NORMAL CLOSURES OF BOUNDING PAIR MAPS

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Abstract. Johnson showed that bounding pair maps of genus 1 normally generate the Torelli group when the genus of the surface is at least 3. We show that this generalizes: bounding pair maps of any genus k also normally generate the Torelli group.

0. Introduction

Birman gave the first normal generating set for the Torelli group of a surface as a consequence of producing a presentation for $\mathrm{Sp}(2g;\mathbb{Z})$ [1]. Powell realized Birman's generators geometrically, showing that they are genus 1 and genus 2 separating twists and genus 1 bounding pair maps [10]. Johnson shortly thereafter trimmed down this normal generating set: for $g \geq 3$, the normal closure of a genus 1 bounding pair map in its mapping class group is the Torelli group [6]. In his survey on the Torelli group, Johnson remarks that his theorem was "the final result along these lines" [4]; this economical normal generating set for such a complicated group is certainly both remarkable and satisfying. A few sentences later, Johnson also points out that "this result is the starting point for several others." Some 40 years later, Johnson's work on Torelli groups continues to be a rich source of tools and inspiration.

Given Johnson's economical normal generating set for Torelli groups, a natural question presents itself: what is the normal closure of a genus k bounding pair map in its mapping class group? Our main result answers this question.

Theorem 1. *For $g \geq 3$ and $1 \leq k < g-1$, the normal closure of a genus k bounding pair map in the mapping class group of a closed or once-punctured or once-bordered surface of genus g is the Torelli group.*

Regarding the bounds in the theorem, note that the Torelli group is trivial when $g = 1$ and that it contains no bounding pair maps when $g = 2$. For closed surfaces, a bounding pair map of genus $g-1$ is trivial, while for punctured and bordered surfaces such a map has normal closure equal to the kernel of the forgetful or capping homomorphism, respectively, and each is a proper normal subgroup of the Torelli group.

Theorem 1 shows that we can be both economical and flexible in normally generating the Torelli group. In certain proofs about Torelli groups in the literature, care is taken so that constructions involve genus 1 bounding pair maps specifically. For instance, this occurs in a proof by the author and Margalit showing that for $g \geq 3$ there exist pseudo-Anosov mapping classes that normally generate the Torelli group, including ones with arbitrarily large stretch factors [8, Theorem 1.3].

In the remainder of the introduction we set some notation and conventions; explain the strategy of the proof of the main theorem and provide an outline of the paper; prove a special case of Theorem 1 as a warm-up; and pose a problem related to our main theorem.

Notation and conventions. Let $g \geq 3$ throughout and let S_g be the closed, connected, orientable surface of genus g. Let $\mathcal{M}_g$ be the mapping class group of S_g, the group of homotopy classes of orientation-preserving homeomorphisms of S_g. Let $H_g = H_1(S_g;\mathbb{Z})$ and let $\mathcal{I}_g$ be the Torelli group, the kernel of the natural homomorphism $\mathcal{M}_g \to \mathrm{Aut}(H_g)$. Let $\tau_g : \mathcal{I}_g \to \wedge^3 H_g/H_g$ be the Johnson homomorphism, and let J_g be the Johnson kernel. We have the following short exact sequence:

$$1 \to J_g \to \mathcal{I}_g \xrightarrow{\tau_g} \wedge^3 H_g/H_g \to 1$$

Corresponding groups and homomorphisms are defined for surfaces with a single puncture or border (i.e. compact boundary component). We denote these surfaces $S_{g,*}$ and $S_{g,1}$ and their corresponding groups similarly. Usually it will not be important to distinguish between the closed, punctured, and bordered cases, and so we use the general terms S, $\mathcal{M}$, H, $\mathcal{I}$, τ, and J.

Throughout we will not distinguish between homeomorphisms and their mapping classes, nor between curves and their isotopy classes, and curves will always be simple and closed. Let T_c be the left-handed Dehn twist about the curve c. A Dehn twist is separating if it is a twist about a separating curve; a separating curve of genus k bounds two subsurfaces, one of genus k and one of genus $g-k$, where the latter contains any puncture or boundary component. Let $\mathcal{T}_k$ be the normal closure of a genus k separating twist in $\mathcal{M}$. A bounding pair map of genus k is a product Dehn twists $T_cT_d^{-1}$ about a pair of disjoint nonhomotopic nonseparating simple closed curves c and d that together bound subsurfaces of genus k and $g-k-1$; again, the latter contains any puncture or boundary component. Let W_k be the normal closure of a genus k bounding pair map in $\mathcal{M}$. Separating twists and bounding pair maps lie in the subgroup $\mathcal{I}$ of $\mathcal{M}$. Further, separating twists lie in J. In these terms, Johnson's result can be phrased as $W_1 = \mathcal{I}$.

Strategy and outline. By the short exact sequence stated above and Johnson's result that $\langle \mathcal{T}_1, \mathcal{T}_2 \rangle = J$ [5], proving the following three statements suffices to show that $W_k = \mathcal{I}$, the conclusion of our main theorem:

(1) The image of W_k under τ is full: $\mathrm{im}\,\tau(W_k) = \mathrm{im}\,\tau(\mathcal{I})$.
(2) $\mathcal{T}_1 < W_k$.
(3) $\mathcal{T}_2 < W_k$.

In Sections 1 and 2 we show that for all $g \geq 3$ and any $1 \leq k < g-1$, W_k satisfies the first and second properties, respectively. With this work in hand, we start Section 3 by showing that the third property holds when $k = 2$, and so $W_2 = \mathcal{I}$ for $g \geq 4$. We go on to show that W_k contains at least one of W_1, W_2 or W_3 by way of a reduction procedure. Since the first two options imply that $W_k = \mathcal{I}$, we conclude the theorem by showing that $W_2 < W_3$ for all $g \geq 5$. Our arguments often closely follow constructions and ideas found in Johnson's papers, especially [2], [7], and [3].

A warm-up result. We offer here the proposition that gave the initial hope that something in the direction of the main theorem might be true.

Proposition 2. *When k is relatively prime to $g-1$, $W_k = \mathcal{I}_g$.*

Proof. Since k is relatively prime to $g-1$, there exists a positive integer n such that $nk = 1 \pmod{g-1}$. We may take a sequence of disjoint homologous nonseparating curves $c_0, \ldots, c_n$ on S_g so that consecutive curves cobound subsurfaces of genus k. Because of the modular condition, these curves can be arranged so that c_0 and c_n cobound a subsurface of genus 1; this is illustrated in Figure 1. Then the product of the n bounding pair maps of genus k on the consecutive c_i is a bounding pair map of genus 1. As $W_1 = \mathcal{I}_g$ and $W_1 < W_k < \mathcal{I}_g$, we have $W_k = \mathcal{I}_g$. □

Our main theorem says that $W_k = \mathcal{I}$, independent of the number theoretic condition assumed in Proposition 2 and without the assumption that the surface is closed.

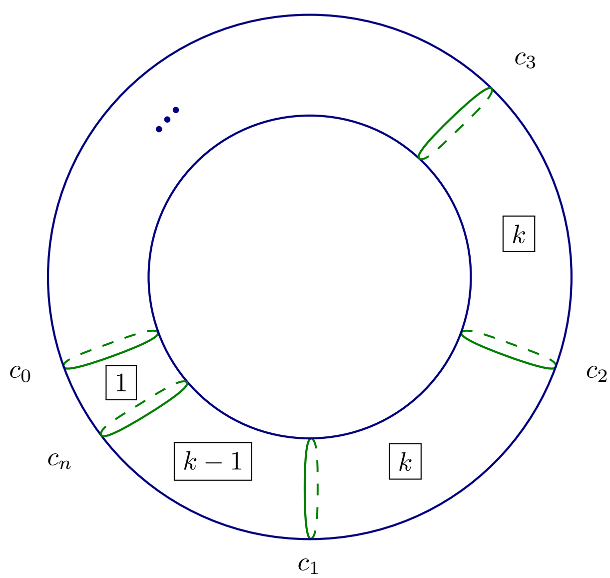

FIGURE 1. The curves c_i. The boxed values in the subsurfaces indicate their genus.

Open problem. The warm-up proposition demonstrates how to write a genus 1 bounding pair map as a product of genus 2 bounding pair maps when g is even and the surface is closed. As a component of our proof of Theorem 1, we show in Proposition 7 that $W_2 = \mathcal{I}$ for all $g \geq 4$. While our other proofs implicitly show how to write a genus 2 bounding pair map as a product in genus k bounding pair maps for any $1 \leq k < g-1$, our proof that $W_2 = \mathcal{I}$ for $g \geq 4$ is non-constructive. Indeed, it instead follows the strategy outlined above. This leads us to pose the following problem.

Problem 3. *For $g \geq 4$, write a genus 1 bounding pair map as a product of genus 2 bounding pair maps in $\mathcal{M}_{g,1}$.*

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1. The image of W_k under τ

In this section we carry out the first step in our strategy, showing that the image of W_k under τ is full.

Proposition 4. *For $1 \leq k < g-1$, the image of W_k under τ is full: $\operatorname{im}\tau(W_k) = \operatorname{im}\tau(\mathcal{I})$.*

Proof. With the standard setup for τ, the image under τ of the standard bounding pair map of genus k is

$$x = \sum_{i=1}^{k}(a_i \wedge b_i) \wedge b_{k+1} \in \operatorname{im}\tau(W_k).$$

Let ϕ be the factor mix element of $\operatorname{Aut}(H)$ given by the following map on generators of H:

$$a_1 \mapsto a_1 - b_{k+2}$$
$$a_{k+2} \mapsto a_{k+2} - b_1$$

and that fixes all other basis elements; this exists by the hypothesis on k.

Then

$$\phi(x) = (-b_{k+2} \wedge b_1 \wedge b_{k+1}) + \sum_{i=1}^{k}(a_i \wedge b_i) \wedge b_{k+1} \in \operatorname{im}\tau(W_k).$$

And so therefore also

$$\phi(x) - x = (-b_{k+2} \wedge b_1 \wedge b_{k+1}) \in \operatorname{im}\tau(W_k).$$

Now following an argument of Johnson [2, Theorem 1], we have that the orbit of $\phi(x) - x$ under the action of $\operatorname{Aut}(H)$ forms a basis for $\operatorname{im}\tau(\mathcal{I})$. Therefore $\operatorname{im}\tau(W_k) = \operatorname{im}\tau(\mathcal{I})$. □

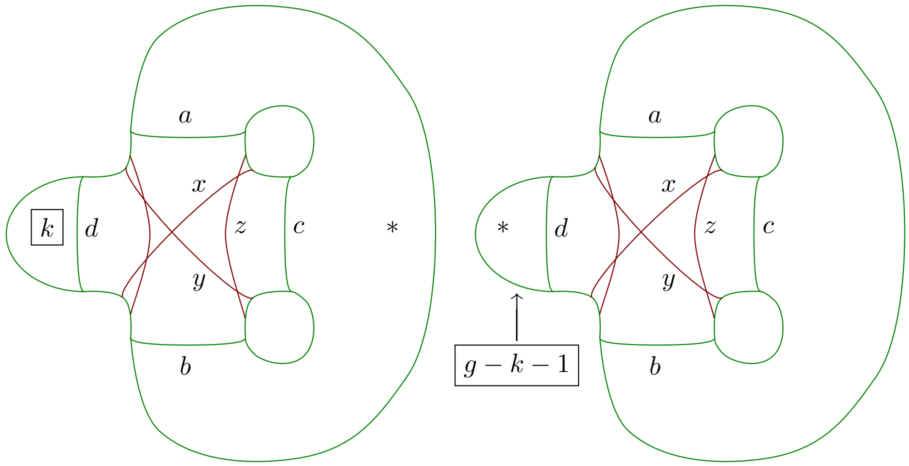

FIGURE 2. Two lantern configurations used to write genus k and $k+1$ separating twists as products in genus k bounding pair maps. The asterisks indicate any puncture or border, and the boxed values in the subsurfaces indicate their genus.

2. SEPARATING TWISTS IN W_k

In this section we show that W_k always contains certain separating twists, namely those of genus k, $k+1$, and 1. The proofs in this section are similar to arguments made by Johnson in showing that $\langle \mathcal{T}_1, \mathcal{T}_2 \rangle$ is equal to the subgroup generated by all separating twists and that $W_1 = \mathcal{I}$ [7, Theorems 1 and 2].

Proposition 5. *For $1 \leq k < g-1$, $\mathcal{T}_k < W_k$ and $\mathcal{T}_{k+1} < W_k$.*

Proof. Take a lantern in S in each of the two configurations depicted in Figure 2. These differ only in the genus of the subsurfaces complimentary to the lantern along with the location of the puncture or border, if any. In both figures, we have by the lantern relation:

$$T_a T_b T_c T_d = T_x T_y T_z$$

or

$$T_d = (T_x T_a^{-1})(T_y T_b^{-1})(T_z T_c^{-1}).$$

In the first configuration, we have that T_d is a separating twist of genus k, while each of the three terms on the right is a bounding pair map of genus k. We therefore conclude that $\mathcal{T}_k < W_k$. In the second configuration, we have that T_d is a separating twist of genus $k+1$, while each of the three terms on the right is a bounding pair map of genus k. We therefore conclude that $\mathcal{T}_{k+1} < W_k$. $\square$

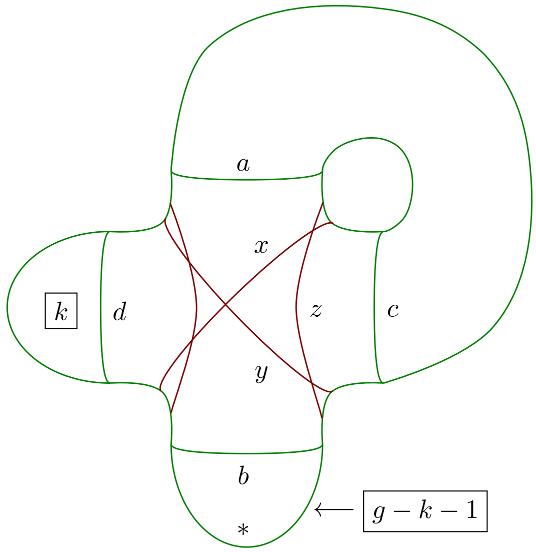

FIGURE 3. The lantern configuration used to show that $\mathcal{T}_1$ is a subgroup of W_k. The asterisk indicates any puncture or border.

Proposition 6. *For* $1 \leq k < g - 1$, $\mathcal{T}_1 < W_k$.

Proof. We may take a lantern in S as depicted in Figure 3. The separating curves b, d, and y cut off subsurfaces of genus $k+1$, k, and 1, respectively. By the lantern relation, we have

$$T_a T_b T_c T_d = T_x T_y T_z$$

or

$$T_y = (T_a T_x^{-1})(T_c T_z^{-1}) T_b T_d.$$

We have that T_y is a separating twist of genus 1. Each of the first two terms on the right is a bounding pair map of genus k and so lie in W_k. The other two terms are separating twists of genus $k+1$ and k, respectively; these lie in W_k by Proposition 5. We therefore conclude that $\mathcal{T}_1 < W_k$. □

3. $\mathcal{T}_2$ AND REDUCTIONS

The third and final step in our strategy to prove $W_k = \mathcal{I}$ is to show that $\mathcal{T}_2 < W_k$ for all k. We start with an easy case where $k = 2$ and then go on to show W_k contains either W_1 or W_2 for all larger k.

Proposition 7. *For all* $g \geq 4$, $\mathcal{T}_2 < W_2$ *and* $W_2 = \mathcal{I}$.

Proof. By Proposition 4, $\operatorname{im} \tau(W_2) = \operatorname{im} \tau(\mathcal{I})$. By Proposition 6, $\mathcal{T}_1 < W_2$. And by Proposition 5, $\mathcal{T}_2 < W_2$. By our stated strategy, we have $W_2 = \mathcal{I}$ for all $g \geq 4$. □

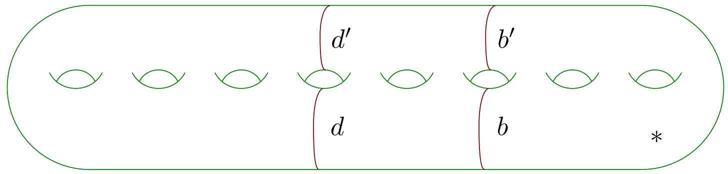

FIGURE 4. Two properly nested bounding pairs, corresponding to the tuple (5, 3) in a surface of genus 8. The map $T_bT_{b'}^{-1}T_dT_{d'}^{-1}$ is a properly nested bounding pair product.

The next two propositions construct some useful elements of W_k and also provide the reduction procedure for the proof of our main theorem. These propositions were inspired by a construction of Johnson [3, Lemma 10]. For a fixed S, let b, b' be a bounding pair that bounds a genus g_1 subsurface and let d, d' be a second bounding pair of genus $g_2 < g_1$ within this subsurface, such that that the homology class $[d] \in H$ does not equal $\pm[b]$. An example of this setup is illustrated in Figure 4. Given S, g_1, and g_2, there is a unique such configuration in S up to homeomorphism by the classification of surfaces. The same construction works just as well with longer sequences of properly nested bounding pairs, and the configuration in S is again unique up to homeomorphism and is determined by it sequence of genus values. Similarly, the conjugacy class in $\mathcal{M}$ of the product of bounding pair maps along a sequence of properly nested bounding pairs is fully determined by its sequence of genus values. We denote such a map comprised of n bounding pair maps as a $(g_1, \ldots, g_n)$ properly nested bounding pair product, where the g_i form a strictly decreasing sequence of positive integers less than $g-1$.

Proposition 8. *For $2 < k < g-1$, W_k contains all $(k+1,\ k-1)$ properly nested bounding pair products.*

Proof. Since W_k is a normal subgroup, by the classification of surfaces it suffices to show that a single element of this form lies in W_k. A subsurface of genus $k+1$ bounded by a bounding pair d, d' can be subdivided into three further subsurfaces: a subsurface of genus $k-1$ with two boundary components b, b', as well as two lanterns. This is depicted in Figure 5.

By the lantern relation, we have in one lantern

$$T_aT_bT_cT_d = T_xT_yT_z$$

and in the other lantern, after inverting, we have

$$T_{d'}^{-1}T_c^{-1}T_{b'}^{-1}T_a^{-1} = T_{x'}^{-1}T_{y'}^{-1}T_{z'}^{-1}$$

since the cyclic order of x, y, z is reversed by mirror reflection.

After multiplying these two relations together and taking advantage of the fact that Dehn twists about disjoint curves commute, as well as some cancellations of inverse pairs, we have

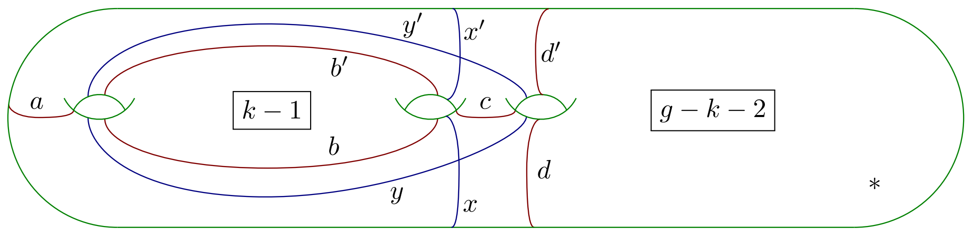

FIGURE 5. A configuration involving two lanterns that yields a relation among boundary pair maps. Curves z and z' (not pictured) complete the two lantern configurations and form a bounding pair of genus k.

$$T_b T_{b'}^{-1} T_d T_{d'}^{-1} = (T_x T_{x'}^{-1})(T_y T_{y'}^{-1})(T_z T_{z'}^{-1})$$

Each of the three terms on the righthand side of this equality is a bounding pair map of genus k, since each pair of curves bounds a subsurface comprised two further subsurfaces: one a genus $k-1$ subsurface with two boundary components, the other a four-holed sphere. Therefore the $(k+1,\ k-1)$ properly nested bounding pair product $T_b T_{b'}^{-1} T_d T_{d'}^{-1}$ lies in W_k, as desired. □

We next prove a kind of reduction formula by carrying out the construction idea in Proposition 8 several steps further.

Proposition 9. *For $3 \le k < g-1$, W_k contains every $(k+1,\ k-2)$ properly nested bounding pair product. Further, $W_{k-3} < W_k$.*

Proof. By Proposition 8, W_k contains all $(k+1,\ k-1)$ properly nested bounding pair products. We begin by taking a product of three such maps we now describe, f_1, f_2, and f_3. Consider the configuration of curves in Figure 6. The pairs of curves e, e' and d, d' and b, b' yield a properly nested sequence of bounding pairs of genus $k+1$, k, and $k-2$, respectively. In the figure there are additionally three bounding pairs of genus $k-1$; these are formed by the pairs of lantern curves x, x' and y, y' and z, z'. Since each f_i expressed below is a $(k+1,\ k-1)$ properly nested bounding pair product, the following products lie in W_k:

$$\begin{aligned} f_1 f_2 f_3 =& (T_x T_{x'}^{-1} T_e T_{e'}^{-1})(T_y T_{y'}^{-1} T_e^{-1} T_{e'})(T_z T_{z'}^{-1} T_e T_{e'}^{-1}) \\ =& (T_e T_{e'}^{-1})(T_x T_{x'}^{-1})(T_y T_{y'}^{-1})(T_z T_{z'}^{-1}) \\ =& (T_e T_{e'}^{-1})(T_d T_{d'}^{-1})(T_b T_{b'}^{-1}) \end{aligned}$$

(Notice that we have chosen the f_i so that there is a cancelling pair of e, e' bounding pair maps in the product.)

By this calculation we have that all $(k+1,\ k,\ k-2)$ properly nested bounding pair products lie in W_k. Since $T_d T_{d'}^{-1}$ also lies in W_k and commutes with each of these three bounding pairs, we have that the product $(T_b T_{b'}^{-1})(T_e T_{e'}^{-1})$ also lies in W_k, and so also every $(k+1,\ k-2)$ properly nested bounding pair product.

We may apply the same construction again, with $(k+1,\ k-2)$ playing the role of $(k+1,\ k-1)$, so that the triple product yields a $(k+1,\ k-1,\ k-3)$ properly nested bounding pair product in W_k. Since every $(k+1,\ k-1)$ properly nested bounding pair product also lies in W_k, we may form a product that, after commuting and cancelling, shows that $W_{k-3} < W_k$. □

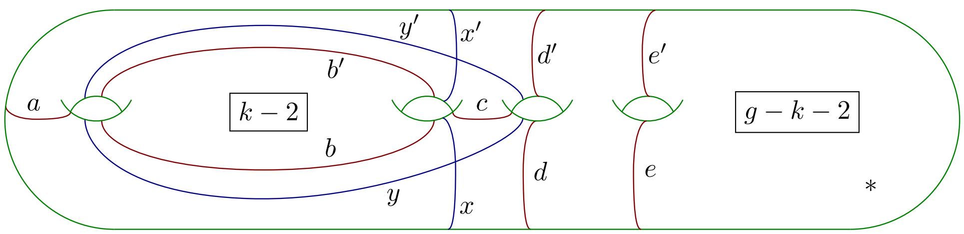

FIGURE 6. A second configuration involving two lanterns that yields a relation among boundary pair maps. Again, curves z and z' (not pictured) complete the two lantern configurations and form a bounding pair of genus $k-1$.

We now justify the final base case for our reduction argument.

Proposition 10. *For $g \geq 5$, $W_2 < W_3$.*

Proof. By the hypothesis on g and Proposition 9, W_3 contains all $(4,\ 1)$ properly nested bounding pair products. Then the following product of two such maps lies in W_3, where the bounding pairs are depicted in Figure 7:

$$\begin{aligned} f_1 f_2 =& (T_a T_{a'}^{-1} T_b T_{b'}^{-1})(T_a^{-1} T_{a'} T_{b'} T_{b''}^{-1}) \\ =& T_b T_{b''}^{-1} \end{aligned}$$

Since $T_b T_{b''}^{-1}$ is a bounding pair map of genus 2, we have $W_2 < W_3$ for $g \geq 5$. □

Our main theorem now follows in a straightforward manner.

Proof of Theorem 1. By repeated application of Proposition 9, W_k contains at least one of W_1, W_2, or W_3. If $W_1 < W_k$, then $W_k = \mathcal{I}$ by Johnson's theorem. If $W_2 < W_k$, then $W_k = \mathcal{I}$ by Proposition 7. Finally, if $W_3 < W_k$, then $W_2 < W_k$ by Proposition 10 and again $W_k = \mathcal{I}$. □

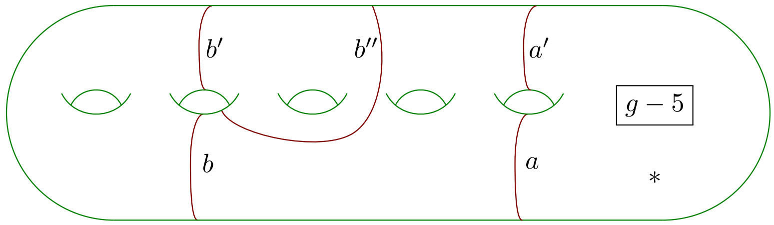

FIGURE 7. Writing a genus 2 bounding pair map as a product of (4, 1) properly nested bounding pair products.

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