CONSTRAINING MAPPING CLASS GROUP HOMOMORPHISMS USING FINITE SUBGROUPS

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Abstract. We give a new proof of a result of Aramayona–Souto that constrains homomorphisms between mapping class groups of closed surfaces. The proof involves an analysis of finite subgroups. Using the same approach, we also constrain homomorphisms from mapping class groups to homeomorphism groups of low-dimensional spheres, extending work of Franks–Handel.

0. Introduction

Let S_g be the connected, closed, orientable surface of genus g and let $\mathbb{S}^n$ be the n-sphere. The groups $\mathrm{Homeo}^+(X)$ and $\mathrm{Diff}^+(X)$ are, respectively, the groups of orientation-preserving homeomorphisms and diffeomorphisms of a given orientable manifold X. The mapping class group $\mathrm{Mod}(X)$ is $\pi_0(\mathrm{Homeo}^+(X))$, the group of isotopy classes of orientation-preserving homeomorphisms of X.

In this paper we analyze torsion to prove theorems about homomorphisms from $\mathrm{Mod}(S_g)$ to mapping class groups and to homeomorphism and diffeomorphism groups of spheres. Much has been learned about homomorphisms of mapping class groups by analyzing torsion. Several examples appear later in the introduction, and other examples include work by Markovic, Mann–Wolff, and the first author [22, 21, 8]. Our approach here is similar in spirit, but for the problems we consider it is not sufficient to analyze individual periodic elements in isolation. We explain this obstacle below. As a consequence, we proceed instead by analyzing non-cyclic finite subgroups of $\mathrm{Mod}(S_g)$. Our theorems are as follows.

Theorem 1. *For $g \geq 3$ and $0 \leq h < 2g-1$ with $h \neq g$, every homomorphism $\phi : \mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$ is trivial. When $g \geq 3$ is odd, the same conclusion holds for the bounds $0 \leq h < 2g+1$ with $h \neq g$.*

Theorem 2. *For $g \geq 3$, every homomorphism from $\mathrm{Mod}(S_g)$ to either $\mathrm{Homeo}^+(\mathbb{S}^2)$ or $\mathrm{Homeo}^+(\mathbb{R}^3)$ is trivial. For $g \geq 42$, every homomorphism from $\mathrm{Mod}(S_g)$ to either $\mathrm{Homeo}^+(\mathbb{S}^3)$ or $\mathrm{Diff}^+(\mathbb{S}^4)$ is trivial.*

Aramayona–Souto proved a rigidity theorem about homomorphisms between mapping class groups [2]. Theorem 1 recovers their theorem in the case of closed surfaces of different genus. Our proof also yields a small improvement in their genus bounds. Theorem 2 is an extension of a theorem of Franks–Handel, who consider maps from $\mathrm{Mod}(S_g)$ to $\mathrm{Diff}^+(\mathbb{S}^2)$ [13]. We give a more detailed comparison of our results and approaches later in the introduction.

Our proofs of Theorems 1 and 2 use the following result of the second author and Margalit. A group element is said to normally generate if its normal closure equals the group.

Theorem 3 (Theorem 1.1, [19])**.** *For $g \geq 3$, every nontrivial periodic mapping class that is not a hyperelliptic involution normally generates $\mathrm{Mod}(S_g)$.*

It follows immediately from this theorem that any homomorphism from $\mathrm{Mod}(S_g)$ that has a nontrivial nonhyperelliptic periodic element in its kernel is trivial. The strong constraints on homomorphisms provided by Theorem 3 are not, however, immediately applicable to the settings of Theorems 1 and 2. Under the hypotheses of each of these theorems, there are cases where, a priori, there could be a homomorphism to the target group where no nontrivial periodic mapping class of $\mathrm{Mod}(S_g)$ would lie in the kernel. For instance, for every periodic element in $\mathrm{Mod}(S_7)$, there is an element of $\mathrm{Mod}(S_9)$ of the same order. For Theorem 2, the situation is more extreme, as homeomorphism groups of spheres contain elements of every finite order.

Despite these obstacles, we are still able to apply Theorem 3 to these situations by analyzing non-cyclic finite subgroups. Our analysis builds upon prior work on these finite subgroups by a number of authors: May–Zimmerman, Müller–Sarkar, Accola, Maclachlan, and Weaver. Our work was facilitated by the cataloging of finite subgroup of $\mathrm{Mod}(S_g)$ up to $g = 48$ by Breuer and by Paulhus; see [4, 30, 29].

Homomorphisms between mapping class groups.

Aramayona–Souto proved a theorem constraining homomorphisms between the mapping class groups of surfaces $S_{g,n,b}$, which are surfaces of genus g with n punctures and b boundary components [2]. The mapping class groups they consider are pure, in that they require mapping classes to fix punctures and boundary components pointwise.

Theorem 4 (Aramayona–Souto, Theorem 1.1, [2])**.** *Let $S = S_{g,n,b}$ and $S' = S'_{g',n',b'}$, such that $g \geq 6$ and $g' \leq 2g-1$. If $g' = 2g-1$, suppose also that S' is not closed. Then every nontrivial homomorphisms $\phi : \mathrm{Mod}(S) \to \mathrm{Mod}(S')$ is induced by an embedding $S \to S'$.*

Aramayona–Souto also prove that the conclusion of this theorem holds when $g = g' \in \{4, 5\}$. They explain the necessity of their upper bound of $2g - 1$ by observing that there is a "double embedding" homomorphism $\mathrm{Mod}(S_{g,0,1}) \to \mathrm{Mod}(S_{2g,0,0})$.

When restricted to the case of closed surfaces, Theorem 4 says that for $g \geq 6$ and $h < 2g - 1$, every homomorphism $\phi : \mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$ is trivial, except for the possibility of an isomorphism in the case $g = h$. Some cases of this result for closed surfaces were previously known. The range $h < g$ was previously handled by Harvey–Korkmaz [15], while the case $h = g$ was treated by Ivanov–McCarthy under the further hypothesis that the homomorphism is injective [16].

Theorem 1 gives a new proof of Theorem 4 when restricted to the case where S and S' are closed surfaces and $g \neq g'$. Further, it is an easy consequence of the proof of Theorem 1 that we may replace S_h with a surface of genus h and arbitrarily many punctures and boundary components, since $\mathrm{Mod}(S_{h,n,b})$ has no additional finite subgroups compared to $\mathrm{Mod}(S_h)$. For this replacement, note that it is not necessary for us to restrict to $\mathrm{Mod}(S_{h,n,b})$ being pure. Our Theorem 1 also covers the additional small values of g of 3, 4, and 5, confirming an expectation that Aramayona–Souto state in their paper. We also extend their upper bounds for g' slightly when g is odd. This is possible because, for closed surfaces, their upper bound of $2g-1$ does not have the same natural justification that appears for surfaces with boundary.

It is important to note that, even for closed surfaces, some upper bound is necessary on h to ensure that the homomorphism is trivial. First, since $\mathrm{Mod}(S_g)$ is residually finite, it has a rich supply of finite quotients [14]. As every finite group is a subgroup for some $\mathrm{Mod}(S_h)$, we obtain for all $g > 0$ non-trivial homomorphisms $\mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$ that factor through finite quotients. An even more striking reason why some upper bound on h is necessary is a result of Aramayona–Leininger–Souto: that for all $g \geq 2$, there exists a nontrivial connected cover S_h of the surface S_g such that $\mathrm{Mod}(S_g)$ injects into $\mathrm{Mod}(S_h)$ [1]. All of these sources of nontrivial homomorphisms between mapping class groups are in accord with the conjectural picture proposed by Mirzakhani and recorded in [2] that every homomorphism between mapping class groups of sufficiently high genus has either finite image or is induced by some manipulation of surfaces. Regarding finite quotients, see also Section 3 of Birman's problem paper [3].

The preceding results raise the following natural question:

Question 5. *For each $g \geq 3$, what is the smallest $h > g$ such that there exists a nontrivial homomorphism $\phi : \mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$?*

We point out that while our Theorem 1 gives approximately the same lower bound on h as that of Aramayona–Souto of about $2g$, our approach suggests that the true bound ought to be higher. Using the work of Breuer and Paulhus that catalogues the finite subgroups of $\mathrm{Mod}(S_g)$ for $g \leq 48$, the following chart indicates for small values of g the smallest h for which $\mathrm{Mod}(S_h)$ contains all of the finite subgroups that are contained in $\mathrm{Mod}(S_g)$. By Theorem 3 and our Corollary 11, for all smaller h we have that $\phi : \mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$ is trivial as long as $h \leq 3^{g-1}$. This last condition holds for all $g \geq 4$ in the table, while for $\mathrm{Mod}(S_3)$ the table implies that the first candidate target for a non-trivial homomorphism is $\mathrm{Mod}(S_{10})$. We observe that for these limited data points, the first values for h that are candidates for a nontrivial homomorphism $\phi : \mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$ are notably larger than $2g$.

g	h
3	15
4	16
5	21
6	> 48
7	> 48
8	40

Homomorphisms to homeomorphism groups of spheres.

Franks–Handel prove a number of theorems showing that homomorphisms from mapping class groups to homeomorphism and diffeomorphism groups are trivial [13]. Their main result is that homomorphisms $\mathrm{Mod}(S) \to \mathrm{GL}(n, \mathbb{C})$ are trivial for S a finite-type surface whenever $g \geq 3$ and $n < 2g$. They also apply the theorem of Aramayona–Souto to show that, with the same genus bounds and the added requirement that $g' > 1$, every homomorphism $\mathrm{Mod}(S_g) \to \mathrm{Homeo}(S_{g'})$ is trivial. As these bounds exclude $g' = 0, 1$, they go on to show that for $g \geq 3$, all homomorphisms from $\mathrm{Mod}(S_g)$ to each of $\mathrm{Homeo}(\mathbb{S}^1)$ and $\mathrm{Homeo}(S_1)$ are trivial, and they also show the following theorem.

Theorem 6 (Franks–Handel, Theorem 1.4, [13])**.** *For $g \geq 7$, every homomorphism $\phi : \mathrm{Mod}(S_g) \to \mathrm{Diff}^+(\mathbb{S}^2)$ is trivial.*

Our Theorem 2 includes an extension of this theorem of Franks–Handel to the target $\mathrm{Homeo}^+(\mathbb{S}^2)$ and also covers several additional small genus cases. Note that the lower bound of $g \geq 3$ in the statement of Theorem 2 is necessary, since in the cases $g = 1, 2$ there are nontrivial homomorphisms that factor through the abelianization of $\mathrm{Mod}(S_g)$.

In Theorem 2, one of our target groups is a diffeomorphism group rather than a homeomorphism group. Unlike in the case of $\mathrm{Homeo}^+(\mathbb{S}^2)$, for higher-dimensional spheres there exist actions by finite groups that have wildly embedded fixed point sets, and such actions cannot be smooth; see the survey article by Zimmermann for a discussion [35]. Recent work of Pardon has as a consequence that there do not exist any isomorphism types of finite subgroups of $\mathrm{Homeo}^+(\mathbb{S}^3)$ that do not also occur in $\mathrm{Diff}^+(\mathbb{S}^3)$ [28]; this result allows us to drop the assumption of smoothness in the case of $\mathbb{S}^3$. On the other hand, the finite subgroups of $\mathrm{Homeo}^+(\mathbb{S}^4)$ are not yet classified. Under the further hypothesis of smoothness, however, there are results classifying finite group actions on $\mathbb{S}^4$; we use these in our proof of Theorem 2.

We close with two questions. Zimmermann observes in the same survey article that just by considering faithful, real, linear representations of finite groups, it is clear that every finite group appears as a subgroup of $\mathrm{Diff}^+(\mathbb{S}^d)$ for some d. This fact suggests that it may be difficult to constrain homomorphisms to these groups in a uniform way through finite subgroup obstructions.

Question 7. *For $g \geq 3$ and $d \geq 5$, when are there nontrivial homomorphisms $\phi : \mathrm{Mod}(S_g) \to \mathrm{Diff}^+(\mathbb{S}^d)$?*

While it is not clear whether or not $\mathrm{Mod}(S_g)$ can act smoothly on higher-dimensional spheres, it does act nontrivially—and, in fact, faithfully—by homeomorphisms on $\mathcal{PMF}(S_g) \cong \mathbb{S}^{6g-7}$, the space of projective measured foliations on S_g. By a suspension construction, $\mathrm{Mod}(S_g)$ acts faithfully by homeomorphisms on each sphere $\mathbb{S}^d$ for $d \geq 6g - 7$. We therefore ask the following:

Question 8. *For $g \geq 3$ and $3 \leq d < 6g-7$, are there any nontrivial homomorphisms $\phi : \mathrm{Mod}(S_g) \to \mathrm{Homeo}^+(\mathbb{S}^d)$?*

It would be remarkable if the answer to this question were "no" and interesting if it were "yes".

Outline. After proving some preliminary lemmas, we prove Theorem 1 in Section 1. We then prove Theorem 2 in Section 2.

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1. Homomorphisms between mapping class groups

The strategy for proving Theorem 1 is straightforward. We first show that some periodic element is in the kernel of the given homomorphism $\mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$. We then show that this implies that the homomorphism is trivial. We lay the groundwork for these two steps in two lemmas, Lemmas 9 and 10. We then prove Theorem 1.

We first show that a nontrivial periodic element must lie in the kernel of the homomorphism, for the simple reason that there exists a finite subgroup of $\mathrm{Mod}(S_g)$ that does not exist in any $\mathrm{Mod}(S_h)$ for h in the specified range. The finite subgroups that we use lie in two infinite families, one for when g is even, the other for when g is odd. These families of subgroups were studied by May–Zimmerman; we will draw out the salient features of their work and will repeat some of their arguments for the sake of clarity, but we refer the reader to their papers for full details.

For $n \geq 2$, let DC_n be the dicyclic group of order $4n$, given by the presentation

$$\langle\ x, y \mid x^{2n} = 1, x^n = y^2, y^{-1}xy = x^{-1}\ \rangle.$$

May–Zimmerman showed that when n is even, DC_n has *strong symmetric genus* n; that is, the first genus g for which DC_n appears as a subgroup of $\mathrm{Mod}(S_g)$ is when $g = n$ [23, Theorem 1]. Similarly, they show that when n is odd, $C_4 \times D_n$ has strong symmetric genus n, where C_n is the cyclic group of order n and D_n is the dihedral group of order $2n$ [24, Theorem 3]. Let G stand for any one of these finite groups. In proving their results, May–Zimmerman first show that G does in fact appear as a subgroup of the specified mapping class group $\mathrm{Mod}(S_g)$. To guarantee that this is the first appearance, they then give lower bounds on h for any other $\mathrm{Mod}(S_h)$ containing G as a subgroup, showing that $h > g$. In the proof of Lemma 9, we follow their method and keep track of lower bounds on h, showing that there exists a gap between the first appearance of G as a subgroup and its next appearance within the family of mapping class groups.

Lemma 9. *When $g \geq 2$ is even, DC_g appears as a subgroup of* $\mathrm{Mod}(S_g)$ *and does not appear in any other* $\mathrm{Mod}(S_h)$ *with $h < 2g - 1$. When $g \geq 3$ is odd, $C_4 \times D_g$ appears as a subgroup of* $\mathrm{Mod}(S_g)$ *and does not appear in any other* $\mathrm{Mod}(S_h)$ *with $h < 2g + 1$.*

Before proving Lemma 9, we require some preliminaries; see the article of Broughton as a reference [5]. Recall that for any faithful orientation-preserving action of a finite group G on a hyperbolic surface S_g by isometries, we have that

$$A = 2g_0 - 2 + \sum_{i=1}^{r} \left(1 - \frac{1}{\lambda_i}\right) \tag{1}$$

where the normalized area A is the hyperbolic area of the quotient orbifold scaled by $\frac{1}{2\pi}$, the g_0 is the genus of the quotient orbifold, the λ_i are the orders of the cone points, and the r is the number of cone points. The data of this action is called its signature and it is often encoded as $(g_0; \lambda_1, \ldots, \lambda_r)$. The normalized area, group order, and genus g of the original surfaces are related by the equation

$$|G| \cdot A = 2g - 2.$$

Finally, for a signature of $(g_0; \lambda_1, \ldots, \lambda_r)$ to arise from an action of G, these values must satisfy the Riemann–Hurwitz equation and there must be elements $a_1, \ldots, a_g$, $b_1, \ldots, b_g$, $c_1, \ldots, c_r \in G$ that together generate G, where $|c_i| = \lambda_i$, and that satisfy

$$\prod_{i=1}^{g} [a_i, b_i] \cdot \prod_{j=1}^{r} c_j = 1. \tag{2}$$

This equation follows from the fact that a finite group acting on a hyperbolic surface must arise as a quotient of a Fuchsian group.

The methods used here to constrain finite group actions on surfaces are similar to those used in the proofs of the classical $84(g-1)$ and $4g+2$ theorems; see, for instance, [11, Theorems 7.4, 7.5].

Proof of Lemma 9. May–Zimmerman argue that when $g \geq 2$ is even and DC_g is a subgroup of $\mathrm{Mod}(S_h)$, either $h = g$ or $h > g$. We will sharpen their second bound to show that in fact either $h = g$ or $h \geq 2g - 1$. Their arguments run by showing a dichotomy that either the normalized area A of a fundamental domain of the action of DC_g on S_h satisfies either $A = \frac{1}{2} - \frac{1}{2g}$ or else $A \geq \frac{1}{2}$. The value of h can then be computed from the Riemann–Hurwitz equation: $h = 1 + 2gA$. We will improve the latter bound to $A \geq \frac{g-1}{g}$, which implies that $h \geq 2g - 1$.

We consider cases based on the values of g_0 and r. By Equation (1), if $g_0 \geq 2$, then $A \geq 2$. Next, if $g_0 = 1$, then $r \geq 1$ since g is assumed to be at least 2. If $g_0 = 1$ and $r \geq 2$, then $A \geq 1$. If $g_0 = 1$ and $r = 1$, then DC_g has a generating set $\langle a, b\rangle$ where $[a, b] = c^{-1}$ and $|c| = \lambda_1$. It is straightforward to check that $|[a, b]| = g$ for any generating pair for DC_g. This yields $A \geq \frac{g-1}{g}$.

Finally, consider the case $g_0 = 0$. Since $A > 0$, we know that $r \geq 3$. If $r \geq 5$, then we have $A \geq \frac{3}{4} \cdot 2 + \frac{1}{2} \cdot 3 - 2 = 1$. Now assume that $r \leq 4$. There is a generating set of DC_g satisfying $\prod_{j=1}^{r} c_j = 1$. Any generating set for DC_g contains at least one generator outside of $\langle x\rangle$, and by the product restriction there are an even number of these in one of our generating sets. Further, each element outside of $\langle x\rangle$ has order 4. We now treat the cases $r = 3$ and $r = 4$ in turn.

Let $r = 3$. Then exactly two c_i lie outside of $\langle x\rangle$ and these have order 4. These must generate DC_g, and a short computation shows that their product has order $2g$. Therefore if $r = 3$, the signature is $(0; 4, 4, 2g)$, $A = \frac{1}{2} - \frac{1}{2g}$, and $h = g$, as shown by May–Zimmerman.

Finally, let $r = 4$. There are either two or four λ_i equal to 4 that correspond to elements outside of $\langle x\rangle$. If there are four, then $A = 1$. If there are two and the corresponding c_i together form a generating pair for DC_g, then the order of their product is relatively prime to $2g$. Then the product of the remaining c_i must equal the inverse of this product and so has the same order. If one of the remaining λ_i is either 2 or 3, then the last λ_i must be $2g$ or $2g/3$, respectively. (The latter case is only possible when g is a multiple of 3.) These yield $A = \frac{2g-1}{2g} > \frac{g-1}{g}$ and $A = \frac{7}{6} - \frac{3}{2g}$. The latter is at least 1 when $g \geq 12$ and also satisfies the required bound when $g = 6$.

The final possibility is that there are exactly two λ_i equal to 4 that correspond to elements outside of $\langle x\rangle$ and so that the corresponding c_i together do not form a generating pair for DC_g. Again, the product of the remaining c_i must equal the inverse of this product and so have the same order. Since a product in these remaining two c_i must generate $\langle x\rangle$, if one of the remaining λ_i is either 2 or 3, then the last λ_i must be again be $2g$ or $2g/3$, respectively. The result follows.

Similarly, May–Zimmerman argue that when $g \geq 3$ is odd and $C_4 \times D_g$ is a subgroup of $\mathrm{Mod}(S_h)$, either $h = g$ or $h > g$. We will show how their arguments imply that $h = g$ or $h \geq 2g + 1$. Their proof proceeds by showing that the normalized area A of a fundamental domain of the action on S_h by $C_4 \times D_g$ is either exactly $\frac{1}{4} - \frac{1}{4g}$ or else $A \geq \frac{1}{2}$, leaving a few of the final cases as an exercise. These bounds imply that either

$h = 1 + 4g(\frac{1}{4} - \frac{1}{4g}) = g$ or $h \geq 1 + 4g \cdot \frac{1}{2} = 2g + 1$ whenever $C_4 \times D_g$ is a subgroup of $\mathrm{Mod}(S_h)$, as desired.

In what follows, we recap the arguments of May–Zimmerman and fill in the cases they leave as an exercise. If $g_0 \geq 1$, then $A \geq \frac{1}{2}$. Further, if $g_0 = 0$, then $r \geq 3$, and also the λ_i are all even. If $r \geq 4$, then $A \geq \frac{1}{2}$. If $r = 3$, they argue that if $\lambda_1 = 2$ then $h = g$, and if $\lambda_1 \geq 6$ then $A \geq \frac{1}{2}$. It remains to treat $\lambda_1 = 4$. For $\lambda_2 \geq 8$, they show that $A \geq \frac{1}{2}$; they leave $\lambda_2 = 4, 6$ as an exercise.

If $\lambda_1 = 4$ and $\lambda_2 = 4$, then each generator c_1, c_2 must be a generator of C_4 times a reflection in D_n, and necessarily the second factor of $c_1 \cdot c_2$ must be a rotation of order n. But a pair of such elements does not generate $C_4 \times D_g$; instead they generate an index 2 normal subgroup corresponding to a quotient to C_2 given by the parity of word length.

Similarly, if $\lambda_1 = 4$ and $\lambda_2 = 6$, then c_1 must be a generator of C_4 times a reflection in D_n, and c_2 must be either the identity or the square of a generator in the first factor and a rotation of order 3 in the second factor. This last condition is impossible unless n is a multiple of 3. But a rotation that does not generate the rotation subgroup in D_n cannot be a member of a generating pair for D_n. So this case is impossible as well. □

We now prepare the second step, showing that a periodic element in the kernel of $\mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$ implies that the homomorphism is trivial whenever h is in the specified range, even in the case when it contains a hyperelliptic involution. The following lemma is similar to a result proved and applied by Harvey–Korkmaz to show their result on homomorphisms $\mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$ where $g > h$ [15, Theorem 7]. Lemma 10 has the advantage of giving a uniform treatment for all $g \geq 3$.

Lemma 10. *Let $g \geq 3$ and let $\psi : \mathrm{Sp}(S_g) \to G$ be a homomorphism. If G does not contain $(\mathbb{Z}/3\mathbb{Z})^g$ as a subgroup, then ψ is trivial.*

Proof. We proceed by constructing normal generators of $\mathrm{Sp}(2g, \mathbb{Z})$ and showing that one must be in the kernel of ψ.

The group $\mathrm{Sp}(2g, \mathbb{Z})$ contains $M = (\mathbb{Z}/3\mathbb{Z})^g$ as a subgroup, with generators m_1 to m_g of the form

$$m_i = \begin{bmatrix} I_{2i-2} & 0 & 0 \\ 0 & A & 0 \\ 0 & 0 & I_{2g-2i} \end{bmatrix}$$

Here A is the 2×2 matrix

$$\begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix}$$

of order 3 and I_n denotes the $n \times n$ identity matrix. Every nontrivial element in M is a normal generator of $\mathrm{Sp}(2g, \mathbb{Z})$, since it is the image of a normal generator of $\mathrm{Mod}(S_g)$, as we now show. The desired elements of $\mathrm{Mod}(S_g)$ are products of roots of Dehn twists about disjoint separating curves, as illustrated in Figure 1. For any nontrivial element $m \in M$, there is a corresponding mapping class $\tilde{m}$ and a nonseparating curve c so that c and $\tilde{m}(c)$ intersect exactly once. By the well-suited curve criterion, it follows that the mapping class $\tilde{m}(c)$ is a normal generator of $\mathrm{Mod}(S_g)$ [18, Lemma 2.2]; so too is its image m in $\mathrm{Sp}(2g, \mathbb{Z})$, since normal generators descend to quotients.

As G contains no subgroup isomorphic to M, some normal generator of $\mathrm{Sp}(2g, \mathbb{Z})$ lies in the kernel of ψ; therefore ψ is trivial. □

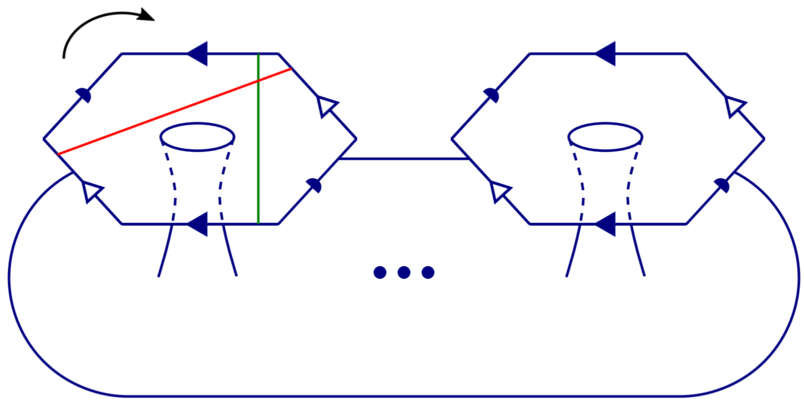

FIGURE 1. The surface S_g, where each of g hexagons with sides identified as indicated yields a handle. Each nontrivial element $\tilde{m} \in \tilde{M}$ consists of $1/3$ or $2/3$ rotations about some of the hexagons; we have $i(c, \tilde{m}(c)) = 1$ for some nonseparating simple closed curve c.

The following corollary can be thought of as an amplification of a theorem of Farb–Masur in the case of $\mathrm{Sp}(2g, \mathbb{Z})$; they show that for any irreducible lattice Γ in a semisimple lie group G of $\mathbb{R}$-rank at least two, the image of any homomorphism $\phi : \Gamma \to \mathrm{Mod}(S)$ is finite [12, Theorem 1.1].

Corollary 11. *Let $g \geq 3$ and let $h \leq 3^{g-1}$. Then every homomorphism $\psi : \mathrm{Sp}(2g, \mathbb{Z}) \to \mathrm{Mod}(S_h)$ is trivial.*

Proof. This follows immediately from Lemma 10 by a result of Müller–Sarkar, who showed that the strong symmetric genus of $(\mathbb{Z}/3\mathbb{Z})^g$ is $1+3^{g-1} \cdot \mu_0(g)$, where $\mu_0(g) \geq 1$ when $g \geq 3$ [27, Section 9.1]. □

With these preliminaries established, the proof of Theorem 1 is straightforward.

Proof of Theorem 1. Let $g \geq 3$ be even and $g < h < 2g - 1$. By Lemma 9, $\mathrm{Mod}(S_g)$ contains DC_g as a subgroup and $\mathrm{Mod}(S_h)$ does not. Then for any homomorphism $\phi : \mathrm{Mod}(S_g) \to \mathrm{Mod}(S_h)$, a nontrivial periodic element f lies in the kernel. If f is not a hyperelliptic involution, we conclude that ϕ is trivial by Theorem 3. If f is a hyperelliptic involution, then ϕ factors through the symplectic representation. Since $h < 2g - 1 \leq 3^{g-1}$, we conclude that ϕ is trivial by Corollary 11.

The proof for odd $g \geq 3$ proceeds in the same way using the finite group $C_4 \times D_g$. □

2. Homomorphisms to homeomorphism groups of spheres

In this section we prove Theorem 2. We consider each of the four target groups in turn: $\mathrm{Homeo}^+(\mathbb{S}^2)$, $\mathrm{Homeo}^+(\mathbb{R}^3)$, $\mathrm{Homeo}^+(\mathbb{S}^3)$, and then $\mathrm{Diff}^+(\mathbb{S}^4)$. The constraining finite subgroups are the same for the first two target groups; when the target groups are $\mathrm{Homeo}^+(\mathbb{S}^3)$ and $\mathrm{Diff}^+(\mathbb{S}^4)$, a deeper analysis of finite subgroups is required.

Proof of Theorem 2. Let $g \geq 3$ and let $\phi : \mathrm{Mod}(S_g) \to \mathrm{Homeo}^+(\mathbb{S}^2)$ be a homomorphism. It is a classical result of Brouwer, Eilenberg, and de Kerékjártó [6, 9, 10] that every finite subgroup of $\mathrm{Homeo}^+(\mathbb{S}^2)$ is conjugate to a finite subgroup of $\mathrm{SO}(3)$. These are the cyclic groups C_n, the dihedral groups D_n, and the tetrahedral, octahedral, and icosahedral groups A_4, Σ_4, and A_5. This classification goes back to the work of Klein; see [7, Chapter 19] or [4, Sect.I.3.4] for treatments.

When $g \geq 3$, $\mathrm{Mod}(S_g)$ contains a finite subgroup that is not isomorphic to any subgroup of SO(3). For instance, we have the Accola–Maclachlan subgroup $C_2 \times C_{2g+2}$ of $\mathrm{Mod}(S_g)$, which attains the $4g+4$ bound on its largest abelian subgroup (see, for instance, [34]). This implies that ϕ has a nontrivial periodic element in its kernel. If this element is not a hyperelliptic involution, Theorem 3 implies that ϕ is trivial. Otherwise, ϕ factors through the symplectic representation. Since SO(3) does not contain $(C_3)^g$ as a subgroup for $g \geq 3$, Lemma 10 implies that ϕ is again trivial.

We next consider homomorphisms to $\mathrm{Homeo}^+(\mathbb{R}^3)$. Zimmermann showed that every orientation-preserving action by homeomorphisms of a finite group on $\mathbb{S}^3$ that has a global fixed point is a finite subgroup of SO(3); he therefore also concludes that every finite subgroup of $\mathrm{Homeo}^+(\mathbb{R}^3)$ is a finite subgroup of SO(3) [36, Corollary 1]. This was also shown independently by Kwasik–Sun [17]. Therefore we may conclude just as we did for $\mathrm{Homeo}^+(\mathbb{S}^2)$ above that ϕ is trivial.

Next we consider homomorphisms to $\mathrm{Homeo}^+(\mathbb{S}^3)$. By recent work of Pardon, every continuous action of a finite group on a smooth 3-manifold is a uniform limit of smooth actions [28]. Pardon's result guarantees that the isomorphism types of the finite subgroups of $\mathrm{Homeo}^+(\mathbb{S}^3)$ are identical to those of $\mathrm{Diff}^+(\mathbb{S}^3)$. By the Geometrization Theorem, every finite group acting smoothly or locally linearly on $\mathbb{S}^3$ is geometric, that is, it is conjugate to a finite subgroup of SO(4). Every finite subgroup of SO(4) is a subgroup of some central product $P_1 \times_{C_2} P_2$, where each P_i is one of the binary polyhedral groups: the cyclic groups C_{2n}, the binary dihedral groups D_n^*, and the binary tetrahedral, binary octahedral, and binary icosahedral groups A_4^*, Σ_4^*, and A_5^*. These facts are exposited in a survey article by Zimmermann [35], and a list of subgroups of the binary groups is in the Appendix of [20]. (Note that $D_n^* \cong DC_n$.)

We must therefore produce for each $g \geq 42$ a finite subgroup G_g of $\mathrm{Mod}(S_g)$ that is not a subgroup of any $P_1 \times_{C_2} P_2$. (Note that the Accola–Maclachlan subgroup $C_2 \times C_{2g+2}$ no longer suffices.) Producing such subgroups proves the theorem, for then either a nontrivial nonhyperelliptic periodic element is in the kernel of ϕ, so that ϕ is trivial; otherwise ϕ factors through the symplectic representation, and since the group $(C_3)^g$ for $g \geq 3$ is not a subgroup of any $P_1 \times_{C_2} P_2$, we conclude by Lemma 10 that ϕ is trivial. Indeed, the finite abelian subgroups of SO(4) can be written as the direct product of at most two finite cyclic groups.

For $g \geq 42$, we make take G_g to be the split metacyclic group $D_{3,7}$. The split metacyclic group of order pq with p and q prime is the group with presentation

$$D_{p,q} = \langle a, b \mid a^q = b^p = 1, bab^{-1} = a^r \rangle \tag{3}$$

where r is a solution (other than 1) to the congruence $r^p \equiv 1 \pmod{q}$. Such a solution exists exactly when q is 1 mod p, and different solutions yield isomorphic groups. Note that $D_{2,q}$ is a dihedral group. Weaver computed the stable upper genus of all split metacyclic groups, and in particular he showed that $\mathrm{Mod}(S_g)$ contains $D_{3,7}$ for all $g \geq 42$ [33, Corollary 4.8]. On the other hand, $D_{3,7}$ is a not a subgroup of $\mathrm{Diff}^+(\mathbb{S}^3)$, since all finite subgroups of SO(4) of odd order are abelian. The classification of finite subgroups of SO(4) goes back to work of Seifert–Threlfall [31, 32]; see the paper of Mecchia–Seppia for a contemporary treatment [25].

Finally, let $\phi : \mathrm{Mod}(S_g) \to \mathrm{Diff}^+(\mathbb{S}^4)$ be a homomorphism. A theorem of Mecchia–Zimmermann [26] states that every finite group that acts smoothly on $\mathbb{S}^4$ and preserves orientation lies on the following list:

(1) orientation-preserving finite subgroups of $\mathrm{O}(3) \times \mathrm{O}(2)$ and of $\mathrm{O}(4) \times \mathrm{O}(1)$,
(2) orientation-preserving subgroups of the Weyl group $W = (C_2)^5 \rtimes \Sigma_5$,
(3) A_5, Σ_5, A_6, Σ_6, and
(4) finite subgroups of $\mathrm{SO}(4)$ and 2-fold extensions thereof.

Since the group $(C_3)^g$ is not on this list for $g \geq 42$, to show that ϕ is trivial it suffices to show that $D_{3,7}$ is also not on this list. We first claim that $D_{3,7}$ does not lie in (1) on the list. Since $D_{3,7}$ is a semidirect product with prime-order non-commuting generators, we conclude that any isomorphism to a direct product restricts to an isomorphism on a factor. But $D_{3,7}$ is not a subgroup of any of $\mathrm{SO}(4)$, $\mathrm{SO}(3)$, $\mathrm{SO}(2)$ or $\mathrm{SO}(1)$, and since G has odd order it is also not a subgroup of the corresponding degree 2 extensions $\mathrm{O}(4)$, $\mathrm{O}(3)$, $\mathrm{O}(2)$ or $\mathrm{O}(1)$.

The group G does not lie within (2) on the list above, since it contains an element of order 7. It is not isomorphic to any of the four permutation groups within (3) on the list. Finally, we argued above that G is not a subgroup of $\mathrm{SO}(4)$, and G is not a 2-fold extension of any group, since it has odd order; it therefore does not lie within (4) on the list above. $\square$

Weaver additionally shows how to compute the genus spectra of all split metacyclic groups [33, Theorem 4.7]. Using his formulas, it is straightforward to compute that $\mathrm{Mod}(S_g)$ contains at least one of $D_{3,7}$, $D_{3,13}$, $D_{3,19}$, $D_{3,31}$, or $D_{3,37}$ as a subgroup for all $g \geq 3$ except for the following ten values:

$$\{4, 5, 7, 11, 13, 16, 23, 25, 34, 41\} \tag{4}$$

It follows by similar arguments to those given in the proof of Theorem 2 that these split metacyclic groups are also not subgroups of either $\mathrm{Homeo}^+(\mathbb{S}^3)$ or $\mathrm{Diff}^+(\mathbb{S}^4)$. Therefore all homomorphisms from $\mathrm{Mod}(S_g)$ to either $\mathrm{Homeo}^+(\mathbb{S}^3)$ or $\mathrm{Diff}^+(\mathbb{S}^4)$ are trivial for all $g \geq 3$ except perhaps for these ten values of g. It is possible that further analysis of finite subgroups would allow for a sharpening of the lower bound of 42 in the statement of Theorem 2.

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