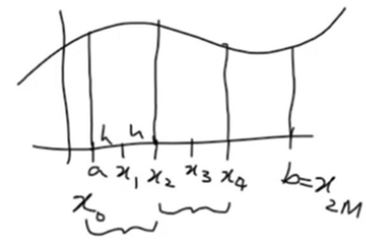
7.2 Composite Simpson Rule



$x_0, x_1, x_2 \to$ Simpson rule

$x_2, x_3, x_4 \to$ Simpson rule.

We divide into 2M subintervals of equal length $h = \frac{b-a}{2M}$

So, the nodes are equally spaced $x_k = x_0 + kh$

or $x_k = a + kh$

Where $k = 0,1,2, \dots ,2M$ We apply Simpson's rule & add them to get

$$\int_a^b f(x)\ dx = \frac{h}{3}[f_0 + 4f_1 + f_2] + \frac{h}{3}[f_2 + 4f_3 + f_4] + \cdots + \frac{h}{3}[f_{2M-2} + \underbrace{f_{2M-1}}_{\text{odd}} + f_{2M}]$$

$$\int_a^b f(x)\ dx = \frac{h}{3}[f_0 + 4\underbrace{(f_1 + f_3 + f_5 + \cdots + f_{2M-1})}_{\text{odd subscripts}} + 2\underbrace{(f_2 + f_4 + f_6 + \cdots + f_{2M-2})}_{\text{even subscripts}} + f_{2M}]$$

Where $h = \frac{b-a}{2M}$ & we have $2M + 1$ Points.

Or

$$\int_a^b f(x)\, dx = \frac{h}{3}\left[f_0 + 4\sum_{k=1}^{M} \underbrace{f_{2k-1}}_{\text{odd}} + 2\sum_{k=1}^{M-1} \underbrace{f_{2k}}_{\text{even}}\right]$$

The error for Composite Simpson rule is of $O(h^4)$

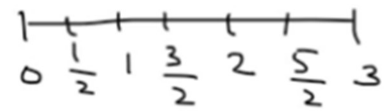
error is $\frac{-(b-a)f^{(4)}(c)h^4}{180}$

Example 1:

Consider the integral $\int_0^3 x^3\, dx$. Approximate the integral using Composite trapezoidal rule with $M = 6$. Also, find the exact error & relative error.

Solution:

$\int_a^b f(x)\ dx = \frac{h}{2}[f_0 + 2(f_1 + \cdots + f_{M-1}) + f_M]$ where $h = \frac{b-a}{M}$



Here $h = \frac{3-0}{6} = \frac{1}{2}$

The nodes are $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2, x_5 = \frac{5}{2}, x_6 = 3$

The f values at the nodes are (Here $f(x) = x^3$)

$$f_0 = (0)^3 = 0$$

$$f_1 = (x_1)^3 = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

$$f_2 = f(x_2) = f(1) = (1)^3 = 1$$

$$f_3 = f(x_3) = f\left(\frac{3}{2}\right) = \left(\frac{3}{2}\right)^3 = \frac{27}{8}$$

$$f_4 = f(x_4) = f(2) = (2)^3 = 8$$

$$f_5 = f(x_5) = f\left(\frac{5}{2}\right) = \left(\frac{5}{2}\right)^3 = \frac{125}{8}$$

$$f_6 = f(x_6) = f(3) = (3)^3 = 27$$

$$\int_0^3 x^3\, dx = \frac{h}{2}[f_0 + 2(f_1 + f_2 + \cdots + f_5) + f_6]$$

$$= \frac{\frac{1}{2}}{2}\left[0 + 2\left(\frac{1}{8} + 1 + \frac{27}{8} + 8 + \frac{125}{8}\right) + 27\right]$$

$$= 20.8125$$

Exact value is $\int_0^3 x^3\,dx = \left.\frac{x^4}{4}\right|_0^3 = \frac{(3)^4-0}{4} = \frac{81}{4} = 20.25$

Exact error is $|20.25 - 20.8125| = 0.5625$
& the relative error is $\frac{0.5625}{20.25} = 0.0278$

b) Approximate the integral using Composite Simpson rule with $M = 3$

$$h = \frac{b-a}{2M} = \frac{3-0}{2(3)} = \frac{3}{6} = \frac{1}{2}$$

The nodes are $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1, x_3 = \frac{3}{2}, x_4 = 2\,, x_5 = \frac{5}{2}, x_6 = 3$
We can use the values of f from above calculations.

$$\int_a^b f(x)\,dx = \frac{h}{3}[f_0 + 4(f_1 + f_3 + \cdots + f_{2M-1}) + 2(f_2 + f_4 + \cdots + f_{2M-2}) + f_{2M}]$$

$$\int_0^3 x^3\,dx = \frac{h}{3}[f_0 + 4(f_1 + f_3 + f_5) + 2(f_2 + f_4) + f_6]$$

$$= \frac{\frac{1}{2}}{3}\left[0 + 4\left(\frac{1}{8} + \frac{27}{8} + \frac{125}{8}\right) + 2(1+8) + 27\right]$$

$$= \frac{1}{6}\left[4\left(\frac{1+27+125}{8}\right) + 2(9) + 27\right]$$

$$= 20.25$$

The exact error is $|20.25 - 20.25| = 0$
& the relative error is $\frac{0}{20.25} = 0$

(Note that for $f(x) = x^3$, we have $f'(x) = 3x^2, f''(x) = 6x, f'''(x) = 6, f^{(4)}(x) = 0$, that is why Composite Simpson rule have zero error).

Example 2:

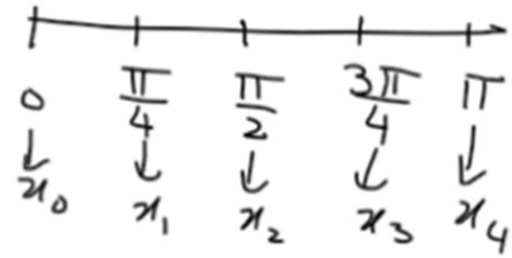
Consider the integral $\int_0^{\pi} \sin(2x)\, e^{-x}\, dx$

(a) Approximate the integral using composite trapezoidal rule with 5 points.
(b) Approximate the integral using composite Simpson rule with 5 points.

Solution:

(a) For composite trapezoidal rule 5 points mean that M = 4

$$M + 1$$



So, $h = \frac{b-a}{M} = \frac{\pi - 0}{4} = \frac{\pi}{4}$

The values at the nodes are

$$f_0 = f(0) = \sin(0)e^{-0} = (0)(1) = 0$$

$$f_1 = f\left(\frac{\pi}{4}\right) = \sin\left(\frac{2\pi}{4}\right) e^{\frac{-\pi}{4}} = (1)e^{\frac{-\pi}{4}} = 0.455938$$

$$f_2 = f\left(\frac{\pi}{2}\right) = \sin\left(\frac{2\pi}{2}\right) e^{\frac{-\pi}{2}} = (1)e^{\frac{-\pi}{4}} = 0$$

$$f_3 = f\left(\frac{3\pi}{4}\right) = \sin\left(2\left(\frac{3\pi}{4}\right)\right) e^{\frac{-3\pi}{4}} = (-1)e^{\frac{-3\pi}{4}} = -0.09478$$

$$f_4 = f(\pi) = \sin(2\pi)\, e^{-\pi} = (0)e^{-\pi} = 0$$

The Composite trapezoidal rule gives

$$\int_0^{\pi} \sin(2x)\, e^{-x}\, dx = \frac{h}{2}[f_0 + 2(f_1 + f_2 + f_3) + f_4]$$

$$= \frac{\frac{\pi}{4}}{2}[0 + 2(0.455938 + 0 - 0.09478) + 0] = 0.2836515$$

(b) For Composite Simpson rule with $\underbrace{\text{5 points}}_{2M = 1}$ means that $2M = 4$ (or M = 2)

So, $h = \frac{b-a}{2M} = \frac{\pi - 0}{4} = \frac{\pi}{4}$ So, the nodes are same as in part (a).

Composite Simpson rule gives

$$\int_0^{\pi} \sin(2x)e^{-e}\,dx = \frac{h}{3}[f_0 + 4(f_1 + f_3) + 2(f_2) + f_4]$$

$$= \frac{\pi/4}{3}[0 + 4(0.455938 + 0.09478) + 2(0) + 0]$$

$$= 0.37820338$$

Note: The exact value is 0.382714432

Textbook has program for Composite trapezoidal & Simpson rule.

```
function s= trapcom(f, a, b, M)
   ⋮
>> trapcom('f' 0, π, 10)
   ans = 0.36695122
```

```
function s = simprl(f, a, b, M)
   ⋮
>> simprl('f', 0, π, 5)
   ans = 0.382793073
```