Cholesky Factorization

This factorization can only be applied to positive define matrices.

Definition: A matrix A is a positive definite matrix if it <u>is symmetric</u> and <u>if $x^T Ax > 0$</u> for any nonzero vector x.

A <u>matrix is symmetric</u> if $A^T = A$ e.g. $A = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = A \Rightarrow Sym.$

$A = \begin{bmatrix} 1 & 0 \\ 2 & 5 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 1 & 2 \\ 0 & 5 \end{bmatrix} \neq A \Rightarrow not\ Sym.$

A matrix A is a positive definite matrix if it is symmetric and all its eigen values are positive (+ve) (>0)

If **<u>A is a positive definite matrix</u>**, then A can be factored in the form

$A = L\,L^T \rightarrow Cholesky\ factorization$

$L \rightarrow Lower\ triangular\ matrix$

$L^T \rightarrow Upper\ triangular\ matrix$

The elements of L are found by equating elements in Cholesky factorization.

Example 1: Find the Cholesky decomposition (factorization) of the matrix A

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 0 \\ 3 & 0 & 9 \end{bmatrix}$$

Solution: $A = L\,L^T$

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 0 \\ 3 & 0 & 9 \end{bmatrix} = \underbrace{\begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}}_{L^T}$$

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 5 & 0 \\ 3 & 0 & 9 \end{bmatrix} = \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{21}l_{11} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{31}l_{11} & l_{31}l_{21} + l_{32}l_{22} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Equating elements on both sides, we obtain

$(1) \leftarrow l_{1\,1}^2 = 2 \Rightarrow l_{1\,1} = \sqrt{2}$

$(2) \leftarrow l_{1\,1} l_{2\,1} = 1 \Rightarrow (\sqrt{2}) l_{2\,1} = 1 \Rightarrow l_{2\,1} = \dfrac{1}{\sqrt{2}}$

$(3) \leftarrow l_{1\,1} l_{3\,1} = 3 \Rightarrow (\sqrt{2}) l_{3\,1} = 3 \Rightarrow l_{3\,1} = \dfrac{3}{\sqrt{2}}$

$(4) \leftarrow l_{2\,1}^2 + l_{2\,2}^2 = 5 \Rightarrow \left(\dfrac{1}{\sqrt{2}}\right)^2 + l_{2\,2}^2 = 5 \Rightarrow l_{2\,2}^2 = 5 - \dfrac{1}{2} = \dfrac{9}{2} \Rightarrow l_{2\,2} = \sqrt{\dfrac{9}{2}} = \dfrac{3}{\sqrt{2}}$

$(5) \leftarrow l_{2\,1} l_{3\,1} + l_{2\,2} l_{3\,2} = 0 \Rightarrow \left(\dfrac{1}{\sqrt{2}}\right)\left(\dfrac{3}{\sqrt{2}}\right) + \left(\dfrac{3}{\sqrt{2}}\right)(l_{3\,2}) = 0 \Rightarrow \dfrac{3}{\sqrt{2}} l_{3\,2} = -\dfrac{3}{2} \Rightarrow l_{3\,2}$

$= -\dfrac{3}{2} \cdot \dfrac{\sqrt{2}}{3} = -\dfrac{1}{\sqrt{2}}$

$(6) \leftarrow l_{3\,1}^2 + l_{3\,2}^2 + l_{3\,3}^2 = 9 \Rightarrow \left(\dfrac{3}{\sqrt{2}}\right)^2 + \left(\dfrac{-1}{\sqrt{2}}\right)^2 + l_{3\,3}^2 = 9 \Rightarrow l_{3\,3}^2 = 9 - \dfrac{9}{2} - \dfrac{1}{2} = 9 - \dfrac{10}{2} = 4$

$\Rightarrow l_{3\,3} = \sqrt{4} = 2$

So, the Cholesky decomposition (factorization) is

$$A = LL^T = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} & 0 \\ \frac{3}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 2 \end{bmatrix} \begin{bmatrix} \sqrt{2} & \frac{1}{\sqrt{2}} & \frac{3}{\sqrt{2}} \\ 0 & \frac{3}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & 0 & 2 \end{bmatrix} \quad \text{Ans}$$

The computational complexity of Cholesky factorization is $O(N^3)$.

If we want to solve a system $Ax = B$ where A is a positive definite matrix then we can find Cholesky decomposition of A and get

$Ax = B \Rightarrow LL^T X = B$

Let $L^T X = Y$

First, we solve $LY = B$ for Y using forward substitution and then we solve $L^T X = Y$ for X using backward substitution.

MATLAB has a built-in command to find Cholesky factorization.

$chol(A) \rightarrow the\ upper\ triangular\ matrix \rightarrow L^T \rightarrow U$

$U^T \rightarrow L$

```
>>A=[2 1 3; 1 5 0; 3 0 9];
>>U=chol(A)
>>L=U^T
```

3.6 Iterative Methods for Linear Systems

A fundamental method in computer science is ITERATION i.e., a process is repeated until an answer is obtained. Iterative methods are used to solve linear and nonlinear systems, finding roots of nonlinear equations and solution of differential equations.

An iterative technique to solve the linear system $Ax = B$ is that we start with an intital approximation $P_o \rightarrow (\ldots)$ and we do iterations to find new values of $x \Rightarrow x_{k+1} = Tx_k + C$, we check if $\|x_{k+1} - x_k\|$ is getting smaller or larger. **<u>If $\|x_{k+1} - x_k\| \rightarrow 0$ after some iterations</u>**, then we say that the method **<u>converges</u>** and **x_{k+1} will be the solution**. However, if **<u>$\|x_{k+1} - x_k\| \rightarrow \infty$ or iterations are going back and forth</u>** (i.e., the sequence X_k is not converging) then **the method diverges.**

<u>JACOBI Method:</u> We start with an initial guess and perform iterations by a certain pattern. We find x_1 from the 1st equation, x_2 from the 2nd equation and so on.

Example 1: Consider the system

$$\begin{aligned} 5x - y + z &= 10 \rightarrow (1) \\ 2x + 8y - z &= 11 \rightarrow (2) \\ -x + y + 4z &= 3 \rightarrow (3) \end{aligned}$$

Starting with a zero vector, perform 3 iteration of Jacobi method.

Solution:

$$1st\ eq \Rightarrow x_{k+1} = \frac{10 + y_k - z_k}{5}$$

$$2st\ eq \Rightarrow y_{k+1} = \frac{11 - 2x_k + z_k}{8}$$

$$3st\ eq \Rightarrow z_{k+1} = \frac{3 + x_k - y_k}{4}$$

$$x_o = (x_o, y_o, z_o) = (0, 0, 0)$$

***For k* = 0** $\Rightarrow$

$$\left.\begin{aligned} x_1 &= \frac{10+Y_0-Z_0}{5} = \frac{10+0-}{5} = 2 \\ y_1 &= \frac{11-2x_o+Z_0}{8} = \frac{11-0+}{8} = \frac{11}{8} \quad = 1.375 \\ z_1 &= \frac{3+x_0-Y_0}{4} = \frac{3+0-0}{4} = \frac{3}{4} = 0.75 \end{aligned}\right\} \quad \begin{aligned} &\Rightarrow 1^{st}\ iteration \\ &(x_1, y_1, z_1) = (2, 1.375, 0.75) \end{aligned}$$

***For k* = 1** $\Rightarrow$

$$\left.\begin{aligned}
x_2 &= \frac{10+Y_1-Z_1}{5} = \frac{10+1.375- \ .75}{5} = 2.125 \\
y_2 &= \frac{11-2x_1+Z_1}{8} = \frac{11-2(2)+(0.75)}{8} = \frac{11}{8} \ = 0.96875 \\
z_2 &= \frac{3+x_1-Y_1}{4} = \frac{3+2-1.375}{4} = 0.90625
\end{aligned}\right\}$$

$\Rightarrow 2^{nd}\ \ iteration$

$(x_2, y_2, z_2) = (2.125, 0.96875, 0.90625)$

***Fork* = 2** $\Rightarrow$

$$\left.\begin{aligned}
x_3 &= \frac{10+Y_2-Z_2}{5} = \frac{10+0.96875 \quad .90625}{5} = 2.0125 \\
y_3 &= \frac{11-2 \ \ _2+Z_2}{8} = \frac{11-2(2.125)+0.90625}{8} = 0.957013125 \\
z_3 &= \frac{3+x_2-Y_2}{4} = \frac{3+2.125-0.96875}{4} = 1.0390625
\end{aligned}\right\}$$

$\Rightarrow 3^{rd}\ iteration$

$(x_3, y_3, z_3) = (2.0125, 0.957, 1.039)$

Using Jacobi method program, it converges to (2, 1, 1) in 12 iterations with tolerance 10^{-5}.
Textbook has program on page 163-165

if $\|x_{k+1} - x_k\| < tol$
$step \rightarrow end$

function x = Jacobi (A, B, P, tol, max ite)

max number of iterations

output

Coeff. matrix

right hand side

initial approximation

Ax = B

Can be changed to
$[x, k]$

number of iterations

Example 2: Consider the system.

$$
\begin{aligned}
2x + 8y - z &= 11\\
5x - y + z &= 10\\
-x + y + 4z &= 3
\end{aligned}
$$

starting with a zero vector, perform 3 iterations of Jacobi method.

Solution:

$$x_{k+1} = \frac{11 - 8y_k + z_k}{2} \leftarrow from\ 1st\ eq.$$

$$y_{k+1} = \frac{10 - 5x_k - z_k}{-1} \leftarrow from\ 2nd\ eq.$$

$$z_{k+1} = \frac{3 + x_k - y_k}{4} \leftarrow from\ 3rd\ eq.$$

$$(x_0, y_0, z_0) = (0, 0, 0)$$

$$
\left.
\begin{aligned}
x_1 &= \frac{11 - 8y_0 + z_0}{2} = \frac{11 - 0 + 0}{2} = 5.5\\
y_1 &= \frac{10 - 5x_0 - z_0}{-1} = \frac{10 - 0 + 0}{-1} = -10\\
z_1 &= \frac{3 + x_0 - y_0}{4} = \frac{3 + 0 - 0}{4} = \frac{3}{4} = 0.75
\end{aligned}
\right\} \text{1st iteration}
$$

$$
\left.
\begin{aligned}
x_2 &= \frac{11 - 8y_1 + z_1}{2} = \frac{11 - 8(-10) + 0.75}{2} = 45.875\\
y_2 &= \frac{10 - 5x_1 - z_1}{-1} = \frac{10 - 5(5.5) - (0.75)}{-1} = 18.25\\
z_2 &= \frac{3 + x_1 - y_1}{4} = \frac{3 + 5.5 - (-10)}{4} = 4.625
\end{aligned}
\right\} \text{2nd iteration}
$$

$$
\left.
\begin{aligned}
x_3 &= \frac{11 - 8y_2 + z_2}{2} = \frac{11 - 8(-18.25) + 4.625}{2} = -65.1875\\
y_3 &= \frac{10 - 5x_2 - z_2}{-1} = \frac{10 - 5(45.875) - (4.625)}{-1} = 224\\
z_3 &= \frac{3 + x_2 - y_2}{4} = \frac{3 + 45.875 - 18.25}{4} = 7.6525
\end{aligned}
\right\} \begin{array}{l}\text{3rd iteration}\\ (x_3, y_3, z_3) = -65.1875, 224, 7.6525)\end{array}
$$

Using computers, it will diverge after 15 iterations answer is $[-0.3030\ \ 1.5299\ \ 0.0557] \times 10^{10}$

Condition of Convergence

An $N \times N$ matrix A is said to be strictly diagonally dominant provided that

$$|a_{k\,k}| > \sum_{\substack{j=1 \\ j \neq k}}^{N} |a_{k\,j}| \text{ for } k = 1,2, \dots, N$$

This means that in each row of the matrix the magnitude of the diagonal element is bigger than the sum of the magnitude of all other elements in the row.

If A is a strictly diagonally dominant matrix, then Jacobi method to solve linear system Ax=B converges. (Condition of convergence)

This condition is sufficient condition but not necessary. If A is not strictly diagonally dominant, then Jacobi method may or may not converge.

In Example 1, the system is

$$\begin{aligned} 5x - y + z &= 10 \\ 2x + 8y - z &= 11 \\ -x + y + 4z &= 3 \end{aligned}$$

$$A = \begin{bmatrix} 5 & -1 & 1 \\ 2 & 8 & -1 \\ -1 & 1 & 4 \end{bmatrix}$$

$$\begin{aligned} |5| > |-1| + |1| &\Rightarrow 5 > 2 \checkmark \\ |8| > |2| + |-1| &\Rightarrow 8 > 3 \checkmark \\ |4| > |-1| + |1| &\Rightarrow 4 > 2 \checkmark \end{aligned}$$

A is strictly diagonally dominant
Jacobi method will converge.

In Example 2, the system is

$$\begin{aligned} 2x + 8y - z &= 11 \\ 5x - y + z &= 10 \\ -x + y + 4z &= 3 \end{aligned}$$

$$A = \begin{bmatrix} 2 & 8 & -1 \\ 5 & -1 & 1 \\ -1 & 1 & 4 \end{bmatrix}$$

$$\begin{aligned} |2| > |8| + |1| &\Rightarrow 2 \ngtr 9 \quad \times \\ |-1| > |5| + |1| &\Rightarrow 1 \ngtr 6 \quad \times \\ |4| > |-1| + |1| &\Rightarrow 4 > 2 \quad \checkmark \end{aligned}$$

A is not strictly diagonally dominant. We have seen that Jacobi method diverges.