MATH 3940 Numerical Analysis for Computer Scientists
Assignment 3 Solutions Fall 2020

1. Consider the equation: $x^7 + 8x^6 + 16x^5 = 0$.

   (a) (4 marks) Solve the equation using hand calculations. What is the order of each root of the equation?

   (b) (4 marks) Find an interval $[a, b]$ such that the bisection method can be used to find a solution of the equation. Justify your answer.

Solution. (a) $x^7 + 8x^6 + 16x^5 = 0$
$\Rightarrow\ x^5(x^2 + 8x + 16) = 0\ \Rightarrow x^5(x+4)^2 = 0\ \Rightarrow\ x = 0, -4$.

$x = 0$ is a root of order 5 and $x = -4$ is a root of order 2.

(b) Here $f(x) = x^7 + 8x^6 + 16x^5$.
$f(x)$ is a polynomial and continuous on $\mathbb{R}$.
Now $f(-1) = -1 + 8 - 16 = -9 < 0$ and $f(1) = 1 + 8 + 16 = 25 > 0$
Since $f(-1)$ and $f(1)$ have opposite signs and f is continuous on $[-1, 1]$, the bisection method can be used for the interval $[-1, 1]$. $\square$

2. Consider the equation: $3x^3 - x^2 + 2 = 0$.

   (a) (5 marks) Using hand calculations, perform 3 iterations of the bisection method starting with the interval $[-1, 0]$.

   (b) (4 marks) Using hand calculations, perform 2 iterations of the regula falsi method starting with the interval $[-1, 0]$.

   (c) (4 marks) Using hand calculations, perform 3 iterations of the secant method starting with the interval $[-1, 0]$.

Solution. (a) Here $a_0 = -1$ and $b_0 = 0$ and $f(x) = 3x^3 - x^2 + 2$.
Now $f(-1) = -3 - 1 + 2 = -2 < 0$ and $f(0) = 0 + 0 + 2 = 2 > 0$.
So $c_0 = \dfrac{a_0 + b_0}{2} = \dfrac{-1}{2} = -0.5$.
$f(c_0) = f(-0.5) = -0.375 - 0.25 + 2 = 1.375 = \frac{11}{8} > 0$.
So the new interval is $[-1, -0.5]$, then $c_1 = \dfrac{a_1 + b_1}{2} = \dfrac{-1.5}{2} = -\dfrac{3}{4} = -0.75$.
$f(c_1) = f(-0.75) = -1.2656 - 0.5625 + 2 = 0.1719 = \frac{11}{64} > 0$.
So the new interval is $[-1, -0.75]$, then $c_2 = \dfrac{a_2 + b_2}{2} = \dfrac{-1.75}{2} = -0.875 = -\dfrac{7}{8}$.
$f(c_2) = f(-0.875) = -2.0098 - 0.7656 + 2 = -0.7754 = -\frac{397}{512} < 0$.
So the new interval is $[-0.875, -0.75]$.

(b) Here $a_0 = -1$ and $b_0 = 0$, where $f(-1) = -2 < 0$ and $f(0) = 2 > 0$.

$$c_0 = b_0 - \frac{f(b_0)(b_0 - a_0)}{f(b_0) - f(a_0)} = 0 - \frac{f(0)(0+1)}{f(0) - f(-1)} = -\frac{2(1)}{2 - (-2)} = -\frac{2}{4} = -\frac{1}{2} = -0.5$$

$$f(c_0) = f(-0.5) = -0.375 - 0.25 + 2 = 1.375 = \frac{11}{8} > 0$$

So the new interval is $[-1, -0.5]$, then

$$c_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)} = -0.5 - \frac{f(-0.5)(-0.5+1)}{f(-0.5) - f(-1)} = -0.5 - \frac{1.375(0.5)}{1.375 - (-2)} = -0.7037 = -\frac{19}{27}$$

$$f(c_1) = f(-0.7037) = 3(-0.3485) - 0.4952 + 2 = 0.4593 = \frac{3014}{6561} > 0$$

So the new interval is $[-1, -0.7037]$.

(c) Here $p_0 = -1$ and $p_1 = 0$, where $f(p_0) = f(-1) = -2$ and $f(p_1) = f(0) = 2$.

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 0 - \frac{f(0)(0+1)}{f(0) - f(-1)} = -\frac{2(1)}{2 - (-2)} = -\frac{2}{4} = -\frac{1}{2} = -0.5$$

$$f(p_2) = f(-0.5) = -0.375 - 0.25 + 2 = 1.375 = \frac{11}{8}$$

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} = -0.5 - \frac{f(-0.5)(-0.5-0)}{f(-0.5) - f(0)} = -0.5 - \frac{1.375(-0.5)}{1.375 - 2} = -1.6 = -\frac{8}{5}$$

$$f(p_3) = f(-1.6) = 3(-4.096) - 2.56 + 2 = -12.848$$

$$p_4 = p_3 - \frac{f(p_3)(p_3 - p_2)}{f(p_3) - f(p_2)} = -1.6 - \frac{f(-1.6)(-1.6+0.5)}{f(-1.6) - f(-0.5)} = -1.6 - \frac{-12.848(-1.1)}{-12.848 - 1.375} = -0.6063 = -\frac{784}{1293}$$

□

3. Consider the equation: $e^x + 2x - 2 = 0$

(a) (2 marks) Use the Matlab built-in function to find the root near 0.

(b) (2 marks) Use Matlab to perform 20 iterations of the bisection method with initial values $a = 0$ and $b = 1$ and tolerance 10^{-5}.

(c) (2 marks) Use Matlab to perform 20 iterations of the regula falsi method with initial values $a = 0$ and $b = 1$, tolerance 10^{-5} and epsilon= 10^{-7}.

(d) (2 marks) Use Matlab to perform 20 iterations of the secant method starting with the initial values $p_0 = 0$, and $p_1 = 1$, tolerance= 10^{-5} and epsilon= 10^{-7}.

(e) (2 marks) Use Matlab to perform 20 iterations of Newton's method with the initial approximation $p_0 = 0$, tolerance= 10^{-5} and epsilon= 10^{-7}.

(f) (2 marks) Based on your results from parts (b) - (e), which method is more successful. Explain your answer using the convergence rates.

Solution. See matlab sheets for solutions of all parts. □

4. Consider the equation: $\tan x = x$

(a) (3 marks) Use the Matlab built-in function to find the root near -2. Do you get the correct solution?

(b) (4 marks) Find the order of the root $x = 0$? Justify your answer.

(c) (5 marks) Using hand calculations, perform 2 iterations of Newton's method starting with the initial approximation $p_0 = -1$.

(d) (3 marks) Use Matlab to perform 30 iterations of Newton's method with the initial approximation $p_0 = -1$, tolerance$= 10^{-7}$ and epsilon$= 10^{-15}$. Is the convergence linear or quadratic?

(e) (4 marks) Using the method discussed in class modify Newton's iterations to accelerate the convergence. Repeat part (d) with the modified iterations using Matlab. Do you get faster convergence? (Provide program for modified Newton's method)

Solution. (a) See Matlab sheets for the solution.
(b) $f(x) = \tan x - x \Rightarrow f(0) = \tan 0 - 0 = 0.$
$f'(x) = \sec^2 x - 1 \Rightarrow f'(0) = \sec^2 0 - 1 = 1 - 1 = 0$
$f''(x) = 2\sec^2 x \tan x \Rightarrow f''(0) = 2\sec^2 0 \tan 0 = 0$
$f'''(x) = 2\sec^4 x + 4\sec^2 x \tan^2 x \Rightarrow f'''(0) = 2\sec^4 0 + 4\sec^2 0 \tan^2 0 = 2 \neq 0$

So $x = 0$ is root of order 3.

(c) Let $f(x) = \tan x - x$, then $f'(x) = \sec^2 x - 1$
The Newton's iterations are $p_{k+1} = p_k - \dfrac{f(p_k)}{f'(p_k)} = p_k - \dfrac{\tan p_k - p_k}{\sec^2 p_k - 1}$

Now $p_0 = -1$, and we need to use radians, so

$$p_1 = p_0 - \frac{\tan p_0 - p_0}{\sec^2 p_0 - 1} = -1 - \frac{\tan(-1) - (-1)}{\sec^2(-1) - 1} = -1 - \frac{-1.557 + 1}{3.426 - 1} = -0.7704$$
$$p_2 = p_1 - \frac{\tan p_1 - p_1}{\sec^2 p_1 - 1} = -0.7704 - \frac{\tan(-0.7704) - (-0.7704)}{\sec^2(-0.7704) - 1} = -0.5578$$

(d) See Matlab sheets for the solution.
(e) Since 0 is root of order 3, we use $M = 3$ for modified Newton method.
See Matlab sheets for the iterations using accelerated Newton's method. □

5. Consider the system of nonlinear equations

$$\begin{aligned} 2x + 2y &= 3 \\ 3x^2 + 2y &= 4 \end{aligned}$$

(a) (6 marks) Using hand calculations find the exact solutions.

(b) (4 marks) Using hand calculations, perform 2 iterations of Gauss-Seidel method starting with the initial values $x_0 = 0$ and $y_0 = 0$.

(c) (4 marks) Use Jacobi method with $x_0 = 0$ and $y_0 = 0$, tolerance$= 10^{-5}$, perform 10 iterations using Matlab. Does it converge? If yes, how many iterations does it take to converge? (Provide program for Jacobi method).

(d) (3 marks) Use Gauss-Seidel method with $x_0 = 0$ and $y_0 = 0$, tolerance$= 10^{-5}$, perform 10 iterations using Matlab. Does it converge? If yes, how many iterations does it take to converge?

(e) (6 marks) Explain the reason for the convergence/divergence in parts (c) and (d) using the conditions of convergence.

Solution. (a) We have to solve the system of equations

$$\begin{array}{rcrcr} 2x & + & 2y & = & 3 \\ 3x^2 & + & 2y & = & 4 \end{array}$$

Multiplying second equation by -1 and adding to the first equation, we have

$$\begin{array}{rcrcr} 2x & + & 2y & = & 3 \\ -3x^2 & - & 2y & = & -4 \\ \hline -3x^2 & + & 2x & = & -1 \end{array}$$

$$3x^2 - 2x - 1 = 0 \Rightarrow (3x+1)(x-1) = 0 \Rightarrow x = -\frac{1}{3}, 1$$

Substituting $x = -\frac{1}{3}$ into the first equation we have

$$2x + 2y = 3 \Rightarrow -\frac{2}{3} + 2y = 3 \Rightarrow 2y = \frac{11}{3} \Rightarrow y = \frac{11}{6}$$

Substituting $x = 1$ into the first equation we have

$$2x + 2y = 3 \Rightarrow 2 + 2y = 3 \Rightarrow 2y = 1 \Rightarrow y = \frac{1}{2}$$

Thus the solution is $(-\frac{1}{3}, \frac{11}{6})$ and $(1, \frac{1}{2})$.

(b) Solving the first equation for x and the second equation for y, Gauss-Seidel iterations are

$$\begin{aligned} x_{k+1} &= \frac{-2y_k + 3}{2} \\ y_{k+1} &= \frac{-3x_{k+1}^2 + 4}{2} \end{aligned}$$

Initial values are $x_0 = 0$ and $y_0 = 0$. Setting these values in the iterations, we obtain

$$\begin{aligned} x_1 &= \frac{0+3}{2} = \frac{3}{2} = 1.5 \\ y_1 &= \frac{-3(9/4)+4}{2} = -\frac{11}{8} = -1.375 \end{aligned}$$

The next iteration will give

$$\begin{aligned} x_2 &= \frac{-2(-1.375)+3}{2} = \frac{5.75}{2} = 2.875 \\ y_2 &= \frac{-3(2.875)^2+4}{2} = -\frac{20.796875}{2} = -10.3984375 \end{aligned}$$

(c) and (d) See Matlab sheets for the solution of parts (c) and (d).

(e) Solving the first equation for x and the second equation for y, we have

$$\begin{aligned} x &= \frac{-2y+3}{2} \quad \Rightarrow g_1(x,y) = \frac{-2y+3}{2} \\ y &= \frac{-3x^2+4}{2} \quad \Rightarrow g_2(x,y) = \frac{-3x^2+4}{2} \end{aligned}$$

$$\frac{\partial g_1}{\partial x} = 0, \quad \frac{\partial g_1}{\partial y} = -1, \quad \frac{\partial g_2}{\partial x} = -3x, \quad \frac{\partial g_2}{\partial y} = 0$$

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |0| + |-1| = 1$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = |-3x| + |0| = |-3x|$$

The value of $|-3x| \geq 1$ for both values of x, which are the solutions of the given system. The sufficient condition for the convergence is not satisfied and the iterations may converge or diverge. We have seen in parts (c) and (d) that the iterations are diverging. $\square$