Assignment 2 Solutions Fall 2019

1. Consider the linear system

$$\begin{array}{rcrcrcr} x & + & 2y & - & z & = & 0 \\ 2x & + & 8y & - & 4z & = & 6 \\ -x & - & 4y & + & 3z & = & -2 \end{array}$$

(a) (4 marks) Starting with the zero vector, use hand calculations to perform two iterations of Jacobi method.

(b) (3 marks) Starting with the zero vector and tolerance of 10^{-6}, use Matlab to perform a maximum of 35 iterations of Jacobi method. Does it converge? If yes, how many iterations does it take to converge?

(c) (4 marks) Starting with the zero vector, use hand calculations to perform two iterations of Gauss-Seidel method.

(d) (4 marks) Starting with the zero vector and tolerance of 10^{-6}, use Matlab to perform a maximum of 35 iterations of Gauss-Seidel method. Does it converge? If yes, how many iterations does it take to converge?

Solution. (a) The jacobi iterations for the above system are:

$$\begin{array}{rclcrcl} x_{k+1} & = & & - & 2y_k & + & z_k \\ y_{k+1} & = & (6 & - & 2x_k & + & 4z_k)/8 \\ z_{k+1} & = & (-2 & + & x_k & + & 4y_k)/3 \end{array}$$

Starting vector is $\mathbf{P_0} = (x_0, y_0, z_0) = (0, 0, 0)$. Setting these values we obtain

$$x_1 = 0,\ y_1 = \frac{6}{8} = \frac{3}{4} \text{ or } 0.75, \text{ and } z_1 = -\frac{2}{3} \text{ or } -0.6667$$

The next iteration will give

$$x_2 = -2\left(\frac{3}{4}\right) + \left(-\frac{2}{3}\right) = -\frac{3}{2} - \frac{2}{3} = -\frac{13}{6} = -2.1667$$

$$y_2 = \frac{6 - 2(0) + 4(-\frac{2}{3})}{8} = \frac{6 - \frac{8}{3}}{8} = \frac{5}{12} = 0.4167$$

$$z_2 = \frac{-2 + 0 + 4(\frac{3}{4})}{3} = \frac{-2 + 3}{3} = \frac{1}{3} = 0.3333$$

(b) See Matlab sheets for solution.

(c) The Gauss-Seidel iterations for the above system are:

$$\begin{array}{rclcrcl} x_{k+1} & = & & - & 2y_k & + & z_k \\ y_{k+1} & = & (6 & - & 2x_{k+1} & + & 4z_k)/8 \\ z_{k+1} & = & (-2 & + & x_{k+1} & + & 4y_{k+1})/3 \end{array}$$

Starting vector is $\mathbf{P_0} = (x_0, y_0, z_0) = (0, 0, 0)$. Setting these values we obtain

$$x_1 = 0, \; y_1 = \frac{6}{8} = \frac{3}{4}, \text{ and } z_1 = \frac{-2+0+4(\frac{3}{4})}{3} = \frac{-2+3}{3} = \frac{1}{3} \text{ or } 0.3333$$

The next iteration will give

$$\begin{aligned}
x_2 &= -2\left(\frac{3}{4}\right) + \frac{1}{3} = -\frac{3}{2} + \frac{1}{3} = -\frac{7}{6} = -1.1667 \\
y_2 &= \frac{6 - 2(-\frac{7}{6}) + 4(\frac{1}{3})}{8} = \frac{6 + \frac{7}{3} + \frac{4}{3}}{8} = \frac{29}{24} = 1.2083 \\
z_2 &= \frac{-2 + (-\frac{7}{6}) + 4(\frac{29}{24})}{3} = \frac{-2 - \frac{7}{6} + \frac{29}{6}}{3} = \frac{10}{18} = \frac{5}{9} = 0.5556
\end{aligned}$$

(d) See Matlab sheets for solution. □

2. Let $A = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix}$, and the initial approximation is $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) (9 marks) Using hand calculations, find the eigenvalues and eigenvectors of A.
(b) (2 marks) Use the Matlab built-in function to find all the eigenvalues and eigenvectors of the matrix A.

(c) (4 marks) Using hand calculations, perform two iterations of the power method for matrix A starting with X_0.

(d) (3 marks) Use Matlab to find the dominant eigenvalue of A and the associated eigenvector using the power method with a tolerance of 10^{-5}, starting with X_0.

(e) (5 marks) Use Matlab to find all eigenvalues and eigenvectors of the matrix A using the shifted-inverse power method with a tolerance of 10^{-5}, starting with X_0. (take $\alpha = 1.5, 4.5$, and 6.5).

Solution. (a) To find the eigenvalues, we have to solve $|A - \lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} = 0$$

Expanding along the first column, we obtain

$$\begin{aligned}
&(2-\lambda)[(10-\lambda)(2-\lambda) - 20] - 5[-7(2-\lambda) - 0] = 0 \\
\Rightarrow\; &(2-\lambda)[(10-\lambda)(2-\lambda) - 20 + 35] = 0 \\
\Rightarrow\; &(2-\lambda)[\lambda^2 - 12\lambda + 20 + 15] = 0 \\
\Rightarrow\; &(2-\lambda)[\lambda^2 - 12\lambda + 35] = 0 \\
\Rightarrow\; &(2-\lambda)[(\lambda-5)(\lambda-7)] = 0
\end{aligned}$$

Thus the eigenvalues are 2, 5, and 7. Now we will find the eigenvectors.

For $\lambda_1 = 2$, we have

$$\left[\begin{array}{ccc|c} 0 & -7 & 0 & 0 \\ 5 & 8 & 4 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right] R_1 \leftrightarrow R_2 \left[\begin{array}{ccc|c} 5 & 8 & 4 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right] R_3 + \frac{5}{7}R_2 \to R_3 \left[\begin{array}{ccc|c} 5 & 8 & 4 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have

$$\begin{array}{rcrcrcl} 5x_1 & + & 8x_2 & + & 4x_3 & = & 0 \\ & - & 7x_2 & & & = & 0 \end{array}$$

x_3 is a free variable (parameter). The second equation gives $x_2 = 0$ and the first equation gives $x_1 = -4/5x_3$. If we take $x_3 = 1$, then the eigenvector is $\mathbf{v}_1 = [-4/5\ 0\ 1]'$

For $\lambda_2 = 5$, we have

$$\left[\begin{array}{ccc|c} -3 & -7 & 0 & 0 \\ 5 & 5 & 4 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right] R_2 + \frac{5}{3}R_1 \to R_2 \left[\begin{array}{ccc|c} -3 & -7 & 0 & 0 \\ 0 & -\frac{20}{3} & 4 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right] R_3 + \frac{3}{4}R_2 \to R_3 \left[\begin{array}{ccc|c} -3 & -7 & 0 & 0 \\ 0 & -\frac{20}{3} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have

$$\begin{array}{rcrcrcl} -3x_1 & - & 7x_2 & & & = & 0 \\ & - & \frac{20}{3}x_2 & + & 4x_3 & = & 0 \end{array}$$

x_3 is a free variable. If we take $x_3 = 1$, the second equation gives $x_2 = 3/5x_3 = 3/5$. The first equation gives $x_1 = -\frac{7}{3}x_2 = -(\frac{7}{3})(\frac{3}{5}) = -\frac{7}{5}$ Thus the eigenvector is $\mathbf{v}_2 = [-7/5\ 3/5\ \ 1]'$.

For $\lambda_3 = 7$, we have

$$\left[\begin{array}{ccc|c} -5 & -7 & 0 & 0 \\ 5 & 3 & 4 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] R_2 + R_1 \to R_2 \left[\begin{array}{ccc|c} -5 & -7 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] R_3 + \frac{5}{4}R_2 \to R_3 \left[\begin{array}{ccc|c} -5 & -7 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have

$$\begin{array}{rcrcrcl} -5x_1 & - & 7x_2 & & & = & 0 \\ & - & 4x_2 & + & 4x_3 & = & 0 \end{array}$$

x_3 is a free variable. If we take $x_3 = 1$, the second equation gives $x_2 = x_3 = 1$. The first equation gives $x_1 = -\frac{7}{5}x_2 = -\frac{7}{5}$ Thus the eigenvector is $\mathbf{v}_3 = [-7/5\ 1\ 1]'$.

(b) See Matlab sheets.

(c) The initial approximation is $X_0 = [1\ 1\ 1]'$. Using the power method

$$Y_0 = AX_0 = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 19 \\ 7 \end{bmatrix}$$

The element of largest magnitude is 19 so $c_0 = 19$.

$$X_1 = \frac{1}{c_0}Y_0 = \begin{bmatrix} -5/19 \\ 1 \\ 7/19 \end{bmatrix} \text{ or } \begin{bmatrix} -0.2632 \\ 1 \\ 0.3684 \end{bmatrix}$$

$$Y_1 = AX_1 = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} -5/19 \\ 1 \\ 7/19 \end{bmatrix} = \begin{bmatrix} -143/19 \\ 193/19 \\ 109/19 \end{bmatrix} \text{ or } \begin{bmatrix} -7.5263 \\ 10.1578 \\ 5.7368 \end{bmatrix}$$

Now $c_1 = 193/19$ or 10.1578 and $X_2 = \dfrac{1}{c_1} Y_1 = \begin{bmatrix} -143/193 \\ 1 \\ 109/193 \end{bmatrix}$ or $\begin{bmatrix} -0.7409 \\ 1 \\ 0.5648 \end{bmatrix}$

Parts (d) and (e) See Matlab sheets. □

3. Let $A = \begin{bmatrix} -5 & 1 & -2 \\ 0 & 1 & 1 \\ 0 & 0 & 5 \end{bmatrix}$, and the initial approximation be $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) (6 marks) Using hand calculations, find the eigenvalues and eigenvectors of A.

(b) (2 marks) Use Matlab to find the dominant eigenvalue and the associated eigenvector of A using the power method with a tolerance of 10^{-5}, starting with X_0.

(c) (3 marks) Use Matlab to find all eigenvalues and eigenvectors of A using the shifted-inverse power method with a tolerance of 10^{-5}, starting with X_0. (take $\alpha = 0, 4$, and -4).

(d) (2 marks) What is your conclusions about the performance of power and shifted-inverse power methods. Explain the reason for their convergence/divergence.

Solution. (a) To find the eigenvalues, we have to solve $|A - \lambda I| = 0$.

$$\begin{vmatrix} -5-\lambda & 1 & -2 \\ 0 & 1-\lambda & 1 \\ 0 & 0 & 5-\lambda \end{vmatrix} = 0$$

Expanding along the first column, we obtain $(-5-\lambda)(1-\lambda)(5-\lambda) = 0$. Thus the eigenvalues are -5, 1, and 5. Note that A is an upper triangular matrix and we can say right away that eigenvalues are the entries on the main diagonal. Now we will find the eigenvectors.

For $\lambda_1 = -5$, we have

$$\left[\begin{array}{ccc|c} 0 & 1 & -2 & 0 \\ 0 & 6 & 1 & 0 \\ 0 & 0 & 10 & 0 \end{array}\right] \Rightarrow \begin{array}{rcrcl} x_2 & - & 2x_3 & = & 0 \\ 6x_2 & + & x_3 & = & 0 \\ & & 10x_3 & = & 0 \end{array}$$

x_1 is a free variable. The last equation gives $x_3 = 0$, substituting $x_3 = 0$ into the first or second equation we get $x_2 = 0$. If we take $x_1 = 1$, then the eigenvector is $\mathbf{v}_1 = [1\ 0\ 0]'$

For $\lambda_2 = 1$, we have

$$\left[\begin{array}{ccc|c} -6 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] R_3{-}4R_2 \to R_3 \left[\begin{array}{ccc|c} -6 & 1 & -2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{rcrcrcl} -6x_1 & + & x_2 & - & 2x_3 & = & 0 \\ & & & & x_3 & = & 0 \end{array}$$

x_2 is a free variable. Let $x_2 = 1$, the first equation gives $x_1 = \frac{1}{6}x_2 = \frac{1}{6}$. Thus the eigenvector is $\mathbf{v}_2 = [\frac{1}{6}\ 1\ 0]'$.

For $\lambda_3 = 5$, we have

$$\left[\begin{array}{rrr|r} -10 & 1 & -2 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{rcrcrcl} -10x_1 & + & x_2 & - & 2x_3 & = & 0 \\ & - & 4x_2 & + & x_3 & = & 0 \end{array}$$

x_3 is a free variable. Let $x_3 = 1$, the second equation gives $x_2 = \frac{1}{4}x_3 = \frac{1}{4}$. Substituting x_2 and x_3 into the first equation we obtain $x_1 = \dfrac{1/4 - 2}{10} = -\dfrac{7}{40}$. Thus the eigenvector is $\mathbf{v}_3 = [-\frac{7}{40}\ \frac{1}{4}\ \ 1]'$.

(b) See Matlab sheets.

(c) See Matlab sheets.

(d) We note that the power method fails while the inverse power method gives all the eigenvalues and eigenvectors. The reason of the failure of the power method is that A does not have a single dominant eigenvalue; both 5 and -5 have largest magnitude. It seems that the power method is good in the case of a single dominant eigenvalue, however the inverse power methods works because the values of alpha were chosen closer to the eigenvalues. □

4. Let $g(x) = \dfrac{x^2}{4} + \dfrac{5x}{4} - 3$.

(a) (4 marks) Using hand calculations, solve $x = g(x)$.

(b) (3 marks) Use Matlab to plot the functions $y = x$ and $y = g(x)$ in the same window. Your graph should show both points of intersections.

(c) (3 marks) Using hand calculations, find 3 iterations of the fixed point method starting with $p_0 = -0.25$.

(d) (3 marks) Do you expect fixed point method to converge with an initial approximation $p_0 = -0.25$? Justify your answer using the condition of convergence.

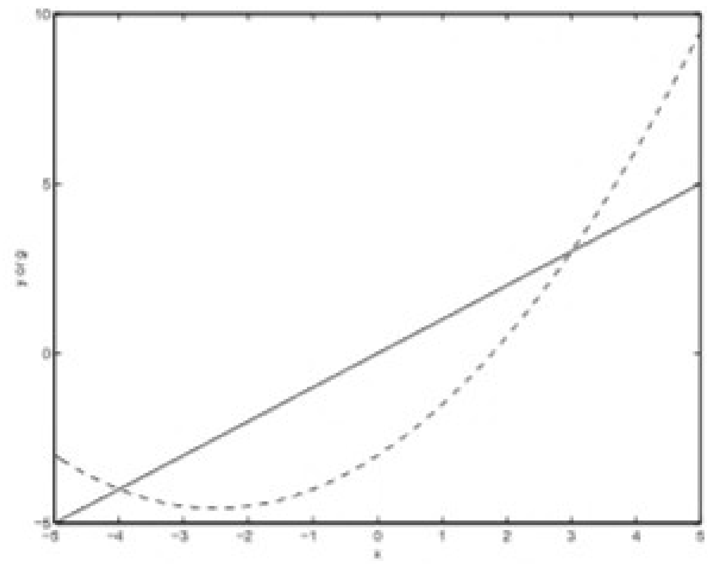
(e) (3 marks) Use Matlab to perform 40 iterations of the fixed point method to solve $x = g(x)$, starting with $p_0 = -0.25$, and a tolerance of 10^{-5}.

Solution. (a) $x = g(x) \Rightarrow x = \dfrac{x^2}{4} + \dfrac{5x}{4} - 3 \Rightarrow 4x = x^2 + 5x - 12$

$\Rightarrow x^2 + x - 12 = 0 \Rightarrow (x+4)(x-3) = 0 \Rightarrow x = -4, 3.$

(b) >> $x = -5 : 0.1 : 5;$

```
>> y=x;
>> g=(x.*x)./4+5*x./4-3;
>> plot(x,y,x,g,'--')
>> xlabel('x')
```



>> ylabel('y or g')

(c) The fixed point iterations are $p_{k+1} = g(p_k)$. Let $p_0 = -0.25$, then

$p_1 = g(-0.25) = \frac{(-0.25)^2}{4} + \frac{5(-0.25)}{4} - 3 = -3.296875 \approx -3.2969$

$p_2 = g(-3.2969) = \frac{(-3.2969)^2}{4} + \frac{5(-3.2969)}{4} - 3 = -4.4037$

$p_3 = g(-4.4037) = \frac{(-4.4037)^2}{4} + \frac{5(-4.4037)}{4} - 3 = -3.6565$

(d) Yes, I expect the iterations will converge with $p_0 = -0.25$. The reason follows:

Here $g'(x) = \dfrac{2x}{4} + \dfrac{5}{4} = \dfrac{2x+5}{4}$. Now $|g'(-4)| = |\frac{-8+5}{4}| = |\frac{-3}{4}| = 0.75 < 1$

The functions $g(x)$ and $g'(x)$ are continuous on $[-5, 0]$. The solution $-4 \in [-5, 0]$ and the initial guess $-0.25 \in [-5, 0]$ and $g(x) \in [-5, 0]$ for all $x \in [-5, 0]$. (Note that here $g(-5) = -3$ and $g(0) = -3$ and the graph of $g(x)$ is an upward parabola with vertex at -2.5, where $g(-2.5) = -4.5625$). And $|g'(-4)| < 1$, thus we expect that the fixed point method will converge to -4 starting with $p_0 = -0.25$.
(we have seen in part(e) that the iterations converge to -4)

(e) See Matlab sheets for solution. □

5. Given the equation $x^3 + x^2 - 3x - 3 = 0$.

(a) (2 marks) Use the Matlab built-in function to find all roots of the above equation.

(b) (6 marks) Use Matlab to perform 25 iterations of the fixed point method for each of the following functions, starting with $p_0 = 1$ and a tolerance of 10^{-5}. In the case of convergence, mention the number of iterations when the convergence is achieved.

(i) $g_1(x) = \sqrt{\dfrac{3 + 3x - x^2}{x}}$

(ii) $g_2(x) = -1 + \dfrac{3x+3}{x^2}$

(iii) $g_3(x) = \dfrac{x^3 + x^2 - x - 3}{2}$.

Solution. See Matlab sheets for the solutions of all parts. $\square$