# 3.7 System of Nonlinear Equations

$$2x + y = 1 \quad \Rightarrow \quad x = \frac{1-y}{2} \quad \Rightarrow \quad g_1(x,y) = \frac{1-y}{2}$$

$$x^2 + y^2 = 4 \quad \Rightarrow \quad y = \sqrt{4 - x^2} \quad \Rightarrow \quad g_{2(x,y)} = \sqrt{4 - x^2}$$

For +ve y-coordinates

m file for the function

(x, y, …)

function $z = G(x)$ → saved as G.m file

$x = x(1);$

$y = x(2);$

$z = \text{zeroes}(1,2); \longrightarrow z = [0\ 0];$ means 1 row and 2 colounms

$x(1) = (1 - y) / 2;$

$z(2) = (4 - x\text{^}2)\text{^}(1/2);$

Program for Gauss Seidel method for nonlinear systems is on page 179 of the textbook.

function[P, iter] = seidel(G, P, tol, maxite)

for j=1:N

A=feval('~~G~~', X) Correction: Use A = feval(G, X)

inputs tolerance max iterations

In Matlab, we write [P, iter] = seidel$\overbrace{(\text{'G', (initial guess)}}^{\text{inputs}}$, $10^{-5}$, 20) to use the program

Textbook does not have program for Jacobi method for nonlinear systems, you need to modify Gauss-Seidel program to write a program for Jacobi method.

**Example 2:** from textbook page 167

System is
$$\begin{aligned} x^2 - 2x - y + 0.5 &= 0 - (1) \\ x^2 + 4y^2 - 4 &= 0 - (2) \end{aligned}$$

It can not be solved by hand calculations.
**Solutions** are (-0.222146, 0.9938) and (1.900677, 0.31122)

If we start with (0, 0)

$$(1) \Rightarrow 2x = x^2 - y + 0.5 \Rightarrow x = \frac{x^2 - y + 0.5}{2}$$

$$(2) \Rightarrow 4y^2 = 4 - x^2 \Rightarrow y = \frac{\pm\sqrt{4 - x^2}}{2}$$

$$\frac{1}{2}(4 - x^2)^{1/2} \qquad \frac{1}{2}x^2 - \frac{y}{2} + \frac{0.5}{2}$$

$$x_{k+1} = \underbrace{\frac{x_k^2 - y_k + 0.5}{2}}_{g_1}$$

&

$$y_{k+1} = \underbrace{\frac{\sqrt{4 - x_k^2}}{2}}_{g_2}$$

Jacobi & Gauss Seidel converge to (-0.222146, 0.9938)

$$g_1(x, y) = \frac{x^2 - y + 0.5}{2} \ \& \ g_2(x, y) = \frac{\sqrt{4 - x^2}}{2}$$

$$\frac{\partial g_1}{\partial x} = \frac{1}{2}(2x) = x \ \& \ \frac{\partial g_2}{\partial x} = \frac{1}{2} \cdot \frac{1}{2}(4 - x^2)^{-\frac{1}{2}} \cdot (-2x) = \frac{-x}{2\sqrt{4 - x^2}}$$

$$\boxed{\begin{array}{l} 4 - x^2 > 0 \Rightarrow x^2 < 4 \\ -2 < x < 2 \end{array}}$$

$$\frac{\partial g_1}{\partial y} = \frac{-1}{2} \ \& \ \frac{\partial g_2}{\partial y} = 0$$

$$R = \{(x, y) | -1 \leq x \leq 1, -1 \leq y \leq 1\}$$

$g_1, \ g_2, \frac{\partial g_1}{\partial x}, \frac{\partial g_1}{\partial y}, \frac{\partial g_2}{\partial x} \& \frac{\partial g_2}{\partial y}$ are cont. on R. Also |? = (0, 0) for the sol (-0.222146, 0.9938) are in R.

$$\left|\frac{\partial g_1}{\partial x}(sol)\right| + \left|\frac{\partial g_1}{\partial y}(sol)\right| < 1 \Rightarrow \overset{x}{\overset{\downarrow}{|-0.222146|}} + \left|\frac{-1}{2}\right| = 0.722146 < 1 \quad ✓$$

&

$$\left|\frac{\partial g_2}{\partial x}(sol)\right| + \left|\frac{\partial g_2}{\partial y}(sol)\right| < 1 \Rightarrow \overset{\frac{-x}{\sqrt{4-x^2}}}{\overset{\downarrow}{\left|\frac{-(-0.222146)}{\sqrt{4-(-0.222146)^2}}\right|}} + |0| = 0.11177 < 1 \quad ✓$$

we expect it to converge.

To find the 2nd solution (1.900677, 0.31122)
if we start with $p_0 = 1.9$ & $q_0 = 0$

Jacobi & Gauss Seidel diverges.

$$\left|\frac{\partial g_1}{\partial x}(sol)\right| + \left|\frac{\partial g_1}{\partial y}(sol)\right| = |1.900677| + \left|\frac{-1}{2}\right| = 2.400677 \not< 1 \quad \text{not satisfied}$$

$\Rightarrow$ Jacobi & Gauss Seidel may or may not converge.

$$\left|\frac{\partial g_2}{\partial x}(sol)\right| + \left|\frac{\partial g_2}{\partial y}(sol)\right| = \left|\frac{-1.900677}{\sqrt{4-(1.900677)^2}}\right| + |0| = 1.526995 > 1$$

We can try finding other iterations.

For eq (1) $\Rightarrow \quad x^2 - 2x - y + 0.5 = 0$

$$x^2 \underbrace{- 2x - y + 0.5 - 2x}_{-4x} = -2x \Rightarrow \underbrace{\frac{x^2 - 4x - y + 0.5}{-2}}_{g_1(x,y)} = x$$

$$\frac{\partial g_1}{\partial x} = \frac{-1}{2}(2x-4) = \underset{\downarrow\, 2-x}{-x+2}, \quad \frac{\partial g_1}{\partial y} = \frac{-1}{2}(-1) = \frac{1}{2}$$

$$\left|\frac{\partial g_1}{\partial x}(sol)\right| + \left|\frac{\partial g_1}{\partial y}(sol)\right| = |2 - 1.900677| + \left|\frac{1}{2}\right| = 0.5993 < 1$$

$$\left|\frac{\partial g_2}{\partial x}(sol)\right| + \left|\frac{\partial g_2}{\partial y}(sol)\right| = 1.526995 \not< 1$$

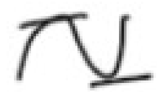
With new $g_1(x, y)$ & the same $g_2(x, y)$, Jacobi converges to (1.90067, 0.31122) in 71 iterations
Gauss Seidel converges to (1.90067, 0.311227) in 26 iterations

## Section 4.1 Taylor Polynomials

# Ch4 Interpolation and Polynomial Approximation

Given a function $f(x)$, such that $f, f', f'', \ldots$ are cont. on some interval $[a, b]$.
We want to approximate $f(x)$ by a polynomial.

$\sin x, \cos x$
$\ln(1-x)$
$e^{x^2}$ $etc$

Simple functions
$x^3 + x^2 + 1 \longrightarrow$ 

In this chapter we will learn different kinds of interpolating polynomials. (In some cases, we have a data, and we want to find an interpolating polynomial for the data).

## Taylor Polynomials

Assume that $f \in c^{n+1}[a, b]$ and $x_0 \in [a, b]$. If $x \in [a, b]$ then Taylor polynomial of degree n is

$$P_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!}(x-x_0)^k = f(x_0) + \frac{f'(x_0)}{1!}(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots$$

$$+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n + \underbrace{\frac{f^{(n+1)}(x_0)}{(n+1)!}(x-x_0)^{n+1}}_{\text{error } E_n(x) \text{ or } R_n(x)}$$

Textbook on page 190

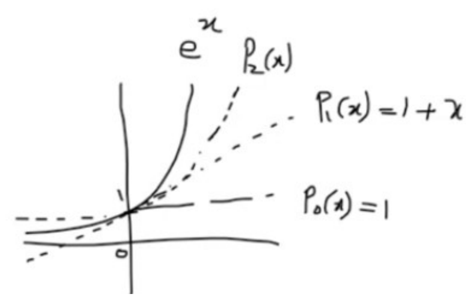
$f(x) = e^x \qquad x_0 = 0$

$f'(x) = e^x, f''(x) = e^x \ldots, f^{(n)}(x) = e^x$
$f'(0) = 1, f''(0) = \cdots = f^{(n)}(0) = 1$

$P_0(x) = 1$

$P_1(x) = 1 + \frac{1}{1!}(x-0) = 1 + x$

$P_2(x) = 1 + x + \frac{1}{2}(x-0)^2 = 1 + x + \frac{x^2}{2}$




Approximations get better as n increases.

$e^{0.3} = \underbrace{1.3498588}_{\text{Actual Value}}$ using calculator

$P_0(0.3) = 1$
$P_1(0.3) = 1 + 0.3 = 1.3$
$P_2(0.3) = 1.345$

When n increases, we need to calculate higher order derivatives.
The accuracy of Taylor polynomial will generally decrease as the values of x moves away from $x_0$.

**Example:**
Find the Taylor polynomial of degree 4 expanded about $x_0 = 1$ to approximate the function $f(x) = \ln x$.

**Solution:**

$f(x) = \ln x$ $\qquad$ $f(1) = \ln 1 = 0$

$f'(x) = \frac{1}{x}$ $\qquad$ $f'(1) = 1$

$f''(x) = \frac{-1}{x^2}$ $\qquad$ $f''(1) = -1$

$f'''(x) = \frac{-(-2)}{x^3} = \frac{2}{x^3}$ $\qquad$ $f'''(1) = 2$

$f^{(4)}(x) = \frac{-6}{x^4}$ $\qquad$ $f^{(4)}(1) = -6$

$$P_4(x) = f(x_0) + \frac{f'(x_0)(x-x_0)}{1!} + \frac{f''(x_0)(x-x_0)^2}{2!} + \frac{f'''(x_0)(x-x_0)^3}{3!} + \frac{f^{(4)}(x_0)(x-x_0)^4}{4!}$$

$$= 0 + (1)(x-1) + \frac{(-1)(x-1)^2}{2} + \frac{\cancel{(2)}(x-1)^3}{3\cancel{(2)}} + \frac{(-\cancel{6})(x-1)^4}{4\cancel{(3)(2)}(1)}$$

$$= x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

$\ln(1.05) = 0.048790164 \longrightarrow$ actual value

$p_4(1.05) = 0.05 - \frac{(0.05)^2}{2} + \frac{(0.05)^2}{3} - \frac{(0.05)^2}{4} = 0.048790104$ $\qquad$ Good accuracy

$p_2(1.05) = 0.04875$ & $p_3(1.05) = 0.048791667$

If you try $\ln(2) = 0.693147181 \longrightarrow$ actual value
$P_2(2) = 0.5,\ P_3(2) = 0.833,\ P_4(2) = 0.5833$ $\qquad$ Not as good

Here 2 is not closer to $x_0 = 1$

We found $P_4(x) = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$

If you are asked to write the polynomial in simplified form.

$$P_4(x) = x - 1 - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$$

$$P_4(x) = x - 1 - \left(\frac{x^2 - 2x + 1}{2}\right) + \frac{x^3 - 3x^2 + 3x - 1}{3} - \frac{x^4 - 4x^3 + 6x^2 - 4x + 1}{4}$$

$$= \frac{-x^4}{4} + x^3\left(\frac{1}{3} + \frac{4}{4}\right) + x^2\left(\frac{-1}{2} - \frac{3}{3} - \frac{6}{4}\right) + x\left(1 + \frac{2}{2} + \frac{3}{3} + \frac{4}{4}\right) + \overbrace{\left(-1 - \frac{1}{2} - \frac{1}{3} - \frac{1}{4}\right)}^{\frac{-12 - 6 - 4 - 3}{12} = \frac{-25}{12}}$$

$= \frac{-x^4}{4} + \frac{4}{3}x^3 - 3x^2 + 4x - \frac{25}{12} \longrightarrow$ Simplified