MATH 3940 Numerical Analysis for Computer Scientists
Assignment 1 Solutions Fall 2021

1. Consider the following system

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & 5x_3 & = & -9 \\ & & x_2 & - & 3x_3 & = & 7 \\ 2x_1 & - & 2x_2 & + & 3x_3 & = & 0 \end{array}$$

(a) (5 marks) Use hand calculations to solve the system using Gaussian elimination method with no pivoting.

(b) (7 marks) Use hand calculations to solve the system using Gaussian elimination method with partial pivoting.

***Solution*.** (a) We will perform Gaussian elimination on the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & -3 & 5 & -9 \\ 0 & 1 & -3 & 7 \\ 2 & -2 & 3 & 0 \end{array}\right] \; R_3 - 2R_1 \to R_3 \; \left[\begin{array}{rrr|r} 1 & -3 & 5 & -9 \\ 0 & 1 & -3 & 7 \\ 0 & 4 & -7 & 18 \end{array}\right]$$

$$R_3 - 4R_2 \to R_3 \left[\begin{array}{rrr|r} 1 & -3 & 5 & -9 \\ 0 & 1 & -3 & 7 \\ 0 & 0 & 5 & -10 \end{array}\right]$$

Using Gaussian elimination the matrix A is reduced to an upper triangular matrix. Now back substitution will be used to find the solution to the following system.

$$\begin{array}{rcrcrcr} x_1 & - & 3x_2 & + & 5x_3 & = & -9 \\ & & x_2 & - & 3x_3 & = & 7 \\ & & & - & 5x_3 & = & -10 \end{array}$$

Last equation gives $x_3 = -2$. Putting $x_3 = -2$ into the second equation we have $x_2 - 3(-2) = 7 \;\Rightarrow\; x_2 = 7 - 6 = 1$.
Finally, substituting the value of x_2 and x_3 into the first equation we find $x_1 - 3(1) + 5(-2) = -9 \;\Rightarrow\; x_1 = -9 + 3 + 10 \;\Rightarrow\; x_1 = 4$.
Thus the solution is $(x_1, x_2, x_3) = (4, 1, -2)$.

(b) We will perform Gaussian elimination on the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & -3 & 5 & -9 \\ 0 & 1 & -3 & 7 \\ 2 & -2 & 3 & 0 \end{array}\right] \; R_3 \leftrightarrow R_1 \; \left[\begin{array}{rrr|r} 2 & -2 & 3 & 0 \\ 0 & 1 & -3 & 7 \\ 1 & -3 & 5 & -9 \end{array}\right]$$

$$R_3 - \frac{1}{2}R_1 \to R_3 \; \left[\begin{array}{rrr|r} 2 & -2 & 3 & 0 \\ 0 & 1 & -3 & 7 \\ 0 & -2 & \frac{7}{2} & -9 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \; \left[\begin{array}{rrr|r} 2 & -2 & 3 & 0 \\ 0 & -2 & \frac{7}{2} & -9 \\ 0 & 1 & -3 & 7 \end{array}\right] \; R_3 + \frac{1}{2}R_2 \to R_3 \; \left[\begin{array}{rrr|r} 2 & -2 & 3 & 0 \\ 0 & -2 & \frac{7}{2} & -9 \\ 0 & 0 & -\frac{5}{4} & \frac{5}{2} \end{array}\right]$$

Now we will use back substitution to find the solution to the following system.

$$\begin{array}{rcrcrcr} 2x_1 & - & 2x_2 & + & 3x_3 & = & 0 \\ & - & 2x_2 & + & \frac{7}{2}x_3 & = & -9 \\ & & & - & \frac{5}{4}x_3 & = & \frac{5}{2} \end{array}$$

The third equation gives $-\frac{5}{4}x_3 = \frac{5}{2} \Rightarrow x_3 = -2$. Putting $x_3 = -2$ into the second equation we obtain $-2x_2 + \frac{7}{2}(-2) = -9 \Rightarrow -2x_2 = -2 \Rightarrow x_2 = 1$.
Finally, substituting the value of x_2 and x_3 into the first equation we obtain $2x_1 - 2(1) + 3(-2) = 0 \Rightarrow 2x_1 = 2 + 6 \Rightarrow x_1 = 4$.
Thus the solution is $(x_1, x_2, x_3) = (4, 1, -2)$. □

2. Consider the system of linear equations

$$\begin{array}{rcrcrcrcrcr} -x_1 & + & 2x_2 & + & 2x_3 & + & 5x_4 & + & x_5 & = & 7 \\ & & 3x_2 & + & x_3 & + & 2x_4 & + & x_5 & = & 5 \\ x_1 & - & 4x_2 & + & x_3 & & & - & 2x_5 & = & 9 \\ & & 5x_2 & + & 3x_3 & + & x_4 & + & x_5 & = & 2 \\ 3x_1 & - & 6x_2 & & & + & 4x_4 & + & 3x_5 & = & -1 \end{array}$$

(a) (2 marks) Use Matlab to find the determinant and the inverse of the coefficient matrix A.

(b) (2 marks) Use Matlab built in command (mentioned during lectures) to solve the linear system $AX = B$

Solution. See Matlab Sheets for solutions. □

3. Consider the system of linear equations

$$\begin{array}{rcrcrcrcr} & & & - & x_3 & + & x_4 & = & -1 \\ x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & -1 \\ x_1 & + & x_2 & & & + & 3x_4 & = & 2 \\ x_1 & + & 2x_2 & - & x_3 & + & 3x_4 & = & 1 \end{array}$$

(a) (11 marks) Use hand calculations to find the LU decomposition of the coefficient matrix A and then solve the resulting triangular system.
(b) (8 marks) Use Matlab to find the LU decomposition of the coefficient matrix A and then solve the resulting triangular system using forward and backward substitutions programs. (You need to provide program for forward substitution)

Solution. (a) Here we have

$$A = \begin{bmatrix} 0 & 0 & -1 & 1 \\ 1 & 1 & -1 & 2 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $a_{11} = 0$, we have to interchange the first and the second row, which gives

$$R_2 \leftrightarrow R_1 \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & 3 \\ 1 & 2 & -1 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The multipliers are $m_{21} = 0,\ m_{31} = 1,\ m_{41} = 1$.

$$\begin{matrix} R_3 - R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_4 \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

The multipliers are $m_{32} = 0,\ m_{42} = 0$, no elimination is required. Next $m_{43} = -1$.

$$R_4 + R_3 \to R_4 \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Thus we have

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$$

Note that we have interchanged m_{21} and m_{41} in L because R_2 and R_4 were interchanged.

$$PB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

$$LY = PB \ \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} y_1 & & & & & & & = & -1 \\ y_1 & + & y_2 & & & & & = & 1 \\ y_1 & & & + & y_3 & & & = & 2 \\ & & & - & y_3 & + & y_4 & = & -1 \end{array}$$

First equation gives $y_1 = -1$. The second equation gives $y_2 = 1 + 1 = 2$. The third equation gives $y_3 = 2 + 1 = 3$. The fourth equation gives $y_4 = -1 + 3 = 2$. Now we have to find x by solving

$$UX = Y \;\Rightarrow\; \begin{bmatrix} 1 & 1 & -1 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{array}{ccccccccc} x_1 & + & x_2 & - & x_3 & + & 2x_4 & = & -1 \\ & & x_2 & & & + & x_4 & = & 2 \\ & & & & x_3 & + & x_4 & = & 3 \\ & & & & & & 2x_4 & = & 2 \end{array}$$

Last equation gives $x_4 = 1$. The third equation gives $x_3 = 3 - 1 = 2$ The second equation gives $x_2 = 2 - 1 = 1$. Finally, substituting the value of x_2, x_3 and x_4 into the first equation, we have $x_1 = -1 - 1 + 2 - 2 = -2$.
Therefore, the solution is $(x_1, x_2, x_3, x_4) = (-2, 1, 2, 1)$.

(b) See Matlab sheets for solution of part(b). $\square$