Newton Polynomials

We did Lagrange polynomials $\rightarrow P_1(x), P_2(x), \ldots$
Which one is a good approximation? P_1, P_2, P_3 *or* P_4

It is sometimes helpful to find several approximating polynomials $P_1(x), P_2(x), \ldots, P_N(x)$ & then choose that one that suits our needs.
If the Lagrange polynomials are used, then there is no recursive relationship between $P_{k-1}(x)$& $P_k(x)$. Each polynomial has to be constructed separately and the work required to compute the higher order polynomials involved many computations. (The work done in calculating P_{N-1} does not lessen the work needed to calculate P_N).

For example,
Let $f(x) = x^3$ & the nodes are $x_0 = 0, x_1 = 1, x_2 = 2, \& \; x_3 = 3$

$P_1(x)$

x_k	0	1	2	3
$f(x_k) \leftarrow y_k$	0	1	8	27

$(0)^3 \quad (1)^3 \quad (2)^3 \quad (3)^3$

(i) Calculate $P_1(x)$ using nodes x_0& x_1.

$$P_1(x) = y_0 \frac{(x - x_1)}{(x_0 - x_1)} + y_1 \frac{x - x_0}{x_1 - x_0}$$
$$= \cancel{0 \frac{(x-1)}{(0-1)}} + 1 \frac{x-0}{1-0}$$
$$= x$$

(ii) Calculate $P_2(x)$ using nodes $x_0, x_1, \& \; x_2$

$$P_2(x) = y_0 \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + y_1 \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + y_2 \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= \cancel{0 \frac{(x-1)(x-2)}{(0-1)(0-2)}} + 1 \frac{(x-0)(x-2)}{(1-0)(1-2)} + 8 \frac{(x-0)(x-1)}{(2-0)(2-1)}$$

$$= \frac{x^2 - 2x}{(1)(-1)} + 8 \frac{(x^2 - x)}{(2)(1)}$$
$$= -x^2 + 2x + 4x^2 - 4x$$
$$= 3x^2 - 2x$$

(iii) Calculate $P_3(x)$ using nodes x_0, x_1, x_2 & x_3

$$P_3(x) = y_0 \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} + y_1 \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} + y_2 \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

$$+y_3 \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}$$

$$= \cancel{0\frac{(\ \ )(\ \ )(\ \ )}{(\ \ )(\ \ )(\ \ )}} + 1\frac{(x-0)(x-2)(x-3)}{(1-0)(1-2)(1-3)} + 8\frac{(x-0)(x-2)(x-3)}{(2-0)(2-1)(2-3)} + 27\frac{(x-0)(x-1)(x-2)}{(3-0)(3-1)(3-2)}$$

$$= \frac{x(x^2-5x+6)}{(1)(-1)(-2)} + 8\frac{x(x^2-4x+3)}{(2)(1)(-1)} + 27\frac{x(x^2-3x+2)}{(3)(2)(1)}$$

$$= \frac{x^3-5x^2+6x}{2} + 4(x^3-4x^2+3x) + \frac{9(x^3-3x^2+2x)}{2}$$

$$= x^3\left(\frac{1}{2} - 4 + \frac{9}{2}\right) + \cancel{x^2\left(-\frac{5}{2} + 16 - \frac{27}{2}\right)} + \cancel{x\left(\frac{6}{2} - 12 + \frac{18}{2}\right)}$$

$$= x^3$$

Newton polynomials have a recursive relationship between $P_{N-1}(x)$ & $P_N(x)$

$$P_1(x) = a_0 + a_1(x-x_0)$$

$$P_2(x) = \overbrace{a_0 + a_1(x-x_0)}^{P_1(x)} + a_2(x-x_0)(x-x_1) = P_2(x0 + a_2(x-x_0)(x-x_1)$$

$$P_3(x) = \overbrace{a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1)}^{P_2(x)} + a_3(x-x_0)(x-x_1)(x-x_2)$$

$$= P_2(x) + a_3(x-x_0)(x-x_1)(x-x_2)$$

$\vdots$

$$P_N(x) = P_{N-1}(x) + a_N(x-x_0)(x-x_1)(x-x_2)\dots(x-x_{N-1})$$

$$P_N(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + \cdots + a_N(x-x_0)(x-x_1)\dots(x-x_{N-1})$$

Called Newton polynomials where the coefficients $a_0, a_1, \dots, a_N$ are found by using divided differences.

For interpolation polynomial (Newton polynomials) $P_N(x_k) = y_k \; or \; f(x_k)$

$P_1(x_0) = f(x_0)$ Now $P_1(x) = a_0 + a_1(x - x_0)$

2 nodes x_0 & x_1

$P_1(x_0) = f(x_0)$

$P_1(x_1) = f(x_1)$

$$a_0 + a_1\underbrace{(x_0 - x_0)}_{0} = f(x_0) \Rightarrow a_0 = f(x_0)$$

$$a_0 + a_1(x_1 - x_0) = f(x_1)$$
$$f(x_0) + a_1(x_1 - x_0) = f(x_1)$$
$$a_1(x_1 - x_0) = f(x_1) - f(x_0)$$
$$a_1 = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

one already found

$$P_2(x) = \underbrace{a_0 + a_1(x - x_0)}_{P_1(x)} + a_2(x - x_0)(x - x_1)$$

$$P_2(x_0) = f(x_0), P_2(x_1) = f(x_1), \underbrace{P_2(x_2) = f(x_2)}$$

$$a_0 + a_1(x_2 - x_0) + a_2(x_2 - x_0)(x_2 - x_1) = f(x_2)$$

$f(x_0)$ $\qquad$ $\frac{f(x_1) - f(x_0)}{x_1 - x_0}$

The Divided Difference for a function $f(x)$ are defined as follows.

$$f[x_k] = f(x_k)$$

$$f[x_{k-1}, x_k] = \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}$$

$$f[x_{k-2}, x_{k-1}, x_k] = \frac{f[x_{k-1}, x_k] - f(x_{k-2}, x_{k-1})}{x_k - x_{k-2}}$$

$\vdots$

$$f\left[x_{k-j}, x_{k-j+1}, \dots, x_k\right] = \frac{f\left[x_{k-j+1}, x_{k-j+2}, \dots, x_k\right] - f\left[x_{k-j}, x_{k-j+1}, \dots, x_{k-1}\right]}{x_k - x_{k-j}}$$

Example 1:

Let $f(x) = x^3$ & the nodes are $x_0 = 0, x_1 = 1, x_2 = 2, x_3 = 3$

(i) Calculate Newton polynomial $P_1(x)$ using nodes x_0 & x_1

(ii) Calculate Newton polynomial $P_2(x)$ using x_0, x_1, x_2

(iii) Find the divided difference table & Newton polynomial $P_3(x)$ using $x_0, x_1, x_2,$ & x_3.

x_k	0	1	2	3
$f(x_k) \leftarrow y_k$	0	1	8	27
	$(0)^3$	$(1)^3$	$(2)^3$	$(3)^3$

Solution:

The divided difference table is

x_k	$f[x_k]$	$f[x_{k-1}, x_k]$	$f[x_{k-2}, x_{k-1}, x_k]$	$f[\ ,\ ,\ ,\ ]$	$f[\ ,\ ,\ ,\ ,\ ]$
0	$\boxed{0} \to a_0$	$\frac{1-0}{1-0} = \boxed{1} \to a_1$	$\frac{7-1}{2-0} = \boxed{3} \to a_2$	$\frac{6-3}{3-0} = \boxed{1} \to a_3$	$\frac{1-1}{4-0} = \boxed{0} \to a_4$
1	1	$\frac{8-1}{2-1} = 7$	$\frac{19-7}{3-1} = 6$	$\frac{9-6}{4-1} = 1$	
2	8	$\frac{27-8}{3-2} = 19$	$\frac{37-19}{4-2} = 9$		
3	27	$\frac{64-27}{4-3} = 37$			
4	64				

$$P_1(x) = a_0 + a_1(x - x_0) = 0 + 1(x - 0) = x$$

$$\begin{aligned} P_2(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1) \\ &= 0 + 1(x - 0) + 3(x - 0)(x - 1) \\ &= x + 3(x^2 - x) \\ &= 3x^2 - 2x \end{aligned}$$

OR

$$\begin{aligned} P_2(x) &= P_1(x) + a_2(x - x_0)(x - x_1) \\ &= x + 3(x - 0)(x - 1) \\ &= x + 3x^2 - 3x \\ &= 3x^2 - 2x \end{aligned}$$

$$P_3(x) = \underbrace{a_0 + a_1(x - x_0) + a_2(x - x_0)(x - x_1)}_{P_2(x)} + a_3(x - x_0)(x - x_1)(x - x_2)$$

$$\begin{aligned} &= (3x^2 - 2x) + (1)(x - 0)\underbrace{(x - 1)(x - 2)}_{(x^2 - 3x + 2)} \\ &= 3x^2 - 2x + x^3 - 3x^2 + 2x \\ &= x^3 \end{aligned}$$

If you try to find $P_4(x)$

$$P_4(x) = P_3(x) + a_4(x - x_0)(x - x_1)(x - x_2)(x - x_3)$$

Example 2:

Consider the data

x_k	0	1	2	3	4	5
y_k	-2	2	4	4	2	-2

Calculate the divided difference table.

Find Newton polynomial using all nodes.

Solution: The divided difference table is

x_k	$f[\,]$	$f[\,,\,]$	$f[\,,\,,\,]$	$f[\,,\,,\,,\,]$	$f[\,,\,,\,,\,,\,]$	$f[\,,\,,\,,\,,\,,\,]$
0	$\boxed{-2} \to a_0$					
		$\frac{2+2}{1-0}=\boxed{4} \to a_1$				
1	2		$\frac{2-4}{2-0}=\boxed{-1} \to a_2$			
		$\frac{4-2}{2-1}=2$		$\frac{-1+1}{3-0}=\boxed{0} \to a_3$		
2	4		$\frac{0-2}{3-1}=-1$		$\frac{0-0}{4-0}=\boxed{0} \to a_4$	
		$\frac{4-4}{3-2}=0$		$\frac{-1+1}{4-1}=0$		$\frac{0-0}{5-0}=\boxed{0} \to a_5$
3	4		$\frac{-2-0}{4-2}=-1$		$\frac{0-0}{5-1}=0$	
		$\frac{2-4}{4-3}=-2$		$\frac{-1+1}{5-2}=0$		
4	2		$\frac{-4+2}{5-3}=-1$			
		$\frac{-2-2}{5-4}=-4$				
5	-2					

Newton polynomial is

$$P_5(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) + a_3(x-x_0)(x-x_1)(x-x_2) + a_4(x-x_0)(x-x_1)(x-x_2)(x-x_3) + a_5(x-x_0)(x-x_1)(x-x_2)(x-x_3)(x-x_4)$$

$$= -2 + 4(x-0) - 1(x-0)(x-1) + \cancel{0(x-0)(x-1)(x-2)} + 0 + 0$$

$$= -2 + 4x - x^2 + x$$

$$= -x^2 + 5x - 2$$

Textbook has program on Page 227

$$\underbrace{function[c\ d]}_{} = newpoly\underbrace{(x,y)}_{\text{Data}}$$

Coefficients of x'a — Divide difference table — Data

>> X=[0 1 2 3 4 5];
>>Y=[-2 2 4 4 2 -2];
>>[C D]=newpoly(X,Y)

C = 0 0 0 -1 5 2 $\longrightarrow -x^2 + 5x + 2$

$$D = \begin{matrix} -2 & 0 & 0 & 0 & 0 & 0 \\ 2 & 4 & 0 & 0 & 0 & 0 \\ 4 & 2 & -1 & 0 & 0 & 0 \\ 4 & 0 & -1 & 0 & 0 & 0 \\ 2 & -2 & -1 & 0 & 0 & 0 \\ -2 & -4 & -1 & 0 & 0 & 0 \end{matrix}$$

If $x_0, x_1, \ldots, x_N$ are $N+1$ nodes then $f(x) = P_N(x) + E_N(x)$

Newton polynomial Error

Where the error is

$E_N(x) = \frac{(x-x_0)(x-x_1)\cdots(x-x_N)f^{N+1}(c)}{(N+1)!}$ for some c in $[x_0, x_N]$

Same error as for Lagrange polynomial.
Called Approximated error. Lagrange is easier to calculate.

If we know that which $P_N(x)$ is needed, then Lagrange is preferred.