MATH 3940 Numerical Analysis for Computer Scientists
Assignment 5 Solutions Fall 2020

1. Consider the data

x_k	1	2	3	4	5
y_k	1.3	3.5	4.2	5	7

(a) (9 marks) Using hand calculations, find the least-squares line $y = f(x) = Ax + B$ for the data and calculate the root-mean-square error $E_2(f)$.
(b) (3 marks) Using Matlab, find the least-squares line $y = f(x) = Ax + B$ for the data.

Solution. (a)

	x_k	y_k	x_k^2	$x_k y_k$
	1	1.3	1	1.3
	2	3.5	4	7.0
	3	4.2	9	12.6
	4	5.0	16	20.0
	5	7.0	25	35.0
Sum	15	21	55	75.9

The normal equations are

$$\begin{array}{rcrcr} 55A & + & 15B & = & 75.9 \\ 15A & + & 5B & = & 21 \end{array}$$

Multiplying second equation by -3 and adding to the first equation, we have

$$\begin{array}{rcrcr} 55A & + & 15B & = & 75.9 \\ -45A & - & 15B & = & 63 \\ \hline 10A & & & = & 12.9 \end{array}$$

Thus $A = \dfrac{12.9}{10} = 1.29$. Substituting $A = 1.29$ into the second equation we obtain

$$15A+5B = 21 \Rightarrow 15(1.29)+5B = 21 \Rightarrow 5B = 21-19.35 = 1.65 \Rightarrow B = \frac{1.65}{5} = 0.33$$

Therefore the least-squares line is $y = f(x) = Ax + B = 1.29x + 0.33$.
The error $E_2(f)$ is given by

$$\begin{aligned} E_2(f) &= \left[\frac{1}{5}\sum_{k=1}^{5}(f(x_k) - y_k)^2\right]^{\frac{1}{2}} = \left[\frac{1}{5}\sum_{k=1}^{5}(1.29x_k + 0.33 - y_k)^2\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{5}\left\{(1.62-1.3)^2 + (2.91-3.5)^2 + (4.2-4.2)^2 + (5.49-5)^2 + (6.78-7)^2\right\}\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{5}\left\{(0.32)^2 + (-0.59)^2 + (0)^2 + (0.49)^2 + (-0.22)^2\right\}\right]^{\frac{1}{2}} \\ &= \left[\frac{1}{5}(0.1024 + 0.3481 + 0 + 0.2401 + 0.0484)\right]^{\frac{1}{2}} = \left(\frac{0.739}{5}\right)^{\frac{1}{2}} = 0.3845 \end{aligned}$$

(b) See Matlab sheets for the solution of part (b).

□

2. Consider the data

x_k	0.5	1	1.5	2
y_k	7.1	3.7	2.6	2

(a) (5 marks) Using hand calculations, find the least-squares power fit $y = f(x) = \dfrac{A}{x}$ for the data.

(b) (3 marks) Using Matlab, find the least-squares power fit $y = \dfrac{A}{x}$ for the data.(Provide program for power fit).

Solution. (a) For the power fit $y = \dfrac{A}{x}$, we have $M = -1$.

We construct the following table

	x_k	y_k	x_k^{-1}	x_k^{-2}	$x_k^{-1}y_k$
	0.5	7.1	2	4	14.2
	1	3.7	1	1	3.7
	1.5	2.6	0.6667	0.4445	1.7334
	2	2.0	0.5	0.25	1
Sum				5.6945	20.6334

The coefficient A is given by $A = \dfrac{20.6334}{5.6945} = 3.6234$

Therefore the power fit is $y = \dfrac{3.6234}{x}$

(b) See Matlab sheets for the solution.

□

3. Consider the data

x_k	1	2	3	4
y_k	0.6	1.9	4.3	7.6

(a) (4 marks) Find the least-squares curve $y = f(x) = Cx^A$ by nonlinear method using Matlab (Hint: Use Matlab built-in function for minimization of a function).

(b) (10 marks) Using hand calculations, find the least-squares curve $y = f(x) = Cx^A$ by data linearization method (the change of variable method).

Solution. (a) See Matlab sheets for the solution of part (a).

(b) The change of variables will be $X = \ln x$, $Y = \ln y$ and $B = \ln C$, or $C = e^B$. We will calculate the following table of values

	x_k	y_k	$X_k = \ln x_k$	$Y_k = \ln y_k$	X_k^2	X_kY_k
	1	0.6	0	−0.5108	0	0
	2	1.9	0.6931	0.6419	0.4804	0.4449
	3	4.3	1.0986	1.4586	1.2069	1.6024
	4	7.6	1.3863	2.0281	1.9218	2.8116
Sum			3.178	3.6178	3.6091	4.8589

The normal equations are

$$\begin{array}{rcrcl} 3.6091A & + & 3.178B & = & 4.8589 \\ 3.178A & + & 4B & = & 3.6178 \end{array}$$

Multiplying first equation by 4 and second equation by -3.6178 and adding, we have

$$\begin{array}{rcrcr} 14.4364A & + & 12.712B & = & 19.4356 \\ -10.09968A & - & 12.712B & = & -11.49737 \\ \hline 4.33672A & & & = & 7.93823 \end{array}$$

Thus $A = \dfrac{7.93823}{4.33672} = 1.83047$.
Substituting $A = 1.83047$ into the second equation we obtain
$3.178A + 4B = 3.6178 \Rightarrow 3.178(1.83047) + 4B = 3.6178$
$\Rightarrow 3B = 3.6178 - 5.81723 = -2.19943 \Rightarrow B = -0.54986$
$C = e^B = e^{-0.546986} = 0.57703$.

Therefore the least-squares fit is $y = f(x) = Cx^A = 0.57703x^{1.83047}$.

$\square$

4. Let $f(x) = \sin x$, and $h = 0.2$ in each case.

(a) (2 marks) Find $f'(1.5)$ using the formula $f'(x) = \dfrac{f(x+h) - f(x)}{h}$

(b) (2 marks) Find $f''(1.5)$ using the formula $f''(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

(c) (4 marks) Find the exact and relative error for the approximation obtained in part (b).

Solution. (a) $f'(x) = \dfrac{f(x+h) - f(x)}{h}$

$$\begin{aligned} f'(1.5) &= \frac{f(1.5+0.2) - f(1.5)}{0.2} \\ &= \frac{f(1.7) - f(1.5)}{0.2} \\ &= \frac{\sin(1.7) - \sin(1.5)}{0.2} \\ &= \frac{0.991664 - 0.997494}{0.2} \\ &= \frac{-0.00583}{0.2} = -0.02915 \end{aligned}$$

(b) $f''(x) = \dfrac{f(x+h) - 2f(x) + f(x-h)}{h^2}$

$$\begin{aligned} f''(1.5) &= \frac{f(1.5+0.2) - 2f(1.5) + f(1.5-0.2)}{(0.2)^2} \\ &= \frac{\sin(1.7) - 2\sin(1.5) + \sin(1.3)}{0.04} \\ &= \frac{0.99166481 - 2(0.997494986) + 0.963558185}{0.04} \\ &= \frac{-0.039766977}{0.04} = -0.994174433 \end{aligned}$$

(c) $f(x) = \sin x$, $f'(x) = \cos x$ and $f''(x) = -\sin x$.

Thus exact value of $f''(1.5)$ is $f''(1.5) = -\sin(1.5) = -0.997494986$.

The exact error is $|-0.997494986 - (-0.994174433)| = 0.003320553$.

The relative error is $\dfrac{0.003320553}{|-0.997494986|} = 0.00332889$ □

5. (7 marks) Find the order of error in the following approximation (show your steps)

$$f'(x) = \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h}$$

Solution. We use Taylor polynomial expansion for the functions and obtain

$$f(x+2h) = f(x) + 2\frac{f'(x)}{1!}h + 4\frac{f''(x)}{2!}h^2 + 8\frac{f'''(x)}{3!}h^3 + 16\frac{f^{(4)}(x)}{4!}h^4 + 32\frac{f^{(5)}(c_1)}{5!}h^5$$

$$f(x+h) = f(x) + \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 + \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(x)}{4!}h^4 + \frac{f^{(5)}(c_2)}{5!}h^5$$

$$f(x-h) = f(x) - \frac{f'(x)}{1!}h + \frac{f''(x)}{2!}h^2 - \frac{f'''(x)}{3!}h^3 + \frac{f^{(4)}(x)}{4!}h^4 - \frac{f^{(5)}(c_3)}{5!}h^5$$

$$f(x-2h) = f(x) - 2\frac{f'(x)}{1!}h + 4\frac{f''(x)}{2!}h^2 - 8\frac{f'''(x)}{3!}h^3 + 16\frac{f^{(4)}(x)}{4!}h^4 - 32\frac{f^{(5)}(c_1)}{5!}h^5$$

Multiplying the first equation by -1, the second equation by 8, the third equation by -8, and the fourth equation by 1 and adding them

$$\begin{aligned} -f(x+2h) &= -f(x) - 2\tfrac{f'(x)}{1!}h - 4\tfrac{f''(x)}{2!}h^2 - 8\tfrac{f'''(x)}{3!}h^3 - 16\tfrac{f^{(4)}(x)}{4!}h^4 - 32\tfrac{f^{(5)}(c_1)}{5!}h^5 \\ 8f(x+h) &= 8f(x) + 8\tfrac{f'(x)}{1!}h + 8\tfrac{f''(x)}{2!}h^2 + 8\tfrac{f'''(x)}{3!}h^3 + 8\tfrac{f^{(4)}(x)}{4!}h^4 + 8\tfrac{f^{(5)}(c_2)}{5!}h^5 \\ -8f(x-h) &= -8f(x) + 8\tfrac{f'(x)}{1!}h - 8\tfrac{f''(x)}{2!}h^2 + 8\tfrac{f'''(x)}{3!}h^3 - 8\tfrac{f^{(4)}(x)}{4!}h^4 + 8\tfrac{f^{(5)}(c_3)}{5!}h^5 \\ f(x-2h) &= f(x) - 2\tfrac{f'(x)}{1!}h + 4\tfrac{f''(x)}{2!}h^2 - 8\tfrac{f'''(x)}{3!}h^3 + 16\tfrac{f^{(4)}(x)}{4!}h^4 - 32\tfrac{f^{(5)}(c_1)}{5!}h^5 \\ \hline -f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h) &= 12\tfrac{f'(x)}{1!}h + O(h^5) \end{aligned}$$

Thus we obtain

$$\begin{aligned} 12hf'(x) &= -f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h) + O(h^5) \\ \Rightarrow\ f'(x) &= \frac{-f(x+2h) + 8f(x+h) - 8f(x-h) + f(x-2h)}{12h} + O(h^4) \end{aligned}$$

Thus the order of error in the given approximation is $O(h^4)$. □

6. (6 marks) Consider the data

x_k	0.1	0.13	0.16
$f(x_k)$	0.0415	−0.0583	−0.1577

Find the approximations to $f'(0.1)$, $f'(0.13)$, and $f'(0.16)$ of order $\mathbf{O}(h^2)$. (Hint: use appropriate difference formulas for each value of x)

Solution. Looking at the table, we should take $h = 0.13-0.1 = 0.03$. For $f'(0.1)$, we will use forward-difference formula.

$$\begin{aligned} f'(x) &= \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h} \\ f'(0.1) &= \frac{-3f(0.1)+4f(0.1+0.03)-f(0.1+0.06)}{2(0.03)} \\ &= \frac{-3f(0.1)+4f(0.13)-f(0.16)}{0.06} \\ &= \frac{-3(0.0415)+4(-0.0583)-(-0.1577)}{0.06} \\ &= \frac{-0.2}{0.06} = -3.3333 \end{aligned}$$

For $f'(0.13)$, we will use central-difference formula.

$$\begin{aligned} f'(x) &= \frac{f(x+h)-f(x-h)}{2h} \\ f'(0.13) &= \frac{f(0.13+0.03)-f(0.13-0.03)}{2(0.03)} \\ &= \frac{f(0.16)-f(0.1)}{0.06} \\ &= \frac{-0.1577-0.0415}{0.06} = \frac{-0.1992}{0.06} = -3.32 \end{aligned}$$

For $f'(0.16)$, we will use backward-difference formula.

$$\begin{aligned} f'(x) &= \frac{3f(x)-4f(x-h)+f(x-2h)}{2h} \\ f'(0.16) &= \frac{3f(0.16)-4f(0.16-0.03)+f(0.16-0.03)}{2(0.03)} \\ &= \frac{3f(0.16)-4f(0.13)+f(0.1)}{0.06} \\ &= \frac{3(-0.1577)-4(-0.0583)+0.0415}{0.06} \\ &= \frac{-0.1984}{0.06} = -3.3067 \end{aligned}$$

□

MATH 3940 Assignment 5 Solutions - Matlab   Fall 2020

Question 1: (b) Using the least-square line program in Matlab

```
>> X=[1 2 3 4 5];
>> Y=[1.3 3.5 4.2 5.0 7.0];
>> [A B]=lsline(X,Y)
A =   1.2900
B =   0.3300
```

The least square line is y=1.29 x + 0.33

Question 2: (b) M file for the power fit is

```
function A=lspower(X,Y,M)
% Input  - X is the 1xn abscissa vector   - Y is the 1xn ordinate vector
% Output - A is the coefficient of x^M
 XM=X.^M;
sumx2m=(XM)*(XM)';
sumxmy=(XM)*(Y)';
A=sumxmy/sumx2m;
>> X=[0.5 1 1.5 2];
>> Y=[7.1 3.7 2.6 2];
>> A=lspower(X,Y,-1)
A =   3.6234
```

Thus the least squares power fit is y=3.6234/x.

Question 3: (a) M file for the error function is

```
function Z=EQ3(U)
C=U(1);
A=U(2);
Z=zeros(1,2);
Z=(C*(1^A)-0.6)^2+(C*(2^A)-1.9)^2+(C*(3^A)-4.3)^2+(C*(4^A)-7.6)^2;
>> fminsearch('EQ3',[1 1]) or >> fminunc('EQ3',[1 1])
ans =     1.96541  0.49756.
```

Thus the least-squares fit is y=f(x)= 0.49756 x^(1.96541)