MATH 3940 Numerical Analysis for Computer Scientists

Problem Set 7 Solutions

1. Let $f(x) = x^4 + 4x^3 + 70$

   (a) Find local minimum value of f.
   (b) Can Golden Ratio search method be used to find a local minimum of f starting with the interval $[-5, -1]$? Justify your answer using the conditions of convergence.

   Solution. (a) $f(x) = x^4 + 4x^3 + 40 \Rightarrow f'(x) = 4x^3 + 12x^2$.

   $f'(x) = 0 \Rightarrow 4x^3 + 12x^2 = 0 \Rightarrow 4x^2(x+3) = 0 \Rightarrow x = -3, 0$

   The critical numbers are $x = -3, 0$.

intervals	test value	$f'(x) = 4x^3 + 12x^2$	Increase/decrease
$x < -3$	$x = -4$	$-$	decreasing on $(-\infty, -3)$
$-3 < x < 0$	$x = -1$	$+$	increasing on $(-3, 0)$
$x > 0$	$x = 1$	$+$	increasing on $(0, \infty)$

   Since f' changes from negative to positive at $x = -3$, there is a local minimum at $x = -3$ and the local minimum value is $f(-3) = (-3)^4 + 4(-3)^3 + 70 = 43$.

   (b) We need to check if f is unimodal on $[-5, -1]$.

   The function f is continuous on $[-5, -1]$.

   $f'(x) = 0 \Rightarrow 4x^3 + 12x^2 = 0 \Rightarrow 4x^2(x+3) = 0 \Rightarrow x = -3, 0$

   Since 0 is not in the interval $[-5, -1]$, the only critical number is $x = -3$.

intervals	test value	$f'(x) = 4x^3 + 12x^2$	Increase/decrease
$-5 \le x < -3$	$x = -5$	$-$	decreasing on $[-5, -3)$
$-3 < x \le -1$	$x = -1$	$+$	increasing on $(-3, -1]$

   So there exists a unique number -3 in the interval $[-5, -1]$ such that f is decreasing on $[-5, -3)$ and f is increasing on $(-3, -1]$. Thus f is unimodal on $[-5, -1]$. Therefore Golden Ratio search method can be used to find a local minimum of f starting with the interval $[-5, -1]$ $\square$

2. Let $f(x) = x^4 + 4x^3 + 40$. Perform 3 iterations of the golden ratio search method starting with the interval $[-4, -2]$.

   Solution. The iterations are
   $c_k = a_k + (1-r)(b_k - a_k) = a_k + (1 - 0.61803)(b_k - a_k) = a_k + 0.38197(b_k - a_k)$
   $d_k = b_k - (1-r)(b_k - a_k) = b_k - (1 - 0.61803)(b_k - a_k) = b_k - 0.38197(b_k - a_k)$

   <u>First Iteration:</u> Here $a_0 = -4$, $b_0 = -2$, and $r = 0.61803$

   $c_0 = a_0 + 0.38197(b_0 - a_0) = -4 + 0.38197(-2+4) = -4 + 0.76394 = -3.23606$
   $d_0 = b_0 - 0.38197(b_0 - a_0) = -2 - 0.38197(-2+4) = -2 - 0.76394 = -2.7639$
   $f(c_0) = f(-3.23606) = (-3.23606)^4 + 4(-3.23606)^3 + 40 = 14.11137$
   and $f(d_0) = f(-2.7639) = (-2.7639)^4 + 4(-2.7639)^3 + 40 = 13.901$

   Since $f(d_0) < f(c_0)$, we take $a_1 = c_0 = -3.23606$ and $b_1 = b_0 = -2$.

Second Iteration:

$c_1 = a_1 + 0.38197(b_1 - a_1) = -3.23606 + 0.38197(-2 + 3.23606) = -2.7639$

$d_1 = b_1 - 0.38197(b_1 - a_1) = -2 - 0.38197(-2 + 3.23606) = -2.472138$

$f(c_1) = f(-2.7639) = (-2.7639)^4 + 4(-2.7639)^3 + 40 = 13.901$
and $f(d_1) = f(-2.472138) = (-2.472138)^4 + 4(-2.472138)^3 + 40 = 16.916$

Since $f(c_1) \le f(d_1)$, we take $a_2 = a_1 = -3.23606$ and $b_2 = d_1 = -2.472138$.

Third Iteration:

$c_2 = a_2 + 0.38197(b_2 - a_2) = -3.23606 + 0.38197(-2.472138 + 3.23606) = -2.94427$

$d_2 = b_2 - 0.38197(b_2 - a_2) = -2.472138 - 0.38197(-2.472138 + 3.23606) = -2.7639$

$f(c_2) = f(-2.94427) = (-2.94427)^4 + 4(-2.94427)^3 + 40 = 13.0545$
and $f(d_2) = f(-2.7639) = (-2.7639)^4 + 4(-2.7639)^3 + 40 = 13.901$

Since $f(c_2) \le f(d_2)$, we take $a_3 = a_2 = -3.23606$ and $b_3 = d_2 = -2.7639$. $\square$

3. Let $f(x) = x^5 - 7\sin x + e^x$. Perform 1 iteration of the golden ratio search method starting with the interval $[-0.1, 1.3]$.

Solution. First Iteration: Here $a_0 = -0.1$, $b_0 = 1.3$, and $r = 0.618$

The iterations are
$c_k = a_k + (1 - r)(b_k - a_k) = a_k + (1 - 0.618)(b_k - a_k) = a_k + 0.382(b_k - a_k)$
$d_k = b_k - (1 - r)(b_k - a_k) = b_k - (1 - 0.618)(b_k - a_k) = b_k - 0.382(b_k - a_k)$

$c_0 = a_0 + 0.382(b_0 - a_0) = -0.1 + 0.382(1.3 + 0.1) = -0.1 + 0.5348 = 0.4348$

$d_0 = b_0 - 0.382(b_0 - a_0) = 1.3 - 0.382(1.3 + 0.1) = 1.3 - 0.5348 = 0.7652$

$f(c_0) = f(0.4348) = (0.4348)^5 - 7\sin 0.4348 + e^{0.4348} = -1.3884$
and $f(d_0) = f(0.7652) = (0.7652)^5 - 7\sin 0.7652 + e^{0.7652} = -2.4370$

Since $f(d_0) < f(c_0)$, we take $a_1 = c_0 = 0.4348$ and $b_1 = b_0 = 1.3$, that is, the new interval is $[0.4348, 1.3]$. $\square$