NORMAL GENERATORS OF TORELLI GROUPS

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Abstract. Johnson showed that the genus 1 bounding pair maps generate the Torelli group of a surface when its genus is at least 3. We show that this generalizes: apart from straightforward exceptions, the bounding pair maps of any fixed genus also generate the Torelli group. We give some applications of this result to normal subgroups of mapping class groups acting on curve complexes.

0. Introduction

Birman gave the first generating sets for the Torelli groups of closed surfaces as a consequence of producing a finite presentation for $\mathrm{Sp}(2g;\mathbb{Z})$ [3]. Powell realized Birman's generators geometrically and showed that the Torelli group of a closed surface is generated by genus 1 bounding pair maps along with genus 1 and genus 2 separating twists [13]. Johnson shortly thereafter trimmed down this generating set, showing that for closed and also once-bordered surfaces of genus at least 3, the genus 1 bounding pair maps suffice [11]. In his survey on the Torelli group, Johnson remarks that his theorem was "the final result along these lines" [8]; indeed, the fact that the Torelli group is normally generated in the mapping class group by a single element is remarkable and satisfying. A few sentences later, Johnson also points out that "this result is the starting point for several others." Some 40 years later, Johnson's work on Torelli groups continues to be a rich source of tools and inspiration.

Given Johnson's result, a natural question presents itself: what is the normal closure of a genus k bounding pair map in its mapping class group? Our main result answers this question.

Theorem 1. *For $g \geq 3$ and $1 \leq k < g-1$, the normal closure of a genus k bounding pair map in the mapping class group of a closed or once-punctured or once-bordered surface of genus g is the Torelli group of the surface.*

We remark the following about the bounds in the theorem. For closed surfaces, the Torelli group is trivial when $g = 1$ and contains no nontrivial bounding pair maps when $g = 2$. A genus $g-1$ bounding pair map is trivial for closed surfaces. For once-punctured and once-bordered surfaces with $g \geq 1$, the genus $g-1$ bounding pair maps generate proper subgroups of their respective Torelli groups, except in the $g = 1$ punctured case. For once-punctured surfaces the genus $g-1$ bounding pair maps generate the kernel of the forgetful homomorphism, which is the point-pushing subgroup; for once-bordered surfaces they generate an index $2g-2$ subgroup of the kernel of the capping homomorphism.

In the remainder of the introduction we set some definitions, notations, and conventions; explain the strategy of the proof of the main theorem and provide an outline of the paper; and discuss some consequences of the main theorem.

Definitions, notations, and conventions. We will sometimes use standard results about Torelli groups without too much comment; as a background reference, see [5, Chapter 6]. Let $g \geq 3$ throughout and let S_g be the closed, connected, orientable surface of genus g. Let $\mathcal{M}_g$ be the mapping class group of S_g, the group of homotopy classes of orientation-preserving homeomorphisms of S_g. Let $H_g = H_1(S_g;\mathbb{Z})$ and let $\mathcal{I}_g$ be the Torelli group, the kernel of the natural homomorphism $\mathcal{M}_g \to \mathrm{Aut}(H_g)$. Letting $\tau_g : \mathcal{I}_g \to \wedge^3 H_g/H_g$ be the Johnson homomorphism, we have the following short exact sequence:

$$1 \to \mathcal{K}_g \to \mathcal{I}_g \xrightarrow{\tau_g} \wedge^3 H_g/H_g \to 1$$

where $\mathcal{K}_g$ is the Johnson kernel.

Corresponding groups and homomorphisms are defined for surfaces with a single puncture or border (i.e. compact boundary component). We denote these surfaces $S_{g,*}$ and $S_{g,1}$ and their corresponding groups similarly. Often it will not be important to distinguish between the closed, punctured, and bordered cases, and in these places we use the general terms S, $\mathcal{M}$, H, $\mathcal{I}$, τ, and $\mathcal{K}$.

Throughout we will not distinguish between homeomorphisms and their mapping classes, nor between curves and their isotopy classes, and curves will always be simple and closed. Two types of Torelli elements that we will encounter frequently are separating twists and bounding pair maps. A Dehn twist is separating if it is a twist about a separating curve. A separating curve of genus k bounds two subsurfaces, one of genus k and one of genus $g-k$, where the latter contains any puncture or boundary component. For a closed surface, a genus k separating twist is also a genus $g-k$ separating twist. Separating twists lie not only in $\mathcal{I}$ but also in $\mathcal{K}$. Let $\mathcal{T}_k$ be the normal closure of a genus k separating twist in $\mathcal{M}$. A bounding pair map of genus k is a product Dehn twists $T_c T_d^{-1}$ about a pair of disjoint nonhomotopic nonseparating simple closed curves c and d that together bound subsurfaces of genus k and $g-k-1$; again, the latter contains any puncture or boundary component. For a closed surface, a genus k bounding pair map is also a genus $g-k-1$ bounding pair map. Let $\mathcal{W}_k$ be the normal closure of a genus k bounding pair map in $\mathcal{M}$. In these terms, Johnson's normal generation result for Torelli groups can be phrased as $\mathcal{W}_1 = \mathcal{I}$ for $g \geq 3$.

Strategy and outline. By the short exact sequence for the Johnson homomorphism and Johnson's result that $\mathcal{K}_{g,1}$ is generated by genus 1 and genus 2 separating twists for $g \geq 3$ [9], the following three properties suffice to show that $\mathcal{W}_k = \mathcal{I}$, the conclusion of our main theorem:

(1) The restriction of τ to $\mathcal{W}_k$ is surjective: $\tau(\mathcal{W}_k) = \tau(\mathcal{I})$.
(2) $\mathcal{T}_1 < \mathcal{W}_k$.
(3) $\mathcal{T}_2 < \mathcal{W}_k$.

In Sections 1 and 2 we show that for all $g \geq 3$ and any $1 \leq k < g-1$, $\mathcal{W}_k$ satisfies the first and second properties, respectively. At the end of Section 2 we show that the third property also holds in the special case of a punctured surface when $k = g-2$ and $g \geq 4$, so in this case we have $\mathcal{W}_{g-2} = \mathcal{I}_{g,*}$. We also provide a second proof that $\mathcal{W}_k = \mathcal{I}_{g,*}$ in this special case, and this second proof does not use Johnson's result about generating $\mathcal{K}_{g,1}$ with separating twists. It is more elementary and proceeds by analyzing the Birman exact sequence.

In order to upgrade this punctured special case to the bordered setting, we take a preparatory step in Section 3, showing that the restriction of σ to $\mathcal{W}_k$ is surjective, where σ is the Birman–Craggs–Johnson homomorphism: $\sigma(\mathcal{W}_k) = \sigma(\mathcal{I})$. We begin Section 4 by proving the bordered special case. We then conclude the general case, Theorem 1, by embedding a bordered surface of genus $k+2$ into the given surface of genus g and invoking the bordered special case.

We note that there is a path through the logic of the paper that avoids ever using Johnson's $\mathcal{K}_{g,1}$ generation result. Still, the three-property strategy explained above is a helpful conceptual throughline. Additionally, while we can avoid the $\mathcal{K}_{g,1}$ generation result, the paper does use a number of deep results about Torelli groups in an essential way; by way of example, we rely on Johnson's calculation of the abelianization of $\mathcal{I}$ in the proof of Proposition 9. Finally, in addition to using many of Johnson's results, our arguments often closely follow constructions and ideas found in his papers, especially [6] and [11].

Consequences. In certain proofs about Torelli groups in the literature, constructions are made using genus 1 bounding pair maps specifically because of their generating property. One example occurs in a proof by the second author and Margalit, showing that for $g \geq 3$ there exist pseudo-Anosov mapping classes whose normal closures in $\mathcal{M}_g$ are equal to $\mathcal{I}_g$, including ones with arbitrarily large stretch factors [12, Theorem 1.3]. Theorem 1 allows for greater flexibility in making such constructions.

In conjunction with results from the same paper with Margalit, Theorem 1 implies the following corollary; for simplicity we only give the statement for closed surfaces.

Corollary 2. *Let $g \geq 3$ and let f be an element of $\mathcal{M}_g$ such that there exists a curve c where c and $f(c)$ are distinct and disjoint. Then the normal closure of f contains $\mathcal{I}_g$.*

Proof. If c is separating or if c and $f(c)$ do not form a bounding pair, then the normal closure of f is $\mathcal{M}_g$ by the well-suited curve criterion [12, Lemmas 2.3 and 2.4]. Otherwise, the normal closure of f contains the bounding pair map $T_c T_{f(c)}^{-1}$, and so by Theorem 1 it contains $\mathcal{I}_g$. □

In particular, this shows that any normal subgroup containing an impure mapping class (in the sense of Ivanov) also contains the Torelli group. Corollary 2 can be viewed as a well-suited curve criterion for the Torelli group.

We also have the following application concerning color preserving subgroups of $\mathcal{M}_g$. Bestvina–Bromberg–Fujiwara give a construction of a finite-index color preserving subgroup of $\mathcal{M}_g$, where each simple closed curve on S_g is assigned one of finitely many colors, curves of the same color intersect, and the color preserving subgroup preserves the colors of curves [1, Lemma 5.7]. Theorem 1 implies a simple sufficient property for producing such colorings and color-preserving subgroups.

Corollary 3. *Let G be a normal subgroup of $\mathcal{M}_g$ with finite index k such that G does not contain $\mathcal{I}_g$. Then there is a coloring of curves on S_g with k colors where distinct curves of the same color pairwise intersect and where the action of G preserves the coloring.*

Proof. There are k orbits of curves under the action of G; assign to each orbit a distinct color. This coloring is preserved by G by definition. As G does not contain $\mathcal{I}_g$, any two distinct curves in the same orbit intersect, by Corollary 2. $\square$

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1. The image of $\mathcal{W}_k$ under the Johnson homomorphism τ

In this section we carry out the first step in our strategy, showing that the restriction of τ to $\mathcal{W}_k$ is surjective.

Proposition 4. *For $g \geq 3$ and $1 \leq k < g-1$, the restriction of τ to $\mathcal{W}_k$ is surjective: $\tau(\mathcal{W}_k) = \tau(\mathcal{I})$.*

Proof. With the standard setup for τ, the image under τ of the standard bounding pair map of genus k is

$$x = \sum_{i=1}^{k} (a_i \wedge b_i) \wedge b_{k+1} \in \tau(\mathcal{W}_k).$$

Let ϕ be the factor mix element of $\mathrm{Aut}(H)$ given by the following map on generators of H:

$$\begin{aligned} a_1 &\mapsto a_1 - b_{k+2} \\ a_{k+2} &\mapsto a_{k+2} - b_1 \end{aligned}$$

and that fixes all other basis elements; this exists by the hypothesis on k.

Then

$$\phi(x) = (-b_{k+2} \wedge b_1 \wedge b_{k+1}) + \sum_{i=1}^{k} (a_i \wedge b_i) \wedge b_{k+1} \in \tau(\mathcal{W}_k).$$

And so therefore also

$$\phi(x) - x = (-b_{k+2} \wedge b_1 \wedge b_{k+1}) \in \tau(\mathcal{W}_k).$$

Now following an argument of Johnson [6, Theorem 1], we have that there are elements in the orbit of $\phi(x) - x$ under the action of $\mathrm{Aut}(H)$ that form a basis for $\tau(\mathcal{I})$. (Johnson begins with $a_1 \wedge b_1 \wedge b_2$ and builds up a basis for $\tau(\mathcal{I})$ by applying simple elements of $\mathrm{Aut}(H)$.) Therefore $\tau(\mathcal{W}_k) = \tau(\mathcal{I})$. $\square$

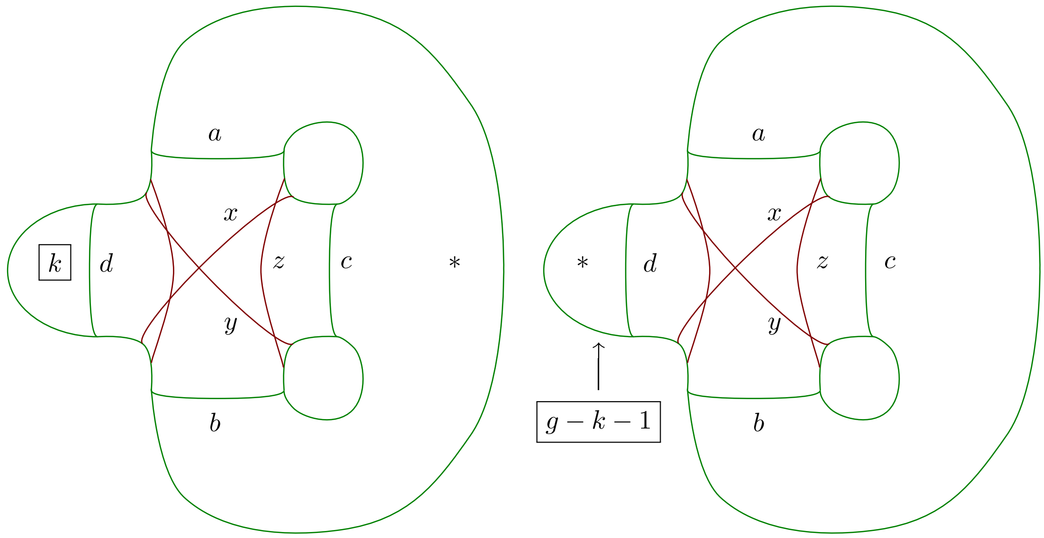

FIGURE 1. Two lantern configurations used to show that $\mathcal{T}_k$ and $\mathcal{T}_{k+1}$ respectively are subgroups of $\mathcal{W}_k$. The asterisks indicate any puncture or border, and the boxed values in the subsurfaces indicate their genus.

2. SEPARATING TWISTS IN $\mathcal{W}_k$

In this section we show that $\mathcal{W}_k$ always contains certain separating twists, namely those of genus k, $k+1$, and 1. The proofs in this section are similar to arguments made by Johnson in showing that $\langle \mathcal{T}_1, \mathcal{T}_2 \rangle$ is equal to the subgroup generated by all separating twists and that $\mathcal{W}_1 = \mathcal{I}$ [11, Theorems 1 and 2]. We end the section by giving two proofs that a genus 2 separating twist is contained in W_k when $k = g-2$, and that in this case $\mathcal{W}_k = \mathcal{I}_{g,*}$. In the remainder of the paper we bootstrap from this special case to conclude the general case.

Proposition 5. *For $g \geq 3$ and $1 \leq k < g-1$, $\mathcal{T}_k < \mathcal{W}_k$ and $\mathcal{T}_{k+1} < \mathcal{W}_k$.*

Proof. Take a lantern in S in each of the two configurations depicted in Figure 1. These differ only in the genus of the subsurfaces complementary to the lantern along with the location of the puncture or border, if any. In both figures, we have by the lantern relation:

$$T_a T_b T_c T_d = T_x T_y T_z$$

or

$$T_d = (T_x T_a^{-1})(T_y T_b^{-1})(T_z T_c^{-1}).$$

In the first configuration, we have that T_d is a separating twist of genus k, while each of the three terms on the right is a bounding pair map of genus k. We therefore conclude that $\mathcal{T}_k < \mathcal{W}_k$. In the second configuration, we have that T_d is a separating twist of genus $k+1$, while each of the three terms on the right is a bounding pair map of genus k. We therefore conclude that $\mathcal{T}_{k+1} < \mathcal{W}_k$. □

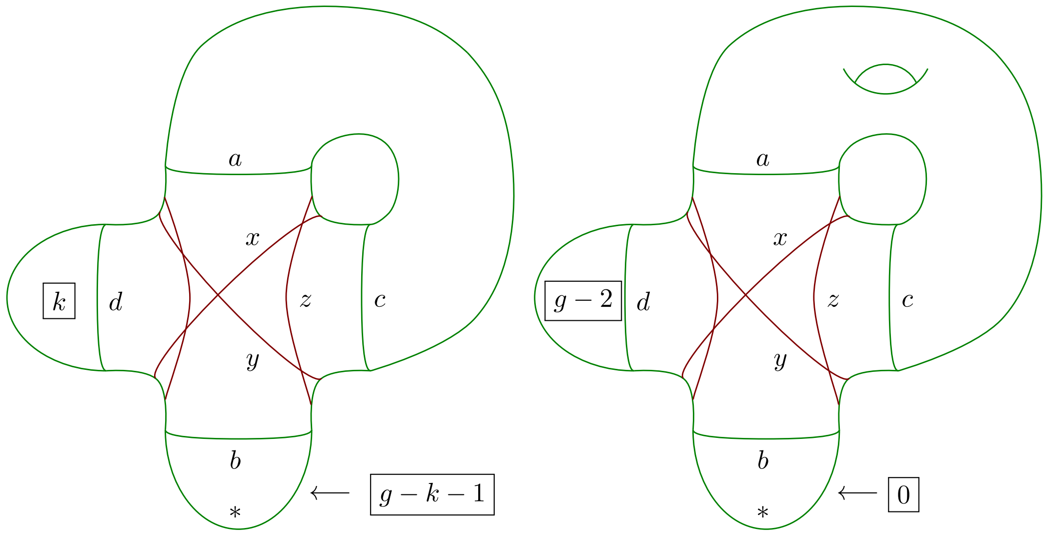

FIGURE 2. Left: the lantern configuration used to show that $\mathcal{T}_1$ is a subgroup of $\mathcal{W}_k$. The asterisk indicates any puncture or border. Right: the lantern configuration used to show that $\mathcal{T}_2$ is a subgroup of $\mathcal{W}_{g-2}$ in a punctured surface; the asterisk indicates the puncture.

Proposition 6. *For $g \geq 3$ and $1 \leq k < g-1$, $\mathcal{T}_1 < \mathcal{W}_k$.*

Proof. We may take a lantern in S as depicted on the left in Figure 2. The separating curves b, d, and y cut off subsurfaces of genus $k+1$, k, and 1, respectively. By the lantern relation, we have

$$T_y = (T_a T_x^{-1})(T_c T_z^{-1}) T_b T_d.$$

We have that T_y is a separating twist of genus 1. Each of the first two terms on the right is a bounding pair map of genus k and so these lie in $\mathcal{W}_k$. The other two terms are separating twists of genus $k+1$ and k, respectively; these lie in $\mathcal{W}_k$ by Proposition 5. We therefore conclude that $\mathcal{T}_1 < \mathcal{W}_k$. □

Proposition 7. *For a punctured surface with $g \geq 3$, $\mathcal{T}_2 < \mathcal{W}_{g-2}$ and $\mathcal{W}_{g-2} = \mathcal{I}_{g,*}$.*

Proof. For $g = 3$, this is Johnson's result $\mathcal{W}_1 = \mathcal{I}_{3,*}$. Otherwise, we may take a lantern in $S_{g,*}$ as depicted on the right in Figure 2. The separating curves b, d, and y cut off subsurfaces of genus g, $g-2$, and 2, respectively. The other subsurface bounded by b has genus 0 and is a punctured disk. By the lantern relation, we have

$$T_y = (T_a T_x^{-1})(T_c T_z^{-1}) T_b T_d.$$

We have that T_y is a separating twist of genus 2. Each of the first two terms on the right is a bounding pair map of genus $g-2$ and so these lie in $\mathcal{W}_{g-2}$. The separating twist T_d of genus $g-2$ lies in $\mathcal{W}_{g-2}$ by Proposition 5, while the twist T_b is trivial. We therefore conclude that $\mathcal{T}_2 < \mathcal{W}_{g-2}$.

Since we also have $\tau(\mathcal{W}_{g-2}) = \tau(\mathcal{I}_{g,*})$ and $\mathcal{T}_1 < \mathcal{W}_{g-2}$ by Propositions 4 and 6, we conclude by our strategy that $\mathcal{W}_{g-2} = \mathcal{I}_{g,*}$. □

Note that if the surface in Proposition 7 were bordered instead of punctured, T_b would be a nontrivial twist about the boundary curve. The argument then yields that $T_y T_b^{-1}$ lies in $\mathcal{W}_{g-2}$, rather than T_y alone. Our goal going forward is to remove this obstacle.

The proof of Proposition 7 uses a major result of Johnson: that $\mathcal{K}_{g,1}$ is generated by genus 1 and genus 2 separating twists for $g \geq 3$ [9]. We provide another proof of Proposition 7 that avoids using Johnson's result.

Second proof of Proposition 7. Let $\pi := \pi_1(S_g)$. By the Birman exact sequence for $\mathcal{I}_{g,*}$, we have that $\mathcal{W}_{g-2} \leq \mathcal{I}_{g,*}$ satisfies a short exact sequence

$$1 \to \pi \cap \mathcal{W}_{g-2} \to \mathcal{W}_{g-2} \to \mathcal{I}_g \to 1$$

(See [2] and [5, Chapter 6.4].) The surjection of the last map is given by Johnson's result that $\mathcal{W}_1 = \mathcal{I}_g$, plus the fact that in $\mathcal{I}_g$ a genus $g-2$ bounding pair map is also a genus 1 bounding pair map. It remains to show that $\pi \cap \mathcal{W}_{g-2} = \pi$.

The commutator point-push $[a_1, b_1] \in \pi' < \pi$ is equal to a product $T_c T_d^{-1}$ where c is a genus $g-1$ separating curve and d is a genus 1 separating curve. Then we have that the element $[a_1, b_1]$ lies in $\mathcal{W}_{g-2}$ by Propositions 5 and 6. The commutator subgroup π' is generated by $\mathcal{M}_{g,*}$ conjugates of $[a_1, b_1]$, as proved by Johnson [9, Section 4]. Therefore $\pi' < \mathcal{W}_{g-2}$.

Let $K := \pi \cap \mathcal{W}_{g-2}$, We now only need to compute

$$K/\pi' < \pi/\pi' = H_1(S_g; \mathbb{Z}).$$

We have the following commutative diagram [9, Section 7].

$$\begin{tikzcd}
1 \arrow[r] & \pi \arrow[r] \arrow[d] & \mathcal{I}_{g,*} \arrow[r] \arrow[d, "\tau_{g,*}"] & \mathcal{I}_g \arrow[r] \arrow[d, "\tau_g"] & 1 \\
1 \arrow[r] & H_g \arrow[r, "\wedge\omega"] & \wedge^3 H_g \arrow[r] & \wedge^3 H_g / H_g \arrow[r] & 1
\end{tikzcd}$$

The restriction $\tau_{g,*}|_\pi$ is the abelianization map. Since the restriction of $\tau_{g,*}$ on $\mathcal{W}_{g-2}$ is surjective by Proposition 4, we have that the abelianization of K is H_g. We conclude that $K = \pi$ and therefore $\mathcal{W}_{g-2} = \mathcal{I}_{g,*}$. As as consequence we have $\mathcal{T}_2 < \mathcal{W}_{g-2}$. □

3. The image of $\mathcal{W}_k$ under the Birman–Craggs homomorphisms σ

In order to prove our main theorem, we will use information about the abelianization of $\mathcal{I}$. Prior to Johnson's work on the Johnson homomorphism τ, Birman–Craggs produced homomorphisms $\mathcal{I} \to \mathbb{Z}/2\mathbb{Z}$ by way of the Rokhlin invariant of 3-manifolds [4]. Johnson showed how to encode all of the Birman–Craggs homomorphisms into a single surjective homomorphism:

$$\sigma : \mathcal{I}_{g,1} \to B^3_{g,1}.$$

(See [7] and [10]). Here $B^d_{g,1}$ is the subalgebra of terms of degree at most d of the following algebra:

$$B(\overline{a_1}, \overline{b_1}, ..., \overline{a_g}, \overline{b_g}) = \mathbb{F}_2[\overline{a_1}, \overline{b_1}, ..., \overline{a_g}, \overline{b_g}] / \langle \overline{a_i}^2 = \overline{a_i}, \overline{b_i}^2 = \overline{b_i} \rangle$$

For any $a, b, c \in H_1(S_g; \mathbb{Z})$, the relation satisfies

$$\overline{a+b} = \overline{a} + \overline{b} + i(a,b) \text{ and } \overline{c}^2 = \overline{c}.$$

After defining the homomorphism σ, Johnson calculated the images under σ of bounding pair maps and separating twists [7, Lemma 8a-c]. For a separating twist T_γ, we have the formula

$$\sigma(T_\gamma) = \sum_{i=1}^{k} \overline{a_i}\overline{b_i}$$

where $\{a_1, b_1, \ldots, a_k, b_k\}$ forms a symplectic basis for the subsurface bounded by γ that does not contain the boundary component.

Proposition 8. *For $g \geq 3$ and $1 \leq k < g-1$, the restriction of σ to $\mathcal{W}_k$ is surjective: $\sigma(\mathcal{W}_k) = \sigma(\mathcal{I}_{g,1})$.*

Proof. Since $\mathcal{W}_k$ contains all genus 1 separating twists by Proposition 6, by Johnson's formula we have that $\overline{a_1}\overline{b_1} \in \sigma(\mathcal{W}_k)$. The group $\mathcal{W}_k$ is normal and Johnson showed that $\overline{a_1}\overline{b_1}$ Sp-generates $B^2_{g,1}$ [7, Proof of Lemma 13]. Therefore $B^2_{g,1} < \sigma(\mathcal{W}_k)$.

It remains to consider the image of $\mathcal{W}_k$ under σ in

$$B^3_{g,1}/B^2_{g,1} \cong \wedge^3 H^1(S_g; \mathbb{F}_2).$$

By [10, Theorem 3], this is the same as considering the image of $\mathcal{W}_k$ under the mod 2 Johnson homomorphism. By Proposition 4 the restriction of τ to $\mathcal{W}_k$ is surjective, so we conclude that the restriction of σ to $\mathcal{W}_k$ is also surjective. $\square$

4. Proof of the main theorem

In this section we first prove a bordered special case by leveraging the punctured special case from Section 2 and the surjectivity result of Section 3. We then use this special case to conclude the main theorem.

Proposition 9. *For a bordered surface with $g \geq 3$, $\mathcal{W}_{g-2} = \mathcal{I}_{g,1}$.*

Proof. For $g = 3$, this is Johnson's result $\mathcal{W}_1 = \mathcal{I}_{3,1}$. For $g \geq 4$, consider the following short exact sequence

$$1 \to \langle T_b \rangle \to \mathcal{I}_{g,1} \to \mathcal{I}_{g,*} \to 1$$

where $\langle T_b \rangle \cong \mathbb{Z}$ and T_b is the Dehn twist about the boundary curve b. The restriction of the homomorphism to $\mathcal{I}_{g,*}$ on the subgroup $\mathcal{W}_{g-2} < \mathcal{I}_{g,1}$ is surjective by Proposition 7. Thus we only need to consider the intersection $\mathcal{W}_{g-2} \cap \langle T_b \rangle$ and show that it is equal to $\langle T_b \rangle$.

By Johnson's calculation of the abelianization of $\mathcal{I}_{g,1}$ [10, Theorem 3], the image of T_b in the abelianization of $\mathcal{I}_{g,1}$ has order 2. (It has trivial image under τ and nontrivial image under σ). We therefore have that T_b^2 is in the kernel of the abelianization and so it is a commutator in $\mathcal{I}_{g,1}$. Given a factorization $T_b^2 = \Pi[r_i, s_i]$, the fact that T_b is central in $\mathcal{I}_{g,1}$ can be used to produce a factorization $T_b^2 = \Pi[r_i', s_i']$ with $r_i', s_i' \in \mathcal{W}_{g-2}$. This implies $T_b^2 \in \mathcal{W}_{g-2}$, so $\mathcal{W}_{g-2}$ is a subgroup of $\mathcal{I}_{g,1}$ of index at most 2.

Each index 2 subgroup of $\mathcal{I}_{g,1}$ is the kernel of a homomorphism $\mathcal{I}_{g,1} \to \mathbb{Z}/2\mathbb{Z}$. By Johnson's calculation of the abelianization of $\mathcal{I}_{g,1}$, we have that every homomorphism $\mathcal{I}_{g,1} \to \mathbb{Z}/2\mathbb{Z}$ factors through at least one of τ or σ. However, by Propositions 4 and 8, the restriction of each of τ and σ to $\mathcal{W}_{g-2}$ is surjective and so $\mathcal{W}_{g-2}$ is not equal to the kernel of any surjective map $\mathcal{I}_{g,1} \to \mathbb{Z}/2\mathbb{Z}$. We conclude that $\mathcal{W}_{g-2} \cap \langle T_b \rangle = \langle T_b \rangle$ and $\mathcal{W}_{g-2} = \mathcal{I}_{g,1}$. $\square$

Our main theorem now follows in a straightforward manner.

Proof of Theorem 1. Since $\mathcal{I}_{g,*}$ and $\mathcal{I}_g$ are quotients of $\mathcal{I}_{g,1}$, it suffices to consider the bordered setting. The case where $k = g - 2$ is treated by Proposition 9. Otherwise $k < g-2$ and we may take a subsurface Σ of $S_{g,1}$ that is homeomorphic to $S_{k+2,1}$. By Proposition 9, every element of $\mathcal{I}_{g,1}$ supported in Σ is contained in $\mathcal{W}_k$. In particular, a genus 1 bounding pair map lies in $\mathcal{W}_k$. Since $\mathcal{W}_k$ is normal in $\mathcal{M}_{g,1}$, we may conclude that $\mathcal{W}_k = \mathcal{I}_{g,1}$ by our strategy or by invoking Johnson's theorem that $\mathcal{W}_1 = \mathcal{I}_{g,1}$. $\square$

References

[1] Mladen Bestvina, Ken Bromberg, and Koji Fujiwara. Constructing group actions on quasi-trees and applications to mapping class groups. *Publ. Math. Inst. Hautes Études Sci.*, 122:1–64, 2015.
[2] Joan S. Birman. Mapping class groups and their relationship to braid groups. *Comm. Pure Appl. Math.*, 22:213–238, 1969.
[3] Joan S. Birman. On Siegel's modular group. *Math. Ann.*, 191:59–68, 1971.
[4] Joan S. Birman and R. Craggs. The μ-invariant of 3-manifolds and certain structural properties of the group of homeomorphisms of a closed, oriented 2-manifold. *Trans. Amer. Math. Soc.*, 237:283–309, 1978.
[5] Benson Farb and Dan Margalit. *A Primer on Mapping Class Groups*. Princeton University Press, 2011.
[6] Dennis Johnson. An abelian quotient of the mapping class group $\mathcal{I}_g$. *Math. Ann.*, 249(3):225–242, 1980.
[7] Dennis Johnson. Quadratic forms and the Birman-Craggs homomorphisms. *Trans. Amer. Math. Soc.*, 261(1):235–254, 1980.
[8] Dennis Johnson. A survey of the Torelli group. In *Low-dimensional topology (San Francisco, Calif., 1981)*, volume 20 of *Contemp. Math.*, pages 165–179. Amer. Math. Soc., Providence, RI, 1983.
[9] Dennis Johnson. The structure of the Torelli group. II. A characterization of the group generated by twists on bounding curves. *Topology*, 24(2):113–126, 1985.
[10] Dennis Johnson. The structure of the Torelli group. III. The abelianization of $\mathcal{T}$. *Topology*, 24(2):127–144, 1985.
[11] Dennis L. Johnson. Homeomorphisms of a surface which act trivially on homology. *Proc. Amer. Math. Soc.*, 75(1):119–125, 1979.
[12] Justin Lanier and Dan Margalit. Normal generators for mapping class groups are abundant. *Comment. Math. Helv.*, 97(1):1–59, 2022.
[13] Jerome Powell. Two theorems on the mapping class group of a surface. *Proc. Amer. Math. Soc.*, 68(3):347–350, 1978.

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