MATH544 Notes

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Contents

Chapter 1

Pre-Class

1.1 Number Fields

Definition 1.1.1. A field is any set $\mathbb{K}$ which follow the Field Axioms:

1. $\alpha + \beta = \beta + \alpha$ (addition is commutative).
2. $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$ (addition is associative).
3. There exists an element 0 such that $\alpha + 0 = \alpha$.
4. For every $\alpha \in \mathbb{K}$, there exists β such that $\alpha + \beta = 0$.
5. $\alpha\beta = \beta\alpha$ (multiplication is commutative).
6. $(\alpha\beta)\gamma = \alpha(\beta\gamma)$ (multiplication is associative).
7. There exists an element 1 such that $1 \cdot \alpha = \alpha$.
8. For every α there exists a γ such that $\alpha\gamma = 1$.
9. $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$, multiplication is distributive over addition.

Definition 1.1.2. Two fields, $\mathbb{K}$ and $\mathbb{K}'$ are said to be **isomorphic** if we can setup a one to one correspondence between $\mathbb{K}$ and $\mathbb{K}'$.

The most common fields are, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$.

1.2 Theory of Linear Algebra

Involving space, and the most general case, a series of linear equations.

$$\begin{gathered} a_{11}x_{11} + a_{12}x_{12} + \cdots + a_{1n}x_{1n} = b_1 \\ a_{21}x_{21} + a_{22}x_{22} + \cdots + a_{2n}x_{2n} = b_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ a_{n1}x_{n1} + a_{n2}x_{n2} + \cdots + a_{nn}x_{nn} = b_n \end{gathered}$$

1.3 Determinant

The determinant is the scalar that a 1 area, volume, hypervolume etc. gets scaled by. Even a one length can be scaled or reduced to 0. When a determinant is 0, it means that all space, gets squished to a lower dimension and thus has 0 volume, or 0 area, and so on.

$$D = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{2n} & \dots & a_{nn} \end{vmatrix} = \det ||a||$$

1.4 Acknowledgements

Almost none of this work is of my own. This work is from *Linear Algebra* by George Shilov. And lecture notes from Dr. Matt Miller's MATH544H class taught at the University of South Carolina in the Spring of 2019.

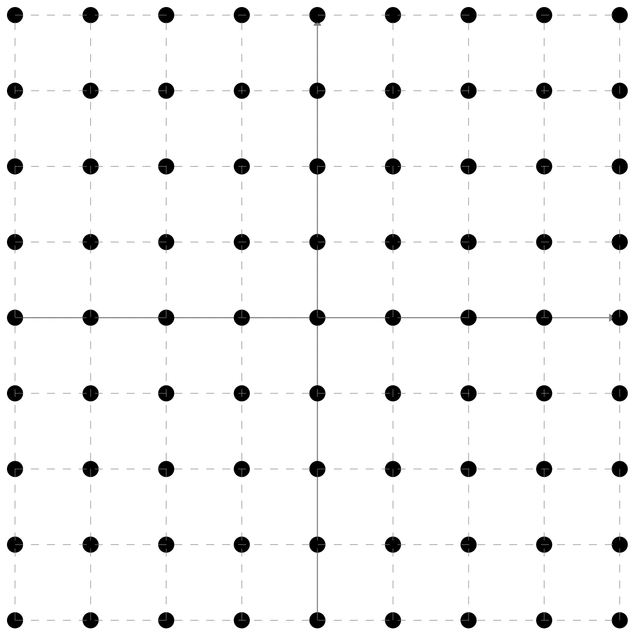

Figure 1.1: Vector Space $\mathbb{R}^2$ for $\forall(x, y); -4 \leq x \leq 4 and -4 \leq y \leq 4$.

Chapter 2

Lectures

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Definition 2.1.1. Linear Algebra is the study of vector spaces and the maps between them.

Definition 2.1.2. A function $T : V \to W$, from vector space V, the domain, to vector space W, the codomain, is called a linear transformation if

$$T(c\vec{u} + \vec{v}) = cT(\vec{u}) + T(\vec{v});\ \forall \vec{u}, \vec{v} \in V \wedge c \in \mathbb{R}$$

Example 2.1.1.

$$T : \mathbb{R} \to \mathbb{R};\ T(x) = 2x$$

$$T(cu + v) = 2cu + v = cT(u) + T(v)$$

So this is a linear transformation.

Example 2.1.2.

$$C^0[0, 2\pi] = \{f : [0, 2\pi] \to \mathbb{R} \mid f\ is\ continuous\}$$

$$S : C^0[0, 2\pi] \to \mathbb{R}$$

$$S(f) = f(\pi)$$

Check linearity.

$$S(cf+g) = (cf+g)(\pi) = cf(\pi) + g(\pi) = cS(f) + S(g)$$

Example 2.1.3.

$$\text{Let } T : C^0[0,2\pi] \to \mathbb{R}$$
$$T(f) = \bar{f}$$
$$T(f) = \frac{1}{2\pi}\int_0^{2\pi} f(x)\ dx$$

Check linearity.

$$T(cf+g) = \frac{1}{2\pi}\int_0^{2\pi}(cf+g)(x)\ dx$$
$$T(cf+g) = \frac{1}{2\pi}(c\int_0^{2\pi} f(x)\ dx + \int_0^{2\pi} g(x)\ dx)$$
$$T(cf+g) = cT(f) + T(g)$$

$\mathbb{R}^2$ and $\mathbb{R}^3$ are vector spaces, (x,y) is mapped to the vector $\begin{bmatrix} x \\ y \end{bmatrix}$ and (x,y,z) is mapped to the vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

Example 2.1.4. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x+3y \\ -x+y \\ 2x+y \end{bmatrix}$$

Verify linearity.

$$T(c\begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} z \\ w \end{bmatrix}) = T(\begin{bmatrix} cx+z \\ cy+w \end{bmatrix} = \begin{bmatrix} cx+z+3cu+3w \\ -cx-z+cy+w \\ 2cx+2z+cy+w \end{bmatrix}$$

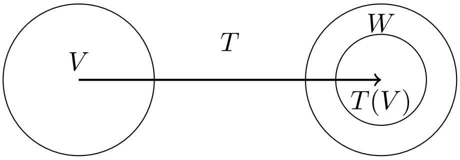
$$T(c\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}z\\w\end{bmatrix})=\begin{bmatrix}cx+3cu\\-cx+cy\\2cx+cy\end{bmatrix}+\begin{bmatrix}z+3w\\-z+w\\2z+w\end{bmatrix}$$

$$T(c\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}z\\w\end{bmatrix})=c\begin{bmatrix}x+3u\\-x+y\\2x+y\end{bmatrix}+\begin{bmatrix}z+3w\\-z+w\\2z+w\end{bmatrix}$$

$$T(c\begin{bmatrix}x\\y\end{bmatrix}+\begin{bmatrix}z\\w\end{bmatrix})=cT(\begin{bmatrix}x\\y\end{bmatrix})+T(\begin{bmatrix}z\\w\end{bmatrix})$$

This transformation is the same as the matrix transformation, $\begin{bmatrix}1&3\\-1&1\\-2&1\end{bmatrix}$.

Definition 2.1.3. Let $T : V \to W$ be a linear transformation. The range or image of T is the is the set range (T).



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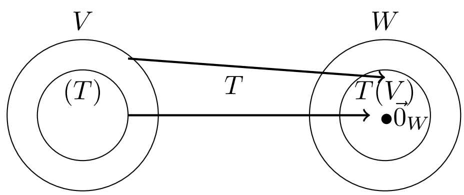
Definition 2.2.1. Let $T : V \to W$ be a linear transformation. The range or image of T is the is the set range $(T) = T(V)$=im(T).

$$\{\vec{w} \in W \mid (\exists \vec{v} \in V)\vec{w} = T(\vec{v})\}$$

Definition 2.2.2. The kernel or nullspace of T is $ker(T) = (T)$.

$$\{\vec{v} \in V \mid T(V) = \vec{0}_W\}$$

Note: The kernel is a vector space.



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Example 2.2.1. Let $T : \mathbb{R}^2 \to \mathbb{R}^3;\ T(\begin{bmatrix} x \\ y \end{bmatrix}) = T(\begin{bmatrix} x+3y \\ -x+y \\ 2x+y \end{bmatrix})$.

$$ker(T) = \{\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \mid T(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in \mathbb{R}^3\}$$

$$x + 3y = 0$$

$$-x + y = 0$$

$$4y = 0 \implies y = 0 \implies x = 0 \implies ker(T) = \{\begin{bmatrix} 0 \\ 0 \end{bmatrix}\}$$

$$T(V) = \{\begin{bmatrix} x+3y \\ -x+y \\ 2x+y \end{bmatrix} \mid x, y \in \mathbb{R}\}$$

$$T(V) = \{x \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix} \mid x, y \in \mathbb{R}\} = \{t\vec{a} + s\vec{b} \mid t, s \in \mathbb{R}\}$$

Which plane is it? Contrast $L(\begin{bmatrix} x \\ y \end{bmatrix}) = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$.

This is the plane, $3x - 5y - 4z = 0$. This proves $T(V) \subseteq P$. To prove $T(V) = P$, we also need to prove $P \subseteq T(V)$.