MATH544
Dr. Miller
HW 3

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Problem 1 (1.4.7)**.** *Show that $ad - bc \neq 0$, then the linear system*

$$ax + by = e$$

$$cx + dy = f$$

over a field $\mathbb{F}$ has a unique solution.

Proof. Multiply the first equation by c and the second by a.

$$acx + bcy = ce$$

$$acx + ady = af$$

$$ady - bcy = af - ce$$

$$(ad - bc)y = af - ce$$

$$y = \frac{af - ce}{ad - bc}$$

We can then extend this argument again with d and b.

$$adx + bdy = de$$

$$bcx + bdy = bf$$

$$adx - bcx = de - bf$$

$$x = \frac{de - bf}{ad - bc}$$

Thus there exists a unique solution for a 2x2 linear system as long as $ad - bc \neq 0$. $\square$

Problem 2 (Supp A).

$$a_{11}x_1 + \cdots + a_{1n}x_n = \lambda x_1$$
$$\vdots$$
$$a_{n1}x_1 + \cdots + a_{nn}x_n = \lambda x_n$$

This can be written as,

$$(a_{11} - \lambda)x_1 + a_{21}x_2 + \cdots + a_{1n}x_n = 0$$
$$\vdots$$
$$a_{b1}x_1 + \cdots + (a_{bb} - \lambda)x_b + \cdots + a_{bn}x_n = 0$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 + \cdots + (a_{nn} - \lambda)x_n = 0$$

If $x_1 = c_1, \dots, x_n = c_n$ and $x_1 = d_1, \dots, x_n = d_n$ are solutions, is $x_1 = rc_n, \dots, x_n = rc_n$ a solution?

$$(a_{11} - \lambda)rc_1 + a_{21}rc_2 + \cdots + a_{1n}rc_n = r((a_{11} - \lambda)c_1 + a_{21}c_2 + \cdots + a_{1n}c_n) = r \cdot 0 = 0$$

This argument can be extended to each equation in the system. Thus this works for all $r \in \mathbb{R}$.
Assume $x_1 = c_1, \dots, x_n = c_n$ and $x_1 = d_1, \dots, x_n = d_n$ are solutions.
Let $x_1 = c_1 + d_1, \dots, x_n = c_n + d_n$.

$$(a_{11} - \lambda)(c_1 + d_1) + a_{21}(c_1 + d_1) + \cdots + a_{1n}(c_n + d_n)$$
$$\vdots$$
$$a_{b1}(c_1 + d_1) + \cdots + (a_{bb} - \lambda)(c_b + d_b) + \cdots + a_{bn}(c_n + d_n)$$
$$\vdots$$
$$a_{n1}(c_1 + d_1) + a_{n2}(c_2 + d_2) + \cdots + (a_{nn} - \lambda)(c_n + d_n)$$

This can be rewritten as

$$((a_1 1 - \lambda)c_1 + \cdots + a_{1n}c_n) + ((a_1 1 - \lambda)c_1 + \cdots + a_{1n}c_n)) = 2 \cdot 0$$

Which is the addition of two solutions where all the equations are homogeneous. This argument can be extended to each equation.

Problem 3. *Find the values of λ does the equations have a non trivial solution. In each case give a sample of a nontrivial solution.*

$$2x + y = \lambda x$$
$$x + 2y = \lambda y$$

Proof.

$$(2-\lambda)x + y = 0$$

$$x + (2-\lambda)y = 0$$

Thus $y = \frac{x}{\lambda-2}$ and $y = (\lambda-2)x$. We can say then,

$$\frac{x}{\lambda-2} = (\lambda-2)x$$

$$x = (\lambda^2 - 4\lambda + 4)x$$

$$-x + (\lambda^2 - 4\lambda + 4)x = 0 = (\lambda^2 - 4\lambda + 3)x$$

The trivial solurion is when $x = 0$, so instead we will try when $\lambda^2 - 4\lambda + 3 = 0$, which is true when $\lambda = 1$ and $\lambda = 3$.
When $\lambda = 1$, the equations become,

$$x + y = 0$$

$$x + y = 0$$

Which is solved when for all $y = -x$ for all $x \in \mathbb{R}$, and when $\lambda = 3$

$$-x + y = 0$$

$$x - y = 0$$

Which is solved when $y = x$ for all $x \in \mathbb{R}$. $\square$