Thesis Rough Work

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1 Setting

Let $L \in \mathbb{N}$, and let $\Lambda_L = \mathbb{Z}_L \times [0, L]$ be the discrete cylinder, equipped with a metric d. To each site $x \in \Lambda_L$, we associate a Hilbert space $\mathcal{H}_x$ whose dimension is bounded uniformly in L, i.e. there exists some $N > 0$ such that for all $L \in \mathbb{N}$, we have $\dim(\mathcal{H}_x) \leq N$ for all $x \in \Lambda_L$. For a subset $X \subseteq \Lambda_L$, we define the Hilbert space $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$, and we set $\mathcal{H}_L := \mathcal{H}_{\Lambda_L} = \otimes_{x \in \Lambda_L} \mathcal{H}_x$. For simplicity, throughout we take $L = 2^n$ for some large n.

The algebra $\mathcal{U}_L \subset \mathcal{B}(\mathcal{H}_L)$ of observables on Λ_L is the set of bounded operators on $\mathcal{H}_L$ which are self-adjoint. For an operator $A_X \in \mathcal{U}_X$, we identify its extension to $\mathcal{U}_L$ by taking its tensor product with copies of the identity, $A_X \otimes_{x \in X^c} \mathbb{I}_x$. Conversely, we say that an operator $A_X \in \Lambda_L$ has support X if $A_X = A_X|_X \otimes_{x \in X^c} \mathbb{I}_x$, and write $A_X \in \mathcal{U}_X$.

A *local interaction* is a map $\Phi : \mathcal{P}(\Lambda_L) \to \mathcal{U}_L$ such that $\Phi(X) = 0$ whenever $\operatorname{diam}(X) > R$, $\Phi(X)$ is supported in X, and $\|\Phi(X)\| \leq C$ for all $X \subset \Lambda_L$, for all L.

We consider a region as depicted in Figure 1. In the left white region $[0, L/2] \times [0, L]$, H_0 is a trivial Hamiltonian which we take to be the void (for example $H_0 = 0$), and in the right blue region $[L/2, L] \times [0, L]$, H_1 is a nontrivial *local Hamiltonian*, in the sense that $H_1 = \sum_{X \subseteq \Lambda} \Phi(X)$, where Φ is a local interaction. We define the Hamiltonian of the full system to be

$$H(\mu) = H_1 + \mu Q_h,$$

where $Q_h = \sum_{x \in \Gamma_h} a_x^* a_x$ is the number operator for the region $\Gamma_h = [L/4, 3L/4] \times [0, L]$.
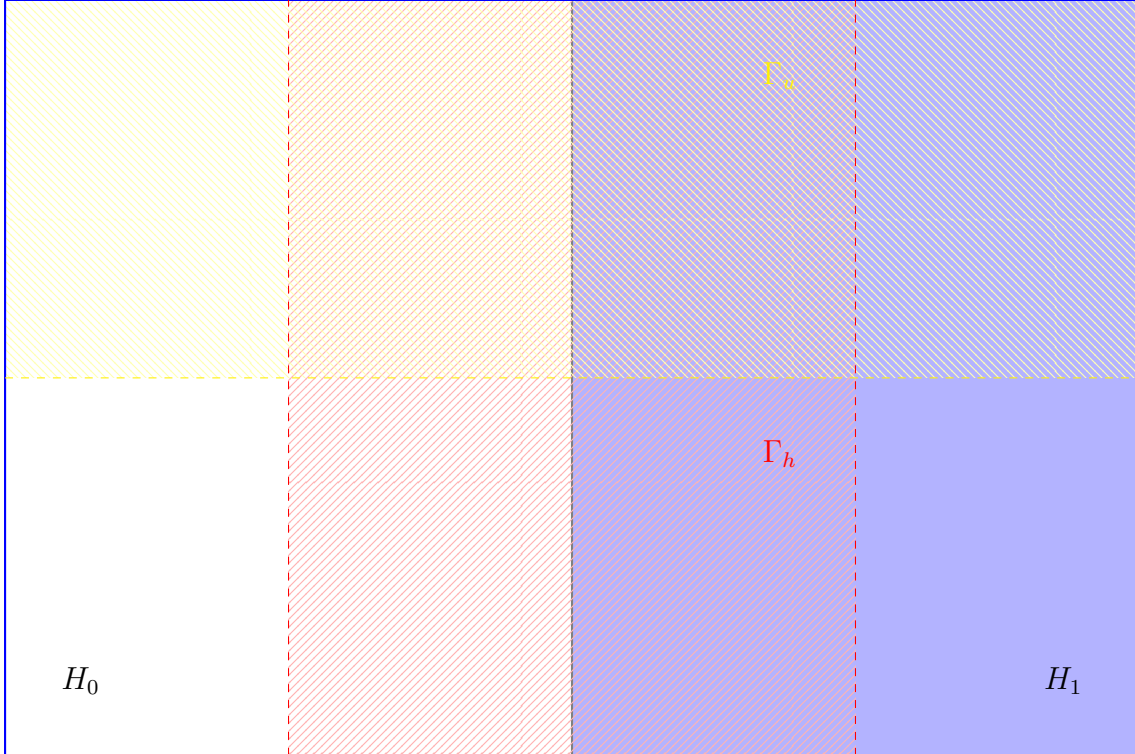

Figure 1: The setup for Λ_L. The left and right edges are identified to form a cylinder.

We also consider the plane $\mathbb{Z}^2$. In this setting, there are no edge states, and so the associated "bulk" Hamiltonian $H_B(\mu)$ is assumed to have a *gapped* spectrum, in the sense that

Assumption 1.
$$\sigma(H_B) = \mathcal{S}_- \cup \mathcal{S}_+,$$
where $\inf \mathcal{S}_+ - \sup \mathcal{S}_- \geq \gamma$ *uniformly in* L *and* μ *for some* $\gamma > 0$.

In the case of the cylinder, this effect does not necessarily occur due to the presence of the edge. We also assume that the Hamiltonian is *charge-conserving.*

Assumption 2. $[H(\mu), Q] = 0$, *where* Q *is the total charge in* Λ_L.

Let P_B be the ground state projection of H_B (the system without an edge), and let P be the ground state projection of H (the system with an edge). We assume that states far from the edge are essentially bulk states, up to tails that vanish quickly in L.

Assumption 3. *Define the bulk region* $\Gamma_B = [L/2 + k, L] \times [0, L]$ *for some* $k > 2R$. *For any operator* $A \in \mathcal{U}_{\Gamma_B}$,
$$\mathrm{Tr}(PA) = \mathrm{Tr}(P_B A) + \mathcal{O}(L^{-\infty}).$$

(need to add justification)
For ease of notation, we omit the subscript L wherever there is no risk of confusion.

2 Equality of Bulk and Edge Currents

2.1 Cylinder Geometry

Let $P(\mu)$ be the (possibly degenerate) ground state projection of $H(\mu)$. Let $Q_u = \sum_{x \in \Gamma_u} a_x^* a_x$ be the charge in the upper half of the cylinder $\Gamma_u = [0, L] \times [L/2, L]$, and define current operator

$$J_u = i[H(\mu), Q_u].$$

For simplicity, we drop the subscript u and simply write $J = J_u$. Charge conservation 2 implies that this current operator is supported on the strip $[L/2, L] \times [L/2 - R, L/2 + R]$, i.e. along a strip of width $2R$ centred at the line $y = L/2$. Indeed, if we inspect a local interaction $\Phi(X)$ of range R with support $S \subset (\Gamma_u)_R$, where $(\Gamma)_\alpha$ is the α-shrinking of Γ, then clearly $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$, by assumption 2. Similarly, if $\Phi(X)$ has support $S \subset ((\Gamma_u)^c)_R$, then $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$. It follows that for an interaction $\Phi(X)$ with range R and arbitrary support, $[\Phi(X), Q_u]$ must be supported on a set which is contained in (or equal to) the strip $[L/2, L] \times [L/2 - R, L/2 + R]$, and so $[H, Q_u]$ must be as well, since H is a sum of such local interactions.

Lemma 1. *The ground state expectation of the current J is zero.*

Proof. By cyclicity of the trace and commutativity of $P(\mu)$ with $H(\mu)$,

$$
\begin{aligned}
\mathrm{Tr}(P(\mu)J) &= \mathrm{Tr}(P(\mu)i[H(\mu), Q_u]) \\
&= \mathrm{Tr}(iP(\mu)H(\mu)Q_u) - \mathrm{Tr}(iP(\mu)Q_uH(\mu)) \\
&= \mathrm{Tr}(iH(\mu)P(\mu)Q_u) - \mathrm{Tr}(iH(\mu)P(\mu)Q_u) \\
&= 0.
\end{aligned}
$$

$\square$

Define the operators

$$
K(\mu) = \mathcal{I}_\mu(\dot{H}(\mu)),
$$

where

$$
\mathcal{I}_\mu(A) = \int_{\mathbb{R}} W(t)e^{itH(\mu)}Ae^{-itH(\mu)}dt.
$$

More explicitly, in this setting we see that $K(\mu) = \mathcal{I}_\mu(Q_h)$. As a shorthand, we use the notation $\dot{A}(\mu_0) = \frac{d}{d\mu}A|_{\mu_0}$.

We present two important properties of the map $\mathcal{I}_\mu$ in the following lemmas, and leave their proofs to the appendix (need to add). We will also need a definition: an *off-diagonal* operator is an operator A such that $A = \overline{A} := PAP^\perp + P^\perp AP$, where $P^\perp = \mathbb{I} - P$ is the projection onto the excited states above the gap.

Lemma 2. *For any off-diagonal operator A, $\mathcal{I}_\mu(\cdot)$ and $[H(\mu), \cdot]$ act as inverses of each other, up to a factor of i:*

$$
\mathcal{I}_\mu\left([H(\mu), A]\right) = [H(\mu), \mathcal{I}_\mu(A)] = iA
$$

It is easy to verify that any operator A behaves as an off-diagonal operator when taking a commutator with P, in the sense that

$$
[\overline{A}, P] = [PA(1 - P) + (1 - P)AP, P] = [A, P].
$$

It follows that for any (not necessarily off-diagonal) operator A,

$$
[\mathcal{I}([H, A]), P] = i[A, P]
$$

Lemma 3. *$\mathcal{I}_\mu$ is local in the sense that for any $A \in \mathcal{U}_X$,*

$$
\|\mathcal{I}(A)_{(X^r)^c}\| \leq \|A\||X|\mathcal{O}(r^{-\infty}).
$$

Proposition 1. *The operator $K(\mu)$ is the* generator of parallel transport, *satisfying*

$$
\dot{P}(\mu) = i[K(\mu), P(\mu)].
$$

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Proof. By the product rule and the fact that $H(\mu)$ and $P(\mu)$ commute,

$$[\dot{H}(\mu), P] = -[H, \dot{P}(\mu)].$$

Now we show that $\dot{P}(\mu)$ is off-diagonal. Taking the derivative on both sides of $P^2 = P$, we see that $\dot{P}P + P\dot{P} = \dot{P}$. Acting on the left and right with P on both sides of this equation gives

$$P\dot{P}P + P\dot{P}P = P\dot{P}P,$$

which implies that $P\dot{P}P = 0$. Now,

$$
\begin{aligned}
\overline{\partial_\mu P} &= P(\partial_\mu P)(1 - P) + (1 - P)(\partial_\mu P)P \\
&= P(\partial_\mu P) - P(\partial_\mu P)P + (\partial_\mu P)P - P(\partial_\mu P)P \\
&= P(\partial_\mu P) + (\partial_\mu P)P \\
&= \partial_\mu(P^2) \\
&= \partial_\mu P,
\end{aligned}
$$

as claimed. It therefore follows from Lemma 2 that

$$
\begin{aligned}
\dot{P}(\mu) &= -i\mathcal{I}_\mu([H(\mu), \dot{P}(\mu)]) \\
&= i\mathcal{I}_\mu([\dot{H}(\mu), P(\mu)]) \\
&= i[\mathcal{I}_\mu(\dot{H}(\mu)), P(\mu)] \\
&= i[K(\mu), P(\mu)].
\end{aligned}
$$

$\square$

Increasing the "electric potential" by a small amount $d\mu Q_h$ and expanding to linear order, the change in ground state current is given by

$$\mathrm{Tr}(P(\mu + d\mu)J) - \mathrm{Tr}(P(\mu)J) = \kappa d\mu + \mathcal{O}(d\mu^2).$$

Dividing by $d\mu$ and taking a limit, we see that the linear response coefficient is given by

$$\kappa = \mathrm{Tr}\left(\dot{P}(\mu)J\right).$$

The *Hall conductivity* of the system on a subset $V \subseteq \Lambda$ is defined to be $\kappa_V := \mathrm{Tr}\left(\dot{P}(\mu)J_V\right)$, where J_V is the restriction of the operator J to V.

Proposition 2. *The Hall conductivity is independent of the driving strength μ.*

Proof. We see by cyclicity of the trace and the formula $\dot{H}(\mu) = Q_h$ that for any μ_1 and μ_2,

$$\kappa(\mu_1) - \kappa(\mu_2) = \text{Tr}\left(\dot{P}(\mu_1)i[H(\mu_1), Q_u] - \dot{P}(\mu_2)i[H(\mu_1), Q_u]\right)$$
$$= i\text{Tr}\left(\left([\dot{P}(\mu_1), H(\mu_1)] - [\dot{P}(\mu_2), H(\mu_2)]\right)Q_u\right)$$
$$= i\text{Tr}\left(\left([\dot{H}(\mu_1), P(\mu_1)] - [\dot{H}(\mu_2), P(\mu_2)]\right)Q_u\right)$$
$$= i\text{Tr}\left([Q_h, P(\mu_1) - P(\mu_2)]Q_u\right)$$
$$= i\text{Tr}\left(Q_h(P(\mu_1) - P(\mu_2))Q_u - (P(\mu_1) - P(\mu_2))Q_uQ_h\right)$$
$$= 0,$$

since Q_h and Q_u commute, indicating that the Hall conductivity is independent of μ as one would expect physically. $\qquad\square$

The following is the main result:

Theorem 1. *The ground state current in the strip $[L/2 + k, 3L/4 - k] \times [0, L]$ between the edge and the bulk current vanishes, in the sense that $\kappa_V = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty})$ for any $V \subseteq [L/2 + R, 3L/4 - R] \times [0, L]$ "in between" the bulk and edge strips, where*

$$r = \text{dist}(V, [L/2 - R, 3L/4 + R] \times [0, L] \cup [3L/4 - R, 3L/4 + R] \times [0, L])$$

is the distance from V to one of the edge or bulk strips.

Proof. By Proposition 1, the Hall conductivity can also be written by the formula $\kappa_V^B = \text{Tr}\left(i[K(\mu), P_B(\mu)]J_V^B\right) = \text{Tr}\left(i[\mathcal{I}_\mu(Q_h), P_B(\mu)]J_V^B\right)$, where $J_V^B = (i[H_B, Q_u])_V$ is the current arising from the bulk Hamiltonian. From commutativity of P_B and H_B along with cyclicity of the trace, we compute

$$\kappa_V^B = \text{Tr}\left(i[\mathcal{I}_\mu(Q_h), P_B(\mu)]J_V^B\right)$$
$$= \text{Tr}\left(i\int_\mathbb{R} W(t)e^{itH_B(\mu)}[Q_h, P_B(\mu)]e^{-itH_B(\mu)}dt\, J_V^B\right)$$
$$= \int_\mathbb{R} W(t)\text{Tr}\left(i[Q_h, P_B(\mu)]e^{-itH_B(\mu)}J_V^B e^{itH_B(\mu)}\right)dt$$
$$= -\int_\mathbb{R} W(t)\text{Tr}\left(i[Q_h, P_B(\mu)]e^{itH_B(\mu)}J_V^B e^{-itH_B(\mu)}\right)dt$$
$$= -\text{Tr}\left(i[Q_h, P_B(\mu)]\mathcal{I}_\mu(J_V^B)\right),$$

since $W(t)$ is odd. Again by cyclicity of trace combined with the fact that $\mathcal{I}_\mu(\cdot)$ is an inverse of $[H_B(\mu), \cdot]$ for commutators with $P_B(\mu)$ (by the remark after lemma 2), we obtain

$$\kappa_V^B = -\text{Tr}([\mathcal{I}_\mu([H_B(\mu), Q_h]), P_B(\mu)]\mathcal{I}_\mu(J_V^B))$$
$$= -\text{Tr}(\mathcal{I}_\mu([H_B(\mu), Q_h])P_B(\mu)\mathcal{I}_\mu(J_V^B) - P_B(\mu)\mathcal{I}_\mu([H_B(\mu), Q_h])\mathcal{I}_\mu(J_V^B))$$
$$= -\text{Tr}(P_B(\mu)\mathcal{I}_\mu(J_V^B)\mathcal{I}_\mu([H_B(\mu), Q_h]) - P_B(\mu)\mathcal{I}_\mu([H_B(\mu), Q_h])\mathcal{I}_\mu(J_V^B))$$
$$= \text{Tr}\left(P_B(\mu)[\mathcal{I}_\mu([H_B(\mu), Q_h]), \mathcal{I}_\mu(J_V^B)]\right).$$

Now, $[H_B(\mu), Q_h]$ is a local operator, supported on the "bulk line" $[3L/4 - R, 3L/4 + R] \times [0, L]$, while J_V^B is a local operator supported on $V \subseteq [L/2 + k, 3L/4 - k] \times [0, L]$.

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Since $\mathcal{I}_\mu$ preserves locality up to tails, in the sense that $\|\mathcal{I}_\mu(A)_{(S^r)^c}\| \leq \|A\|\|S\|\mathcal{O}(r^{-\infty})$ (Lemma 3), it follows that the commutator $[\mathcal{I}_\mu([H_B(\mu), Q_h]), \mathcal{I}_\mu(J_V^B)] = C\mathcal{O}(r^{-\infty})$ whenever $V \cap ([3L/4 - R, 3L/4 + R] \times [L/2, L]) = \varnothing$.

The previous fact applies to the bulk setting with H_B and P_B. To extend this to the setting with an edge, it is enough to use Assumption 3 to conclude the same result, except with equality up to $\mathcal{O}(L^{-\infty})$, i.e.

$$\kappa_V = \mathrm{Tr}\left(\dot{P}J_V\right) = \mathrm{Tr}\left(\dot{P}(J_V^B + \mathcal{O}(L^{-\infty}))\right) = \kappa_V^B + \mathcal{O}(L^{-\infty}) = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

$\square$

The intuitive picture from the previous result is that, in the bulk region, the Hall conductivity is essentially only nonzero along the bulk line $[3L/4 - R, 3L/4 + R] \times [L/2, L]$. Since the ground state expectation of the current is zero (by lemma 1), it must be that there is an equal current flowing along the edge, but in the opposite direction, see figure (need to add).

2.2   Torus Geometry

Our goal is to show the same result on the discrete torus $\mathbb{T}_L := \mathbb{Z}_L \times \mathbb{Z}_L$. We define the same regions Γ_u and Γ_h, and the same current operator $J_u = i[H(\mu), Q_u]$. This time, however, Lemma 1 does not apply. Intuitively, it does not apply because electrons can now flow through both the bottom and the top of the region Γ_u, rather than just the bottom. Mathematically, the lemma fails because our definition of the current is slightly changed.

We use charge conservation and the fact that H is finite range to split the current J_u into two components, $J_u = i[H_-, Q_u] + i[H_+, Q_u] = J_- - J_+$, supported on strips of width $2R$ at $y = L/2$ and $y = L$, respectively. We then define the current operator to be $J = J_-$, which is the current on the lower strip. This is the mathematical reason that the proof in Lemma 1 fails on the torus; we have replaced H by H_-, which may no longer commute with P. We instead proceed by a different approach. We will need a few auxiliary results first.

Lemma 4. *$K_\pm$ is supported on $\partial_\pm$ up to tails.*

Proof. $\square$

Proposition 3. *The operator $Q_h - K$ leaves the ground state space invariant, i.e. $[Q_h - K, P] = 0$.*

Proof. $\square$

Lemma 5. *Show that $\mathrm{Tr}(A, [Q_h, P]) = 0$ for all $A \in \mathcal{U}_{edge}$. This shows that Q_h commutes with P "along the edge".*

Proof. Let $A \in \mathcal{U}_{\mathrm{edge}}$. Since H is charge conserving, we may choose a simultaneous eigenbasis of H and the total charge Q, in which case P and Q commute. It follows that

$$\mathrm{Tr}(A[Q_h, P]) = \mathrm{Tr}([A, Q_h]P) = \mathrm{Tr}([A, Q]P) = \mathrm{Tr}(A[Q, P]) = 0.$$

$\square$

Finally, we will prove that in the bulk system with Hamiltonian $H_B(\mu)$, the ground state expectation of the current vanishes faster than any power as $L \to \infty$.

Lemma 6. *The ground state expectation of the current $J_B := i[(H_B)_-, Q_h]$ (of the system without an edge) is $\mathrm{Tr}(P_B J_B) = \mathcal{O}(L^{-\infty})$.*

Proof. First, $K = \mathcal{I}(i[H_B, Q])$ splits into $K = K_- - K_+$, with the support of $K_\pm$ contained in $\partial_\pm$ up to tails:

$$[K_\pm, A_X] = \mathcal{O}(p^{-\infty}),$$

for every $A_X \in \mathcal{U}_X$ such that $\|A_X\| = 1$, and where $p = \mathrm{dist}(X, \partial_\pm)$ (need to add). Using the fact that $K_\pm$ is supported in $\partial_\pm$ up to tails (Lemma 4), we see that

$$i[H_B, K_-] = i[(H_B)_-, K_-] + \mathcal{O}(L^{-\infty}),$$

and similarly $i[(H_B)_-, K_+] = \mathcal{O}(L^{-\infty})$. Putting these facts together, it follows that the current can be rewritten as

$$\begin{aligned} J_B &= i[H_B, Q_h + K_- - K_- + K_+] + \mathcal{O}(L^{-\infty}) \\ &= i[H_B, K_-] + i[(H_B)_-, Q_h - K_- + K_+)] + \mathcal{O}(L^{-\infty}). \end{aligned}$$

From here, we use the fact that H_B and $Q_h - K_- + K_+$ both commute with P_B to write

$$P_B J_B P_B = i[H_B, P_B K_- P_B] + i[P_B(H_B)_- P_B, Q_h - K_- + K_+)] + P_B \mathcal{O}(L^{-\infty}) P_B.$$

Since the trace of any commutator is zero,

$$\mathrm{Tr}(P_B J_B) = \mathrm{Tr}(P_B J_B P_B) = \mathcal{O}(L^{-\infty}).$$

$\square$

Using this, we can show a simple proof of the analogue of Lemma 1 on the torus, in the case of non-interacting systems.

Proposition 4. *Let $H = \sum_{x \in \mathbb{T}} h_x$ be a non-interacting Hamiltonian, i.e. a sum of single site Hamiltonians h_x. The ground state expectation of the current $J = i[H_-, Q_h]$ (of the system with an edge) is $\mathrm{Tr}(PJ) = \mathcal{O}(L^{-\infty})$.*

Proof. Since H is a sum of single site Hamiltonians, we can split H_- into the restrictions $H_- = (H_-)_{\mathrm{edge}} + (H_-)_{\mathrm{bulk}}$, with no fear of any terms which are in both the edge region and the bulk region. By Assumption 3,

$$\begin{aligned} \mathrm{Tr}(PJ) &= \mathrm{Tr}(Pi[H_-, Q_h]) \\ &= i\mathrm{Tr}([H_-, Q_h]P) \\ &= i\mathrm{Tr}((H_-)_{\mathrm{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\mathrm{bulk}}[Q_h, P]) \\ &= i\mathrm{Tr}((H_-)_{\mathrm{edge}}[Q_h, P]) + i\mathrm{Tr}((H_-)_{\mathrm{bulk}}[Q_h, (P)_{\mathrm{bulk}}]) \\ &= i\mathrm{Tr}((H_-)_{\mathrm{edge}}[Q_h, P]) + i\mathrm{Tr}((H_B)_-[Q_h, P_B]) + \mathcal{O}(L^{-\infty}) \\ &= i\mathrm{Tr}((H_-)_{\mathrm{edge}}[Q_h, P]) + \mathrm{Tr}(i[(H_B)_-, Q_h]P_B) + \mathcal{O}(L^{-\infty}). \end{aligned}$$

By Lemma 5, the first term is zero. By Lemma 6, the second term is $\mathcal{O}(L^{-\infty})$. $\square$

A   Properties of $\mathcal{I}_\mu$

Proof. (Of Lemma 2). Let $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi}} \int_\mathbb{R} W(t) e^{-2\pi i t \xi} dt$ be the Fourier transform of W. One can show that for $|\xi| \geq \gamma$, $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi i \xi}}$ (need to add). Let A be an observable. First, we show that $\mathcal{I}([H, PAP^\perp]) = iPAP^\perp$.

Decomposing

$$e^{itH}P = \sum_{j=0}^\infty \frac{(itH)^j}{j!} P$$

$$= \sum_{j=0}^\infty \frac{(it)^j}{j!} \left(\sum_n E_n^j P_n \right) P$$

$$= \sum_{j=0}^\infty \frac{(it)^j}{j!} \sum_{n:E_n=0} E_n^j P_n$$

$$= \sum_{n:E_n=0} e^{itE_n} P_n,$$

and similarly

$$P^\perp e^{-itH} = \sum_{m:E_m \geq \gamma} P_m e^{-itE_m},$$

we see that

$$\mathcal{I}([H, PAP^\perp]) = \mathcal{I}(P[H, A]P^\perp)$$

$$= \int_\mathbb{R} W(t) e^{itH} P[H, A] P^\perp e^{-itH} dt$$

$$= \int_\mathbb{R} W(t) \sum_{n:E_n=0} e^{itE_n} P_n[H, A] \sum_{m:E_m \geq \gamma} P_m e^{-itE_m} dt$$

$$= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} \int_\mathbb{R} W(t) e^{itE_n} P_n A(E_n - E_m) P_m e^{-itE_m} dt$$

$$= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \int_\mathbb{R} W(t) e^{-it(E_m - E_n)} dt$$

$$= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \sqrt{2\pi} \widehat{W}(E_m - E_n)$$

$$= i \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m$$

$$= iPAP^\perp.$$

(need to check the 2π factor)

By the same argument, $\mathcal{I}([H, P^\perp AP]) = iP^\perp AP$ as well, and so $\mathcal{I}([H, \overline{A}]) = i\overline{A}$. $\qquad\square$

Proof. (Of Lemma 3). We break the integral into two parts,

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$$\|\mathcal{I}(A)\| \leq \left\| \int_{-T}^{T} W(t)e^{itH}Ae^{-itH}dt \right\| + \left\| \int_{\mathbb{R}\setminus[-T,T]} W(t)e^{itH}Ae^{-itH}dt \right\|.$$

The first term can be estimated using the Lieb-Robinson bound found in Appendix B.

$\square$

B Lieb-Robinson Bound

Let N be a uniform upper bound for the dimensions of the Hilbert spaces at each site, i.e. $\dim(\mathcal{H}_x) \leq N$ for all sites x.

The following is a version of the Lieb-Robinson. For any operators $A \in \mathcal{U}_X$ and $B \in \mathcal{U}_Y$ having disjoint supports $X \cap Y = \varnothing$,

$$\|[e^{itH}Ae^{-itH}, B]\| \leq C\|A\|\|B\||X||Y|N^{2|X|}e^{2t\|\Phi\|_\lambda - \lambda d(X,Y)}.$$

C Grönwall's Inequality and Uniqueness

Theorem 2. *(Grönwall's Inequality). Let $\alpha : I \to (0, \infty)$ be positive and continuous on I^o for some interval of the form $[a, b)$, $[a, b]$, or $[a, \infty)$. Suppose $u : \mathbb{R} \to \mathcal{U}$ is a Banach-valued, differentiable function. If $\|u'(t)\| \leq \alpha(t)\|u(t)\|$ for all $t \in I$, then*

$$\|u(t)\| \leq \|u(a)\|e^{\int_a^t \alpha(s)ds} \quad \forall t \in I$$

Proof. Let $f(t) = e^{\int_a^t \alpha(s)ds}$, which is nonzero, has initial value $f(a) = 1$, and has derivative $f'(t) = \alpha(t)f(t)$. Then by the quotient rule,

$$\left(\frac{\|u(t)\|}{f(t)} \right)' = \frac{\|u'(t)\|f(t) - \|u(t)\|\alpha(t)f(t)}{f(t)^2} \leq 0,$$

where the inequality follows from the assumption $\|u'(t)\| \leq \|\alpha(t)u(t)\|$. Thus $\frac{\|u(t)\|}{f(t)}$ is decreasing, so that

$$\frac{\|u(t)\|}{f(t)} \leq \frac{\|u(a)\|}{f(a)} = \|u(a)\|,$$

which is the desired inequality. $\square$

Theorem 3. *(ODE Uniqueness). Let $F : \mathcal{U} \to \mathcal{U}$ be Lipschitz and consider the differential equaton $u'(t) = F(u(t))$ with initial condition $u(a) = u_a$ for some function $u : I \to \mathcal{U}$, where $I = [a, b]$, or $[a, b)$, or $[a, \infty)$. Solutions to this equation are unique.*

Proof. Suppose there are two solutions $u(t)$ and $v(t)$, and let $g(t) = \|u(t) - v(t)\|^2$. By assumption, there exists a constant L_F such that $\|F(u(t)) - F(v(t))\| \leq L_F \|u(t) - v(t)\|$, so that

$$\begin{aligned} g'(t) &= 2\|u(t) - v(t)\| \|u'(t) - v'(t)\| \\ &= 2\|u(t) - v(t)\| \|F(u(t)) - F(v(t))\| \\ &\leq 2L_F \|u(t) - v(t)\|^2 \\ &= 2L_F g(t). \end{aligned}$$

Notice that $\alpha := 2L_F$ is a positive continuous function, so we may apply Grönwall's inequality to $g(t)$ to conclude

$$g(t) \leq g(a)e^{2L_f(t-a)} = 0,$$

since $g(a) = 0$. $\qquad\square$

D   Note on Generators of Parallel Transport

Consider the differential equation $\dot{\rho}(\mu) = i[K_B, \rho(\mu)]$ with initial condition $\rho(0) = P_B(0)$. Here $K_B = \int_{\mathbb{R}} W_\gamma(t)e^{-itH_B}\dot{H}_B e^{itH_B}dt$, and recall that in our setting, $\dot{H}_B = Q_h$. We know that the solution is $\rho(\mu) = P_B(\mu)$ (proposition 1). Notice that the map $F : \mathcal{U} \to \mathcal{U}$ defined by $F(A) = i[K_B, A]$ is Lipschitz, since

$$\|F(A) - F(B)\| = \|[K_B, A - B]\| \leq 2\|K_B\| \|A - B\|.$$

The Lipschitz constant is $2\|K_B\|$, which is finite since K_B is a bounded operator:

$$\|K_B\| \leq \int_{\mathbb{R}} |W_\gamma(t)| \|e^{-itH_B}Q_h e^{itH_B}\|dt \leq \int_{\mathbb{R}} |W_\gamma(t)|dt \|Q_h\|.$$

Indeed, since Q_h is the number operator on a finite volume, by charge conservation and the fact that the dimension of the Hilbert space is bounded uniformly by d, there can only be a finite number of charges in the region Γ_h.

Thus, by Grönwall's uniqueness theorem (appendix C), we see that the solution to the equation $\dot{\rho}(\mu) = F(\rho(\mu)) = i[K_B, \rho(\mu)]$ is unique.

Now define

$$K_E := \int_{\mathbb{R}} W_\gamma(t)e^{-itH_E}Q_h e^{itH_E}dt,$$

which is using the gap γ of H_B to define W_γ, but also using the edge Hamiltonian in the time evolution operators. Consider $\sigma : [0, \infty) \to \mathcal{U}$ defined by

$$\dot{\sigma}(\mu) = i[K_E, \sigma(\mu)] \qquad\qquad \sigma(0) = P_B(0).$$

We now show that, similar to how ρ is an approximation of P_B, σ is also a good approximation of P_E (up to $\mathcal{O}(L^{-\infty})$) "in the bulk". Let $A \in \Gamma_B$ be an operator localized in the bulk of the edge system. Then

$$\text{Tr}(\dot{\sigma}A) = \text{Tr}(i[K_E, \sigma]A)$$
$$= \text{Tr}(i[A, K_E]\sigma)$$
$$= \int_{\mathbb{R}} W_\gamma(t)\text{Tr}([e^{-itH_E}Q_h e^{itH_E}, A]\sigma)dt$$
$$= \int_{\mathbb{R}} W_\gamma(t)\text{Tr}(e^{-itH_E}[Q_h, e^{itH_E}Ae^{-itH_E}]e^{itH_E}\sigma)dt$$
$$= \int_{\mathbb{R}} W_\gamma(t)\text{Tr}(e^{-itH_E}[Q_h, e^{itH_B}Ae^{-itH_B}]e^{itH_E} + \mathcal{O}(L^{-\infty})\sigma)dt$$
$$= \int_{\mathbb{R}} W_\gamma(t)\text{Tr}(e^{-itH_B}[Q_h, e^{itH_B}Ae^{-itH_B}]e^{itH_B} + \mathcal{O}(L^{-\infty})\sigma)dt$$
$$= \int_{\mathbb{R}} W_\gamma(t)\text{Tr}([e^{-itH_B}Q_h e^{itH_B}, A]\sigma)dt + \mathcal{O}(L^{-\infty})$$
$$= \text{Tr}(i[A, K_B]\sigma) + \mathcal{O}(L^{-\infty})$$
$$= \text{Tr}(i[K_B, \sigma]A) + \mathcal{O}(L^{-\infty}),$$

since σ is trace-class (?) and $W_\gamma \in L^1$. By linearity of the trace, we see that $\text{Tr}((\dot{\sigma} - i[K_B, \sigma])A) = \mathcal{O}(L^{-\infty})$ for any operator $A \in \Gamma_B$ (does this mean $\dot{\sigma} - i[K_E, \sigma] = 0$?). But the solution of $\dot{\sigma} - i[K_B, \sigma] = 0$ (with initial condition $\sigma(0) = P_B(0)$) is unique; it is $\rho(\mu)$, or $P_B(\mu)$. Hence

$$\text{Tr}(P_E A) = \text{Tr}(P_B A) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\rho A) + \mathcal{O}(L^{-\infty}) = Tr(\sigma A) + \mathcal{O}(L^{-\infty})$$

for any operator $A \in \Gamma_B$. In particular, this gives another local formula for the Hall conductivity in the bulk of an edge system, by taking $A = J_V$, where J is the current operator and $V \subset \Gamma_B$ is a set localized in the bulk. The Hall conductivity is given by $\text{Tr}(\dot{P}_E J_V)$, and this can be approximated by

$$\text{Tr}(\dot{P}_E J_V) = \text{Tr}(\dot{P}_B J_V) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\dot{\rho} J_V) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\dot{\sigma} J_V) + \mathcal{O}(L^{-\infty}).$$

Want to pick a norm s.t. Gronwall gives $\|\rho(\mu) - \sigma(\mu)\|_G \le \|P_B(0) - P_E(0)\|_G e^{2L_F\mu}$. Need $\|P_B(0) - P_E(0)\|_G$ to be small enough to kill the exponential which depends on $L_F = 2\|K_B\|_G \le \|W_\gamma\|_{L^1}\|Q_h\|_G$. If we use the operator norm for $\|\cdot\|_G$, we would get $\|Q_h\|_G = d|\Gamma_h|$ in the exponent. Need $\|\cdot\|_G$ to be an actual norm so that $\|\rho - \sigma\|_G = 0 \implies \rho = \sigma$.