MATH 3940 Numerical Analysis for Computer Scientists

Problem Set 4 Solutions

1. Consider the system of nonlinear equations

$$\begin{array}{rcrcr} 2x & + & 2y & = & 3 \\ 3x^2 & + & 2y & = & 4 \end{array}$$

(a) Find the exact solutions.
(b) Perform 2 iterations of Jacobi method starting with the initial values $x_0 = 0$ and $y_0 = 0$.
(c) Perform 2 iterations of Gauss-Seidel method starting with the initial values $x_0 = 0$ and $y_0 = 0$.
(d) Use Matlab to perform 10 iterations of Jacobi method starting with $x_0 = 0$ and $y_0 = 0$, and tolerance 10^{-5}. Does it converge?
(e) Use Matlab to perform 10 iterations of Gauss-Seidel method starting with $x_0 = 0$ and $y_0 = 0$, and tolerance 10^{-5}. Does it converge?
(f) Do you expect Jacobi or Gauss Seidel method to converge starting with $x_0 = 0$ and $y_0 = 0$? Justify your answer using the conditions of convergence.

Solution. (a) We have to solve the system of equations

$$\begin{array}{rcrcr} 2x & + & 2y & = & 3 \\ 3x^2 & + & 2y & = & 4 \end{array}$$

Multiplying second equation by -1 and adding to the first equation, we have

$$\begin{array}{rcrcr} 2x & + & 2y & = & 3 \\ -3x^2 & - & 2y & = & -4 \\ \hline -3x^2 & + & 2x & = & -1 \end{array}$$

$3x^2 - 2x - 1 = 0 \Rightarrow (3x+1)(x-1) = 0 \Rightarrow x = -\frac{1}{3}, 1.$
Substituting $x = -\frac{1}{3}$ into the first equation we have

$$2x + 2y = 3 \Rightarrow -\frac{2}{3} + 2y = 3 \Rightarrow 2y = \frac{11}{3} \Rightarrow y = \frac{11}{6}$$

Substituting $x = 1$ into the first equation we have

$$2x + 2y = 3 \Rightarrow 2 + 2y = 3 \Rightarrow 2y = 1 \Rightarrow y = \frac{1}{2}$$

Thus the solution is $(-\frac{1}{3}, \frac{11}{6})$ and $(1, \frac{1}{2})$.
(b) Solving the first equation for x and second equation for y, Jacobi iterations are

$$x_k = \frac{-2y_{k-1} + 3}{2} \quad \text{and} \quad y_k = \frac{-3x_{k-1}^2 + 4}{2}$$

Initial values are $x_0 = 0$ and $y_0 = 0$. Setting these values in the iterations, we get

$$x_1 = \frac{-2(0)+3}{2} = \frac{3}{2} = 1.5 \quad \text{and} \quad y_1 = \frac{-3(0)^2+4}{2} = 2$$

The next iteration will give

$$x_2 = \frac{-2(2)+3}{2} = -\frac{1}{2} = -0.5 \quad \text{and} \quad y_2 = \frac{-3(1.5)^2+4}{2} = -\frac{2.75}{2} = -1.375$$

(c) Gauss-Seidel method uses the recent values, so the iterations are

$$\begin{aligned} x_{k+1} &= \frac{-2y_k+3}{2} \\ y_{k+1} &= \frac{-3x_{k+1}^2+4}{2} \end{aligned}$$

Initial values are $x_0 = 0$ and $y_0 = 0$. Setting these values in the iterations, we get

$$\begin{aligned} x_1 &= \frac{0+3}{2} = \frac{3}{2} = 1.5 \\ y_1 &= \frac{-3(9/4)+4}{2} = -\frac{11}{8} = -1.375 \end{aligned}$$

The next iteration will give

$$\begin{aligned} x_2 &= \frac{-2(-1.375)+3}{2} = \frac{5.75}{2} = 2.875 \\ y_2 &= \frac{-3(2.875)^2+4}{2} = -\frac{20.796875}{2} = -10.3984375 \end{aligned}$$

(d) and (e) See Matlab sheets for the solution of parts (d) and (e).

(f) Solving the first equation for x and the second equation for y, we have

$$x = \frac{-2y+3}{2} \quad \Rightarrow g_1(x,y) = \frac{-2y+3}{2}$$

$$y = \frac{-3x^2+4}{2} \quad \Rightarrow g_2(x,y) = \frac{-3x^2+4}{2}$$

$$\frac{\partial g_1}{\partial x} = 0, \quad \frac{\partial g_1}{\partial y} = -1, \quad \frac{\partial g_2}{\partial x} = -3x, \quad \frac{\partial g_2}{\partial y} = 0$$

The functions g_1, g_2 and their first order partial derivatives are continuous on the region $R = \{(x,y) | -2 \le x \le 2, -2 \le y \le 2\}$, the initial guess and both solutions are also in this region.

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |0| + |-1| = 1$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = |-3x| + |0| = |-3x|$$

For $x = -\dfrac{1}{3}$, we have $\left|\dfrac{\partial g_2}{\partial x}\right| + \left|\dfrac{\partial g_2}{\partial y}\right| = |-3x| = \left|-3\left(-\dfrac{1}{3}\right)\right| = 1 \not< 1$

Also, for $x = 1$, we have $\left|\dfrac{\partial g_2}{\partial x}\right| + \left|\dfrac{\partial g_2}{\partial y}\right| = |-3x| = |-3\,(1)\,| = 3 \not< 1$

The sufficient condition for the convergence is not satisfied, so the iterations may or may not converge. We have seen in parts (d) and (e) that the iterations are diverging. □

2. (a) Find the Taylor polynomial of degree 3 for $f(x) = \dfrac{1}{x+1}$ expanded about $x_0 = 0$.
(b) Does $f(x) = \dfrac{1}{x+1}$ have a Taylor polynomial expansion about $x_0 = -1$? Justify your answer.

Solution. (a) Here $f(x) = \dfrac{1}{x+1} = (x+1)^{-1} \;\Rightarrow\; f(0) = 1.$

$$f'(x) = -1(x+1)^{-2} = -\frac{1}{(x+1)^2} \quad\Rightarrow\; f'(0) = -1$$
$$f''(x) = 2(x+1)^{-3} = \frac{2}{(x+1)^3} \quad\Rightarrow\; f''(0) = 2$$
$$f'''(x) = -6(x+1)^{-4} = -\frac{6}{(x+1)^4} \quad\Rightarrow\; f'''(0) = -6$$

Taylor polynomial of degree 3 expanded about $x_0 = 0$ is given by

$$P_3(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3$$
$$P_3(x) = f(0) + \frac{f'(0)}{1!}(x - 0) + \frac{f''(0)}{2!}(x - 0)^2 + \frac{f'''(0)}{3!}(x - 0)^3$$
$$= 1 + \frac{-1}{1}(x) + \frac{2}{2}(x)^2 - \frac{6}{6}(x)^3 = 1 - x + x^2 - x^3$$

(b) Since the function f and its derivatives are not defined at $x_0 = -1$, the function $f(x)$ does not have a Taylor polynomial expansion about $x_0 = -1$. □

3. Consider the data

x_i	-2	-1	0	1	2	3
y_i	1	4	11	16	13	-4

(a) Use Matlab built in functions to find an interpolation polynomial of degree 5 for the data and to approximate the value at $x = -1.5$.
(b) Find Lagrange polynomial $P_2(x)$ using the nodes x_0, x_1, and x_2.
(c) Use Matlab to find Lagrange polynomial $P_2(x)$ using the nodes x_0, x_1, and x_2.
(d) Find divided difference table and Newton polynomial using all nodes in the above table.
(e) Use Matlab to find divided difference table and Newton polynomial using all nodes in the above table.

Solution. (a) See Matlab sheets for the solution of part (a).

(b) Lagrange polynomial $P_2(x)$ is given by

$$P_2(x) = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

Substituting the values from the given data, we obtain

$$\begin{aligned}
P_2(x) &= 1 \frac{[x - (-1)](x - 0)}{[-2 - (-1)](-2 - 0)} + 4 \frac{[x - (-2)](x - 0)}{[-1 - (-2)](-1 - 0)} + 11 \frac{[x - (-2)][x - (-1)]}{[0 - (-2)][0 - (-1)]} \\
&= 1 \frac{(x + 1)(x)}{(-2 + 1))(-2)} + 4 \frac{(x + 2)(x)}{(-1 + 2)(-1)} + 11 \frac{(x + 2)(x + 1)}{(2)(1)} \\
&= \frac{x^2 + x}{2} + \frac{4x^2 + 8x}{-1} + \frac{11(x^2 + 3x + 2)}{2} \\
&= \frac{x^2 + x - 8x^2 - 16x + 11x^2 + 33x + 22}{2} = \frac{4x^2 + 18x + 22}{2} \\
&= 2x^2 + 9x + 11
\end{aligned}$$

(c) See Matlab sheets for the solution of part (c), the polynomial obtained is $P_2(x) = 2x^2 + 9x + 11$.

(d)

x_i	y_i	$f[,]$	$f[,\ ,]$	$f[,\ ,\ ,]$	$f[,\ ,\ ,\ ,]$	$f[,\ ,\ ,\ ,\ ,]$
-2	1					
-1	4	$\frac{4-1}{-1+2} = 3$				
0	11	$\frac{11-4}{0+1} = 7$	$\frac{7-3}{0+2} = 2$			
1	16	$\frac{16-11}{1-0} = 5$	$\frac{5-7}{1+1} = -1$	$\frac{-1-2}{1+2} = -1$		
2	13	$\frac{13-16}{2-1} = -3$	$\frac{-3-5}{2-0} = -4$	$\frac{-4+1}{2+1} = -1$	$\frac{-1+1}{2+2} = 0$	
3	-4	$\frac{-4-13}{3-2} = -17$	$\frac{-17+3}{3-1} = -7$	$\frac{-7+4}{3-0} = -1$	$\frac{-1+1}{3+1} = 0$	0

From divided differences table we have $a_0 = 1$, $a_1 = 3$, $a_2 = 2$, $a_3 = -1$, $a_4 = 0$, and $a_5 = 0$. Thus the Newton interpolation polynomial will be of degree 3 and is

$$\begin{aligned}
P(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) + a_3(x - x_0)(x - x_1)(x - x_2) \\
&= 1 + 3(x + 2) + 2(x + 2)(x + 1) - 1(x + 2)(x + 1)(x - 0) \\
&= 1 + 3x + 6 + 2(x^2 + 3x + 2) - x(x^2 + 3x + 2) \\
&= 1 + 3x + 6 + 2x^2 + 6x + 4 - x^3 - 3x^2 - 2x \\
&= -x^3 - x^2 + 7x + 11
\end{aligned}$$

(e) See Matlab sheets for the solution of part (e), the polynomial obtained is $P(x) = -x^3 - x^2 + 7x + 11$. □

4. Let $f(x) = x + e^{-x}$. The nodes are $x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, and $x_3 = 1.5$.

(a) Use Matlab to find lagrange polynomial $P_3(x)$ using the above nodes.

(b) Use Matlab to find Newton polynomial $P_3(x)$ using the above nodes.

(c) Use the error formula to find a bound for the error for $P_3(1.2)$ and compare the bound to the actual error. (use the Lagrange or Newton polynomial $P_3(x)$ obtained from Matlab in part(a) or (b)).

Solution. Using $f(x_i) = x_i + e^{-x_i}$, the values at the nodes are

x_i	0	0.5	1	1.5
$f(x_i)$	1	1.1065	1.3679	1.7231

See Matlab sheets for the solution of parts (a) and (b).
The Lagrange and Newton polynomial obtained in part (a) and part (b) is
$P_3(x) = -0.081467x^3 + 0.432x^2 + 0.017367x + 1$.

(c) The error formula for both Lagrange and Newton polynomial $P_3(x)$ is same and it is $E_3(x) = \dfrac{f^{(4)}(c)(x-0)(x-0.5)(x-1)(x-1.5)}{4!}$ for some $c \in [0, 1.5]$.

Now $f(x) = x + e^{-x}$, $f'(x) = 1 - e^{-x}$, $f''(x) = e^{-x}$, $f'''(x) = -e^{-x}$, $f^{(4)}(x) = e^{-x}$

For the error bound, we need to find the maximum value of $|f^{(4)}(x)|$.

Since $f^{(5)}(x) = -e^{-x}$ is never zero. We calculate the values of $f^{(4)}(x)$ at the end points $f^{(4)}(0) = e^{-0} = 1$, and $f^{(4)}(1.5) = e^{-1.5} = 0.22313$. So the maximum value is 1.

Thus the error bound for $P_3(1.2)$ is

$$
\begin{aligned}
|E_3(1.2)| &= \left| \frac{(e^{-c})(1.2)(1.2-0.5)(1.2-1)(1.2-1.5)}{4!} \right| \\
&\le \frac{(1)(1.2)(0.7)(0.2)(-0.3)}{24} = 0.0021
\end{aligned}
$$

The exact value at 1.2 is $f(1.2) = 1.2 + e^{-1.2} = 1.5011942$.

The polynomial obtained in part (a) and part (b) using Matlab is
$P_3(x) = -0.081467x^3 + 0.432x^2 + 0.017367x + 1$.

So $P_3(1.2) = -0.081467(1.2)^3 + 0.432(1.2)^2 + 0.017367(1.2) + 1 = 1.502145424$.

The exact error is $|P_3(1.2) - f(1.2)| = |1.502145424 - 1.5011942| = 0.000951224$.

So the exact error is smaller than the bound. $\square$