Computational Physics - Project 4

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1 Introduction

In this report, we investigate the 1-dimensional and 2-dimensional diffusion equations, given respectively below:

$$\frac{\partial^2 U}{\partial x^2} = D\frac{\partial U}{\partial t}$$

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = D\frac{\partial U}{\partial t}$$

with boundary conditions $U(0,t) = 0$, $U(L,t) = 1$, and initial condition $U(x,0) = 0$. We will analyze this PDE using three methods:

- The forward Euler algorithm (explicit)
- The backward Euler algorithm (implicit)
- The Crank-Nicholson algorithm (implicit)

2 Analytical Results

We first solve the 1D solution analytically. Assume that the solution can be written as the sum of a transient part W and a steady-state part V:

$$U(x,t) = W(x,t) + V(x)$$

So V must, on its own, satisfy the PDE and hence $V''(x) = 0$ which implies that the solution takes the form $V(x) = Ax + B$. The boundary conditions imply that $B = 0$ and $A = 1/L$, giving the unsurprising steady state

$$V(x) = \frac{x}{L}$$

Now, we will solve for $W = U - V$ as it turns out this will be simpler than solving for U. the boundary conditions from W become:

- $U(0,t) = 0 \implies W(0,t) = 0$
- $U(L,t) = 1 \implies W(L,t) = 0$
- $U(x,0) = 0 \implies W(x,0) = -x/L$

Now, we shall solve for W using separation of variables. Suppose that a solution takes the form $W(x,t) = X(x)T(t)$. Then the differential equation for U (and hence for W) reads

$$\frac{\partial^2 X}{\partial x^2} T = X \frac{\partial T}{\partial t}$$

Rearranging, we find that

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} = \frac{1}{T}\frac{\partial T}{\partial t} = -\lambda$$

The left side is purely a function of x and the right side is purely a function of t, so we set both sides equal to a constant which I've denoted by $-\lambda$.

We begin by solving the X equation. It is simple to check that the cases $\lambda < 0$ and $\lambda = 0$ lead to trivial solutions $X(x) = 0$ due to the boundary conditions. So the only case of interest is $\lambda > 0$. In this case, the solution is just

$$X(x) = A \sin(\sqrt{\lambda} x) + B \cos(\sqrt{\lambda} x)$$

The boundary conditions give

$$X(0) = 0 \implies B = 0$$

$$X(L) = 0 \implies \sqrt{\lambda} = \frac{n\pi}{L}; \quad n = 1, 2, 3, \ldots$$

Thus there are infinitely many solutions of the form $X_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right)$.

Now, for the $T(t)$ solution, the $\lambda > 0$ case gives an exponential solution

$$T(t) = Ce^{-\lambda t} + De^{+\lambda t}$$

Again, the boundary conditions give $D = 0$ and $\lambda = n^2\pi^2/L^2$, for $n \in \mathbb{Z}_+$. Hence there are infinitely many solutions of the form $T(t) = c_n e^{-\frac{n^2\pi^2}{L^2}t}$. The solution for W is thus the linear combination

$$W(x,t) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right) e^{-\frac{n^2\pi^2}{L^2}t}$$

But we still need to satisfy the initial condition $W(0,t) = -x/L$, i.e.

$$-\frac{x}{L} = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi}{L}x\right)$$

We use the orthogonality relation of sines to do this. Recall that

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx = \frac{1}{2}\delta_{mn}$$

So, multiplying our condition equation by $\sin\left(\frac{m\pi x}{L}\right)$ and integrating both sides gives

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\left(-\frac{x}{L}\right) = \int_0^L \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx$$

Which implies that

$$B_m = -2\frac{sin(m\pi) - m\pi\cos{(m\pi)}}{\pi^2 m^2 L} = 2\frac{(-1)^{m+1}}{\pi m L}$$

And thus we have our full solution for W, and therefore for $U = V + W$, which is

$$U(x,t) = \frac{x}{L} + \sum_{n=1}^{\infty} 2\frac{(-1)^{n+1}}{\pi n L}\sin\left(\frac{n\pi x}{L}\right)e^{-\frac{n^2\pi^2}{L^2}t}$$

Later, we will compare this to the numerical solutions.

3 Numerical Analysis

In this section we discretize time and space into grids, in which case I'll adopt the notation $U_i^j := U(x_i, t_j)$.

3.1 Forward Euler Method

The forward Euler method is forward in time, centred in space:

$$\frac{\partial U}{\partial t} \approx \frac{U_i^{j+1} - U_i^j}{\Delta t}$$

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{\Delta x^2}$$

In which case the one-dimensional heat equation reads (approximately)

$$\frac{U_i^{j+1} - U_i^j}{\Delta t} = \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{\Delta x^2}$$

Rearranging to solve for the $j+1$ time step and introducing $\alpha := D\Delta t/\Delta x^2$:

$$U_i^{j+1} = U_i^j + \alpha \left(U_{i+1}^j - 2U_i^j + U_{i-1}^j \right)$$

3.1.1 Truncation Error of the Forward Euler Method

We begin by substituting the true function $U(x,t)$ into our difference equation, and finding the difference between the left and right sides. I'll denote this difference by $D(x,t)$.

$$D(x,t) = \frac{U(x,t+\Delta t) - U(x,t)}{\Delta t} - \frac{U(x+\Delta x,t) - 2U(x,t+\Delta t) + U(x-\Delta x,t)}{\Delta x^2}$$

Taylor expanding the derivatives of U we get:

$$\frac{\partial U}{\partial t} + \mathcal{O}(\Delta t) = \frac{U(x,t+\Delta t) - U(x,t)}{\Delta t}$$

$$\frac{\partial^2 U}{\partial x^2} + \mathcal{O}(\Delta x^2) = \frac{U(x+\Delta x,t) - 2U(x,t) + U(x-\Delta x,t)}{\Delta x^2}$$

Substituting these into the difference equation and recalling that $U_t = U_{xx}$:

$$\begin{aligned} D(x,t) &= \frac{\partial U}{\partial t} + \mathcal{O}(\Delta t) - \frac{\partial^2 U}{\partial x^2} + \mathcal{O}(\Delta x^2) \\ &= \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2) \end{aligned}$$

And thus the local truncation error of the forward Euler method is $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$.

3.1.2 Stability Analysis of the Forward Euler Method

Write $U_i^j = \hat{U}^j e^{ikx_i}$. Substituting this into the forward Euler algorithm and applying some basic trigonometry gives

$$\begin{aligned}\hat{U}_i^{j+1} &= \hat{U}^j\left((1-2\alpha) + 2\alpha\cos\left(k\Delta x\right)\right) \\ &= (1 - 4\alpha\sin^2\left(k\Delta x/2\right))\hat{U}^j\end{aligned}$$

Thus $g = 1 - 4\alpha\sin^2\left(k\Delta x/2\right)$, hence for stability we require

$$|1 - 4\alpha\sin^2\left(k\Delta x/2\right)| \leq 1$$

Now, $\sin^2(k\Delta x/2)$ attains extreme values of 0 and 1. So for the above inequality to hold, we require that $0 \leq \alpha \leq 1/2$. Recall that $\alpha = D\Delta t/\Delta x^2$, hence the stability condition for the forward Euler method is

$$\Delta t \leq \frac{1}{2}\frac{\Delta x^2}{D}$$

3.2 Backward Euler Method

The backward Euler method is backward in time, centered in space:

$$\frac{\partial U}{\partial t} \approx \frac{U_i^j - U_i^{j-1}}{\Delta t}$$

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{\Delta x^2}$$

In which case the one-dimensional heat equation reads (approximately)

$$\frac{U_i^j - U_i^{j-1}}{\Delta t} = \frac{U_{i+1}^j - 2U_i^j + U_{i-1}^j}{\Delta x^2}$$

Rearranging so that the $(j-1)^{\text{th}}$ time step is on the left, again with the same definition of α,

$$-U_i^{j-1} = \alpha U_{i+1}^j - (1+2\alpha)U_i^j + \alpha U_{i-1}^j$$

Observe that if we define the following matrices and vectors, the problem can be written in linear algebra terms

$$A = \begin{pmatrix} 1+2\alpha & -\alpha & 0 & \dots & 0 \\ -\alpha & 1+2\alpha & -\alpha & & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & & -\alpha & 1+2\alpha & -\alpha \\ 0 & \dots & 0 & -\alpha & 1+2\alpha \end{pmatrix}$$

$$\mathbf{U}_j = \begin{pmatrix} U_{2,j} \\ U_{3,j} \\ \vdots \\ U_{N-2,j} \\ U_{N-1,j} \end{pmatrix}$$

$$\mathbf{b}_j = \begin{pmatrix} \alpha U_{1,j} \\ 0 \\ \vdots \\ 0 \\ \alpha U_{N,j} \end{pmatrix}$$

We see that the equation can be rewritten in matrix form as

$$A\mathbf{U}_{j+1} = \mathbf{U}_j + \mathbf{b}_j$$

Where A is a tridiagonal matrix.

3.2.1 Truncation Error of the Backward Euler Method

Again we substitute the true solution U into our difference equation and taking the difference of the left and right sides:

$$D(x,t) = \frac{U(x,t+\Delta t) - U(x,t)}{\Delta t} - \frac{U(x+\Delta x, t+\Delta t) - 2U(x,t+\Delta t) + U(x-\Delta x, t+\Delta t)}{\Delta x^2}$$

Taylor expanding the derivatives of U we get:

$$\frac{\partial U}{\partial t} + \mathcal{O}(\Delta t) = \frac{U(x,t+\Delta t) - U(x,t)}{\Delta t}$$

$$\frac{\partial^2 U}{\partial x^2} + \mathcal{O}(\Delta x^2) = \frac{U(x+\Delta x, t+\Delta t) - 2U(x, t+\Delta t) + U(x-\Delta x, t+\Delta t)}{\Delta x^2}$$

Substituting these into the difference equation and recalling that $U_t = U_{xx}$:

$$\begin{aligned} D(x,t) &= \frac{\partial U}{\partial t} + \mathcal{O}(\Delta t) - \frac{\partial^2 U}{\partial x^2} + \mathcal{O}(\Delta x^2) \\ &= \mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2) \end{aligned}$$

Hence the local truncation error of the backward Euler method is the same as the forward Euler method: $\mathcal{O}(\Delta t) + \mathcal{O}(\Delta x^2)$.

3.2.2 StabilityAnalysis of the Backward Euler Method

Again write $U_i^j = \hat{U}^j e^{ikx_i}$. Substituting this into the backward Euler algorithm and applying some basic trigonometry gives

$$\begin{aligned} -\hat{U}_i^j e^{ikx_i} &= \alpha e^{ikx_{i+1}} \hat{U}^{j+1} - (1+2\alpha) e^{ikx_i} \hat{U}^{j+1} + \alpha e^{ikx_{i-1}} \hat{U}^{j+1} \\ &= \alpha e^{ik(x+\Delta x)} \hat{U}^{j+1} - (1+2\alpha) e^{ikx} \hat{U}^{j+1} + \alpha e^{ik(x-\Delta x)} \hat{U}^{j+1} \\ &= e^{ikx} \hat{U}^{j+1} \left(\alpha e^{ik\Delta x} - (1+2\alpha) + \alpha e^{ik\Delta x} \right) \end{aligned}$$

Cancelling the exponentials and recognizing that $e^{ik\Delta x} + e^{-ik\Delta x}$ is a cosine, we can rearrange for $\hat{U}^{j+1}$:

$$\begin{aligned} \hat{U}^{j+1} &= \frac{1}{1 - 2\alpha(\cos(k\Delta x) - 1)} \hat{U}^j \\ &= \frac{1}{1 + 4\alpha \sin^2(k\Delta x/2)} \hat{U}^j \end{aligned}$$

So we have

$$g = \frac{1}{1 + 4\alpha \sin^2(k\Delta x/2)}$$

But notice that $\sin^2(k\Delta x/2)$ can only take values between 0 and 1, hence we always have (for any value of α) that

$$|g|^2 = \left| \frac{1}{1 + 4\alpha \sin^2(k\Delta x/2)} \right|^2 \leq 1$$

Hence the backward Euler method is *unconditionally stable.*

3.3 Crank-Nicolson Method

The Crank-Nicolson Method essentially averages the forward and backward Euler methods by using a time-centered scheme. The scheme is based on half-time steps and is

$$\frac{U_i^{j+1} - U_i^j}{\Delta t} = D \frac{U_{i+1}^{j+1/2} - 2U_i^{j+1/2} + U_{i-1}^{j+1/2}}{\Delta x^2}$$

Or

$$\frac{U_i^{n+1} - U_i^n}{\Delta t} = D \frac{(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) + (U_{i+1}^n - 2U_i^n + U_{i-1}^n)}{2\Delta x^2}$$

This can again be rearranged into the following form, with the $(j+1)^{\text{th}}$ terms on the left

$$-\alpha U_{i-1}^{j+1} + (2 + 2\alpha) U_i^{j+1} - \alpha U_{i+1}^{j+1} = \alpha U_{i-1}^j + (2 - 2\alpha) U_i^j + \alpha U_{i+1}^j$$

Again in matrix form we see this is a tridiagonal problem. Defining

$$A = \begin{pmatrix} 2+2\alpha & -\alpha & 0 & \dots & 0 \\ -\alpha & 2+2\alpha & -\alpha & & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & & -\alpha & 2+2\alpha & -\alpha \\ 0 & \dots & 0 & -\alpha & 2+2\alpha \end{pmatrix}$$

$$B = \begin{pmatrix} 2-2\alpha & \alpha & 0 & \dots & 0 \\ \alpha & 2-2\alpha & \alpha & & \vdots \\ 0 & & \ddots & & 0 \\ \vdots & & \alpha & 2-2\alpha & \alpha \\ 0 & \dots & 0 & \alpha & 2-2\alpha \end{pmatrix}$$

$$\mathbf{U}_j = \begin{pmatrix} U_{2,j} \\ U_{3,j} \\ \vdots \\ U_{N-2,j} \\ U_{N-1,j} \end{pmatrix}$$

$$\mathbf{b}_j = \begin{pmatrix} \alpha U_{1,j} \\ 0 \\ \vdots \\ 0 \\ \alpha U_{N,j} \end{pmatrix}$$

We see that the equation can be rewritten in matrix form as

$$A\mathbf{U}_{j+1} = B\mathbf{U}_j + \mathbf{b}_j$$

Where A and B are tridiagonal matrices.

3.3.1 Truncation Error of the Crank-Nicolson Method

Notice that the Crank-Nicolson method difference is

$$D(x,t) = \frac{U_i^{n+1} - U_i^n}{\Delta t} - \frac{(U_{i+1}^{n+1} - 2U_i^{n+1} + U_{i-1}^{n+1}) + (U_{i+1}^n - 2U_i^n + U_{i-1}^n)}{2\Delta x^2}$$

Which is exactly the same as the forward and backward Euler methods, except evaluated at half time steps for the U_{xx} derivatives, and thus it gains an extra order of magnitude in accuracy in time, $\mathcal{O}(\Delta x^2) + \mathcal{O}(\Delta t^2)$.

3.3.2 Stability Analysis of the Crank-Nicolson Method

Once again, we use Von-Neumann stability analysis by writing $U_i^j = e^{ikx_i}\hat{U}^j$. Putting this into the Crank-Nicolson algorithm gives

$$\begin{aligned}(1+\alpha)\hat{U}^{j+1}e^{ikx} - (1-\alpha)\hat{U}^j e^{ikx} \\ - \frac{\alpha}{2}e^{ikx}\left(e^{ik\Delta x}\hat{U}^{j+1} + e^{-ik\Delta x}\hat{U}^{j+1} + e^{ik\Delta x}\hat{U}^j + e^{-ik\Delta x}\hat{U}^j\right) = 0\end{aligned}$$

collecting the $j+1$ terms on the left and the j terms on the right,

$$\hat{U}^{j+1}\left((1+\alpha)-\frac{\alpha}{2}\left(e^{ik\Delta x}+e^{-ik\Delta x}\right)\right)=\hat{U}^{j}\left((1-\alpha)+\frac{\alpha}{2}\left(e^{ik\Delta x}+e^{-ik\Delta x}\right)\right)$$

Again recognizing that these complex exponentials are a cosine and rearranging,

$$\hat{U}^{j+1}=\frac{1-\alpha(1-\cos{(k\Delta x)})}{1+\alpha(1-\cos{(k\Delta x)})}\hat{U}^{j}$$

Hence

$$g=\frac{1-\alpha(1-\cos{(k\Delta x)})}{1+\alpha(1-\cos{(k\Delta x)})}$$

From which it is evident that $|g^2|\leq 1$ always, so the Crank-Nicolson method is *unconditionally stable.*

4 Numerical Results

In figures 1 and 2 we see the results of all three algorithms and the numerical solution in one spatial dimension. They are plotted for $\Delta x = 1/100$ and $\Delta x = 1/10$ in figures 1 and 2, respectively. Using the stability condition for the forward Euler stability condition, I determined the appropriate values for Δt to be 10^{-5} and $1/500$, respectively.

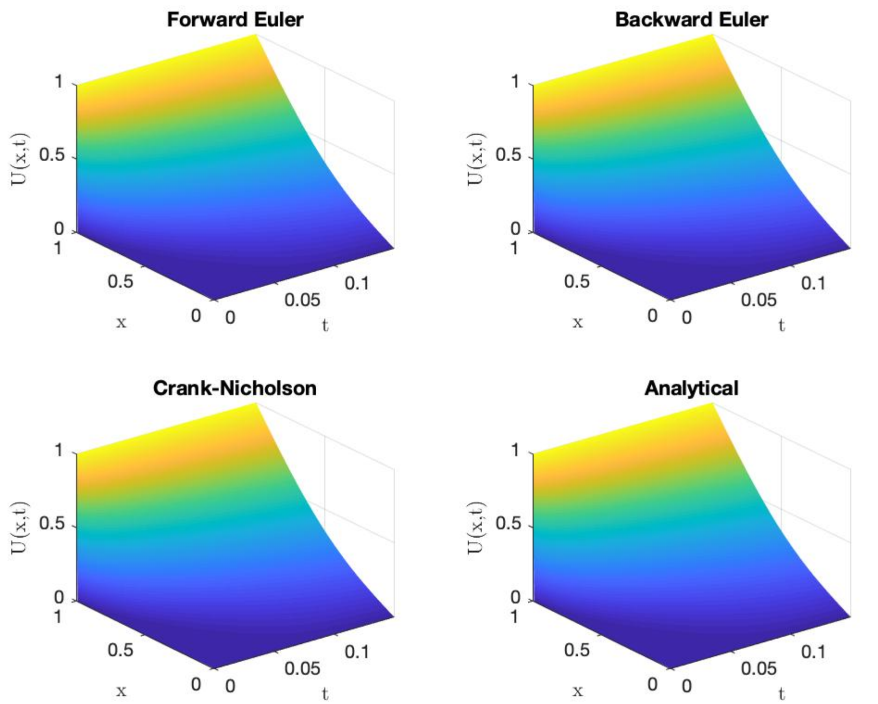

Figure 1: The algorithms, $\Delta x = 1/100$ and $\Delta t = 10^{-5}$. I've removed the grid lines as they get too cluttered with this fine of a mesh.

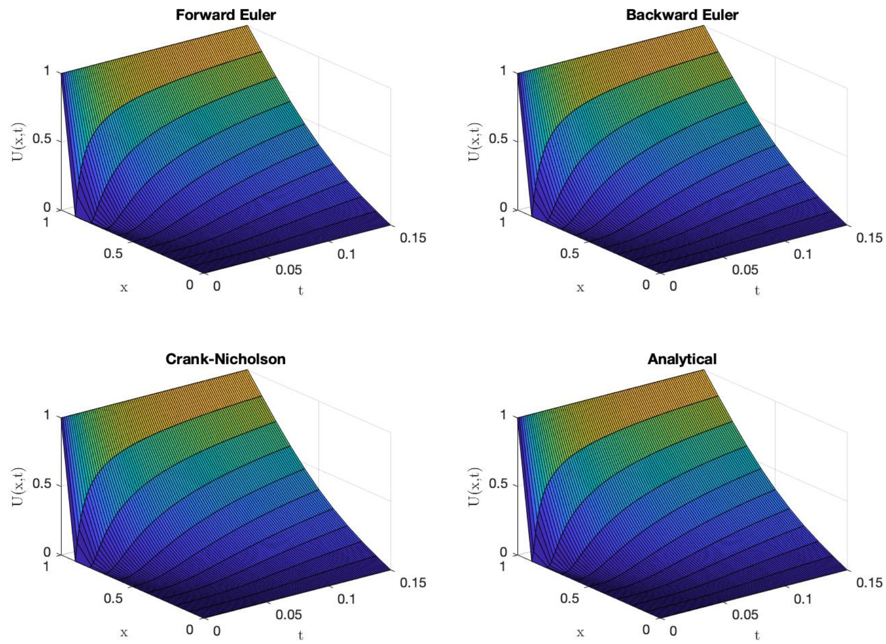

Figure 2: The algorithms, $\Delta x = 1/10$ and $\Delta t = 1/500$. This time I've added a grid to emphasize the how the curvature of U changes at different times

The results are as expected; early in time, we have the condition of the rod being heated to 1 at the end $L = 1$ and nearly 0 everywhere else, resulting in a higher curvature. As time passes, the heat diffuses along the rod, approaching a steady state solution of a straight line from 1 to 0 as the heat goes from $x = L$ to $x = 0$. A close match is observed between all three algorithms and the analytical result.

In figures 3 and 4, we see a section in time of the results, again with $\Delta x = 1/100$ and $\Delta x = 1/10$ respectively. Both figures include a section at an early time $t_1 = 0.01335$, while the heat is still significantly curved, and at another later time $t_2 = 0.1335$ when the solution is more linear (i.e. closer to its steady state).

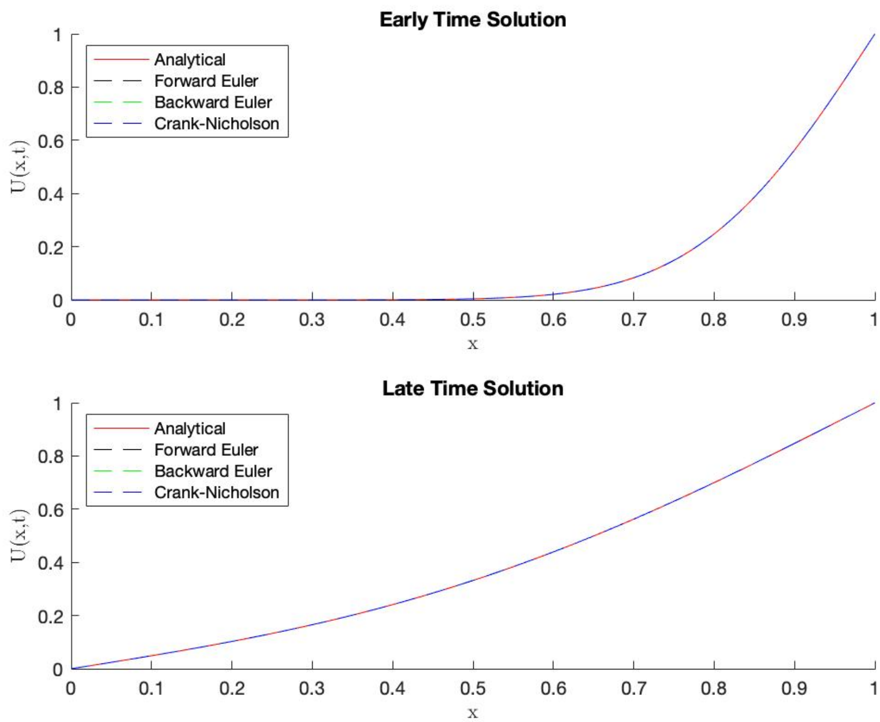

Figure 3: The algorithms compared to the numerical result at an early time $t_1 = 0.01335$, and a later time $t_2 = 0.1335$. $\Delta x = 1/100$.

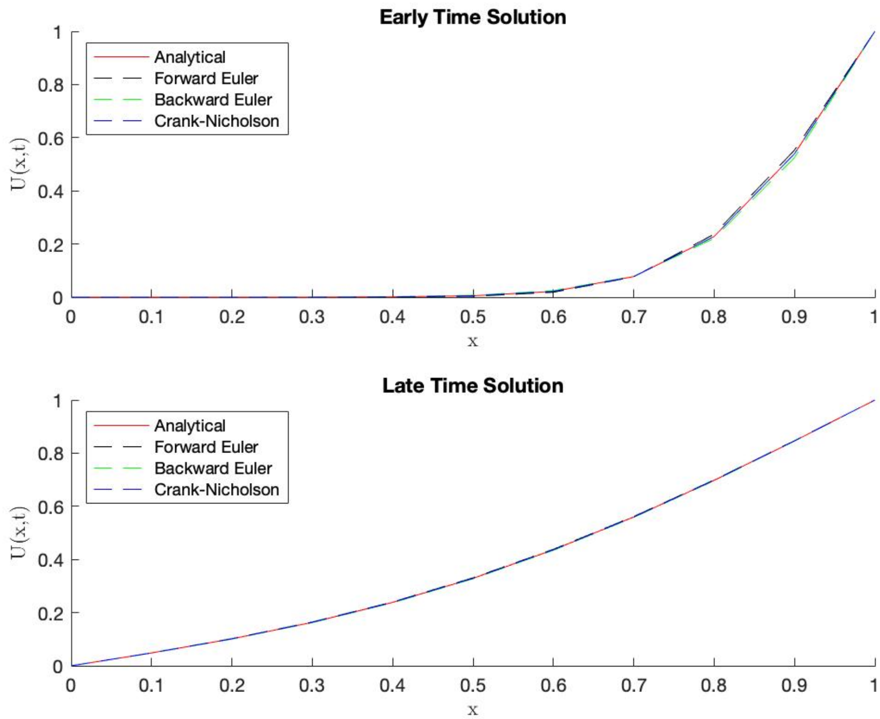

Figure 4: The algorithms compared to the numerical result at an early time $t_1 = 0.01335$, and a later time $t_2 = 0.1335$. $\Delta x = 1/10$.

Again we see an excellent match between the all three numerical results and the analytical results. However, zooming in (see figure 5) reveals that there is a clear preferred method, which is Crank-Nicolson. This is not surprising, as forward Euler overestimates the result while backward Euler underestimates it, and the Crank-Nicholson algorithm is an average, of sorts, between the two.

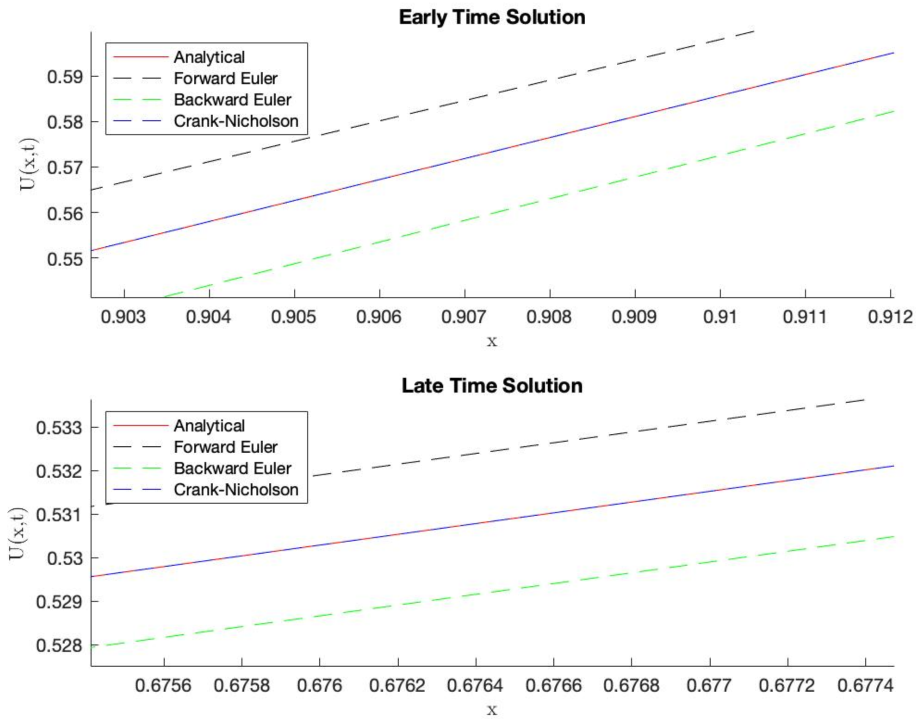

Figure 5: Figure 4 zoomed in very close to see the advantage of Crank-Nicolson.

5 Two-Dimensional Heat Equation

5.1 Analytical Solution

We now turn our attention to the 2D heat equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = D\frac{\partial U}{\partial t}$$

I'll choose the following boundary and initial conditions:

- $U(x, 0, t) = U(0, y, t) = 0$ and $U(L, 0, t) = U(0, L, t) = 1$
- $f(x, y) := U(x, y, 0) = 0$

This can be solved in a method exactly analogous to the 1D solution. We assume a separable solution $U(x,y,t) = X(x)Y(y)T(t)$. Substituting this into the 2D diffusion equation, and playing the usual trick of arranging so that we have each equation equal to a constant, we obtain three differential equations

$$X'' = -\lambda_1 X$$
$$Y'' = -\lambda_2 Y$$
$$T' = -(\lambda_1 + \lambda_2)T$$

So X and Y are linear combinations of sin and cos and T is a linear combination of exponentials. It is simple to check that the initial conditions lead to the following families of solutions

$$X_n(x) = A_n \sin\left(\frac{n\pi}{L}x\right)$$
$$Y_m(y) = B_n \sin\left(\frac{m\pi}{L}y\right)$$
$$T_{mn}(t) = C_{mn}e^{-\frac{(n^2+m^2)\pi^2}{L^2}t}$$

So that

$$U(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} D_{nm}\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)C_{mn}e^{-\frac{(n^2+m^2)\pi^2}{L^2}t}$$

Where the Fourier coefficients are determined by the usual orthogonality relation, leading to

$$D_{nm} = 4\int_0^1\int_0^1 f(x,y)\sin\left(\frac{n\pi}{L}x\right)\sin\left(\frac{m\pi}{L}y\right)dxdy$$

Where in this case, the initial condition is $f(x,y) = 0$.

We solve this by making the same assumption as in the one-dimensional case; write $U(x,y,t) = W(x,y,t) + V(x,y)$ as a linear combination of transient and steady-state solutions. The solution is exactly analogous to the 1D case so I'll spare the details, but the boundary conditions show that

$$V(x,y) = \frac{x}{L} + \frac{y}{L}$$

Now we rearrange and solve for W, which should also obey the diffusion equation (but with slightly modified boundary conditions, just like in the 1D case). Evaluating the integral for the Fourier coefficients gives the following:

$$D_{nm} = \frac{8}{nmL^2\pi^2}\left[\left(1-(-1)^m\right)(-1)^{n+1} + \left(1-(-1)^n\right)(-1)^{m+1}\right]$$

Thereby completing the solution:

$$U(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \frac{8}{nmL^2\pi^2}\left[\left(1-(-1)^m\right)(-1)^{n+1} + \left(1-(-1)^n\right)(-1)^{m+1}\right]\sin\left(\frac{n\pi}{L}x\right) \sin\left(\frac{m\pi}{L}y\right)C_{mn}e^{-\frac{(n^2+m^2)\pi^2}{L^2}t}$$

5.2 Forward Euler in 2D

The forward Euler algorithm can be extended to two spatial dimensions as follows. Let $U_{i,j}^k = U(x_i, y_j, t_k)$. For each time in the time array, we compute an entirely new matrix at time $k+1$ by

$$U_{i,j}^{k+1} = U_{i,j}^k + \alpha\left(U_{i+1,j}^k + U_{i-1,j}^k + U_{i,j+1}^k + U_{i,j-1}^k - 4U_{i,j}\right)$$

Then, for each k, we compute the corresponding matrix which is the solution U on the entire $x-y$ mesh at time k. Some spatial plots are shown in figures 6, 7, and 8 at different points along its temporal evolution. Each plot has a different boundary or initial condition.

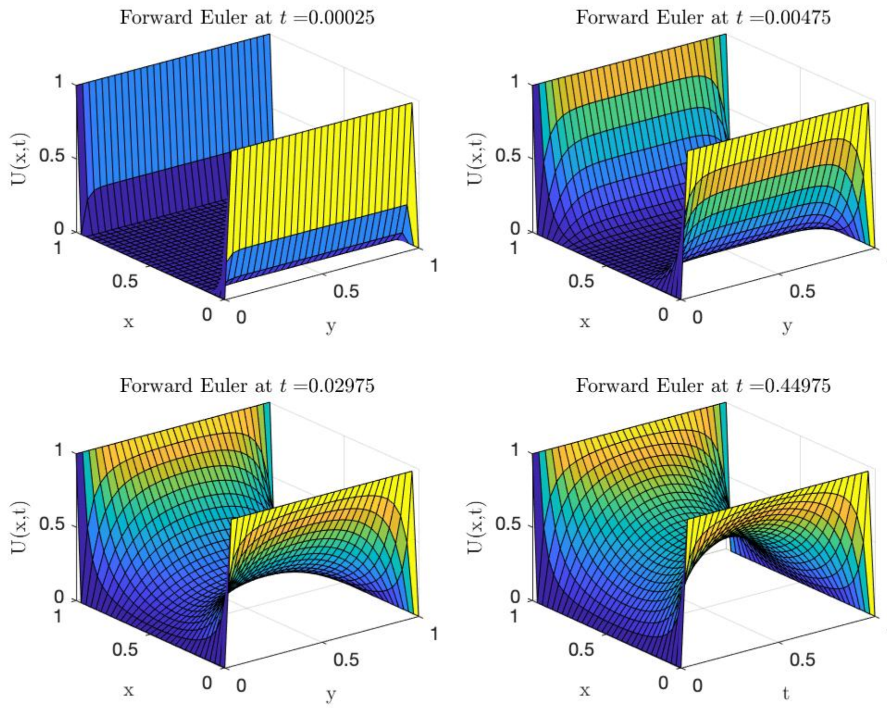

Figure 6: The 2D solution with initial conditions $U(0, y) = 1$, $U(x, 0) = 0$, $U(y, L) = 0$, and $U(y, L) = 1$. All boundaries are held at this constant heat.

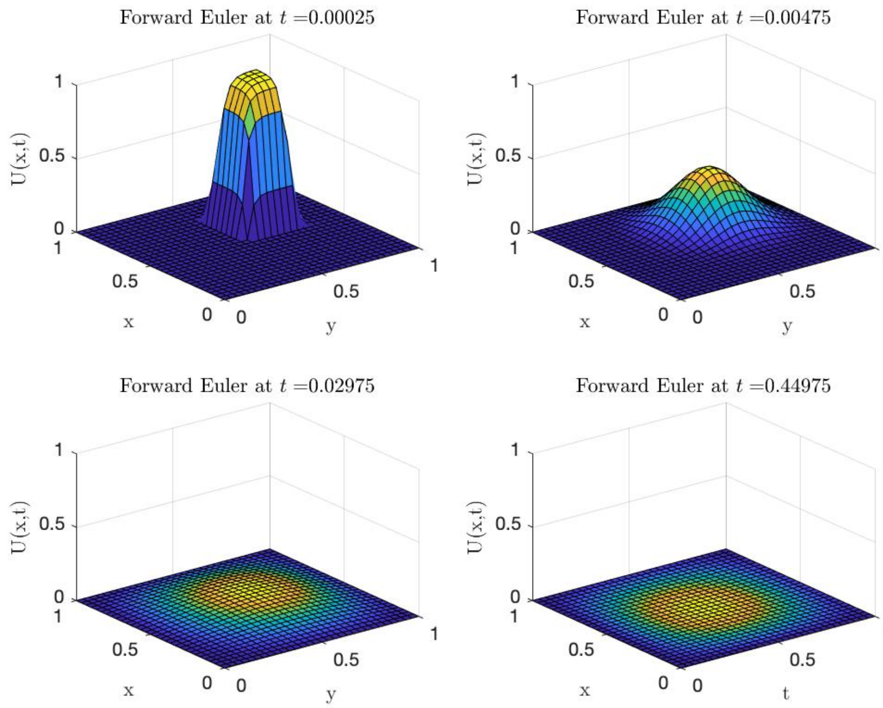

Figure 7: The 2D solution with initial conditions $U(0, y) = U(x, 0) = U(y, L) = U(y, L) = 0$, but with a some of the points near the center of the mesh set to $U = 1$. The non-zero ends are not held at constant heat, and are thus allowed to go to zero.

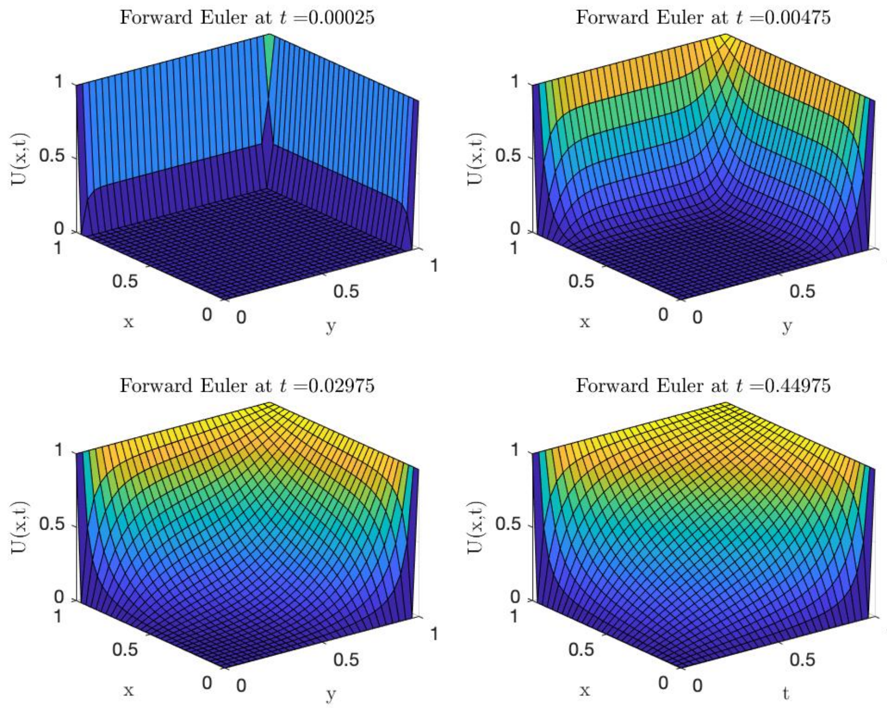

Figure 8: The 2D solution with initial conditions $U(0, y) = 0$, $U(x, 0) = 0$, $U(y, L) = 1$, and $U(y, L) = 1$. All boundaries are held at this constant heat.

I've attached (in the email) an animation I've created of figures 6, 7, and 8 in MATLAB® 2018b.

5.3 Stability Analysis of 2D Forward Euler

We start by setting $U_{i,j}^n := e^{-ikx_i} e^{ikx_j} \hat{U}^n$. Substituting this into the 2D forward Euler equation:

$$\begin{aligned} e^{-ikx} e^{iky} \hat{U}^{n+1} &= (1-4\alpha) e^{-ikx} e^{iky} \hat{U}^n + \alpha \left(e^{-ik(x+\Delta x)} e^{iky} + e^{-ik(x-\Delta x)} e^{iky} \right) \hat{U}^n \\ &\quad + \alpha \left(e^{-ik(y+\Delta y)} e^{ikx} + e^{-ik(y-\Delta y)} e^{ikx} \right) \hat{U}^n \\ \hat{U}^{n+1} &= (1-4\alpha)\hat{U}^n + 2\alpha \cos k\Delta x + 2\alpha \cos k\Delta y \\ &= 1 - 4\alpha \left(\sin^2 \left(\frac{k\Delta x}{2} \right) + \sin^2 \left(\frac{k\Delta y}{2} \right) \right) \end{aligned}$$

And so

$$g = 1 - 4\alpha \left(\sin^2\left(\frac{k\Delta x}{2}\right) + \sin^2\left(\frac{k\Delta y}{2}\right) \right)$$

Which gives us the stability condition

$$\left| 1 - 4\alpha \left(\sin^2\left(\frac{k\Delta x}{2}\right) + \sin^2\left(\frac{k\Delta y}{2}\right) \right) \right|^2 \leq 1$$

Or, since we're assuming that $\Delta y = \Delta x$,

$$\left| 1 - 8\alpha \sin^2\left(\frac{k\Delta x}{2}\right) \right|^2 \leq 1$$

Which is satisfied if $0 < \alpha < 1/4$, or

$$\Delta t \leq \frac{1}{4} \frac{\Delta x^2}{D}$$