Power Fit

Depending on the data, sometimes it is good to approximate the data by a power fit. This means that we want to have a least-square curve of the form $y = f(x) = Ax^M$ where M is a known constant (M can be +ve or -ve integer).

$Ax^2, Ax^3, Ax^4, \ldots \rightarrow$ +ve powers

$\frac{A}{x}, \frac{A}{x^2}, \ldots \rightarrow$ -ve powers

The only constant need to be determined is A.

The root mean square error is $E_2(f) = \left[\frac{1}{N}\sum_{k=1}^{N}|f(x_k) - y_k|^2\right]^{\frac{1}{2}}$

$M = -1$ $M = -2$

$E_2(f)$ will be minimum if $\underbrace{\sum_{k=1}^{N}(f(x_k) - y_k)^2}_{E(f)}$ is minimum.

$$E(f) = \sum_{k=1}^{N}(Ax_k^M - y_k)^2$$

$$\frac{dE}{dA} = 0 \Rightarrow \sum_{k=1}^{N} 2(Ax_k^M - y_k) \cdot \frac{d}{dA}(Ax_k^M - y_k) = 0$$

$$\sum_{k=1}^{N}(Ax_k^M - y_k)(x_k^M) = 0$$

$$\sum_{k=1}^{N}(Ax_k^{2M} - x_k^M y_k) = 0 \Rightarrow \sum_{k=1}^{N} Ax_k^{2M} - \sum_{k=1}^{N} x_k^M y_k = 0$$

$$A\sum_{k=1}^{N} x_k^{2M} - \sum_{k=1}^{N} x_k^M y_k = 0 \Rightarrow \boldsymbol{A}\sum_{k=1}^{N} x_k^{2M} = \sum_{k=1}^{N} x_k^M\, y_k$$

$$\boxed{A = \frac{\sum_{k=1}^{N} x_k^M y_k}{\sum_{k=1}^{N} x_k^{2M}}}$$

Example: Consider the data

x_k	1	2	3	4
y_k	2	5	8	15

Find the least-squares power fit $\underbrace{y = Ax^2}_{\text{M=2}}$ for the data. Also find the root mean square error $E_2(f)$.

Solution:

The formula for $A = \dfrac{\sum_{k=1}^{N} x_k^M y_k}{\sum_{k=1}^{N} x_k^{2M}} \Rightarrow A = \dfrac{\sum_{k=1}^{4} x_k^2 y_k}{\sum_{k=1}^{4} x_k^4}$

x_k^M $\quad x_k^M y_k$ $\quad (x_k^M)^2$

x_k	y_k	x_k^2	$x_k^2 y_k$	x_k^4
1	2	1	2	1
2	5	4	20	16
3	8	9	72	81
4	15	16	240	256
Sum:			334	354

$$\Rightarrow A = \frac{334}{354} = 0.9435$$

The least-squares power fit is $f(x) = y = Ax^2 = 0.9435x^2$

The root mean square error $E_2(f) = \left[\frac{1}{N}\sum_{k=1}^{N}(f(x_k) - y_k)^2\right]^{\frac{1}{2}}$

$$E_2(f) = \left[\frac{1}{4}\sum_{k=1}^{4}(0.9435x_k^2 - y_k)^2\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{4}\{(0.9435(1)^2 - 2)^2 + (0.9435(2)^2 - 2)^2 + (0.9435(3)^2 - 2)^2 + (0.9435(4)^2 - 2)^2\}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{4}\{(0.9435 - 2)^2 + (3.774 - 5)^2 + (8.4915 - 8)^2 + (15.096 - 15)^2\}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{4}\{(-1.0565)^2 + (-1.226)^2 + (0.4915)^2 + (0.096)^2\}\right]^{\frac{1}{2}}$$

$$= \left[\frac{1}{4}\{1.11619 + 1.50308 + 0.24157 + 0.00922\}\right]^{\frac{1}{2}}$$

$$= \left[\frac{2.87006}{4}\right]^{\frac{1}{2}}$$

$$= \underbrace{0.847}_{\text{3 decimal digits}}062571$$