Triangular Factorization (Matrix Factorization)

If A is a non-singular matrix, then the linear system Ax=B can be solved by using Gaussian elimination. Another method is that we factor the matrix A in terms of a lower triangular matrix and an upper triangular matrix. Then we can use forward and backward substitution to solve the system.

Case 1: **The Gaussian elimination** is performed **without any row interchanges**. That is, no pivoting and $a_{k\,k} \neq 0$ for all k= 0, 1, 2, …, N.

In this case the matrix A can be factored as A=LU where

$$\underbrace{\begin{bmatrix} a_{1\,1} & a_{1\,2} & a_{1\,3} & \cdots & a_{1\,N} \\ a_{2\,1} & a_{2\,2} & a_{2\,3} & \cdots & a_{2\,N} \\ a_{3\,1} & a_{3\,2} & a_{3\,3} & \cdots & a_{3\,N} \\ \vdots & \vdots & \vdots & & \\ a_{N\,1} & a_{N\,2} & a_{N\,3} & \cdots & a_{N\,N} \end{bmatrix}}_{\mathbf{A}} \mathbf{=} \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ m_{2\,1} & 1 & 0 & \cdots & 0 \\ m_{3\,1} & m_{3\,2} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \\ m_{N\,1} & m_{N\,2} & m_{N\,3} & \cdots & 1 \end{bmatrix}}_{\mathbf{L}} \underbrace{\begin{bmatrix} u_{1\,1} & u_{1\,2} & u_{1\,3} & \cdots & u_{1\,N} \\ 0 & u_{2\,2} & u_{2\,3} & \cdots & u_{2\,N} \\ 0 & 0 & u_{3\,3} & \cdots & u_{3\,N} \\ \vdots & \vdots & \vdots & & \\ 0 & 0 & 0 & \cdots & u_{N\,N} \end{bmatrix}}_{\mathbf{U}}$$

$U_{k\,k} \neq 0 \; for \; k = 1,2, \dots, N$

All diagonal entries of L are 1, ($l_{k\,k} = 1 \; for \; k = 1,2, \dots, N$).
m's are multipliers calculated during the process of Gaussian elimination.

To solve the system Ax=B (x is a solution)

$$L\underbrace{Ux}_{y}=B \text{ – } (*)$$

Let $UX = y \overset{(*)}{\Rightarrow} Ly = B$

First, we solve $Ly = B$ by using forward substitution (b/c lower triangular matrix) and then solve $Ux = y$ by using back substitution.

Example 1: Consider the system

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 4x_3 &= 4 \\ x_1 - 3x_2 - 2x_3 &= 5 \end{aligned}$$

Find LU decomposition of the coefficient matrix A with no pivoting.
Then **solve** the resulting system using forward and back substitution.

Solution:

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{bmatrix}$$

$$\begin{array}{ccccc} R_3 & 0 & -7 & -5 \\ \frac{7}{3}R_2 & 0 & -7 & -\frac{14}{7} \\ & - & + & + \\ \hline & 0 & 0 & -\frac{1}{3} \end{array}$$

$m_{2\,1} = \frac{2}{1} = 2 \qquad R_2 - 2R_1 \to R_2$

$m_{3\,1} = \frac{1}{1} = 1 \qquad R_3 - R_1 \to R_3 \qquad \begin{bmatrix} 1 & 4 & 3 \\ 0 & -3 & -2 \\ 0 & -7 & -5 \end{bmatrix}$

$m_{3\,2} = -\frac{7}{-3} = \frac{7}{3} \qquad R_3 - \frac{7}{3}R_2 \to R_3 \qquad \underbrace{\begin{bmatrix} 1 & 4 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}}_{U}$

We have $A = LU$ where $L = \begin{bmatrix} 1 & 0 & 0 \\ m_{2\,1} & 1 & 0 \\ m_{3\,1} & m_{3\,2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{7}{3} & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & 4 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$

First, we solve $Ly = B \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 1 & \frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix}$

$$\begin{array}{lclcl} y_1 = 1 & \Rightarrow & y_1 = 1 & & \\ 2y_1 + y_2 = 4 & \Rightarrow & 2(1) + 4 & \Rightarrow & y_2 = 2 \\ y_1 + \frac{7}{3}y_2 + y_3 = 5 & \Rightarrow & 1 + \frac{7}{3}(2) + y_3 = 5 & \Rightarrow & y_3 = 5 - 1 - \frac{14}{3} = -\frac{2}{3} \end{array}$$

$$\Rightarrow y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -\frac{2}{3} \end{bmatrix}$$

Now we solve $UX = y \Rightarrow \begin{bmatrix} 1 & 4 & 3 \\ 0 & -3 & -2 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -\frac{2}{3} \end{bmatrix}$

$$x_1 + 4x_2 + 3x_3 = 1$$
$$-3x_2 - 2x_3 = 2$$
$$-\frac{1}{3}x = -\frac{2}{3} \Rightarrow x_3 = 2$$

(2) $\Rightarrow -3x_2 - 2(2) = 2 \Rightarrow -3x_2 = 6 \Rightarrow x_2 = -2$

(1) $\Rightarrow x_1 + 4(-2) + 3(2) = 1 \Rightarrow x_1 = 1 + 8 - 6 = 3$

The solution is $(x_1, x_2, x_3) = (3, -2, 2)$ or $x = \begin{bmatrix} 3 \\ -2 \\ 2 \end{bmatrix}$ Ans

50 x 50 system

N=50

$(50)^2 = 2500$ operations

$(50)^3 = 125000$ operations using more storage and time and more roundoff errors.

Computational complexity of backward substitution is of $O(N^2)$.
Computational complexity of forward substitution is of $O(N^2)$.
Computational complexity of Gaussian elimination is $O(N^3)$.
Computational complexity of LU decomposition is $O(N^3)$.

<u>In Gaussian elimination</u>, we start with $[A|B]$

If B changes then we need to do all calculations again.

<u>In Lu decomposition</u>, we do not use B. The B is only used in forward substitution.

If a linear <u>system is to be solved many times</u> with same coefficient matrix A but with different B, then <u>LU decomposition is more efficient</u> in terms of computational complexity.

However, if a <u>system is to be solved only one time, then Gaussian elimination</u> is used.

<u>Case 2:</u> When the **Gaussian elimination** is performed **with row interchanged**, that is, partial pivoting or any of $a_{k\,k} = 0$

$$I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

A permutation matrix P is an N x N matrix obtained by row interchanges of the identity matrix I_N.

P has only one nonzero entry in each row and row and that entry is 1.

In the case of row interchanges, we can factor PA in terms of lower triangular matrix and upper triangular matrix.

$$R_2 \leftrightarrow R_4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

P

$$PA = LU \text{ where } L = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ m_{2\,1} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ m_{N\,1} & m_{N\,2} & \cdots & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} u_{1\,1} & u_{1\,2} & \cdots & u_{1\,N} \\ 0 & u_{2\,2} & \cdots & u_{2\,N} \\ 0 & 0 & \cdots & u_{N\,N} \end{bmatrix}$$

In this case we need to switch rows of L, but we can only interchange the elements in the row which are permissible, i.e., they do not change the patterns of L.

To solve Ax=B, we multiply by $P \Rightarrow PAX = PB$

$$LUX = PB$$

Let $UX = Y \Rightarrow LY = PB$

First, we <u>solve $LY = PB$ by forward substitution</u>
and **then** we <u>solve $UX = Y$ by backward substitution</u>.

Example 2: Consider the system

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 4x_3 &= 4 \\ x_1 - 3x_2 - 2x_3 &= 5 \end{aligned}$$

Find LU decomposition of the coefficient matrix A **<u>using partial pivoting</u>**, **then** solve the system using forward and backward substitution.

Solution:

$$A = \begin{bmatrix} 1 & 4 & 3 \\ 2 & 5 & 4 \\ 1 & -3 & -2 \end{bmatrix} \qquad \overset{P}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$R_1 \leftrightarrow R_2 \begin{bmatrix} 2 & 5 & 4 \\ 1 & 4 & 3 \\ 1 & -3 & -2 \end{bmatrix} \qquad R_1 \leftrightarrow R_2 \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\begin{aligned} m_{2\,1} &= \frac{1}{2} \quad R_2 - \frac{1}{2}R_1 \to R_2 \\ m_{3\,1} &= \frac{1}{2} \quad R_3 - \frac{1}{2}R_1 \to R_3 \end{aligned} \quad \begin{bmatrix} 2 & 5 & 4 \\ 0 & \frac{3}{2} & 1 \\ 0 & -\frac{11}{2} & -4 \end{bmatrix}$$

$$\left|-\frac{11}{2}\right| > \left|\frac{3}{2}\right| \quad R_2 \leftrightarrow R_3 \begin{bmatrix} 2 & 5 & 4 \\ 0 & -\frac{11}{2} & -4 \\ 0 & \frac{3}{2} & 1 \end{bmatrix} \qquad \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}}_{P}$$

$$m_{3\,2} = \frac{\frac{3}{2}}{-\frac{11}{2}} = -\frac{3}{11}$$

$$\begin{aligned} & R_3 - \left(-\frac{3}{11}\right)R_2 \to R_3 \\ \text{or} \quad & R_3 + \frac{3}{11}R_2 \to R_3 \end{aligned} \quad \underbrace{\begin{bmatrix} 2 & 5 & 4 \\ 0 & -\frac{11}{2} & -4 \\ 0 & 0 & -\frac{1}{11} \end{bmatrix}}_{U} \qquad 1 - \left(\frac{12}{11}\right) = \frac{11-12}{11}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ m_{2\,1} & 1 & 0 \\ m_{3\,1} & m_{3\,2} & 1 \end{bmatrix} \qquad \begin{aligned} & R_1 \leftrightarrow R_2\, will\ not\ change\ anything. \\ & R_2 \leftrightarrow R_3\, will\ change\ m_{2\,1}\, and\ m_{3\,1} \end{aligned}$$

So here $L = \begin{bmatrix} 1 & 0 & 0 \\ m_{3\,1} & 1 & 0 \\ m_{2\,1} & m_{3\,2} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{3}{11} & 1 \end{bmatrix}$

To solve, we first solve $LY = PB$

$$PB = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & -\frac{3}{11} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

$$y_1 = 4$$

$$\frac{1}{2}y_1 + y_2 = 5 \quad \Rightarrow \quad \frac{1}{2}(4) + y_2 = 5 \quad \Rightarrow \quad y_2 = 3$$

$$\frac{1}{2}y_1 - \frac{3}{11}y_2 + y_3 = 1 \quad \Rightarrow \quad \frac{1}{2}(4) - \frac{3}{11}(3) + y_3 \quad \Rightarrow \quad y_3 = 1 - 2 + \frac{9}{11} = -\frac{2}{11}$$

$$\boxed{\Rightarrow y = \begin{bmatrix} 4 \\ 3 \\ -\frac{2}{11} \end{bmatrix}}$$

Now we solve $Ux = Y \Rightarrow \begin{bmatrix} 2 & 5 & 4 \\ 0 & -\frac{11}{2} & -4 \\ 0 & 0 & -\frac{1}{11} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ -\frac{2}{11} \end{bmatrix}$

$$\begin{aligned} 2x_1 + 5x_2 + 4x_3 &= 4 - (1) \\ -\frac{11}{2}x_2 - 4x_3 &= 3 - (2) \\ -\frac{1}{11}x_3 &= -\frac{2}{11} \quad \Rightarrow \quad x_3 = 2 \end{aligned}$$

$$(2) \Rightarrow -\frac{11}{2}x_2 - 4(2) = 3 \quad \Rightarrow \quad -\frac{11}{2}x_2 = 11 \quad \Rightarrow \quad x_2 = -2$$

$$(1) \Rightarrow 2x_1 + 5(-2) + 4(2) = 4 \quad \Rightarrow \quad 2x_1 = 4 + 10 - 8 = 6 \quad \Rightarrow \quad x_1 = 3$$

The solution is $(x_1, x_2, x_3) = (3, -2, 2)$ Ans

By default, MATLAB do partial pivoting for LU decomposition. MATLAB has a built-in command [L U P] = lu(A)

```
>> A = [1 4 3; 2 5 4; 1 -3 -2];
>>[L U P] = lu(A)
```

output

$$L = \begin{matrix} 1 & 0 & 0 \\ 0.5 & 1 & 0 \\ 0.5 & -0.2727 & 1 \end{matrix}$$

$$U = \begin{matrix} 2 & 5 & 4 \\ 0 & -5.5 & -4 \\ 0 & 0 & -0.0909 \end{matrix}$$

$$P = \begin{matrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{matrix}$$

Do not write $[L\ U] = \ln\,(A)$ it will not give correct L and U

Also, if you only write
>>lu(A)
then it will return correct U but not L (not lower triangular)

Textbook has program for back substitution on page 123.
You have to modify that program for forward substitution.

using the x, you will solve the system