Numerical Differentiation

We have data & want to find the rate of change (derivative).

Taylor's polynomial

$f = f(x_0) + hf'(x_0) + \frac{h^2}{2}f''(x_0) + \frac{h^2}{3!}f'''(x_0) + \cdots + \overbrace{\frac{h^N}{N!}f^{(N)}(c)}^{\text{error term}}$ for some c in $[a, b]$

If we think that $x = x_0 + h$

$f(x_0 + h) = f(x_0) + hf'(x_0) + \cdots$

Taylor polynomial

$f(x + h) = f(x) + hf'(x) + \underbrace{\frac{h^2}{2}f''(c)}_{\text{error}}$ for some c

$$hf'(x) = f(x + h) + f(x) - \frac{h^2}{2}f''(c)$$

$$f'(x) = \frac{f(x + h) + f(x)}{h} - \frac{\frac{h^2}{2}f''(c)}{h}$$

$$f'^{(x)} = \frac{f(x + h) + f(x)}{h} + 0(h) \rightarrow \textbf{Forward Difference}$$

$$f(x - h) = f(x) - hf'(x) + \frac{h^2}{2}f''(c)$$

$$\frac{hf'(x)}{h} = \frac{f(x) - f(x - h)}{h} + \frac{\frac{h^2}{2}f''(c)}{h}$$

$$f'(x) = \frac{f(x) - f(x - h)}{h} + 0(h) \rightarrow \textbf{Backward Difference}$$

$$
\begin{array}{rl}
 & f(x+h) = f(x) + hf'(x) + \dfrac{h^2}{2!}f''(x) + \dfrac{h^3}{3!}f'''(c_1) \\
- & f(x-h) = f(x) - hf'(x) + \dfrac{h^2}{2!}f''(x) - \dfrac{h^3}{3!}f'''(c_2) \\
\hline
 & f(x+h) - f(x-h) = 2hf'(x) - \dfrac{h^3}{3!}\left(f'''(c_1) + f'''(c_2)\right)
\end{array}
$$

$$\frac{2hf'(x)}{2h} = \frac{f(x+h) - f(x-h)}{2h} - \frac{\frac{h^3}{3!}\left(f'''(c_1) + f'''(c_2)\right)}{2h}$$

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + 0(h^2) \rightarrow \textbf{Central Difference}$$

Difference Formulas

Central Difference:

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} + O(h^2)$$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h^2)$$

only f ’’ (x) formula

Forward Difference:

$$f'(x) = \frac{f(x+h) - f(x)}{h} + O(h)$$

$$f'(x) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} + O(h^2)$$

Backward Difference:

$$f'(x) = \frac{f(x) - f(x-h)}{h} + O(h)$$

$$f'(x) = \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h} + O(h^2)$$

Example 1:

Let $f(x) = \ln x$ and $h = 0.1$ Find $f'(1.8)$ using forward, backward & central difference formulas. Find the exact error and relative error.

Solution:

Forward Difference

$$f'(x) = \frac{f(x+h) - f(x)}{h} \Rightarrow f'(1.8) = \frac{\mathrm{f}(1.8+0.1) - \mathrm{f}(1.8)}{0.1} = \frac{\ln(1.9) - \ln(1.8)}{0.1} = 0.541$$

exact value: $f'(x) = \frac{1}{x} \Rightarrow f'(1.8) = \frac{1}{1.8} = 0.556$

Exact error is $|0.556 - 0.541| = 0.015$

Relative error is $\frac{exact\ error}{|exact\ value|} = \frac{0.015}{0.556} = 0.026978 \approx 0.027$

Backward difference

$$f'(x) = \frac{f(x) - f(x-h)}{h} \Rightarrow f'(1.8) = \frac{\mathrm{f}(1.8) - \mathrm{f}(1.8-0.1)}{0.1} = \frac{\ln(1.8) - \ln(1.7)}{0.1} = 0.572$$

Exact error is $|0.556 - 0.572| = 0.016$

Relative error is $\frac{0.016}{0.556} = 0.028776978 \approx 0.029$

Central Difference

$$f'(x) = \frac{f(x+h) - f(x-h)}{h} \Rightarrow f'(1.8) = \frac{\mathrm{f}(1.8+0.1) - \mathrm{f}(1.8-0.1)}{h} = \frac{\ln(1.9) - \ln(1.7)}{0.1} = 0.5565$$

Exact error is $|0.5565 - 0.5556| = 0.0009$

Relative error is $\frac{0.0009}{0.5556} = 0.0016$

We can not take h to be very small.

1) We may not have data available for smaller values of h.
2) $f(x+h) = y_1 + e_1 \rightarrow round\ off\ error$
   $f(x-h) = y_2 + e_2 \rightarrow round\ off\ error$

Central Difference formula

$$f'(x) = \frac{f(x+h) - f(x-h)}{2h} = \frac{(y_1 + e_1) - (y_2 + e_2)}{2h} = \frac{y_1 - y_2}{2h} + \frac{e_1 - e_2}{2h}$$

If h is small than $\frac{e_1 - e_2}{2h}$ will be large.

So, we need to have formulas whose error are small (like $O(h^4), O(h^8), etc.$)

Example 2:

Find the order of error in the following approximation.

$$f'(x) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h}$$

Solution:

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(c_1)$$

multiply by 4

$$f(x+2h) = f(x) + 2hf'(x) + \frac{(2h)^2}{2!}f''(x) + \frac{(2h)^3}{3!}f'''(c_2)$$

When they eliminate, the next derivative is the error. We write $c_1 \& c_2$ for the last term.

$$4f(x+h) = 4f(x) + 4hf'(x) + \cancel{\frac{4h^2}{2}f''(x)} + \frac{4h^3}{3!}f'''(c_1)$$

$$- \quad f(x+2h) = f(x) + 2hf'(x) + \cancel{\frac{4h^2}{2}f''(x)} + \frac{8h^3}{3!}f'''(c_2)$$

$$4f(x+h) - f(x+2h) = 3f(x) + 2hf'(x) + \frac{4h^3}{3!}\left(f'''(c_1) - 2f'''(c_2)\right)$$

$$2hf'(x) = 4f(x+h) - f(x+2h) - 3f(x) - \frac{4h^3}{3!}\left(f'''(c_1) - 2f'''(c_2)\right)$$

$$f'(x) = \frac{4f(x+h) - f(x+2h) - 3f(x)}{2h} - \frac{\frac{4h^3}{3!}\left(f'''(c_1) - 2f'''(c_2)\right)}{2h}$$

$$f'(x) = \frac{-3f(x) + 4f(x+h) - f(x+2h)}{2h} + 0(h^2)$$

error is $0(h^2)$