**Problem 2.3.4.** Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the map defined by first rotating counterclockwise by $\theta$, and then reflecting across the line $y = x$. Find the matrix of $T$.

*Solution:* The rotation and relection can be represented as matrices (example 2.3.2), and we can multiply the matrices in the correct order.

$$L_T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}$$

□

**Problem 2.3.8.** The norm of a vector $\vec{v} \in \mathbb{R}^n$ is defined (in analogy with $\mathbb{R}^2$ and $\mathbb{R}^3$) as $||\vec{v}|| - \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$. Show that if $\vec{v} \in \mathbb{R}^n$ then $||\vec{v}||^2 = \vec{v}^T\vec{v}$.

*Solution:*

$$\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \text{ and } \vec{v}^T = \begin{bmatrix} v_1 & v_2 & \dots & v_n \end{bmatrix}$$

Thus the product $\vec{v}^T\vec{v} = v_1^2 + v_2^2 + \cdots + v_n^2$. By definition, the square of the norm, $||\vec{v}|| = v_1^2 + v_2^2 + \cdots + v_n^2$, thus they are equal. □

**Problem 2.3.10.** Suppose that $A \in M_n(\mathbb{F})$ is invertible. Show that $A^T$ is also invertible, and that $(A^T)^{-1} = (A^{-1})^T$.

*Solution:* By using the 6 propositions from class.

$$A \in M_n(\mathbb{F}),\ A^{-1} \in M_n(\mathbb{F}),\ A^{-1}A = I_n$$

$$(A^{-1}A)^T = I_n^T$$

$$A^T(A^{-1})^T = I_n$$

$$(A^T)^{-1}A^T(A^{-1})^T = (A^T)^{-1}I_n$$

$$(A^{-1})^T = (A^T)^{-1}$$

Because $A^{-1}$ exists, we know that $(A^T)^{-1}$ exists because it is computationally $(A^{-1})^T$. □

**Problem 2.5.4d.** Determine whether each list of vectors span $\mathbb{F}^n$, this case $\mathbb{F} = \mathbb{R}$.

$$\left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

*Solution:* We need to prove that any $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ has a solution for $a, b, c \in \mathbb{F}$, for all $x, y, z \in \mathbb{F}$. We can turn this into a system of equations. $\begin{cases} x = a + b \\ y = a + c \\ z = b + c \end{cases}$ We can turn this into an augmented matrix, and after using Gaussian elimination, we can turn this into RREF.

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & x \\ 1 & 0 & 1 & y \\ 0 & 1 & 1 & z \end{array}\right] \xrightarrow{R1:=(R2-R3)+R1} \left[\begin{array}{ccc|c} 2 & 0 & 0 & x+y-z \\ 1 & 0 & 1 & y \\ 0 & 1 & 1 & z \end{array}\right] \xrightarrow{\frac{1}{2}R1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(x+y-z) \\ 1 & 0 & 1 & y \\ 0 & 1 & 1 & z \end{array}\right]$$

$$\xrightarrow{R2:=-R1+R2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(x+y-z) \\ 0 & 0 & 1 & y - \frac{1}{2}(x+y-z) \\ 0 & 1 & 1 & z \end{array}\right]$$

$$\xrightarrow{R3:=-R2+R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(x+y-z) \\ 0 & 0 & 1 & y - \frac{1}{2}(x+y-z) \\ 0 & 1 & 0 & z - y + \frac{1}{2}(x+y-z) \end{array}\right]$$

$$\xrightarrow{R2 \leftrightarrow R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{1}{2}(x+y-z) \\ 0 & 1 & 0 & z - y + \frac{1}{2}(x+y-z) \\ 0 & 0 & 1 & y - \frac{1}{2}(x+y-z) \end{array}\right]$$

So for all $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$ we know there exists a linear combination, one that $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = a \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + c \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ where $a = \frac{1}{2}(x+y-z)$, $b = z - y + \frac{1}{2}(x+y-z)$ and $c = y - \frac{1}{2}(x+y-z)$. □

**Problem 2.5.4e.** Determine whether each list of vectors spans $\mathbb{F}^n$, in this case $\mathbb{F} = \mathbb{F}_2$.

$$\left( \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right)$$

*Solution:* $\mathbb{F}_2 = \{0, 1\}$.

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

So $< \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} > = < \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} >$. This cannot span all of $F_2^3$, as there are only two independent vectors. □

**Problem 2.5.18.** Suppose that $T : V \to W$ is linear and that $U$ is a subspace of W, let

$$X = \{v \in V \mid T(v) \in U\}.$$

Show that X is a subspace of V.

*Solution:* Because T is linear, $\vec{0}_V \in X$, because $\vec{0} \in V$, and $T(\vec{0}_V) = \vec{0}_W$(T is linear).
Let $\vec{x}, \vec{y} \in X$, we know that $T(\vec{x}+\vec{y}) = T(\vec{x})+T(\vec{y}) \in U$, because U is a subspace, so $(\vec{x}+\vec{y}) \in X$(T is linear). Thus $X$ is closed under addition.
Let $c \in \mathbb{F}$, and $\vec{x} \in X$ and we know that $T(c\vec{x}) \in U$, so $c\vec{x} \in X$, and X is closed under scalar multiplication. And finally because $X \subseteq V$ by definition, X is a subspace of V. □

**Problem K.** We have seen that if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\Delta = ad - bc = 0$, then $A^{-1}$ exists and there's a nice formula. What happens if $\Delta = 0$? Could A still be invertible, but with a different formula? Let's take $\vec{v} = \begin{bmatrix} -b \\ a \end{bmatrix}$. Compute $A\vec{v}$. If $\vec{v} = \vec{0}$ what can you conclude? If $\vec{v} \neq \vec{0}$, pick another vector $\vec{w}$ in a clever way, and compute $A\vec{w}$. Analyze the cases $\vec{w} = \vec{0}$ and $\vec{w} \neq \vec{0}$. What is the upshot of all of this?

*Solution:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$$

When $a = c = 0$, we know that $ad - bc = 0$, and this can be reduced to $y \begin{bmatrix} b \\ d \end{bmatrix}$, this means that all of $\mathbb{R}^2$ gets mapped to all scalar multiples of $\begin{bmatrix} b \\ d \end{bmatrix}$, which is a line. Because A is not bijective in this case, it is no longer invertible. By the same argument, $b = d = 0$ is also no longer invertible.
The case that $a = b = 0$, means this can be reduced to $\begin{bmatrix} 0 \\ cx + dy \end{bmatrix}$, which means that is no longer bijective as well, because $v_1 \neq 0$ can never be mapped to. By the same argument, except $v_2$ is always 0, $c = d = 0$ is no longer bijective.
For the case $ad = bc \neq 0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ \frac{ad}{b} & d \end{bmatrix} = a \begin{bmatrix} 1 & \frac{b}{a} \\ \frac{d}{b} & \frac{d}{a} \end{bmatrix}$$

Now after the transformation.

$$a \begin{bmatrix} 1 & \frac{b}{a} \\ \frac{d}{b} & \frac{d}{a} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = a \begin{bmatrix} x + \frac{b}{a}y \\ \frac{d}{b}x + \frac{d}{a}y \end{bmatrix} = ax \begin{bmatrix} 1 \\ \frac{d}{b} \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$$

These are collinear, $b \neq 0$, so

$$\frac{1}{b}ax \begin{bmatrix} b \\ d \end{bmatrix} + y \begin{bmatrix} b \\ d \end{bmatrix}$$

Thus $L_A$ is not bijective, and cannot have an inverse. □