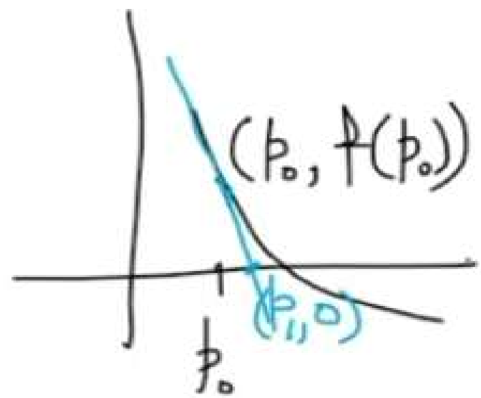
Newton's Method

Also known as Newton-Raphson method. This one of the most powerful and well-known numerical methods for solving a nonlinear eq. $f(x) = 0$.
For Newton's method, we need f to be continuous and also f' & f'' to be continuous
we say that $\underbrace{f \in c^2[a,b]}$

f, f', f'' are cont. on [a, b]



We will start with an initial approximation P_0.

Slope of the tangent line at $(P_0, f(P_0))$ is $m = f'(P_0) \leftarrow$ **(1)**
Also, we can find the slope of tangent line using the points $\big(P_0, f(P_0)\big)$ & $(P_1, 0) \Rightarrow m = \frac{0-f(P_0)}{P_1-P_0} \leftarrow$ **(2)**

Equating **(1)** & **(2)**, we get $f'(P_0) = \frac{-f(P_0)}{P_1-P_0}$

$$\Rightarrow f'(P_0)(P_1 - P_0) = -f(P_0)$$

$$(P_1 - P_0) = -\frac{f(P_0)}{f'(P_0)} \Rightarrow P_1 = P_0 - \frac{f(P_0)}{f'(P_0)} \qquad if\ f'(P_0) \neq 0$$

$P_k = P_{k-1} - \frac{f(P_{k-1})}{f'(P_{k-1})}$ provided that $f'(P_{k-1}) \neq 0$ for $k = 1, 2, \ldots.$

Example (1): Consider the eq. $x^2 + 23 = 10x$

(a) Can we use Newton's method starting with $P_0 = 5$? Justify your answer.

$x^2 + 23 = 10x \Rightarrow x^2 - 10x + 23 = 0$

Let $f(x) = x^2 - 10x + 23$

$f'(x) = 2x - 10$

$f'(5) = 2(5) - 10 = 0$ **<u>So, we can not</u>** use Newton's method.

(b) Perform 3 iterations of Newton's method starting with $P_0 = 3$.

$$P_k = P_{k-1} - \frac{f(P_{k-1})}{f'(P_{k-1})}$$

$$P_1 = P_0 - \frac{f(P_0)}{f'(P_0)} = 3 - \frac{f(3)}{f'(3)} = 3 - \frac{(3)^2 - 10(3) + 23}{2(3) - 10} = 3 + \frac{1}{2} = \frac{7}{2} = 3.5$$

$$P_2 = 3.5 - \frac{f(3.5)}{f'(3.5)} = 3.5 - \frac{(3.5)^2 - 10(3.5) + 23}{2(3.5) - 10} = 3.5 + \frac{0.25}{-3} = 3.5833$$

$$P_3 = 3.5833 - \frac{f(3.5833)}{f'(3.5833)} = 3.58784$$

Using Matlab it converges to 3.585788 in 4 iterations with tol 10^{-5}

Example 2:

Perform 2 iterations of Newton's method to find the +ve root of the equation $\sin x = x^2$ starting with $P_0 = 1$

Solution:

$\sin x = x^2 \Rightarrow \sin x - x^2 = 0$

Let $f(x) = \sin x - x^2$

$f'(x) = cosx - 2x$

$$P_1 = P_0 - \frac{f(P_0)}{f'(P_0)} = 1 - \frac{f(1)}{f'(1)} = 1 - \frac{\sin 1 - (1)^2}{\cos 1 - 2(1)}$$

use calculator in Radian's mode

$$P_1 = 1 - \frac{0.84147}{0.5403 - 2} = 0.8914$$

$$P_2 = 0.8914 - \frac{f(0.8914)}{f'(0.8914)} = 0.8914 - \frac{\overbrace{\sin(0.8914)}^{\boxed{0.77795}} - (0.8914)^2}{\underbrace{\cos(0.8914)}_{\boxed{0.62832}} - 2(0.8914)} = 0.87698$$

Textbook has program on page 84

err= abs($P_1 - P_0$)
y = feval(f, P_0)
if(err<tol | relerror <tol | abs(y) < epsilon) break;

10^{-5} $\qquad\qquad$ 10^{-7}

Newton's method $P_k = P_{k-1} - \frac{f(P_{k-1})}{f'(P_{k-1})}$ $\qquad$ fixed point: $P_k = g(P_{k-1})$

$g(P_{k-1})$

$$\Rightarrow g(x) = x - \frac{f(x)}{f'(x)} \quad \& \quad g'(x) = 1 - \frac{f'(x)f'(x) - f'(x)f''(x)}{[f'(x)]^2} = \frac{\cancel{[f'(x)]^2} - \cancel{[f'(x)]^2} + f'(x)f''(x)}{[f'(x)]^2}$$

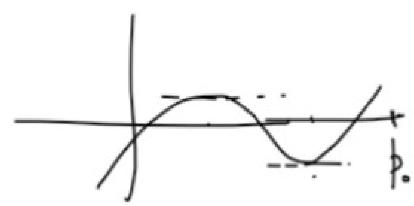
$$g'(x) = \frac{f'(x)f''(x)}{[f'(x)]^2}$$

So, Newton's method will converge if f, f', f'' are cont. on $[a, b]$ & $P_0 \in [a, b]$.
& $|g'(x)| = \left|\frac{f'(x)f''(x)}{[f'(x)]^2}\right| < 1$ for all $c \in (a, b)$

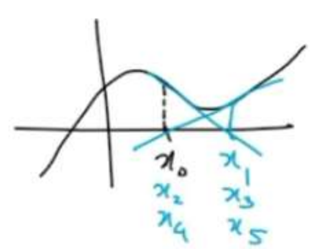
Newton's method has few problems.

(i) We can have division by zero. $P_k = P_{k-1} - \frac{f(P_{k-1})}{f'(P_{k-1})}$ $\qquad$ $f'(P_{k-1}) \neq 0$ for $k = 1,2, \ldots$

we can see that $f'(P_0) \neq 0$
but may be f' is zero at certain iteration later on.

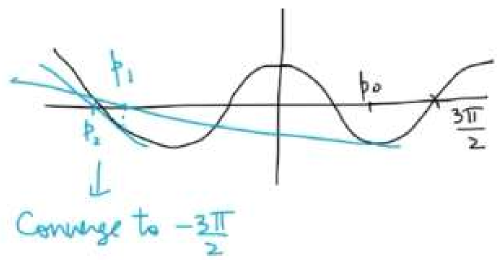


(ii) It is possible that values of iterations goes back & forth.



(iii) $\cos x = 0 \Rightarrow x = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

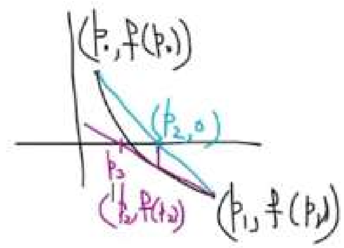
Suppose you want to find $x = \frac{3\pi}{2} = 4.71239$



Newton's method is fast method, but we need to have a good initial guess and graphs are always helpful to give us an idea of the initial guess P_0

Secant Method

In Newton's method we need to find $f's$ it may be complicated to find f'. Secant method is quite faster method, and it does not require to calculate f'.



We need to start with an interval$[P_0, P_1]$
such that f is continuous on $[P_0, P_1]$
We draw secant line & the new value P_2 is the
x-intercept of the secant line.

Slope of the secant line is $m = \frac{f(P_1)-f(P_0)}{P_1-P_0}$ $\leftarrow$ **(1)**

also, the slope of the secant line is $m = \frac{0-f(P_1)}{P_2-P_1}$ $\leftarrow$ **(2)**

From **(1)** & **(2)** $\frac{f(P_1)-f(P_0)}{P_1-P_0} = \frac{0-f(P_1)}{P_2-P_1}$

$$(P_2 - P_1)(f(P_1) - f(P_0)) = -f(P_1)(P_1 - P_0)$$

$$P_2 - P_1 = \frac{-f(P_1)(P_1 - P_0)}{f(P_1) - f(P_0)}$$

$$P_2 = P_1 - \frac{f(P_1)(P_1 - P_0)}{f(P_1) - f(P_0)}$$

$$P_k = P_{k-1} - \frac{f(P_{k-1})(P_{k-1}-P_{k-2})}{f(P_{k-1})-f(P_{k-2})} \qquad k = 2,3, \ldots$$

Example:

Consider the equation $x^2 + 23 = 10x$. Perform 3 iterations of secant method starting with $P_0 = 3$ & $P_1 = 4$ or starting with interval [3,4].

Solution:

$x^2 + 23 = 10x \Rightarrow x^2 - 10x + 23 = 0$ Let $f(x) = x^2 - 10x + 23$

$f(P_0) = f(3) = (3)^2 - 10(3) + 23 = 2$

$f(P_1) = f(4) = (4)^2 - 10(4) + 23 = -1$

$$P_k = P_{k-1} - \frac{f(P_{k-1})(P_{k-1} - P_{k-2})}{f(P_{k-1}) - f(P_{k-2})}$$

$$P_2 = P_1 - \frac{f(P_1)(P_1 - P_0)}{f(P_1) - f(P_0)} = 4 - \frac{f(4)(4-3)}{f(4) - f(3)} = 4 - \frac{-1(1)}{-1-2} = 4 - \frac{1}{3} = \frac{11}{3} = 3.6667$$

$f(3.6667) = (3.6667)^2 - 10(3.6667) + 23 = -0.2222$

$$P_3 = P_2 - \frac{f(P_2)(P_2 - P_1)}{f(P_2) - f(P_1)} = 3.6667 - \frac{f(3.6667)(3.6667 - 4)}{f(3.6667) - f(4)} = 3.6667 - \frac{(-0.2222)(-0.3333)}{-0.2222 - (-1)}$$

$$= 3.57143$$

$$P_4 = P_3 - \frac{f(P_3)(P_3 - P_2)}{f(P_3) - f(P_2)} = 3.57143 - \frac{f(3.57143)(3.57143 - 3.6667)}{f(3.57143) - f(3.6667)} = \underbrace{3.586207}_{3^{\text{rd}}\text{ iteration}}$$

Using Matlab it converges to 3.58788 in 5 iterations with tol 10^{-5}.

$$\left.\begin{array}{ll}\text{Bisection} & \to 17 \text{ iterations}\\ \text{Regula Falsi} & \to 8 \text{ iterations}\end{array}\right\} \boxed{\begin{array}{l}\text{slower}\\ \text{methods}\end{array}} \qquad \left.\begin{array}{ll}\text{Newton} & \to 4 \text{ iterations}\\ \text{Secant} & \to 5 \text{ iterations}\end{array}\right\} \boxed{\begin{array}{l}\text{faster}\\ \text{methods}\end{array}}$$

The textbook has program for secant method on page 84

function $\underbrace{[p1, err, k, y]}$ = secant $(f, P0, P1, delta, epsilon, max1)$

we can change outputs depending how many outputs we want to see

need to have an in? file for function

2 initial guesses

tolerance

max. number of iterations

$(f \cdot m)$

myfunction or fQ2A etc.

>> secant('f', 0, 2, 10^(-5), 10^(-7), 20)

ans = P, here   so it will not show all outputs

Bracketing methods → Bisection method, Regula Falsi (Method of False position) } Convergence rate is linear **(Slower methods but globally convergent)**

$$|E_{k+1}| \approx \underbrace{A}_{\text{a constant}}|E_k|$$

$$|P_{k+1} - P_k| \approx A|P_k - P_{k-1}|$$

If we start with an interval [a,b] such that f is cont. on [a,b] and f(a) f(b) < 0 then it **always converges**

Newton's method and secant method are faster methods.

Newton's method: $|E_{k+1}| \approx \underbrace{A}_{\text{a constant}}|E_k|^2$

Secant method: $|E_{k+1}| \approx \underbrace{A}_{\text{a constant}}|E_k|^{1.618}$

Convergence rate is quadratic. → if the root is a simple root ← Convergence rate is nearly quadratic

Both Newton's method and secant methods are locally convergent

Their convergence depends on a good initial guess (approximation P_0).
We need initial guess closer to the solution.

We have many hybrid algorithms used in computers where the program use bisection or regula Falsi method in the beginning and then it will use Newton or secant method once it is closer to the solution. ($f(c)$ is small or the interval $b - a$ is small so we can choose a guess in [a,b]).

Newton's method converges quadratically if the root is a <u>simple root</u>
it is linear convergence rate if root is not a simple root
(root is a multiple root)
Secant method convergence rate is nearly quadratic of the root is a <u>simple root</u> & it is linear convergence rate if root is a multiple root.
In next lecture we will discuss about simple & multiple roots.