

Airport Scheduling Optimization Formulation

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1. Introduction

The airport scheduling formulation uses a modified job shop scheduling scheme where the planes are the objects being processed and the processors are the resources the planes use during their stay at the airport. The airport is modeled to have 4 runways, 2 taxi lanes, and 16 gates. The formulation deviates from a typical job shop problem since planes have to use the same taxi lanes and runways upon entry and exit.

2. Sets, Variables, Parameters

2.1 Sets

- $a = \{\text{Airplane 1, Airplane 2, ...}\} = \text{Airplane}$
- $r = \{1, 2, ... \} = \text{Runway}$
- $l = \{1, 2, ... \} = \text{Taxi Lane}$
- $g = \{A, B, ... \} = \text{Gate}$
- $t = \{\text{landing, taxi in, unloading, taxi out, takeoff}\} = \text{Task}$
- $s = \{\text{small, medium, large}\}$

2.2 Variables/Parameters

- $PT_{a,t}$ estimate process time for airplane a on task t
- $RT_{a,t}$ the time at which a plane a is released from task t
- $DD_{a,t}$ the due date for airplane a to complete task t
- $ST_{a,t}$ the time at which we start task t for airplane a
 - Arrival time of airplane a is $x_{a,1}$
 - $t = 1 = \text{First task} = \text{Landing}$
- $z_{a,t}$ the tardiness of airplane a on task t

2.3 Constants

- $Arrival_a = \text{Arrival time of airplane A}$
- $PlaneSize_{a,s} = \text{Plane size s of airplane a}$
- $AllowG_{s,g} = \text{if plane of size s can use gate g (1) or not (0)}$
- $AllowR_{s,r} = \text{if plane of size s can use runway r (1) or not (0)}$
- $AllowL_{s,l} = \text{if plane of size s can use taxi lane l (1) or not (0)}$

2.4 Decision Variables

2.4.1 Continuous Decision Variables

- $k_{a,t}$ the time at which plane a starts task t

2.4.2 Binary Decision Variables

2.4.2.1 Order

- $yr_{a,a',r}^{in}$ if plane a uses runway r (1) or not (0) during landing before plane a'
- $yr_{a,a',r}^{out}$ if plane a uses runway r (1) or not (0) during takeoff before plane a'
- $yl_{a,a',l}^{in}$ if plane a uses taxi lane l (1) or not (0) after landing before plane a'
- $yl_{a,a',l}^{out}$ if plane a uses taxi lane l (1) or not (0) before takeoff before plane a'
- $yg_{a,a',g}$ if plane a uses gate g (1) or not (0) before plane a'
- $yr_{a,a',r}^{io}$ if plane a 's landing precedes airplane a' 's takeoff on runway r (1) or not (0)
- $yl_{a,a',r}^{io}$ if plane a 's taxi in precedes airplane a' 's taxi out on taxi lane l (1) or not (0)

2.4.2.2 Utilization

- $xr_{a,r}^{in}$ if plane a uses runway r (1) or not (0) during landing
- $xr_{a,r}^{out}$ if plane a uses runway r (1) or not (0) during takeoff
- $xl_{a,l}^{in}$ if plane a uses taxi lane l (1) or not (0) after landing
- $xl_{a,l}^{out}$ if plane a uses runway r (1) or not (0) before takeoff
- $xg_{a,g}$ if plane a uses gate g (1) or not (0)

3. Objective Function

Minimize the maximum tardiness

$$\min_{x,y,z} \varphi = \sum_{t=1}^T \max\{0, x_{a,takeoff} + PT_{a,takeoff} - DD_{a,takeoff}\}$$

4. Constraints

4.1 Arrival

The airplane must land after the arrival time:

$$k_{a,landing} \geq Arrival_a$$

The arrival time is the actual time of arrival. Since we did not include random variability, which would alter the objective function to minimize the worst-case scenario and add complexity, we must know when the plane arrives with certainty. If a plane cannot land right away it must circle until a runway has opened.

4.2 Circling Limit

Each plane can only circle for a fixed number of minutes:

$$z_{a,landing} \leq TimeLimit$$

4.3 Size Restrictions

4.3.1 Runway

Only allow a plane on runway r if it is of an acceptable size:

$$xr_{a,r}^{in} \leq \sum_{s=1}^S PlaneSize_{a,s} \cdot AllowR_{s,r}, \forall a, r$$
$$xr_{a,r}^{out} \leq \sum_{s=1}^S PlaneSize_{a,s} \cdot AllowR_{s,r}, \forall a, r$$

Since $PlaneSize_{a,s}$ and $AllowR_{s,r}$ are both binary variables, the product of them results in a binary variable too. Each plane is assigned a single size only one of the summation terms is possibly non-zero. The $AllowR_{s,r}$ variable indicates whether a plane of size s can use runway r or not. This constraint specifies that for each runway, a plane may only be assigned to it if it has the correct size. There are two constraints – one for the runway assignment upon landing and another for the runway assignment for takeoff.

Consider a problem where there are three sizes of airplanes available: small, medium, and large. If you have a large plane (A1) and three runways where only R1 can accommodate medium and large planes, R2 can accommodate small and medium planes, and R3 can accommodate all planes:

$$xr_{A1,R1}^{in} \leq 0 \cdot 0 + 0 \cdot 1 + 1 \cdot 1 = 1$$

$$xr_{A1,R2}^{in} \leq 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 0$$

$$xr_{A1,R3}^{in} \leq 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 1 = 1$$

Here you can see that this constraint specifies that only R1 and R3 can accommodate the large plane. This is not enough to constrain the runway selection as a single airplane may be assigned to multiple runways. To fix this, we add resource constraints. The taxi and gate constraints follow the same logic, and an analogous explanation is omitted.

4.3.2 Taxi

$$xl_{a,l}^{in} \leq \sum_{s=1}^S PlaneSize_{a,s} \cdot AllowL_{s,l}, \forall a, l$$

$$xl_{a,l}^{out} \leq \sum_{s=1}^S PlaneSize_{a,s} \cdot AllowL_{s,l}, \forall a, r$$

4.3.3 Gate

Here there is only one constraint since an airplane is only assigned to one gate during its time in the airport.

$$xg_{a,g}^{in} \leq \sum_{s=1}^S PlaneSize_{a,s} \cdot AllowG_{s,g}, \forall a, g$$

4.4 Resource Limits

The resource limits coupled with the size restriction completely constrain the runway, taxi lane, and gate selection for both landing and takeoff of an airplane. This does not ensure that multiple airplanes will not be assigned to the same task simultaneously. This will be accounted for in the disjunctive constraints.

4.4.1 Runway

Each airplane must be assigned to exactly one runway for landing and takeoff:

$$\sum_{r=1}^R xr_{a,r}^{in} = 1, \forall a$$

$$\sum_{r=1}^R xr_{a,r}^{out} = 1, \forall a$$

4.4.2 Taxi

Each airplane must be assigned to exactly one taxi lane after landing and prior to takeoff:

$$\sum_{l=1}^L xl_{a,l}^{in} = 1, \forall a$$

$$\sum_{l=1}^L xl_{a,l}^{out} = 1, \forall a$$

4.4.3 Gate

Each airplane can only use one gate:

$$\sum_{g=1}^G xg_{a,g} = 1, \forall a$$

4.5 Disjunctive

4.5.1 Runway

The disjunctive constraints are implemented to ensure that two airplanes are not scheduled to the same resource at a time. The implementation deviates slightly from the traditional method as a variable capable of tracking the runway, taxi lane, and gate assignment was necessary.

$$k_{a,landing} + PT_{a,landing} \leq k_{a',landing} + M(1 - yr_{a,a',r}^{in}) + M(1 - xr_{a,r}^{in}) + M(1 - xr_{a',r}^{in})$$

$$k_{a',landing} + PT_{a',landing} \leq k_{a,landing} + M(yr_{a,a',r}^{in}) + M(1 - xr_{a,r}^{in}) + M(1 - xr_{a',r}^{in})$$

Both equations in the disjunctive constraint pair include the terms $M(1 - xr_{a,r}^{in})$ and $M(1 - xr_{a',r}^{in})$. These were added so that the precedence constraint would be “turned off” if either airplane a or a' was not on the runway of interest. This is necessary as it would be incorrect to assign a precedence if two airplanes are completing a task independently. When either $xr_{a,r}^{in}$ or $xr_{a',r}^{in}$ is 0, then the right-hand size of the equation becomes essentially infinity meaning that planes a and a' must land at some point. Analogous constraints are added for planes that use a runway for takeoff:

$$k_{a,takeoff} + PT_{a,takeoff} \leq k_{a',takeoff} + M(1 - yr_{a,a',r}^{out}) + M(1 - xr_{a,r}^{out}) + M(1 - xr_{a',r}^{out})$$

$$k_{a',takeoff} + PT_{a',takeoff} \leq k_{a,takeoff} + M(yr_{a,a',r}^{out}) + M(1 - xr_{a,r}^{out}) + M(1 - xr_{a',r}^{out})$$

To ensure that landing and takeoff do not occur at the same time, another set of disjunctive constraints are added so that two airplanes do not compete for the same resource at the same time.

$$k_{a,landing} + PT_{a,landing} \leq k_{a',takeoff} + M(1 - yr_{a,a',r}^{io}) + M(1 - xr_{a,r}^{in}) + M(1 - xr_{a',r}^{out})$$

$$k_{a',takeoff} + PT_{a',takeoff} \leq k_{a,landing} + M(yr_{a,a',r}^{io}) + M(1 - xr_{a,r}^{in}) + M(1 - xr_{a',r}^{out})$$

Here if plane a lands before a' takes off on runway r then $yr_{a,a',r}^{io} = 1$. This entails that:

$$k_{a,landing} + PT_{a,landing} \leq k_{a',takeoff}$$

- iff airplane a uses runway r for landing and airplane a' uses runway r for takeoff

$$k_{a',takeoff} + PT_{a',takeoff} \leq \infty$$

Meaning that a is forced to land before a' takes off and that a' must takeoff at some point. If $yr_{a,a',r}^{io} = 0$ then plane a lands after airplane a' takes off. This results in:

$$k_{a,landing} + PT_{a,landing} \leq \infty$$

$$k_{a',takeoff} + PT_{a',takeoff} \leq k_{a,landing}$$

- iff airplane a uses runway r for landing and airplane a' uses runway r for takeoff

This does not need to be formulated with planes a and a' swapped since this is expressed for each pair of planes a and a' where $a \neq a'$.

- $yr_{a,a',r}^{io}$ if plane a 's landing precedes airplane a' 's takeoff on runway r (1) or not (0)

4.5.2 Taxi

$$k_{a,taxi\ in} + PT_{a,taxi\ in} \leq k_{a',taxi\ in} + M(1 - yl_{a,a',l}^{in}) + M(1 - xl_{a,l}^{in}) + M(1 - xl_{a',l}^{in})$$

$$k_{a',taxi\ in} + PT_{a',taxi\ in} \leq k_{a,taxi\ in} + M(yl_{a,a',l}^{in}) + M(1 - xl_{a,l}^{in}) + M(1 - xl_{a',l}^{in})$$

$$k_{a,taxi\ out} + PT_{a,taxi\ out} \leq k_{a',taxi\ out} + M(1 - yl_{a,a',l}^{out}) + M(1 - xl_{a,l}^{out}) + M(1 - xl_{a',l}^{out})$$

$$k_{a',taxi\ out} + PT_{a',taxi\ out} \leq k_{a,taxi\ out} + M(yl_{a,a',l}^{out}) + M(1 - xl_{a,l}^{out}) + M(1 - xl_{a',l}^{out})$$

$$k_{a,taxi\ in} + PT_{a,taxi\ in} \leq k_{a',taxi\ out} + M(1 - yl_{a,a',l}^{io}) + M(1 - xl_{a,l}^{in}) + M(1 - xl_{a',l}^{out})$$

$$k_{a',taxi\ out} + PT_{a',taxi\ out} \leq k_{a,taxi\ in} + M(yl_{a,a',l}^{io}) + M(1 - xl_{a,l}^{in}) + M(1 - xl_{a',l}^{out})$$

4.5.3 Gate

$$k_{a,unload} + PT_{a,unload} \leq k_{a',unload} + M(1 - yg_{a,a',g}) + M(1 - xg_{a,g}) + M(1 - xg_{a',g})$$

$$k_{a',unload} + PT_{a',unload} \leq k_{a,unload} + M(yg_{a,a',g}) + M(1 - xg_{a,g}) + M(1 - xg_{a',g})$$

4.6 Linking

The linking constraint is necessary to ensure that the two sets of decision variables agree with one another. Since there are two sets of binary variables implemented which, both depend on each other, an expression must be added to describe their relationship. The assignment binary variables were added to keep track of the assignment of an airplane to a resource. The precedence binary variables were added to account for the order in which airplanes complete a task on a resource. To enforce agreement, the linking constraints ensure that these assignment and precedence binary variables refer to the same resource.

4.6.1 Runway

$$yr_{a,a',r}^{in} \leq xr_{a,r}^{in}, \forall a, a', r$$

$$yr_{a,a',r}^{out} \leq xr_{a,r}^{out}, \forall a, a', r$$

$$yr_{a,a',r}^{io} \leq xr_{a,r}^{io}, \forall a, a', r$$

This constraint ensures that $yr_{a,a',r}^{in}$ can only be 1 if airplanes a, a' are chosen to be on the same runway. If the airplanes a and a' are chosen to be on different runways then the precedence constraint becomes meaningless since they are independent on one another. If they are chosen to land on the same runway, the GAMS has the freedom to choose either airplane a or a' to go first.

4.6.2 Taxi

$$yl_{a,a',l}^{in} \leq xl_{a,l}^{in}, \forall a, a', l$$

$$yl_{a,a',l}^{out} \leq xl_{a,l}^{out}, \forall a, a', l$$

4.6.3 Gate

$$yg_{a,a',g}^{in} \leq xg_{a,g}^{in}, \forall a, a', g$$

$$yg_{a,a',g}^{out} \leq xg_{a,g}^{out}, \forall a, a', g$$

$$yg_{a,a',g}^{io} \leq xg_{a,g}^{io}, \forall a, a', g$$

4.7 Order

$$x_{a,t+1} \geq x_{a,t} + PT_{a,t} \forall a$$

For example, the taxiing can only occur when the landing is complete. Taxiing can start $PT_{a,t}$ minutes after the landing has begun.

5. Test Cases

Test Case 1: No Bottleneck

Two airplanes land at the same time.

Airplane	Size	Runway (Landing)	Taxi Lane In	Gate	Taxi Lane Out	Runway (Takeoff)
A1	Large	R2	L1	G7	L1	R1
A2	Small	R1	L2	G1	L2	R1

Objective function value of 0

Test Case 2: Bottleneck in Taxi Lanes

Here all airplanes land at the same time:

Airplane	Size	Runway (Landing)	Taxi Lane In	Gate	Taxi Lane Out	Runway (Takeoff)
A1	Small	R4	L2	G11	L2	R1
A2	Small	R1	L2	G12	L2	R4
A3	Medium	R3	L1	G13	L1	R1
A4	Large	R2	L1	G1	L1	R2

Start Time (k)					
Airplane	Landing	Taxi In	Unload	Taxi Out	Takeoff
A1	0	6	7	37	38
A2	0	5	6	36	37
A3	0	10	12	72	74
A4	0	15	18	108	111

Here all airplanes begin to land at the same time. The two small airplanes land in 5 minutes and only have taxi lane L2 available to them so A1 waits its turn. Both small aircrafts are taxied by the time A3 and A4 land so there is no tardiness associated with landing or taxiing. A1 is now a minute behind each step (taxi in, unload, taxi out, takeoff) so the objective function evaluates to 4. Since the taxiing of A2 only takes a minute, A1 does not have to wait for a taxi lane to take off since it just freed up.

Test Case 3: More Planes Than Runways

Airplane	Size	Runway (Landing)	Taxi Lane In	Gate	Taxi Lane Out	Runway (Takeoff)
A1	small	R1	L2	G6	L2	R4
A2	small	R4	L2	G12	L2	R1
A3	small	R4	L2	G4	L2	R4

A4	medium	R1	L2	G1	L2	R3
A5	medium	R3	L1	G16	L2	R4
A6	large	R2	L1	G15	L1	R3
A7	large	R4	L1	G14	L2	R2
A8	large	R3	L2	G7	L1	R4

Start Time (k)					
Airplane	Landing	Taxi In	Unload	Taxi Out	Takeoff
A1	0	5	6	36	37
A2	0	6	7	37	38
A3	5	10	11	41	42
A4	5	15	17	77	79
A5	0	10	12	72	74
A6	0	15	18	108	111
A7	10	25	28	118	121
A8	10	25	28	118	121

Only two runways accommodate small airplanes so A3 is forced to wait even though it would be faster to land all the small airplanes right away. This is because the small airplanes are processed faster, and the objective function does not weight the tardiness of the larger aircrafts over the smaller aircrafts. Processing a small aircraft first would minimize the objective function since the large aircraft would be slightly late. Since the large aircraft must be landed first it forces the small airplane to wait longer than the previous case causing a larger objective function value. By this logic, GAMS chooses to land one of the medium planes (A5) on R3 after the small planes land. One of the large planes (A6) is forced to land instead of the medium one (A4) since R4 only accommodates large planes immediately.

When the two small planes (A1, A2) finish their landing procedure, they are taxied sequentially since only L2 allows small planes. A2 is in a holding area off the runway so other planes can use it while it is waiting to be taxied.

Since A5 (medium) was landed at time 0, and A3 (small) had to wait to use a runway, they end up being taxied simultaneously on different taxi lanes at time = 10. The same scenario occurs with planes A6 and A4 which are taxied at time = 15. The two large planes are taxied last at time = 25.

Each plane is assigned to its own gate, so they operate independently of one another. Plane A2 was the only plane which had to wait for a taxi lane in the beginning, so it is now 1 minute behind A1. This means it does not need to wait for a taxi lane on the way out. Planes

A1, A5, and A6 are on time since they landed first and had no delays getting processed in the airport. A2 landed first but had to wait for a taxi lane causing a tardiness penalty.

Test Case 4: Planes Arriving as Planes are Scheduled for Takeoff

Airplane	Size	Runway (Landing)	Taxi Lane In	Gate	Taxi Lane Out	Runway (Takeoff)
A1	large	R2	L1	G1	L2	R3
A2	large	R3	L1	G2	L2	R4
A3	large	R3	L2	G3	L1	R2
A4	large	R4	L2	G7	L1	R2
A5	large	R3	L2	G8	L1	R4
A6	large	R4	L1	G9	L2	R3
A7	large	R3	L2	G10	L2	R2
A8	large	R4	L2	G13	L1	R3

Start Time (k)						End Time
Airplane	Landing	Taxi In	Unload	Taxi Out	Takeoff	Takeoff
A1	0	15	18	108	111	126
A2	15	30	33	123	126	141
A3	0	15	18	108	111	126
A4	0	18	21	111	126	141
A5	141	156	159	249	252	267
A6	141	156	159	249	252	267
A7	126	141	144	234	237	252
A8	111	126	129	219	222	237

The first three planes which come in are immediately landed. Since only R2-R4 accommodate large airplanes, one of them (A2) must wait for a runway. A1, A3, A4 finish landing at the same time but only two taxi lanes are available so one of them (A4) has to wait for a taxi lane. When A1 and A3 taxi then A2 lands immediately. When A1 and A2 are done taxiing then A4 immediately begins to taxi, then A2 shortly after.

A1 and A3 begin takeoff at $t=111$ minutes on runways R3 and R2 respectively, which is when planes A5-A8 are scheduled to arrive. This leaves one runway (R4) open for one of the incoming airplanes to land since A2 is running late. As these planes are landing/taking off, airplanes A2 and A4 are taxied. After 15 minutes, all three runways open since A1 and A2 have taken off and A8 has landed. GAMs chooses A2 and A4 to takeoff instead of letting the remaining three airplanes land. Since only 2 airplanes must take off, one of the incoming planes (A7) can land simultaneously. A5 and A6 are forced to wait another 15 minutes before they are able to land – at exactly the circling time limit.

Test Case 5: Only Small Planes

Airplane	Size	Runway (Landing)	Taxi Lane In	Gate	Taxi Lane Out	Runway (Takeoff)
A1	small	R1	L1	G1	L2	R1
A2	small	R1	L1	G11	L2	R4
A3	small	R4	L2	G4	L2	R4
A4	small	R4	L2	G3	L2	R4
A5	small	R4	L2	G6	L2	R4
A6	small	R1	L1	G5	L2	R1
A7	small	R1	L1	G2	L2	R1
A8	small	R4	L2	G12	L2	R1
A9	small	R1	L1	G9	L2	R4
A10	small	R1	L1	G10	L2	R1

Start Time (k)						End Time
Airplane	Landing	Taxi In	Unload	Taxi Out	Takeoff	Takeoff
A1	6	12	13	43	44	49
A2	1	6	7	37	38	43
A3	21	27	38	68	69	74
A4	1	7	8	38	39	44
A5	21	26	37	67	68	73
A6	11	16	17	47	48	53
A7	11	17	18	48	49	54
A8	16	22	23	53	54	59
A9	6	11	12	42	43	48
A10	16	21	22	52	53	58

In this test case, all ten airplanes are small and subject to restricted access to infrastructure. Small aircraft are allowed to use runways R1, R2, and R4 (but not R3), taxi lanes L1 and L2, and a subset of gates including G1 through G6, G9 through G12. The planes arrive within a narrow window, and due to limited taxi lanes and shared gate availability, delays occur throughout the process. Although small aircraft are processed quickly, resource conflicts lead to waiting times. For instance, A3 and A5 land later and are forced to wait for gates and taxi lanes to become available, resulting in takeoffs around 30 minutes after landing. Taxi lane L2 is highly utilized during the taxi-out phase, further contributing to scheduling conflicts. The model balances these delays to minimize average system-wide tardiness, but the overlap in tasks and competition for limited infrastructure causes a noticeable accumulation of lateness. The final objective function value is 125, reflecting the impact of tight scheduling and constrained infrastructure on operational performance.

Test Case 6: Same Number of Planes as Gates

Airplane	Size	Runway (Landing)	Taxi Lane In	Gate	Taxi Lane Out	Runway (Takeoff)
A1	medium	R4	L1	G2	L2	R1
A2	large	R1	L2	G6	L2	R2
A3	small	R1	L2	G5	L2	R2
A4	large	R3	L1	G1	L2	R3
A5	small	R2	L1	G4	L2	R1
A6	medium	R2	L1	G3	L2	R4
A7	small	R4	L1	G8	L2	R2
A8	small	R2	L1	G13	L2	R2
A9	small	R3	L1	G9	L2	R3
A10	small	R2	L2	G10	L2	R2
A11	small	R2	L2	G11	L2	R1
A12	small	R1	L2	G12	L2	R1
A13	Large	R1	L2	G14	L2	R3
A14	Medium	R2	L2	G15	L2	R4
A15	Medium	R1	L2	G15	L2	R2
A16	Large	R3	L1	G7	L2	R4

Start Time (k)						End Time
Airplane	Landing	Taxi In	Unload	Taxi Out	Takeoff	Takeoff
A1	1	16	19	109	112	127
A2	16	26	28	88	90	105
A3	1	6	7	37	38	53
A4	36	51	54	144	147	162
A5	6	19	21	81	83	98
A6	16	33	36	126	129	144
A7	1	28	30	90	92	107
A8	11	26	29	119	122	137
A9	31	46	49	139	142	157
A10	41	56	59	149	152	167
A11	16	21	22	52	53	68
A12	26	41	44	134	137	152
A13	26	36	38	98	100	115
A14	46	61	64	154	157	172
A15	21	31	33	93	95	110
A16	1	16	19	109	112	127

In this test case, the number of airplanes matches the number of available gates, allowing for a one-to-one assignment. However, the planes vary in size, including small, medium, and large aircraft, each with specific access restrictions for gates, taxi lanes, and runways.

All planes can use both taxi lanes, and each size group has clearly defined runway compatibility, most notably, large planes cannot use R1. Despite the balanced gate availability, delays emerge due to bottlenecks at taxi lanes and overlapping task times. The model prioritizes sequencing to reduce conflict, but aircraft such as A4, A6, A8, and A10 experience extended turnaround times due to limited runway and taxi lane availability at their specific processing windows. The final objective function value of 302 reflects the cumulative impact of these overlaps and infrastructure sharing. This scenario demonstrates that while gate constraints were fully satisfied, runway and taxi lane competition still played a significant role in system-wide tardiness.