MATH 3940 Numerical Analysis for Computer Scientists
Test 2 Solutions Fall 2023

1. (6 marks) Answer each of the following parts. Each part is worth one mark.

   (a) What is the built-in function in Matlab to find interpolation polynomial of degree 7 for a given data of row vectors X and Y?
   polyfit(X,Y,7)

   (b) What is the built-in function in Matlab to find the points where a nonlinear function f has minimum value?
   fminsearch('f', initial guess).

   (c) Does $f(x) = \ln x$ have a Taylor polynomial expansion about $x_0 = 1$?
   Yes, f, f', f", ... are continuous about $x_0 = 1$.

   (d) Suppose we are given the nodes $x_0, x_1, \ldots, x_8$. What will be the highest possible degree of Lagrange polynomial for these nodes?
   Since there are 9 nodes, the highest possible degree of Lagrange polynomial will be 8.

   (e) Suppose $f(x)$ is a least-squares approximation for the given data (x_i, y_i), do you expect that $f(x_i) = y_i$ for each of the nodes x_i?
   No.

   (f) Suppose you have to find the least-squares curve $y = a_1 \ln x + a_0$ by data linearization method, what would be the change of variable formulas?
   The change of variables are $X = \ln x$ and $Y = y$.

2. (6 marks) Consider the function: $f(x) = \dfrac{1}{1-x}$. Find the Taylor polynomial of degree 2 expanded about $x_0 = 0$.

Solution. Here $f(x) = \frac{1}{1-x}$, so $f(0) = \frac{1}{1-0} = 1$

$$f'(x) = -\frac{-1}{(1-x)^2} \;\Rightarrow\; f'(0) = \frac{1}{(1-0)^2} = 1$$
$$f''(x) = -2\frac{-1}{(1-x)^3} \;\Rightarrow\; f''(0) = \frac{2}{(1-0)^3} = 2$$

Taylor polynomial of degree 2 is given by

$$P_2(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2$$

For $x_0 = 0$, we have

$$\begin{aligned} P_2(x) &= f(0) + \frac{f'(0)}{1!}(x-0) + \frac{f''(0)}{2!}(x-0)^2 \\ &= 1 + 1(x) + \frac{2}{2}(x)^2 = 1 + x + x^2 \end{aligned}$$

□

3. Consider the system of nonlinear equations: $\begin{array}{ccccccc} x & + & y^2 & - & 4y & = & 5 \\ x & & & + & 2y & = & 13 \end{array}$

(a) (5 marks) Solve the system to find the exact solutions.
(b) (3 marks) Perform one iteration of Jacobi method starting with $x_0 = 7$, $y_0 = 1$.
(c) (7 marks) Suppose Gauss-Seidel method is used to solve the system with $x_0 = 7$ and $y_0 = 1$. Do you expect it to converge to any solution? Justify your answer using the conditions of convergence.

Solution. (a) Multiplying second equation by -1 and adding to first equation, we have

$$\begin{array}{ccccccc} x & + & y^2 & - & 4y & = & 5 \\ -x & & & - & 2y & = & 13 \\ \hline & & y^2 & - & 6y & = & -8 \end{array}$$

$y^2 - 6y + 8 = 0 \ \Rightarrow (y-2)(y-4) = 0 \ \Rightarrow \ y = 2, 4$
Substituting $y = 2$ into the second equation we have
$x + 2y = 13 \ \Rightarrow \ x + 4 = 13 \ \Rightarrow x = 9$
Substituting $y = 4$ into the second equation we have
$x + 2y = 513 \Rightarrow \ x + 8 = 13 \ \Rightarrow x = 5.$
Thus the solution is $(9, 2)$ and $(5, 4)$.

(b) Solving the first equation for x and the second equation for y, we have Jacobi iterations as

$$x_k = -y_{k-1}^2 + 4y_{k-1} + 5 \quad \text{and} \quad y_k = \frac{13 - x_{k-1}}{2}$$

Using $x_0 = 7$ and $y_0 = 1$, we have

$$x_1 = -(1)^2 + 4(1) + 5 = 8 \quad \text{and} \quad y_1 = \frac{13-7}{2} = \frac{6}{2} = 3$$

(c) Solving the first equation for x and the second equation for y, we have

$$\begin{aligned} x &= -y^2 + 4y + 5 & \Rightarrow g_1(x,y) &= -y^2 + 4y + 5 \\ y &= \frac{-x+13}{2} & \Rightarrow g_2(x,y) &= \frac{-x+13}{2} \end{aligned}$$

$$\frac{\partial g_1}{\partial x} = 0, \quad \frac{\partial g_1}{\partial y} = -2y + 4, \quad \frac{\partial g_2}{\partial x} = \frac{-1}{2}, \quad \frac{\partial g_2}{\partial y} = 0$$

The functions g_1, g_2 and their first order partial derivatives are continuous on the region $R = \{(x,y)|0 \le x \le 10, 0 \le y \le 5\}$ that contains the solutions and the intial guess.

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |0| + |-2y+4| = |-2y+4|$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = |\frac{-1}{2}| + |0| = \frac{1}{2} < 1$$

For $y = 2$, the value of $|-2y+4| = |-2(2)+4| = 0 < 1$. So the method will converge to $(9, 2)$.

For $y = 4$, the value of $|-2y+4| = |-2(4)+4| = 4 \neq 1$. The sufficient condition for the convergence is not satisfied and the iterations may converge or diverge. $\square$

4. Consider the data

x_i	-1	0	1
$f(x_i)$	0	-1	-2

obtained from $f(x) = x^3 - 2x - 1$.

(a) (4 marks) Find Lagrange polynomial $P_2(x)$ using the nodes x_0, x_1, and x_2.
(b) (4 marks) Find divided difference table and Newton polynomial $P_2(x)$ using the nodes x_0, x_1, and x_2.
(c) (6 marks) Use the error formula to find a bound for the error $P_2(0.3)$ and compare the bound to the actual error.

Solution. (a) Lagrange polynomial $P_2(x)$ is given by

$$P_2(x) = f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

Substituting the values from the given data, we obtain

$$\begin{aligned} P_2(x) &= 0\frac{(x-0)(x-1)}{(-1-0)(-1-1)} - 1\frac{(x+1)(x-1)}{(0+1)(0-1)} - 2\frac{(x+1)(x-0)}{(1+1)(1-0)} \\ &= 0 - \frac{1(x^2-1)}{-1} - \frac{2(x^2+x)}{2} \\ &= x^2 - 1 - x^2 - x = -x - 1 \end{aligned}$$

(b)

x_i	$f[x_i]$	$f[\ ,\ ]$	$f[\ ,\ ,\ ]$
-1	0		
0	-1	$\frac{-1-0}{0+1} = -1$	
1	-2	$\frac{-2+1}{1-0} = -1$	$\frac{-1+2}{1+1} = 0$

The Newton interpolation polynomial is

$$\begin{aligned} P(x) &= a_0 + a_1(x-x_0) + a_2(x-x_0)(x-x_1) \\ &= 0 - 1(x+1) + 0(x+1)(x-0) \\ &= -x - 1 \end{aligned}$$

(c) The error formula is $E_2(x) = \dfrac{f'''(c)(x-x_0)(x-x_1)(x-x_2)}{3!} = \dfrac{f'''(c)(x+1)(x)(x-1)}{3!}$ for some $c \in [-1, 1]$.

Now $f(x) = x^3 - 2x - 1,\ f'(x) = 3x^2 - 2,\ f''(x) = 6x,\ f'''(x) = 6$
Thus the error bound for $P_2(0.3)$ is

$$|E_2(0.3)| = \left|\frac{(6)(0.3+1)(0.3)(0.3-1)}{6}\right| = |(1.3)(0.3)(-0.7)| = 0.273.$$

The error bound is 0.273.

The exact value at 0.3 is $f(0.3) = (0.3)^3 - 2(0.3) - 1 = -1.573$.
The polynomial obtained in part (a) and (b) is $P_2(x) = -x - 1$.
So $P_2(0.3) = -0.3 - 1 = -1.3$.

The exact error is $|P_2(0.3) - f(0.3)| = |-1.3 + 1.573| = 0.273$.

Note that error bound is same as the exact error.

□

5. (9 marks) Consider the data

x_i	1	2	3
y_i	4	7	12

Find the least-squares line $y = a_1 x + a_0$ for the data and calculate the error.

Solution.

	x_i	y_i	x_i^2	$x_i y_i$
	1	4	1	4
	2	7	4	14
	3	12	9	36
Sum	6	23	14	54

The normal equations are

$$\begin{array}{rcrcr} 3a_0 & + & 6a_1 & = & 23 \\ 6a_0 & + & 14a_1 & = & 54 \end{array}$$

Multiplying the first equation by -2 and adding to the second equation, we have

$$\begin{array}{rcrcr} -6a_0 & - & 12a_1 & = & -46 \\ 6a_0 & + & 14a_1 & = & 54 \\ \hline & - & 2a_1 & = & -8 \end{array}$$

Thus $a_1 = \frac{-8}{-2} = 4$.

Substituting $a_1 = 4$ into the first equation we obtain
$3a_0 + 6a_1 = 23 \;\Rightarrow\; 3a_0 + 6(4) = 23 \;\Rightarrow 3a_0 = 23 - 24 = -1 \;\Rightarrow\; a_0 = \frac{-1}{3} = -0.3333$.
Therefore the least-squares line is $y = f(x) = a_1 x + a_0 = 4x - \frac{1}{3}$ or $y = 4x - 0.3333$.

The error $E = \displaystyle\sum_{i=1}^{3} (y_i - f(x_i))^2 = \sum_{i=1}^{3} (y_i - (4x_i - \frac{1}{3}))^2$

$$\begin{aligned} E &= \left(4 - \frac{11}{3}\right)^2 + \left(7 - \frac{23}{3}\right)^2 + \left(12 - \frac{35}{3}\right)^2 \\ &= \left(\frac{1}{3}\right)^2 + \left(-\frac{2}{3}\right)^2 + \left(\frac{1}{3}\right)^2 \\ &= \frac{1}{9} + \frac{4}{9} + \frac{1}{9} = \frac{6}{9} = \frac{2}{3} = 0.667 \end{aligned}$$

$\square$