Homework 5

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AST 381: Star Formation
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Problem 1. The α and the Ω.

Consider a disk having a dimensionless viscosity α. The disk accretes at a steady rate $\dot{M}$, where $\dot{M} = -3\pi\Sigma\nu$ and for an "alpha" disk with scale height h, $\nu = c_s h\alpha$. The disk cools radiatively. Neglect the difference between the effective temperature of the disk T_{eff} (which is nothing more than a convenient way of stating what the emitted flux is) and the actual gas kinetic temperature T. Take the gas to have sound speed c_s and angular frequency Ω, both of which vary with disk radius r.

You will need to make use of the relation between the power emitted per unit area and the accretion rate:

$$\sigma_{sb} T_{\text{eff}}^4 = \frac{3}{8\pi}\frac{GM_*\dot{M}}{r^3}.$$

(a) Find how h/r scales with r, where h is the disk vertical scale height. Sketch how the disk looks.

(b) Find how the surface density Σ scales with r.

(c) Find how the disk mid-plane density ρ scales with r.

(d) Find an approximate expression for how long it takes a pressure disturbance to equilibrate away. Call this time t_z, and express it as simply as possible.

(e) Find an approximate expression for how long it takes a temperature disturbance to equilibrate away. Call this time t_{cool} and express it in terms of α and Ω.

(f) Find an approximate expression for how long it takes a mass disturbance (say, a local bunching of material) to viscously diffuse away. Call this time t_{visc}, and express it in terms of α, Ω, and h/r. Arrange t_z, t_{cool}, and t_{visc} in increasing order.

(g) Find an expression for the critical radius r_{crit} beyond which Toomre's $Q < 1$. Express the result in terms of α, M, and $\dot{M}$. Alpha disks are generically unstable at large radii, leading some to surmise that the outer peripheries of quasar accretion disks/protostellar accretion disks are fertile breeding grounds for starbursts/binary companion stars or brown dwarfs.

Solution 1

(a) We follow the same derivation as in class. Assume that the disk is axisymmetric ($\frac{\partial}{\partial\phi} = 0$), in steady state ($\frac{\partial u}{\partial t} = 0$), $M_{disk} \ll M_*$, and isothermal ($P = \rho c_s^2$). The z-component of the momentum equation gives us

$$\frac{1}{\rho}\frac{dP}{dz} = \frac{GM}{r^2}\sin\theta.$$

Let $\int_{-\infty}^{\infty} dz \equiv 2h$ and $\int_{-\infty}^{\infty} \rho dz = 2h\rho \equiv \Sigma$. Using these and our assumptions, the momentum equation reduces to

$$\frac{P}{\rho c_s^2} = \frac{GM}{r^2}\frac{h}{r} \implies \frac{c_s^2}{h} = \frac{GM}{r^2}\frac{h}{r} \implies c_s^2 = \frac{GM}{r}\frac{h^2}{r^2}.$$

Rearranging and taking the square root, we arrive at our desired equation.

$\boxed{\therefore \frac{h}{r} = c_s\sqrt{\frac{r}{GM}}.}$

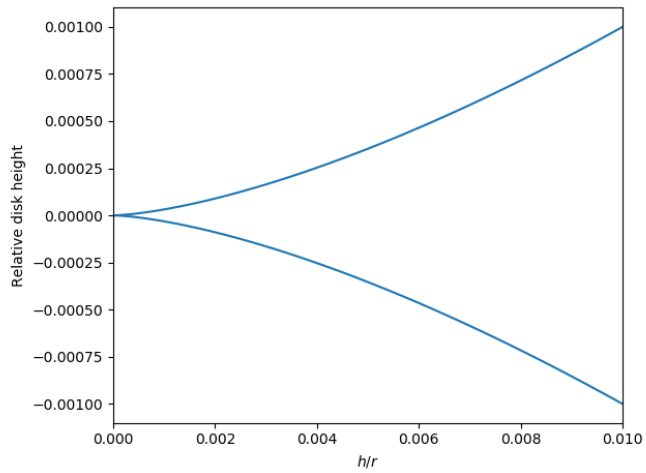

Figure 1: The general shape of the disk. Here h/r is set with h as a constant and varying r. The y-axis captures only the general shape of the disk, with no regard for constants.

(b) We know that $\dot{M} = -3\pi\Sigma\nu$, and $\nu = \alpha c_s h$. From (a), we know that $h = c_s\Omega = c_s\sqrt{\frac{r^3}{GM}}$. Substituting in ν to $\dot{M}$, we find that

$$\dot{M} = -3\pi\Sigma\alpha c_s^2\sqrt{\frac{r^3}{GM}}.$$

We can then solve for Σ to find our final equation.

$\boxed{\therefore \Sigma(r) = -\frac{\dot{M}}{3\pi\alpha c_s^2}\sqrt{\frac{GM}{r^3}}.}$

(c) From (a), we can derive that $\rho(z) = \rho_0 e^{-z^2/2h^2}$, where ρ_0 is the disk mid-plane density. Plugging this into our definition for Σ, we see that

$$\Sigma = \int_{-\infty}^{\infty} \rho dz = \int_{-\infty}^{\infty} \rho_0 e^{-z^2/2h^2} dz = \sqrt{2\pi}\rho_0 h.$$

Using our expression for h from (a) and Σ from (b), we can then solve for ρ_0.

$\boxed{\therefore \rho_0(r) = -\frac{1}{3\sqrt{2}\pi^{3/2}}\frac{GM\dot{M}}{\alpha c_s^3 r^3}.}$

(d) Physically, we know that the pressure disturbance will travel at the sound speed c_s. It has to travel through the disk scale height to equilibrate away, thus this time is simply given by

$$\frac{h}{c_s} = \frac{c_s/\Omega}{c_s} = \frac{1}{\Omega}.$$

$\boxed{\therefore t_z \approx \Omega^{-1}.}$

(e) The time for a thermal disturbance to equilibrate away, the thermal time scale, is given by the heat content per unit area divided by the local dissipation rate. Here the local dissipation rate is just the power emitted per unit area. This gives us an equation of the form

$$\frac{\Sigma c_s^2}{\sigma_{sb}T_{\text{eff}}^4} \sim \frac{\dot{M}c_s^2/3\pi\nu}{3GM\dot{M}/8\pi r^3} \approx \frac{r^3 c_s^2}{GM\nu} = \frac{1}{\Omega^2}\frac{c_s^2}{\alpha c_s h} = \frac{1}{\Omega^2}\frac{c_s}{\alpha c_s/\Omega} = \frac{1}{\alpha\Omega}.$$

$\boxed{\therefore t_{cool} \approx (\alpha\Omega)^{-1}.}$

(f) The time for a mass disturbance to viscously diffuse away, the viscous time scale, is equivalent to the accretion time scale. Let v_r be the radial velocity of the disk material. The accretion rate is then given by

$$\dot{M} = 2\pi r(-v_r)\Sigma = -3\pi\Sigma\nu,$$

making use of our definition for $\dot{M}$. Solving for v_r, $v_r = \frac{3}{2}\frac{\nu}{r}$. The accretion time scale is then

$$\frac{r}{v_r} \sim \frac{r^2}{\nu} = \frac{r^2}{\alpha c_s h} = \left(\frac{h}{r}\right)^{-2} \frac{1}{\alpha\Omega}.$$

$\boxed{\therefore t_{visc} \approx (h/r)^{-2}(\alpha\Omega)^{-1}.}$

Knowing that $0 < \alpha < 1$ and $h \ll r$, we can then order these time scales.

$\boxed{\therefore t_z < t_{cool} < t_{visc}.}$

(g) The Toomre Q is defined as $Q \equiv \frac{c_s\kappa}{\pi G\Sigma}$, where $\kappa = \Omega$ is the epicyclic frequency for a Keplerian disk. Making use of our definition of Σ, the Toomre criterion for instability is then

$$Q = \frac{c_s\Omega(-3\pi\alpha c_s h)}{\pi G\dot{M}} = -\frac{3\alpha c_s^3}{G\dot{M}} < 1.$$

Rearranging, we find that equivalently the criterion is given by

$$\dot{M} \gtrsim \frac{3\alpha c_s^3}{GM}.$$

From our definition for $\dot{M}$, we can see that $\dot{M} = 3\pi\alpha c_s^2\Sigma/\Omega$. Equating these two, we find that this critical radius occurs when

$$\pi\Sigma/\Omega \approx c_s/G \implies \Sigma \approx \frac{c_s\Omega}{\pi G}.$$

Now we can redefine the instability criterion as when $h \sim r_{crit}$. Using our expression from (a), we can write this as

$$c_s\sqrt{\frac{r_{crit}}{GM}} \sim 1 \implies c_s = \sqrt{\frac{GM}{r_{crit}}},$$

which gives us that the sound speed is simply the Keplerian orbit speed. Making use of our definition for Σ, we can also write

$$c_s = \frac{\dot{M}}{-3\pi\Sigma\alpha r_{crit}}.$$

Plugging in our expression for Σ from above, we can rewrite this as

$$c_s \approx \frac{\dot{M}}{-3\pi(c_s\Omega/\pi G)\alpha r_{crit}} = -\frac{G\dot{M}}{3\alpha c_s}\sqrt{\frac{r_{crit}}{GM}}.$$

Setting the two equations for c_s equal to each other, we find that

$$\frac{M}{r_{crit}} = -\frac{\dot{M}\sqrt{r_{crit}}}{3\alpha\sqrt{GM}} \implies r_{crit}^{3/2} = -\frac{3\alpha\sqrt{GM^3}}{\dot{M}} \implies r_{crit} = \left(\frac{9GM^3\alpha^2}{\dot{M}^2}\right)^{1/3}.$$

$\boxed{\therefore r_{crit} = \left(\frac{9GM^3\alpha^2}{\dot{M}^2}\right)^{1/3}.}$

Problem 2. Obsessing Over Outflows...

This problem uses CO(1-0) interferometric observations of a protostellar outflow to explore outflow launching and properties. The observations were carried out using the OVRO millimeter array of six 10.4 m telescopes. This source is a member of L1589, in the λ Orionis molecular shell. Assume $d = 460$ pc, the beam size is 4", and the velocity channel $\Delta v = 0.325$ $\mathrm{km\,s^{-1}}$. The intensity units are (sadly) in Jy/beam.

(a) Produce an integrated intensity map of the outflow. Identify the velocity range of the blue and red-shifted lobes. What is the outflow extent and opening angle? Based on the data, do you think this outflow is launched by a younger or older source?

(b) Make a "Hubble diagram" (p–v) of the outflow, and estimate the dynamical age of the outflow. Is this a good measure of the source's actual age? What does the Hubble diagram suggest about the accretion history of the source?

(c) Use the CO emission to calculate the outflow mass. Offner & Chaban (2017) used simulations to show that molecular outflows contain about three times as much entrained core mass as is actually launched. Use this, together with the outflow mass, to estimate the source accretion rate and source mass. Discuss any assumptions you make.

Solution 2

(a) The integrated intensity map is shown in Figure 2. The velocity ranges of the blueshifted and redshifted lobes seem to be $[-9884.5, -4032.7]$ m/s and $[843.8, 8321.1]$ m/s, respectively. The outflow extents are approximately 0.441 pc and 0.288 pc, and the opening angles are 31.2930° and 35.8195°. Based on the data, we know that the lobes are highly collimated, as they have an opening angle of <45°, which suggests that this is a class 0 source. Thus we expect this outflow to be launched by a younger source, with an age of at most 0.2 Myr.

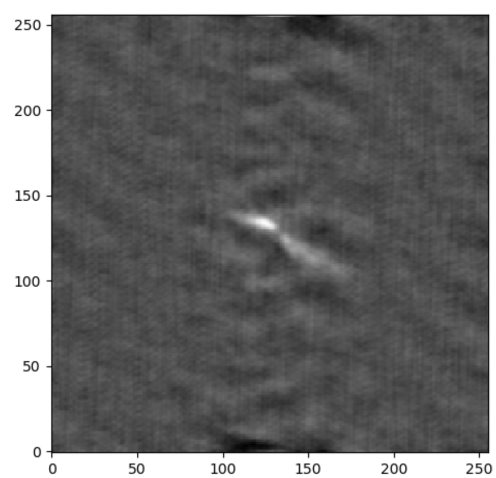

Figure 2: The integrated intensity map of the outflow. Here the units of the axes are pixels in the FITS file and the values themselves are brightness temperature (K). The bottom right lobe is blueshifted and the top left is redshifted.

(b) We generate the Hubble diagram using the `pvextractor` package in Python. Here the path was chosen to be the end of the redshifted lobe, through the central point between the lobes, and the end of the blue blueshifted lobe. Looking at the p–v diagram (Figure 3), we notice that there seems to be somewhat of a bimodal distribution in the velocity corresponding to pixels 24 and 36. From the FITS file pixel 36 in the velocity roughly corresponds to $v = 0$, so we can assume this to be noise from the interferometer. If we take pixel 24 as the true signal, then we have that the velocity peaks at around 3.7 km/s. Dividing the combined length of the outflows ($\sim$0.729 pc) by this velocity, we obtain a dynamical age of roughly 0.192 Myr. This seems to be a good measure of the source's actual age, as it falls within the age range of a class 0 protostar. This Hubble diagram suggests that accretion is relatively constant, as this velocity peak is roughly uniform throughout the outflow. However we also note that at farther distances there is a blend of higher velocities, which suggests that overall the accretion is slowing down. This agrees with our expectations, as accretion rate should be the highest during the class 0 stage and go down as the envelope gets consumed or blown away by the forming star.

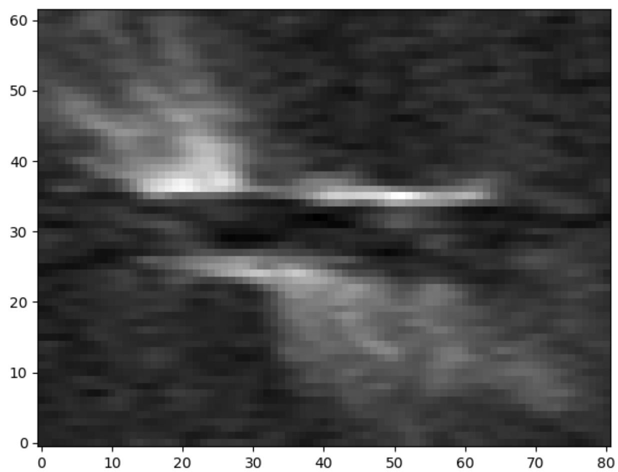

Figure 3: The Hubble diagram (p–v) of the outflow. Again the axes here represent pixels; the x-axis represents the pixel distance and the y-axis represent the FITS file's velocity in terms of pixels.

(c) Using our old code from Homework 2, we obtain an outflow mass of 0.0669 $M_\odot$ for the outflow. Assuming that this outflow is actually about 0.2 Myr old, we can assume a mass ejection rate (from outflows) of $3.475 \cdot 10^{-7}$ $M_\odot$/yr. The outflow mass rate is related to the accretion rate by the equation $\dot{m}_0 \approx f_0 \dot{m}_*$, where $f_0 \approx 0.1$–0.3. This gives us an estimated accretion rate of $(1.158$–$3.475) \cdot 10^{-6}$ $M_\odot$/yr, which is reasonable. Finally, from Offner & Arce (2014) we see that after about 0.2 Myr, a star's mass is roughly four times greater than the mass launched in outflows (although this simulation admittedly had much higher collimation). If only a third of the calculated outflow mass is actually launched, this gives an estimate of roughly 0.0891 $M_\odot$, which although on the low side is not entirely out of the realm of possibility for a class 0 protostar.