Problem 2.5.9d. Show that for any $a, b \in \mathbb{R}$ there is a unique solution of the differential equation(2.22) which satisfies $f(0) = a$ and $f(\frac{\pi}{2}) = b$.

$$2.22: \qquad \frac{d^2 f}{dt^2} + f(t) = 4e^{-t}$$

Problem 2.5.14. Show that $T \in L(V)$ is injective if and only if 0 is not an eigenvalue of T.

Solution: Let's assume the contrapositive of the right hand direction. If 0 is an eigenvalue, then there exists a $\vec{v} \neq \vec{0}$ such that $T(\vec{v}) = 0\vec{v}$. Thus this fails to be injective as $T(\vec{0}) = T(\vec{v}) = \vec{0}$ but $\vec{v} \neq \vec{0}$.
Let's assume the contrapositive of the left hand direction. If T is not injective then there exists distinct $\vec{v_1}, \vec{v_2}$, such that $T(\vec{v_1}) = T(\vec{v_2})$. Thus $T(\vec{v_1} - \vec{v2}) = T(\vec{v_1}) - T(\vec{v_2}) = \vec{0}$, however $(\vec{v_1} - \vec{v_2}) \neq \vec{0}$, and thus $(\vec{v_1} - \vec{v_2}$ have an eigenvalue of 0, because $T(\vec{v_1} - \vec{v_2}) = 0(\vec{v_1} - \vec{v_2})$. □ [

Problem 3.1.2a. Determine whether this list of vectors is linearly independent in $\mathbb{C}$.

$$\left(\begin{bmatrix} i \\ 1+i \end{bmatrix}, \begin{bmatrix} 1-i \\ 2 \end{bmatrix} \right)$$

Solution: We can put this into matrix form $A = [\vec{v2}|\vec{v1}]$, and convert this to RREF, and use a previous theorem depending on the leading pivots to find linear independence or dependence.

$$A = \begin{bmatrix} i & 1-i \\ 1+i & 2 \end{bmatrix} \xrightarrow[-1 \cdot R_1]{-R_2 + R_1} \begin{bmatrix} 1 & 1+i \\ 1+i & 2 \end{bmatrix} \xrightarrow{(-1-i)R_1 + R_2} \begin{bmatrix} 1 & 1+i \\ 0 & 2-2i \end{bmatrix}$$

$$\xrightarrow{-\frac{1}{2}R_2 + R_1} \begin{bmatrix} 1 & 2 \\ 0 & 2-2i \end{bmatrix} \xrightarrow[\frac{1}{8}R_2]{(2+2i)R_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \xrightarrow{-2R_2 + R_1} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

This matrix can be reduced to the identity, and thus it is linearly independent. □

Problem 3.1.2b. Determine whether this list of vectors is linearly independent.

$$\left(\begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -5 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ -1 \end{bmatrix} \right)$$

Solution: We can use our method of turning this into a 4 by 3 matrix with every column vector acting as a column in the matrix. Then by corollary from class(February 20), we know that this list is dependent.

Corollary (February 20)**.** *Suppose that* $S = \{\vec{v_1}, \vec{v_2}, \vec{v_3}, \ldots, \vec{v_n}\} \subset \mathbb{F}^m$.

1. *If* $n > m$*, then* S *is dependent.*
2. *If* $m > n$*, then* S *fails to span* $\mathbb{F}^n$.

□

Problem 3.1.2c. Determine whether this list of vectors is linearly independent.

$$\left(\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Solution: Using the same method as (3.1.2a)

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 2 & 3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This matrix has only two pivots. So one of the vectors is a linear combination of the two others.

□

Problem 3.1.2d. Determine whether this list of vectors is linearly independent.

$$\left(\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right)$$

Solution:

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \end{bmatrix}$$

These 4 vectors are linearly independent. □

Problem 3.1.8. Show that the list of vectors $\left(\begin{bmatrix} 1 \\ i \end{bmatrix}, \begin{bmatrix} i \\ -1 \end{bmatrix} \right)$ in $\mathbb{C}^2$ is linearly independent over $\mathbb{R}$, but is linearly dependent over $\mathbb{C}$.

Solution: Over $\mathbb{R}$, these vectors are $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right)$, which is trivially independent because if we scale the second vector by -1, then we get the normal standard vectors $\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right)$.

Over $\mathbb{C}$, we will use the technique used previously. Let $A = [\vec{v_1}|\vec{v_2}]$.

$$A = \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix} \xrightarrow{-iR_1+R_2} \begin{bmatrix} 1 & i \\ 0 & 0 \end{bmatrix}$$

Thus we have linear dependence, by the theorem in class dealing with pivots and linear dependence. We also know that

$$\begin{bmatrix} 1 \\ i \end{bmatrix} = -i \begin{bmatrix} i \\ -1 \end{bmatrix}$$

□

Problem 3.1.10a. Let $A \in M_n(\mathbb{F})$. Show that $ker\ A = \{\vec{0}\}$ iff $C(A) = \mathbb{F}^n$.

Solution: If $ker\ A = \{\vec{0}\}$ then the columns of the matrix A are linearly independent(proposition 3.1). Then by corollary 3.2, because the columns of A are linearly independent, then A is injective from $\mathbb{F}^n \to \mathbb{F}^n$, meaning there exists a solution, $\vec{x}$ for $A\vec{x} = \vec{y}$ for all $\vec{y} \in \mathbb{F}^n$. If we break down matrix multiplication, we get a linear combination of A's column vectors.

$$A\vec{x} = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{11} \\ \vdots \\ a_{n1} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{n1} \\ \vdots \\ a_{nn} \end{bmatrix} = \vec{y}$$

Thus if $ker\ A = \{\vec{0}\}$, then $C(A) = \mathbb{F}^n$.
If $C(A) = \mathbb{F}^n$, then by the proposition from class, the set of column vectors of A are linearly independent. By proposition 3.1, then $ker\ A = \{\vec{0}\}$. □

Proposition. *(February 20) If $A \in M_n(\mathbb{F})$, then the set of column vectors of A is independent if and only if $A\vec{x} = \vec{0}$ has only the trivial solution if and only if $ker\ A = \{\vec{0}\}$.*

Problem 3.1.10b. Let $T \in \mathcal{L}(\mathbb{F}^n)$. Show that T is injective iff T is surjective.

Solution: If T is injective, then $T(\vec{v}) = T(\vec{w}) \implies \vec{v} = \vec{w}$. □

Problem 3.1.14. Let $n \geq 1$ be an integer, and suppose there are constants such that

$$\sum_{k=1}^{n} a_k \sin kx = 0$$

for every $x \in \mathbb{R}$. Prove that $a_1 = \dots = a_n = 0$.

Solution: Let $D^2 : C^\infty(\mathbb{R}) \to c^\infty(\mathbb{R})$ be the linear transformation $D^2 f = f''$. Then

$$D^2(\sin kx) = -k^2 \sin kx$$

for $k = 1, \dots, k = n$, thus these are eigenvectors, with eigenvalues from $\lambda_{k=1} = -1, \dots, \lambda_{k=n} = -n^2$ respectively. By theorem 3.8, these are linearly independent. By definition of linear independence,

$$\sum_{k=1}^{n} a_k \vec{v_k} = \vec{0} \iff a_1 = \dots = a_n = 0$$

where $\vec{v_k} = \sin kx$. □

Definition. Matrix Exponential

$$e^A = I_n + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \cdots + \frac{1}{n!}A^n + \ldots$$

Problem M.1. Why is convergence not an issue for $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$? What is e^A in this case?

Solution:

$$e^A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \ldots$$

It converges very fast because $A^2 = 0_{2,2}$, thus $A^n = 0_{2,2}$ for $n \geq 2$. Which means we only need to calculate the first two terms and sum with the identity. for e^A. Thus $e^A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. ☐

Problem M.2. Why is convergence not an issue for $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$? What is e^A in this case?

Solution:

$$e^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \ldots$$

Thus this too converges fast because $A^2 = 0_{3,3}$, and $A^n = 0_{3,3}$ for $n \geq 2$. Thus it adds infinite 0 matrices and we only need to calculate the sum of the first 2 terms and the identity.

Thus $e^A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. ☐

Problem M.5. Why is convergence not an issue if $A = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$. What is e^A in this case?

Solution:

$$e^A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} \lambda_1^2 & 0 & 0 \\ 0 & \lambda_2^2 & 0 \\ 0 & 0 & \lambda_3^2 \end{bmatrix} + \cdots + \frac{1}{n!}\begin{bmatrix} \lambda_1^n & 0 & 0 \\ 0 & \lambda_2^n & 0 \\ 0 & 0 & \lambda_3^n \end{bmatrix} + \ldots$$

Thus each diagonal,

$$A_{ii} = \sum_{k=0}^{\infty} \frac{1}{k!}\lambda_i^k \text{ and } A_{ij} = 0 \text{ when } i \neq j.\ A_{ii} \text{ converges for all } \lambda \in \mathbb{R}.$$

To prove this we can use the Ratio Test, $x_n = \frac{1}{n!}\lambda_i^n$. We cannot generalize this, we can use Taylor's Remainder Theorem to get an arbitrary precision estimate.

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \frac{\lambda_i^{n+1}}{(n+1)!} \cdot \frac{n!}{\lambda_i^n} = \lim_{n\to\infty} \frac{\lambda_i}{n+1} = 0$$

☐