MATH 3940 Numerical Analysis for Computer Scientists

Problem Set 2 Solutions

1. Let $A = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix}$ and the initial approximation is $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) Find all the eigenvalues and eigenvectors of matrix A.

(b) Use the Matlab built-in function to find all the eigenvalues and eigenvectors of matrix A.

(c) Perform two iterations of the power method for matrix A starting with X_0.

(d) Use Matlab to find the dominant eigenvalue of A and the associated eigenvector using the power method with a tolerance of 10^{-5}, starting with X_0.

Solution. (a) To find the eigenvalues, we have to solve $|A - \lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & -7 & 0 \\ 5 & 10-\lambda & 4 \\ 0 & 5 & 2-\lambda \end{vmatrix} = 0$$

Expanding along the first column, we obtain

$$\begin{aligned}
&(2-\lambda)[(10-\lambda)(2-\lambda) - 20] - 5[-7(2-\lambda) - 0] = 0 \\
\Rightarrow\ &(2-\lambda)[(10-\lambda)(2-\lambda) - 20 + 35] = 0 \\
\Rightarrow\ &(2-\lambda)[\lambda^2 - 12\lambda + 20 + 15] = 0 \\
\Rightarrow\ &(2-\lambda)[\lambda^2 - 12\lambda + 35] = 0 \\
\Rightarrow\ &(2-\lambda)[(\lambda-5)(\lambda-7)] = 0
\end{aligned}$$

Thus the eigenvalues are 2, 5, and 7. Now we will find the eigenvectors.

For $\lambda_1 = 7$, we have

$$\left[\begin{array}{rrr|r} -5 & -7 & 0 & 0 \\ 5 & 3 & 4 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] R_2{+}R_1 \to R_2 \left[\begin{array}{rrr|r} -5 & -7 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 5 & -5 & 0 \end{array}\right] R_3{+}\frac{5}{4}R_2 \to R_3 \left[\begin{array}{rrr|r} -5 & -7 & 0 & 0 \\ 0 & -4 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have

$$\begin{array}{rcrcrcl} -5x_1 & - & 7x_2 & & & = & 0 \\ & - & 4x_2 & + & 4x_3 & = & 0 \end{array}$$

x_3 is a free variable. If we take $x_3 = 1$, the second equation gives $x_2 = x_3 = 1$. The first equation gives $x_1 = -\frac{7}{5}x_2 = -\frac{7}{5}$ Thus the eigenvector is $\mathbf{v}_1 = [-7/5\ 1\ 1]'$.

For $\lambda_2 = 5$, we have

$$\left[\begin{array}{rrr|r} -3 & -7 & 0 & 0 \\ 5 & 5 & 4 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right] R_2{+}\frac{5}{3}R_1 \to R_2 \left[\begin{array}{rrr|r} -3 & -7 & 0 & 0 \\ 0 & -\frac{20}{3} & 4 & 0 \\ 0 & 5 & -3 & 0 \end{array}\right] R_3{+}\frac{3}{4}R_2 \to R_3 \left[\begin{array}{rrr|r} -3 & -7 & 0 & 0 \\ 0 & -\frac{20}{3} & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have

$$\begin{array}{rcrcrcl} -3x_1 & - & 7x_2 & & & = & 0 \\ & - & \frac{20}{3}x_2 & + & 4x_3 & = & 0 \end{array}$$

x_3 is a free variable. If we take $x_3 = 1$, the second equation gives $x_2 = 3/5x_3 = 3/5$. The first equation gives $x_1 = -\frac{7}{3}x_2 = -(\frac{7}{3})(\frac{3}{5}) = -\frac{7}{5}$ Thus the eigenvector is $\mathbf{v}_2 = [-7/5\ 3/5\ \ 1]'$.

For $\lambda_3 = 2$, we have

$$\left[\begin{array}{rrr|r} 0 & -7 & 0 & 0 \\ 5 & 8 & 4 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right] R_1 \leftrightarrow R_2 \left[\begin{array}{rrr|r} 5 & 8 & 4 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{array}\right] R_3 + \frac{5}{7}R_2 \rightarrow R_3 \left[\begin{array}{rrr|r} 5 & 8 & 4 & 0 \\ 0 & -7 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have

$$\begin{array}{rcrcrcl} 5x_1 & + & 8x_2 & + & 4x_3 & = & 0 \\ & - & 7x_2 & & & = & 0 \end{array}$$

x_3 is a free variable (parameter). The second equation gives $x_2 = 0$ and the first equation gives $x_1 = -4/5x_3$. If we take $x_3 = 1$, then the eigenvector is $\mathbf{v}_3 = [-4/5\ 0\ 1]'$

(b) See Matlab sheets.

(c) The initial approximation is $X_0 = [1\ 1\ 1]'$. Using the power method

$$Y_1 = AX_0 = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ 19 \\ 7 \end{bmatrix}$$

The element of largest magnitude is 19 so $\mu_1 = 19$.

$$X_1 = \frac{1}{\mu_1}Y_1 = \begin{bmatrix} -5/19 \\ 1 \\ 7/19 \end{bmatrix} \text{ or } \begin{bmatrix} -0.2632 \\ 1 \\ 0.3684 \end{bmatrix}$$

$$Y_2 = AX_1 = \begin{bmatrix} 2 & -7 & 0 \\ 5 & 10 & 4 \\ 0 & 5 & 2 \end{bmatrix} \begin{bmatrix} -5/19 \\ 1 \\ 7/19 \end{bmatrix} = \begin{bmatrix} -143/19 \\ 193/19 \\ 109/19 \end{bmatrix} \text{ or } \begin{bmatrix} -7.5263 \\ 10.1578 \\ 5.7368 \end{bmatrix}$$

Now $\mu_2 = 193/19$ or 10.1578 and $X_2 = \frac{1}{\mu_2}Y_2 = \begin{bmatrix} -143/193 \\ 1 \\ 109/193 \end{bmatrix}$ or $\begin{bmatrix} -0.7409 \\ 1 \\ 0.5648 \end{bmatrix}$

(d) See Matlab sheets. $\square$

2. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ and the initial approximation is $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) Find all the eigenvalues of A and find the eigenvector associated with the dominant eigenvalue of the matrix A.

(b) Perform two iterations of the power method starting with X_0.

(c) Use Matlab to find the dominant eigenvalue of B and the associated eigenvector using the power method with a tolerance of 10^{-5}, starting with X_0.

Solution. (a) To find the eigenvalues, we have to solve $|A - \lambda I| = 0$.

$$\begin{vmatrix} 2-\lambda & 1 & 3 \\ 0 & -3-\lambda & 1 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

Expanding along the first column, we obtain $(2-\lambda)(-3-\lambda)(1-\lambda) = 0$. Thus the eigenvalues are 2, -3, and 1. Note that A is an upper triangular matrix and we can say right away that eigenvalues are the entries on the main diagonal.

The dominant eigenvalue is -3, so we will find the eigenvector for $\lambda = -3$.

$$\left[\begin{array}{ccc|c} 5 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 4 & 0 \end{array}\right] R_3 - 4R_2 \to R_3 \left[\begin{array}{ccc|c} 5 & 1 & 3 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{array}{rcrcrcl} 5x_1 & + & x_2 & + & 3x_3 & = & 0 \\ & & & & x_3 & = & 0 \end{array}$$

x_2 is a free variable. The first equation gives $x_1 = -\frac{1}{5}x_2$. If we take $x_2 = 5$, then $x_1 = -1$, and the eigenvector is $\mathbf{v} = [-1\ 5\ 0]'$.

(b) The initial approximation is $X_0 = [1\ 1\ 1]'$. Using the power method

$$Y_1 = AX_0 = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ 1 \end{bmatrix}$$

The element of largest magnitude is 6 so $\mu_1 = 6$ and $X_1 = \frac{1}{\mu_1} Y_1 = \begin{bmatrix} 1 \\ -1/3 \\ 1/6 \end{bmatrix}$

$$Y_2 = AX_1 = \begin{bmatrix} 2 & 1 & 3 \\ 0 & -3 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1/3 \\ 1/6 \end{bmatrix} = \begin{bmatrix} 13/6 \\ 7/6 \\ 1/6 \end{bmatrix}$$

The element of largest magnitude is 13/6 so $\mu_2 = 13/6$ and $X_2 = \frac{1}{\mu_2} Y_2 = \begin{bmatrix} 1 \\ 7/13 \\ 1/13 \end{bmatrix}$

(c) See Matlab sheets. □

3. Let $A = \begin{bmatrix} 4 & -1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -4 \end{bmatrix}$ and the initial approximation is $X_0 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

(a) Find all the eigenvalues and eigenvectors of A.

(b) Use the Matlab built-in function to find all the eigenvalues and eigenvectors of matrix A.

(c) Use Matlab to find the dominant eigenvalue and the associated eigenvector of A using the power method with a tolerance of 10^{-5}, starting with X_0. Does it converge or diverge? Explain the reason for its convergence/divergence.

Solution. (a) To find the eigenvalues, we have to solve $|A-\lambda I| = 0$. Since A is an upper triangular matrix, the eigenvalues will be the entries on the main diagonal. Thus the eigenvalues are 4, 2, and -4. Now we will find the eigenvectors.

For $\lambda_1 = 4$, we have

$$\left[\begin{array}{rrr|r} 0 & -1 & 1 & 0 \\ 0 & -2 & -1 & 0 \\ 0 & 0 & -8 & 0 \end{array}\right] \Rightarrow \begin{array}{rcrcl} -x_2 & + & x_3 & = & 0 \\ -2x_2 & - & x_3 & = & 0 \\ & - & 8x_3 & = & 0 \end{array}$$

x_1 is a free variable. The last equation gives $x_3 = 0$, substituting $x_3 = 0$ into the first equation or the second equation, we get $x_2 = 0$. If we take $x_1 = 1$, then the eigenvector is $\mathbf{v}_1 = [1\ 0\ 0]'$

For $\lambda_2 = -4$, we have

$$\left[\begin{array}{rrr|r} 8 & -1 & 1 & 0 \\ 0 & 6 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{rcrcrcl} 8x_1 & - & x_2 & + & x_3 & = & 0 \\ & & 6x_2 & - & x_3 & = & 0 \end{array}$$

x_3 is a free variable. If we take $x_3 = 1$, then the second equation gives $x_2 = 1/6$. Substituting $x_2 = 1/6$ and $x_3 = 1$ into the first equation we obtain
$8x_1 = (1/6) - 1 = -5/6 \ \Rightarrow\ x_1 = -5/48$. Thus the eigenvector is $\mathbf{v}_2 = [-\frac{5}{48}\ \frac{1}{6}\ 1]'$.

For $\lambda_3 = 2$, we have

$$\left[\begin{array}{rrr|r} 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -6 & 0 \end{array}\right] R_3{-}6R_2 \to R_3 \left[\begin{array}{rrr|r} 2 & -1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{rcrcrcl} 2x_1 & - & x_2 & + & x_3 & = & 0 \\ & & & - & x_3 & = & 0 \end{array}$$

x_2 is a free variable. The second equation gives $x_3 = 0$. If we take $x_2 = 1$, then the first equation gives $x_1 = x_2/2 = 1/2$, and the eigenvector is $\mathbf{v}_3 = [1/2\ 1\ 0]'$.

Parts (b) and (c) see Matlab sheets.

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