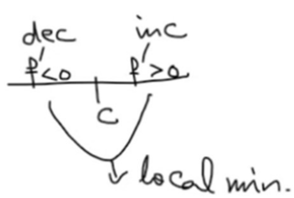
Minimization of a Function of One Variable

A function of f has a local min. value at $x = c$ if $f(c) \leq f(x)$ for all x near c.
A function f has a local max value at $x = c$ if $f(c) \geq f(x)$ for x near c.

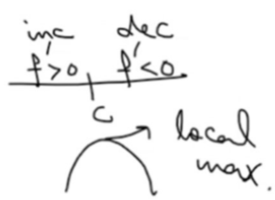
If $f'(x) > 0$ on I then f is increasing on I.
If $f'(x) < 0$ on I then f is decreasing on I.
The critical numbers are c such that $f'(c) = 0$ or $f'(c)$ does not exists. The critical numbers are possible local max & min.

1st derivative test for local max/min: Suppose c is a critical number in the domain of f.

(i) If $f'(x)$ changes sign from -ve to +ve at $x = c$ then f has local min at $x = c$



(ii) If $f'(x)$ changes sign from +ve to -ve at $x = c$ then f has local max at $x = c$



(iii) If f' does not change sign at $x = c$ then f has neither local max nor local min at $x = c$

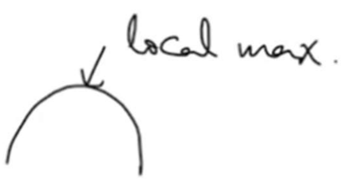
Second derivative test for local max/min: Suppose c is a critical number in the domain of f.

(i) If $f''(c) > 0$ then f has a local min at $x = c$

$f''(x) > 0 \Rightarrow$
Concave upward.

(ii) If $f''(c) > 0$ then f has a local max at $x = c$

Concave downward

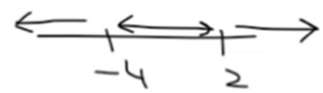


(iii) If $f''(c) = 0$ the test gives no information.

Example 1:
Let $f(x) = x^3 + 3x^2 - 24x$ Using hand calculation, find the local min value of f.

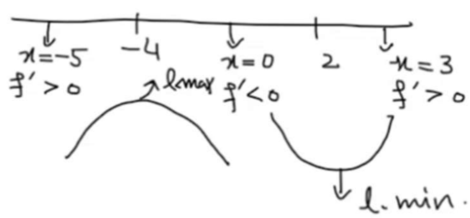
Solution:
$f'(x) = 0 \Rightarrow 3x^2 + 6x - 24 = 0 \Rightarrow x^2 + 2x - 8 = 0 \Rightarrow (x+4)(x-2) = 0 \Rightarrow x = -4, 2$

So, the critical numbers are -4 & 2.



Intervals	Test value	$f'(x) = 3(x^2 + 2x - 8)$	increase/decrease
$(-\infty, -4)$	$x = -5$	$3((-5)^2 + 2(-5) - 8)$ $= 3(25 - 10 - 8)$ $= 21 > 0$	increasing on $(-\infty, -4)$
$(-4,2)$	$x = 0$	$3(0 + 0 - 8)$ $= -24 < 0$	decreasing on $(-4,2)$
$(2, \infty)$	$x = 3$	$3((3)^2 + 2(3) - 8)$ $= 3(9 + 6 - 8)$ $= 21 > 0$	increasing on $(2, \infty)$

f has a local min at $x = 2$



Alternate:
$f'(x) = 3x^2 + 6x - 24 \Rightarrow f''(x) = 6x + 6$
critical numbers are -4 & 2
$f''(-4) = 6(-4) + 6 = -18 < 0 \Rightarrow$ local max at $x = -4$
$f''(2) = 6(2) + 6 = 18 > 0 \Rightarrow$ local max at $x = 2$

The local min value of f is $f(2) = (2)^3 + 3(2)^2 - 24(2) = 8 + 12 - 48 = -28$

Using Octave if you write function y=f(x)

y=x^3+3*x^2-24*x;

initial guess

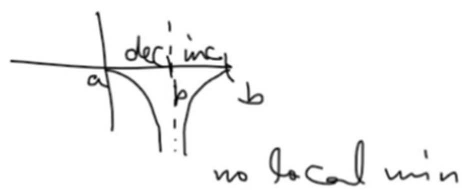
```
>>fminunc('f', 0)
      ans = 2.0000

>>fminunc('f', -3)
      ans = 2.0000
```

```
>>fminunc('f',-4)
      ans=-4.0000 which is not correct
                                    L. min.
>>fminunc('f',-5)
      ans=-40485694.70598 which is not correct
                                    L. min.
```

If we have to find a minimum value of f & we can not solve for critical number, then we can find values of f for different values of x and compare them to find a local min. We need to have a good strategy to reduce the number of function evaluations. In Bracketing Search methods for finding local min values, we need the condition that f is UNIMODAL on the interval [a, b] because this will guarantee that the function f has a local min in (a, b).

> [a, b] & then we cut it into smaller & smaller with each iteration

A function f if UNIMODAL on [a, b] if f is continuous on [a,b] and there exists a unique number p in (a, b) such that f is decreasing on [a, p] and f is increasing on [p, b]

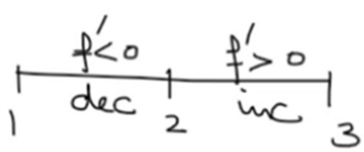


Let $f(x) = x^3 + 3x^2 - 24x$. Is f unimodal on $[1, 3]$?

We have seen in Example 1 that $f'(x) = 3x^2 + 6x - 24$

$f'(x) = 0 \Rightarrow x = -4, 2$

f is continuous on [1, 3]



$f'(1) = 3 + 6 - 24 < 0$

$f'(3) = 3(3)^2 + 6(3) - 24 < 0$
$\quad = 27 + 18 - 24 > 0$

f is decreasing on [1, 2] or [1, 2)

f is increasing on [2, 3] or (2, 3]

So, f is unimodal o [1, 3]

For Unimodal functions, two efficient bracketing methods to find local min values of the function are the golden ratio search method and Fibonacci Search method.