Gauss-Seidel Iterative Method

To solve a linear system, we start with an initial guess (approximation). Find x_1 from the 1st equation, x_2 form the 2nd equation and so on. In Gauss-Seidel method the most recent calculated values of the elements $x_i\,'s$ are used.

Note: On Exam she will state the starting point.

Example 1: Consider the following system

$$\begin{aligned} 5x - y + z &= 10 \\ 2x + 8y - z &= 11 \\ -x + y + 4z &= 3 \end{aligned}$$

Starting with a zero vector, perform 3 iteration s of Gauss-Seidel method.

Solution:

$$1^{st} eq \Rightarrow x_{k+1} = \frac{10 + y_k - z_k}{5}$$

$$2^{nd} eq \Rightarrow y_{k+1} = \frac{11 - 2x_{k+1} + z_k}{8}$$

$$3^{rd} eq \Rightarrow z_{k+1} = \frac{3 + x_{k+1} - y_{k+1}}{4}$$

initial approximation is $(x_0, y_0, z_0) = (0, 0, 0)$

$$\left.\begin{aligned} x_1 &= \frac{10 + y_0 - z_0}{5} = \frac{10 + 0 - 0}{5} = 2 \\ y_1 &= \frac{11 - 2x_1 + z_0}{8} = \frac{11 - 2(2) + 0}{8} = \frac{7}{8} = 0.875 \\ z_1 &= \frac{3 + x_1 - y_1}{4} = \frac{3 + 2 - 0.875}{4} = 1.03125 \end{aligned}\right\} \text{1}^{\text{st}} \text{ iteration}$$

$$\left.\begin{aligned} x_2 &= \frac{10 + y_1 - z_1}{5} = \frac{10 + 0.875 - 1.03125}{5} = 1.96875 \\ y_2 &= \frac{11 - 2x_2 + z_1}{8} = \frac{11 - 2(1.96875) + 1.03125}{8} = 1.011719 \\ z_2 &= \frac{3 + x_2 - y_2}{4} = \frac{3 + 1.96875 - 1.011719}{4} = 0.989258 \end{aligned}\right\} \text{2}^{\text{nd}} \text{ iteration}$$

$$\left.\begin{aligned}
x_3 &= \frac{10 + y_2 - z_2}{5} = \frac{10 + 1.011719 - 0.989258}{5} = 2.0045 \\
y_3 &= \frac{11 - 2x_3 + z_2}{8} = \frac{11 - 2(2.0045) + 0.989258}{8} = 0.997535 \\
z_3 &= \frac{3 + x_3 - y_3}{4} = \frac{3 + 2.0045 - 0.997535}{4} = 1.001739
\end{aligned}\right\} \text{3}^{\text{rd}} \text{ iteration}$$

$(x_3, y_3, z_3) = (2.0045, 0.997535, 1.001739)$ **Ans**

Using MATLAB, it converges to (2, 1, 1) in 8 iterations with tolerance 10^{-5} (previously, we have seen that Jacobi converges to (2, 1, 1) in 12 iterations with tolerance 10^{-5})

Example 2: Consider the system

$$\begin{aligned} 2x + 8y - z &= 11 \\ 5x - y + z &= 10 \\ -x + y + 4z &= 3 \end{aligned}$$

Starting with a zero vector, perform 2 iterations of Gauss-Seidel method.

Solution:

$$1^{st} eq \Rightarrow \quad x_{k+1} = \frac{11 - 8y_k + z_k}{2}$$

$$2^{nd} eq \Rightarrow \quad y_{k+1} = \frac{10 - 5x_{k+1} - z_k}{-1}$$

$$3^{rd} eq \Rightarrow \quad z_{k+1} = \frac{3 + x_{k+1} - y_{k+1}}{4}$$

Starting with $(x_0, y_0, z_0) = (0, 0, 0)$

$$x_1 = \frac{11 - 8y_0 + z_0}{2} = \frac{11 + 0 - 0}{2} = 5.5$$

$$y_1 = \frac{10 - 5x_1 - z_0}{-1} = \frac{10 - 5(5.5) - 0}{-1} = 17.5$$

$$z_1 = \frac{3 + x_1 - y_1}{4} = \frac{3 + 5.5 - 17.5}{4} = -2.25$$

$$x_2 = \frac{11 - 8y_1 + z_1}{2} = \frac{11 - 8(17.5) + (-2.25)}{2} = -65.625$$

$$y_2 = \frac{10 - 5x_2 - z_1}{-1} = \frac{10 - 5(-65.625) - (-2.25)}{-1} = -340.375$$

$$z_2 = \frac{3 + x_2 - y_2}{4} = \frac{3 + (-65.625) - (-340.375)}{4} = 69.4375$$

$(x_2, y_2, z_2) = (-65.625, -340.375, 69.4375)$ **Ans**

Using MATLAB, it diverges. If we do 15 iterations, the answer is $(0.8675, 4.3783, -0.8778) \times 10^{19}$.

If A is strictly diagonally dominant, then Gauss-Seidel method for solving linear system converges. This condition is sufficient not necessary. If A is not strictly diagonally the Gauss-Seidel method may or may not converge.

For system in Example 1

$$A = \begin{bmatrix} 5 & -1 & 1 \\ 2 & 8 & -1 \\ -1 & 1 & 4 \end{bmatrix}$$

$|5| \quad > \quad |-1| + |1|$
$|8| \quad > \quad |2| + |-1|$
$|4| \quad > \quad |-1| + |1|$

A is strictly diagonally dominant
Gauss-Seidel will converge.

For system in Example 2

$$A = \begin{bmatrix} 2 & 8 & -1 \\ 5 & -1 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

$|2| \quad > \quad |8| + |1|$ NOT TRUE

A is not strictly diagonally dominant.
Gauss-Seidel may or may not converge.

Iterative methods are seldom used for solving linear system of equations with small dimension (N is small). For small N, direct methods are more efficient.

(Gaussian elimination
triangular factorization etc)

However, when solving partial differential equations, we need to solve system of equations that have a high percentage of zero entries (sparce matrices), the iterative methods are efficient in terms of both computer storage and computational time.

Textbook has program on page 164

function x = gausseid(A, B, P, tol, max ite) → max iterations

$10^{-5}, 10^{-6}, etc.$

change to $[x, k]$ initial approximation

iteration in out

Gauss-Seidel seems superior (faster to Jacobi) method, but this is not always true. Also, strictly diagonally dominant condition is sufficient not necessary.

Example 3: Consider the system

The solution is (1,1,1)

$$\begin{aligned} x + z &= 2 \\ -x + y &= 0 \\ x + 2y - 3z &= 0 \end{aligned}$$

Here $A = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 1 & 2 & -3 \end{bmatrix}$ $\qquad |1| > |0| + |1| \implies 1 > 1 \quad \mathit{NOT\ TRUE}$

⇒ A is not strictly diagonally dominant

Start with (0, 0, 0)

$$\boldsymbol{Jacobi} \Rightarrow \begin{aligned} x_{k+1} &= 2 - z_k \\ y_{k+1} &= x_k \\ z_{z+1} &= \tfrac{-x_k - 2y_k}{-3} \end{aligned}$$

$$\begin{aligned} x_1 &= 2 - 0 = 2 \\ y_1 &= 0 \\ z_1 &= 0 \end{aligned}$$

$$\begin{aligned} x_2 &= 2 - z_1 & &= 2 \\ y_2 &= x_1 & &= 2 \\ z_2 &= \tfrac{-x_1 - 2y_1}{-3} = \tfrac{-2-0}{-3} & &= \tfrac{2}{3} \end{aligned}$$

Using MATLAB, it converges to (1,1,1) in 203 iterations with tolerance 10^{-5}

$$\boldsymbol{Gauss \cdot Seidel\ method} \Rightarrow \begin{aligned} x_{k+1} &= 2 - z_k \\ y_{k+1} &= x_k + 1 \\ z_{z+1} &= \tfrac{-x_{k+1} - 2y_{k+1}}{-3} \end{aligned}$$

$$\begin{aligned} x_1 &= 2 - 0 = 2 \\ y_1 &= x_1 = 2 \\ z_1 &= \tfrac{-x_1 - 2y_1}{-3} = \tfrac{-2-2(2)}{-3} = +2 \end{aligned}$$

$$\begin{aligned} x_2 &= 2 - z_1 = 2 - 2 = 0 \\ y_2 &= x_2 = 0 \\ z_2 &= \tfrac{-x_1 - 2y_1}{-3} = 0 \end{aligned}$$

It will oscillate between (0,0,0) & (1,1,1) ⇒ diverges

Start with (0.5, 0.5, 0.5) then Jacobi converges to (1,1,1) in 190 iterations with tol 10^{-5}.
While Gauss-Seidel method diverges, it oscillates between (0.5, 0.5, 0.5)& (1.5, 1.5, 1.5).

Start with (0, 0.5, 0) → Gauss Seidel oscillates between (0, 0, 0) & (2, 2, 2)

diverges

11.1 Eigenvalues and Eigenvectors

Let A be an n x n matrix, then λ is an eigen value of A if there exists a nonzero vector $\vec{v}$ such that $A\vec{v} = \lambda\vec{v}$

$\vec{v}$ is the eigen vector corresponding to eigenvalue λ.

$$\vec{v} \neq \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

In linear algebra, the system $Ax = 0$ has a trivial solution $x = 0$ if $|A| \neq 0$
If $|A| = 0$ then the system has <u>infinite solutions</u>.

$$A\vec{v} = \lambda\vec{v} \Rightarrow (A - \lambda I)\vec{v} = 0$$

$$\vec{v} \neq 0 \text{ if } \underbrace{\det(A - \lambda I) = 0}_{\text{called characteristic equation}}$$

To find eigenvalues of A, we solve $|A - \lambda I| = 0$ for λ.
For each λ, we solve $(A - \lambda I)\vec{v} = 0$ using Gaussian elimination and get nonzero $\vec{v}$.

we will always have at least one row of zeros in reduced upper triangular matrix

Example: Find the $\underbrace{\text{eigenpairs}}_{\text{(Eigen values \& Eigen vectors)}}$ for the matrix

$$A = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix}$$

$$A - \lambda I = \begin{bmatrix} -4 & 1 & 1 \\ 1 & 5 & -1 \\ 0 & 1 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}$$
$$= \begin{bmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{bmatrix}$$

Solution:

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} -4-\lambda & 1 & 1 \\ 1 & 5-\lambda & -1 \\ 0 & 1 & -3-\lambda \end{vmatrix} = 0$$

Expanding along the 1st column

$$(-4-\lambda)\begin{vmatrix}5-\lambda & -1\\ 1 & -3-\lambda\end{vmatrix} - 1\begin{vmatrix}1 & 1\\ 1 & -3-\lambda\end{vmatrix} + 0\overset{0}{\begin{vmatrix}1 & 1\\ 5-\lambda & 1\end{vmatrix}} = 0$$

$$(-4-\lambda)[(5-\lambda)(-3-\lambda)-(-1)] - 1[(-3-\lambda)-1] = 0$$

$$(-4-\lambda)[-15-2\lambda+\lambda^2+1] - 1[-4-\lambda] = 0$$

$$(-4-\lambda)(\lambda^2-2\lambda-14) - 1(-4-\lambda) = 0$$

$$(-4-\lambda)(\lambda^2-2\lambda-14-1) = 0$$

$$(-4-\lambda)(\lambda^2-2\lambda-15) = 0$$

$$(-4-\lambda)(\lambda-5)(\lambda+3) = 0$$

$$\Rightarrow \underbrace{\lambda = -4, 5, -3}_{\textbf{Eigen values}}$$

To find Eigen vectors,

For $\lambda = 5$

$$(A-5I)\vec{v} = 0 \quad \Rightarrow \quad \begin{bmatrix}-4-5 & 1 & 1\\ 1 & 5-5 & -1\\ 0 & 1 & -3-5\end{bmatrix}\begin{bmatrix}v_1\\ v_2\\ v_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

Gaussian elimination $\Rightarrow \left[\begin{array}{ccc|c}-9 & 1 & 1 & 0\\ 1 & 0 & -1 & 0\\ 0 & 1 & -8 & 0\end{array}\right] R_2 - \left(-\frac{1}{9}\right)R_1 \to R_2 \left[\begin{array}{ccc|c}-9 & 1 & 1 & 0\\ 0 & \frac{1}{9} & -\frac{8}{9} & 0\\ 0 & 1 & -8 & 0\end{array}\right]$

$$R_3 - 9R_2 \to R_3 \left[\begin{array}{ccc|c}-9 & 1 & 1 & 0\\ 0 & \frac{1}{9} & -\frac{8}{9} & 0\\ \boxed{0} & \boxed{0} & \boxed{0} & \boxed{0}\end{array}\right]$$

$\boxed{\Rightarrow \text{infinite solution}}$

$a_{3\,3} = 0 \Rightarrow v_3$ is an arbitrary real number.

$$\begin{aligned} -9v_1 + v_2 + v_3 &= 0\\ \frac{1}{9}v_2 + \frac{8}{9}v_3 &= 0\\ v_3 &\in \mathbb{R}\end{aligned}$$

If $\vec{v}$ is an eigen vector for λ then $c\vec{v}$ is also an eigen vector for λ.

$Let\ v_3 = 1 \Rightarrow \frac{1}{9}v_2 = \frac{+8}{9} \Rightarrow v_2 = +8$

$-9V_1 + (+8) + (1) = 0 \Rightarrow v_1 = 1$

$\Rightarrow$Eigen vector for $\lambda = 5$ is $\vec{v} = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$

For $\lambda = -3 \Rightarrow (A - \lambda I)\vec{v} = 0$

$$(A - \lambda I)\vec{v} = 0 \Rightarrow \begin{bmatrix} -4+3 & 1 & 1 \\ 1 & 5+3 & -1 \\ 0 & 1 & -3+3 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 1 & 8 & -1 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] R_2 + R_1 \to R_2 \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{array}\right] R_3 - \frac{1}{9}R_2 \to R_3 \left[\begin{array}{ccc|c} -1 & 1 & 1 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$-v_1 + v_2 + v_3 = 0 - (*)$

$9v_2 = 0 \Rightarrow v_2 = 0$

$v_3 \in \mathbb{R}$

Let $v_3 = 1 \Rightarrow (*) - v_1 + 0 + 1 = 0 \Rightarrow v_1 = 1 \Rightarrow egein\ vector\ \lambda = -3\ is\ \vec{v} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

For $\lambda = -4$

$$\left[\begin{array}{ccc|c} 0 & 1 & 1 & 0 \\ 1 & 9 & -1 & 0 \\ 0 & 1 & -0 & 0 \end{array}\right] R_2 \leftrightarrow R_1 \left[\begin{array}{ccc|c} 1 & 9 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] R_3 - R_2 \to R_3 \left[\begin{array}{ccc|c} 1 & 9 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$v_1 + 9v_2 - v_3 = 0 \Rightarrow v_1 - 9 - 1 = 0 \Rightarrow v_1 = 10$

$v_2 + v_3 = 0 \quad Let\ v_3 = 1 \quad v_2 = -v_3 = -1$

$v_3 \in \mathbb{R}$

$\Rightarrow$Eigen vector for $\lambda = -4\ is\ \vec{v} = \begin{bmatrix} 10 \\ -1 \\ 1 \end{bmatrix}$