MATH 3940 Numerical Analysis for Computer Scientists
Assignment 4 Solutions Fall 2020

1. (a) (6 marks) Find the Taylor polynomial of degree 3 for $f(x) = x^{3/2}$ expanded about $x_0 = 4$.
(b) (2 marks) Does $f(x) = x^{3/2}$ have a Taylor polynomial expansion about $x_0 = 0$? Justify your answer.

Solution. (a) Here $f(x) = x^{3/2} \Rightarrow f(4) = (4)^{3/2} = 8$.

$$f'(x) = \frac{3}{2}x^{1/2} \Rightarrow f'(4) = \frac{3}{2}(4)^{1/2} = \frac{3}{2}(2) = 3$$
$$f''(x) = \left(\frac{3}{2}\right)\left(\frac{1}{2}\right)x^{-1/2} = \frac{3}{4}x^{-1/2} \Rightarrow f''(4) = \frac{3}{4}(4)^{-1/2} = \left(\frac{3}{4}\right)\left(\frac{1}{2}\right) = \frac{3}{8}$$
$$f'''(x) = \left(\frac{3}{4}\right)\left(\frac{-1}{2}\right)x^{-3/2} = \frac{-3}{8}x^{-3/2} \Rightarrow f'''(4) = \frac{-3}{8}(4)^{-3/2} = \left(\frac{-3}{8}\right)\left(\frac{1}{8}\right) = -\frac{3}{64}$$

Taylor polynomial of degree 3 around $x_0 = 4$ is given by

$$\begin{aligned}
f(x) &= f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 \\
&= f(4) + \frac{f'(4)}{1!}(x-4) + \frac{f''(4)}{2!}(x-4)^2 + \frac{f'''(4)}{3!}(x-4)^3 \\
&= 8 + 3(x-4) + \frac{3}{8(2)}(x-4)^2 - \frac{3}{64(6)}(x-4)^3 \\
&= 8 + 3(x-4) + \frac{3}{16}(x^2 - 8x + 16) - \frac{1}{128}(x^3 - 12x^2 + 48x - 64) \\
&= -\frac{1}{128}x^3 + \frac{9}{32}x^2 + \frac{9}{8}x - \frac{1}{2}
\end{aligned}$$

(b) Since the second order derivative f'' and higher order derivatives of f are not defined at $x_0 = 0$, the function $f(x)$ does not have a Taylor polynomial expansion about $x_0 = 0$. The only possible polynomial is of degree zero. □

2. Consider the data

x	-2	-1	0	1	2
y	4	-1	3	1	8

(a) (5 marks) Find an interpolation polynomial of degree 4 by solving the system $AX = B$ (Use Matalb to solve the system $AX = B$).
(b) (3 marks) Use Matlab built in functions to find an interpolation polynomial of degree 4 and then to approximate the value at $x = -1.4$.

Solution. (a) Let the interpolation polynomial be
$P_4(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$.
Since the interpolation polynomial satisfy $P_4(x_k) = y_k$, for $k = 0, 1, 2, 3, 4, 5$, we have the following system of equations to solve for the coefficients a_j.

$$\begin{aligned}
P_4(-2) = 4 &\Rightarrow a_0 + a_1(-2) + a_2(-2)^2 + a_3(-2)^3 + a_4(-2)^4 = 4 \\
P_4(-1) = -1 &\Rightarrow a_0 + a_1(-1) + a_2(-1)^2 + a_3(-1)^3 + a_4(-1)^4 = -1 \\
P_4(0) = 3 &\Rightarrow a_0 + a_1(0) + a_2(0)^2 + a_3(0)^3 + a_4(0)^4 = 3 \\
P_4(1) = 1 &\Rightarrow a_0 + a_1(1) + a_2(1)^2 + a_3(1)^3 + a_4(1)^4 = 1 \\
P_4(2) = 8 &\Rightarrow a_0 + a_1(2) + a_2(2)^2 + a_3(2)^3 + a_4(2)^4 = 8
\end{aligned}$$

See Matlab sheets for the solution of this system using Matlab, the polynomial is $P_4(x) = 3.0000 + 1.0000x - 4.2500x^2 + 0x^3 + 1.2500x^4 = 3 + x - 4.25x^2 + 1.25x^4$.
(b) See Matlab sheets for the solution. □

3. Consider the data

x_k	0	1	2	3	4	5
y_k	5	5	3	5	17	45

(a) (4 marks) Using hand calculations, find Lagrange polynomial $P_2(x)$ using the nodes x_0, x_1, x_2.
(b) (2 marks) Use Matlab to find Lagrange polynomial $P_2(x)$ using the nodes x_0, x_1, and x_2.
(c) (6 marks) Using hand calculations, find divided difference table and Newton polynomial using all nodes in the above table.
(d) (2 marks) Use Matlab to find divided difference table and Newton polynomial using all nodes in the above table.

Solution. (a) Lagrange polynomial $P_2(x)$ is given by

$$\begin{aligned}
P_2(x) &= y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} \\
&= 5\frac{(x-1)(x-2)}{(0-1)(0-2)} + 5\frac{(x-0)(x-2)}{(1-0)(1-2)} + 3\frac{(x-0)(x-1)}{(2-0)(2-1)} \\
&= \frac{5(x^2-3x+2)}{2} + \frac{5(x^2-2x)}{-1} + \frac{3(x^2-x)}{2} \\
&= \frac{5x^2-15x+10-10x^2+20x+3x^2-3x}{2} = -x^2+x+5
\end{aligned}$$

(b) See Matlab sheets for the solution of part(b).

(c)

x_k	y_k	$f[,]$	$f[,\,,]$	$f[,\,,\,,]$	$f[,\,,\,,\,,]$	$f[,\,,\,,\,,\,,]$
0	5					
1	5	$\frac{5-5}{1-0}=0$				
2	3	$\frac{3-5}{2-1}=-2$	$\frac{-2-0}{2-0}=-1$			
3	5	$\frac{5-3}{3-2}=2$	$\frac{2+2}{3-1}=2$	$\frac{2+1}{3-0}=1$		
4	17	$\frac{17-5}{4-3}=12$	$\frac{12-2}{4-2}=5$	$\frac{5-2}{4-1}=1$	$\frac{1-1}{4-0}=0$	
5	45	$\frac{45-17}{5-4}=28$	$\frac{28-12}{5-3}=8$	$\frac{8-5}{5-2}=1$	$\frac{1-1}{5-1}=0$	0

The Newton interpolation polynomial is

$$\begin{aligned}P(x) &= 5 + 0(x-0) - 1(x-0)(x-1) + 1(x-0)(x-1)(x-2)\\ &\quad + 0(x-0)(x-1)(x-2)(x-3) + 0(x-0)(x-1)(x-2)(x-3)(x-4)\\ &= 5 - 1(x^2 - x) + (x^3 - 3x^2 + 2x) = x^3 - 4x^2 + 3x + 5\end{aligned}$$

(d) See Matlab sheets for the solution of part (d). □

4. Let $f(x) = xe^x$. The nodes are $x_0 = -1$, $x_1 = -0.5$, $x_2 = 0$, $x_3 = 0.5$ and $x_4 = 1$.

(a) (3 marks) Use Matlab built in functions to find an interpolation polynomial of degree 4 and to approximate the value at $x = 0.2$.

(b) (2 marks) Use Matlab to find lagrange polynomial $P_4(x)$ using all nodes.

(c) (2 marks) Use Matlab to find Newton polynomial $P_4(x)$ using all nodes.

(d) (8 marks) Using hand calculations, calculate the exact error and the approximated error for Lagrange polynomial $P_4(x)$ at $x = -0.25$ (use $c = 0.1$).

Solution. See Matlab sheets for the solution of parts (a), (b), and (c).

(d) The exact value at -0.25 is $f(-0.25) = -0.25e^{-0.25} = -0.194700195$.
Substituting $x = -0.25$ in the Lagrange polynomial obtained in part (b), we have $P_4(-0.25) = 0.1773(-0.25)^4 + 0.5539(-0.25)^3 + 0.9979(-0.25)^2 + 0.9892(-0.25) = -0.192893$.
The exact error is $|P_4(-0.25) - f(-0.25)| = |-0.192893 + 0.194700| = 0.001807$.
Now $f(x) = xe^x$, $f'(x) = xe^x + e^x$, $f''(x) = xe^x + 2e^x$, $f'''(x) = xe^x + 3e^x$, $f^{(4)}(x) = xe^x + 4e^x$, $f^{(5)}(x) = xe^x + 5e^x$
For $x = -0.25$ and $c = 0.1$, the approximated error is

$$\begin{aligned}E_4(-0.25) &= \frac{(x+1)(x+0.5)(x)(x-0.5)(x-1)}{5!} f^{(5)}(c)\\ &= \frac{(-0.25+1)(-0.25+0.5)(-0.25)(-0.25-0.5)(-0.25-1)}{5!}(0.1+5)e^{0.1}\\ &= \frac{(0.75)(0.25)(-0.25)(-0.75)(-1.25)}{120}(5.1e^{0.1}) = -0.0020641\end{aligned}$$

□