MATH 3940 Numerical Analysis for Computer Scientists

Problem Set 3 Solutions

1. Let $g(x) = \dfrac{x^2}{4} + \dfrac{5x}{4} - 3$.

(a) Solve the equation $x = g(x)$.

(b) Perform 3 iterations of the fixed point method starting with $p_0 = -3.5$.

(c) Do you expect fixed point method to converge with an initial approximation $p_0 = -3.5$? Justify your answer using the condition of convergence.

(d) Use Matlab to perform 40 iterations of fixed point method to solve $x = g(x)$, starting with $p_0 = -3.5$, and a tolerance of 10^{-5}. Do you get the expected convergence/divergence as your answer in part (c)?

(e) Do you expect fixed point method to converge with an initial approximation $p_0 = -0.25$? Justify your answer using the condition of convergence.

Solution. (a) $x = g(x) \;\Rightarrow\; x = \dfrac{x^2}{4} + \dfrac{5x}{4} - 3 \;\Rightarrow\; 4x = x^2 + 5x - 12$

$\Rightarrow\; x^2 + x - 12 = 0 \;\Rightarrow\; (x+4)(x-3) = 0 \;\Rightarrow\; x = -4, 3.$

(b) The fixed point iterations are $p_n = g(p_{n-1})$. For $p_0 = -3.5$, we have

$p_1 = g(-3.5) = \frac{(3.5)^2}{4} + \frac{5(3.5)}{4} - 3 = -4.3125$

$p_2 = g(-4.3125) = \frac{(-4.3125)^2}{4} + \frac{5(-4.3125)}{4} - 3 = -3.7412$

$p_3 = g(-3.7412) = \frac{(-3.7412)^2}{4} + \frac{5(-3.7412)}{4} - 3 = -4.17735$

(c) Yes, I expect the iterations will converge with $p_0 = -3.5$. The reason follows:

Here $g'(x) = \dfrac{2x}{4} + \dfrac{5}{4} = \dfrac{2x+5}{4}$.

Now $|g'(-4)| = |\frac{-8+5}{4}| = |\frac{-3}{4}| = 0.75 < 1$

The functions $g(x)$ and $g'(x)$ are continuous on $[-5, 0]$. The solution $-4 \in [-5, 0]$ and the initial guess $-3.5 \in [-5, 0]$. Also $|g'(-4)| < 1$, thus we expect that the fixed point method will converge to -4 starting with $p_0 = -3.5$.

(d) Using Matlab we see that the iterations converge to -4 in 36 iterations with tolerance 10^{-5} (See Matlab sheets for fixed point iterations).

(e) Yes, I expect the iterations will converge with $p_0 = -0.25$. The reason follows:

Here $g'(x) = \dfrac{2x}{4} + \dfrac{5}{4} = \dfrac{2x+5}{4}$.

Now $|g'(-4)| = |\frac{-8+5}{4}| = |\frac{-3}{4}| = 0.75 < 1$

The functions $g(x)$ and $g'(x)$ are continuous on $[-5, 0]$. The solution $-4 \in [-5, 0]$ and the initial guess $-0.25 \in [-5, 0]$ Also $|g'(-4)| < 1$, thus we expect that the fixed point method will converge to -4 starting with $p_0 = -0.25$.

(Actually with Matlab it did converge. You can see on Matlab sheets that the iterations converge to -4 in 38 iterations with tolerance 10^{-5}) $\square$

2. Given the equation $x^3 + x^2 - 3x - 3 = 0$.
   (a) Use the Matlab built-in function to find all roots of the above equation.
   (b) Use Matlab to perform 25 iterations of the fixed point method for each of the following functions, starting with $p_0 = 1$ and a tolerance of 10^{-5}. In the case of convergence, mention the number of iterations when the convergence is achieved.

   (i) $g_1(x) = \sqrt{\dfrac{3 + 3x - x^2}{x}}$

   (ii) $g_2(x) = -1 + \dfrac{3x + 3}{x^2}$

   (iii) $g_3(x) = \dfrac{x^3 + x^2 - x - 3}{2}$.

   Solution. See Matlab sheets for the solutions of all parts. $\square$

3. Consider the equation: $x^3 + 2x = 1$.
   (a) Can we use bisection method to find a solution of the equation starting with the interval $[0, 1]$? Justify your answer using the conditions of convergence.
   (b) Using hand calculations, perform 3 iterations of the bisection method starting with the interval $[0, 1]$.
   (c) Using hand calculations, perform 2 iterations of the method of false position starting with the interval $[0, 1]$.
   (d) Using hand calculations, perform 3 iterations of the secant method starting with the initial values $p_0 = 0$ and $p_1 = 1$.

   Solution. (a) The equation $x^3 + 2x = 1$ can be rewritten as $x^3 + 2x - 1 = 0$.
   So $f(x) = x^3 + 2x - 1$.

   Now $f(0) = 0 + 0 - 1 = -1 < 0$ and $f(1) = 1 + 2 - 1 = 2 > 0$.
   $f(x)$ is a polynomial and continuous on $\mathbb{R}$. Since $f(0)$ and $f(1)$ have opposite signs and f is continuous on $[0, 1]$, the bisection method can be used for the interval $[0, 1]$.

   (b) Here $a_1 = 0$, $b_1 = 1$, $f(x) = x^3 + 2x - 1$, $f(0) = -1 < 0$ and $f(1) = 2 > 0$.
   So $p_1 = \dfrac{a_1 + b_1}{2} = \dfrac{1}{2} = 0.5$.

   $f(p_1) = f(0.5) = 0.125 + 1 - 1 = 0.125 > 0$

   So we should take $a_2 = 0$ and $b_2 = 0.5$, the new interval is $[0, 0.5]$

   $p_2 = \dfrac{a_2 + b_2}{2} = \dfrac{0.5}{2} = 0.25$.

   $f(p_2) = f(0.25) = 0.015625 + 0.5 - 1 = -0.484375 < 0$

   So we should take $a_3 = 0.25$ and $b_3 = 0.5$, the new interval is $[0.25, 0.5]$

   $p_3 = \dfrac{a_3 + b_3}{2} = \dfrac{0.75}{2} = 0.375$.

   $f(p_3) = f(0.375) = 0.05273 + 0.75 - 1 = -0.19727 < 0$

   So we should take $a_4 = 0.375$ and $b_4 = 0.5$, the new interval is $[0.375, 0.5]$.

(c) Here $a_1 = 0$ and $b_1 = 1$, where $f(0) = -1 < 0$ and $f(1) = 2 > 0$.

$$p_1 = b_1 - \frac{f(b_1)(b_1 - a_1)}{f(b_1) - f(a_1)} = 1 - \frac{f(1)(1-0)}{f(1) - f(0)} = 1 - \frac{2(1)}{2-(-1)} = \frac{1}{3} = 0.3333$$

$f(p_1) = f(0.3333) = 0.0370 + 0.6666 - 1 = -0.2964 < 0$

So we should take $a_2 = 0.3333$ and $b_2 = 1$, the new interval is $[0.333, 1]$.

$$p_2 = b_2 - \frac{f(b_2)(b_2 - a_2)}{f(b_2) - f(a_2)} = 1 - \frac{f(1)(1 - 0.3333)}{f(1) - f(0.3333)} = 1 - \frac{2(0.6667)}{2 - (-0.2964)} = 0.4194$$

$$f(p_2) = f(0.4194) = 0.07377 + 0.8388 - 1 = -0.08743 < 0$$

So we should take $a_3 = 0.4194$ and $b_3 = 1$, the new interval is $[0.4194, 1]$.

(d) Here $p_0 = 0$ and $p_1 = 1$, where $f(p_0) = f(0) = -1$ and $f(p_1) = f(1) = 2$.

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 1 - \frac{f(1)(1-0)}{f(1) - f(0)} = 1 - \frac{2(1)}{2-(-1)} = \frac{1}{3} = 0.3333$$

$$f(p_2) = f(0.3333) = 0.0370 + 0.6666 - 1 = -0.2964$$

$$p_3 = p_2 - \frac{f(p_2)(p_2 - p_1)}{f(p_2) - f(p_1)} = 0.3333 - \frac{f(0.3333)(0.3333 - 1)}{f(0.3333) - f(1)}$$

$$= 0.3333 - \frac{-0.2964(-0.6667)}{-0.2964 - 2} = 0.41935$$

$$f(p_3) = f(0.41935) = 0.07374 + 2(0.41935) = -0.08756$$

$$p_4 = p_3 - \frac{f(p_3)(p_3 - p_2)}{f(p_3) - f(p_2)} = 0.41935 - \frac{f(0.41935)(0.41935 - 0.3333)}{f(0.41935) - f(0.3333)}$$

$$= 0.41935 - \frac{-0.08756(0.08605)}{-0.08756 - (-0.2964)} = 0.4554$$

$\square$

4. Consider the equation: $x - 2^{-x} = 0$

(a) Use the Matlab built-in function to find the root near 0.

(b) Use Matlab to perform 20 iterations of the bisection method with initial values $a = 0$, $b = 1$ and tolerance 10^{-5}.

(c) Use Matlab to perform 20 iterations of the method of false position with initial values $a = 0$, $b = 1$, tolerance$= 10^{-5}$ and epsilon$= 10^{-10}$.

(d) Use Matlab to perform 20 iterations of the secant method starting with the initial values $p_0 = 0$, $p_1 = 1$, tolerance$= 10^{-5}$ and epsilon$= 10^{-10}$.

(e) Use Matlab to perform 20 iterations of Newton's method with the initial approximation $p_0 = 1$, tolerance$= 10^{-5}$ and epsilon$= 10^{-10}$.

(f) Based on your results from parts (b) - (e), which method is more successful. Explain your answer using the convergence rates.

Solution See Matlab sheets for the solutions of all parts.

5. Consider the equation: $x \cos x = x$

(a) Using hand calculations find the exact solution(s) in the interval $[-\pi, \pi]$.

(b) Using hand calculations, perform 2 iterations of Newton's method starting with the initial approximation $p_0 = 1$.

(c) Use Matlab to perform 15 iterations of Newton's method with the initial approximation $p_0 = 1$, tolerance$= 10^{-5}$ and epsilon$= 10^{-7}$.

Solution. (a) $x \cos x = x \;\Rightarrow\; x \cos x - x = 0 \;\Rightarrow\; x(\cos x - 1) = 0$
$\Rightarrow\; x = 0,$ or $\cos x = 1$
$\Rightarrow\; x = 0,$ or $x = 2\pi$, which is not in $[-\pi, \pi]$.

So $x = 0$ is the only solution.

(b) $x \cos x = x \;\Rightarrow\; x \cos x - x = 0$.

Let $f(x) = x \cos x - x$, then $f'(x) = \cos x - x \sin x - 1$

The Newton's iterations are $p_n = p_{n-1} - \dfrac{f(p_{n-1})}{f'(p_{n-1})} = p_{n-1} - \dfrac{p_{n-1} \cos p_{n-1} - p_{n-1}}{\cos p_{n-1} - p_{n-1} \sin p_k - 1}$

Now $p_0 = 1$, so

$$
\begin{aligned}
p_1 &= p_0 - \frac{p_0 \cos p_0 - p_0}{\cos p_0 - p_0 \sin p_0 - 1} = 1 - \frac{\cos 1 - 1}{\cos 1 - \sin 1 - 1} \\
&= 1 - \frac{0.5403 - 1}{0.5403 - 0.84147 - 1} = 0.6467 \\
p_2 &= p_1 - \frac{p_1 \cos p_1 - p_1}{\cos p_1 - p_1 \sin p_1 - 1} = 0.6467 - \frac{0.6467 \cos 0.6467 - 0.6467}{\cos 0.6467 - 0.6467(\sin 0.6467) - 1} \\
&= 0.6467 - \frac{-0.13058}{-0.5916} = 0.426.
\end{aligned}
$$

(c) See Matlab sheets for the solution. □