Section 11.1 and 11.2

Computers are used to find eigenvalues and eigenvector for matrixes of higher dimensions. MATLAB has a built-in command.

```
>>A=[-4 1 1; 1 5 -1; 0 1 -3];
>>eig(A)
```

$$ans = \begin{matrix} 5.0000 \\ -4.0000 \\ -3.0000 \end{matrix}$$

To have both outputs of Eigen values and Eigen vectors, we type

```
>>[ V D] = eig(A)
```

eigen vectors

eigen values on the diagonal

$$V = \begin{array}{ccc} \text{eigen vector for } \lambda = 5 & \text{eigen vector for } \lambda = -4 & \text{eigen vector for } \lambda = -3 \\ \overbrace{\qquad\qquad} & \overbrace{\qquad\qquad} & \overbrace{\qquad\qquad} \\ -0.1231 & 0.9901 & 0.7071 \\ -0.9847 & -0.0990 & 0.0000 \\ -0.1231 & 0.0990 & 0.7071 \end{array}$$

$$D = \begin{matrix} 5.0000 & 0 & 0 \\ 0 & -4.0000 & 0 \\ 0 & 0 & -3.000 \end{matrix}$$

For $\lambda = 5$, the eigen vector in the above example is

$$\vec{V} = \begin{bmatrix} 1 \\ 8 \\ 1 \end{bmatrix}$$

$$\left\|\vec{V}\right\|_2 = \sqrt{1^2 + 8^2 + 1^2} = \sqrt{1 + 64 + 1} = \sqrt{66}$$

The normalized eigen vector is

$$\begin{bmatrix} 1/\sqrt{66} \\ 8/\sqrt{66} \\ 1/\sqrt{66} \end{bmatrix} = \begin{bmatrix} 0.1231 \\ 0.9847 \\ 0.1231 \end{bmatrix}$$

11.2 Power Method

Power method is used to find the dominant eigen value and its corresponding eigen vector.
λ_1 is a dominant eigen value of A if $|\lambda_1| > |\lambda_i|$ for all i

$|\lambda_1| > |\lambda_i| > |\lambda_1| \ldots$

Dominant eigen value and the corresponding eigen vector is called dominant eigen vector.

An **eigenvector $\vec{V}$** is said to be **normalized** if the coordinate of the largest magnitude is equal to one.

It is easy **to normalize an eigen vector** $\vec{V} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ by dividing it by its norm. (Can be norm1, norm2, infinite norm).

Matlab by default norm2 is used

$\|\vec{V}\|_\infty = \max_{1 \le i \le n} |V_i|$

Power method is an iterative method used to find the dominant eigen pair (eigen value and vector). Start with $X_0 = [1\ 1 \ldots 1]'$ (we can choose some other X_0)

We find $Y_k = AX_k$ for $k = 0, 1, 2, \ldots,$
And $X_{k+1} = \frac{1}{C_{k+1}} Y_k$

where C_{k+1} is the coordinate of Y_k of largest magnitude.
(In the case of tie, choose the coordinate that comes first)

We find the absolute values to compare but c_{k+1} is the actual coordinate not the absolute value.

e.g., $Y = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$ $|-3| = |3| > 1$
so $C = -3$

If the method converges then c_k will converge to λ the dominant eigen value and X_k will converge to eigen vector $\vec{V}$.

NOTE: If X_0 is an eigen vector of any other eigen value (which is not dominant) then it will always converge to that eigen value, so it will not give dominant eigen value. So, we need to start with some other X_o.

Example 1:

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{bmatrix}$

a) Find the dominant eigen value and eigen vector.

b) Use power method starting with $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, perform 2 iterations.

Solution:

Here A is an upper triangular matrix so its eigen values are the entries on the diagonal.

$\Rightarrow \lambda = 1, 2, -3$

$|-3| > |2| > |1| \Rightarrow$ The dominant eigen value is $\lambda = -3$

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} 1-\lambda & -1 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & -3-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ 0 & -3-\lambda \end{vmatrix} - 0\left|\quad\right| + 0\left|\quad\right| = 0$$

$$(1-\lambda)(2-\lambda)(-3-\lambda) = 0$$

$$\lambda = 1 \quad \lambda = 2 \quad \lambda = -3$$

For eigen vector $(A - \lambda I)\vec{V} = 0 \Rightarrow \underbrace{(A - (-3)I)}\vec{V} = 0$

$$(A + 3I)\vec{V} = 0$$

$$\left[\begin{array}{ccc|c} 4 & -1 & 1 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$a_{3\,3} = 0 \Rightarrow x_3$ is a free variable

$4x_1 - x_2 + x_3 = 0 - \textbf{(1)}$

$5x_2 - x_3 = 0 - (2)$

$x_3 \in \mathbb{R}, \quad Let\ x_3 = 1 \quad \textbf{(2)} \Rightarrow \quad 5x_2 = 1 \quad \Rightarrow \quad x_2 = \dfrac{1}{5}$

$\textbf{(1)} \Rightarrow \quad 4x_1 - \dfrac{1}{5} + 1 = 0 \quad \Rightarrow \quad -\dfrac{4}{5} \quad \Rightarrow \quad x_1 = -\dfrac{4}{5(4)} = -\dfrac{1}{5}$

So, the dominant eigen vector is $\vec{V} = \begin{bmatrix} -1/5 \\ 1/5 \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -0.2 \\ 0.2 \\ 1 \end{bmatrix}$

b) $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$Y_0 = AX_0 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1-1+1 \\ 0+2-1 \\ 0+0-3 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} \qquad |-3| > |1| \Rightarrow C_1 = -3$$

$$X_1 = \frac{1}{C_1} Y_0 = \frac{1}{-3} \begin{bmatrix} 1 \\ 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} \longrightarrow \text{1}^{\text{st}} \text{ iteration}$$

$$Y_1 = AX_1 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} -\frac{1}{3} \\ -\frac{1}{3} \\ 1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} + \frac{1}{3} + 1 \\ 0 - \frac{2}{3} - 1 \\ 0 + 0 - 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{5}{3} \\ -3 \end{bmatrix} \qquad \text{So } C_2 = -3$$

$$X_2 = \frac{1}{C_2} Y_1 = \frac{1}{-3} \begin{bmatrix} 1 \\ -\frac{5}{3} \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{9} \\ -1 \end{bmatrix} \longrightarrow \text{2}^{\text{nd}} \text{ iteration}$$

Using MATLAB power method converges to $\lambda = -3$ and $\vec{V} = \begin{bmatrix} -0.2 \\ 0.2 \\ 1 \end{bmatrix}$ in 23 iterations with tolerance 10^{-5}.

Textbook has program on page 606. (Need one correction)

tolerance

$function\underbrace{[lambda, V]} = power1(A, x, epsilon, max1)$

in output your can have comma
or not
$[lambda\ V] is\ also\ find$

initial
guess

max. number of iterations

Y=A*X;
[m j] = max(abs(Y))

max. absolute
value

is the coordinate of Y which has largest absolute value.

$\cancel{C_1 =} m$ → in textbook and is WRONG

replace it by C1 = Y(j);

end

Example 2:

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & -3 \end{bmatrix}$

The eigen values of A are $\lambda = 1, -3, -3$ $\qquad$ $|-3| > 1 \Rightarrow$ The dominant eigen value is -3.

For $\lambda = -3$ $\quad (A + 3I)\vec{V} = 0$

$$\left[\begin{array}{ccc|c} 4 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$4x_1 - x_2 + x_3 = 0 - (1)$

x_2 is free, $x_2 \in \mathbb{R} \Rightarrow$ Let $x_2 = 1$ then $(1) \Rightarrow 4x_1 - 1 = 0 \Rightarrow x_1 = \frac{1}{4}$

$0x_2 - x_3 = 0 \Rightarrow x_3 = 0$

eigen vector is

$\vec{V} = \begin{bmatrix} 1/4 \\ 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0.25 \\ 1 \\ 0 \end{bmatrix}$

Using power method starting with $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$Y_0 = AX_0 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 1 + 1 \\ 0 - 3 - 1 \\ 0 + 0 - 3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} C_1 = -4 \Rightarrow X_1 = \frac{1}{-4} \begin{bmatrix} 1 \\ -4 \\ -3 \end{bmatrix} = \begin{bmatrix} -1/4 \\ 1 \\ 3/4 \end{bmatrix} \longrightarrow 1^{st} \text{ iteration}$$

Using MatLab the power method

after 3000 iterations, it gives $\lambda = -3.0017$ and $\vec{V} = \begin{bmatrix} 0.2495 \\ 1.0000 \\ 0.0017 \end{bmatrix}$ with tol 10^{-6}

it is converging to the dominant eigen value and eigen vector.

The speed of Convergence of X_k to the dominant eigen vector is governed by the terms

$\left(\frac{\lambda_2}{\lambda_1}\right)^k \nearrow$ in 1st example it is $\left(\frac{2}{-3}\right)^k$ so it will approach zero as k gets large because $\left|\frac{2}{-3}\right| < 1$

$\left(\frac{\lambda_2}{\lambda_1}\right)^k \searrow$ in 2nd example it is $\left(\frac{-3}{-3}\right)^k \to (1)^k \to$ it takes large number of iterations to converge.

So, the power method will converge slowly if the dominant eigen value is repeated.

Note: If we start with $X_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ then

$$Y_0 = AX_0 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1-1+0 \\ 0-3+0 \\ 0+0+0 \end{bmatrix} = \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} \Rightarrow C_1 = -3 \text{ and } X_1 = \frac{1}{-3} \begin{bmatrix} 0 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Using MATLAB with $X_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ power method converges to $\lambda = -3$ and $\vec{V} = \begin{bmatrix} 0.25 \\ 1 \\ 0 \end{bmatrix}$ in 8 iterations with tol 10^{-5}

Example 3:

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix}$

The eigen values are $\lambda = 1, 3, -3$ $\quad |-3| = |3| \Rightarrow$ both -3 and 3 are dominant eigen values

So, the dominant eigen value is not unique.

$\underline{\boldsymbol{\lambda = 3}}$

$(A - 3I)\vec{V} = 0 \Rightarrow \left[\begin{array}{ccc|c} -2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right]$

$a_{22} = 0 \Rightarrow x_2 is\ free$

$$\begin{aligned} -2x_1 - x_2 + x_3 &= 0 - (1) \\ -x_3 &= 0 \\ -6x_3 &= 0 \end{aligned} \Rightarrow x_3 = 0 \qquad \text{Let } x_2 = 1 \qquad (1) \Rightarrow -2x_1 - 1 = 0 \Rightarrow x_1 = -\frac{1}{2} \Rightarrow \vec{V} = \begin{bmatrix} -\frac{1}{2} \\ 1 \\ 0 \end{bmatrix}$$

$\underline{\boldsymbol{\lambda = -3}}$

$(A + 3I)\vec{V} = 0 \Rightarrow \left[\begin{array}{ccc|c} 4 & -1 & 1 & 0 \\ 0 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$

Let $x_3 = 1$

$$\begin{aligned} 4x_1 - x_2 + x_3 &= 0 - (*) \\ 6x_2 - x_3 &= 0 \end{aligned} \Rightarrow 6x_2 = 1 \Rightarrow x_2 = \frac{1}{6}$$

$$(*) \Rightarrow 4x_1 \underbrace{- \frac{1}{6} + 1}_{\frac{5}{6}} = 0 \Rightarrow x_1 = -\frac{5}{6(4)} = -\frac{5}{24}$$

The eigen vector is $\vec{V} = \begin{bmatrix} -\frac{5}{24} \\ \frac{1}{6} \\ 1 \end{bmatrix}$ or $\begin{bmatrix} -0.20833 \\ 0.166667 \\ 1 \end{bmatrix}$

Using power method starting with $x_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$Y_0 = AX_0 = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & -1 \\ 0 & 0 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 - 1 + 1 \\ 0 + 3 - 1 \\ 0 + 0 - 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} \Rightarrow C_1 = -3 \ \&\ X_1 = \frac{1}{-3} \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}$$

Power method using MATLAB gives $\lambda = -3$ and $\vec{V} = \begin{bmatrix} 0.2083 \\ -0.6667 \\ 1.0000 \end{bmatrix}$ after 20 iterations ← even

and it gives $\lambda = -3$ and $\vec{V} = \begin{bmatrix} -0.6250 \\ 1.0000 \\ 1.0000 \end{bmatrix}$ after 21 iterations. ← odd

It is going back and forth for even and odd number of iterations ⇒ it diverges.

If you choose $X_0 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ then power method using MATLAB again oscillates for even and odd number of iterations.

$$30\ iterations \quad \rightarrow \quad \lambda = 2.5\ and\ \vec{V} = \begin{bmatrix} -0.4250 \\ 1.0000 \\ -0.6000 \end{bmatrix}$$

$(even)$

it diverges

$$31\ iterations \quad \rightarrow \quad \lambda = 3.6\ and\ \vec{V} = \begin{bmatrix} -0.5625 \\ 1.0000 \\ 0.5000 \end{bmatrix}$$

(odd)

(unique)

Power method fails because we do not have a single dominant eigen value.

For this particular upper triangular matrix, we can choose $X_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and then

power method will converge to $\lambda = 3$ and $\vec{V} = \begin{bmatrix} -0.5 \\ 1 \\ 0 \end{bmatrix}$ in 9 iterations with tol 10^{-5}