Shifted Inverse Power Method

We have seen that power method find the dominant eigen values and eigen vector. We can do some modification to power method so that it has faster convergence and also, we can find all the eigen values and eigen vectors of the matrix by choosing an initial proximation α. (It will converge to an eigen value close to α).

If λ is an eigen value of A & $\vec{V}$ is the corresponding eigen vector, then
$(\lambda - \alpha)$ is an eigen value of $(A - \alpha I)$ & $\vec{V}$ is the corresponding eigen vector.
$\frac{1}{\lambda}$ is an eigen value of A^{-1} & $\vec{V}$ is the corresponding eigen vector.
$\frac{1}{\lambda-\alpha}$ is an eigen value of $(A - \alpha I)^{-1}$ & $\vec{V}$ is the corresponding eigen vector.

How are the iterations are performed?

We start with an initial approximation $X_0 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}$ (X_0 can be chosen other vectors)

and a value of α. Then the iterations are

$Y_k = (A - \alpha I)^{-1} X_k$ $\longrightarrow$ In programs we solve the system $(A - \alpha I)Y_k = X_k$

& $X_{K+1} = \frac{1}{C_{k+1}} Y_k$

where C_{k+1} is the coordinate of Y_k of largest magnitude (we compare the magnitude but C_{k+1} is not the absolute value, it is the actual coordinate of Y_k. In the case of a tie, choose the coordinate that comes first).
$X_k \longrightarrow \vec{V}$ the eigen vector & $C_k \rightarrow \mu = \frac{1}{\lambda-\alpha}$

$$\Rightarrow \lambda - \alpha = \frac{1}{\mu} \Rightarrow \lambda = \frac{1}{\mu} + \alpha$$

λ is the eigen value of A

Textbook has program on page 608

$function[lambda,\ V] = invpower(A, X, alpha, tol, maxite)$

⋮

$[m,\ j] = \max\,(abs(y))$

~~c1 = m~~ not true

correct is c1 = y(j);

Example 1:

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{bmatrix}$

a) Find all the eigen values and eigen vectors of A.

b) Use MATLAB to find eigen values and eigen vectors of A starting with $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and shifted inverse power method (use $\alpha = 1, 2.5, -2.5, -1.5, 0$)

Solution:

A is an upper triangular matrix so the diagonal entries are the eigen values.

$So\ \lambda = 1, 2, -3$

For $\boldsymbol{\lambda} = 1$ $\begin{bmatrix} 0 & -1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$-x_2 + x_3 = 0 \Rightarrow x_2 = x_3 = 0$$

$$x_2 - x_3 = 0 \Rightarrow x_2 = x_3 = 0 \Rightarrow \vec{V} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$-4x_3 = 0 \Rightarrow x_3 = 0$$

$a_{1\,1} = 0 \Rightarrow X_1$ is arbitrary $\Rightarrow$ Let $x_1 = 1$

For $\boldsymbol{\lambda} = 2$ $\begin{bmatrix} -1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & -5 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$-x_1 - x_2 + x_3 = 0 - (*)$$

$$\left.\begin{matrix} -x_3 = 0 \\ -5x_3 = 0 \end{matrix}\right\} \Rightarrow x_3 = 0$$

$a_{2\,2} = 0 \Rightarrow X_2 \in \mathbb{R}$

Let $x_2 = 1 \overset{(*)}{\Rightarrow} -x_1 - 1 + 0 = 0 \Rightarrow -x_1 = 1 \Rightarrow x_1 = -1$

So $\overrightarrow{V_2} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

For $\boldsymbol{\lambda} = -3$ $\begin{bmatrix} 4 & -1 & 1 \\ 0 & 5 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$$\overrightarrow{V_3} = \begin{bmatrix} -\frac{1}{5} \\ \frac{1}{5} \\ 1 \end{bmatrix} or \begin{bmatrix} -0.2 \\ 0.2 \\ 1 \end{bmatrix}$$

$a_{3\,3} = 0 \Rightarrow X_3 \in \mathbb{R}$ Let $x_3 = 1$

$$4x_1 - x_2 + x_3 = 0 - (1)$$

$$5x_2 - x_3 = 0 \Rightarrow 5x_2 = 1 \Rightarrow x_2 = \frac{1}{5}$$

$$(1) \Rightarrow 4x_1 - \frac{1}{5} + 1 = 0 \Rightarrow 4x_1 = -1 + \frac{1}{5} = -\frac{4}{5} \Rightarrow x_1 = -\frac{1}{5}$$

b) Using MATLAB starting with $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\alpha = 1 \longrightarrow$ error (because $\lambda = 1$ is an eigen value)

$\alpha = 2.5$ converges to $\lambda = 2$ & $\vec{V} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ in 9 iterations with tol 10^{-5}

$\alpha = -2.5$ converges to $\lambda = 3$ & $\vec{V} = \begin{bmatrix} -0.2 \\ 0.2 \\ 1 \end{bmatrix}$ in 5 iterations with tol 10^{-5}

$\alpha = -1.5$ converges to $\lambda = -3$ & $\vec{V} = \begin{bmatrix} -0.2 \\ 0.2 \\ 1 \end{bmatrix}$ in 20 iterations with tol 10^{-5}

$\alpha = 0$ converges to $\lambda = 1$ & $\vec{V} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in 13 iterations with tol 10^{-5}

Example 2:

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}$ $and\ X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

eigen values are $\lambda = 1, 3, -3$

$\lambda = 1 \rightarrow$ eigen vector is $\vec{V} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

from Example 3 of power method

$\lambda = 3 \rightarrow$ eigen vector is $\vec{V} = \begin{bmatrix} -0.5 \\ 1 \\ 0 \end{bmatrix}$

$\lambda = -3 \rightarrow$ eigen vector is $\vec{V} = \begin{bmatrix} -0.2083 \\ 0.1667 \\ 1 \end{bmatrix}$

Shifted power inverse method starting with X_0
(Use $\alpha = 2.5, -3.5, 0$)

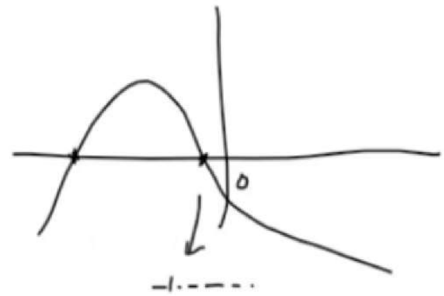
$\alpha = 2.5$ converges to $\lambda = 3$ & $\vec{V} = \begin{bmatrix} -0.5 \\ 1.0 \\ 0.0 \end{bmatrix}$ in 12 iterations with tol 10^{-5}

$\alpha = -3.5$ converges to $\lambda = -3$ & $\vec{V} = \begin{bmatrix} -0.2083 \\ 0.1667 \\ 1.0000 \end{bmatrix}$ in 13 iterations with tol 10^{-5}

$\alpha = 0$ converges to $\lambda = 1$ & $\vec{V} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ in 9 iterations with tol 10^{-5}

Ch2 Solution of Non-Linear Equations f(x)=0

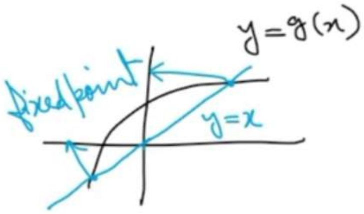
In this chapter we will learn many iterative methods to find numerical approximation of the solution of the eq. $f(x) = 0$.
Graphically the solution of $f(x) = 0$ are x-intercepts of the graph.
These graphs are goo to give an idea of initial approximation.



Fixed Point Method

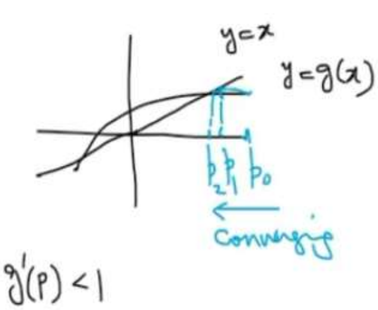
A fixed point for a function is a number at which the value of the function does not change when the function is applied. That is, the number p is a fixed point of g if $g(p) = p$
$f(x) = 0$, we can change to $x = g(x)$

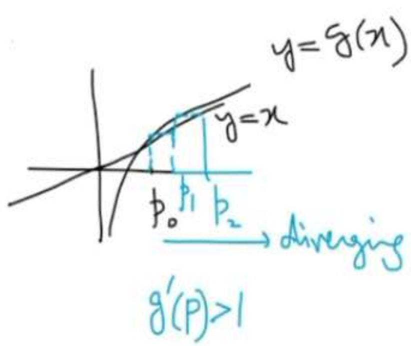
If we are given $x = g(x)$, we can change it to? $\overbrace{x - g(x)}^{f(x)} = 0$ or $\overbrace{g(x) - x}^{f(x)} = 0$
Graphically, the fixed points of g are the points of intersection of the graphs of $y = g(x)$ & $y = x$

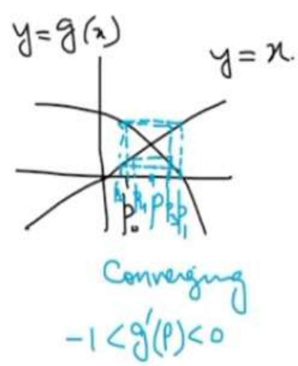


We start with an initial approximation P_0 & the iterations are
$P_{(k+1)} = g(P_k)$ for $k = 0, 1, 2, \dots$
If the method converges then $P_k \to P$ the solution of the equation.

$\downarrow$
P







Theorem 2.3

g & g' are continuous on [a, b]

Let $\underbrace{g, g' \in C[a,b]}$, $P_0 \in (a,b)$ and $g(x) \in [a,b]$

i. **If $|g'(x)| \le k < 1$ for all $x \in [a,b]$ then fixed-point method converges to a unique fixed point.**

ii. **If $|g'(x)| > 1$ for all $x \in [a,b]$ then fixed-point iterations will not converge**.

Note: If P is a solution (a fixed point), we can use if $|g'(P)| < 1$ then it converges & if

of $x = g(x)$

$|g'(P)| > 1$ then it diverges.

Example 1:

Let $g(x) = 1 + \frac{2}{x}$

a) Using hand calculations, solve $x = g(x)$

$$x = g(x) \Rightarrow x = 1 + \frac{2}{x} \Rightarrow x^2 = x + 2 \Rightarrow x^2 - x - 2 = 0 \Rightarrow (x-2)(x+1) = 0 \Rightarrow \underbrace{x = 2, -1}$$

$$\boxed{\begin{gathered} g(2) = 1 + \frac{2}{2} = 2 \\ g(-1) = 1 + \frac{2}{-1} = 1 - 2 = -1 \end{gathered}}$$

b) Using hand calculations, perform 3 iterations of the fixed-point method starting with initial approximation $P_0 = 4$

$$P_{k+1} = g(P_k)$$

$$P_1 = g(4) = 1 + \frac{2}{4} = 1 + \frac{1}{2} = \frac{3}{2} = 1.5$$

$$P_2 = g(1.5) = 1 + \frac{2}{\frac{3}{2}} = 1 + \frac{4}{3} = \frac{7}{3} = 2.333$$

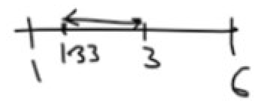
$P_3 = g(2.333) = 1 + \frac{2}{2.333} = 1.857 \longrightarrow$ 3rd iteration

c) Do you expect fixed point method to converge with an initial approximation $p_0 = 4$? Justify your answer using the conditions of convergence.

We can choose the interval $[1, 6]$ $\quad P_0 = 4 \in [1,6]$ & $2 \in [1,6]$

$g(x) = 1 + \frac{2}{x}$ & $g'(x) = -\frac{2}{x^2}$ are continuous on [1, 6]

$g(1) = 1 + \frac{2}{1} = 3$



$g(6) = 1 + \frac{2}{6} = \frac{4}{3} = 1.333$

$g'(x) = -\frac{2}{x^2} < 0$ for all $x \in [1,6]$ so $g(x)$is decreasing on [1, 6] $\Rightarrow g(x)[1.33,3]$

$\Rightarrow g(x) \in [1,6]$ for all $x \in [1,6]$

Solution is 2 & $|g'(2)| = \left|\frac{-2}{(2)^2}\right| = \left|\frac{1}{2}\right| < 1 \Rightarrow$ the fixed-point iterations will converge.

Using MATLAB fixed point converges to 2 in 19 iterations with tol 10^{-5}

$|g'(-1)| = \left|-\frac{2}{(-1)^2}\right| = 2 > 1$ So, it will not converge to -1.

Textbook has program on page 49.

$function\ [k, p, err, P] = fixpt(g, P_0, tol, maxi)$

You need to have a g.m file

$function \quad y = g(x)$

$y = 1 + \frac{2}{x};$

In MATLAB

$[\qquad] = fixpt('g', 4, 10^{-5}, 20)$

octave ‘g’ or “g”

To draw the graph in MATLAB

\>> x = -3 : +0.1 : 5

initial increment final

\>> y = exp(x);

\>> plot(x, y)

\>> title(‘exponential function’)

\>> xlabel(‘x’)

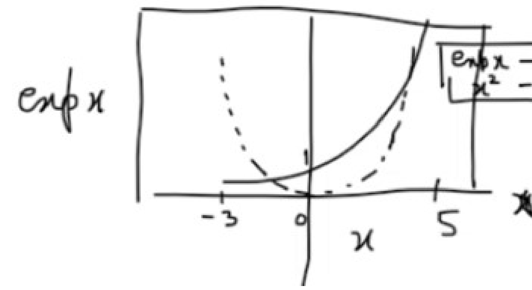
\>> ylabel(‘exp x’)

\>> z = x.^2; or z = x.*x;

\>> plot(x, y, ‘ –‘, x, z, ‘.’)

or colour r, b, g etc.

\>>legend(‘emp x’, ‘x^2’)



exponential function

$$y = e^x$$

$$y = x^2$$

$x = [-1\ 2\ 3]$

$x * x =$ can not be multiplied

$x .* x = [(-1)^2 (2)^2 (3)^2]$

multiplication of elements of x with corresponding elements of x.

(a.*b, a./b, a.^b)