MATH 3940 Numerical Analysis for Computer Scientists
Test 1 Solutions Fall 2023

1. (6 marks) Answer the following. Each part is worth one mark.

   (a) In terms of O( ) notations, what is the computational complexity of the backward substitution?
   $O(n^2)$

   (b) What is the built in command in Matlab to solve the linear system $AX = B$?
   $X = A \setminus B$

   (c) What is the built in command in Matlab to find Cholesky factorization of A?
   chol(A)

   (d) Suppose you have to find all eigenvalues and eigenvectors of A, what would you type in Matlab?
   [V D]=eig(A)

   (e) Suppose you have to find a root of any nonlinear equation $f(x) = 0$, what would you type in Matlab?
   fzero('f', x_0) will find a root near x_0 for function f saved in f.m file.

   (f) If you type feval('f', a) in Matlab, what will it return?
   The value of f at a.

2. (6 marks) Find Cholesky factorization of the matrix $A = \begin{bmatrix} 1 & -3 & 0 \\ -3 & 25 & 4 \\ 0 & 4 & 11 \end{bmatrix}$

Solution. Let $A = LL^T$ where L is a lower triangular matrix. Then we have

$$\begin{aligned}\begin{bmatrix} 1 & -3 & 0 \\ -3 & 25 & 4 \\ 0 & 4 & 11 \end{bmatrix} &= \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix} \\ &= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}\end{aligned}$$

Equating the corresponding entries of the matrices, we obtain following equations.

$$\begin{aligned}
& l_{11}^2 = 1 \ \Rightarrow\ l_{11} = 1 \\
& l_{11}l_{21} = -3 \ \Rightarrow\ l_{21} = -3 \\
& l_{11}l_{31} = 0 \ \Rightarrow\ l_{31} = 0 \\
& l_{21}^2 + l_{22}^2 = 20 \ \Rightarrow\ (-3)^2 + l_{22}^2 = 25 \ \Rightarrow\ l_{22}^2 = 16 \ \Rightarrow\ l_{22} = 4 \\
& l_{21}l_{31} + l_{22}l_{32} = 4 \ \Rightarrow\ 0 + 4(l_{32}) = 4 \ \Rightarrow\ l_{32} = 1 \\
& l_{31}^2 + l_{32}^2 + l_{33}^2 = 11 \ \Rightarrow\ 0 + (1)^2 + l_{33}^2 = 11 \Rightarrow\ l_{33}^2 = 10 \Rightarrow\ l_{33} = \sqrt{10}
\end{aligned}$$

Thus we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 4 & 0 \\ 0 & 1 & \sqrt{10} \end{bmatrix} \begin{bmatrix} 1 & -3 & 0 \\ 0 & 4 & 1 \\ 0 & 0 & \sqrt{10} \end{bmatrix}$$

□

3. (9 marks) Consider the system:
$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 6 \\ 4x_1 & + & 4x_2 & + & x_3 & = & 9 \\ & - & 4x_2 & - & 7x_3 & = & 21 \end{array}$$

(a) (7 marks) Solve the system using Gaussian elimination method with partial pivoting.
(b) (2 marks) Do you expect that the iterations of Gauss-Seidel method for the above system will converge? Justify your answer using the condition of convergence.

Solution. (a) We will perform Gaussian elimination on the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 6 \\ 4 & 4 & 1 & 9 \\ 0 & -4 & -7 & 21 \end{array}\right] \quad R_2 \leftrightarrow R_1 \quad \left[\begin{array}{rrr|r} 4 & 4 & 1 & 9 \\ 1 & 2 & -1 & 6 \\ 0 & -4 & -7 & 21 \end{array}\right]$$

$$R_2 - \frac{1}{4}R_1 \to R_2 \quad \left[\begin{array}{rrr|r} 4 & 4 & 1 & 9 \\ 0 & 1 & -\frac{5}{4} & \frac{15}{4} \\ 0 & -4 & -7 & 21 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{rrr|r} 4 & 4 & 1 & 9 \\ 0 & -4 & -7 & 21 \\ 0 & 1 & -\frac{5}{4} & \frac{15}{4} \end{array}\right]$$

$$R_3 + \frac{1}{4}R_2 \to R_3 \quad \left[\begin{array}{rrr|r} 4 & 4 & 1 & 9 \\ 0 & -4 & -7 & 21 \\ 0 & 0 & -3 & 9 \end{array}\right]$$

Now we will use back substitution to find the solution to the following system.

$$\begin{array}{rcrcrcr} 4x_1 & + & 4x_2 & + & x_3 & = & 9 \\ & - & 4x_2 & - & 7x_3 & = & 21 \\ & & & - & 3x_3 & = & 9 \end{array}$$

The third equation gives $-3x_3 = 9 \;\Rightarrow\; x_3 = -3$.

Putting $x_3 = -3$ into the second equation we obtain
$-4x_2 - 7(-3) = 21 \Rightarrow -4x_2 = 0 \Rightarrow x_2 = 0$.
Finally, substituting the value of x_2 and x_3 into the first equation we obtain
$4x_1 + 4(0) + (-3) = 9 \;\Rightarrow\; 4x_1 = 12 \;\Rightarrow\; x_1 = 3$.
Thus the solution is $(x_1, x_2, x_3) = (3, 0, -3)$.

(b) If A is strictly diagonally dominant, then Gauss-Seidel method will converge.

Here $|1| > |2| + |-1|$ is not true for the first row so A is not strictly diagonally dominant, Gauss-Seidel method may or may not converge. □

4. (10 marks) Consider the matrix $A = \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix}$ and $X_0 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

(a) (4.5 marks) Find all the eigenvalues and eigenvectors of A.
(b) (4 marks) Perform one iteration of the power method starting with X_0.
(c) (1.5 marks) What can you say about the convergence of power method for the matrix A? Justify your answer.

Solution. (a) To find the eigenvalues, we have to solve $|A - \lambda I| = 0$.

$$\begin{vmatrix} -3-\lambda & 1 & 1 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & -2-\lambda \end{vmatrix} = 0$$

Expanding along the first column, we obtain $(-3-\lambda)(2-\lambda)(-2-\lambda) = 0$. Thus the eigenvalues are -3, 2 and -2. Note that A is an upper triangular matrix and we can say right away that eigenvalues are the entries on the main diagonal.

For $\lambda_1 = -3$, we have

$$\left[\begin{array}{rrr|r} 0 & 1 & 1 & 0 \\ 0 & 5 & -1 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_2{-}5R_1 \to R_2 \left[\begin{array}{rrr|r} 0 & 1 & 1 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right] R_3{+}1/6R_2 \to R_3 \left[\begin{array}{rrr|r} 0 & 1 & 1 & 0 \\ 0 & 0 & -6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus we have

$$\begin{array}{rcrcl} x_2 & + & x_3 & = & 0 \\ & - & 6x_3 & = & 0 \end{array}$$

x_1 is a free variable. The second equation gives $x_3 = 0$, substituting $x_3 = 0$ into the first equation, we get $x_2 = 0$. If we take $x_1 = 1$, then the eigenvector is $\mathbf{v}_1 = [1\ 0\ 0]'$

For $\lambda_2 = 2$, we have

$$\left[\begin{array}{rrr|r} -5 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -4 & 0 \end{array}\right] R_3{+}4R_2 \to R_3 \left[\begin{array}{rrr|r} -5 & 1 & 1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{rcrcrcl} -5x_1 & + & x_2 & + & x_3 & = & 0 \\ & & & - & x_3 & = & 0 \end{array}$$

x_2 is a free variable. The first equation gives $x_1 = \frac{1}{5}x_2$. If we take $x_2 = 5$, then $x_1 = 1$, and the eigenvector is $\mathbf{v}_2 = [1\ 5\ 0]'$.

For $\lambda_3 = -2$, we have

$$\left[\begin{array}{rrr|r} -1 & 1 & 1 & 0 \\ 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{rcrcrcl} -x_1 & + & x_2 & + & x_3 & = & 0 \\ & & 4x_2 & - & x_3 & = & 0 \end{array}$$

x_3 is a free variable. The second equation gives $x_2 = x_3/4$. If we take $x_3 = 4$, then $x_2 = 1$. Substituting $x_2 = 1$ and $x_3 = 4$ into the first equation we obtain $x_1 = 1 + 4 = 5$. Thus the eigenvector is $\mathbf{v}_3 = [5\ 1\ \ 4]'$.

(b) The initial approximation is $X_0 = [2\ 1\ 1]'$. Using the power method

$$Y_1 = AX_0 = \begin{bmatrix} -3 & 1 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -6+1+1 \\ 0+2-1 \\ 0+0-2 \end{bmatrix} = \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix}$$

The element of largest magnitude is -4 so $\mu_1 = -4$.

Thus $X_1 = \dfrac{1}{\mu_1} Y_0 = \dfrac{1}{-4} \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1/4 \\ 1/2 \end{bmatrix}$

(c) Here the dominant eigenvalue is -3, and it is unique, so the power method will converge. $\square$

5. (a) (3 marks) Can we use the method of false position to solve the equation: $\dfrac{2x}{x-2} + x = 0$ starting with the interval $[1, 3]$? Justify your answer using the condition of convergence.

(b) (3 marks) Consider the equation: $x^3 - x - 3 = 0$. Perform one iteration of bisection method starting with the interval $[0, 2]$.

Solution. Here (a) $f(x) = \dfrac{2x}{x-2} + x$

The function is not continuous at $x = 2$ and $2 \in [1, 3]$. therefore we cannot use the method of false position starting with the interval $[1, 3]$ because the function is not continuous on the interval $[1, 3]$.

(b) $a = 0$ and $b = 2$, where $f(0) = 0 - 3 = -3 < 0$ and $f(2) = (2)^3 - 2 - 3 = 3 > 0$.

$$p = \frac{a+b}{2} = \frac{0+2}{2} = 1$$

$f(p) = f(1) = (1)^3 - 1 - 3 = -3 < 0$.

So the new interval will be $[1, 2]$.

□

6. (6 marks) Consider the equation: $1 + \sin x = x$

(a) (5 marks) Perform one iteration of Newton's method starting with $p_0 = 1$.

(b) (1 mark) What would be the convergence rate of Newton's method?

Solution. (a) $1 + \sin x = x \Rightarrow 1 + \sin x - x = 0$.

Let $f(x) = 1 + \sin x - x$, then $f'(x) = \cos x - 1$

The Newton's iterations are $p_n = p_{n-1} - \dfrac{f(p_{n-1})}{f'(p_{n-1})}$

$p_0 = 1$, thus

$p_1 = p_0 - \dfrac{f(p_0)}{f'(p_0)} = 1 - \dfrac{1 + \sin 1 - 1}{\cos 1 - 1} = 1 - \dfrac{\sin 1}{\cos 1 - 1}$

Using calculator in radian mode we obtain

$p_1 = 1 - \dfrac{\sin 1}{\cos 1 - 1} = 1 - \dfrac{0.84147}{0.5403 - 1} = 1 - \dfrac{0.84147}{-0.45969} = 2.8305$

(b) The convergence rate of Newton's method will be quadratic. □

7. (7 marks) Let $g(x) = \dfrac{x^2 + 3x - 4}{3}$

(a)(2 marks) Perform two iterations of the fixed point method starting with $p_0 = 0.5$.

(b) (5 marks) Do you expect fixed point method to converge to the solution $p = -2$ starting with the initial guess $p_0 = 0.5$? Justify your answer using the conditions of convergence.

Solution. (a) The fixed point iterations are $p_n = g(p_{n-1})$.

For $p_0 = 0.5$, we have

$$p_1 = g(0.5) = \frac{(0.5)^2 + 3(0.5) - 4}{3} = -0.75$$

$$p_2 = g(-0.75) = \frac{(-0.75)^2 + 3(-0.75) - 4}{3} = -1.8958$$

(b) Here $g(x) = \dfrac{x^2 + 3x - 4}{3}$ and $g'(x) = \dfrac{2x + 3}{3}$.

$g(x)$ and $g'(x)$ are continuous for all real numbers, so we can choose any interval containing the solution $p = -2$ and the initial guess $p_0 = 0.5$, say $[-3, 1]$. Then $g(x)$ and $g'(x)$ are continuous on the interval $[-3, 1]$.

Also $|g'(-2)| = \left|\dfrac{2(-2) + 3}{3}\right| = \left|\dfrac{-1}{3}\right| = 0.3333 < 1$

Thus the fixed point method will converge to the solution $p = -2$. $\square$