MATH 3940-1 Numerical Analysis for Computer Scientists

Problem Set 5 Solutions

Note: You can use Octave or Matlab for the questions that says to use Matlab.

1. Consider the data

x_i	1	2	3	4	5
y_i	1.4	2	2.3	3	3.4

Find the least-squares line $f(x) = a_1 x + a_0$ for the data and calculate the error.

Solution.

	x_i	y_i	x_i^2	$x_i y_i$
	1	1.4	1	1.4
	2	2.0	4	4.0
	3	2.3	9	6.9
	4	3.0	16	12.0
	5	3.4	25	17.0
Sum	15	12.1	55	41.3

The normal equations are

$$\begin{array}{rcrcl} 5a_0 & + & 15a_1 & = & 12.1 \\ 15a_0 & + & 55a_1 & = & 41.3 \end{array}$$

Multiplying first equation by -3 and adding to the second equation, we have

$$\begin{array}{rcrcl} -15a_0 & - & 45a_1 & = & 36.3 \\ 15a_0 & + & 55a_1 & = & 41.3 \\ \hline & & 10a_1 & = & 5 \end{array}$$

Thus $a_1 = \dfrac{5}{10} = 0.5$. Substituting $a_1 = 0.5$ into the first equation we obtain

$$5a_0 + 15a_1 = 12.1 \Rightarrow 5a_0 + 15(0.5) = 12.1 \Rightarrow 5a_0 = 12.1 - 7.5 = 4.6 \Rightarrow a_0 = \frac{4.6}{5} = 0.92$$

Therefore the least-squares line is $f(x) = a_1 x + a_0 = 0.5x + 0.92$.

The error $E = \displaystyle\sum_{i=1}^{m} (y_i - f(x_i))^2 = \sum_{i=1}^{5} (y_i - (0.5x_i + 0.92))^2$

Substituting the values from the data

$$\begin{aligned} E &= (1.4 - 1.42)^2 + (2 - 1.92)^2 + (2.3 - 2.42)^2 + (3 - 2.92)^2 + (3.4 - 3.42)^2 \\ &= (-0.02)^2 + (0.08)^2 + (-0.12)^2 + (0.08)^2 + (-0.02)^2 = 0.028 \end{aligned}$$

□

2. Consider the data

x_i	-1	0	1	2	3
y_i	6.62	3.94	2.17	1.35	0.89

(a) Find the least-squares approximation $y = be^{ax}$ by data linearization method (the change of variable method) and calculate the error.

(b) Find the least-squares approximation $y = be^{ax}$ by nonlinear method using Matlab (Hint: Use Matlab built-in function to minimize error).

Solution. (a) $y = be^{ax} \Rightarrow \ln y = \ln b + \ln e^{ax} \Rightarrow \ln y = \ln b + ax$.

The change of variables are: $X = x$, $Y = \ln y$, and $a_0 = \ln b$ (this means $b = e^{a_0}$).

	x_i	y_i	$X_i = x_i$	$Y_i = \ln y_i$	X_i^2	X_iY_i
	-1	6.62	-1	1.890095	1	-1.890095
	0	3.94	0	1.371181	0	0
	1	2.17	1	0.774727	1	0.774727
	2	1.35	2	0.300105	4	0.60021
	3	0.89	3	-0.116533	9	-0.349599
Sum			5	4.219574	15	-0.864757

The normal equations are

$$\begin{array}{rcrcr} 5a_0 & + & 5a_1 & = & 4.219574 \\ 5a_0 & + & 15a_1 & = & -0.864757 \end{array}$$

Subtracting second equation from the first equation, we have

$$-10a_1 = 5.084327 \Rightarrow a_1 = \frac{5.084327}{-10} = -0.50843$$

Substituting $a_1 = -0.50843$ into the first equation we obtain

$5a_0 + 5(-0.50843) = 4.219574 \Rightarrow 5a_0 = 6.761738 \Rightarrow a_0 = 1.35235$

Thus $b = e^{a_0} = e^{1.35235} = 3.8665$.

Therefore the least-squares approximation is $y = be^{ax} = 3.8665e^{-0.50843x}$.

The error is $E = \sum_{i=1}^{m}(y_i - f(x_i))^2 = \sum_{i=1}^{5}(y_i - 3.8665e^{-0.50843x_i})^2$

Substituting the values from the data

$$\begin{aligned} E &= (6.62 - 6.4287)^2 + (3.94 - 3.8665)^2 + (2.17 - 2.3255)^2 + (1.35 - 1.3986)^2 + (0.89 - 0.8412)^2 \\ &= 0.03658 + 0.00540 + 0.02417 + 0.00236 + 0.00238 = 0.07089 \end{aligned}$$

(b) See Matlab sheets for the solution. $\square$

3. Consider the data

x_i	1	3	5
y_i	0.465	0.202	0.129

Find the least-squares approximation $y = \dfrac{1}{a_1x + a_0}$ by data linearization method (the change of variable method).

Solution. $y = \dfrac{1}{a_1x + a_0} \;\Rightarrow\; \dfrac{1}{y} = a_1x + a_0.$

Thus the change of variables will be $X = x$ and $Y = \dfrac{1}{y}$.

	x_i	y_i	$X_i = x_i$	$Y_i = \frac{1}{y_i}$	X_i^2	X_iY_i
	1	0.465	1	2.1505	1	2.1505
	3	0.202	3	4.9505	9	14.8515
	5	0.129	5	7.7519	25	38.7595
Sum			9	14.8529	35	55.7615

The normal equations are

$$\begin{array}{rcrcl} 3a_0 & + & 9a_1 & = & 14.8529 \\ 9a_0 & + & 35a_1 & = & 55.7615 \end{array}$$

Multiplying first equation by -3 and adding to the second equation, we have

$$\begin{array}{rcrcr} -9a_0 & - & 27a_1 & = & -44.5587 \\ 9a_0 & + & 35a_1 & = & 55.7615 \\ \hline & & 8a_1 & = & 11.2028 \end{array}$$

Thus $a_1 = \dfrac{11.2028}{8} = 1.40035.$

Substituting $a_1 = 1.40035$ into the first equation we obtain

$3a_0+9(1.40035) = 14.8529 \Rightarrow\; 3a_0 = 14.8529-12.60315 = 2.24975 \;\Rightarrow\; a_0 = 0.7499$

Therefore the least-squares approximation is $y = f(x) = \dfrac{1}{a_1x + a_0} = \dfrac{1}{1.40035x + 0.7499}$. □

4. Suppose you have to find the least-squares approximation $y = \dfrac{x}{a_1 + a_0x}$ by data linearization method, what would be the change of variable formulas?

Solution. $y = \dfrac{x}{a_1 + a_0x} \;\Rightarrow\; \dfrac{1}{y} = \dfrac{a_1 + a_0x}{x} = \dfrac{a_1}{x} + \dfrac{a_0x}{x} = \dfrac{a_1}{x} + a_0$

The change of variables will be $X = \dfrac{1}{x}$ and $Y = \dfrac{1}{y}$. □