

Thesis Rough Work

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1 Introduction

2 Noninteracting Setting

Consider the lattice \mathbb{Z}^2 , on which we define a bulk Hamiltonian H_B , whose matrix elements follow a short-range assumption:

$$\sup_{y \in \mathbb{Z}^2} \sum_{x \in \mathbb{Z}^2} |H_B(x, y)| (e^{\mu|x-y|} - 1) < \infty$$

for some $\mu > 0$. We define the bulk conductivity

$$\sigma_B(\lambda) = -i \text{Tr}(P_\lambda [[P_\lambda, \Lambda_1], [P_\lambda, \Lambda_2]])$$

where P_λ is the projection onto the eigenstates of H_B with energy lies in $(-\infty, \lambda)$, and where

$$\Lambda_i(x) = \begin{cases} 1 & x_i < 0 \\ 0 & x_i \geq 0 \end{cases}$$

are characteristic functions. We construct an edge Hamiltonian on the lattice $\mathbb{Z}_a^2 = \{x \in \mathbb{Z}^2 : x_2 > -a\}$. We denote the edge Hamiltonian by $H_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}_a^2)$, requiring only that the edge operator $E_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}_a^2)$ define by

$$E_a = J_a H_a - H_B J_a$$

satisfies the edge assumption

$$\sup_{x \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}_a^2} E_a(x, y) |e^{\mu(|x_2+a|-|x_1-y_1|)}| < \infty$$

for some $\mu > 0$, where $|x| := |x_1| + |x_2|$. The interpretation

Each site $x \in \mathbb{Z}^2$ get an associated Hilbert space \mathcal{H}_x . The dimension of these Hilbert spaces is bounded uniformly in x . We consider the Hilbert space $\ell^2(\mathbb{Z}^2, \mathbb{C}^n) = \{(x_1, x_2, \dots) \in \mathbb{C}^n : \sum_{i \in \mathbb{Z}^2} \|x_i\|^2 < \infty\}$. For example, one might consider a system of spins at the lattice sites, in which case the Hilbert space \mathcal{H}_x at each site would be \mathbb{C}^2 , and the total Hilbert space $\mathcal{H} = \otimes_x \mathcal{H}_x$ would then be the space of summable wavefunctions $\psi = \otimes_x \psi_x \in \ell^2(\mathbb{Z}^2, \mathbb{C}^2)$.

The Hilbert space $\ell^2(\mathbb{Z}^2)$ is the “bulk” setting, i.e. the setting in which we consider an infinite two-dimensional medium with no edges, and we consider a “bulk Hamiltonian” H_B on this Hilbert space. We also define the “edge” Hilbert space $\ell^2(\mathbb{Z}_a^2)$ and an associated “edge Hamiltonian” H_a , where $\mathbb{Z}_a^2 := \{(n, m) \in \mathbb{Z}^2 : n \geq -a\}$. The bulk and edge Hamiltonians are related by the edge operator $E_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}^2)$ defined by

$$E_a := J_a H_a - H_B J_a,$$

where $J_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}^2)$ denotes extension by zeroes. We assume that

Assumption 1. *The edge operator satisfies*

$$\sup_{z \in \mathbb{Z}^2} \sum_{y \in \mathbb{Z}_a^2} |E_a(x, y)| e^{\mu(|x_2+a|+|x_1-y_1|)} < \infty.$$

The interpretation is that $E_a = J_a H_a - H_B J_a$ is the difference between first applying H_a on $\ell^2(\mathbb{Z}_a^2)$, and then making everything below $-a$ into zeroes, versus first making all $x \in \mathbb{Z}^2$ such that $x_2 < -a$ zeroes, and then applying H_B . The assumption ensures that the effects from introducing the edge at $-a$ die exponentially as we move upward away from the edge (due to the $|x_2 - (-a)|$ term in the exponent), and also terms do not interact too much as their x_1 distance increases (due to the $|x_1 - y_1|$ term in the exponent).

We also make the following assumption about both the bulk and edge Hamiltonians:

Assumption 2. *The Hamiltonians have a spectral gap. There exists an interval Δ such that $\Delta \cap \sigma(H) = \emptyset$.*

Remark: The spectral gap assumption can be relaxed to a “mobility gap” assumption,

$$\sup_{f \in B_c(\Delta)} |f(H_B)(x, y)| (1 + |x|)^{-\nu} e^{\mu|x-y|} < \infty$$

for some $\nu > 0$, where $B_c(\Delta)$ is the set of Borel functions f which are constant on $(-\infty, \inf \Delta)$ and on $(\sup \Delta, \infty)$ such that $|f(x)| \leq 1$ for all x . See ? for details.

An example of an edge Hamiltonian satisfying the assumption on E_a is $H_a = J_a^* H_B J_a$, where $J_a : \ell^2(\mathbb{Z}_a^2) \rightarrow \ell^2(\mathbb{Z}^2)$ denotes extension by zeros. The idea is that for a state $\psi \in \ell^2(\mathbb{Z}_a^2)$, we have $\langle \psi, H_a \psi \rangle = \langle (J_a \psi), H_B (J_a \psi) \rangle$, which we interpret as the edge Hamiltonian having the same expectation as the bulk Hamiltonian if we just turned all the states ψ_x with $x_2 < -a$ into zeroes. The edge operator is

$$E_a = J_a J_a^* H_B J_a - H_B J_a = (J_a J_a^* - \mathbb{1}) H_B J_a = \begin{cases} -H_B(x, y) & \text{if } x_2 < -a \\ 0 & \text{if } x_2 \geq -a \end{cases}$$

Intuitively, there is no difference between H_B and H_a on \mathbb{Z}_a^2 . The bound in assumption ? is satisfied by the short range assumption ?.

We define the *bulk conductivity* at Fermi energy μ as follows. Suppose we subject the system to an external electric potential difference V in the x_2 direction. We write this as $-V_0 \Lambda_2$, where Λ_i are multiplication operators $\Lambda_i |\psi\rangle = \Lambda(x_i) |\psi\rangle$ which are *switch functions*, satisfying

$$\Lambda(x_i) = \begin{cases} 1 & \text{if } x_i < 0 \\ 0 & \text{if } x_i > 1 \end{cases}$$

and are smooth and monotonically decreasing on $[0, 1]$. This gives $\vec{E} = -\nabla V = V_0 \frac{\partial \Lambda_2}{\partial x_2}$, so that \vec{E} has compact support $\text{supp}(\Lambda_2')$. We introduce a function which grows slowly in time as t grows from $-\infty$ to 0, so as to invoke the adiabatic principle. Here, we choose $e^{\varepsilon t}$, and we will let $\varepsilon \rightarrow 0$ at the end. The Hamiltonian therefore experiences a perturbation,

$$\tilde{H}_B(t) = H_B - V_0 \Lambda_2 e^{\varepsilon t}.$$

We define the Hall current operator $J_H = i[\tilde{H}_B(t), \Lambda_1] = i[H_B, \Lambda_1]$, which is related to the Hall conductivity by $J_H = \sigma_H V$.

Proposition 1. *The Hall conductivity σ_H in the bulk system is equal to*

$$\sigma_B = -i \text{Tr} (P_\mu [[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]),$$

where $P_\mu := P((-\infty, \mu))$ is the projection-valued measure associated with H_B onto states with energy less than the Fermi energy μ .

Proof. We begin with the Heisenberg equation of motion for the density matrix, $\dot{\rho}(t) = -i[\tilde{H}_B(t), \rho(t)]$, with initial condition $\lim_{t \rightarrow -\infty} \|\rho(t) - e^{-itH_B} P_\mu e^{itH_B}\| = 0$, which also implies $\lim_{t \rightarrow -\infty} \|e^{itH_B} \rho(t) e^{-itH_B} - P_\mu\| = 0$.

We work in the interaction picture, and define $\rho_I(t) = e^{itH_B} \rho(t) e^{-itH_B}$, and $\Delta H_B(t) = -e^{itH_B} V_0 \Lambda_2 e^{\varepsilon t} e^{-itH_B}$. Thus

$$\dot{\rho}_I(t) = -i[\Delta H_B(t), \rho_I(t)]$$

The solution to this differential equation is readily verified to be

$$\rho(t) = i \int_{-\infty}^t [\Delta H_B(s), P_\mu] ds + P_\mu$$

where $\Delta H_B(t)$ depends on s . Indeed, taking the derivative of the right hand side gives $i[\Delta H_B(t), P_\mu] = i[\Delta H_B(t), \rho_I(t)] + \mathcal{O}(V_0^2)$, but P_μ and $\rho_I(t)$ are equal up to zeroth order in V_0 . The initial condition is also satisfied.

Using $J_H = i[H_B, \Lambda_1] = \sigma_H V$, we obtain $\sigma_H = \frac{1}{V_0} \lim_{\varepsilon \rightarrow 0} \text{Tr}(\rho(0) i[H_B, \Lambda_1])$. Since the expectation of the ground state current is zero, $\text{Tr}(P_\mu J_H) = 0$, we have

$$\begin{aligned} \sigma_H &= \frac{i}{V_0} \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(i \int_{-\infty}^0 [\Delta H_B(t), P_\mu] [H_B, \Lambda_1] ds \right) \\ &= -\frac{1}{V_0} \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 [-e^{isH_B} V_0 \Lambda_2 e^{\varepsilon s} e^{-isH_B}, P_\mu] [H_B, \Lambda_1] ds \right) \\ &= -\lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 e^{isH_B} [\Lambda_2, P_\mu] e^{-isH_B} [H_B, \Lambda_1] e^{\varepsilon s} ds \right) \\ &= -\lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 (e^{-isH_B} [H_B, \Lambda_1] e^{isH_B}) \cdot ([\Lambda_2, P_\mu] e^{\varepsilon s}) ds \right) \end{aligned}$$

Where we used the fact that P_μ and H_B commute. Using integration by parts on the two terms in brackets, and noting that $\frac{d}{ds}(e^{isH_B} [H_B, \Lambda_1] e^{-isH_B}) = -(ie^{isH_B} \Lambda_1 e^{-isH_B} - \Lambda_1)$, we obtain

$$\begin{aligned} \sigma_H &= i \lim_{\varepsilon \rightarrow 0} \text{Tr} \left(\int_{-\infty}^0 (e^{-isH_B} \Lambda_1 e^{isH_B} - \Lambda_1) \frac{d}{ds} ([\Lambda_2, P_\mu] e^{\varepsilon s}) ds \right) \\ &= i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \Lambda_1^s [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right) \end{aligned}$$

where $\Lambda_1^s := e^{-isH_B}\Lambda_1 e^{isH_B} - \Lambda_1$. Using the notation $\overline{A} := P_\mu A P_\mu^\perp + P_\mu^\perp A P_\mu$, it is readily verified that the commutator $[\Lambda_2, P_\mu]$ is an *off-diagonal* operator, in the sense that $[\Lambda_2, P_\mu] = \overline{[\Lambda_2, P_\mu]}$. Furthermore, a simple computation reveals that for any two operators A and B , $\text{Tr}(\overline{AB}) = \text{Tr}(A\overline{B})$. It therefore follows that

$$\sigma_H = i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \overline{\Lambda_1^s} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right)$$

The integrand can be broken into two terms,

$$\overline{\Lambda_1^s} [\Lambda_2, P_\mu] e^{\varepsilon s} = e^{-isH_B} \overline{\Lambda_1} e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} - \overline{\Lambda_1} [\Lambda_2, P_\mu] e^{\varepsilon s}$$

by commutativity of P_μ and H_B . We show that the integral of the first term vanishes. We begin by breaking the first term down further into

$$e^{-isH_B} P_\mu \Lambda_1 P_\mu^\perp e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} + e^{-isH_B} P_\mu^\perp \Lambda_1 P_\mu e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s}.$$

We treat the first of these two terms; the other is handled in an identical manner. We use the spectral theorem to write $e^{-isH_B} P_\mu = \int_{-\infty}^\mu e^{-is\lambda} dP_\lambda$, and similarly $P_\mu^\perp e^{isH_B} = (\text{Id} - P_\mu) e^{isH_B} = \int_\mu^\infty e^{is\nu} dP_\nu$.

We remark that, since the Fermi energy μ is assumed to lie in a spectral gap, there must exist a neighbourhood $(\mu - \delta, \mu + \delta)$ in which there are no states. We exploit this fact to rewrite the limits of integration, $\int_{-\infty}^{\mu-\delta} e^{-is\lambda} dP_\lambda$ and $\int_{\mu+\delta}^\infty e^{is\nu} dP_\nu$. We therefore obtain

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \varepsilon \int_{-\infty}^0 e^{-isH_B} P_\mu \Lambda_1 P_\mu^\perp e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s} ds &= \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \int_{-\infty}^{\mu-\delta} e^{-is\lambda} dP_\lambda \Lambda_1 \int_{\mu+\delta}^\infty e^{is\nu} dP_\nu [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right) \\ &= \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \int_{-\infty}^{\mu-\delta} \int_{\mu+\delta}^\infty e^{s(\varepsilon - i\lambda + i\nu)} dP_\lambda \Lambda_1 dP_\nu [\Lambda_2, P_\mu] ds \right) \end{aligned}$$

Performing the integral over s yields

$$\lim_{\varepsilon \rightarrow 0} \varepsilon \int_{-\infty}^0 e^{s(\varepsilon - i\lambda + i\nu)} ds = - \lim_{\varepsilon \rightarrow 0} \frac{\varepsilon}{i\varepsilon + \lambda - \nu}$$

This limit is zero, since $\lambda \neq \nu$. Indeed, due to the spectral gap, the integration variables live in $\lambda \in (-\infty, \mu - \delta)$ and $\nu \in (\mu + \delta, \infty)$. The case for the $e^{-isH_B} P_\mu^\perp \Lambda_1 P_\mu e^{isH_B} [\Lambda_2, P_\mu] e^{\varepsilon s}$ term (where the P_μ and P_μ^\perp swap places) is treated analogously. Hence the first term in the integrand for σ_H vanishes, as claimed.

Finally, we return to our expression for the Hall conductivity, which now reads

$$\sigma_H = i \lim_{\varepsilon \rightarrow 0} \varepsilon \text{Tr} \left(\int_{-\infty}^0 \overline{\Lambda_1} [\Lambda_2, P_\mu] e^{\varepsilon s} ds \right).$$

It is a basic algebraic calculation to show that $\overline{\Lambda_1} = [[\Lambda_1, P_\mu], P_\mu]$. Evaluating the integral over s is now trivial; $\int_{-\infty}^0 e^{\varepsilon s} ds = \varepsilon^{-1}$. Thus

$$\sigma_H = -i \text{Tr}([[\Lambda_1, P_\mu], P_\mu] [\Lambda_2, P_\mu]).$$

Shifting the commutator completes the proof:

$$\begin{aligned}\sigma_H &= -i\text{Tr}(P_\mu[[\Lambda_2, P_\mu], [\Lambda_1, P_\mu]]) \\ &= i\text{Tr}(P_\mu[[\Lambda_1, P_\mu], [\Lambda_2, P_\mu]]) \\ &= i\text{Tr}(P_\mu[[P_\mu, \Lambda_1], [P_\mu, \Lambda_2]]).\end{aligned}$$

□

Remark: This is reminiscent of the well-known adiabatic curvature formula,

$$\kappa = \text{Tr}(P[\partial_1 P, \partial_2 P]) = \text{Tr}(P[[P, K_1], [P, K_2]]) = \text{Tr}(P[K_1, K_2]),$$

where K_i are called *generators of parallel transport*. We will see the adiabatic curvature formula again later in the interacting setting.

For the *edge conductivity*, we need the current operator across the line $x_1 = 0$, which is given by $-i[H_a, \xi_1]$. We define

$$\sigma_E = -i \lim_{a \rightarrow \infty} \text{Tr}(\rho'(H_a)[H_a, \xi_1]),$$

where $\rho \in C^\infty(\mathbb{R})$ satisfies

$$\rho(r) = \begin{cases} 1 & \text{if } r < \inf \Delta \\ 0 & \text{if } r > \sup \Delta \end{cases}$$

and decreases smoothly and monotonically in Δ . The definition of σ_E is reminiscent of another formula we will see later in the interacting setting, $\text{Tr}(\dot{P}J)$, where J is the current operator. The interpretation of σ_E is that if we apply a small potential difference V across $x_2 = -a$ to $x_2 = \infty$, there will be a net current

$$\begin{aligned}I &= -i\text{Tr}(\rho(H_a + V)[H_a + V, \xi_1] - \rho(H_a)[H_a, \xi_1]) \\ &= -i\text{Tr}((\rho(H_a + V) - \rho(H_a))[H_a, \xi_1])\end{aligned}$$

Thus we obtain the conductivity

$$\sigma_E = \frac{I}{V} = -i\text{Tr}\left(\frac{(\rho(H_a + V) - \rho(H_a))}{V}[H_a, \xi_1]\right) \rightarrow -i\text{Tr}(\rho'(H_a)[H_a, \xi_1])$$

in the limit as $V \rightarrow 0$. As we shall see, it turns out that σ_E is independent of the choice of ρ , and σ_B is independent of λ .

The main result of this section is

Theorem 1. $\sigma_E = \sigma_B$.

2.1 Outline of the Proof

First, let

$$\tilde{\sigma}_E(a, t) = -i\text{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_{2,a}(t))$$

where $\Lambda_{2,a}(t) = e^{itH_a}\Lambda_2e^{-itH_a}$ is the time evolution of Λ_2 . One can show that, while

$$\sigma_E = \lim_{T \rightarrow \infty} \lim_{a \rightarrow \infty} \frac{1}{T} \int_0^T \operatorname{Re}(\tilde{\sigma}_E(a, t)) dt,$$

it is unfortunately the case that $\lim_{a \rightarrow \infty} \|\rho'(H_a)[H_a, \Lambda_1]\Lambda_{2,a}(t)\|_1 = \infty$. However, even though the trace norm diverges, it turns out that the trace itself does not, so we will instead subtract a clever choice of zero-trace operator $Z(a, t)$ to define

$$\sigma_E(a, t) = -i\operatorname{Tr}(\rho'(H_a)[H_a, \Lambda_1]\Lambda_{2,a}(t) - Z(a, t))$$

so that the equation $\sigma_E = \lim_{T \rightarrow \infty} \lim_{a \rightarrow \infty} \frac{1}{T} \int_0^T \operatorname{Re}(\sigma_E(a, t)) dt$ still holds, but we also have $\lim_{a \rightarrow \infty} \|\rho'(H_a)[H_a, \Lambda_1]\Lambda_{2,a}(t) - Z(a, t)\|_1 < \infty$. The correct choice of Z will become apparent after writing $\rho(H_a)$ and $\rho'(H_a)$ in terms of their Hellfer-Sjostrand representations,

$$\rho(H_a) = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z)$$

$$\rho'(H_a) = -\frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z)^2$$

where $R(z) = (H_a - z)^{-1}$ is the resolvent. Using $[R(z), \Lambda_i] = R(z)[H_a, \Lambda_i]R(z)$, we obtain the representations of the following useful operators:

$$[\rho(H_a), \Lambda_1] = \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z)[H_a, \Lambda_1]R(z)$$

$$\rho'(H_a)[H_a, \Lambda_1] = -\frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z)^2[H_a, \Lambda_1]$$

From here, we define the zero-trace operator

$$Z(a, t) = [\rho(H_a), \Lambda_1]\Lambda_2 - \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z)(R(z)[H_a, \Lambda_1]\Lambda_{2,a}(t) - [H_a, \Lambda_1]\Lambda_{2,a}(t)R(z))$$

from which we obtain

$$\begin{aligned} \sigma_E(a, t) &= \tilde{\sigma}_E(a, t) - Z(a, t) \\ &= \operatorname{Tr} \left(-[\rho(H_a), \Lambda_1]\Lambda_2 - \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z)[H_a, \Lambda_1]\Lambda_{2,a}(t)R(z) \right) \\ &= \operatorname{Tr} \left([\rho(H_a), \Lambda_1](\Lambda_{2,a}(t) - \Lambda_2) - \frac{1}{2\pi} \int_{\mathbb{C}} \frac{\partial \tilde{\rho}(z)}{\partial \bar{z}} R(z)[H_a, \Lambda_1]R(z)[H_a, \Lambda_{2,a}(t)]R(z) \right) \end{aligned}$$

All of the statements so far can be verified by calculations. The difficult part of the theorem (aside from proving that the relevant operators are trace-class) is proving that

$$\|J_a \Sigma'_a J_a^* - \Sigma'_B\|_1, \|J_a \Sigma''_a J_a^* - \Sigma''_B\|_1 \rightarrow 0$$

as $a \rightarrow \infty$, where Σ'_B and Σ''_B are the same as with the subscript a , except using the bulk Hamiltonian H_B in their definition rather than H_a . It follows that

$$\sigma_E(a, t) = \text{Tr}(J_a \Sigma'_a J_a^* + J_a \Sigma''_a J_a^*) = \text{Tr}(\Sigma'_a + \Sigma''_a) \rightarrow \text{Tr}(\Sigma'_B + \Sigma''_B)$$

From there, a calculation shows that $\lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \text{Tr}(\Sigma'_B + \Sigma''_B) dt = \sigma_B$, concluding the proof.

3 Interacting Setting

Let $L \in \mathbb{N}$, and let $\Lambda_L = \mathbb{Z}_L \times [0, L]$ be the discrete cylinder, equipped with a metric d . To each site $x \in \Lambda_L$, we associate a Hilbert space \mathcal{H}_x whose dimension is bounded uniformly in L , i.e. there exists some $N > 0$ such that for all $L \in \mathbb{N}$, we have $\dim(\mathcal{H}_x) \leq N$ for all $x \in \Lambda_L$. For a subset $X \subseteq \Lambda_L$, we define the Hilbert space $\mathcal{H}_X = \otimes_{x \in X} \mathcal{H}_x$, and we set $\mathcal{H}_L := \mathcal{H}_{\Lambda_L} = \otimes_{x \in \Lambda_L} \mathcal{H}_x$. For simplicity, throughout we take $L = 2^n$ for some large n .

The algebra $\mathcal{U}_L \subset \mathcal{B}(\mathcal{H}_L)$ of observables on Λ_L is the set of bounded operators on \mathcal{H}_L which are self-adjoint. For an operator $A_X \in \mathcal{U}_X$, we identify its extension to \mathcal{U}_L by taking its tensor product with copies of the identity, $A_X \otimes_{x \in X^c} \mathbb{I}_x$. Conversely, we say that an operator $A_X \in \mathcal{U}_L$ has support X if $A_X = A_X|_X \otimes_{x \in X^c} \mathbb{I}_x$, and write $A_X \in \mathcal{U}_X$.

A *local interaction* is a map $\Phi : \mathcal{P}(\Lambda_L) \rightarrow \mathcal{U}_L$ such that $\Phi(X) = 0$ whenever $\text{diam}(X) > R$, $\Phi(X)$ is supported in X , and $\|\Phi(X)\| \leq C$ for all $X \subset \Lambda_L$, for all L .

We consider a region as depicted in Figure 1. In the left white region $[0, L/2] \times [0, L]$, H_0 is a trivial Hamiltonian which we take to be the void (for example $H_0 = 0$), and in the right blue region $[L/2, L] \times [0, L]$, H_1 is a nontrivial *local Hamiltonian*, in the sense that $H_1 = \sum_{X \subseteq \Lambda} \Phi(X)$, where Φ is a local interaction. We define the Hamiltonian of the full system to be

$$H(\mu) = H_1 + \mu Q_h,$$

where $Q_h = \sum_{x \in \Gamma_h} a_x^* a_x$ is the number operator for the region $\Gamma_h = [L/4, 3L/4] \times [0, L]$.

We also consider the plane \mathbb{Z}^2 . In this setting, there are no edge states, and so the associated “bulk” Hamiltonian $H_B(\mu)$ is assumed to have a *gapped* spectrum, in the sense that

Assumption 3.

$$\sigma(H_B) = \mathcal{S}_- \cup \mathcal{S}_+,$$

where $\inf \mathcal{S}_+ - \sup \mathcal{S}_- \geq \gamma$ uniformly in L and μ for some $\gamma > 0$.

In the case of the cylinder, this effect does not necessarily occur due to the presence of the edge. We also assume that the Hamiltonian is *charge-conserving*.

Assumption 4. $[H(\mu), Q] = 0$, where Q is the total charge in Λ_L .

Let P_B be the ground state projection of H_B (the system without an edge), and let P be the ground state projection of H (the system with an edge). We assume that states far from the edge are essentially bulk states, up to tails that vanish quickly in L .

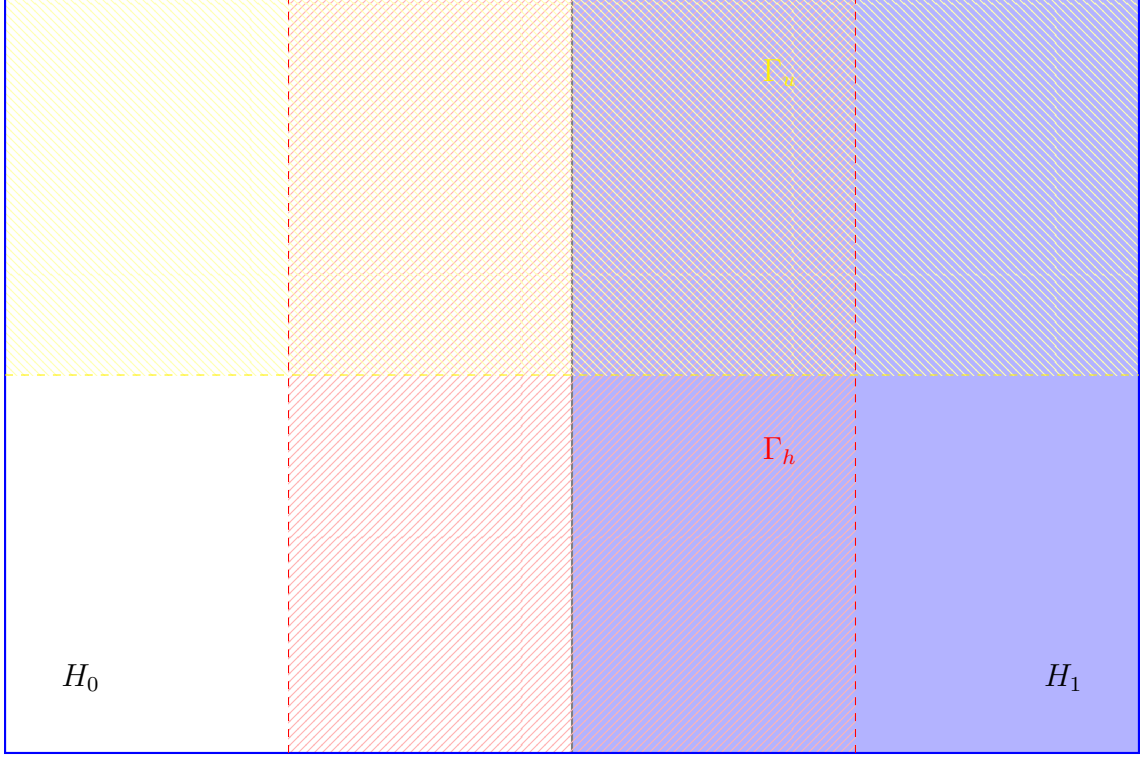


Figure 1: The setup for Λ_L . The left and right edges are identified to form a cylinder.

Assumption 5. Define the bulk region $\Gamma_B = [L/2 + k, L] \times [0, L]$ for some $k > 2R$. For any operator $A \in \mathcal{U}_{\Gamma_B}$,

$$\text{Tr}(PA) = \text{Tr}(P_B A) + \mathcal{O}(L^{-\infty}).$$

(need to add justification)

For ease of notation, we omit the subscript L wherever there is no risk of confusion.

4 Equality of Bulk and Edge Currents

4.1 Cylinder Geometry

Let $P(\mu)$ be the (possibly degenerate) ground state projection of $H(\mu)$. Let $Q_u = \sum_{x \in \Gamma_u} a_x^* a_x$ be the charge in the upper half of the cylinder $\Gamma_u = [0, L] \times [L/2, L]$, and define current operator

$$J_u = i[H(\mu), Q_u].$$

For simplicity, we drop the subscript u and simply write $J = J_u$. Charge conservation 4 implies that this current operator is supported on the strip $[L/2, L] \times [L/2 - R, L/2 + R]$, i.e. along a strip of width $2R$ centred at the line $y = L/2$. Indeed, if we inspect a local interaction $\Phi(X)$ of range R with support $S \subset (\Gamma_u)_R$, where $(\Gamma)_\alpha$ is the α -shrinking of Γ ,

then clearly $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$, by assumption 4. Similarly, if $\Phi(X)$ has support $S \subset ((\Gamma_u)^c)_R$, then $[\Phi(X), Q_u] = [\Phi(X), Q] = 0$. It follows that for an interaction $\Phi(X)$ with range R and arbitrary support, $[\Phi(X), Q_u]$ must be supported on a set which is contained in (or equal to) the strip $[L/2, L] \times [L/2 - R, L/2 + R]$, and so $[H, Q_u]$ must be as well, since H is a sum of such local interactions.

Lemma 1. *The ground state expectation of the current J is zero.*

Proof. By cyclicity of the trace and commutativity of $P(\mu)$ with $H(\mu)$,

$$\begin{aligned} \text{Tr}(P(\mu)J) &= \text{Tr}(P(\mu)i[H(\mu), Q_u]) \\ &= \text{Tr}(iP(\mu)H(\mu)Q_u) - \text{Tr}(iP(\mu)Q_uH(\mu)) \\ &= \text{Tr}(iH(\mu)P(\mu)Q_u) - \text{Tr}(iH(\mu)P(\mu)Q_u) \\ &= 0. \end{aligned}$$

□

Define the operators

$$K(\mu) = \mathcal{I}_\mu(\dot{H}(\mu)),$$

where

$$\mathcal{I}_\mu(A) = \int_{\mathbb{R}} W(t) e^{itH(\mu)} A e^{-itH(\mu)} dt.$$

More explicitly, in this setting we see that $K(\mu) = \mathcal{I}_\mu(Q_h)$. As a shorthand, we use the notation $\dot{A}(\mu_0) = \frac{d}{d\mu} A|_{\mu_0}$.

We present two important properties of the map \mathcal{I}_μ in the following lemmas, and leave their proofs to the appendix (need to add). We will also need a definition: an *off-diagonal* operator is an operator A such that $A = \bar{A} := PAP^\perp + P^\perp AP$, where $P^\perp = \mathbb{I} - P$ is the projection onto the excited states above the gap.

Lemma 2. *For any off-diagonal operator A , $\mathcal{I}_\mu(\cdot)$ and $[H(\mu), \cdot]$ act as inverses of each other, up to a factor of i :*

$$\mathcal{I}_\mu([H(\mu), A]) = [H(\mu), \mathcal{I}_\mu(A)] = iA$$

It is easy to verify that any operator A behaves as an off-diagonal operator when taking a commutator with P , in the sense that

$$[\bar{A}, P] = [PA(1 - P) + (1 - P)AP, P] = [A, P].$$

It follows that for any (not necessarily off-diagonal) operator A ,

$$[\mathcal{I}([H, A]), P] = i[A, P]$$

Lemma 3. \mathcal{I}_μ is local in the sense that for any $A \in \mathcal{U}_X$,

$$\|\mathcal{I}(A)_{(X^r)^c}\| \leq \|A\| |X| \mathcal{O}(r^{-\infty}).$$

Proposition 2. The operator $K(\mu)$ is the generator of parallel transport, satisfying

$$\dot{P}(\mu) = i[K(\mu), P(\mu)].$$

Proof. By the product rule and the fact that $H(\mu)$ and $P(\mu)$ commute,

$$[\dot{H}(\mu), P] = -[H, \dot{P}(\mu)].$$

Now we show that $\dot{P}(\mu)$ is off-diagonal. Taking the derivative on both sides of $P^2 = P$, we see that $\dot{P}P + P\dot{P} = \dot{P}$. Acting on the left and right with P on both sides of this equation gives

$$P\dot{P}P + P\dot{P}P = P\dot{P}P,$$

which implies that $P\dot{P}P = 0$. Now,

$$\begin{aligned} \overline{\partial_\mu P} &= P(\partial_\mu P)(1 - P) + (1 - P)(\partial_\mu P)P \\ &= P(\partial_\mu P) - P(\partial_\mu P)P + (\partial_\mu P)P - P(\partial_\mu P)P \\ &= P(\partial_\mu P) + (\partial_\mu P)P \\ &= \partial_\mu(P^2) \\ &= \partial_\mu P, \end{aligned}$$

as claimed. It therefore follows from Lemma 2 that

$$\begin{aligned} \dot{P}(\mu) &= -i\mathcal{I}_\mu([H(\mu), \dot{P}(\mu)]) \\ &= i\mathcal{I}_\mu([\dot{H}(\mu), P(\mu)]) \\ &= i[\mathcal{I}_\mu(\dot{H}(\mu)), P(\mu)] \\ &= i[K(\mu), P(\mu)]. \end{aligned}$$

□

Increasing the “electric potential” by a small amount $d\mu Q_h$ and expanding to linear order, the change in ground state current is given by

$$\text{Tr}(P(\mu + d\mu)J) - \text{Tr}(P(\mu)J) = \kappa d\mu + \mathcal{O}(d\mu^2).$$

Dividing by $d\mu$ and taking a limit, we see that the linear response coefficient is given by

$$\kappa = \text{Tr} \left(\dot{P}(\mu)J \right).$$

The *Hall conductivity* of the system on a subset $V \subseteq \Lambda$ is defined to be $\kappa_V := \text{Tr} \left(\dot{P}(\mu)J_V \right)$, where J_V is the restriction of the operator J to V .

Proposition 3. *The Hall conductivity is independent of the driving strength μ .*

Proof. We see by cyclicity of the trace and the formula $\dot{H}(\mu) = Q_h$ that for any μ_1 and μ_2 ,

$$\begin{aligned}
\kappa(\mu_1) - \kappa(\mu_2) &= \text{Tr} \left(\dot{P}(\mu_1) i[H(\mu_1), Q_u] - \dot{P}(\mu_2) i[H(\mu_1), Q_u] \right) \\
&= i \text{Tr} \left(\left([\dot{P}(\mu_1), H(\mu_1)] - [\dot{P}(\mu_2), H(\mu_2)] \right) Q_u \right) \\
&= i \text{Tr} \left(\left([\dot{H}(\mu_1), P(\mu_1)] - [\dot{H}(\mu_2), P(\mu_2)] \right) Q_u \right) \\
&= i \text{Tr} ([Q_h, P(\mu_1) - P(\mu_2)] Q_u) \\
&= i \text{Tr} (Q_h (P(\mu_1) - P(\mu_2)) Q_u - (P(\mu_1) - P(\mu_2)) Q_u Q_h) \\
&= 0,
\end{aligned}$$

since Q_h and Q_u commute, indicating that the Hall conductivity is independent of μ as one would expect physically. \square

The following is the main result:

Theorem 2. *The ground state current in the strip $[L/2 + k, 3L/4 - k] \times [0, L]$ between the edge and the bulk current vanishes, in the sense that $\kappa_V = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty})$ for any $V \subseteq [L/2 + R, 3L/4 - R] \times [0, L]$ “in between” the bulk and edge strips, where*

$$r = \text{dist}(V, [L/2 - R, 3L/4 + R] \times [0, L] \cup [3L/4 - R, 3L/4 + R] \times [0, L])$$

is the distance from V to one of the edge or bulk strips.

Proof. By Proposition 2, the Hall conductivity can also be written by the formula $\kappa_V^B = \text{Tr} (i[K(\mu), P_B(\mu)] J_V^B) = \text{Tr} (i[\mathcal{I}_\mu(Q_h), P_B(\mu)] J_V^B)$, where $J_V^B = (i[H_B, Q_u])_V$ is the current arising from the bulk Hamiltonian. From commutativity of P_B and H_B along with cyclicity of the trace, we compute

$$\begin{aligned}
\kappa_V^B &= \text{Tr} (i[\mathcal{I}_\mu(Q_h), P_B(\mu)] J_V^B) \\
&= \text{Tr} \left(i \int_{\mathbb{R}} W(t) e^{itH_B(\mu)} [Q_h, P_B(\mu)] e^{-itH_B(\mu)} dt J_V^B \right) \\
&= \int_{\mathbb{R}} W(t) \text{Tr} (i[Q_h, P_B(\mu)] e^{-itH_B(\mu)} J_V^B e^{itH_B(\mu)}) dt \\
&= - \int_{\mathbb{R}} W(t) \text{Tr} (i[Q_h, P_B(\mu)] e^{itH_B(\mu)} J_V^B e^{-itH_B(\mu)}) dt \\
&= - \text{Tr} (i[Q_h, P_B(\mu)] \mathcal{I}_\mu(J_V^B)),
\end{aligned}$$

since $W(t)$ is odd. Again by cyclicity of trace combined with the fact that $\mathcal{I}_\mu(\cdot)$ is an inverse of $[H_B(\mu), \cdot]$ for commutators with $P_B(\mu)$ (by the remark after lemma 2), we obtain

$$\begin{aligned}
\kappa_V^B &= - \text{Tr} ([\mathcal{I}_\mu([H_B(\mu), Q_h]), P_B(\mu)] \mathcal{I}_\mu(J_V^B)) \\
&= - \text{Tr} (\mathcal{I}_\mu([H_B(\mu), Q_h]) P_B(\mu) \mathcal{I}_\mu(J_V^B) - P_B(\mu) \mathcal{I}_\mu([H_B(\mu), Q_h]) \mathcal{I}_\mu(J_V^B)) \\
&= - \text{Tr} (P_B(\mu) \mathcal{I}_\mu(J_V^B) \mathcal{I}_\mu([H_B(\mu), Q_h]) - P_B(\mu) \mathcal{I}_\mu([H_B(\mu), Q_h]) \mathcal{I}_\mu(J_V^B)) \\
&= \text{Tr} (P_B(\mu) [\mathcal{I}_\mu([H_B(\mu), Q_h]), \mathcal{I}_\mu(J_V^B)]).
\end{aligned}$$

Now, $[H_B(\mu), Q_h]$ is a local operator, supported on the “bulk line” $[3L/4 - R, 3L/4 + R] \times [0, L]$, while J_V^B is a local operator supported on $V \subseteq [L/2 + k, 3L/4 - k] \times [0, L]$. Since \mathcal{I}_μ preserves locality up to tails, in the sense that $\|\mathcal{I}_\mu(A)_{(S^r)^c}\| \leq \|A\|S|\mathcal{O}(r^{-\infty})$ (Lemma 3), it follows that the commutator $[\mathcal{I}_\mu([H_B(\mu), Q_h]), \mathcal{I}_\mu(J_V^B)] = C\mathcal{O}(r^{-\infty})$ whenever $V \cap ([3L/4 - R, 3L/4 + R] \times [L/2, L]) = \emptyset$.

The previous fact applies to the bulk setting with H_B and P_B . To extend this to the setting with an edge, it is enough to use Assumption 5 to conclude the same result, except with equality up to $\mathcal{O}(L^{-\infty})$, i.e.

$$\kappa_V = \text{Tr}(\dot{P}J_V) = \text{Tr}(\dot{P}(J_V^B + \mathcal{O}(L^{-\infty}))) = \kappa_V^B + \mathcal{O}(L^{-\infty}) = \mathcal{O}(r^{-\infty}) + \mathcal{O}(L^{-\infty}).$$

□

The intuitive picture from the previous result is that, in the bulk region, the Hall conductivity is essentially only nonzero along the bulk line $[3L/4 - R, 3L/4 + R] \times [L/2, L]$. Since the ground state expectation of the current is zero (by lemma 1), it must be that there is an equal current flowing along the edge, but in the opposite direction, see figure (need to add).

4.2 Torus Geometry

Our goal is to show the same result on the discrete torus $\mathbb{T}_L := \mathbb{Z}_L \times \mathbb{Z}_L$. We define the same regions Γ_u and Γ_h , and the same current operator $J_u = i[H(\mu), Q_u]$. This time, however, Lemma 1 does not apply. Intuitively, it does not apply because electrons can now flow through both the bottom and the top of the region Γ_u , rather than just the bottom. Mathematically, the lemma fails because our definition of the current is slightly changed.

We use charge conservation and the fact that H is finite range to split the current J_u into two components, $J_u = i[H_-, Q_u] + i[H_+, Q_u] = J_- - J_+$, supported on strips of width $2R$ at $y = L/2$ and $y = L$, respectively. We then define the current operator to be $J = J_-$, which is the current on the lower strip. This is the mathematical reason that the proof in Lemma 1 fails on the torus; we have replaced H by H_- , which may no longer commute with P . We instead proceed by a different approach. We will need a few auxiliary results first.

Lemma 4. K_\pm is supported on ∂_\pm up to tails.

Proof.

□

Proposition 4. The operator $Q_h - K$ leaves the ground state space invariant, i.e. $[Q_h - K, P] = 0$.

Proof.

□

Lemma 5. Show that $\text{Tr}(A, [Q_h, P]) = 0$ for all $A \in \mathcal{U}_{\text{edge}}$. This shows that Q_h commutes with P “along the edge”.

Proof. Let $A \in \mathcal{U}_{\text{edge}}$. Since H is charge conserving, we may choose a simultaneous eigenbasis of H and the total charge Q , in which case P and Q commute. It follows that

$$\text{Tr}(A[Q_h, P]) = \text{Tr}([A, Q_h]P) = \text{Tr}([A, Q]P) = \text{Tr}(A[Q, P]) = 0.$$

□

Finally, we will prove that in the bulk system with Hamiltonian $H_B(\mu)$, the ground state expectation of the current vanishes faster than any power as $L \rightarrow \infty$.

Lemma 6. *The ground state expectation of the current $J_B := i[(H_B)_-, Q_h]$ (of the system without an edge) is $\text{Tr}(P_B J_B) = \mathcal{O}(L^{-\infty})$.*

Proof. First, $K = \mathcal{I}(i[H_B, Q])$ splits into $K = K_- - K_+$, with the support of K_{\pm} contained in ∂_{\pm} up to tails:

$$[K_{\pm}, A_X] = \mathcal{O}(p^{-\infty}),$$

for every $A_X \in \mathcal{U}_X$ such that $\|A_X\| = 1$, and where $p = \text{dist}(X, \partial_{\pm})$ (need to add). Using the fact that K_{\pm} is supported in ∂_{\pm} up to tails (Lemma 4), we see that

$$i[H_B, K_-] = i[(H_B)_-, K_-] + \mathcal{O}(L^{-\infty}),$$

and similarly $i[(H_B)_-, K_+] = \mathcal{O}(L^{-\infty})$. Putting these facts together, it follows that the current can be rewritten as

$$\begin{aligned} J_B &= i[H_B, Q_h + K_- - K_- + K_+] + \mathcal{O}(L^{-\infty}) \\ &= i[H_B, K_-] + i[(H_B)_-, Q_h - K_- + K_+] + \mathcal{O}(L^{-\infty}). \end{aligned}$$

From here, we use the fact that H_B and $Q_h - K_- + K_+$ both commute with P_B to write

$$P_B J_B P_B = i[H_B, P_B K_- P_B] + i[P_B (H_B)_- P_B, Q_h - K_- + K_+] + P_B \mathcal{O}(L^{-\infty}) P_B.$$

Since the trace of any commutator is zero,

$$\text{Tr}(P_B J_B) = \text{Tr}(P_B J_B P_B) = \mathcal{O}(L^{-\infty}).$$

□

Using this, we can show a simple proof of the analogue of Lemma 1 on the torus, in the case of non-interacting systems.

Proposition 5. *Let $H = \sum_{x \in \mathbb{T}} h_x$ be a non-interacting Hamiltonian, i.e. a sum of single site Hamiltonians h_x . The ground state expectation of the current $J = i[H_-, Q_h]$ (of the system with an edge) is $\text{Tr}(P J) = \mathcal{O}(L^{-\infty})$.*

Proof. Since H is a sum of single site Hamiltonians, we can split H_- into the restrictions $H_- = (H_-)_{\text{edge}} + (H_-)_{\text{bulk}}$, with no fear of any terms which are in both the edge region and the bulk region. By Assumption 5,

$$\begin{aligned} \text{Tr}(P J) &= \text{Tr}(P i[H_-, Q_h]) \\ &= i \text{Tr}([H_-, Q_h] P) \\ &= i \text{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i \text{Tr}((H_-)_{\text{bulk}}[Q_h, P]) \\ &= i \text{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i \text{Tr}((H_-)_{\text{bulk}}[Q_h, (P)_{\text{bulk}}]) \\ &= i \text{Tr}((H_-)_{\text{edge}}[Q_h, P]) + i \text{Tr}((H_B)_-[Q_h, P_B]) + \mathcal{O}(L^{-\infty}) \\ &= i \text{Tr}((H_-)_{\text{edge}}[Q_h, P]) + \text{Tr}(i[(H_B)_-, Q_h] P_B) + \mathcal{O}(L^{-\infty}). \end{aligned}$$

By Lemma 5, the first term is zero. By Lemma 6, the second term is $\mathcal{O}(L^{-\infty})$. □

A Properties of \mathcal{I}_μ

Proof. (Of Lemma 2). Let $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} W(t) e^{-2\pi i t \xi} dt$ be the Fourier transform of W . One can show that for $|\xi| \geq \gamma$, $\widehat{W}(\xi) = \frac{1}{\sqrt{2\pi i \xi}}$ (need to add). Let A be an observable. First, we show that $\mathcal{I}([H, PAP^\perp]) = iPAP^\perp$.

Decomposing

$$\begin{aligned} e^{itH}P &= \sum_{j=0}^{\infty} \frac{(itH)^j}{j!} P \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \left(\sum_n E_n^j P_n \right) P \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \sum_{n:E_n=0} E_n^j P_n \\ &= \sum_{n:E_n=0} e^{itE_n} P_n, \end{aligned}$$

and similarly

$$P^\perp e^{-itH} = \sum_{m:E_m \geq \gamma} P_m e^{-itE_m},$$

we see that

$$\begin{aligned} \mathcal{I}([H, PAP^\perp]) &= \mathcal{I}(P[H, A]P^\perp) \\ &= \int_{\mathbb{R}} W(t) e^{itH} P[H, A] P^\perp e^{-itH} dt \\ &= \int_{\mathbb{R}} W(t) \sum_{n:E_n=0} e^{itE_n} P_n [H, A] \sum_{m:E_m \geq \gamma} P_m e^{-itE_m} dt \\ &= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} \int_{\mathbb{R}} W(t) e^{itE_n} P_n A(E_n - E_m) P_m e^{-itE_m} dt \\ &= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \int_{\mathbb{R}} W(t) e^{-it(E_m - E_n)} dt \\ &= \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m (E_n - E_m) \sqrt{2\pi} \widehat{W}(E_m - E_n) \\ &= i \sum_{n:E_n=0} \sum_{m:E_m \geq \gamma} P_n A P_m \\ &= iPAP^\perp. \end{aligned}$$

(need to check the 2π factor)

By the same argument, $\mathcal{I}([H, P^\perp A P]) = iP^\perp A P$ as well, and so $\mathcal{I}([H, \overline{A}]) = i\overline{A}$. \square

Proof. (Of Lemma 3). We break the integral into two parts,

$$\|\mathcal{I}(A)\| \leq \left\| \int_{-T}^T W(t) e^{itH} A e^{-itH} dt \right\| + \left\| \int_{\mathbb{R} \setminus [-T, T]} W(t) e^{itH} A e^{-itH} dt \right\|.$$

The first term can be estimated using the Lieb-Robinson bound found in Appendix B. \square

B Lieb-Robinson Bound

Let N be a uniform upper bound for the dimensions of the Hilbert spaces at each site, i.e. $\dim(\mathcal{H}_x) \leq N$ for all sites x .

The following is a version of the Lieb-Robinson. For any operators $A \in \mathcal{U}_X$ and $B \in \mathcal{U}_Y$ having disjoint supports $X \cap Y = \emptyset$,

$$\| [e^{itH} A e^{-itH}, B] \| \leq C \|A\| \|B\| |X| |Y| N^{2|X|} e^{2t\|\Phi\|_\lambda - \lambda d(X, Y)}.$$

C Grönwall's Inequality and Uniqueness

Theorem 3. (*Grönwall's Inequality*). Let $\alpha : I \rightarrow (0, \infty)$ be positive and continuous on I° for some interval of the form $[a, b)$, $[a, b]$, or $[a, \infty)$. Suppose $u : \mathbb{R} \rightarrow \mathcal{U}$ is a Banach-valued, differentiable function. If $\|u'(t)\| \leq \alpha(t)\|u(t)\|$ for all $t \in I$, then

$$\|u(t)\| \leq \|u(a)\| e^{\int_a^t \alpha(s) ds} \quad \forall t \in I$$

Proof. Let $f(t) = e^{\int_a^t \alpha(s) ds}$, which is nonzero, has initial value $f(a) = 1$, and has derivative $f'(t) = \alpha(t)f(t)$. Then by the quotient rule,

$$\left(\frac{\|u(t)\|}{f(t)} \right)' = \frac{\|u'(t)\|f(t) - \|u(t)\|\alpha(t)f(t)}{f(t)^2} \leq 0,$$

where the inequality follows from the assumption $\|u'(t)\| \leq \alpha(t)\|u(t)\|$. Thus $\frac{\|u(t)\|}{f(t)}$ is decreasing, so that

$$\frac{\|u(t)\|}{f(t)} \leq \frac{\|u(a)\|}{f(a)} = \|u(a)\|,$$

which is the desired inequality. \square

Theorem 4. (*ODE Uniqueness*). Let $F : \mathcal{U} \rightarrow \mathcal{U}$ be Lipschitz and consider the differential equation $u'(t) = F(u(t))$ with initial condition $u(a) = u_a$ for some function $u : I \rightarrow \mathcal{U}$, where $I = [a, b]$, or $[a, b)$, or $[a, \infty)$. Solutions to this equation are unique.

Proof. Suppose there are two solutions $u(t)$ and $v(t)$, and let $g(t) = \|u(t) - v(t)\|^2$. By assumption, there exists a constant L_F such that $\|F(u(t)) - F(v(t))\| \leq L_F \|u(t) - v(t)\|$, so that

$$\begin{aligned} g'(t) &= 2\|u(t) - v(t)\| \|u'(t) - v'(t)\| \\ &= 2\|u(t) - v(t)\| \|F(u(t)) - F(v(t))\| \\ &\leq 2L_F \|u(t) - v(t)\|^2 \\ &= 2L_F g(t). \end{aligned}$$

Notice that $\alpha := 2L_F$ is a positive continuous function, so we may apply Grönwall's inequality to $g(t)$ to conclude

$$g(t) \leq g(a) e^{2L_F(t-a)} = 0,$$

since $g(a) = 0$. □

D Note on Generators of Parallel Transport

Consider the differential equation $\dot{\rho}(\mu) = i[K_B, \rho(\mu)]$ with initial condition $\rho(0) = P_B(0)$. Here $K_B = \int_{\mathbb{R}} W_\gamma(t) e^{-itH_B} \dot{H}_B e^{itH_B} dt$, and recall that in our setting, $\dot{H}_B = Q_h$. We know that the solution is $\rho(\mu) = P_B(\mu)$ (proposition 2). Notice that the map $F : \mathcal{U} \rightarrow \mathcal{U}$ defined by $F(A) = i[K_B, A]$ is Lipschitz, since

$$\|F(A) - F(B)\| = \|[K_B, A - B]\| \leq 2\|K_B\| \|A - B\|.$$

The Lipschitz constant is $2\|K_B\|$, which is finite since K_B is a bounded operator:

$$\|K_B\| \leq \int_{\mathbb{R}} |W_\gamma(t)| \|e^{-itH_B} Q_h e^{itH_B}\| dt \leq \int_{\mathbb{R}} |W_\gamma(t)| dt \|Q_h\|.$$

Indeed, since Q_h is the number operator on a finite volume, by charge conservation and the fact that the dimension of the Hilbert space is bounded uniformly by d , there can only be a finite number of charges in the region Γ_h .

Thus, by Grönwall's uniqueness theorem (appendix C), we see that the solution to the equation $\dot{\rho}(\mu) = F(\rho(\mu)) = i[K_B, \rho(\mu)]$ is unique.

Now define

$$K_E := \int_{\mathbb{R}} W_\gamma(t) e^{-itH_E} Q_h e^{itH_E} dt,$$

which is using the gap γ of H_B to define W_γ , but also using the edge Hamiltonian in the time evolution operators. Consider $\sigma : [0, \infty) \rightarrow \mathcal{U}$ defined by

$$\dot{\sigma}(\mu) = i[K_E, \sigma(\mu)] \quad \sigma(0) = P_E(0).$$

We now show that, similar to how ρ is an approximation of P_B , σ is also a good approximation of P_E (up to $\mathcal{O}(L^{-\infty})$) “in the bulk”. Let $A \in \Gamma_B$ be an operator localized in the bulk of the edge system. Then

$$\begin{aligned}
\text{Tr}(\dot{\sigma}A) &= \text{Tr}(i[K_E, \sigma]A) \\
&= \text{Tr}(i[A, K_E]\sigma) \\
&= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}([e^{-itH_E} Q_h e^{itH_E}, A]\sigma) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}(e^{-itH_E} [Q_h, e^{itH_E} A e^{-itH_E}] e^{itH_E} \sigma) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}(e^{-itH_E} [Q_h, e^{itH_B} A e^{-itH_B}] e^{itH_E} + \mathcal{O}(L^{-\infty})\sigma) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}(e^{-itH_B} [Q_h, e^{itH_B} A e^{-itH_B}] e^{itH_B} + \mathcal{O}(L^{-\infty})\sigma) dt \\
&= \int_{\mathbb{R}} W_\gamma(t) \text{Tr}([e^{-itH_B} Q_h e^{itH_B}, A]\sigma) dt + \mathcal{O}(L^{-\infty}) \\
&= \text{Tr}(i[A, K_B]\sigma) + \mathcal{O}(L^{-\infty}) \\
&= \text{Tr}(i[K_B, \sigma]A) + \mathcal{O}(L^{-\infty}),
\end{aligned}$$

since σ is trace-class (?) and $W_\gamma \in L^1$. By linearity of the trace, we see that $\text{Tr}((\dot{\sigma} - i[K_B, \sigma])A) = \mathcal{O}(L^{-\infty})$ for any operator $A \in \Gamma_B$ (does this mean $\dot{\sigma} - i[K_B, \sigma] = 0$?). But the solution of $\dot{\sigma} - i[K_B, \sigma] = 0$ (with initial condition $\sigma(0) = P_B(0)$) is unique; it is $\rho(\mu)$, or $P_B(\mu)$. Hence

$$\text{Tr}(P_E A) = \text{Tr}(P_B A) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\rho A) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\sigma A) + \mathcal{O}(L^{-\infty})$$

for any operator $A \in \Gamma_B$. In particular, this gives another local formula for the Hall conductivity in the bulk of an edge system, by taking $A = J_V$, where J is the current operator and $V \subset \Gamma_B$ is a set localized in the bulk. The Hall conductivity is given by $\text{Tr}(\dot{P}_E J_V)$, and this can be approximated by

$$\text{Tr}(\dot{P}_E J_V) = \text{Tr}(\dot{P}_B J_V) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\dot{\rho} J_V) + \mathcal{O}(L^{-\infty}) = \text{Tr}(\dot{\sigma} J_V) + \mathcal{O}(L^{-\infty}).$$

Want to pick a norm s.t. Gronwall gives $\|\rho(\mu) - \sigma(\mu)\|_G \leq \|P_B(0) - P_E(0)\|_G e^{2L_F \mu}$. Need $\|P_B(0) - P_E(0)\|_G$ to be small enough to kill the exponential which depends on $L_F = 2\|K_B\|_G \leq \|W_\gamma\|_{L^1} \|Q_h\|_G$. If we use the operator norm for $\|\cdot\|_G$, we would get $\|Q_h\|_G = d|\Gamma_h|$ in the exponent. Need $\|\cdot\|_G$ to be an actual norm so that $\|\rho - \sigma\|_G = 0 \implies \rho = \sigma$.

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Let $r(t) = \rho(t) - \sigma(t)$. Notice that

$$\frac{d}{dt} e^{itK_B} \sigma_0 e^{-itK_B} = e^{itK_B} i[K_B, \sigma_0] e^{-itK_B} + e^{-itK_B} \dot{\sigma}_0 e^{itK_B}.$$