Textbook has program on page 217

$function\ [c\ L] = lagram(x,y)$
⋮

for k=1:N
⋮

>>x=[1 2 3];

>> y=[3   3   3.5];

>>[C L]=lagran(x,y)

C = 0.2500   -0.7500   3.500 $\longrightarrow 0.25x^2 - 0.75x + 3.5$

$$L = \left.\begin{matrix} 0.5000 & -2.5000 & 3.0000 \\ -1.0000 & 4.0000 & -3.0000 \\ 0.50000 & -1.5000 & 1.0000 \end{matrix}\right\}$$

Lagrange Coefficients
$3(0.5x^2 - 2.5x + 3) + 3(-x^2 + 4x - 3) + 3.5(0.5x^2 - 1.5x + 1)$

>>p=poly(1)
p=1   -1 $\rightarrow (x-1)$

>>q=ploy(2)
q=1   -2$\rightarrow (x-2)$

then use the command
>>conv(p,q)   (x-1)(x-2)
ans = 1 -3 2
$x^2 - 3x + 2$

x_k	-1	0	1
y_k	-1	0	1

$y_0 = f(x_0) = f(-1) = (-1)^3 = -1$
$y_1 = f(x_1) = f(-1) = (-1)^3 = -1$
$y_2 = f(x_2) = f(-1) = (-1)^3 = -1$

Example 2:
Let $f(x) = x^3$. The nodes are $x_0 = -1, x_1 = 0, and\ x_2 = 1$
Find Lagrange Polynomial $P_2(x)$.

Solution:

$$P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_1)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_1)(x-x_2)}{(x_2-x_0)(x_2-x_1)}$$

$$= -1 \frac{(x-0)(x-1)}{(-1-0)(-1-1)} + 0 \cancelto{0}{\frac{(x-x_1)(x-x_2)}{(x_1-x_0)(x_1-x_2)}} + 1 \frac{(x-1)(x-0)}{(1+1)(1-0)}$$

$$= -1 \frac{(x^2-x)}{(-1)(-2)} + 1 \frac{(x^2+x)}{(2)(1)}$$

$= \frac{1}{2}(-x^2 + x + x^2 + x) = \frac{1}{2}(2x) = x$ **Ans**

There are 3 nodes, so the highest possible degree of the polynomial is $P_2(x)$.

ERROR: At the nodes error is zero. (Because at nodes $P(x\ k) = y_k$ or $f(x_k)$)

Let $f \in C^{N+1}[a, b]$ and that $x_0, x_1, \ldots, x_n$ are $N + 1$ nodes in $[a, b]$. If $x \in [a, b]$ then

$f, f', f'', \ldots, f^{N+1}$ are cont. on $[a, b]$.

$f(x) = P_N(x) + E_N(x)$ $where$ $P_N(x)$ is $Lagrange$ $Polynomial$ & $E_N(x)$ is the approximated error.

$E_N(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_N)}{(N+1)!} f^{(N+1)}(c)$ for some c that lies in the interval [a, b]

Example 3:

Let $f(x) = x^3$. The nodes are $x_0 = 0, x_1 = 1,$ and $x_2 = 2$.

(a) Find Lagrange Polynomial $P_2(x)$ using all nodes.

Solution:

x_k	0	1	2
y_k	0	1	8

$(0)^3 \quad (1)^3 \quad (2)^3$

$$P_2(x) = y_0\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1\frac{(x-x_1)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2\frac{(x-x_1)(x-x_2)}{(x_2-x_0)(x_2-x_1)}$$

$$P_2(x) = \cancelto{0}{0\frac{(x-1)(x-2)}{(0-1)(0-2)}} + 1\frac{(x-0)(x-2)}{(1-0)(1-2)} + 8\frac{(x-0)(x-1)}{(2-0)(2-1)}$$

$$= \frac{(x^2-2x)}{(1)(-1)} + 8\frac{(x^2-x)}{(2)(1)}$$

$$= -x^2 + 2x + 4x^2 - 4x$$

$$= 3x^2 - 2x$$

<u>Check</u>

$P_2(0) = 0 - 0 = 0$

$P_2(1) = 3(1)^2 - 2(1) = 1$

$P_2(2) = 3(2)^2 - 2(2) = 12 - 4 = 8$

(b) Calculate the exact error and the approximated error for Lagrange polynomials
$P_2(x)$ at x=1.2
$f(1.2) = (1.2)^3 = 1.728$
&
$P_2(1.2) = 3(1.2)^2 - 2(1.2) = 1.92$

The exact error is $|f(1.2) - P_2(1.2)| = |1.7828 - 1.92| = 0.192$

The approximated error is

$$E_N(x) = \frac{(x - x_0)(x - x_1) \cdots (x - x_N) f^{(N+1)}(c)}{(N+1)!}$$

For $P_2(x)$, we have

$$E_2(x) = \frac{(x-x_0)(x-x_1)(x-x_2)}{3!} \underbrace{f'''(c)} \text{ for some } c \in [0, 2]$$

$f(x) = x^3$
$f'(x) = 3x^2$
$f''(x) = 6x$
$f'''(x) = 6$

For $x = 1.2$

$$E_2(1.2) = \frac{(1.2 - 0)(1.2 - 1)(1.2 - 2)}{3(2)(1)}(6)$$

$$= (1.2)(0.2)(-0.8)$$

$$= -0.192$$