MATH544
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HW 4

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Problem 1 (2.2.2)**.** *Give an explicit isomorphism between* $\mathbb{R}^2$ *and the set of solutions of the linear system over(*$\mathbb{R}$*)*

$$\begin{aligned} w - x + 3z &= 0 \\ w - x + y + 5z &= 0 \\ 2w - 2x - y + 4z &= 0 \end{aligned}$$

Proof. The set of solutions(after throwing this into an augmented matrix and using Gaussian row elimination) is,

$$\left\{ \begin{bmatrix} s - 3t \\ -2s \\ s \\ t \end{bmatrix} \;\middle|\; s, t \in \mathbb{R} \right\}$$

We want a transformation that hits all solutions, and is bijective. Let the transformation $= A$.

$$A \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - 3y \\ -2y \\ x \\ y \end{bmatrix}$$

The following is an isomorphism from $\mathbb{R}^2$ to the solutions.

$$\begin{bmatrix} 1 & -3 \\ 0 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

□

Problem 2 (2.2.6)**.** *Find the matrix of the orthogonal projection onto the line $y = x$ in $\mathbb{R}^2$.*

Proof. The orthogonal projection transformation should look like,

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} y \\ x \end{bmatrix}$$

We can solve this as,

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$$

When expanded, and solved we get, $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. $\square$

Problem 3 (2.2.8)**.** *Show that if 0 is an eigenvalue of $T \in L(V)$, then T is not injective and therefore not invertible.*

Proof. If 0 is an eigenvalue of T, then for some $\vec{v} \in V$, and $\vec{v} \neq \vec{0}$(definition of eigenvector), $T(\vec{v}) = 0\vec{v}$. This is equal to $\vec{0}$. We know that because $T \in \mathcal{L}(V)$, that $T(\vec{0}) = \vec{0}$, thus it is not injective. In essence, $\vec{v} \neq \vec{0}$, and $T(\vec{v}) = T(\vec{0})$. $\square$

Problem 4 (G)**.** *Let $V = \mathbb{F}^\infty$ be the infinite sequence space. Define the shift left operator on V by, $L(a_0, a_1, a_2, \dots) = (a_1, a_2, a_3, \dots)$ and the shift right operator as $R(a_0, a_1, a_2, \dots) = (0, a_0, a_1, \dots)$.*

Problem 5 (G.ii)**.** *Is L injective? Surjective? If yes prove it, if no, why not?*

Proof. L is not injective, because say we have $A := \begin{bmatrix} 1 \\ 2 \\ 2 \\ \vdots \\ 2 \end{bmatrix}$ and $B := \begin{bmatrix} 2 \\ 2 \\ 2 \\ \vdots \\ 2 \end{bmatrix}$. Both of these vectors are distinct, but $L(A) = L(B)$.

L is surjective, for all $\vec{v} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} \in \mathbb{F}^\infty$, so $T\left(\begin{bmatrix} x \\ a_0 \\ a_1 \\ \vdots \end{bmatrix}\right) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \vec{v}$ for all $x \in \mathbb{F}$. $\square$

Problem 6 (G.iii)**.** *Is R injective? Surjective? If yes prove it, if no, why not?*

Proof. R is injective, suppose $T\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}\right) = T\left(\begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix}\right)$. Then $\begin{bmatrix} 0 \\ a_0 \\ a_1 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ b_0 \\ b_1 \\ \vdots \end{bmatrix}$. And so, $a_0 = b_0$, $a_1 = b_1$, $a_2 = b_2$, $\dots$, $a_n = b_n$, thus $\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix} = \begin{bmatrix} b_0 \\ b_1 \\ b_2 \\ \vdots \end{bmatrix}$

R is not surjective, because $\begin{bmatrix} 1 \\ \vdots \end{bmatrix} \in V^\infty$ but there does not exist $\vec{v}$, such that $T(\vec{v}) = \begin{bmatrix} 1 \\ \vdots \end{bmatrix}$, because all elements in the image have the first row as zero. □

Problem 7 (G.iv)**.** *Are the compositions, LR and RL the same? If yes prove it; if no, why not?*

Proof. $LR = L \circ R$, so a transformation of such,

$$L\left(R\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}\right)\right) = L\left(\begin{bmatrix} 0 \\ a_0 \\ a_1 \\ \vdots \end{bmatrix}\right) = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

And RL $= R \circ L$, so a transformation of such,

$$R\left(L\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}\right)\right) = R\left(\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \end{bmatrix}\right) = \begin{bmatrix} 0 \\ a_1 \\ a_2 \\ \vdots \end{bmatrix}$$

And so these transformations are not the same. □

Problem 8 (I)**.** *Let* $V = C^\infty(R) = \{f : R \to R \mid f$ *and all of its derivatives exist and are continuous*$\}$. *This is a vector space over* R

Problem 9 (I.i)**.** *Define* $D : V \to V by D(f) = f'$. *Convince yourself that D is linear. Which functions are in ker(D)?*

Proof. Let $z(x) = 0$. This is $\vec{0}$ in this space.
By definition, $ker(D) = \{f \mid f' = z\}$. By the fundamental theorem of calculus, $ker(D) = \{f \mid \forall x \in \mathbb{R}; f(x) = 0x + C\}$. □

Problem 10 (I.ii)**.** *Let* $g(x) = e^{-3x}$
By definition for an eigenvector, the eigenvectors for which the eigenvalue, -3, *would solve the equation.*

$$D(f) = f' = -3f$$

The eigenvector g, works in this equation because,

$$D(g) = g' = -3 \cdot e^{-3x} = -3g$$

There exists a function f *for every real number,* λ, *such that* λ *is the eigenvalue because by differentiation in calculus, let* $f = e^{\lambda x}$ *then* $\frac{df}{dx} = D(f) = \lambda f$

Problem 11 (I.iii)**.** *The composition* $D^2 = D \circ D : V \to V$ *is given by* $D^2(f) = f''$. *Show that* -4 *is an eigenvalue for* D^2. *Is every negative real number an eigenvalue for* D^2*?*

Proof. Every negative real number is an eigenvalue, because for all $\lambda \in \mathbb{R}$ and $\lambda < 0$, let $\beta = -\lambda$, we can construct a function, f that $D^2(f) = \lambda f = -\beta f$. Let $f = e^{\sqrt{\beta} i}$, then $D(D(f)) = D(\sqrt{\beta} i f) = i^2 \beta f = -\beta f = \lambda f$, by differentiation rules of calculus. We can now find a function that has an eigenvalue of -4, let $f = e^{\sqrt{4} i}$, then $D^2(f) = -4f$. $\square$