Thesis Rough Work

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1 Introduction

2 Nonnteracting Setting

Consider the lattice $\mathbb{Z}^2$, on which we define a bulk Hamiltonian H_B, whose matrix elements follow a short-range assumption:

$$\sup_{y\in\mathbb{Z}^2}\sum_{x\in\mathbb{Z}^2}|H_B(x,y)|(e^{\mu|x-y|}-1)<\infty$$

for some $\mu>0$. We define the bulk conductivity

$$\sigma_B(\lambda)=-i\mathrm{Tr}(P_\lambda[[P_\lambda,\Lambda_1],[P_\lambda,\Lambda_2]])$$

where P_λ is the projection onto the eigenstates of H_B with energy lies in $(-\infty,\lambda)$, and where

$$\Lambda_i(x)=\begin{cases}1 & x_i<0\\ 0 & x_i\geq 0\end{cases}$$

are characteristic functions. We construct an edge Hamiltonian on the lattice $\mathbb{Z}^2_a=\{x\in\mathbb{Z}^2 : x_2>-a\}$. We denote the edge Hamiltonian by $H_a:\ell^2(\mathbb{Z}^2_a)\to\ell^2(\mathbb{Z}^2_a)$, requiring only that that the edge operator $E_a:\ell^2(\mathbb{Z}^2_a)\to\ell^2(\mathbb{Z}^2_a)$ define by

$$E_a=J_aH_a-H_BJ_a$$

satisfies the edge assumption

$$\sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2_a}E_a(x,y)|e^{\mu(|x_2+a|-|x_1-y_1|)}\leq C<\infty$$

for some $\mu>0$, where $|x|:=|x_1|+|x_2|$. The interpretation

Each site $x\in\mathbb{Z}^2$ get an associated Hilbert space $\mathcal{H}_x$. The dimension of these Hilbert spaces is bounded uniformly in x. We consider the Hilbert space $\ell^2(\mathbb{Z}^2,\mathbb{C}^n)=\{(x_1,x_2,\ldots)\subset\mathbb{C}^n : \sum_{i\in\mathbb{Z}^2}\|x_i\|^2<\infty\}$. For example, one might consider a system of spins at the lattice sites, in which case the Hilbert space $\mathcal{H}_x$ at each site would be $\mathbb{C}^2$, and the total Hilbert space $\mathcal{H}=\otimes_x\mathcal{H}_x$ would then be the space of summable wavefunctions $\psi=\otimes_x\psi_x\in\ell^2(\mathbb{Z}^2,\mathbb{C}^2)$.

The Hilbert space $\ell^2(\mathbb{Z}^2)$ is the "bulk" setting, i.e. the setting in which we consider an infinite two-dimensional medium with no edges, and we consider a "bulk Hamiltonian" H_B on this Hilbert space. We also define the "edge" Hilbert space $\ell^2(\mathbb{Z}^2_a)$ and an associated "edge Hamiltonian" H_a, where $\mathbb{Z}^2_a:=$

$\{(n,m) \in \mathbb{Z}^2 : n \geq -a\}$. The bulk and edge Hamiltonians are related by the edge operator $E_a : \ell^2(\mathbb{Z}_a^2) \to \ell(\mathbb{Z}^2)$ defined by

$$E_a := J_a H_a - H_B J_a,$$

where $J_a : \ell^2(\mathbb{Z}_a^2) \to \ell(\mathbb{Z}^2)$ denotes extension by zeroes. We assume that

Assumption 1. *The edge operator satisfies*

$$\sup_{z\in\mathbb{Z}^2} \sum_{y\in\mathbb{Z}_a^2} |E_a(x,y)| e^{\alpha(|x_2+a|+|x_1-y_1|)} \leq C < \infty$$

for some $\alpha > 0$, where $|x| = |x_1| + |x_2|$ is the taxicab metric.

The interpretation is that $E_a = J_a H_a - H_B J_a$ is the difference between first applying H_a on $\ell^2(\mathbb{Z}_a^2)$, and then making everything below $-a$ into zeroes, versus first making all $x \in \mathbb{Z}^2$ such that $x_2 < -a$ zeroes, and the applying H_B. The assumption ensures that the effects from introducing the edge at $-a$ die exponentially as we move upward away from the edge (due to the $|x_2 - (-a)|$ term in the exponent), and also terms do not interact too much as their x_1 distance increases (due to the $|x_1 - y_1|$ term in the exponent).

We also make the following assumption about both the bulk and edge Hamiltonians:

Assumption 2. *The Hamiltonians have a spectral gap. There exists an interval Δ such that $\Delta \cap \sigma(H) = \varnothing$.*

Remark: The spectral gap assumption can be relaxed to a "mobility gap" assumption,

$$\sup_{f\in B_c(\Delta)} |f(H_B)(x,y)|(1+|x|)^{-\nu} e^{\mu|x-y|} < \infty$$

for some $\nu > 0$, where $B_c(\Delta)$ is the set of Borel functions f which are constant on $(-\infty, \inf\Delta)$ and on $(\sup\Delta, \infty)$ such that $|f(x)| \leq 1$ for all x. See ? for details.

An example of an edge Hamiltonian satisfying the assumption on E_a is $H_a = J_a^* H_B J_a$, where $J_a : \ell^2(\mathbb{Z}_a^2) \to \ell^2(\mathbb{Z}^2)$ denotes extension by zeros. The idea is that for a state $\psi \in \ell^2(\mathbb{Z}_a^2)$, we have $\langle \psi, H_a\psi\rangle = \langle (J_a\psi), H_B(J_a\psi)\rangle$, which we interpret as the edge Hamiltonian having the same expectation as the bulk Hamiltonian if we just turned all the states ψ_x with $x_2 < -a$ into zeroes. The edge operator is

$$E_a = J_a J_a^* H_B J_a - H_b J_a = (J_a J_a^* - \mathbb{1}) H_B J_a = \begin{cases} -H_B(x,y) & \text{if } x_2 < -a \\ 0 & \text{if } x_2 \geq -a \end{cases}$$

Intuitively, there is no difference between H_B and H_a on $\mathbb{Z}_a^2$. The bound in assumption ? is satisfied by the short range assumption ?.

We define the *bulk conductivity* at Fermi energy μ as follows. Suppose we subject the system to an external electric potential difference V in the x_2 direction. We write this as $-V_0\Lambda_2$, where Λ_i are multiplication operators $\Lambda_i|\psi(x_1,x_2)\rangle = \Lambda(x_i)|\psi(x_1,x_2)\rangle$ which are *switch functions*,

$$\Lambda : \mathbb{R} \to \mathbb{R} \qquad\qquad \Lambda(x_i) = \begin{cases} 1 & \text{if } x_i \le 0 \\ 0 & \text{if } x_i \ge 1 \end{cases}$$

and are smooth and monotonically decreasing on $(0,1)$. Note that the ensuing physics (in particular, our definition of the Hall conductivity) is independent of the particular choice of switch function Λ_i, since any two switch functions are exactly equal on the lattice.

This gives $\vec{E} = -\nabla V = V_0\frac{\partial \Lambda_2}{\partial x_2}$, so that $\vec{E}$ is has compact support $\mathrm{supp}(\Lambda_2')$. We introduce a function which grows slowly in time as t grows from $-\infty$ to 0, so as to invoke the adiabatic principle. Here, we choose $e^{\varepsilon t}$, and we will let $\varepsilon \to 0$ at the end. The Hamiltonian therefore experiences a perturbation,

$$\widetilde{H}_B(t) = H_B - V_0\Lambda_2 e^{\varepsilon t}.$$

We define the Hall current operator $J_H = i[\widetilde{H}_B(t),\Lambda_1] = i[H_B,\Lambda_1]$, which is related to the Hall conductivity by $J_H = \sigma_H V$.

Lemma 1. *The ground state expectation $\mathrm{Tr}(P_\mu J_H)$ of the Hall current is zero.*

Proof. Notice that since J_H is trace-class and P_μ is bounded, and since $[H_B,P_\mu] = 0$, we have

$$\mathrm{Tr}(P_\mu J_H) = i\mathrm{Tr}(P_\mu[H_B,\Lambda_1]) = i\mathrm{Tr}(P_\mu[H_B,P_\mu\Lambda_1 P_\mu])$$

□

Proposition 1. *The Hall conductivity σ_H in the bulk system is equal to*

$$\sigma_B = -i\mathrm{Tr}\left(P_\mu\left[[P_\mu,\Lambda_1],[P_\mu,\Lambda_2]\right]\right),$$

where $P_\mu := P((-\infty,\mu))$ is the projection-valued measure associated with H_B onto states with energy less than the Fermi energy μ.

Proof. We begin with the Heisenberg equation of motion for the density matrix, $\dot{\rho}(t) = -i[\widetilde{H}_B(t),\rho(t)]$, with initial condition $\lim_{t\to-\infty}\|\rho(t) - e^{-itH_B}P_\mu e^{itH_B}\| = 0$, which also implies $\lim_{t\to-\infty}\|e^{itH_B}\rho(t)e^{-itH_B} - P_\mu\| = 0$.

We work in the interaction picture, and define $\rho_I(t) = e^{itH_B}\rho(t)e^{-itH_B}$, and $\Delta H_B(t) = -e^{itH_B}V_0\Lambda_2 e^{\varepsilon t}e^{-itH_B}$. Thus

$$\dot{\rho}_I(t) = -i[\Delta H_B(t), \rho_I(t)]$$

The solution to this differential equation is readily verified to be

$$\rho(t) = i\int_{-\infty}^{t}[\Delta H_B(s), P_\mu]ds + P_\mu$$

Indeed, taking the derivative of the right hand side gives $i[\Delta H_B(t), P_\mu] = i[\Delta H_B(t), \rho_I(t)] + \mathcal{O}(V_0^2)$, but P_μ and $\rho_I(t)$ are equal up to zeroth order in V_0. The initial condition is also satisfied.

Using $J_H = i[H_B, \Lambda_1] = \sigma_H V = -\sigma_H V_0 \Lambda_2$, we obtain $\sigma_H = \frac{1}{V_0}\lim_{\varepsilon\to 0}\mathrm{Tr}(\rho(0)i[H_B,\Lambda_1])$. Since the expectation of the ground state current is zero, $\mathrm{Tr}(P_\mu J_H) = 0$, we have

$$\begin{aligned}
\sigma_H &= \frac{i}{V_0}\lim_{\varepsilon\to 0}\mathrm{Tr}\left(i\int_{-\infty}^{0}[\Delta H_B(t), P_\mu][H_B,\Lambda_1]ds\right)\\
&= -\frac{1}{V_0}\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}[-e^{isH_B}V_0\Lambda_2 e^{\varepsilon s}e^{-isH_B}, P\mu][H_B,\Lambda_1]ds\right)\\
&= -\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}e^{isH_B}[\Lambda_2,P_\mu]e^{-isH_B}[H_B,\Lambda_1]e^{\varepsilon s}ds\right)\\
&= -\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}(e^{-isH_B}[H_B,\Lambda_1]e^{isH_B})\cdot([\Lambda_2,P_\mu]e^{\varepsilon s})ds\right)
\end{aligned}$$

Where we used the fact that P_μ and H_B commute. Using integration by parts on the two terms in brackets, and noting that $\frac{d}{ds}(e^{isH_B}[H_B,\Lambda_1]e^{-isH_B}) = -(ie^{isH_B}\Lambda_1 e^{-isH_B} - \Lambda_1)$, we obtain

$$\begin{aligned}
\sigma_H &= i\lim_{\varepsilon\to 0}\mathrm{Tr}\left(\int_{-\infty}^{0}(e^{-isH_B}\Lambda_1 e^{isH_B} - \Lambda_1)\frac{d}{ds}([\Lambda_2,P_\mu]e^{\varepsilon s})ds\right)\\
&= i\lim_{\varepsilon\to 0}\varepsilon\mathrm{Tr}\left(\int_{-\infty}^{0}\Lambda_1^s[\Lambda_2,P_\mu]e^{\varepsilon s})ds\right)
\end{aligned}$$

where $\Lambda_1^s := e^{-isH_B}\Lambda_1 e^{isH_B} - \Lambda_1$. Using the notation $\overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, it is readily verified that the commutator $[\Lambda_2, P_\mu]$ is an *off-diagonal* operator, in the sense that $[\Lambda_2,P_\mu] = \overline{[\Lambda_2,P_\mu]}$. Furthermore, a simple computation reveals that for any two operators A and B, $\mathrm{Tr}(\overline{A}B) = \mathrm{Tr}(A\overline{B})$. It therefore follows that

$$\sigma_H = i \lim_{\varepsilon\to 0} \varepsilon \mathrm{Tr}\left(\int_{-\infty}^{0} \overline{\Lambda_1^s}[\Lambda_2, P_\mu]e^{\varepsilon s})ds\right)$$

The integrand can be broken into two terms,

$$\overline{\Lambda_1^s}[\Lambda_2, P_\mu]e^{\varepsilon s} = e^{-isH_B}\overline{\Lambda_1}e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s} - \overline{\Lambda_1}[\Lambda_2, P_\mu]e^{\varepsilon s}$$

by commutativity of P_μ and H_B. We show that the integral of the first term vanishes. We begin by breaking the first term down further into

$$e^{-isH_B}P_\mu\Lambda_1 P_\mu^\perp e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s} + e^{-isH_B}P_\mu^\perp\Lambda_1 P_\mu e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s}.$$

We treat the first of these two terms; the other is handled in an identical manner. We use the spectral theorem to write $e^{-isH_B}P_\mu = \int_{-\infty}^{\mu} e^{-is\lambda}dP_\lambda$, and similarly $P_\mu^\perp e^{isH_B} = (\mathrm{Id} - P_\mu)e^{isH_B} = \int_{\mu}^{\infty} e^{is\nu}dP_\nu$.

We remark that, since the Fermi energy μ is assumed to lie in a spectral gap, there must exist a neighbourhood $(\mu-\delta, \mu+\delta)$ in which there are no states. We exploit this fact to rewrite the limits of integration, $\int_{-\infty}^{\mu-\delta} e^{-is\lambda}dP_\lambda$ and $\int_{\mu+\delta}^{\infty} e^{is\nu}dP_\nu$. We therefore obtain

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\varepsilon \int_{-\infty}^{0} e^{-isH_B}P_\mu\Lambda_1 P_\mu^\perp e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s}ds \\
&\qquad = \lim_{\varepsilon\to 0}\varepsilon \mathrm{Tr}\left(\int_{-\infty}^{0}\int_{-\infty}^{\mu-\delta} e^{-is\lambda}dP_\lambda\Lambda_1 \int_{\mu+\delta}^{\infty} e^{is\nu}dP_\nu[\Lambda_2, P_\mu]e^{\varepsilon s}ds\right) \\
&\qquad = \lim_{\varepsilon\to 0}\varepsilon \mathrm{Tr}\left(\int_{-\infty}^{0}\int_{-\infty}^{\mu-\delta}\int_{\mu+\delta}^{\infty} e^{s(\varepsilon - i\lambda + i\nu)}dP_\lambda\Lambda_1 dP_\nu[\Lambda_2, P_\mu]ds\right)
\end{aligned}$$

Performing the integral over s yields

$$\lim_{\varepsilon\to 0}\varepsilon \int_{-\infty}^{0} e^{s(\varepsilon - i\lambda + i\nu)}ds = -\lim_{\varepsilon\to 0}\frac{\varepsilon}{i\varepsilon + \lambda - \nu}$$

This limit is zero, since $\lambda \neq \nu$. Indeed, due to the spectral gap, the integration variables live in $\lambda \in (-\infty, \mu-\delta)$ and $\nu \in (\mu+\delta, \infty)$. The case for the $e^{-isH_B}P_\mu^\perp\Lambda_1 P_\mu e^{isH_B}[\Lambda_2, P_\mu]e^{\varepsilon s}$ term (where the P_μ and $P_\mu^\perp$ swap places) is treated analogously. Hence the first term in the integrand for σ_H vanishes, as claimed.

Finally, we return to our expression for the Hall conductivity, which now reads

$$\sigma_H = i \lim_{\varepsilon \to 0} \varepsilon \mathrm{Tr}\left(\int_{-\infty}^{0} \overline{\Lambda_1}[\Lambda_2, P_\mu] e^{\varepsilon s} ds \right).$$

It is a basic algebraic calculation to show that $\overline{\Lambda_1} = [[\Lambda_1, P_\mu], P_\mu]$. Evaluating the integral over s is now trivial; $\int_{-\infty}^{0} e^{\varepsilon s} ds = \varepsilon^{-1}$. Thus

$$\sigma_H = -i\mathrm{Tr}([[\Lambda_1, P_\mu], P_\mu][\Lambda_2, P_\mu]).$$

Shifting the commutator completes the proof:

$$\begin{aligned}
\sigma_H &= -i\mathrm{Tr}(P_\mu[[\Lambda_2, P_\mu], [\Lambda_1, P_\mu]]) \\
&= i\mathrm{Tr}(P_\mu[[\Lambda_1, P_\mu], [\Lambda_2, P_\mu]]) \\
&= i\mathrm{Tr}(P_\mu[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]).
\end{aligned}$$

□

Remark: This is reminiscent of the well-known adiabatic curvature formula,

$$\kappa = \mathrm{Tr}(P[\partial_1 P, \partial_2 P]) = \mathrm{Tr}(P\,[[P, K_1], [P, K_2]]) = \mathrm{Tr}(P[K_1, K_2]),$$

where K_i are called *generators of parallel transport.* We will see the adiabatic curvature formula again later in the interacting setting.

For the *edge conductivity*, we need the current operator across the line $x_1 = 0$, which is given by $-i[H_a, \Lambda_1]$. We define

$$\sigma_E = -i \lim_{a \to \infty} \mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]),$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(r) = \begin{cases} 1 & \text{if } r < \inf \Delta \\ 0 & \text{if } r > \sup \Delta \end{cases}$$

and decreases smoothly and monotonically in Δ. The definition of σ_E is reminiscent of another formula we will see later in the interacting setting, $\mathrm{Tr}(\dot{P}J)$, where J is the current operator. The interpretation of σ_E is that if we apply a small potential difference V across $x_2 = -a$ to $x_2 = \infty$, there will be a net current

$$\begin{aligned}
I &= -i\mathrm{Tr}(\rho(H_a + V)[H_a + V, \Lambda_1] - \rho(H_a)[H_a, \Lambda_1]) \\
&= -i\mathrm{Tr}((\rho(H_a + V) - \rho(H_a))[H_a, \Lambda_1])
\end{aligned}$$

Thus we obtain the conductivity

$$\sigma_E = \frac{I}{V} = -i\mathrm{Tr}\left(\frac{(\rho(H_a+V)-\rho(H_a)}{V}[H_a,\Lambda_1]\right) \to -i\mathrm{Tr}(\rho'(H_a)[H_a,\Lambda_1])$$

in the limit as $V \to 0$. As we shall see, it turns out that σ_E is independent of the choice of ρ, and σ_B is independent of λ.

The main result of this section is

Theorem 1. $\sigma_E = \sigma_B$.

2.1 Outline of the Proof

First, let

$$\tilde{\sigma}_E(a,t) = -i\mathrm{Tr}(\rho'(H_a)[H_a,\Lambda_1]\Lambda_{2,a}(t))$$

where $\Lambda_{2,a}(t) = e^{itH_a}\Lambda_2 e^{-itH_a}$ is the time evolution of Λ_2. One can show that, while

$$\sigma_E = \lim_{T\to\infty}\lim_{a\to\infty}\frac{1}{T}\int_0^T \mathrm{Re}(\tilde{\sigma}_E(a,t))dt,$$

it is unfortunately the case that $\lim_{a\to\infty}\|\rho'(H_a)[H_a,\Lambda_1]\Lambda_{2,a}(t)\|_1 = \infty$. However, even though the trace norm diverges, it turns out that the trace itself does not, so we will instead subtract a clever choice of zero-trace operator $Z(a,t)$ to define

$$\sigma_E(a,t) = -i\mathrm{Tr}(\rho'(H_a)[H_a,\Lambda_1]\Lambda_{2,a}(t) - Z(a,t))$$

so that the equation $\sigma_E = \lim_{T\to\infty}\lim_{a\to\infty}\frac{1}{T}\int_0^T \mathrm{Re}(\sigma_E(a,t))dt$ still holds, but we also have $\lim_{a\to\infty}\|\rho'(H_a)[H_a,\Lambda_1]\Lambda_{2,a}(t) - Z(a,t)\|_1 < \infty$. The correct choice of Z will become apparent after writing $\rho(H_a)$ and $\rho'(H_a)$ in terms of their Hellfer-Sjöstrand representations,

$$\rho(H_a) = \frac{1}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\rho}(z)}{\partial\bar{z}}R(z)$$

$$\rho'(H_a) = -\frac{1}{2\pi}\int_{\mathbb{C}}\frac{\partial\tilde{\rho}(z)}{\partial\bar{z}}R(z)^2$$

where $R(z) = (H_a - z)^{-1}$ is the resolvent. Using $[R(z),\Lambda_i] = R(z)[H_a,\Lambda_i]R(z)$, we obtain the representations of the following useful operators:

$$[\rho(H_a), \Lambda_1] = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)[H_a, \Lambda_1]R(z)$$

$$\rho'(H_a)[H_a, \Lambda_1] = -\frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)^2[H_a, \Lambda_1]$$

From here, we define the zero-trace operator

$$Z(a,t) = [\rho(H_a), \Lambda_1]\Lambda_2 - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)(R(z)[H_a, \Lambda_1]\Lambda_{2,a}(t) - [H_a, \Lambda_1]\Lambda_{2,a}(t)R(z))$$

from which we obtain

$$\begin{aligned}
\sigma_E(a,t) &= \tilde{\sigma}_E(a,t) - Z(a,t) \\
&= \mathrm{Tr}\left(-[\rho(H_a), \Lambda_1]\Lambda_2 - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)[H_a, \Lambda_1]\Lambda_{2,a}(t)R(z)\right) \\
&= \mathrm{Tr}\left([\rho(H_a), \Lambda_1](\Lambda_{2,a}(t) - \Lambda_2) - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}(z)}{\partial\bar{z}} R(z)[H_a, \Lambda_1]R(z)[H_a, \Lambda_{2,a}(t)]R(z)\right)
\end{aligned}$$

All of the statements so far can be verified by calculations. The difficult part of the theorem (aside from proving that the relevant operators are trace-class) is proving that

$$\|J_a\Sigma'_a J_a^* - \Sigma'_B\|_1, \|J_a\Sigma''_a J_a^* - \Sigma''_B\|_1 \to 0$$

as $a \to \infty$, where Σ'_B and Σ''_B are the same as with the subscript a, except using the bulk Hamiltonian H_B in their definition rather than H_a. It follows that

$$\sigma_E(a,t) = \mathrm{Tr}(J_a\Sigma'_a J_a^* + J_a\Sigma''_a J_a^*) = \mathrm{Tr}(\Sigma'_a + \Sigma''_a) \to \mathrm{Tr}(\Sigma'_B + \Sigma''_B)$$

From there, a calculation shows that $\lim_{T\to\infty} \frac{1}{T}\int_0^T \mathrm{Tr}(\Sigma'_B + \Sigma''_B)dt = \sigma_B$, concluding the proof.

2.2 Gap Simplifications

We now assume that H_B has a spectral gap. In this case, the edge condition guarantees that $\sigma_E(a) := \rho'(H_a)[H_a, \Lambda_1]$ is trace-class (need to add).

Need to add section on why $\sigma_E(a) := -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1])$ is equal to

$$-\frac{i}{2}\mathrm{Tr}(\rho'(H_a)\{[H_a, \Lambda_1], \Lambda_2\}).$$

Theorem 2. $\sigma_E = \sigma_B$

Outline of the Proof

Before giving the proof in its entirety, we outline the basic steps. The key ingredient is the use of the funcional calculus given by the Helffer-Sjöstrand representation of self-adjoint operators on a Hilbert space (need to add reference to appendix here).

The Proof

Proof. We posit that the edge conductivity can be rewritten as $\sigma_E = \lim_{a\to\infty} \sigma_E(a)$, where

$$\sigma_E(a) = -i\mathrm{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_2),$$

since we have assumed that there is a spectral gap (as opposed to a mobility gap), so that there are extended states near the edge, and no bound states or resonances far from the edge. Thus, intuitively, the cutoff introduced by Λ_2 is irrelevant as we take $a \to \infty$. We provide a more concrete justification for this later.

"Concrete justificiation": Since σ_B is translation invariant (need to add), we only need to prove to case $-i\mathrm{Tr}(\rho'(H_{a=0})[H_{a=0}, \Lambda_1] = \sigma_B$. We drop the subscript, $H := H_{a=0}$. Since the multiplication operator $\Lambda_2(n)|\psi\rangle := \Lambda(x_2 - n)|\psi\rangle$ converges strongly to the identity as $n \to \infty$, we can write

$$\sigma_E(a) = -i\mathrm{Tr}(\rho'(H)[H, \Lambda_1]) = -i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H)[H, \Lambda_1]\Lambda_2(n)).$$

Instead of completing the shift $(0, -n)$ with the operator $\Lambda_2(n)$, we can consider a shifted Hamiltonian. Indeed, rather than restrict H_B at $x_2 = -n$ to obtain $H_{a=n}$, consider the shifted bulk Hamiltonian $H_B \mapsto H_B(n)$ obtained by the shift $(0, -n)$, and then restricting this at $x_2 = 0$ to obtain $H(n)$. In other words, $H(n)$ is the edge Hamiltonian associated with the shifted bulk Hamiltonian. Comparing this with the expression above, this is exactly equivalent to

$$-i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H)[H, \Lambda_1]\Lambda_2(n)) = -i \lim_{n\to\infty} \mathrm{Tr}(\rho'(H(n))[H(n), \Lambda_1]\Lambda_2).$$

In other words, the difference between whether we cut off everything above $x_2 = n$ and then apply $H_{a=0}$, or instead cut off everything above $x_2 = 0$ and then apply the Hamiltonian $H(n)$ shifted down by $(0, -n)$ is immaterial.

Thus, our goal is to show that $-i\mathrm{Tr}(\rho'(H(n))[H(n),\Lambda_1]\Lambda_2) \to \sigma_B$ as $n \to \infty$.

End of concrete justification.

Define

$$Z(a) = [\rho(H_a),\Lambda_1]\Lambda_2 - \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[R_a(z),[H_a,\Lambda_1]\Lambda_2]dz^2$$

This operator has zero trace. Indeed, the first term has vanishing trace in the position basis, while the second term's integrand involves the trace of $[R,R]=0$. The bounds necessary for shifting the commutator like this are on

$$\|R[H_a,\Lambda_1]e^{\delta|x_1|}\|, \quad \|e^{-\delta|x_1|}e^{-\delta|x_2}\|_1, \quad \|\Lambda_2 R e^{\delta|x_2|}\|,$$

the first two of which are given below, and the third is obvious since $\|R\|$ is bounded and Λ_2 provides a cutoff which ensures $e^{\delta|x_2|}$ is finite. So $\sigma_E(a) = \mathrm{Tr}(\Sigma(a))$, where

$$\begin{aligned}\Sigma(a) &= -i\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 + iZ(a)\\ &= -i\rho'(H_a)[H_a,\Lambda_1]\Lambda_2 + i[\rho(H_a),\Lambda_1]\Lambda_2 - \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[R_a(z),[H_a,\Lambda_1]\Lambda_2]dz^2.\end{aligned}$$

Using the Hellfer-Sjöstrand representations for the first two terms on the right hand side, we obtain

$$\begin{aligned}\Sigma(a) = \frac{i}{2\pi}&\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)^2[H_a,\Lambda_1]\Lambda_2 dz^2 + \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1]R_a(z)\Lambda_2 dz^2\\ &- \frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} (R_a(z)^2[H_z,\Lambda_1]\Lambda_2 - R_a(z)[H_a,\Lambda_1]\Lambda_2 R_a(z))dz^2\\ = -&\frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1][R_a(z),\Lambda_2]dz^2\\ = &\frac{i}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}} R_a(z)[H_a,\Lambda_1]R_a(z)[H_a,\Lambda_2]R_a(z)dz^2,\end{aligned}$$

where we used

$$[R_a(z),\Lambda_i] = -R_a(z)[H_a,\Lambda_i]R_a(z)$$

in the final equality. Next, we must prove that the operator above converges to the corresponding bulk operator in trace-norm,

$$\|\Sigma(a) - \Sigma_B\|_1 \to 0,$$

as $a \to \infty$, which in turn proves that $\mathrm{Tr}(\Sigma(a)) \to \mathrm{Tr}(\Sigma_B)$ because of the bound $|\mathrm{Tr}(A)| \le \|A\|_1$. Here, Σ_B is the same operator as before, but using the bulk operators H_B and $R_B(z)$. Once this limit is established, we shall prove that $\sigma_B = \mathrm{Tr}(\Sigma_B)$ to conclude the proof.

To show that the limit is zero as claimed, we bound the trace norm of the integrand of $\Sigma(a)$ by breaking it into three parts,

$$R[H_a, \Lambda_1]R[H_a, \Lambda_2]R = J_a[R, \Lambda_1]e^{\delta|x_1|}J_a^* \cdot e^{-\delta|x_1|}e^{-\delta|x_2|} \cdot J_a e^{\delta|x_2|}[H_a, \Lambda_2]RJ_a^*,$$

and bounding the norm of each, making use of the fact that $\|AB\|_1 \le \|A\|\|B\|_1$.

1. For the first term, $J_a[R, \Lambda_1]e^{\delta|x_1|}J_a^*$, we bound its operator norm by breaking it down further into

$$\begin{aligned} \|J_a[R, \Lambda_1]e^{\delta|x_1|}J_a^*\| &= \|[R, \Lambda_1]e^{\delta|x_1|}\| \\ &= \|-R[H_a, \Lambda_1]Re^{\delta|x_1|}\| \\ &= \|-R \cdot [H_a, \Lambda_1]e^{\delta|x_1|} \cdot e^{-\delta|x_1|}Re^{\delta|x_1|}\| \\ &\le \|R\| \cdot \|[H_a, \Lambda_1]e^{\delta|x_1|}\| \cdot \|e^{-\delta|x_1|}Re^{\delta|x_1|}\| \end{aligned}$$

   The norm of R is bounded by

$$\|R_a(z)\| \le \frac{1}{|\mathrm{Im}(z)|}$$

   for any $z \notin \mathbb{R}$ since H_a is self-adjoint. The norm of the second operator can be bounded by inspecting its matrix elements.

$$\begin{aligned} \langle x, [H_a, \Lambda_1]e^{\delta|x_1|}y\rangle &= \langle x, H_a\Lambda_1 y\rangle e^{\delta|y_1|} - \langle x, \Lambda_1 H_a y\rangle e^{\delta|y_1|} \\ &= H_a(x, y)e^{\delta|y_1|}(\Lambda(y_1) - \Lambda(x_1)). \end{aligned}$$

   This is zero if $|x_1 - y_1| \le |y_1|$, since this would imply that x_1 and y_1 have the same sign, yielding $\Lambda(x_1) = \Lambda(y_1)$. So either the matrix element is zero, or $|y_1| \le |x_1 - y_1|$, which implies

$$
\begin{aligned}
|H_a(x,y)e^{\delta|y_1|}(\Lambda(y_1)-\Lambda(x_1))| &\leq 2|H_a(x,y)|e^{\delta|x_1-y_1|}\\
&\leq 2|H_a(x,y)|e^{\delta|x-y|}\\
&\leq C|H_a(x,y)|(e^{\delta|x-y|}-1)
\end{aligned}
$$

where the final inequality comes from the fact that the diagonal matrix elements are zero. Hence the assumption

$$
\sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}|H(x,y)|(e^{\mu|x-y|}-1)<\infty,
$$

combined with Holmgren's bound

$$
\|A\|\leq \max\left\{\sup_{x\in\mathbb{Z}^2}\sum_{y\in\mathbb{Z}^2}|A(x,y)|,\ \sup_{y\in\mathbb{Z}^2}\sum_{x\in\mathbb{Z}^2}|A(x,y)|\right\},
$$

implies that the second term is bounded. Finally, for the third term $e^{-\delta|x_1|}Re^{\delta|x_1|}$, we apply the Combes-Thomas bound,

$$
\|e^{-\varepsilon f(x)}R_a(z)e^{\varepsilon f(x)}\|\leq \frac{C}{|\mathrm{Im}(z)|}
$$

where $f:\mathbb{Z}^2\to\mathbb{R}$ is any Lipschitz function, and ε can be chosen as $\varepsilon=\frac{1}{C(1+|\mathrm{Im}(z)|)}$.

Altogether, the bound of the first term takes the form

$$
\frac{C}{\mathrm{Im}(z)^2}.
$$

2. For $e^{-\delta|x_1|}e^{-\delta|x_2|}$, we bound the trace norm by noticing that this is a positive operator satisfying

$$
\langle(n,m),e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m)\rangle=\langle e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m),(n,m)\rangle,
$$

so that its trace norm is equal to its trace. In the position basis, we see that its trace is given by a geometric series

$$\begin{aligned}\mathrm{Tr}(e^{-\delta|x_1|}e^{-\delta|x_2|}) &= \sum_{(n,m)\in\mathbb{Z}^2} \langle (n,m), e^{-\delta|x_1|}e^{-\delta|x_2|}(n,m)\rangle \\ &\leq 2\sum_{n=0}^{\infty}\sum_{m=0}^{\infty} e^{-\delta m}e^{-\delta n} \\ &= 2\left(\frac{1}{1-e^{-\delta}}\right)^2.\end{aligned}$$

3. For $J_a e^{\delta|x_2|}[H_a, \Lambda_2]R_a(z)J_a^*$, note that analogously to 1. above where we bounded $[H_a, \Lambda_1]e^{\delta|x_1|}$, we also have that $e^{\delta|x_2|}[H_a, \Lambda_2]$ is bounded. Again, the resolvent $R_a(z)$ is also bounded.

   The bound of the third term takes the same form as the bound of the first term,

$$\frac{C}{\mathrm{Im}(z)^2}.$$

Altogether, we see that the integrand is bounded by the product of the three bounds from 1., 2., and 3., and is of the form $\frac{C}{\mathrm{Im}(z)^4}$.

For any $n \in \mathbb{N}$, the quasi-analytic extension $\tilde{\rho}$ of ρ in the Helffer-Sjöstrand representation can be chosen so that

$$\int_{\mathbb{C}} \frac{\partial\tilde{\rho}}{\partial\bar{z}}\frac{1}{|\mathrm{Im}(z)|^{p+1}}dz^2 \leq C_0\sum_{k=0}^{n+2}\|\rho^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_m = \int_{-\infty}^{\infty}|f(x)|(x^2+1)^{m/2}dx.$$

Since $|\rho(x)| \leq 1$ and ρ' is compactly supported, these norms are all clearly finite. This fact, combined with the bound

$$\|R_a(z)[H_a, \Lambda_1]R_a(z)[H_a, \Lambda_2]R_a(z)\|_1 \leq \frac{C}{\mathrm{Im}(z)^4}$$

for the trace norm of the integrand of $\Sigma(a)$ provides the necessary bound for Lebesgue dominated convergence. Thus, it suffices to show pointwise convergence in z of the integrand to the associated bulk operator.

In other words, we wish to show

$$J_a[R_a(z),\Lambda_1]e^{\delta|x_1|}J_a^* \xrightarrow{s} [R_B(z),\Lambda_1]e^{\delta|x_1|},$$

$$J_a e^{\delta|x_2|}[H_a,\Lambda_2]J_a^* \xrightarrow{s} e^{\delta|x_2|}[H_B,\Lambda_2],$$

and

$$J_a R_a(z) J_a^* \xrightarrow{s} R_B(z)$$

for each fixed $z \in \mathbb{C}$. Inspecting the bounds we found for the left hand sides of these limits, it is clear that they are uniformly bounded in a. It therefore suffices to show convergence on a dense subspace of $\ell^2(\mathbb{Z}^2)$; in particular, we may choose the dense subspace of compactly supported states, which allows us to ignore the $e^{\delta|x_i|}$ terms. Thus, we need to prove

$$J_a[R_a(z),\Lambda_1]J_a^* \xrightarrow{s} [R_B(z),\Lambda_1],$$

$$J_a[H_a,\Lambda_2]J_a^* \xrightarrow{s} [H_B,\Lambda_2],$$

and

$$J_a R_a(z) J_a^* \xrightarrow{s} R_B(z).$$

In fact, the final statement implies the first two; we appeal to the general fact of functional analysis that strong convergence of the resolvent of a self-adjoint operator implies that $J_a f(H_a) J_a^* \xrightarrow{s} f(H_B)$ for any bounded and continuous function f. In particular, the functions $[(\cdot - z)^{-1},\Lambda_1]$ and $[\cdot,\Lambda_2]$ above are bounded and continuous, so we will have proven the desired limits if we can prove the strong convergence of the resolvent, $J_a R_a(z) J_a^* \xrightarrow{s} R_B(z)$.

To prove this, we use the edge assumption. Recall the edge operator, $E_a = J_a H_a - H_B J_a$. Adding and subtracting zJ_a gives

$$E_a = J_a(H_a - z) - (H_B - z)J_a.$$

Applying R_B from the left and R_a from the right on both sides, we obtain

$$R_B(z)E_a R_a(z) = R_B(z)J_a - J_a R_a(z).$$

Taking the adjoint, and then multiplying from the left by J_a, we see that

$$J_a R_a(z) E_a^* R_B(z) = J_a J_a^* R_B(z) - J_a R_a(z) J_a^*.$$

Thus

$$R_B(z) - J_a R_a(z) J_a^* = (J_a R_a(z) E_a^* - J_a J_a^* + 1) R_B(z) \xrightarrow{s} 0,$$

since $E_a^* \xrightarrow{s} 0$ by Lemma 2, and $-J_a J_a^* + 1 \xrightarrow{s} 0$. This proves that the limits above converge to the desired associated bulk operators, and hence $\|\Sigma(a) - \Sigma_B\|_1 \to 0$.

Finally, it remains to show that

$$\mathrm{Tr}(\Sigma_B) = \sigma_B.$$

First, we manipulate

$$\begin{aligned}
\Sigma_B &= \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1] R_B(z) [H_B, \Lambda_2] R_B(z) dz^2 \\
&= -\frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1][R_B(z), \Lambda_2] dz^2 \\
&= i[\rho(H_B), \Lambda_1]\Lambda_2 - \frac{i}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}}{\partial \bar{z}} R_B(z)[H_B, \Lambda_1]\Lambda_2 R_B(z) dz^2.
\end{aligned}$$

Define $P_+ := P((\sup \Delta, \infty))$ and $P_- := P((-\infty, \inf \Delta))$, the projections onto states above and below the gap, respectively. Since H_B is assumed to have a gap, we have

$$\mathrm{Tr}(\Sigma_B) = \mathrm{Tr}(P_+ \Sigma_B P_+) + \mathrm{Tr}(P_- \Sigma_B P_-).$$

Since $P_\pm R_B(z)$ and $R_B(z) P_\pm$ are analytic on $\mathrm{supp}(\rho(z))$ and $\mathrm{supp}(1-\rho(z))$, the integral in $P_\pm \Sigma_B P_\pm$ vanishes by integration by parts. Thus

$$\Sigma_B = i P_+ [\rho(H_B), \Lambda_1]\Lambda_2 P_+ + i P_- [\rho(H_B), \Lambda_1]\Lambda_2 P_-.$$

By the spectral theorem for projection-valued measures, if the Fermi energy lies in the gap, $\lambda \in \Delta$, we have

$$\rho(H_B) = \int_{-\infty}^{\infty} \rho(\lambda) dP_\nu = \int_{-\infty}^{\lambda} \rho(\lambda) dP_\nu = \int_{-\infty}^{\lambda} dP_\nu = P_\lambda.$$

We may therefore replace $\rho(H_B)$ by P_λ, by which we obtain

$$\mathrm{Tr}(\Sigma_B) = i\mathrm{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+) + i\mathrm{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-).$$

Now, the bulk conductivity is given by

$$
\begin{aligned}
\sigma_B &= i\mathrm{Tr}(P_\lambda[[P_\lambda,\Lambda_1],[P_\lambda,\Lambda_2]]) \\
&= i\mathrm{Tr}(P_\lambda((P_\lambda\Lambda_1-\Lambda_1P_\lambda)(P_\lambda\Lambda_2-\Lambda_2P_\lambda)-(P_\lambda\Lambda_2-\Lambda_2P_\lambda)(P_\lambda\Lambda_1-\Lambda_1P_\lambda))) \\
&= i\mathrm{Tr}(P_\lambda(P_\lambda\Lambda_1P_\lambda\Lambda_2-P_\lambda\Lambda_1\Lambda_2P_\lambda-\Lambda_1P_\lambda\Lambda_2+\Lambda_1P_\lambda\Lambda_2P_\lambda \\
&\qquad\qquad -P_\lambda\Lambda_2P_\lambda\Lambda_1+P_\lambda\Lambda_2\Lambda_1P_\lambda+\Lambda_2P_\lambda\Lambda_1-\Lambda_2P_\lambda\Lambda_1P_\lambda)) \\
&= i\mathrm{Tr}(-P_\lambda\Lambda_1\Lambda_2P_\lambda+\Lambda_1P_\lambda\Lambda_2P_\lambda+P_\lambda\Lambda_2\Lambda_1P_\lambda-\Lambda_2P_\lambda\Lambda_1P_\lambda) \\
&= i\mathrm{Tr}(-P_\lambda\Lambda_1\Lambda_2P_\lambda+P_\lambda\Lambda_1P_\lambda\Lambda_2P_\lambda+P_\lambda\Lambda_2\Lambda_1P_\lambda-P_\lambda\Lambda_2P_\lambda\Lambda_1P_\lambda) \\
&= i\mathrm{Tr}(P_\lambda\Lambda_1P_\lambda^\perp\Lambda_2P_\lambda-P_\lambda\Lambda_2P_\lambda^\perp\Lambda_1P_\lambda) \\
&= i\mathrm{Tr}(P_\lambda\Lambda_1P_\lambda^\perp\Lambda_2P_\lambda-P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2) \\
&= i\mathrm{Tr}(P_\lambda\Lambda_1P_\lambda^\perp\Lambda_2P_\lambda-P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2P_\lambda^\perp).
\end{aligned}
$$

We define $T_\lambda := P_\lambda\Lambda_1P_\lambda^\perp\Lambda_2P_\lambda-P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2P_\lambda^\perp$, so that

$$\sigma_B = i\mathrm{Tr}(T_\lambda),$$

and show that $P_\pm T_\lambda P_\pm = P_\pm[P_\lambda,\Lambda_1]\Lambda_2P_\pm$.

First, notice that because of the gap, we have $P_\lambda^\perp P_- = 0$, and thus also $P_\lambda P_- = P_-$. Thus

$$
\begin{aligned}
P_-T_\lambda P_- &= P_-P_\lambda\Lambda_1P_\lambda^\perp\Lambda_2P_\lambda P_- \\
&= P_-(P_\lambda\Lambda_1\Lambda_2-\Lambda_1P_\lambda\Lambda_2)P_- \\
&= P_-[P_\lambda,\Lambda_1]\Lambda_2P_-,
\end{aligned}
$$

and similarly, for P_+, we have $P_\lambda^\perp P_+ = P_+$, and $P_\lambda P_- = 0$, which implies

$$
\begin{aligned}
P_+T_\lambda P_+ &= -P_+P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2P_\lambda^\perp P_+ \\
&= -P_+P_\lambda^\perp\Lambda_1P_\lambda\Lambda_2P_+ \\
&= -P_+P_\lambda^\perp\Lambda_1\Lambda_2P_+ + P_+P_\lambda^\perp\Lambda_1P_\lambda^\perp\Lambda_2P_+ \\
&= -P_+P_\lambda^\perp\Lambda_1\Lambda_2P_+ + P_+\Lambda_1P_\lambda^\perp\Lambda_2P_+ \\
&= -P_+[P_\lambda^\perp,\Lambda_1]\Lambda_2P_+ \\
&= -P_+[(1-P_\lambda),\Lambda_1]\Lambda_2P_+ \\
&= P_+[P_\lambda,\Lambda_1]\Lambda_2P_+.
\end{aligned}
$$

Finally, we obtain

$$
\begin{aligned}
\sigma_B &= i\mathrm{Tr}(T_\lambda) \\
&= i\mathrm{Tr}(P_- T_\lambda P_-) + i\mathrm{Tr}(P_+ T_\lambda P_+) \\
&= i\mathrm{Tr}(P_-[P_\lambda, \Lambda_1]\Lambda_2 P_-) + i\mathrm{Tr}(P_+[P_\lambda, \Lambda_1]\Lambda_2 P_+) \\
&= \mathrm{Tr}(\Sigma_B),
\end{aligned}
$$

concluding the proof. □

Lemma 2. *E_a and E_a^* converge strongly to zero in the limit $a \to \infty$.*

Proof. Let $\psi \in \ell^2(\mathbb{Z}^2)$. Since E_a has real entries,

$$
\begin{aligned}
\|E_a^*\psi\|^2 &= \langle E_a^*\psi, E_a^*\psi\rangle \\
&= \sum_z \left(\sum_y \overline{E_a^*(z,y)\psi(y)}\right)\left(\sum_x E_a^*(z,x)\psi(x)\right) \\
&= \sum_z \left(\sum_y E_a(y,z)\overline{\psi(y)}\right)\left(\sum_x E_a(x,z)\psi(x)\right)
\end{aligned}
$$

Consider the $\sum_x E_a(x,z)\psi(x)$ term. To take advantage of the edge assumption, we insert the exponentials

$$
\sum_x E_a(x,z) e^{\alpha(|z_2-a|-|z_1-x_1|)} e^{-\alpha(|z_2-a|-|z_1-x_1|)} \psi(x).
$$

By assumption 1, $g(z) := \sup_x E_a(x,z)e^{\alpha(|z_2-a|-|z_1-x_1|)}$ is finite and summable. Precisely the same argument can be used on the sum over y. Altogether,

$$
\|E_a^*\psi\|^2 \leq e^{-\alpha|a|} \sum_z g(z)^2 e^{-2\alpha(|z_1|+|z_2|)} \sum_x e^{-\alpha|x_1|}\psi(x) \sum_y e^{-\alpha|y_1|}\overline{\psi(y)},
$$

where we bounded the exponential by $e^{-\alpha(|z_1|+|z_2|)}e^{-\alpha|x_1|}e^{-\alpha|a|}$. Since g is summable, g^2 is also summable, and so too is the summand over z. The sums over x and y are clearly finite, as they are bounded by the summable state $|\psi|$. Thus E_a^* converges strongly to zero. An exactly analogous argument applies for E_a.

□

3 Interacting Setting

Let $L \in \mathbb{N}$, and let $\Gamma_L = \mathbb{Z}_L \times [0, L]$ be the discrete cylinder, equipped with a metric d. To each site $x \in \Gamma_L$, we associate a Hilbert space $\mathcal{H}_x$ whose dimension is bounded uniformly in L. We denote $N = \sup_L \mathcal{H}_L$. For a subset $X \subseteq \Gamma_L$, we define the Hilbert space $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$, and we set

$$\mathcal{H}_L := \mathcal{H}_{\Gamma_L} = \bigotimes_{x \in \Gamma_L} \mathcal{H}_x.$$

The algebra $\mathcal{U}_X \subset \mathcal{B}(\mathcal{H}_X)$ of observables on $\mathcal{H}_X$ is the set of bounded self-adjoint operators supported in X. For an operator $A_X \in \mathcal{U}_X$, we identify its extension to an operator on $\mathcal{H}_L$ by taking its tensor product with copies of the identity, $(\otimes_{x \in X^c} \mathbb{I}_x) \otimes A_X$. Conversely, we say that an operator $A \in \mathcal{U}_L$ has support X if $A_X := (\otimes_{x \in X^c} \mathbb{I}_x) \otimes (A|_X)$ is equal to A, and write $A_X \in \mathcal{U}_X$. For ease of notation, we omit the subscript L wherever there is no risk of confusion.

A *local interaction* is a map $\Phi : \mathcal{P}(\Gamma_L) \to \mathcal{U}_L$ such that

1. $\Phi(X) = 0$ whenever $\mathrm{diam}(X) > R$ for some $R > 0$.
2. $\Phi(X)$ is supported in X.
3. $\|\Phi(X)\| \leq C$ for all $X \subset \Gamma_L$, for all L.

We consider a region as depicted in Figure 1, with the left and right edges joined together to form a cylinder. In the left white region $[0, L/2] \times [0, L]$, H_0 is a trivial Hamiltonian which we take to be empty space (we take $H_0 = 0$), and in the right blue region $[L/2, L] \times [0, L]$, H_1 is a *local Hamiltonian*, in the sense that $H_1 = \sum_{X \subseteq \Gamma_L} \Phi(X)$, is a sum of local interactions. We define the Hamiltonian of the full system to be

$$H_\mu = H_1 + \mu Q_h,$$

where $Q_h = \sum_{x \in \Gamma_h} a_x^* a_x$ is the number operator for the region $\Gamma_h = [L/4, 3L/4] \times [0, L]$ shown in red. This introduces a driving strength; the μQ_h term can be viewed as a potential difference $V(x)$.

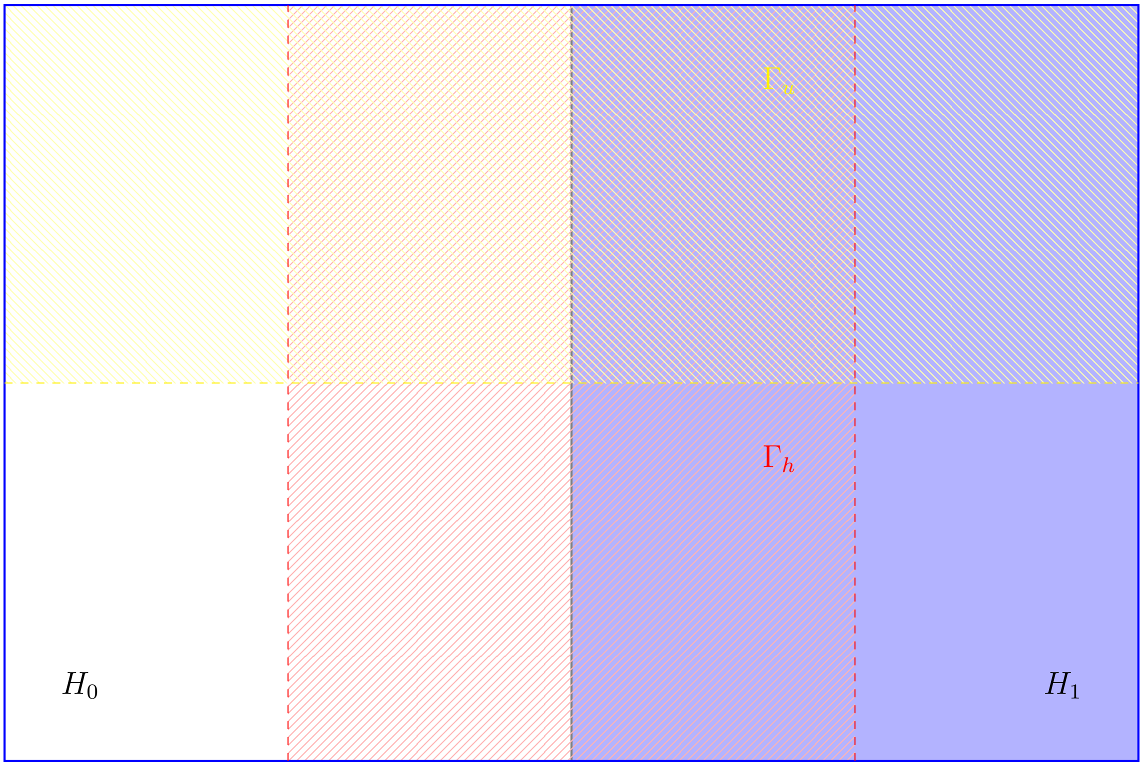

Figure 1: The cylinder Γ_L.

We also consider the plane $\mathbb{Z}^2$. In this setting, there are no edge states, and so the associated "bulk" Hamiltonian H_B is assumed to have a *gapped* spectrum, in the sense that

Assumption 3.
$$\mathrm{Spec}(H_B) = \mathcal{S}_- \cup \mathcal{S}_+,$$
where $\inf \mathcal{S}_+ - \sup \mathcal{S}_- \geq \gamma$ *uniformly in* L *and* μ *for some* $\gamma > 0$.

In the case of the cylinder, this effect does not necessarily occur due to the presence of the edge. We also assume that the Hamiltonian is *charge-conserving*.

Assumption 4. $[H_\mu, Q] = 0$, *where* Q *is the total charge in* Γ_L.

Let P_B be the ground state projection of H_B (the system without an edge), and let P be the ground state projection of H (the system with an edge). We assume that states far from the edge are essentially bulk states, up to tails that vanish quickly in L.

Assumption 5. *Define the* edge region

$$\Gamma_E = [L/2 - k, L/2 + k] \times [0, L] \cup [L - k, k] \times [0, L].$$

for some $k > 0$. For any operator A supported on Γ_E^{c},

$$\mathrm{Tr}(PA) = \mathrm{Tr}(P_B A) + \mathcal{O}(L^{-\infty}).$$

The A on the right hand side is understood to be the extension by zeroes of A to the plane $\mathbb{Z}^2$.

The idea is that observables localized far away from the edge are not affected by the edge of the system. We similarly define the *bulk region*

$$\Gamma_B = [3L/4 - k, 3L/4 + k] \times [0, L],$$

and the *middle region*

$$\Gamma_m = [L/2, L] \cup [0, L] \setminus (\Gamma_E \cup \Gamma_B).$$

The three regions are depicted in figure 2.

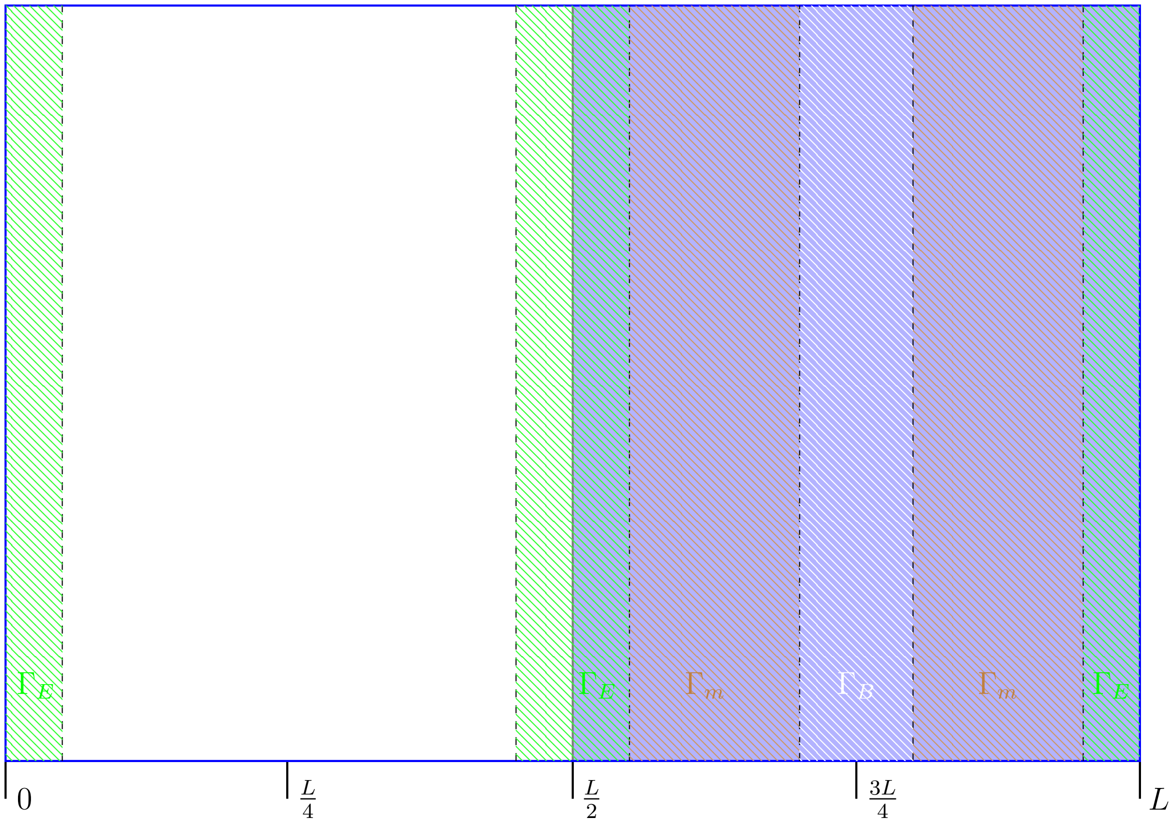

Figure 2: The regions Γ_E, Γ_B, and Γ_m.

3.1 Equality of Bulk and Edge Currents

3.1.1 Cylinder Geometry

Let P_μ be the (possibly degenerate) ground state projection of H_μ. Let $Q_u = \sum_{x\in\Gamma_u} a_x^* a_x$ be the charge in the upper half of the cylinder $\Gamma_u = [0, L]\times[L/2, L]$ (the yellow region in Figure 1), and define current operator

$$J = i[H_\mu, Q_u],$$

which measures the current across the fiducial line $y = L/2$. Charge conservation 4 implies that this current operator is supported along a strip of width $2R$ centred on the fiducial line $y = L/2$. Indeed, if we inspect a local interaction $\Phi(X)$ of range R with support $(\Gamma_u)_R$, where $(X)_\alpha$ is the α-shrinking of the set X, then clearly $\Phi(X)$ commutes with the charge outside Γ_u, so that $[\Phi(X), Q_u] = [\Phi(X), Q]$, which vanishes by the charge conservation assumption 4. Similarly, if $\Phi(X)$ is supported in $((\Gamma_u)^{\mathsf{c}})_R$, then $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$. It follows that for an interaction $\Phi(X)$ with range R and arbitrary support, $[\Phi(X), Q_u]$ must be supported on a set which is contained in (or equal to) the strip $[L/2, L] \times [L/2 - R, L/2 + R]$. There $[H_\mu, Q_u]$ must be supported there as well, since H_μ is a sum of such local interactions.

From this point, we drop the subscript μ wherever it is not needed for context.

Lemma 3. *The ground state expectation of the current J is zero.*

Proof. Assuming linearity and cyclicity of the trace hold, the proof is trivial,

$$\mathrm{Tr}(PJ) = i\mathrm{Tr}(P[H, Q_u]) = i\mathrm{Tr}([P, H]Q_u) = 0.$$

In order for this calculation to hold, we need to prove that

1. PHQ_u and PQ_uH are separately trace-class to apply linearity of the trace, and
2. $\|H\| < \infty$ and $PQ_u \in \mathcal{J}_1$ to apply cyclicity of the trace.

The latter implies the former by the bound $\|AB\|_1 \leq \|A\|_1 \|B\|$. To prove (2), fix a finite L. The Hamiltonian is bounded since it is a finite sum of at most $\mathcal{P}(\Gamma_L)$ local interactions $\Phi(X)$, each of which is uniformly bounded by assumption, along with the μQ_h term. But the number operator for the entire space is bounded by $\|Q\| \leq NL^2$, where N is the uniform bound on the dimension of each Hilbert space. This shows that both Q_u and Q_h are bounded in operator norm. Finally, $\|P\|_1 \leq CL^2$ because the projection is finite-rank, since the dimension of each site is bounded. Therefore $PQ_u \in \mathcal{J}_1$. $\square$

Next, we define a family of operators indexed by μ called *Hastings operators*,

$$K_\mu = \mathcal{I}_\mu(\dot{H}_\mu),$$

where

$$\mathcal{I}_\mu(A) = \int_{\mathbb{R}} W(t) e^{itH_\mu} A e^{-itH_\mu} dt.$$

Here, $W : \mathbb{R} \to \mathbb{R}$ is a function satisfying (need to add). More explicitly, in our setting we see that

$$K_\mu = \mathcal{I}_\mu(Q_h).$$

We present two important properties of the map $\mathcal{I}_\mu : \mathcal{U}_L \to \mathcal{U}_L$ in the following lemmas, and leave their proofs to the appendix (need to add).

First, recall a definition from the non-interacting setting: an *off-diagonal* operator is an operator A such that $A = \overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, where $P_\mu^\perp = \mathbb{I} - P_\mu$ is the projection onto the excited states above the gap.

Lemma 4. *1. For any off-diagonal operator $A = \overline{A}$, $\mathcal{I}_\mu(\cdot)$ and $[H_\mu, \cdot]$ act as inverses of each other, up to a factor of i:*

$$\mathcal{I}_\mu\left([H_\mu, A]\right) = [H_\mu, \mathcal{I}_\mu(A)] = iA.$$

2. For any (not necessarily off-diagonal) operator A,

$$[\mathcal{I}_\mu([H_\mu, A]), P_\mu] = i[A, P_\mu].$$

Another important property of the map $\mathcal{I}_\mu$ is that it preserves locality.

Lemma 5. *$\mathcal{I}_\mu$ is local in the sense that for any $A \in \mathcal{U}_X$,*

$$\|\mathcal{I}(A)_{(X^r)^c}\| \le \|A\||X|\mathcal{O}(r^{-\infty})$$

where $X^r = X \cup \{x : d(x, X) \le r\}$ is the r-fattening of X.

Proposition 2. *The operator K_μ is the* generator of parallel transport*, satisfying*

$$\dot{P}_\mu = i[K_\mu, P_\mu]$$

for all μ.

Proof. First, we show that $\dot{P}$ is off-diagonal. Taking the derivative on both sides of $P^2 = P$, we see that $\dot{P}P + P\dot{P} = \dot{P}$. Acting on the left and right with P on both sides of this equation gives

$$P\dot{P}P + P\dot{P}P = P\dot{P}P,$$

which implies that $P\dot{P}P = 0$. Thus

$$\begin{aligned}
\overline{\partial_\mu P} &= P\dot{P}(1-P) + (1-P)\dot{P}P \\
&= P\dot{P} - P\dot{P}P + \dot{P}P - P\dot{P}P \\
&= P\dot{P} + \dot{P}P \\
&= \partial_\mu(P^2) \\
&= \partial_\mu P,
\end{aligned}$$

as claimed. By the product rule and the fact that H and P commute,

$$[\dot{H}, P] = -[H, \dot{P}].$$

It therefore follows from Lemma 4 that

$$\dot{P} = -i\mathcal{I}_\mu([H, \dot{P}]) = i\mathcal{I}([\dot{H}, P]) = i[\mathcal{I}(\dot{H}), P] = i[K, P].$$

□

Increasing the electric potential by a small amount $d\mu Q_h$ and expanding to linear order, the change in ground state current is given by

$$\mathrm{Tr}(P_{\mu+d\mu}J) - \mathrm{Tr}(P_\mu J) = \kappa d\mu + \mathcal{O}(d\mu^2).$$

Dividing by $d\mu$ and taking a limit, we see that the linear response coefficient is given by

$$\sigma(\mu) = \mathrm{Tr}\left(\dot{P}_\mu J\right).$$

The *Hall conductivity* of the system on a subset $V \subseteq \Gamma_L$ is defined to be $\sigma_V := \mathrm{Tr}\left(\dot{P}J_V\right)$, where J_V is the restriction of J to V.

Proposition 3. *The Hall conductivity is independent of the driving strength μ.*

Proof. For any μ_1 and μ_2,

$$\begin{aligned}
\sigma(\mu_1) - \sigma(\mu_2) &= \mathrm{Tr}\left(\dot{P}_{\mu_1} i[H_{\mu_1}, Q_u] - \dot{P}_{\mu_2} i[H_{\mu_2}, Q_u]\right) \\
&= i\mathrm{Tr}\left(\left([\dot{P}_{\mu_1}, H_{\mu_1}] - [\dot{P}_{\mu_2}, H_{\mu_2}]\right) Q_u\right) \\
&= -i\mathrm{Tr}\left(\left([\dot{H}_{\mu_1}, P_{\mu_1}] - [\dot{H}_{\mu_2}, P_{\mu_2}]\right) Q_u\right) \\
&= i\mathrm{Tr}\left([Q_h, P_{\mu_1} - P_{\mu_2}]Q_u\right) \\
&= i\mathrm{Tr}\left([Q_u, Q_h](P_{\mu_1} - P_{\mu_2})\right) \\
&= 0,
\end{aligned}$$

since H and P commute. Note that $\|\dot{P}\|_1 < \infty$ since we are working in a finite-dimensional space. The proof of Lemma 3 provides the other necessary bounds to invoke linearity and cyclicity of the trace to shift the commutator in the second line and second-last line. $\square$

This indicates that the Hall conductivity is independent of μ as one would expect physically. We simply write $\sigma = \sigma(\mu)$ from this point, in accordance with proposition 3.

The following is the main result:

Theorem 3. *Let $V \subseteq \Gamma_m$ be a set contained within the strip in between the edge region Γ_E and the bulk region Γ_B (see Figure 2), and define the distance*

$$r = \operatorname{dist}(V, \Gamma_E \cup \Gamma_B)$$

from V to the bulk and edge regions. The Hall conductivity in this regions vanishes in the sense that

$$\sigma_V = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

Proof. By Proposition 2, the bulk Hall conductivity can also be written by the formula

$$\sigma_V^B = \operatorname{Tr}\left(i[K, P_B]J_V^B\right) = \operatorname{Tr}\left(i[\mathcal{I}(Q_h), P_B]J_V^B\right),$$

where $J_V^B = i[H_B, Q_u]|_V$ is the current in the region V arising from the bulk Hamiltonian. From commutativity of P_B and H_B along with cyclicity of the trace, we compute

$$\begin{aligned}
\sigma_V^B &= \operatorname{Tr}\left(i\int_{\mathbb{R}} W(t)e^{itH_B}[Q_h, P_B]e^{-itH_B}dt J_V^B\right) \\
&= \int_{\mathbb{R}} W(t)\operatorname{Tr}\left(i[Q_h, P_B]e^{-itH_B}J_V^B e^{itH_B}\right)dt \\
&= -\int_{\mathbb{R}} W(t)\operatorname{Tr}\left(i[Q_h, P_B]e^{itH_B}J_V^B e^{-itH_B}\right)dt \\
&= -\operatorname{Tr}\left(i[Q_h, P_B]\mathcal{I}(J_V^B)\right),
\end{aligned}$$

since $W(t)$ is odd. By part (2) of Lemma 4, we have $i[Q_h, P_B] = [\mathcal{I}([H_B, Q_h]), P_B]$. Therefore

$$\begin{aligned}
\sigma_V^B &= -\operatorname{Tr}([\mathcal{I}([H_B, Q_h]), P_B]\mathcal{I}(J_V^B)) \\
&= \operatorname{Tr}\left(P_B[\mathcal{I}([H_B, Q_h]), \mathcal{I}(J_V^B)]\right).
\end{aligned}$$

Now, $[H_B, Q_h]$ is a local operator supported on Γ_B, while J_V^B is a local operator supported on $V \cap \Gamma_B = \varnothing$. Since $\mathcal{I}$ preserves locality up to tails, in the sense that $\|\mathcal{I}(A)_{(S^r)^c}\| \leq \|A\||S|\mathcal{O}(r^{-\infty})$ for any operator A supported in S (Lemma 5), it follows that the commutator can be written

$$[\mathcal{I}([H_B, Q_h])|_{\Gamma_B} + \mathcal{O}(r^{-\infty})A_1, \mathcal{I}(J_V^B)|_V + \mathcal{O}(r^{-\infty})A_2] = C\mathcal{O}(r^{-\infty}),$$

for some operators A_1 and A_2 supported on Γ_B^c and V^c, respectively. This fact applies to the bulk setting with H_B and P_B. To extend this to the setting with an edge, it is enough to use Assumption 5 to conclude the same result, except with equality up to $\mathcal{O}(L^{-\infty})$, i.e.

$$\sigma_V = \mathrm{Tr}\left(\dot{P}J_V\right) = \mathrm{Tr}\left(\dot{P}(J_V^B + \mathcal{O}(L^{-\infty}))\right) = \sigma_V^B + \mathcal{O}(L^{-\infty}) = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

$\square$

The intuitive picture from the previous result is that, in the bulk region, the Hall conductivity is essentially only nonzero along the bulk strip Γ_B. Since the ground state expectation of the current is zero (by lemma 3), it must be that there is an equal current flowing along the edge strip Γ_E, but in the opposite direction.

3.1.2 Torus Geometry

Our goal is to show the same result on the discrete torus $\mathbb{T}_L := \mathbb{Z}_L \times \mathbb{Z}_L$. We define the same regions Γ_u and Γ_h, and the same current operator $J_u = i[H(\mu), Q_u]$. This time, however, Lemma 3 does not apply. Intuitively, it does not apply because electrons can now flow through both the bottom and the top of the region Γ_u, rather than just the bottom. Mathematically, the lemma fails because our definition of the current is slightly changed.

We use charge conservation and the fact that H is finite range to split the current J_u into two components, $J_u = i[H_-, Q_u] + i[H_+, Q_u] = J_- - J_+$, supported on strips of width $2R$ at $y = L/2$ and $y = L$, respectively. We then define the current operator to be $J = J_-$, which is the current on the lower strip. This is the mathematical reason that the proof in Lemma 3 fails on the torus; we have replaced H by H_-, which may no longer commute with P. We instead proceed by a different approach. We will need a few auxiliary results first.

Lemma 6. *$K_\pm$ is supported on $\partial_\pm$ up to tails.*

Proof. □

Proposition 4. *The operator $Q_h - K$ leaves the ground state space invariant, i.e. $[Q_h - K, P] = 0$.*

Proof. □

Lemma 7. *Show that $\mathrm{Tr}(A, [Q_h, P]) = 0$ for all $A \in \mathcal{U}_{edge}$. This shows that Q_h commutes with P "along the edge".*

Proof. Let $A \in \mathcal{U}_{\text{edge}}$. Since H is charge conserving, we may choose a simultaneous eigenbasis of H and the total charge Q, in which case P and Q commute. It follows that

$$\mathrm{Tr}(A[Q_h, P]) = \mathrm{Tr}([A, Q_h]P) = \mathrm{Tr}([A, Q]P) = \mathrm{Tr}(A[Q, P]) = 0.$$

□

Finally, we will prove that in the bulk system with Hamiltonian $H_B(\mu)$, the ground state expectation of the current vanishes faster than any power as $L \to \infty$.

Lemma 8. *The ground state expectation of the current $J_B := i[(H_B)_-, Q_h]$ (of the system without an edge) is $\mathrm{Tr}(P_B J_B) = \mathcal{O}(L^{-\infty})$.*

Proof. First, $K = \mathcal{I}(i[H_B, Q])$ splits into $K = K_- - K_+$, with the support of $K_\pm$ contained in $\partial_\pm$ up to tails:

$$[K_\pm, A_X] = \mathcal{O}(p^{-\infty}),$$

for every $A_X \in \mathcal{U}_X$ such that $\|A_X\| = 1$, and where $p = \mathrm{dist}(X, \partial_\pm)$ (need to add). Using the fact that $K_\pm$ is supported in $\partial_\pm$ up to tails (Lemma 6), we see that

$$i[H_B, K_-] = i[(H_B)_-, K_-] + \mathcal{O}(L^{-\infty}),$$

and similarly $i[(H_B)_-, K_+] = \mathcal{O}(L^{-\infty})$. Putting these facts together, it follows that the current can be rewritten as

$$\begin{aligned} J_B &= i[H_B, Q_h + K_- - K_- + K_+] + \mathcal{O}(L^{-\infty}) \\ &= i[H_B, K_-] + i[(H_B)_-, Q_h - K_- + K_+)] + \mathcal{O}(L^{-\infty}). \end{aligned}$$

From here, we use the fact that H_B and $Q_h - K_- + K_+$ both commute with P_B to write

$$P_BJ_BP_B = i[H_B, P_BK_-P_B] + i[P_B(H_B)_-P_B, Q_h - K_- + K_+)] + P_B\mathcal{O}(L^{-\infty})P_B.$$

Since the trace of any commutator is zero,

$$\mathrm{Tr}(P_BJ_B) = \mathrm{Tr}(P_BJ_BP_B) = \mathcal{O}(L^{-\infty}).$$

□

Using this, we can show a simple proof of the analogue of Lemma 3 on the torus, in the case of non-interacting systems.

Proposition 5. *Let $H = \sum_{x\in\mathbb{T}} h_x$ be a non-interacting Hamiltonian, i.e. a sum of single site Hamiltonians h_x. The ground state expectation of the current $J = i[H_-, Q_h]$ (of the system with an edge) is $\mathrm{Tr}(PJ) = \mathcal{O}(L^{-\infty})$.*

Proof. Since H is a sum of single site Hamiltonians, we can split H_- into the restrictions $H_- = (H_-)_{\text{edge}} + (H_-)_{\text{bulk}}$, with no fear of any terms which are in both the edge region and the bulk region. By Assumption 5,

$$\begin{aligned}
\mathrm{Tr}(PJ) &= \mathrm{Tr}(Pi[H_-, Q_h]) \\
&= i\mathrm{Tr}([H_-, Q_h]P) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\text{bulk}}[Q_h, P]) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\text{bulk}}[Q_h, (P)_{\text{bulk}}]) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i\mathrm{Tr}((H_B)_-[Q_h, P_B]) + \mathcal{O}(L^{-\infty}) \\
&= i\mathrm{Tr}((H_-)_{\text{edge}}[Q_h, P]) + \mathrm{Tr}(i[(H_B)_-, Q_h]P_B) + \mathcal{O}(L^{-\infty}).
\end{aligned}$$

By Lemma 7, the first term is zero. By Lemma 8, the second term is $\mathcal{O}(L^{-\infty})$. □

A Properties of $\mathcal{I}_\mu$

Proof. (Of Lemma 4). Let $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} W(t) e^{-2\pi i t \xi} dt$ be the Fourier transform of W. One can show that for $|\xi| \geq \gamma$, $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi} i \xi}$ (need to add). Let A be an observable. First, we show that $\mathcal{I}([H, PAP^\perp]) = iPAP^\perp$.

Decomposing

$$
\begin{aligned}
e^{itH} P &= \sum_{j=0}^{\infty} \frac{(itH)^j}{j!} P \\
&= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left(\sum_n E_n^j P_n \right) P \\
&= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \sum_{n: E_n = 0} E_n^j P_n \\
&= \sum_{n: E_n = 0} e^{itE_n} P_n,
\end{aligned}
$$

and similarly

$$
P^\perp e^{-itH} = \sum_{m: E_m \geq \gamma} P_m e^{-itE_m},
$$

we see that

$$
\begin{aligned}
\mathcal{I}([H, PAP^{\perp}]) &= \mathcal{I}(P[H,A]P^{\perp}) \\
&= \int_{\mathbb{R}} W(t) e^{itH} P[H,A]P^{\perp} e^{-itH} dt \\
&= \int_{\mathbb{R}} W(t) \sum_{n:E_n=0} e^{itE_n} P_n [H,A] \sum_{m:E_m \geq \gamma} P_m e^{-itE_m} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} \int_{\mathbb{R}} W(t) e^{itE_n} P_n A (E_n - E_m) P_m e^{-itE_m} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \int_{\mathbb{R}} W(t) e^{-it(E_m - E_n)} dt \\
&= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \sqrt{2\pi} \widehat{W}(E_m - E_n) \\
&= i \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m \\
&= i P A P^{\perp}.
\end{aligned}
$$

(need to check the 2π factor)

By the same argument, $\mathcal{I}([H, P^{\perp}AP]) = iP^{\perp}AP$ as well, and so $\mathcal{I}([H, \overline{A}]) = i\overline{A}$. □

Proof. (Of Lemma 5). We break the integral into two parts,

$$
\|\mathcal{I}(A)\| \leq \left\| \int_{-T}^{T} W(t) e^{itH} A e^{-itH} dt \right\| + \left\| \int_{\mathbb{R}\setminus[-T,T]} W(t) e^{itH} A e^{-itH} dt \right\|.
$$

The first term can be estimated using the Lieb-Robinson bound found in Appendix B.

□

B Lieb-Robinson Bound

Let N be a uniform upper bound for the dimensions of the Hilbert spaces at each site, i.e. $\dim(\mathcal{H}_x) \leq N$ for all sites x.

The following is a version of the Lieb-Robinson. For any operators $A \in \mathcal{U}_X$ and $B \in \mathcal{U}_Y$ having disjoint supports $X \cap Y = \varnothing$,

$$
\|[e^{itH} A e^{-itH}, B]\| \leq C \|A\| \|B\| |X| |Y| N^{2|X|} e^{2t\|\Phi\|_{\lambda} - \lambda d(X,Y)}.
$$

C Grönwall's Inequality and Uniqueness

Theorem 4. *(Grönwall's Inequality). Let $\alpha : I \to (0, \infty)$ be positive and continuous on I^o for some interval of the form $[a, b)$, $[a, b]$, or $[a, \infty)$. Suppose $u : \mathbb{R} \to \mathcal{U}$ is a Banach-valued, differentiable function. If $\|u'(t)\| \leq \alpha(t)\|u(t)\|$ for all $t \in I$, then*

$$\|u(t)\| \leq \|u(a)\| e^{\int_a^t \alpha(s) ds} \quad \forall t \in I$$

Proof. Let $f(t) = e^{\int_a^t \alpha(s) ds}$, which is nonzero, has initial value $f(a) = 1$, and has derivative $f'(t) = \alpha(t) f(t)$. Then by the quotient rule,

$$\left(\frac{\|u(t)\|}{f(t)} \right)' = \frac{\|u'(t)\| f(t) - \|u(t)\| \alpha(t) f(t)}{f(t)^2} \leq 0,$$

where the inequality follows from the assumption $\|u'(t)\| \leq \|\alpha(t) u(t)\|$. Thus $\frac{\|u(t)\|}{f(t)}$ is decreasing, so that

$$\frac{\|u(t)\|}{f(t)} \leq \frac{\|u(a)\|}{f(a)} = \|u(a)\|,$$

which is the desired inequality. □

Theorem 5. *(ODE Uniqueness). Let $F : \mathcal{U} \to \mathcal{U}$ be Lipschitz and consider the differential equaton $u'(t) = F(u(t))$ with initial condition $u(a) = u_a$ for some function $u : I \to \mathcal{U}$, where $I = [a, b]$, or $[a, b)$, or $[a, \infty)$. Solutions to this equation are unique.*

Proof. Suppose there are two solutions $u(t)$ and $v(t)$, and let $g(t) = \|u(t) - v(t)\|^2$. By assumption, there exists a constant L_F such that $\|F(u(t)) - F(v(t))\| \leq L_F \|u(t) - v(t)\|$, so that

$$\begin{aligned} g'(t) &= 2\|u(t) - v(t)\| \|u'(t) - v'(t)\| \\ &= 2\|u(t) - v(t)\| \|F(u(t)) - F(v(t))\| \\ &\leq 2L_F \|u(t) - v(t)\|^2 \\ &= 2L_F g(t). \end{aligned}$$

Notice that $\alpha := 2L_F$ is a positive continuous function, so we may apply Grönwall's inequality to $g(t)$ to conclude

$$g(t) \leq g(a) e^{2L_f(t-a)} = 0,$$

since $g(a) = 0$. □

D Note on Generators of Parallel Transport

Consider the differential equation $\dot{\rho}(\mu) = i[K_B, \rho(\mu)]$ with initial condition $\rho(0) = P_B(0)$. Here $K_B = \int_{\mathbb{R}} W_\gamma(t) e^{-itH_B} \dot{H}_B e^{itH_B} dt$, and recall that in our setting, $\dot{H}_B = Q_h$. We know that the solution is $\rho(\mu) = P_B(\mu)$ (proposition 2). Notice that the map $F : \mathcal{U} \to \mathcal{U}$ defined by $F(A) = i[K_B, A]$ is Lipschitz, since

$$\|F(A) - F(B)\| = \|[K_B, A - B]\| \leq 2\|K_B\|\|A - B\|.$$

The Lipschitz constant is $2\|K_B\|$, which is finite since K_B is a bounded operator:

$$\|K_B\| \leq \int_{\mathbb{R}} |W_\gamma(t)| \|e^{-itH_B} Q_h e^{itH_B}\| dt \leq \int_{\mathbb{R}} |W_\gamma(t)| dt \|Q_h\|.$$

Indeed, since Q_h is the number operator on a finite volume, by charge conservation and the fact that the dimension of the Hilbert space is bounded uniformly by d, there can only be a finite number of charges in the region Γ_h.

Thus, by Grönwall's uniqueness theorem (appendix C), we see that the solution to the equation $\dot{\rho}(\mu) = F(\rho(\mu)) = i[K_B, \rho(\mu)]$ is unique.

Now define

$$K_E := \int_{\mathbb{R}} W_\gamma(t) e^{-itH_E} Q_h e^{itH_E} dt,$$

which is using the gap γ of H_B to define W_γ, but also using the edge Hamiltonian in the time evolution operators. Consider $\sigma : [0, \infty) \to \mathcal{U}$ defined by

$$\dot{\sigma}(\mu) = i[K_E, \sigma(\mu)] \qquad \sigma(0) = P_E(0).$$

We now show that, similar to how ρ is an approximation of P_B, σ is also a good approximation of P_E (up to $\mathcal{O}(L^{-\infty})$) "in the bulk". Let $A \in \mathcal{U}_{\Gamma_B}$ be an operator localized in the bulk of the edge system. Then

$$
\begin{aligned}
\operatorname{Tr}(\dot\sigma A) &= \operatorname{Tr}(i[K_E,\sigma]A)\\
&= \operatorname{Tr}(i[A,K_E]\sigma)\\
&= \int_{\mathbb{R}} W_\gamma(t)\operatorname{Tr}([e^{-itH_E}Q_h e^{itH_E},A]\sigma)dt\\
&= \int_{\mathbb{R}} W_\gamma(t)\operatorname{Tr}(e^{-itH_E}[Q_h, e^{itH_E}Ae^{-itH_E}]e^{itH_E}\sigma)dt\\
&= \int_{\mathbb{R}} W_\gamma(t)\operatorname{Tr}(e^{-itH_E}[Q_h, e^{itH_B}Ae^{-itH_B}]e^{itH_E}+\mathcal{O}(L^{-\infty})\sigma)dt\\
&= \int_{\mathbb{R}} W_\gamma(t)\operatorname{Tr}(e^{-itH_B}[Q_h, e^{itH_B}Ae^{-itH_B}]e^{itH_B}+\mathcal{O}(L^{-\infty})\sigma)dt\\
&= \int_{\mathbb{R}} W_\gamma(t)\operatorname{Tr}([e^{-itH_B}Q_h e^{itH_B},A]\sigma)dt+\mathcal{O}(L^{-\infty})\\
&= \operatorname{Tr}(i[A,K_B]\sigma]+\mathcal{O}(L^{-\infty})\\
&= \operatorname{Tr}(i[K_B,\sigma]A)+\mathcal{O}(L^{-\infty}),
\end{aligned}
$$

since σ is trace-class (?) and $W_\gamma \in L^1$. By linearity of the trace, we see that $\operatorname{Tr}((\dot\sigma - i[K_B,\sigma])A) = \mathcal{O}(L^{-\infty})$ for any operator $A \in \Gamma_B$ (does this mean $\dot\sigma - i[K_E,\sigma] = 0$?). But the solution of $\dot\sigma - i[K_B,\sigma] = 0$ (with initial condition $\sigma(0) = P_B(0)$) is unique; it is $\rho(\mu)$, or $P_B(\mu)$. Hence

$$
\operatorname{Tr}(P_E A) = \operatorname{Tr}(P_B A) + \mathcal{O}(L^{-\infty}) = \operatorname{Tr}(\rho A) + \mathcal{O}(L^{-\infty}) = Tr(\sigma A) + \mathcal{O}(L^{-\infty})
$$

for any operator $A \in \Gamma_B$. In particular, this gives another local formula for the Hall conductivity in the bulk of an edge system, by taking $A = J_V$, where J is the current operator and $V \subset \Gamma_B$ is a set localized in the bulk. The Hall conductivity is given by $\operatorname{Tr}(\dot P_E J_V)$, and this can be approximated by

$$
\operatorname{Tr}(\dot P_E J_V) = \operatorname{Tr}(\dot P_B J_V) + \mathcal{O}(L^{-\infty}) = \operatorname{Tr}(\dot\rho J_V) + \mathcal{O}(L^{-\infty}) = \operatorname{Tr}(\dot\sigma J_V) + \mathcal{O}(L^{-\infty}).
$$

Want to pick a norm s.t. Gronwall gives $\|\rho(\mu) - \sigma(\mu)\|_G \le \|P_B(0) - P_E(0)\|_G e^{2L_F\mu}$. Need $\|P_B(0) - P_E(0)\|_G$ to be small enough to kill the exponential which depends on $L_F = 2\|K_B\|_G \le \|W_\gamma\|_{L^1}\|Q_h\|_G$. If we use the operator norm for $\|\cdot\|_G$, we would get $\|Q_h\|_G = d|\Gamma_h|$ in the exponent. Need $\|\cdot\|_G$ to be an actual norm so that $\|\rho - \sigma\|_G = 0 \implies \rho = \sigma$.

From Dec 13 Meeting

Let $r(t) = \rho(t) - \sigma(t)$. Notice that

$$\frac{d}{dt}e^{itK_B}\sigma_0 e^{-itK_B} = e^{itK_B}i[K_B, \sigma_0]e^{-itK_B} + e^{-itK_B}\dot{\sigma_0}e^{itK_B}.$$

E The Helffer-Sjöstrand Representation

The Helffer-Sjöstrand representation is a functional calculus $f \mapsto f(H)$ for arbitrary (possibly unbounded) operators H on the set of functions

$$\mathcal{A} = \bigcup_{\beta<0}\{f : \mathbb{R} \to \mathbb{C} \ : \ f \in C^\infty(\mathbb{R}), \ |f^{(n)}(x)| \leq c_n(1+x^2)^{\frac{\beta-n}{2}}\}.$$

It has the following properties.

Theorem 6. *For any $f \in \mathcal{A}$,*

1. *$f \mapsto f(H)$ is an algebraic homomorphism (linear and multiplicative).*
2. *$\overline{f}(H) = f(H^*)$.*
3. *$\|f(H)\| \leq \|f\|_\infty$.*
4. *For all $w \notin \mathbb{R}$, if $r_w(s) = \frac{1}{s-w}$ then $r_w(H) = (H-z)^{-1}$.*
5. *For all $f \in C_c^\infty(\mathbb{R})$ with $supp(f) \cap Spec(H) = \varnothing$, we have $f(H) = 0$.*

There is an explicit formula for $f(H)$, which is given by

$$f(H) = \frac{1}{2\pi}\int_{\mathbb{C}} \frac{\partial\tilde{f}}{\partial\bar{z}}(H-z)^{-1}dz \wedge d\bar{z},$$

where $\tilde{f} : \mathbb{C} \to \mathbb{C}$ is a *quasi-analytic extension* of $f : \mathbb{R} \to \mathbb{R}$. It is defined as follows. For any smooth f, we set

$$\tilde{f}(z) = \sum_{r=0}^{n} \tau\left(\frac{y}{(1+x^2)^{1/2}}\right)\frac{(iy)^r}{r!}f^{(r)}(x)$$

where $\tau : \mathbb{R} \to \mathbb{R}$ is any smooth function satisfying

$$\tau(s) = \begin{cases} 1 & |s| < 1 \\ 0 & |s| > 2 \end{cases}.$$

The extension turns out to be independent of the choice of n and τ. Furthermore, as $|\mathrm{Im}(z)| \to 0$, the Wirtinger derivative of the extension obeys the bound

$$\left|\frac{\partial \tilde{f}}{\partial \bar{z}}\right| = \mathcal{O}(|y|^n).$$

Thus $\frac{\partial \tilde{f}}{\partial \bar{z}} = 0$ for all real z, which is why $\tilde{f}$ is called a "quasi"-analytic extension (the Wirtinger derivative would be zero everywhere were $\tilde{f}$ analytic).

A crucial property of the Helffer-Sjöstrand functional calculus is the following bound. For any $n \in \mathbb{N}$, the quasi-analytic extension $\tilde{f}$ can be chosen so that

$$\int_{\mathbb{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} \frac{1}{|\mathrm{Im}(z)|^{p+1}} dz \wedge d\bar{z} \leq C_0 \sum_{k=0}^{n+2} \|f^{(k)}\|_{k-p-1},$$

where the norms on the right hand side are defined by

$$\|f\|_m = \int_{-\infty}^{\infty} |f(x)|(1+x^2)^{m/2} dx.$$

This is often useful because the resolvent obeys the bound $\|(H-z)^{-1}\| \leq |\mathrm{Im}(z)|^{-1}$.

F Spectral Measures and Projection-Values Measures

Projection-valued measures are maps $P : \mathcal{M} \to \mathcal{B}(\mathcal{H})$ from measurable subsets of $\mathbb{R}$ to the space of bounded linear operators on $\mathcal{H}$ satisfying the usual properties of both projections and measures.

1. $P(M) = P(M)^* = P(M)^2$ is an orthogonal projection for all $M \in \mathcal{M}$. Note that this implies that $P(M)$ is a positive operator.

2. $P(\varnothing) = 0$ and $P(\mathbb{R}) = \mathbb{1}_{\mathcal{H}}$.

3. If $\{M_i\}_{i\in\mathbb{N}}$ are pairwise disjoint, then $\sum_{i=1}^{n} P(M_i) \xrightarrow{s} P(\cup_{i\in\mathbb{N}} M_i)$ as $n \to \infty$ (σ-additivity).

4. $P(M_1 \cap M_2) = P(M_1)P(M_2)$ for any $M_1, M_2 \in \mathcal{M}$.

The heuristic motivation is that $P(M)$ projects onto the subspace spanned by states whose energies lie in M. Using these operator-valued measures, one can construct an operator-valued integral with respect to P in the usual fashion (beginning on nonnegative simple functions, extending to nonnegative measurable functions, and finally to real-valued measurable functions).

Theorem 7 (Spectral Theorem for Projection-Valued Measures)**.** *There exists a one-to-one correspondence between self adjoint operators H and projection-valued measures P given by the formula*

$$H = \int_{\mathbb{R}} \lambda dP_\lambda,$$

where $P_\lambda := P((-\infty, \lambda])$. Moreover, if $g : \mathbb{R} \to \mathbb{R}$ is any bounded Borel function, then $g(H)$ defined via the Borel function calculus coincides with the formula

$$g(H) = \int_{\mathbb{R}} g(\lambda) dP_\lambda,$$

and $g(H) = g(H)^*$.

We remark that it follows from the second part of this theorem that if $\mathbb{1}_M$ denotes the characteristic function of a Borel set $M \subseteq \mathbb{R}$, then

$$\mathbb{1}_M(H) = \int_M dP_\lambda = P(M).$$

We also note that $\mathrm{Spec}(H) = \mathrm{supp}(P)$.