No: Cholesky factorization, shifted inverse Power, Taylor polynomial, simpsons $\frac{3}{8}$, adaptive quadrature

Final Exam Sat Dec 17 12-3PM
Chrysler Hall North 6133

Cumulative final.
no programs will be asked, but MATLAB commands might be asked.

1 Question:
(a) computational complexity of gaussian elimination? $O(n^3)$

(b) If you type eig(A) in MATLAB what will it return? eigenvalues

(c) ill condition & norm infinity, convergence rates, order error ~~trapezoidal~~ & Simpson Rules

1 Question: Gauss Elim with no pivoting or with partial pivoting.
LU factorization,
row interchange or no interchange
$PA=LU$ $A=LU$

1 Question: eigenvalues & eigenvectors
$|A-\lambda I|=0$ $(A-\lambda I)\vec{v}=0$

Power Method $Y_k = AX_k$ $X_{k+1} = Y_k / C_{k+1}$
where C_{k+1} is element of Y_k with largest magnitude. If dominant eigenvalue is unique then Power method converges.

Ch2: how to do iterations, conditions of convergence, convergence rates for the following.

Fixed Point Method. $x_{k+1} = g(x_k)$, g & g' continuous on $[a,b]$ $|g'(sol)| < 1 \Rightarrow$ converges $|g'(sol)| > 1 \Rightarrow$ diverges
$|g'(sol)| = 1$ may or may not converge.

Bisection Method. $c = \frac{a+b}{2}$

Regula falsi Method, $c = b - \frac{f(b)(b-a)}{f(b)-f(a)}$

if $f(a)f(c) < 0$ then new interval is $[a, c]$
if >0 then $[c, b]$
if $=0$ then sol $= c$

Both converge if f is continuous on $[a,b]$ & $f(a)f(b)<0$ with linear convergence.

Secant Method, f is cont. on $[a,b]$.

$$p_{k+1} = p_k - \frac{f(p_k)(p_k - p_{k-1})}{f(p_k) - f(p_{k-1})}$$

Newton Method

$$x_k = x_{k-1} - \frac{f(x_{k-1})}{f'(x_{k-1})}$$ given that $f'(x_{k-1}) \neq 0$

Modified Newton Method

$$x_k = x_{k-1} \frac{m f(x_{k-1})}{f'(x_{k-1})}$$

quadratic convergence if solution is simple root, otherwise it is linear.

Jacobi & Gauss-Seidel

Jacobi 1st equation → solve for x $x_{k+1} = f(\quad y_k, z_k)$
2nd equation → solve for y $y_{k+1} = f(x_k \quad , z_k)$
3rd equation → solve for z $z_{k+1} = f(x_k, y_k \quad)$

Gauss-Seidel " " $x_{k+1} = f(\quad y_k, z_k)$
$y_{k+1} = f(x_{k+1}, \quad z_k)$
$z_{k+1} = f(x_{k+1}, y_{k+1} \quad)$

Converge if linear system of equations is strictly diagonally dominant.

Nonlinear system converges if

$x = g_1(x, y)$ & $y = g_2(x, y)$. g_1 g_2 $\frac{\partial g_1}{\partial x}$ $\frac{\partial g_1}{\partial y}$ $\frac{\partial g_2}{\partial x}$ $\frac{\partial g_2}{\partial y}$

are continuous on a region containing solution & initial guess. Converges if:

$$\left|\frac{\partial g_1(s,l)}{\partial x}\right| + \left|\frac{\partial g_1(s,l)}{\partial y}\right| < 1 \quad \& \quad \left|\frac{\partial g_2(s,l)}{\partial x}\right| + \left|\frac{\partial g_2(s,l)}{\partial y}\right| < 1$$

1 question: Lagrange polynomial

$(N+1)$ nodes means degree N polynomial

$$P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$P_N(x_k) = y_k$ for all nodes

Newton polynomial: Divided difference table

$\Rightarrow a_0 \; a_1 \; a_2 \; a_3 \ldots$

$P_N(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) \ldots a_N(x-x_0) \ldots$
$(x-x_{N-1})$

Lagrange & Newton Error formula:

$$E_N(f) = \frac{(x-x_0)(x-x_1)\ldots(x-x_N)}{(N+1)!} f^{N+1}(c) \quad \text{for } c \in [x_0, x_N]$$

1 Question:

Least Squares: error is $E_2(f) = \left[\frac{1}{N}\left(f(x_k) - y_k\right)^2\right]^{1/2}$

line: $y = Ax + B$ normal equations

$$A \Sigma x_k^2 + B \Sigma x_k = \Sigma x_k y_k$$
$$A \Sigma x_k + NB = \Sigma y_k$$

Power $y = Ax^M \Rightarrow A = \frac{\sum x_k^M y_k}{\sum x_k^{2M}}$

Change of variables 7 of them $\boxed{\text{Page 269}}$

$y = Ce^{Ax}$ $\quad y = Cx^A$ $\quad y = \frac{A}{x} + B$ $\quad y = \frac{1}{Ax+B}$

$y = A\ln(x) + B$ $\quad y = \frac{1}{(Ax+B)^2}$ $\quad y = \frac{x}{A+Bx}$

1 or 2 Questions;

Numerical Differentiation

forward Backward & central difference

formula sheet will be given

find the order of error $\quad f'(x) = \frac{f(x+h) - f(x-h)}{2h}$

(find derivation using taylors polynomial) she may ask any formula from our sheet or not from sheet.

Integration; Trapezoidal

$h = b - a$ $\quad \int_b^a f(x)\,dx = \frac{h}{2}\left[f_0 + f_1\right] + O(h^3)$

error

Simpsons $h = \frac{b-a}{2}$ $\quad \int_0^a f(x)\,dx = \frac{h}{3}\left[f_0 + 4f_1 + f_2\right] + O(h^5)$

Composite Trapezoidal rule; $h = \frac{b-a}{m}$

$$\int_a^b f(x)\,dx = \frac{h}{2}\left[f_0 + 2(f_1 + f_2 \ldots f_{m-1}) + f_m\right] + O(h^2)$$

Golden ratio search if f is unimodal then it converges.

$c = (1-r)(b-a) + a \qquad r = 0.618$

$d = b - (1-r)(b-a)$

if $f(c) \leq f(d)$ new interval is $[a, d]$

if $f(d) < f(c)$ then $[c, b]$

Composite simpsons rule: $h = \frac{b-a}{2M}$

$$\int_a^b f(x)\,dx = \frac{h}{3}\Big[f_0 + 4\underbrace{(f_1 + f_3 + f_5 \ldots f_{2M-1})}_{\text{odd subscript}} + 2\underbrace{(f_2 + f_4 \ldots f_{2M-2})}_{\text{even subscript}} + f_{2M}\Big] + O(h^4)$$

Find exact error in derivatives / integration

x^n, $\ln(x)$, $\sin x$, $\cos x$, $\tan x$
product rule, chain rule.

$\int e^x$ $\int x^n \to \frac{x^{n+1}}{n+1}$

$\int \cos(ax) \to \frac{\sin(ax)}{a}$ $\int \frac{1}{x} \to \ln(x)$ $\int \sin(ax) \to \frac{-\cos(ax)}{a}$

Degree of precision:
Trapezoidal $\to$ 1 Simpsons 3/8
Simpsons $\to$ 3 not asked.

Optimization. find local min of $f(x)$
hand calc $\to$ find critical numbers (x such that $f'(x) = 0$) check if $f'(x)$ decreases on the left of c and increases on the right of c.

interval	$f'(x)$	increasing or decreasing
$x < -4$		
$-4 < x < 2$		
$2 < x$		

take sample x from interval & write if it is negative or positive.