MATH 3940 Numerical Analysis for Computer Scientists
Assignment 1 Solutions Fall 2020

1. Consider the following system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & -3 \\ x_1 & + & x_2 & - & x_3 & = & 0 \\ 3x_1 & - & x_2 & - & 9x_3 & = & 2 \end{array}$$

(a) (5 marks) Use hand calculations to solve the system using Gaussian elimination method with no pivoting.

(b) (7 marks) Use hand calculations to solve the system using Gaussian elimination method with partial pivoting.

Solution. (a) We will perform Gaussian elimination on the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & -3 \\ 1 & 1 & -1 & 0 \\ 3 & -1 & -9 & 2 \end{array}\right] \begin{array}{c} R_2 - R_1 \to R_2 \\ R_3 - 3R_1 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & 3 & -3 \\ 0 & -1 & -4 & 3 \\ 0 & -7 & -18 & 11 \end{array}\right]$$

$$R_3 - 7R_2 \to R_3 \left[\begin{array}{rrr|r} 1 & 2 & 3 & -3 \\ 0 & -1 & -4 & 3 \\ 0 & 0 & 10 & -10 \end{array}\right]$$

Using Gaussian elimination the matrix A is reduced to an upper triangular matrix. Now back substitution will be used to find the solution to the following system.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & + & 3x_3 & = & -3 \\ & - & x_2 & - & 4x_3 & = & 3 \\ & & & & 10x_3 & = & -10 \end{array}$$

From the last equation $x_3 = -1$. Setting $x_3 = -1$, the second equation gives $-x_2 - 4(-1) = 3 \Rightarrow -x_2 = 3 - 4 \Rightarrow -x_2 = -1 \Rightarrow x_2 = 1$.

Finally, substituting the value of x_2 and x_3 into the first equation we find $x_1 + 2(1) + 3(-1) = -3 \Rightarrow x_1 = -3 - 2 + 3 \Rightarrow x_1 = -2$.

Thus the solution is $(x_1, x_2, x_3) = (-2, 1, -1)$.

(b) We will perform Gaussian elimination on the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 2 & 3 & -3 \\ 1 & 1 & -1 & 0 \\ 3 & -1 & -9 & 2 \end{array}\right] R_3 \leftrightarrow R_1 \left[\begin{array}{rrr|r} 3 & -1 & -9 & 2 \\ 1 & 1 & -1 & 0 \\ 1 & 2 & 3 & -3 \end{array}\right]$$

$$\begin{array}{c} R_2 - \frac{1}{3}R_1 \to R_2 \\ R_3 - \frac{1}{3}R_1 \to R_3 \end{array} \left[\begin{array}{rrr|r} 3 & -1 & -9 & 2 \\ 0 & \frac{4}{3} & 2 & -\frac{2}{3} \\ 0 & \frac{7}{3} & 6 & -\frac{11}{3} \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \left[\begin{array}{ccc|c} 3 & -1 & -9 & 2 \\ 0 & \frac{7}{3} & 6 & -\frac{11}{3} \\ 0 & \frac{4}{3} & 2 & -\frac{2}{3} \end{array}\right] \quad R_3 - \frac{4}{7}R_2 \rightarrow R_3 \left[\begin{array}{ccc|c} 3 & -1 & -9 & 2 \\ 0 & \frac{7}{3} & 6 & -\frac{11}{3} \\ 0 & 0 & -\frac{10}{7} & \frac{10}{7} \end{array}\right]$$

Using Gaussian elimination we obtain the following system.

$$\begin{array}{rcrcrcr} 3x_1 & - & x_2 & - & 9x_3 & = & 2 \\ & & \frac{7}{3}x_2 & + & 6x_3 & = & -\frac{11}{3} \\ & & & - & \frac{10}{7}x_3 & = & \frac{10}{7} \end{array}$$

From the last equation, $x_3 = -1$. Setting $x_3 = -1$, the second equation gives

$$\frac{7}{3}x_2 + 6(-1) = -\frac{11}{3} \Rightarrow \frac{7}{3}x_2 = \frac{7}{3} \quad \Rightarrow \quad x_2 = 1$$

Finally, substituting the value of x_2 and x_3 into the first equation we obtain $3x_1 - 1 - 9(-1) = 2 \Rightarrow 3x_1 = 2 + 1 - 9 \Rightarrow x_1 = -2$.

Thus the solution is $(x_1, x_2, x_3) = (-2, 1, -1)$. □

2. Consider the system of linear equations

$$\begin{array}{rcrcrcrcrcr} 2x_1 & + & 3x_2 & + & x_3 & + & 2x_4 & + & x_5 & = & 9 \\ x_1 & - & 4x_2 & + & x_3 & & & - & 2x_5 & = & -4 \\ & & 5x_2 & + & 3x_3 & + & x_4 & + & x_5 & = & 16 \\ 3x_1 & - & 6x_2 & & & + & 4x_4 & + & 3x_5 & = & 14 \\ -x_1 & + & 2x_2 & + & 2x_3 & + & 5x_4 & + & x_5 & = & 5 \end{array}$$

(a) (2 marks) Use Matlab to find the determinant and the inverse of the coefficient matrix A.

(b) (2 marks) Use Matlab built in command to solve the system $AX = B$.

Solution. See Matlab Sheets. □

3. Consider the system of linear equations

$$\begin{array}{rcrcrcrcr} & & x_2 & + & 2x_3 & - & x_4 & = & -1 \\ x_1 & + & x_2 & - & x_3 & & & = & 5 \\ -x_1 & - & x_2 & + & x_3 & + & 3x_4 & = & 1 \\ x_1 & + & 2x_2 & & & + & x_4 & = & 9 \end{array}$$

(a) (10 marks) Use hand calculations to find the LU decomposition of the coefficient matrix A and then solve the resulting triangular system.

(b) (9 marks) Use Matlab to find the LU decomposition of the coefficient matrix A and then solve the resulting triangular system using forward and backward substitutions programs. (You need to provide program for forward substitution)

Solution. (a) Here we have

$$A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ 5 \\ 1 \\ 9 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $a_{11} = 0$, we have to interchange the first and the second row, which gives

$$R_2 \leftrightarrow R_1 \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ -1 & -1 & 1 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The multipliers are $m_{21} = 0$, $m_{31} = -1$, and $m_{41} = 1$.

$$\begin{matrix} R_3 + R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

The multipliers are $m_{32} = 0$ and $m_{42} = 1$.

Now $R_4 - R_2 \to R_4 \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

$$R_3 \leftrightarrow R_4 \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The matrix is reduced to an upper triangular matrix so we have

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Note that we have interchanged m_{31} and m_{41} and also m_{32} and m_{42} in L because R_3 and R_4 was interchanged.

First we have to solve $LY = PB$, so we will find PB

$$PB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 9 \\ 1 \end{bmatrix}$$

$$LY = PB \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 9 \\ 1 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} y_1 & & & & & & & = & 5 \\ & & y_2 & & & & & = & -1 \\ y_1 & + & y_2 & + & y_3 & & & = & 9 \\ -y_1 & & & & & + & y_4 & = & 1 \end{array}$$

First equation gives $y_1 = 5$ and the second equation gives $y_2 = -1$. The third equation gives $y_3 = 9 - 5 + 1 = 5$. The fourth equation gives $y_4 = 1 + 5 = 6$.

Now we solve $UX = Y$

$$UX = Y \Rightarrow \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & & & = & 5 \\ & & x_2 & + & 2x_3 & - & x_4 & = & -1 \\ & & & - & x_3 & + & 2x_4 & = & 5 \\ & & & & & & 3x_4 & = & 6 \end{array}$$

Last equation gives $x_4 = 2$. The third equation gives $x_3 = 4 - 5 = -1$. The second equation gives $x_2 = -1 + 2 + 2 = 3$. Finally the first equation gives $x_1 = 5 - 3 - 1 = 1$.

Therefore, the solution is $(x_1, x_2, x_3, x_4) = (1, 3, -1, 2)$.

(b) See Matlab sheets for solution of part(b). $\square$