MATH 3940 Numerical Analysis for Computer Scientists
Midterm Solutions Fall 2020

1. Consider the following system

$$\begin{array}{rcrcrcr} 4x_1 & - & x_2 & + & 2x_3 & = & 12 \\ -4x_1 & + & x_2 & + & 3x_3 & = & 3 \\ 2x_1 & + & 3x_2 & - & 6x_3 & = & -22 \end{array}$$

(a) (10 marks) Find the LU decomposition of the coefficient matrix A and then solve the resulting triangular systems.

(b) (2 marks) Do you expect that the iterations of Jacobi method for the above system will converge? Justify your answer using the condition of convergence.

Solution. (a) Here we have

$$A = \begin{bmatrix} 4 & -1 & 2 \\ -4 & 1 & 3 \\ 2 & 3 & -6 \end{bmatrix} \qquad B = \begin{bmatrix} 12 \\ 3 \\ -22 \end{bmatrix} \quad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The multipliers are $m_{21} = \frac{-4}{4} = -1$, $m_{31} = \frac{2}{4} = \frac{1}{2}$.

$$\begin{array}{c} R_2 + R_1 \to R_2 \\ R_3 - \frac{1}{2}R_1 \to R_3 \end{array} \begin{bmatrix} 4 & -1 & 2 \\ 0 & 0 & 5 \\ 0 & 7/2 & -7 \end{bmatrix}$$

Since $a_{22} = 0$, we have to interchange the second and the third row, which gives

$$R_2 \leftrightarrow R_3 \begin{bmatrix} 4 & -1 & 2 \\ 0 & 7/2 & -7 \\ 0 & 0 & 5 \end{bmatrix} \qquad P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

The multiplier is $m_{32} = 0$, no elimination is required. Thus we have

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \qquad U = \begin{bmatrix} 4 & -1 & 2 \\ 0 & 7/2 & -7 \\ 0 & 0 & 5 \end{bmatrix}$$

Note that we have interchanged m_{21} and m_{31} in L because R_2 and R_3 was interchanged.

$$PB = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 12 \\ 3 \\ -22 \end{bmatrix} = \begin{bmatrix} 12 \\ -22 \\ 3 \end{bmatrix}$$

$$LY = PB \Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -22 \\ 3 \end{bmatrix}$$

Solving these we obtain $y_1 = 12$, $y_2 = -28$, $y_3 = 15$. Now we solve

$$UX = Y \Rightarrow \begin{bmatrix} 4 & -1 & 2 \\ 0 & 7/2 & -7 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 12 \\ -28 \\ 15 \end{bmatrix}$$

Solving these we obtain $x_1 = 1$, $x_2 = -2$, $x_3 = 3$.

Therefore the solution is $(x_1, x_2, x_3) = (1, -2, 3)$.

(b) If A is strictly diagonally dominant, then Jacobi method will converge.

Here $|4| > |-1| + |2|$ is true for the first row but $|1| > |-4| + |3|$ is not true for the second row so A is not strictly diagonally dominant, Jacobi method may or may not converge. □

2. (6 marks) Find Cholesky decomposition of the matrix $A = \begin{bmatrix} 1 & -2 & 0 \\ -2 & 20 & -8 \\ 0 & -8 & 7 \end{bmatrix}$

Solution. Let $A = LL^T$ where L is a lower triangular matrix. Then we have

$$\begin{bmatrix} 1 & -2 & 0 \\ -2 & 20 & -8 \\ 0 & -8 & 7 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$

$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Equating the corresponding entries of the matrices, we obtain following equations.

$$\begin{aligned}
&l_{11}^2 = 1 \Rightarrow l_{11} = 1 \\
&l_{11}l_{21} = -2 \Rightarrow l_{21} = -2 \\
&l_{11}l_{31} = 0 \Rightarrow l_{31} = 0 \\
&l_{21}^2 + l_{22}^2 = 20 \Rightarrow (-2)^2 + l_{22}^2 = 20 \Rightarrow l_{22}^2 = 16 \Rightarrow l_{22} = 4 \\
&l_{21}l_{31} + l_{22}l_{32} = -8 \Rightarrow 0 + 4(l_{32}) = -8 \Rightarrow l_{32} = -2 \\
&l_{31}^2 + l_{32}^2 + l_{33}^2 = 7 \Rightarrow 0 + (-2)^2 + l_{33}^2 = 7 \Rightarrow l_{33}^2 = 3 \Rightarrow l_{33} = \sqrt{3}
\end{aligned}$$

Thus we have

$$A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 4 & 0 \\ 0 & -2 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 4 & -2 \\ 0 & 0 & \sqrt{3} \end{bmatrix}$$

□

3. (7 marks) Consider the matrix $A = \begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 5 \\ 0 & 0 & -2 \end{bmatrix}$ and $X_0 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

(a) Find all the eigenvalues of A and find the eigenvector corresponding to the **dominant eigenvalue**.
(b) Perform one iteration of the power method starting with X_0.

Solution. (a) Note that A is an upper triangular matrix, so the eigenvalues are $1, -2$, the entries on the main diagonal.
The dominant eigenvalue is -2. So for $\lambda = -2$, we have

$$\left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 0 & 3 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] R_2 - 3R_1 \to R_2 \left[\begin{array}{ccc|c} 0 & 1 & 3 & 0 \\ 0 & 0 & -4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{rcrcl} x_2 & + & 3x_3 & = & 0 \\ & - & 4x_3 & = & 0 \end{array}$$

x_1 is a free variable. Let $x_1 = 1$. The second equation gives $x_3 = 0$ and the first equation gives $x_2 = -3x_3 = 0$. Thus the eigenvector is $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

(b) The initial approximation is $X_0 = [1\ 1\ -2]'$. Using the power method

$$Y_0 = AX_0 = \begin{bmatrix} -2 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ -9 \\ 4 \end{bmatrix}$$

The element of largest magnitude is -9 so $c_1 = -9$.

$$X_1 = \tfrac{1}{c_1} Y_0 = \begin{bmatrix} 7/9 \\ 1 \\ -4/9 \end{bmatrix} \text{ or } \begin{bmatrix} 0.778 \\ 1 \\ -0.444 \end{bmatrix}$$

□

4. (6 marks) Let $g(x) = \dfrac{x^2}{5} + \dfrac{3x}{5} - 3$.
(a) Solve the equation $x = g(x)$.
(b) Do you expect fixed point method to converge starting with an initial approximation $p_0 = 0$? Justify your answer using the conditions of convergence.

Solution. (a) $x = g(x) \Rightarrow x = \dfrac{x^2}{5} + \dfrac{3x}{5} - 3 \Rightarrow 5x = x^2 + 3x - 15$
$\Rightarrow x^2 - 2x - 15 = 0 \Rightarrow (x-5)(x+3) = 0 \Rightarrow x = 5, -3$.
Thus -3 and 5 are solutions.

(b) Here $g'(x) = \dfrac{2x}{5} + \dfrac{3}{5}$.

Since $|g'(5)| = \left|\dfrac{2(5)}{5} + \dfrac{3}{5}\right| = \dfrac{13}{5} > 1$ the fixed point method will not converge to 5.

Now $|g'(-3)| = \left|\dfrac{2(-3)}{5} + \dfrac{3}{5}\right| = \dfrac{3}{5} < 1$. We can choose interval $[-3.5, 1]$. Both g and g' are continuous on $[-3.5, 1]$, both p_0 and the solution -3 are in $[-3.5, 1]$. Note that also $g(x) \in [-3.5, 1]$ for all $x \in [-3.5, 1]$. Also $|g'(-3)| < 1$, so the fixed point method will converge to -3.

□

5. (7 marks) Consider the equation: $3\cos x - 1 = x$.

(a) Can we use bisection method to solve the above equation starting with the interval $[0, 1]$? Justify your answer using the conditions of convergence.

(b) Perform one iteration of Newton's method to solve the above equation starting with the initial approximation $p_0 = 1$.

Solution. (a) $3\cos x - 1 = x \;\Rightarrow\; 3\cos x - 1 - x = 0$.

Let $f(x) = 3\cos x - 1 - x$.

$f(0) = 3\cos 0 - 1 - 0 = 2 > 0$ and $f(1) = 3\cos 1 - 1 - 1 = -0.379 < 0$

$f(0)$ and $f(1)$ have opposite signs and the function is continuous on the interval $[0, 1]$, so we can use Bisection method starting with the interval $[0, 1]$.

(b) $3\cos x - 1 = x \;\Rightarrow\; 3\cos x - 1 - x = 0$.

Let $f(x) = 3\cos x - 1 - x$, then $f'(x) = -3\sin x - 1$

The Newton's iterations are $p_{k+1} = p_k - \dfrac{f(p_k)}{f'(p_k)} = p_k - \dfrac{3\cos p_k - 1 - p_k}{-3\sin p_k - 1}$

$p_0 == 1$, thus

$$p_1 = p_0 - \frac{3\cos p_0 - p_0 - 1}{-3\sin p_0 - 1} = 1 - \frac{3\cos 1 - 1 - 1}{-3\sin 1 - 1}$$

$$x_1 = 1 - \frac{3(0.5403) - 2}{-3(0.8415) - 1} = 1 - \frac{-0.3791}{-3.5245} = 1 - 0.1076 = 0.8924 \qquad \square$$

6. (4 marks) Consider the equation: $x^5 - 3x^4 = 0$

(a) Perform one iteration of Secant method starting with $p_0 = -1$ and $p_1 = 1$.

(b) What would be the convergence rate if secant method is used to find the root $x = 0$?

Solution. (a) Here $f(x) = x^5 - 3x^4$. $p_0 = -1$ and $p_1 = 1$.

$f(-1) = -1 - 3 = -4$ and $f(1) = 1 - 3 = -2$.

$$p_2 = p_1 - \frac{f(p_1)(p_1 - p_0)}{f(p_1) - f(p_0)} = 1 - \frac{f(1)(1 - (-1))}{f(1) - f(-1)} = 1 - \frac{(-2)(1+1)}{(-2) - (-4)} = 1 - \frac{-4}{2} = 1 + 2 = 3$$

(b) The convergence will be linear as $x = 0$ is a multiple root. $\square$

7. (8 marks) Consider the system of nonlinear equations

$$\begin{array}{rcrcl} 3x & + & y^2 & = & 6 \\ 3x & + & 4y & = & 9 \end{array}$$

(a) Perform one iteration of Gauss-Seidel method starting with $x_0 = 2$ and $y_0 = 1.2$.
(b) Do you expect Gauss-Seidel method to converge to the solution $(\frac{5}{3}, 1)$ starting with $x_0 = 2$ and $y_0 = 1.2$? Justify your answer using the conditions of convergence.

Solution. (a) Solving the first equation for x and the second equation for y, we have Gauss-Seidel iterations as

$$x_{k+1} = \frac{6 - y_k^2}{3} \quad \text{and} \quad y_{k+1} = \frac{9 - 3x_{k+1}}{4}$$

Using $x_0 = 2$ and $y_0 = 1.2$, we have

$$x_{k+1} = \frac{6 - y_k^2}{3} = \frac{6 - (1.2)^2}{3} = \frac{4.56}{3} = 1.52$$

$$y_{k+1} = \frac{9 - 3x_k}{4} = \frac{9 - 3(1.52)}{4} = \frac{4.44}{4} = 1.11$$

(b) Here

$$g_1(x, y) = \frac{6 - y^2}{3} \quad \text{and} \quad g_2(x, y) = \frac{9 - 3x}{4}$$

$$\frac{\partial g_1}{\partial x} = 0, \quad \frac{\partial g_1}{\partial y} = -\frac{2y}{3}, \quad \frac{\partial g_2}{\partial x} = -\frac{3}{4}, \quad \frac{\partial g_2}{\partial y} = 0$$

The functions g_1, g_2 and their first order partial derivatives are continuous on $\mathbb{R}^2$. So we can take region $R = \{(x, y) | 1 \leq x \leq 3, 0 \leq y \leq 2\}$. The intial guess $(2, 1.2)$ and the solution $(\frac{5}{3}, 1)$ are in R.

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |0| + \left|-\frac{2y}{3}\right| = \left|-\frac{2y}{3}\right|$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = \left|-\frac{3}{4}\right| + |0| = \frac{3}{4} < 1 \text{ for all } x, y$$

For the solution $(\frac{5}{3}, 1)$, the value of $\left|-\frac{2y}{3}\right| = \left|-\frac{2(1)}{3}\right| < 1$. The sufficient condition for the convergence is satisfied and the iterations will converge to the solution $(\frac{5}{3}, 1)$. □