Math 546
Spring 2021
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Problem 1. Consider the operation $a * b = 2a + 5b$ on the set of real numbers.

a) Is the operation associative? *Solution:*
$(a * b) * c = (2a + 5b) * c = 4a + 10b + 5c$
$a * (b * c) = a * (2b + 5c) = 2a + 10b + 25c$
This is not an associative operation. $\square$

b) Explain why there is no identity element. *Solution:*
Assume there exists a value $I \in \mathbb{R}$ that fulfills for $a \in \mathbb{R}$, $a * I = a$. For $a * I = 2a + 5I$, we can solve for I, and $I = -\frac{a}{5}$. Let $b \in \mathbb{R}$, $b \neq a$, when we plug I in for $b * I = b$, we get $b * I = 2b - a$. We are left with the statement, $2b - a = b$, and when reduced, $a = b$, which is a contradiction and there does not exist an identity. $\square$

Problem 2. Consider the operation $a * b = a - b$ on the set of integers.

a) Is the operation associative? *Solution:*
$(a * b) * c = (a - b) - c$
$a * (b * c) = a - (b - c) = (a - b) + c$
Thus this is not an associative operation. $\square$

b) Explain why there is no identity element. *Solution:*
Assume there exists I that fulfills $a * I = a$ for some $a \neq 0$.
$a * I = a - I = a$, thus $I = 0$.
When plugged into $I * a = a$, we get $a = -a$. Thus there does not exist an identity. $\square$

Problem 3. Let G be the set of all integers that are greater than or equal to 10. Consider the operation $a * b = \max\{a, b\}$ on the set G.

a) Is the operation associative? *Solution:*
$\max\{a, \max\{b, c\}\} \stackrel{?}{=} \max\{\max\{a, b\}, c\}$

(a) $a \geq b \geq c$
$\max\{a, \max\{b, c\}\} = \max\{a, b\} = a$
$\max\{\max\{a, b\}, c\} = \max\{a, c\} = a$

(b) $a \geq c \geq b$
$\max\{a, \max\{b, c\}\} = \max\{a, c\} = a$
$\max\{\max\{a, b\}, c\} = \max\{a, c\} = a$

(c) $b \geq a \geq c$
$\max\{a, \max\{b, c\}\} = \max\{a, b\} = b$
$\max\{\max\{a, b\}, c\} = \max\{b, c\} = b$

(d) $b \geq c \geq a$
$\max\{a, \max\{b, c\}\} = \max\{a, b\} = b$
$\max\{\max\{a, b\}, c\} = \max\{b, c\} = b$

(e) $c \geq a \geq b$
$\max\{a, \max\{b, c\}\} = \max\{a, c\} = c$
$\max\{\max\{a, b\}, c\} = \max\{a, c\} = c$

(f) $c \geq b \geq a$
$\max\{a, \max\{b, c\}\} = \max\{a, c\} = c$
$\max\{\max\{a, b\}, c\} = \max\{b, c\} = c$

Thus this operation is associative. $\square$

b) Is there an identity element? If so, find it. *Solution:*
Because every element $x \in G$, $x \geq 10$, $\max\{10, x\} = \max\{x, 10\} = x$. $\square$

c) Is the "inverse" requirement satisfied? *Solution:*
Assume there exists an inverse. Such that $(a * b) * a^{-1} = b$. Let $a > b$, Then $\max\{\max\{a, b\}, a^{-1}\} = \max\{a, a^{-1}\} = c$, where $c \geq a > b$, thus $c > b$. There does not exist an inverse. $\square$

Problem 4. Let G be the set of all the matrices of the form $\begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix}$ with a, b nonzero real numbers, and consider the operation given by multiplication of matrices on the set G.

1. Does the "closure" requirement hold? *Solution:*
Let $A = \begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$, and $B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$, then $A * B = \begin{bmatrix} 2 & 4 \\ 0 & 2 \end{bmatrix}$ This does not satisfy closure. $\square$

2. Explain why there is no identity element. *Solution:*
Assume there exists an identity I that satisfies the equality $A * I = A$ and $I * A = A$, where $A = \begin{bmatrix} a & 1 \\ 0 & b \end{bmatrix}$, let $I = \begin{bmatrix} c & 1 \\ 0 & d \end{bmatrix}$. Let $B = \begin{bmatrix} a+1 & 1 \\ 0 & b \end{bmatrix}$
$$A * I = \begin{bmatrix} ac & a+d \\ 0 & bd \end{bmatrix}$$
$$B * I = \begin{bmatrix} (a+1)c & (a+1)+c \\ 0 & bd \end{bmatrix}$$
We have a contradiction $a+1+c = 1$ and $a+c = 1$, so there does not exist an identity. $\square$