Matlab has built in command to solve nonlinear eq. $\boldsymbol{f(x) = 0}$.

$fzero('f', initial\ guess)$

$function \quad y = f(x)$

$$y = 1 + \frac{2}{x} - x;$$

$$g(x) = 1 = \frac{2}{x}$$

$$x = 1 + \frac{2}{x}$$

$$1 + \frac{2}{x} - x = 0$$

\>> fzero('f', 4)

ans = 2 $\longrightarrow$ got correct answer

Fixed point method converges with $P_0 = 1$, it converges to 2 $\downarrow$ correct sol.

\>> fzero('f', 1)

ans = -5.7895e-16 $\longrightarrow$ 0 not correct answer

If you have to solve $x^4 - 3x^2 - 4 = 0$ $\longrightarrow$ $function \quad y = f1(x)$

$$y = x^4 - 3 * x^2 - 4;$$

\>>fzero('f1', 1) $\longrightarrow$ get answer ans = 2

ans =2

In the case of polynomial, we can use another command.

roots

\>> roots($\overbrace{[1\ 0\ -3\ 0\ -4]}^{\text{coefficient}}$)

ans = 2, -2, 2', -2'

Example 2:

To solve $x^3 - 3x = 0$ by fixed point method. We know that for fixed point method, we need $x = g(x)$

$x^2 - 3x = 0 \longrightarrow x^2 = 3x \Rightarrow x = \boxed{\frac{x^2}{3}} \longrightarrow g(x)$

$x^2 - 3x = 0 \longrightarrow x^2 = 3x \Rightarrow x = \boxed{\sqrt{3x}} \longrightarrow g(x)$

or $x = \boxed{-\sqrt{3x}} \longrightarrow g(x)$ not good choice

$x^2 - 3x = 0$

$x(x-3) = 0$

$\Rightarrow x = 0, 3$ *are solutions.*

$x^2 - 3x = 0 \longrightarrow x^2 - 3x + x = x$

$\boxed{x^2 - 2x} = x$

$\downarrow$

$g(x)$

Let $P_0 = 0.5$ & $g(x) = \frac{x^2}{3}$

$P_1 = g(P_0) = g(0.5) = \dfrac{(0.5)^2}{3} = 0.0833$

$P_2 = g(0.0833) = \dfrac{(0.0833)^2}{3} = 0.0023168$

using Matlab, it converges to 0 in 5 iterations.

Let $P_0 = 3.5$

$P_1 = g(3.5) = \dfrac{(3.5)^2}{3} = 4.0833$

$P_2 = g(4.0833) = \dfrac{(4.0833)^2}{3} = 5.5578$

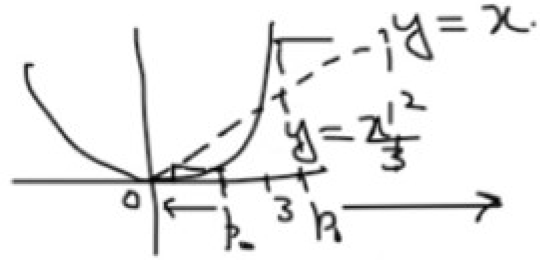
$P_3 = g(5.5578) = \dfrac{(5.5578)^2}{3} = 10.29664$

use Matlab it diverges.

Here $g'(x) = \frac{2x}{3}$

$|g'(0)| = |0| < 1 \longrightarrow$ we expect to **converge**.

$|g'(3)| = \frac{2(3)}{3} > 1 \longrightarrow$ we expect to **diverge we can not get $x = 3$.**



Converge if P_0 is less than 3.
Diverges if P_0 is greater than 3.

Now if we take

$g(x) = \sqrt{3x} = \sqrt{3}\sqrt{x}$

$g'(x) = \sqrt{3}\dfrac{1}{2\sqrt{x}}$

$|g'(0)|$ is not defined $\longrightarrow$ we can not get $x = 0$ from this $g(x)$

Finally,

$|g'(3)| = \frac{\sqrt{3}}{2\sqrt{3}} = 0.5 < 1 \longrightarrow$ it can converge to $x = 3$

$P_0 = 3.5 \to P_1 = g(3.5) = \sqrt{3(3.5)} = 3.24037$

$P_2 = g(3.24037) = \sqrt{3(3.24037)} = 3.11787$

$$\boxed{g'(3.5) = \frac{\sqrt{3}}{2\sqrt{3.5}} = 0.429 < 1}$$

Using Matlab fixed point converges to $x = 3$ in 15 iterations with tol 10^{-5}.

$P_0 = 0.5 \to P_1 = g(0.5) = \sqrt{3(0.5)} = 1.2247$

$P_2 = g(3.24037) = \sqrt{3(3.24037)} = 3.11787$

$g'(0.5) = \dfrac{\sqrt{3}}{2\sqrt{0.5}} = 1.2247 > 1$

but $|g'(3)| < 1$ and g & g' are cont. on $[0.4, 4]$

Using Matlab it converges to $x = 3$ in 19 iterations with tol 10^{-5}.

$g'(x) = \frac{\sqrt{3}}{2\sqrt{x}} > 0$ on $[0.4, 4]$

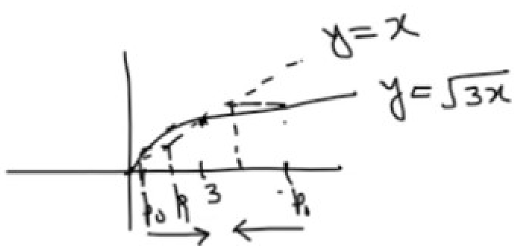
So, g is increasing

$g(0.4) = 1.0954$

$g(3) = 3.464$

$g(x) \in [1.095, 3.5] \in [0.4,4]$

Any $P_0 > 0$ will converge to $x = 3$



Example 3: (Textbook page 48)

Let $g(x) = 2\sqrt{x-1}$

The solution of

$$x = g(x) \Rightarrow x = 2\sqrt{x-1} \Rightarrow (x)^2 = \left(2\sqrt{x-1}\right)^2 \Rightarrow x^2 = 4(x-1) \Rightarrow x^2 - 4x + 4 = 0$$
$$\Rightarrow (x-2)^2 = 0 \Rightarrow x = 2$$

$\lvert g'(sol)\rvert < 1$	Converges
$\lvert g'(sol)\rvert > 1$	Diverges
$\lvert g'(sol)\rvert = 1 \rightarrow$	May or may not converges

$$g'(x) = 2 \cdot \frac{1}{2\sqrt{x-1}} = \frac{1}{\sqrt{x-1}}$$

$|g'(\underset{\text{sol}}{\underbrace{2}})| = \frac{1}{\sqrt{2-1}} = 1$ we may or may not converge.

$P_0 = 1.5$
$P_1 = 2\sqrt{1.5-1} = 1.4142$
$P_3 = 2\sqrt{1.4142-1} = 1.2872$
.
.
.
$P_5 = 2\sqrt{-ve}$ not defined
method diverges.

$P_0 = 2.5$
$P_1 = 2\sqrt{2.5-1} = 2.4495$
$P_2 = 2\sqrt{2.4495-1} = 2.4078$
.
.
.
Using Matlab in 1000 iterations, we get $x = 2.008987$
it will converge to 2.

Bracketing Methods

Start with an interval [a, b] & we will make interval smaller as we do iterations.
To solve the nonlinear eq. $f(x) = 0$, we will do 2 bracketing methods in this course.

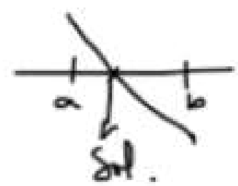
(1) The Bisection method
(2) Method of False position

Bracketing methods are **globally convergent**. (Advantage of these methods)

If we start with a correct interval [a, b] then they will always converge.

The disadvantage of bracketing methods is that they are slower methods.
Their convergence rate is linear.

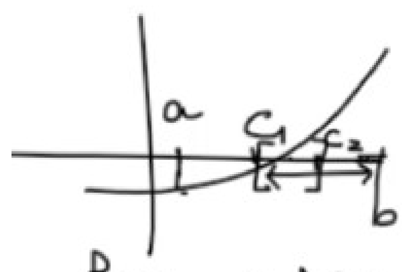
Bisection Method



We will start with an interval [a, b] such that
f is cont. on [a, b] and $f(a)f(b) < 0$.   (Note that we are solving $f(x) = 0$)

Let $c = \frac{a+b}{2}$ & then we decide the new interval by checking the signs of $f(a), f(b), \& f(c)$.

(i) If $f(a)$ & $f(c)$ have opposite signs then the new interval will be [a, c] (take b=c)
(ii) If $f(a)$ & $f(c)$ have same signs then the new interval will be [c, b] (take a=c)
(iii) If $f(c) = 0$ then c is the solution.



$f(a) < 0$ & $f(b) > 0$.

Example 1:

Consider the eq. $x^2 + 23 = 10x$

(a) Find an interval [a, b] such that the bisection method can be used to find a solution of the eq.

Solution:

$x^2 + 23 = 10x \Rightarrow x^2 - 10x + 23 = 0$

Here $f(x) = x^2 - 10x + 23$

$f(1) = 1 - 10 + 23 = 14 > 0$

$f(2) = (2)^2 - 10(2) + 23 = 7 > 0$

$$\left.\begin{array}{l} f(3) = 9 - 30 + 23 = 2 > 0 \\ f(4) = 16 - 40 + 23 = -1 < 0 \end{array}\right\} f(3)f(4) < 0 \;\&\; f \text{ is cont. on } [3, 4] \text{ so we can have interval } [3, 4]$$

(b) Using hand calculations perform 3 iterations of Bisection method starting with $[\underset{a_1}{3}, \underset{b_1}{4}]$

$f(\overset{3}{a_1}) > 0 \;\&\; f(\overset{4}{b_1}) < 0$

$c_1 = \frac{a_1+b_1}{2} = \frac{3+4}{2} = 3.5$, $f(3.5) = (3.5)^2 - 10(3.5) + 23 = 0.25 > 0$

So, the new interval will be $[\underset{a_2}{3.5}, \underset{b_2}{4}]$ i.e., take $a_2 = c_1 = 3.5$ ⟶ 1st iteration

$c_2 = \frac{a_2+b_2}{2} = \frac{3.5+4}{2} = 3.75$ & $f(3.75) = (3.75)^2 - 10(3.75) + 23 = -0.4375 < 0$

So, the new interval will be $[\underset{a_3}{3.5}, \underset{b_3}{3.75}]$ ⟶ 2nd iteration

$c_3 = \frac{3.5+3.75}{2} = 3.675$ & $f(3.675) = (3.675)^2 - 10(3.675) + 23 = -0.109 < 0$

So, the new interval will be $[\underset{a_4}{3.5}, \underset{b_4}{3.675}]$ ⟶ 3rd iteration

Using Matlab it converges to 3.585788 in 17 iterations with tol 10^{-5}

Note: $x^2 - 10x + 23 = 0$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-(-10) \pm \sqrt{(-10)^2 - 4(1)(23)}}{2(1)} = \frac{10 \pm \sqrt{8}}{2} = \frac{10 \pm 2\sqrt{2}}{2} = 5 \pm \sqrt{2}$$

$5 + \sqrt{2} = 6.4142$

$5 - \sqrt{2} = 3.58578$

Textbook has program on page 59.

$function\ [c, k, err, yc] = bisect(f, a, b, tol, \quad)$

add max iteration

add k if you want to know the number of iterations

need f.m file

.

.

.

If b-a < tol

stop

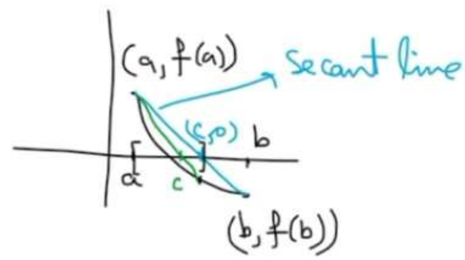
The Method of False Position (Regula Falsi Method)

To solve $f(x) = 0$, we start with an interval [a, b] such that $f(a)f(b) < 0$. (Same as bisection method)

How to find c?

The point (c, 0) is where the secant line joining the points $(a, f(a))$ & $(b, f(b))$ crosses the x-axis.

Slope of the secant line is



$$m = \frac{f(b)-f(a)}{b-a} \leftarrow (1) \quad m = \frac{0-f(a)}{c-a} \leftarrow (2) \quad m = \frac{0-f(b)}{c-b} \leftarrow (3)$$

Equating (1) & (2) $\Rightarrow \frac{f(b)-f(a)}{b-a} = \frac{0-f(a)}{c-a}$

$$(c-a)\big(f(b)-f(a)\big) = -f(a)(b-a)$$

$$c - a = \frac{-f(a)(b-a)}{f(b)-f(a)} \Rightarrow c = a - \frac{f(a)(b-a)}{f(b)-f(a)}$$

Equating (1) & (3) $\Rightarrow \frac{f(b)-f(a)}{b-a} = \frac{0-f(b)}{c-b}$

$$\Rightarrow (c-b)\big(f(b)-f(a)\big) = -f(b)(b-a)$$

$$c - b = \frac{-f(b)(b-a)}{f(b)-f(a)} \Rightarrow c = b - \frac{f(a)(b-a)}{f(b)-f(a)}$$

After finding c, find $f(c)$

(i) If $f(a)$ & $f(c)$ have opposite signs then the new interval will be [a, c]
(ii) If $f(a)$ & $f(c)$ have same signs then the new interval will be [c, b]
(iii) If $f(c) = 0$ then c is the solution.

same as in bisection method

Example:

Consider the eq. $x^2 + 23 = 10\,x$.
Find 2 iterations of regular Falsi method starting with [3, 4].

Solution:

$x^2 - 10x + 23 = 0 \Rightarrow f(x) = x^2 - 10x + 23$

$$f(\underset{a_1}{3}) = 9 - 30 + 23 = 2 > 0 \quad \& \quad f(\underset{b_1}{4}) = 16 - 40 + 23 = -1 < 0$$

$$c_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)} = 4 - \frac{f(4)(4-3)}{f(4) - f(3)} = 4 - \frac{(-1)(1)}{-1-2} = 4 - \frac{-1}{-3} = \frac{11}{3} = 3.6667$$

$$f(c_1) = f(3.6667) = (3.6667)^2 - 10(3.6667) + 23 = -0.222 < 0$$

The new interval will be $[\underset{a_2}{3}, \underset{b_2}{3.6667}]$ $\longrightarrow$ 1st iteration

$$c_2 = 3.6667 - \frac{f(3.6667)(3.6667 - 3)}{f(3.6667) - f(3)} = 3.6667 - \frac{(-0.2222)(0.6667)}{-0.2222 - 2} = 3.59987$$

$$f(c_2) = f(3.59987) = -0.0396 < 0$$

So, the new interval will be [3, 3.59987] $\longrightarrow$ 2nd iteration

Using Matlab it converges to 3.58788 in 8 iterations with tol 10^{-5} & Epsilon 10^{-7}

Textbook has program on Page 60.

if interval $< 10^{-5}$ or if $f(c) < 10^{-7}$ stop