MATH544
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HW 3

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Problem 1 (1.4.2)**.** *Let $\mathbb{F} = \{a + bi|\ a, b \in \mathbb{Q}\}$, show that $\mathbb{F}$ is a field.*

Proof. In order for $\mathbb{F}$ to be a field it would need to follow that,
Note it is redundant that $a, b \in \mathbb{Z} \to (a + b \in \mathbb{Z}$ and $a \cdot b \in \mathbb{Z})$, and $\frac{a}{b} \in \mathbb{Q}$ if $a, b \in \mathbb{Z}$, so it will be omitted from the rest of the proof.

1. $\mathbb{F}$ is closed and commutative under addition.
   Let $u, v \in \mathbb{F}$ then $(a, b, c, d, e, f, g, h \in \mathbb{Z})$

$$u + v = \frac{a}{b} + \frac{c}{d}i + \frac{e}{f} + \frac{g}{h}i = \frac{af + be}{bf} + \frac{ch + fg}{dh}i = v + u$$

2. $\mathbb{F}$ is associative under addition.
   Let $u, v, w \in \mathbb{F}$ then $(a, b, c, d, e, f, g, h, j, k, l, m \in \mathbb{Z})$

$$(u + v) + v = (\frac{a}{b} + \frac{c}{d}i + \frac{e}{f} + \frac{g}{h}i) + \frac{j}{k} + \frac{l}{m}i$$

$$= (\frac{afk + bek + bfj}{bfk} + \frac{chm + fgm + dhl}{dhm}i) = u + (v + w)$$

3. $\mathbb{F}$ is closed and commutative under multiplication.
   Let $u, v \in \mathbb{F}$ then $(a, b, c, d, e, f, g, h \in \mathbb{Z})$

$$u + v = (\frac{a}{b} + \frac{c}{d}i) \cdot (\frac{e}{f} + \frac{g}{h}i) = \frac{ae}{bf} - \frac{cg}{dh} + (\frac{ce}{df} + \frac{ag}{bh})i$$

$$= \frac{adeh - bcgf}{bdfh} + \frac{bceh + adgf}{bdfh}i = v + u$$

4. $\mathbb{F}$ is associative under multiplication.
   Let $u, v, v \in \mathbb{F}$ then $(a, b, c, d, e, f, g, h, j, k, l, m \in \mathbb{Z})$

$$(u \cdot v) \cdot w = ((\frac{a}{b} + \frac{c}{d}i) \cdot (\frac{e}{f} + \frac{g}{h}i)) \cdot (\frac{j}{k} + \frac{l}{m}i)$$

$$(\frac{adeh - bcfg}{bdfh} + \frac{bceh + adgf}{bhdf}i) \cdot (\frac{j}{k} + \frac{l}{m}i)$$

$$= \frac{mjadeh - mjbcfg - lbcehk - ladgf}{bhdfmk} + \frac{mjbceh + mjadgf + kladeh - klbcfg}{bhdfkm}i$$

$$= u \cdot (v \cdot w)$$

5. 0 is an additive identity.
   Let $u \in \mathbb{F}$.

$$u + 0 = (\frac{a}{b} + \frac{c}{d}i) + (0 + 0i) = \frac{a}{b} + \frac{c}{d}i = u$$

$$0 = 0 + 0i$$

6. Every element has an additive inverse. $\frac{a}{b} \in \mathbb{Q} \to -1\frac{a}{b} \in \mathbb{Q}$.
   Let $u \in \mathbb{F}$, $a, b, c, d \in \mathbb{Z}$.

$$u + (-u) = (\frac{a}{b} + \frac{c}{d}i) + (-\frac{a}{b} - \frac{c}{d}i) = 0 + 0i = 0$$

7. 1 is a multiplicative identity. Trivial. $1 = 1 + 0i$

8. Every nonzero element has a multiplicative inverse.
   Let $u \in \mathbb{F}$, $a, b, c, d \in \mathbb{Z}$.

$$u \cdot u^{-1} = 1$$

$$(\frac{a}{b} + \frac{c}{d}i)u^{-1} = 1$$

$$u^{-1} = \frac{1}{(\frac{a}{b} + \frac{c}{d}i)}$$

$$\frac{1}{(\frac{a}{b} + \frac{c}{d}i)} = \frac{1}{(\frac{a}{b} + \frac{c}{d}i)} \cdot \frac{\frac{a}{b} - \frac{c}{d}i}{\frac{a}{b} - \frac{c}{d}i} = \frac{\frac{a}{b} - \frac{c}{d}i}{\frac{a^2}{b^2} + \frac{c^2}{d^2}} = \frac{\frac{a}{b} - \frac{c}{d}i}{\frac{d^2a^2 + b^2c^2}{d^2b^2}} = \frac{ad^2b + cdb^2i}{d^2a^2 + b^2c^2}$$

$$u^{-1} = \frac{ad^2b + cdb^2i}{d^2a^2 + b^2c^2}$$

□

Problem 2 (1.4.3d)**.** *Explain why* $\mathbb{F} = \{\frac{m}{2^n} | \ m, n \in \mathbb{Z}\}$ *is not a field.*

Proof. Fails multiplicative inverse, $u = \frac{3}{2} \in \mathbb{F}$, $u^{-1} = \frac{2}{3} \in \mathbb{Q}$ but $log_2(3) \notin \mathbb{Z} \to u^{-1} \notin \mathbb{F}$. □

Problem 3 (1.5.2). *Show that the set of solutions of a nonhomogeneous m x n linear system is never a subspace of* $\mathbb{F}^n$.

Proof.

$$\begin{aligned} a_{11}x_1 + \cdots + a_{1n}x_n &= b_1 \\ \vdots \quad \ddots \quad \vdots \quad &\quad \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n &= b_m \end{aligned}$$

For this set of equations at least one of the equations has a $b \neq 0$, we'll call this row, row j, where $b_j \neq 0$.

$$a_{j1}x_1 + \cdots + a_{jn} = b_j \mid b_j \neq 0$$

Thus this set of solutions fails closure over additions. Say $x_1 = c_!, \ldots x_n = c_n$ and $x_1 = d_1, \ldots, x_n = d_n$ are solutions. $x_1 = c_1 + d_1, \ldots, c_n + d_n$ is not a solution because $b_j \neq 0$ for the equation j, we have the following contradiction.

$$a_{j1}(c_1 + d_1) + \cdots + a_{jn}(c_1 + d_1) = (a_{j1}c_1 + \cdots + a_{jn}c_n) + (a_{j1}d_1 + \cdots + a_{jn}d_n)$$

$$= b_j + b_j = 2 \cdot b_j \neq b_j$$

Because $b_j \neq 0$, then $2b_j \neq b_j$. Ad absurdum. □

Problem 4 (1.5.4). *Determine which of the following are subspaces of* $C[a, b]$. *Here* $a, b, c \in \mathbb{R}$ *are fixed and* $a < c < b$.

(a) $V = \{f \in C[a, b] \mid f(c) = 0\}$

(b) $V = \{f \in C[a, b] \mid f(c) = 1\}$

(c) $V = \{f \in D[a, b] \mid f'(c) = 0\}$

(d) $V = \{f \in D[a, b] \mid f' \text{ is constant}\}$

(a) *Proof.*
Suppose we did have a 1 element, when $1 \cdot f$, where $f \in V$ would be the same f, this would mean for all input values, $1 \cdot f$ would map to the same values of f. This would mean for all $x \in \mathbb{R}$, $1(x) = 1$, which means there would be no where $1(c) = 0$. Thus V is not a field, because it fails to have a 1 element, ad absurdum.
Side note, what if we had a function $1(x) = \left\{ \begin{array}{c|c} 1 & x \neq c \\ 0 & x = c \end{array} \right\}$ I think this can be the one identity. □

(b) *Proof.*
The function $f(x) = 1$ would be contained in V. However, $(f + f)(x)$ would not, because $(f + f)(x) = f(x) + f(x) = 2$ for all x.
Thus V is not a field, because it fails closure over addition. □

(c) I don't believe this is a field, still thinking about it.

(d) *Proof.*
It is assumed, these are derived, or integrated with respect to x, however this is just a placeholder, it can be any singular variable differentiation.
We have closure over addition. Let $f, g \in \mathbb{F}$, and because f' and g' are constants, then we know that $f'(x) = c$ for all x, and $g'(x) = d$ for all x. Thus $f = \int f' \; dx = \int c \; dx = cx + C_1$, and $g = \int g' \; dx = \int d \; dx = dx + C_2$. And $(f+g)(x) = cx + dx + C_1 + C_2 = (c+d)(x) + C$. Thus $\frac{d}{dx}(c+d) = (c+d)$ for all x. And thus closure is maintained.
However, closure over multiplication does not exist. Suppose it did. $f = \int f' \; dx$ and $g = \int g' \; dx$. By the same argument as before, $f = cx + C_1$ and $g = dx + C_2$, thus $(f \cdot g) = cdx^2 + cC_2x + dC_1x + C_1C_2$, which means $(f+g)' = 2cdx + dC_1 + cC_2$, and it is not constant(when $c \neq 0$ and $d \neq 0$). Thus V is not a field because it fails closure over multiplication. $\square$

Problem 5. *Let V be a vector space, and suppose that U and W are both subspaces of V. Show that*

$$U \cap W := \{v \mid v \in U \wedge v \in W\}$$

is also a subspace.

Proof.

(a) If U and W are subspaces of V, then they both contain $\vec{0}$, and so $\vec{0} \in U \cap W$.

(b) By the same reasoning, $\vec{1}$ is contained in $U \cap W$. This is because $\vec{1} \in V$

(c) Say $\vec{u} \in U \cap W$, then $\vec{u} \in U$ and $\vec{u} \in W$, then because of closure over scalar multiplication for U and W, we know for $c \in \mathbb{F}$, $c\vec{u} \in U$ and $c\vec{u} \in W$, so we have closure over scalar multiplication for $U \cap W$, because $c\vec{u} \in U \cap W$.

(d) Say $\vec{u}, \vec{v} \in U \cap W$, then $\vec{u}, \vec{v} \in U$ and $\vec{u}, \vec{v} \in W$, and because U and W are vector spaces, we know they have closure over vector addition. So $\vec{u} + \vec{v} \in U$ and $\vec{u} + \vec{v} \in W$, so $\vec{u} + \vec{v} \in U \cap W$, thus we have closure over vector addition for $U \cap W$.

Thus $U \cap W$ is a subspace of V, because $U \subseteq V$ and $W \subseteq V$, thus $U \cap W \subseteq V$ and $U \cap W$ is a vector space. $\square$

Problem 6. *Problem 1.5.9 is not so terribly important in its own right, but becomes much more important in the case that A is actually a vector space over the field $\mathbb{F}$. We have to refine the definition just a bit, however. If V is a vector space over a field $\mathbb{F}$, we call $V^* = \{T : V \to \mathbb{F} \mid T \text{ is linear}\}$ the dual space of V. Addition and scalar multiplication of elements of V^* is just like in 1.5.9, but one also has to check that sums and scalar multiples of linear functions are in fact still linear. Worth doing, but* *does not to be written up nicely – you may assume that* V^* *is a vector space. Can you give some geometrically interesting examples of elements of* $(\mathbb{R}^2)*$*? Since* $V*$ *is a vector space, its dual* $V^{**} = (V^*)^*$ *is also a vector space. To get some idea what sort of critters inhabit it, let's fix a vector* $\vec{v} \in V$ *and define a function* $ev_v := V^* \to \mathbb{F}$ *by* $ev_v(T) = T(v)$*. Show that* ev_v *is indeed in* V^{**}*. Then define* $\sigma : V \to V^{**}$ *by* $\sigma(v) = ev_v$*. Show that* σ *is a linear transformation.*