Assume that $f(x)$ and its derivatives $f'(x), f''(x), \ldots, f^{(M)}(x)$ are defined and continuous on an interval $x = p$ where p is a root of $f(x) = 0 \Rightarrow f(P) = 0$.

We say that $f(x) = 0$ has a root of order M at $x = p$ iff $f(P) = 0, f'(P) = 0, \ldots, f^{M-1}(P) = 0$ and $f^{(M)}(P) \neq 0$. A root of order $M = 1$ is often called a <u>simple root.</u>

If $M = 2$ it is called double root, in general if $M > 1$ it is called <u>multiple root.</u>

For polynomials, e.g., $x^3 = 0 \Rightarrow x = \underbrace{0,0,0}$ 0 is a root of order 3.

$f(x) = x^3, \quad f'(x) = 3x^2, \quad f''(x) = 6x, \quad f'''(x) = 6$

$f(0) = 0, \quad f'(0) = 0, \quad f''(0) = 0, \quad f'''(0) \neq 0 \Rightarrow$ 0 is a root of order 3

Consider the eq. $\underbrace{xe^x - x}_{\text{f(x)}} = 0$, $x = 0$ is a root

$f(x) = xe^x - x,$ $\qquad f(0) = 0e^0 - 0 = 0$

$f'(x) = (1)e^x + xe^x - 1;$ $\qquad f'(0) = e^0 + 0 - 1 = 1 - 1 = 0$

$f''(x) = e^x + (1)e^x + xe^x;$ $\qquad f''(0) = e^0 + e^0 + 0 = 2 \neq 0 \Rightarrow x = 0$ is a root of order 2.

Speed of Convergence

(1) If P is a simple root of $f(x) = 0$ Newton's method will converge quadratically that is, the number of accurate decimal places roughly doubles at each iteration.

(2) If P is a multiple root, then Newton's method will have linear convergence that is, the error in each successive iteration is a fraction of the previous error.

E_n is error at n[th] iteration

E_{n+1} is error at $(\mathrm{n}+1)^{\mathrm{th}}$

For simple root Newton's method has $|E_{n+1}| \approx \left|\frac{f''(P)}{2f''(P)}\right| |E_n|^2$

For multiple roots, Newton's method as $|E_{n+1}| \approx \frac{M-1}{M} |E_n|$

$$|E_{n+1}| \approx \frac{M-1}{M}$$

Acceleration of Newton's Method

For multiple root $(M > 1)$ the Newton's method can be modified to

$P_k = P_{k-1} - \frac{M f(P_{k-1})}{f'(P_{k-1})}$ This will converge quadratically.

Example:

We have to solve $x(x-2)^3 = 0$ using Newton's method.

We know that the solutions are $x = 0$ and $\underbrace{x = 2}_{\text{multiple root } M = 3}$

simple root

Suppose we start with $P_0 = 2.1$

$$P_k = P_{k-1} - \frac{f(P_{k-1})}{f'(P_{k-1})}$$

$$P_1 = 2.1 - \frac{f(2.1)}{f'(2.1)} = 2.0671875 \qquad |E_1| = |P_1 - P_0| = 0.0328125$$

$$P_2 = 2.0450317 \qquad |E_2| = |P_2 - P_1| = 0.0221558$$

$$P_3 = 2.03013051 \qquad |E_3| = |P_3 - P_2| = 0.01490119$$

$$\frac{|E_2|}{|E_1|} = 0.675 \qquad \frac{|E_3|}{|E_2|} = 0.6725$$

Using Matlab it converges to $x = 2$ in 24 iterations with tol 10^{-5} & epsilon 10^{-7}.

(Slow as $x = 2$ is a multiple root & the convergence rate is linear)

With $M = 2$ & $P_k = P_{k-1} - \frac{2f(P_{k-1})}{f'(P_{k-1})}$

$P_0 = 2.1$
$P_1 = 2.034375$
$P_2 = 2.0115876$
$P_3 = 2.003877$
Using Matlab it converges to 2 in 9 iterations with tol 10^{-5} & epsilon 10^{-7}.

With $M = 3$, $P_k = P_{k-1} - \frac{3f(P_{k-1})}{f'(P_{k-1})}$
$P_0 = 2.1$
$P_1 = 2.00156$
$P_2 = 2.000000406$
Converged to 2 in 2 iterations with tol 10^{-5} & epsilon 10^{-7}.
(Quadratic convergence with modified Newton method).

Note: $|E_{n+1}| \approx \frac{M-1}{M} |E_n|$

$$\frac{|E_{n+1}|}{|E_n|} \approx \frac{M-1}{M} \Rightarrow \left|\frac{E_{n+1}}{E_n}\right| = \frac{M}{M} - \frac{1}{M} \Rightarrow \frac{1}{M} = 1 - \left|\frac{E_{n+1}}{E_n}\right| \Rightarrow M = \frac{1}{1 - \left|\frac{E_{n+1}}{E_n}\right|}$$

For above example $M = \frac{1}{1 - \left|\frac{E_2}{E_1}\right|} = \frac{1}{1 - 0.675} = 3.079$

So, we can use $M = 3$

3.7 System of Nonlinear Equations

Example 1:

Solve the system
$$2x + y = 1 - (\mathbf{1})$$
$$x^2 + y^2 = 4 - (\mathbf{2})$$

(a) Using hand calculations.

Solution

From eq **(1)** $y = 1 - 2x - (*)$

Sub. in eq **(2)** $x^2 + (1-2x)^2 = 4 \Rightarrow x^2 + \overbrace{1 - 4x + 4x^2}^{(1-2x)^2} = 4 \Rightarrow 5x^2 - 4x - 3 = 0$

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(5)(-3)}}{2(5)} = \frac{4 \pm \sqrt{76}}{10}$$

$a = 5, b = -4, c = -3$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{4+\sqrt{76}}{10}, \frac{4-\sqrt{76}}{10} \Rightarrow x = 1.27178, -0.47178$$

Using **(*)** when $x = 1.27178, y = 1 - 2(1.27178) = -1.54356$

when $x = -0.47178, y = 1 - 2(-0.47178) = 1.94356$

The solutions are (1.27178, -1.54356) & (-0.47178, 1.94356).

(b) Starting with (0,0) perform 2 iterations of Jacobi method to find the solution (-0.47178, 1.94356).

$x_{k+1} = \frac{1-y_k}{2}$ & $y_{k+1} = \sqrt{4 - x_k^2}$

$$y^2 = 4 - x^2$$
$$y = \pm\sqrt{4 - x^2}$$

$x_1 = \frac{1-y_0}{2} = \frac{1-0}{2} = \frac{1}{2}$ & $y_1 = \sqrt{4 - x_0^2} = \sqrt{4-0} = 2$

$x_2 = \frac{1-y_1}{2} = \frac{1-2}{2} = \frac{-1}{2}$ & $y_2 = \sqrt{4 - x_1^2} = \sqrt{4 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{15}{4}} = 1.93649$

Jacobi method is also referred as a fixed-point method

$x_{k+1} = g_1(x_k, y_k)$

$y_{k+1} = g_2(x_k, y_k)$

fixed point is $P_k = g(P_{k-1})$

Using Matlab it converges to (-0.471786, 1.94356) in 10 iterations with tolerance 10^{-5}

(c) Starting with (0,0) perform 2 iterations of Gauss-Seidel method to find the solution whose y component is +ve

$$x_{k+1} = \frac{1-y_k}{2} \ \& \ y_{k+1} = \sqrt{4 - x_k^2}$$

$$x_1 = \frac{1-y_0}{2} = \frac{1-0}{2} = \frac{1}{2} \ \& \ y_1 = \sqrt{4 - x_1^2} = \sqrt{4 - \left(\frac{1}{2}\right)^2} = \sqrt{\frac{15}{4}} = 1.93649$$

$$x_2 = \frac{1-y_1}{2} = \frac{1-1.93649}{2} = -0.468245$$

$$y_2 = \sqrt{4 - x_1^2} = \sqrt{4 - (-0.468245)^2} = 1.94441421$$

Using Matlab it converges to (-0.471786, 1.94356) in 6 iterations with tol 10^{-5}

Conditions of Convergence $x = g_1(x, y) \ \& \ y = g_2(x, y)$

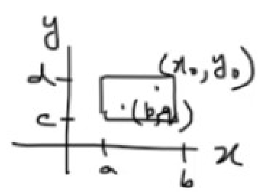
Suppose g_1, g_2 and their 1st order partial derivatives are continuous on a region that contains the solution . If the initial guess (x_0, y_0) is close to the solution (p, q) & (x_0, y_0) is in the region and

if

$$\left|\frac{\partial g_1}{\partial x}(p,q)\right| + \left|\frac{\partial g_1}{\partial y}(p,q)\right| < 1$$

and

$$\left|\frac{\partial g_2}{\partial x}(p,q)\right| + \left|\frac{\partial g_2}{\partial y}(p,q)\right| < 1$$



For fixed point method
$x = g(x)$
g & g' cont. on [a,b], $P_0 \in [a,b]$
$g(x) \in [a,b]$ for all $x[a,b]$

$|g'(x)| < 1$ for all $x \in [a,b]$
then it converges

$|g'(x)| > 1$ for all $x \in [a,b]$
then it diverges
$|g'(p)| < 1 \longrightarrow$ converges
$|g'(p)| > 1 \longrightarrow$ diverges

Then the iterations (Jacobi & Gauss-Seidel) converges to (p, q).

This is a sufficient condition not a necessary condition. So, if one of them is bigger than one then it may or may not converge.

For the system $\left.\begin{array}{l}2x+y=1\\x^2+y^2=4\end{array}\right\}$ **Do you expect Jacobi and Gauss Seidel method to converge starting with (0, 0)? Justify your answer using the conditions of convergence.**

$$x=\frac{1-y}{2} \Rightarrow g_1(x,y)=\frac{1-y}{2}$$
$$y=\sqrt{4-x^2} \Rightarrow g_2(x,y)=\sqrt{4-x^2}$$

if we are doing
+ve one

$g_1(x,y)$ is cont. on $\mathbb{R}^2$
$g_2(x,y)$ is cont. on $\{(x,y)\mid -2\le x\le 2, y\in\mathbb{R}\}$

$$4-x^2\ge 0$$
$$4\ge x^2$$
$$\Rightarrow -2\le x\le 2$$

$\frac{\partial g_1}{\partial x}=0$

$\frac{\partial g_2}{\partial x}=\frac{1}{2}(4-x^2)^{-\frac{1}{2}}(-2x)$

$\frac{\partial g_1}{\partial y}=\frac{-1}{2}\rightarrow \frac{1}{2}-\frac{y}{2}$

$\frac{\partial g_2}{\partial y}=0$

$=\frac{-x}{\sqrt{4-x^2}}$

Partial derivatives

$$f(x,y,z)=x^2+xy^2+e^xz+y^3$$
$\frac{\partial f}{\partial x}\rightarrow$ treat y & z as a constant
$$\frac{\partial f}{\partial x}=2x+(1)y^2+e^xz+0$$

$\frac{\partial f}{\partial y}\rightarrow$ treat x & z as a constant
$$\frac{\partial f}{\partial y}=0+x(2y)+0+3y^2$$

$\frac{\partial g_1}{\partial x},\frac{\partial g_1}{\partial y},\frac{\partial g_2}{\partial y}$ are cont. on $\mathbb{R}^2$

$\frac{\partial g_2}{\partial x}$ is cont. on $\{(x,y)\mid -2\le x<2, y\in\mathbb{R}\}$

$$\boxed{4-x^2>0,\ \ x^2<4,\qquad -2<x<2}$$

Let $R=\{(x,y)\mid \underbrace{-1.99\le x\le 1.99,}_{\substack{\text{or}\\ -1\le x\le 1}}\ -2\le y\le 2\}$

Solution
(-0.47178, 1.9435597)

$g_1,g_2,\frac{\partial g_1}{\partial x},\frac{\partial g_1}{\partial y},\frac{\partial g_2}{\partial x},\frac{\partial g_2}{\partial y}$ are cont. on R. the initial guess $(0,0)\in R$ and the Solution $(-0.47178, 1.9435597)\in R$

$$\left|\frac{\partial g_1}{\partial x}(p,q)\right|+\left|\frac{\partial g_1}{\partial y}(p,q)\right|=|0|+\left|\frac{-1}{2}\right|=\frac{1}{2}<1 \quad ✓$$

&

$$\left|\frac{\partial g_2}{\partial x}(p,q)\right| + \left|\frac{\partial g_2}{\partial y}(p,q)\right| = \underbrace{\left|\frac{-x}{\sqrt{4-x^2}}\right|}_{} + |0|$$

$$\left|\frac{-(-0.47178)}{\sqrt{4-(-0.47178)^2}}\right| = 0.2427 < 1$$

So, we expect that Jacobi & Gauss-Seidel method will converge.