MATH544
Dr. Miller
HW 2

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Problem 1 (1.3.6)**.** *Let $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3} \in \mathbb{R}^3$, and suppose that the linear system*

$$x\vec{v_1} + y\vec{v_2} + z\vec{v_3} = \vec{0}$$

has infinitely many solutions. show that $\vec{v_1}$, $\vec{v_2}$, $\vec{v_3} \in \mathbb{R}^3$ lie in a plane containing the origin in $\mathbb{R}^3$.

Proof.

$$\vec{v_1} = \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix}; \quad \vec{v_2} = \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix}; \quad \vec{v_3} = \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Then

$$x \cdot x_1 + y \cdot x_2 + z \cdot x_3 = 0$$
$$x \cdot y_1 + y \cdot y_2 + z \cdot y_3 = 0$$
$$x \cdot z_1 + y \cdot z_2 + z \cdot z_3 = 0$$

The trivial solution is $x = y = z = 0$, but that only gives us one solution. However we can show that one vector is some linear combination of the other 2.

$$x \cdot x_1 + y \cdot x_2 = -z \cdot x_3$$
$$x \cdot y_1 + y \cdot y_2 = -z \cdot y_3$$
$$x \cdot z_1 + y \cdot z_2 = -z \cdot z_3$$

If $x \neq 0 \wedge y \neq 0 \wedge z \neq 0$, then this can be represented in vector form again.

$$x \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + y \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = -z \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Thus one of the vectors is collinear to the addition of the other two. And so $< \vec{v_1}, \vec{v_2}, \vec{v_3} >=< \vec{v_1}, \vec{v_2} >$. Because $< \vec{v_1}, \vec{v_2} >$ includes the $\vec{0}$(from the given), and all three vectors, because it is the span of 2 and the third is a linear combination of the other two, covered by the definition of span. If this forms a plane, then it will include all 4 vectors. If they both are collinear it will form a line, and thus there are an infinite number of planes that will contain all 4 points. And if it forms a point, then that point must be , and again there are an infinite number of planes from that point

If $x = 0$,

$$0 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + y \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = -z \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix} \implies y \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = -z \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Thus we get two collinear vectors, and the same argument, this can be extended to $y = 0$ and $z = 0$.
Suppose we can have 2 solutions that are 0. Then we get

$$0 \begin{bmatrix} x_1 \\ y_1 \\ z_1 \end{bmatrix} + 0 \begin{bmatrix} x_2 \\ y_2 \\ z_2 \end{bmatrix} = -z \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

Thus

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = -z \begin{bmatrix} x_3 \\ y_3 \\ z_3 \end{bmatrix}$$

And so we have the null vector as one of the vectors in the linear combination, and we can just draw a plane through the 3 points(or line if they are collinear, and a point if they are all the null vector.
If 3 solutions are 0, this is trivially true. $\square$