot{P}].$$

It therefore follows from Lemma 7 that

$$\dot{P} = -i\mathcal{I}_\mu([H, \dot{P}]) = i\mathcal{I}([\dot{H}, P]) = i[\mathcal{I}(\dot{H}), P] = i[K, P].$$

□

Increasing the electric potential by a small amount $d\mu Q_h$ and expanding to linear order, the change in ground state current is given by

$$\mathrm{Tr}(P_{\mu+d\mu}J) - \mathrm{Tr}(P_\mu J) = \kappa d\mu + \mathcal{O}(d\mu^2).$$

Dividing by $d\mu$ and taking a limit, we see that the linear response coefficient is given by

$$\sigma(\mu) = \mathrm{Tr}\left(\dot{P}_\mu J\right).$$

The *Hall conductivity* of the system on a subset $V \subseteq \Gamma_L$ is defined to be $\sigma_V := \mathrm{Tr}\left(\dot{P}J_V\right)$, where J_V is the restriction of J to V.

Proposition 3. *The Hall conductivity is independent of the driving strength μ.*

Proof. For any μ_1 and μ_2,

$$\begin{aligned}\sigma(\mu_1) - \sigma(\mu_2) &= \mathrm{Tr}\left(\dot{P}_{\mu_1} i[H_{\mu_1}, Q_u] - \dot{P}_{\mu_2} i[H_{\mu_2}, Q_u]\right) \\ &= i\mathrm{Tr}\left(\left([\dot{P}_{\mu_1}, H_{\mu_1}] - [\dot{P}_{\mu_2}, H_{\mu_2}]\right) Q_u\right) \\ &= -i\mathrm{Tr}\left(\left([\dot{H}_{\mu_1}, P_{\mu_1}] - [\dot{H}_{\mu_2}, P_{\mu_2}]\right) Q_u\right) \\ &= i\mathrm{Tr}\left([Q_h, P_{\mu_1} - P_{\mu_2}]Q_u\right) \\ &= i\mathrm{Tr}\left([Q_u, Q_h](P_{\mu_1} - P_{\mu_2})\right) \\ &= 0,\end{aligned}$$

since H and P commute. Note that $\|\dot{P}\|_1 < \infty$ since we are working in a finite-dimensional space. The proof of Lemma 6 provides the other necessary bounds to invoke linearity and cyclicity of the trace to shift the commutator in the second line and second-last line. □

This indicates that the Hall conductivity is independent of μ as one would expect physically. We simply write $\sigma = \sigma(\mu)$ from this point, in accordance with proposition 3.

The following is the main result:

Theorem 2. *Let $V \subseteq \Gamma_m$ be a set contained within the strip in between the edge region Γ_E and the bulk region Γ_B (see Figure 3), and define the distance*

$$r = \operatorname{dist}(V, \Gamma_E \cup \Gamma_B)$$

from V to the bulk and edge regions. The Hall conductivity in this regions vanishes in the sense that

$$\sigma_V = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

Proof. By Proposition 2, the bulk Hall conductivity can also be written by the formula

$$\sigma_V^B = \operatorname{Tr}\left(i[K, P_B] J_V^B\right) = \operatorname{Tr}\left(i[\mathcal{I}(Q_h), P_B] J_V^B\right),$$

where $J_V^B = i[H_B, Q_u]|_V$ is the current in the region V arising from the bulk Hamiltonian. From commutativity of P_B and H_B along with cyclicity of the trace, we compute

$$\begin{aligned}
\sigma_V^B &= \operatorname{Tr}\left(i \int_{\mathbb{R}} W(t) e^{itH_B} [Q_h, P_B] e^{-itH_B} dt J_V^B\right) \\
&= \int_{\mathbb{R}} W(t) \operatorname{Tr}\left(i[Q_h, P_B] e^{-itH_B} J_V^B e^{itH_B}\right) dt \\
&= -\int_{\mathbb{R}} W(t) \operatorname{Tr}\left(i[Q_h, P_B] e^{itH_B} J_V^B e^{-itH_B}\right) dt \\
&= -\operatorname{Tr}\left(i[Q_h, P_B] \mathcal{I}(J_V^B)\right),
\end{aligned}$$

since $W(t)$ is odd. By part (2) of Lemma 7, we have $i[Q_h, P_B] = [\mathcal{I}([H_B, Q_h]), P_B]$. Therefore

$$\begin{aligned}
\sigma_V^B &= -\operatorname{Tr}([\mathcal{I}([H_B, Q_h]), P_B] \mathcal{I}(J_V^B)) \\
&= \operatorname{Tr}\left(P_B [\mathcal{I}([H_B, Q_h]), \mathcal{I}(J_V^B)]\right).
\end{aligned}$$

Now, $[H_B, Q_h]$ is a local operator supported on Γ_B, while J_V^B is a local operator supported on $V \cap \Gamma_B = \varnothing$. Since $\mathcal{I}$ preserves locality up to tails, in the sense that $\|\mathcal{I}(A)_{(S^r)^\mathsf{c}}\| \leq \|A\||S|\mathcal{O}(r^{-\infty})$ for any operator A supported in S (Lemma 8), it follows that the commutator can be written

$$[\mathcal{I}([H_B, Q_h])|_{\Gamma_B} + \mathcal{O}(r^{-\infty})A_1, \mathcal{I}(J_V^B)|_V + \mathcal{O}(r^{-\infty})A_2] = C\mathcal{O}(r^{-\infty}),$$

for some operators A_1 and A_2 supported on Γ_B^c and V^c, respectively. This fact applies to the bulk setting with H_B and P_B. To extend this to the setting with an edge, it is enough to use Assumption 6 to conclude the same result, except with equality up to $\mathcal{O}(L^{-\infty})$, i.e.

$$\sigma_V = \operatorname{Tr}\left(\dot{P}J_V\right) = \operatorname{Tr}\left(\dot{P}(J_V^B + \mathcal{O}(L^{-\infty}))\right) = \sigma_V^B + \mathcal{O}(L^{-\infty}) = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

□

The intuitive picture from the previous result is that, in the bulk region, the Hall conductivity is essentially only nonzero along the bulk strip Γ_B. Since the ground state expectation of the current is zero (by lemma 6), it must be that there is an equal current flowing along the edge strip Γ_E, but in the opposite direction.

3.1.2 Torus Geometry

Our goal is to show the same result on the discrete torus $\mathbb{T}_L := \mathbb{Z}_L \times \mathbb{Z}_L$. We define the same regions Γ_u and Γ_h, and the same current operator $J_u = i[H(\mu), Q_u]$. This time, however, Lemma 6 does not apply. Intuitively, it does not apply because electrons can now flow through both the bottom and the top of the region Γ_u, rather than just the bottom. Mathematically, the lemma fails because our definition of the current is slightly changed.

We use charge conservation and the fact that H is finite range to split the current J_u into two components, $J_u = i[H_-, Q_u] + i[H_+, Q_u] = J_- - J_+$, supported on strips of width $2R$ at $y = L/2$ and $y = L$, respectively. We then define the current operator to be $J = J_-$, which is the current on the lower strip. This is the mathematical reason that the proof in Lemma 6 fails on the torus; we have replaced H by H_-, which may no longer commute with P. We instead proceed by a different approach. We will need a few auxiliary results first.

Lemma 9. *$K_\pm$ is supported on $\partial_\pm$ up to tails.*

Proof. □

Proposition 4. *The operator $Q_h - K$ leaves the ground state space invariant, i.e. $[Q_h - K, P] = 0$.*

Proof. □

Lemma 10. *Show that* $\mathrm{Tr}(A, [Q_h, P]) = 0$ *for all* $A \in \mathcal{U}_{edge}$. *This shows that Q_h commutes with P "along the edge".*

Proof. Let $A \in \mathcal{U}_{\text{edge}}$. Since H is charge conserving, we may choose a simultaneous eigenbasis of H and the total charge Q, in which case P and Q commute. It follows that

$$\mathrm{Tr}(A[Q_h, P]) = \mathrm{Tr}([A, Q_h]P) = \mathrm{Tr}([A, Q]P) = \mathrm{Tr}(A[Q, P]) = 0.$$

□

Finally, we will prove that in the bulk system with Hamiltonian $H_B(\mu)$, the ground state expectation of the current vanishes faster than any power as $L \to \infty$.

Lemma 11. *The ground state expectation of the current $J_B := i[(H_B)_-, Q_h]$ (of the system without an edge) is* $\mathrm{Tr}(P_B J_B) = \mathcal{O}(L^{-\infty})$.

Proof. First, $K = \mathcal{I}(i[H_B, Q])$ splits into $K = K_- - K_+$, with the support of $K_\pm$ contained in $\partial_\pm$ up to tails:

$$[K_\pm, A_X] = \mathcal{O}(p^{-\infty}),$$

for every $A_X \in \mathcal{U}_X$ such that $\|A_X\| = 1$, and where $p = \mathrm{dist}(X, \partial_\pm)$ (need to add). Using the fact that $K_\pm$ is supported in $\partial_\pm$ up to tails (Lemma 9), we see that

$$i[H_B, K_-] = i[(H_B)_-, K_-] + \mathcal{O}(L^{-\infty}),$$

and similarly $i[(H_B)_-, K_+] = \mathcal{O}(L^{-\infty})$. Putting these facts together, it follows that the current can be rewritten as

$$\begin{aligned} J_B &= i[H_B, Q_h + K_- - K_- + K_+] + \mathcal{O}(L^{-\infty}) \\ &= i[H_B, K_-] + i[(H_B)_-, Q_h - K_- + K_+)] + \mathcal{O}(L^{-\infty}). \end{aligned}$$

From here, we use the fact that H_B and $Q_h - K_- + K_+$ both commute with P_B to write

$$P_B J_B P_B = i[H_B, P_B K_- P_B] + i[P_B (H_B)_- P_B, Q_h - K_- + K_+)] + P_B \mathcal{O}(L^{-\infty}) P_B.$$

Since the trace of any commutator is zero,

$$\mathrm{Tr}(P_B J_B) = \mathrm{Tr}(P_B J_B P_B) = \mathcal{O}(L^{-\infty}).$$

□

Using this, we can show a simple proof of the analogue of Lemma 6 on the torus, in the case of non-interacting systems.

Proposition 5. *Let $H = \sum_{x \in \mathbb{T}} h_x$ be a non-interacting Hamiltonian, i.e. a sum of single site Hamiltonians h_x. The ground state expectation of the current $J = i[H_-, Q_h]$ (of the system with an edge) is $\mathrm{Tr}(PJ) = \mathcal{O}(L^{-\infty})$.*

Proof. Since H is a sum of single site Hamiltonians, we can split H_- into the restrictions $H_- = (H_-)_{\text{edge}} + (H_-)_{\text{bulk}}$, with no fear of any terms which are in both the edge region and the bulk region. By Assumption 6,

$$\begin{aligned}
\mathrm{Tr}(PJ) &= \mathrm{Tr}(Pi[H_-, Q_h]) \\
&= i\mathrm{Tr}([H_-, Q_h]P) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\text{bulk}}[Q_h, P]) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\text{bulk}}[Q_h, (P)_{\text{bulk}}]) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_B)_-[Q_h, P_B]) + \mathcal{O}(L^{-\infty}) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + \mathrm{Tr}(i[(H_B)_-, Q_h]P_B) + \mathcal{O}(L^{-\infty}).
\end{aligned}$$

By Lemma 10, the first term is zero. By Lemma 11, the second term is $\mathcal{O}(L^{-\infty})$. □

A General Functional Analysis

Lemma 12. *Let A be a bounded linear operator on a Hilbert space $\mathcal{H}$. Suppose $A_n \xrightarrow{s} A$ on a dense subspace $\mathcal{D} \subset \mathcal{H}$. If A_n are bounded uniformly in n, then $A_n \xrightarrow{s} A$ on all of $\mathcal{H}$.*

Proof. Let $\psi_n \in \mathcal{D}$ be a sequence converging in norm to $\psi \in \mathcal{H}$. The result follows from a standard $\frac{\varepsilon}{3}$ argument. Let C be a bound for both $\sup_n \|A_n\|$ and $\|A\|$. Then

$$\begin{aligned}\|A_n\psi - A\psi\| &\leq \|A_n(\psi - \psi_m)\| + \|(A_n - A)\psi_m\| + \|A(\psi - \psi_m)\| \\ &\leq C\|(\psi - \psi_m)\| + \|(A_n - A)\psi_m\| + C\|\psi - \psi_m\|.\end{aligned}$$

There exists an M such that the first and third terms are less than $\frac{\epsilon}{3}$ for all $m > M$. For the middle term, observe that for each m, there exists by hypothesis an N_m such that $\|(A_n - A)\psi_m\| < \frac{\epsilon}{3}$ for all $n > N_m$. Thus, by picking some $m > M$ and fixing a suitably large n, the inequality above is less than ε. □

B Properties of $\mathcal{I}_\mu$

Proof. (Of Lemma 7). Let $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} W(t) e^{-2\pi i t \xi} dt$ be the Fourier transform of W. One can show that for $|\xi| \geq \gamma$, $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi} i \xi}$ (need to add). Let A be an observable. First, we show that $\mathcal{I}([H, PAP^{\perp}]) = iPAP^{\perp}$.

Decomposing

$$\begin{aligned} e^{itH}P &= \sum_{j=0}^{\infty} \frac{(itH)^j}{j!} P \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left(\sum_n E_n^j P_n \right) P \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \sum_{n : E_n = 0} E_n^j P_n \\ &= \sum_{n : E_n = 0} e^{itE_n} P_n, \end{aligned}$$

and similarly

$$P^{\perp}e^{-itH} = \sum_{m:E_m \geq \gamma} P_m e^{-itE_m},$$

we see that

$$\begin{aligned}
\mathcal{I}([H, PAP^{\perp}]) &= \mathcal{I}(P[H,A]P^{\perp}) \\
&= \int_{\mathbb{R}} W(t)e^{itH}P[H,A]P^{\perp}e^{-itH}dt \\
&= \int_{\mathbb{R}} W(t) \sum_{n:E_n=0} e^{itE_n}P_n[H,A] \sum_{m:E_m\geq\gamma} P_m e^{-itE_m}dt \\
&= \sum_{n:E_n=0}\sum_{m:E_m\geq\gamma} \int_{\mathbb{R}} W(t)e^{itE_n}P_nA(E_n-E_m)P_m e^{-itE_m}dt \\
&= \sum_{n:E_n=0}\sum_{m:E_m\geq\gamma} P_nAP_m(E_n-E_m)\int_{\mathbb{R}} W(t)e^{-it(E_m-E_n)}dt \\
&= \sum_{n:E_n=0}\sum_{m:E_m\geq\gamma} P_nAP_m(E_n-E_m)\sqrt{2\pi}\widehat{W}(E_m-E_n) \\
&= i\sum_{n:E_n=0}\sum_{m:E_m\geq\gamma} P_nAP_m \\
&= iPAP^{\perp}.
\end{aligned}$$

(need to check the 2π factor)

By the same argument, $\mathcal{I}([H, P^{\perp}AP]) = iP^{\perp}AP$ as well, and so $\mathcal{I}([H,\overline{A}]) = i\overline{A}$. □

Proof. (Of Lemma 8). Note that any operator can be written as the telescoping sum

$$\mathcal{I}(A) = \mathrm{tr}_X(\mathcal{I}(A)) + \sum_{j=1}^{\infty}\left(\mathrm{tr}_{X^{(j)}}(\mathcal{I}(A)) - \mathrm{tr}_{X^{(j-1)}}(\mathcal{I}(A))\right),$$

where tr_Y denoted the partial trace over Y, and Y^{α} denotes the α-fattening of Y.

Our goal is to prove that $\|\mathrm{tr}_{X^{(j)}}(\mathcal{I}(A)) - \mathcal{I}(A)\| \leq \|A\||X|\mathcal{O}(j^{-\infty})$. To that end, we break the integral $\mathcal{I}(A)$ into two terms,

$$\mathcal{I}(A) = \int_{-T}^{T} W(t)e^{itH}Ae^{-itH}dt + \int_{\mathbb{R}\setminus[-T,T]} W(t)e^{itH}Ae^{-itH}dt.$$

We break the integral into two parts,

$$\|\mathcal{I}(A)\| \leq \left\| \int_{-T}^{T} W(t) e^{itH} A e^{-itH} dt \right\| + \left\| \int_{\mathbb{R}\setminus[-T,T]} W(t) e^{itH} A e^{-itH} dt \right\|.$$

The first term can be estimated using the Lieb-Robinson bound found in Appendix C.

□

C Lieb-Robinson Bound

Let N be a uniform upper bound for the dimensions of the Hilbert spaces at each site, i.e. $\dim(\mathcal{H}_x) \leq N$ for all sites x.

The following is a version of the Lieb-Robinson. For any operators $A \in \mathcal{U}_X$ and $B \in \mathcal{U}_Y$ having disjoint supports $X \cap Y = \varnothing$,

$$\|[e^{itH} A e^{-itH}, B]\| \leq C\|A\|\|B\||X||Y|N^{2|X|} e^{2t\|\Phi\|_\lambda - \lambda d(X,Y)}.$$

D Grönwall's Inequality and Uniqueness

Theorem 3. *(Grönwall's Inequality). Let $\alpha : I \to (0, \infty)$ be positive and continuous on I^o for some interval of the form $[a, b)$, $[a, b]$, or $[a, \infty)$. Suppose $u : \mathbb{R} \to \mathcal{U}$ is a Banach-valued, differentiable function. If $\|u'(t)\| \leq \alpha(t)\|u(t)\|$ for all $t \in I$, then*

$$\|u(t)\| \leq \|u(a)\| e^{\int_a^t \alpha(s) ds} \quad \forall t \in I$$

Proof. Let $f(t) = e^{\int_a^t \alpha(s) ds}$, which is nonzero, has initial value $f(a) = 1$, and has derivative $f'(t) = \alpha(t) f(t)$. Then by the quotient rule,

$$\left(\frac{\|u(t)\|}{f(t)} \right)' = \frac{\|u'(t)\| f(t) - \|u(t)\|\alpha(t) f(t)}{f(t)^2} \leq 0,$$

where the inequality follows from the assumption $\|u'(t)\| \leq \|\alpha(t) u(t)\|$. Thus $\frac{\|u(t)\|}{f(t)}$ is decreasing, so that

$$\frac{\|u(t)\|}{f(t)} \leq \frac{\|u(a)\|}{f(a)} = \|u(a)\|,$$

which is the desired inequality. □

Theorem 4. *(ODE Uniqueness). Let $F : \mathcal{U} \to \mathcal{U}$ be Lipschitz and consider the differential equaton $u'(t) = F(u(t))$ with initial condition $u(a) = u_a$ for some function $u : I \to \mathcal{U}$, where $I = [a, b]$, or $[a, b)$, or $[a, \infty)$. Solutions to this equation are unique.*

Proof. Suppose there are two solutions $u(t)$ and $v(t)$, and let $g(t) = \|u(t) - v(t)\|^2$. By assumption, there exists a constant L_F such that $\|F(u(t)) - F(v(t))\| \leq L_F \|u(t) - v(t)\|$, so that

$$\begin{aligned} g'(t) &= 2\|u(t) - v(t)\|\|u'(t) - v'(t)\| \\ &= 2\|u(t) - v(t)\|\|F(u(t)) - F(v(t))\| \\ &\leq 2L_F \|u(t) - v(t)\|^2 \\ &= 2L_F g(t). \end{aligned}$$

Notice that $\alpha := 2L_F$ is a positive continuous function, so we may apply Grönwall's inequality to $g(t)$ to conclude

$$g(t) \leq g(a) e^{2L_f (t-a)} = 0,$$

since $g(a) = 0$. □

E Note on Generators of Parallel Transport

Consider the differential equation $\dot{\rho}(\mu) = i[K_B, \rho(\mu)]$ with initial condition $\rho(0) = P_B(0)$. Here $K_B = \int_{\mathbb{R}} W_\gamma(t) e^{-itH_B} \dot{H}_B e^{itH_B} dt$, and recall that in our setting, $\dot{H}_B = Q_h$. We know that the solution is $\rho(\mu) = P_B(\mu)$ (proposition 2). Notice that the map $F : \mathcal{U} \to \mathcal{U}$ defined by $F(A) = i[K_B, A]$ is Lipschitz, since

$$\|F(A) - F(B)\| = \|[K_B, A - B]\| \leq 2\|K_B\|\|A - B\|.$$

The Lipschitz constant is $2\|K_B\|$, which is finite since K_B is a bounded operator:

$$\|K_B\| \leq \int_{\mathbb{R}} |W_\gamma(t)| \|e^{-itH_B} Q_h e^{itH_B}\| dt \leq \int_{\mathbb{R}} |W_\gamma(t)| dt \|Q_h\|.$$

Indeed, since Q_h is the number operator on a finite volume, by charge conservation and the fact that the dimension of the Hilbert space is bounded uniformly by d, there can only be a finite number of charges in the region Γ_h.

Thus, by Grönwall's uniqueness theorem (appendix D), we see that the solution to the equation $\dot{\rho}(\mu) = F(\rho(\mu)) = i[K_B, \rho(\mu)]$ is unique.

Now define

$$K_E := \int_{\mathbb{R}} W_\gamma(t) e^{-itH_E} Q_h e^{itH_E} dt,$$

which is using the gap γ of H_B to define W_γ, but also using the edge Hamiltonian in the time evolution operators. Consider $\sigma : [0, \infty) \to \mathcal{U}$ defined by

$$\dot{\sigma}(\mu) = i[K_E, \sigma(\mu)] \qquad\qquad \sigma(0) = P_E(0).$$

We now show that, similar to how ρ is an approximation of P_B, σ is also a good approximation of P_E (up to $\mathcal{O}(L^{-\infty})$) "in the bulk". Let $A \in \mathcal{U}_{\Gamma_B}$ be an operator localized in the bulk of the edge system. Then

$$\begin{aligned}
\mathrm{Tr}(\dot{\sigma} A) &= \mathrm{Tr}(i[K_E, \sigma]A) \\
&= \mathrm{Tr}(i[A, K_E]\sigma) \\
&= \int_{\mathbb{R}} W_\gamma(t) \mathrm{Tr}([e^{-itH_E} Q_h e^{itH_E}, A]\sigma) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \mathrm{Tr}(e^{-itH_E}[Q_h, e^{itH_E} A e^{-itH_E}] e^{itH_E} \sigma) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \mathrm{Tr}(e^{-itH_E}[Q_h, e^{itH_B} A e^{-itH_B}] e^{itH_E} \sigma + \mathcal{O}(L^{-\infty})) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \mathrm{Tr}(e^{-itH_B}[Q_h, e^{itH_B} A e^{-itH_B}] e^{itH_B} \sigma + \mathcal{O}(L^{-\infty})) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \mathrm{Tr}([e^{-itH_B} Q_h e^{itH_B}, A]\sigma) dt + \mathcal{O}(L^{-\infty}) \\
&= \mathrm{Tr}(i[A, K_B]\sigma] + \mathcal{O}(L^{-\infty}) \\
&= \mathrm{Tr}(i[K_B, \sigma]A) + \mathcal{O}(L^{-\infty}),
\end{aligned}$$

since σ is trace-class (?) and $W_\gamma \in L^1$. By linearity of the trace, we see that $\mathrm{Tr}((\dot{\sigma} - i[K_B, \sigma])A) = \mathcal{O}(L^{-\infty})$ for any operator $A \in \Gamma_B$ (does this mean $\dot{\sigma} - i[K_E, \sigma] = 0$?). But the solution of $\dot{\sigma} - i[K_B, \sigma] = 0$ (with initial condition $\sigma(0) = P_B(0)$) is unique; it is $\rho(\mu)$, or $P_B(\mu)$. Hence

$$\mathrm{Tr}(P_E A) = \mathrm{Tr}(P_B A) + \mathcal{O}(L^{-\infty}) = \mathrm{Tr}(\rho A) + \mathcal{O}(L^{-\infty}) = Tr(\sigma A) + \mathcal{O}(L^{-\infty})$$

for any operator $A \in \Gamma_B$. In particular, this gives another local formula for the Hall conductivity in the bulk of an edge system, by taking $A = J_V$, where

J is the current operator and $V \subset \Gamma_B$ is a set localized in the bulk. The Hall conductivity is given by $\mathrm{Tr}(\dot{P}_E J_V)$, and this can be approximated by

$$\mathrm{Tr}(\dot{P}_E J_V) = \mathrm{Tr}(\dot{P}_B J_V) + \mathcal{O}(L^{-\infty}) = \mathrm{Tr}(\dot{\rho} J_V) + \mathcal{O}(L^{-\infty}) = \mathrm{Tr}(\dot{\sigma} J_V) + \mathcal{O}(L^{-\infty}).$$

Want to pick a norm s.t. Gronwall gives $\|\rho(\mu) - \sigma(\mu)\|_G \leq \|P_B(0) - P_E(0)\|_G e^{2L_F\mu}$. Need $\|P_B(0) - P_E(0)\|_G$ to be small enough to kill the exponential which depends on $L_F = 2\|K_B\|_G \leq \|W_\gamma\|_{L^1}\|Q_h\|_G$. If we use the operator norm for $\|\cdot\|_G$, we would get $\|Q_h\|_G = d|\Gamma_h|$ in the exponent. Need $\|\cdot\|_G$ to be an actual norm so that $\|\rho - \sigma\|_G = 0 \implies \rho = \sigma$.

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Let $r(t) = \rho(t) - \sigma(t)$. Notice that

$$\frac{d}{dt} e^{itK_B} \sigma_0 e^{-itK_B} = e^{itK_B} i[K_B, \sigma_0] e^{-itK_B} + e^{-itK_B} \dot{\sigma_0} e^{itK_B}.$$

F The Helffer-Sjöstrand Representation

The Helffer-Sjöstrand representation is a functional calculus $f \mapsto f(H)$ for arbitrary (possibly unbounded) operators H on the set of functions

$$\mathcal{A} = \bigcup_{\beta<0} \{ f : \mathbb{R} \to \mathbb{C} \ : \ f \in C^\infty(\mathbb{R}),\ |f^{(n)}(x)| \leq c_n (1+x^2)^{\frac{\beta-n}{2}} \}.$$

It has the following properties.

Theorem 5. *For any $f \in \mathcal{A}$,*

1. *$f \mapsto f(H)$ is an algebraic homomorphism (linear and multiplicative).*
2. *$\overline{f}(H) = f(H^*)$.*
3. *$\|f(H)\| \leq \|f\|_\infty$.*
4. *For all $w \notin \mathbb{R}$, if $r_w(s) = \frac{1}{s-w}$ then $r_w(H) = (H-z)^{-1}$.*
5. *For all $f \in C_c^\infty(\mathbb{R})$ with supp$(f) \cap$ Spec$(H) = \varnothing$, we have $f(H) = 0$.*

There is an explicit formula for $f(H)$, which is given by

$$f(H) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (H - z)^{-1} dz \wedge d\bar{z},$$

where $\tilde{f} : \mathbb{C} \to \mathbb{C}$ is a *quasi-analytic extension* of $f : \mathbb{R} \to \mathbb{R}$. It is defined as follows. For any smooth f, we set

$$\tilde{f}(z) = \sum_{r=0}^{n} \tau \left(\frac{y}{(1+x^2)^{1/2}} \right) \frac{(iy)^r}{r!} f^{(r)}(x)$$

where $\tau : \mathbb{R} \to \mathbb{R}$ is any smooth function satisfying

$$\tau(s) = \begin{cases} 1 & |s| < 1 \\ 0 & |s| > 2 \end{cases}.$$

The extension turns out to be independent of the choice of n and τ. Furthermore, as $|\mathrm{Im}(z)| \to 0$, the Wirtinger derivative of the extension obeys the bound

$$\left| \frac{\partial \tilde{f}}{\partial \bar{z}} \right| = \mathcal{O}(|y|^n).$$

Thus $\frac{\partial \tilde{f}}{\partial \bar{z}} = 0$ for all real z, which is why $\tilde{f}$ is called a “quasi”-analytic extension (the Wirtinger derivative would be zero everywhere were $\tilde{f}$ analytic).

A crucial property of the Helffer-Sjöstrand functional calculus is the following bound. For any $n \in \mathbb{N}$, the quasi-analytic extension $\tilde{f}$ can be chosen so that

$$\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \frac{1}{|\mathrm{Im}(z)|^{p+1}} dz \wedge d\bar{z} \leq C_0 \sum_{k=0}^{n+2} \|f^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_m = \int_{-\infty}^{\infty} |f(x)|(1+x^2)^{m/2} dx.$$

This is often useful because the resolvent obeys the bound $\|(H-z)^{-1}\| \leq |\mathrm{Im}(z)|^{-1}$.

G Spectral Measures and Projection-Values Measures

Projection-valued measures are maps $P : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ from measurable subsets of $\mathbb{R}$ to the space of bounded linear operators on $\mathcal{H}$ satisfying the usual properties of both projections and measures.

1. $P(M) = P(M)^* = P(M)^2$ is an orthogonal projection for all $M \in \mathcal{M}$. Note that this implies that $P(M)$ is a positive operator.

2. $P(\varnothing) = 0$ and $P(\mathbb{R}) = \mathbb{1}_{\mathcal{H}}$.

3. If $\{M_i\}_{i\in\mathbb{N}}$ are pairwise disjoint, then $\sum_{i=1}^{n} P(M_i) \xrightarrow{s} P(\cup_{i\in\mathbb{N}} M_i)$ as $n \to \infty$ (σ-additivity).

4. $P(M_1 \cap M_2) = P(M_1)P(M_2)$ for any $M_1, M_2 \in \mathcal{M}$.

The heuristic motivation is that $P(M)$ projects onto the subspace of $\mathcal{H}$ spanned by states whose energies lie in M. Using these operator-valued measures, one can construct an operator-valued integral with respect to P in the usual fashion (beginning on nonnegative simple functions, extending to nonnegative measurable functions, and finally to real-valued measurable functions).

Theorem 6 (Spectral Theorem for Projection-Valued Measures)**.** *There exists a one-to-one correspondence between self adjoint operators H and projection-valued measures P given by the formula*

$$H = \int_{\mathbb{R}} \lambda dP_\lambda,$$

where $P_\lambda := P((-\infty, \lambda])$. Moreover, if $g : \mathbb{R} \to \mathbb{R}$ is any bounded Borel function, then $g(H)$ defined via the Borel function calculus coincides with the formula

$$g(H) = \int_{\mathbb{R}} g(\lambda) dP_\lambda,$$

and $g(H) = g(H)^*$.

We remark that it follows from the second part of this theorem that if $\mathbb{1}_M$ denotes the characteristic function of a Borel set $M \subseteq \mathbb{R}$, then

$$\mathbb{1}_M(H) = \int_M dP_\lambda = P(M).$$

We also note that $\mathrm{Spec}(H) = \mathrm{supp}(P)$.