MATH 3940 Numerical Analysis for Computer Scientists

Problem Set 1 Solutions

1. Consider the following system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 6 \\ 3x_1 & + & x_2 & & & = & 5 \\ 2x_1 & + & x_2 & + & x_3 & = & 3 \end{array}$$

(a) Solve the system using Gaussian elimination method with no pivoting.

(b) Solve the system using Gaussian elimination method with partial pivoting.

Solution. (a) We will perform Gaussian elimination on the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 6 \\ 3 & 1 & 0 & 5 \\ 2 & 1 & 1 & 3 \end{array}\right] \begin{array}{c} R_2 - 3R_1 \to R_2 \\ R_3 - 2R_1 \to R_3 \end{array} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 6 \\ 0 & -5 & 3 & -13 \\ 0 & -3 & 3 & -9 \end{array}\right]$$

$$R_3 - \frac{3}{5}R_2 \to R_3 \left[\begin{array}{rrr|r} 1 & 2 & -1 & 6 \\ 0 & -5 & 3 & -13 \\ 0 & 0 & \frac{6}{5} & -\frac{6}{5} \end{array}\right]$$

Using Gaussian elimination the matrix A is reduced to an upper triangular matrix. Now back substitution will be used to find the solution to the following system.

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 6 \\ & - & 5x_2 & + & 3x_3 & = & -13 \\ & & & & \frac{6}{5}x_3 & = & -\frac{6}{5} \end{array}$$

The last equation gives $x_3 = -1$. Putting $x_3 = -1$ into the second equation we have
$-5x_2 + 3(-1) = -13 \;\Rightarrow\; -5x_2 = -10 \;\Rightarrow\; x_2 = 2.$
Finally, substituting the value of x_2 and x_3 into the first equation we find
$x_1 + 2(2) - (-1) = 6 \;\Rightarrow\; x_1 = 6 - 4 - 1 = 1.$
Thus the solution is $(x_1, x_2, x_3) = (1, 2, -1)$.

(b) We will perform Gaussian elimination on the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 6 \\ 3 & 1 & 0 & 5 \\ 2 & 1 & 1 & 3 \end{array}\right] R_2 \leftrightarrow R_1 \left[\begin{array}{rrr|r} 3 & 1 & 0 & 5 \\ 1 & 2 & -1 & 6 \\ 2 & 1 & 1 & 3 \end{array}\right] \begin{array}{c} R_2 - \frac{1}{3}R_1 \to R_2 \\ R_3 - \frac{2}{3}R_1 \to R_3 \end{array} \left[\begin{array}{rrr|r} 3 & 1 & 0 & 5 \\ 0 & \frac{5}{3} & -1 & \frac{13}{3} \\ 0 & \frac{1}{3} & 1 & -\frac{1}{3} \end{array}\right]$$

$$R_3 - \frac{1}{5}R_2 \to R_3 \left[\begin{array}{rrr|r} 3 & 1 & 0 & 5 \\ 0 & \frac{5}{3} & -1 & \frac{13}{3} \\ 0 & 0 & \frac{6}{5} & -\frac{6}{5} \end{array}\right]$$

Using Gaussian elimination the matrix A is reduced to an upper triangular matrix. Now back substitution will be used to find the solution to the following system.

$$\begin{array}{rcrcrcr} 3x_1 & + & x_2 & & & = & 5 \\ & & \frac{5}{3}x_2 & - & x_3 & = & \frac{13}{3} \\ & & & & \frac{6}{5}x_3 & = & -\frac{6}{5} \end{array}$$

The last equation gives $x_3 = -1$. Putting $x_3 = -1$ into the second equation we obtain

$$\frac{5}{3}x_2 - x_3 = \frac{13}{3} \ \Rightarrow\ \frac{5}{3}x_2 - (-1) = \frac{13}{3} \ \Rightarrow\ \frac{5}{3}x_2 = \frac{13}{3} - 1 \ \Rightarrow\ \frac{5}{3}x_2 = \frac{10}{3} \ \Rightarrow x_2 = 2$$

Finally, substituting the value of x_2 and x_3 into the first equation we obtain
$3x_1 + 2 = 5 \ \Rightarrow\ 3x_1 = 3 \ \Rightarrow\ x_1 = 1$.
Thus the solution is $(x_1, x_2, x_3) = (1, 2, -1)$. $\square$

2. Consider the system of linear equations

$$\begin{array}{rcrcrcrcrcr}
-x_1 & + & 2x_2 & + & 2x_3 & + & 5x_4 & + & x_5 & = & 7 \\
 & & 3x_2 & + & x_3 & + & 2x_4 & + & x_5 & = & 5 \\
x_1 & - & 4x_2 & + & x_3 & & & - & 2x_5 & = & 9 \\
 & & 5x_2 & + & 3x_3 & + & x_4 & + & x_5 & = & 2 \\
3x_1 & - & 6x_2 & & & + & 4x_4 & + & 3x_5 & = & -1
\end{array}$$

(a) Use Matlab to find the determinant and the inverse of the coefficient matrix A.

(b) Use Matlab built in command (mentioned during lectures) to solve the linear system $AX = B$

Solution. See Matlab Sheets for solutions. $\square$

3. Consider the system of linear equations

$$\begin{array}{rcrcrcrcr}
 & & x_2 & + & 2x_3 & - & x_4 & = & -1 \\
x_1 & + & x_2 & - & x_3 & & & = & 5 \\
-x_1 & - & x_2 & + & x_3 & + & 3x_4 & = & 1 \\
x_1 & + & 2x_2 & & & + & x_4 & = & 9
\end{array}$$

(a) Find the LU factorization of the coefficient matrix A and then solve the resulting triangular system.
(b) Use Matlab built in command to find the LU factorization of the coefficient matrix A and then solve the resulting triangular system using forward and backward substitutions programs.

Solution. (a) Here we have

$$A = \begin{bmatrix} 0 & 1 & 2 & -1 \\ 1 & 1 & -1 & 0 \\ -1 & -1 & 1 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ 5 \\ 1 \\ 9 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Since $a_{11} = 0$, we have to interchange the first and the second row, which gives

$$R_2 \leftrightarrow R_1 \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ -1 & -1 & 1 & 3 \\ 1 & 2 & 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The multipliers are $m_{21} = 0,\ m_{31} = -1$, and $m_{41} = 1$.

$$\begin{matrix} R_3 + R_1 \to R_3 \\ R_4 - R_1 \to R_4 \end{matrix} \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 1 & 1 & 1 \end{bmatrix}$$

The multipliers are $m_{32} = 0$ and $m_{42} = 1$.

Now $R_4 - R_2 \to R_4$ $\begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & -1 & 2 \end{bmatrix}$

$$R_3 \leftrightarrow R_4 \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The matrix is reduced to an upper triangular matrix so we have

$$P = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

Note that we have interchanged m_{31} and m_{41} and also m_{32} and m_{42} in L because R_3 and R_4 was interchanged.

First we have to solve $LY = PB$, so we will find PB

$$PB = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 5 \\ 1 \\ 9 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 9 \\ 1 \end{bmatrix}$$

$$LY = PB \ \Rightarrow \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 9 \\ 1 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} y_1 & & & & & & & = & 5 \\ & & y_2 & & & & & = & -1 \\ y_1 & + & y_2 & + & y_3 & & & = & 9 \\ -y_1 & & & & & + & y_4 & = & 1 \end{array}$$

First equation gives $y_1 = 5$ and the second equation gives $y_2 = -1$. The third equation gives $y_3 = 9 - 5 + 1 = 5$. The fourth equation gives $y_4 = 1 + 5 = 6$.

Now we solve $UX = Y$

$$UX = Y \;\Rightarrow\; \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 5 \\ 6 \end{bmatrix}$$

$$\begin{array}{rcrcrcrcr} x_1 & + & x_2 & - & x_3 & & & = & 5 \\ & & x_2 & + & 2x_3 & - & x_4 & = & -1 \\ & & & - & x_3 & + & 2x_4 & = & 5 \\ & & & & & & 3x_4 & = & 6 \end{array}$$

Last equation gives $x_4 = 2$. The third equation gives $x_3 = 4 - 5 = -1$. The second equation gives $x_2 = -1 + 2 + 2 = 3$. Finally the first equation gives $x_1 = 5 - 3 - 1 = 1$.

Therefore, the solution is $(x_1, x_2, x_3, x_4) = (1, 3, -1, 2)$.

(b) See Matlab sheets for solution of part(b). □

4. Consider the system of linear equations

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & -1 \\ 2x_1 & + & 8x_2 & - & 4x_3 & = & -10 \\ -x_1 & - & 4x_2 & + & 3x_3 & = & 7 \end{array}$$

(a) Find the Cholesky factorization of the coefficient matrix A and then solve the resulting triangular system.

(b) Use Matlab built in command to find the Cholesky factorization of the coefficient matrix A and then solve the resulting triangular system using forward and backward substitutions programs.

Solution. (a) Here we have

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 8 & -4 \\ -1 & -4 & 3 \end{bmatrix} \qquad B = \begin{bmatrix} -1 \\ -10 \\ 7 \end{bmatrix}$$

Let $A = LL^T$ where L is a lower triangular matrix. Then we have

$$\begin{aligned} \begin{bmatrix} 1 & 2 & -1 \\ 2 & 8 & -4 \\ -1 & -4 & 3 \end{bmatrix} &= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & u_{33} \end{bmatrix} \\ &= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix} \end{aligned}$$

Equating the corresponding entries of the matrices, we obtain following equations.

$$\begin{aligned}
&l_{11}^2 = 1 \;\Rightarrow\; l_{11} = 1\\
&l_{11}l_{21} = 2 \;\Rightarrow\; l_{21} = 2\\
&l_{11}l_{31} = -1 \;\Rightarrow\; l_{31} = -1\\
&l_{21}^2 + l_{22}^2 = 8 \;\Rightarrow\; 4 + l_{22}^2 = 8 \;\Rightarrow\; l_{22}^2 = 4 \;\Rightarrow\; l_{22} = 2\\
&l_{21}l_{31} + l_{22}l_{32} = -4 \;\Rightarrow\; (2)(-1) + 2l_{32} = -4 \;\Rightarrow\; 2l_{32} = -2 \;\Rightarrow\; l_{32} = -1\\
&l_{31}^2 + l_{32}^2 + l_{33}^2 = 3 \;\Rightarrow\; 1 + 1 + l_{33}^2 = 3 \;\Rightarrow\; l_{33}^2 = 1 \;\Rightarrow\; l_{33} = 1
\end{aligned}$$

Thus we have

$$A = LL^T = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

First we find Y by solving

$$LY = B \;\Rightarrow\; \begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -10 \\ 7 \end{bmatrix}$$

$$\begin{array}{rcrcrcr}
y_1 & & & & & = & -1\\
2y_1 & + & 2y_2 & & & = & -10\\
-y_1 & - & y_2 & + & y_3 & = & 7
\end{array}$$

The first equation gives $y_1 = -1$.
The second equation gives $2y_2 = -10 + 2 = -8 \;\Rightarrow\; y_2 = -4$.
The third equation gives $y_3 = 7 - 1 - 4 = 2$.

To find x, we solve

$$L^T X = Y \;\Rightarrow\; \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \\ 2 \end{bmatrix}$$

$$\begin{array}{rcrcrcr}
x_1 & + & 2x_2 & - & x_3 & = & -1\\
 & & 2x_2 & - & x_3 & = & -4\\
 & & & & x_3 & = & 2
\end{array}$$

Last equation gives $x_3 = 2$. Putting $x_3 = 2$ into the second equation we have
$2x_2 - 2 = -4 \;\Rightarrow\; 2x_2 = -4 + 2 = -2 \;\Rightarrow\; x_2 = -1$.
Finally, substituting the value of x_2 and x_3 into the first equation we find
$x_1 + 2(-1) - (2) = -1 \;\Rightarrow\; x_1 = -1 + 2 + 2 = 3$.
Therefore, the solution is $(x_1, x_2, x_3) = (3, -1, 2)$.

(b) See Matlab sheets for solution of part(b). □

5. Consider the linear system

$$\begin{array}{rcrcrcr}
x & + & 2y & - & z & = & 0\\
2x & + & 8y & - & 4z & = & 6\\
-x & - & 4y & + & 3z & = & -2
\end{array}$$

(a) Perform two iterations of Jacobi method starting with the zero vector.
(b) Use Matlab to perform a maximum of 35 iterations of Jacobi method starting with the zero vector and tolerance of 10^{-6}. Does it converge? If yes, how many iterations does it take to converge?
(c) Perform two iterations of Gauss-Seidel method starting with the zero vector.
(d) Use Matlab to perform a maximum of 35 iterations of Gauss-Seidel method starting with the zero vector and tolerance of 10^{-6}. Does it converge? If yes, how many iterations does it take to converge?

Solution. (a) The jacobi iterations for the above system are:

$$\begin{array}{rcl} x_{k+1} & = & -2y_k + z_k \\ y_{k+1} & = & (6 - 2x_k + 4z_k)/8 \\ z_{k+1} & = & (-2 + x_k + 4y_k)/3 \end{array}$$

Starting vector is $\mathbf{P_0} = (x_0, y_0, z_0) = (0,0,0)$. Setting these values we obtain

$$x_1 = 0,\ y_1 = \frac{6}{8} = \frac{3}{4} \text{ or } 0.75, \text{ and } z_1 = -\frac{2}{3} \text{ or } -0.6667$$

The next iteration will give

$$x_2 = -2\left(\frac{3}{4}\right) + \left(-\frac{2}{3}\right) = -\frac{3}{2} - \frac{2}{3} = -\frac{13}{6} = -2.1667$$

$$y_2 = \frac{6 - 2(0) + 4(-\frac{2}{3})}{8} = \frac{6 - \frac{8}{3}}{8} = \frac{5}{12} = 0.4167$$

$$z_2 = \frac{-2 + 0 + 4(\frac{3}{4})}{3} = \frac{-2 + 3}{3} = \frac{1}{3} = 0.3333$$

(b) See Matlab sheets for solution.

(c) The Gauss-Seidel iterations for the above system are:

$$\begin{array}{rcl} x_{k+1} & = & -2y_k + z_k \\ y_{k+1} & = & (6 - 2x_{k+1} + 4z_k)/8 \\ z_{k+1} & = & (-2 + x_{k+1} + 4y_{k+1})/3 \end{array}$$

Starting vector is $\mathbf{P_0} = (x_0, y_0, z_0) = (0,0,0)$. Setting these values we obtain

$$x_1 = 0,\ y_1 = \frac{6}{8} = \frac{3}{4}, \text{ and } z_1 = \frac{-2 + 0 + 4(\frac{3}{4})}{3} = \frac{-2 + 3}{3} = \frac{1}{3} \text{ or } 0.3333$$

The next iteration will give

$$x_2 = -2\left(\frac{3}{4}\right) + \frac{1}{3} = -\frac{3}{2} + \frac{1}{3} = -\frac{7}{6} = -1.1667$$

$$y_2 = \frac{6 - 2(-\frac{7}{6}) + 4(\frac{1}{3})}{8} = \frac{6 + \frac{7}{3} + \frac{4}{3}}{8} = \frac{29}{24} = 1.2083$$

$$z_2 = \frac{-2 + (-\frac{7}{6}) + 4(\frac{29}{24})}{3} = \frac{-2 - \frac{7}{6} + \frac{29}{6}}{3} = \frac{10}{18} = \frac{5}{9} = 0.5556$$

(d) See Matlab sheets for solution. $\square$