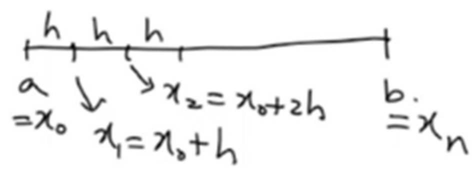
Numerical Integration

Numerical integration is used in many situations, as sometimes the antiderivatives of functions are not available. (e.g. $\int_0^2 e^{x^2}\,dx \to no\ antiderivative\ of\ e^{x^2}$) and sometimes it is hard to obtain an antiderivative such as ($\int_0^\pi e^x \sin 3x\ \ dx$)

$$\int_a^b f(x)\,dx = antiderivative\Big|_a^b$$

Numerical integration is used to approximate the definite integral $\int_a^b f(x)\,dx$ where $f(x)$ is continuouos on $[a,b]$.

The interval $[a,b]$ is divided by using nodes $x_0, x_1, \dots, x_n$ nodes. We will do the methods where the nodes are equally spaced, i.e., $x_i = x_0 + ih$ for $i = 1,2,\dots,n$



If nodes are $x_i, i = 0, \dots, n$

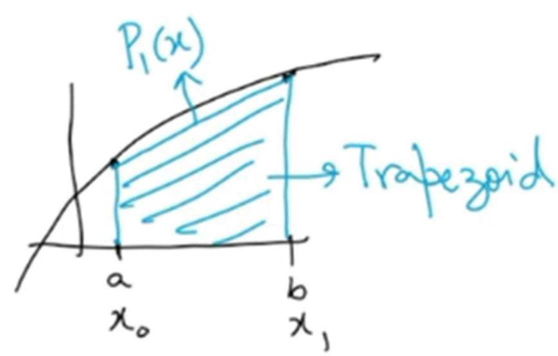
formulas for $\int_a^b f(x)\,dx$ are of the form

$\int_a^b f(x)\,dx = \sum_{i=0}^{n} w_i f(x_i)$

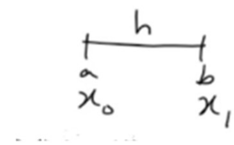
weight Quadratic Formulas

The derivation of quadrature formulas is sometimes based on interpolation polynomial. These types of formulas are called Newton cotes formulas are called Newton cotes formulas. If the end points a & b are also used as nodes, then these are called closed Newton Cotes formulas.

If we integrate Lagrange polynomial, $P_1(x)$ then the formula obtained is called Trapezoidal rule.



Lagrange polynomial $P_1(x)$ is given by



$$P_1(x) = f(x_0)\frac{x-x_1}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0} + \frac{(x-x_0)(x-x_1)}{2!}f''(c)\ where\ c \in [a,b]$$

Here $h = b - a$

$$\int_{a=x_0}^{b=x_1} f(x)dx = \int_{x_0}^{x_1} f(x_0)\frac{(x-x_0)}{x_0-x_1} + f(x_1)\frac{x-x_0}{x_1-x_0} + \frac{(x-x_0)(x-x_1)}{2!}f''(c)$$

$$= \frac{f(x_0)}{x_0-x_1}\frac{(x-x_1)^2}{2}\Bigg|_{x_0}^{x_1} + \frac{f(x_1)}{x_1-x_0}\frac{(x-x_0)^2}{2}\Bigg|_{x_0}^{x_1} + \frac{f''(c)}{2!}\left(\frac{x^3}{3} - (x_0+x_1)\frac{x^2}{2} + x_0x_1x\right)\Bigg|_{x_0}^{x_1}$$

$$= \frac{f(x_0)}{(x_0-x_1)}\cdot\left[\frac{(x_1-x_1)^2}{2} - \frac{(x_0-x_1)^2}{2}\right] + \frac{f(x_1)}{x_1-x_0}\cdot\left[\frac{(x_1-x_0)^2}{2} - \frac{(x_0-x_0)^2}{2}\right] + \frac{f''(c)}{2}$$
$$\cdot\left[\frac{x_1^3}{3} - \frac{(x_0+x_1)(x_1^2)}{2} + x_0x_1x_1 - \frac{x_0^3}{3} + \frac{(x_0+x_1)(x_0^2)}{2} + x_0x_1x_0\right]$$

$$= \frac{-f(x_0)}{x_0-x_1}\frac{(x_0-x_1)^2}{2} + \frac{f(x_1)}{x_1-x_0}\frac{(x_0-x_1)^2}{2} + \frac{f''(c)}{2}\left[\frac{x_1^3}{3} - \frac{x_0x_1^2}{2} - \frac{x^3}{2} + x_0x_1^2 - \frac{x_0^3}{3} + \frac{x_0^3}{3} + \frac{x_1x_0^2}{2} - x_0^2x_1\right]$$

$$= \frac{h}{2}f(x_0) + \frac{h}{2}f(x_1) + \frac{f''(c)}{2}\left[\frac{2x_1^3 - 3x_0x_1^2 - 3x_1^3 + 6x_0x_1^2 - 2x_0^3 + 3x_0^3 + 3x_1x_0^2 - 6x_0^2x_1}{6}\right]$$

$$= \frac{h}{2}[f(x_0) + f(x_1)] + \frac{f''(c)}{12}[\underbrace{-x_1^3 + 3x_0x_1^2 + x_0^3 - 3x_0^2x_1}]$$

$-(x_1^3 - 3x_0x_1^2 + 3x_0^2 + 3x_0^2x_1 - x_0^3)$
$-(x_1 - x_0)^3$

h

$$= \int_a^b f(x)dx = \frac{h}{2}[f(x_0) + f(x_1)] - \underbrace{\frac{h^3 f''(c)}{12}}_{\text{error}}$$

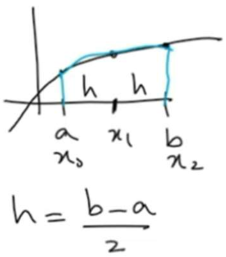
or

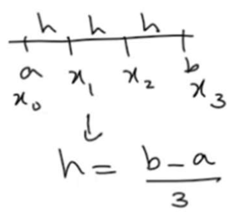
$\frac{h}{2}[f_0 + f_1]$

Trapezoidal Rule
error is of $O(h^3)$

The integration of $P_2(x), P_3(x), \ldots$ *give other rules*.

x_0, x_1, x_2 $\quad$ x_0, x_1, x_2, x_3





$$\underbrace{\int_{a\to x_0}^{b\to x_2} f(x)dx = \frac{h}{3}[f_0 + 4f_1 + f_2] - \underbrace{\frac{h^5}{90}}_{\text{error}} f^{(4)}(c)}\ \textit{where } c \in [a,b]$$

Simpson's rule error is of $O(h^5)$

where $h = \frac{b-a}{2}$

$$\int_{a\to x_0}^{b\to x_3} f(x)dx = \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3] - \frac{3h^5}{80} f^{(4)}(c) \quad \textit{where } c \in [a,b]$$

Simpson's $\frac{3}{8}$ rule error is of $O(h^5)$

where $h = \frac{b-a}{3}$

This h will be smaller than h in Simpson's'