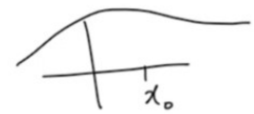
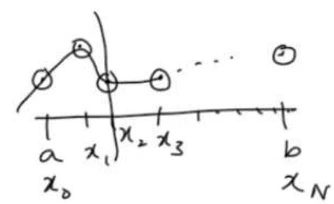
Section 4.2 Interpolation Polynomials

Taylor polynomials limit approximations to the situations in which approximations are needed only at points closer to x_0 (a single point). Another problem is that we need the value of f & its derivatives at x_0 and for some functions higher derivatives are difficult to compute. Taylor polynomials are mostly used to derive different numerical methods. They are not very useful for approximation purpose. For ordinary computational purposes it is more convenient to use methods that includes information at various points.



Suppose that the function $y = f(x)$ is known at $N + 1$ points $(x_0, y_0), (x_1, y_1), \ldots, (x_N, y_N)$ where the values x_k are spread out over an interval $[a, b]$.

$a = x_0 < x_1 < x_2 < \cdots < x_N = b$ & $y_k = f(x_k)$ for $k = 0,1, \ldots, N$



We want to find a polynomial $P_N(x)$ of degree (at most N) that will pass through $N + 1$ points. In the construction only the numerical values of (x_k, y_k) are needed no derivatives need to be evaluated).

$x_0, x_1, \ldots, x_N$ are called NODES & the values of interpolation polynomial $P_N(x)$ are same as $f(x)$ at all nodes, that means for each pair (x_k, y_k) we will have $P(x_k) = f(x_k)$

Situations in statistical and scientific analysis arise when we have data available at $N + 1$ points (x_k, y_k) and we need a method to find a polynomial that approximates the data.

If $x_0 < x < x_N$, the approximation $P_N(x)$ is called an INTERPOLATED VALUE. The polynomial $P_n(x)$ is called an interpolation polynomial of degree N.

First method of finding an interpolation polynomial $P_N(x)$ is

Let $P_N(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_N x^N$

we have to find $a_0, a_1, a_2, \ldots, a_N$ such that $P_N(x)$ satisfy $P_N(x_k) = y_k$ for $k = 0,1,2, \ldots, N$

$P_N(x_0) = y_0 \Rightarrow a_0 + a_1 x_0 + a_2 x_0^2 + \cdots + a_N x_0^N = y_0 \leftarrow (1)$

$P_N(x_1) = y_1 \Rightarrow a_0 + a_1 x_1 + a_2 x_1^2 + \cdots + a_N x_1^N = y_1 \leftarrow (2)$

$P_N(x_2) = y_2 \Rightarrow a_0 + a_1 x_2 + a_2 x_2^2 + \cdots + a_N x_2^N = y_2 \leftarrow (3)$

$\vdots$

$P_N(x_N) = y_N \Rightarrow a_0 + a_1 x_N + a_2 x_N^2 + \cdots + a_N x_N^N = y_N \leftarrow (\text{N}^{\text{th}})$

$$\underbrace{\begin{bmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ 1 & x_2 & x_2^2 & \cdots & x_2^N \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{bmatrix}}_{\text{A}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}}_{\text{x}} = \underbrace{\begin{bmatrix} y_0 \\ y_1 \\ y_2 \\ \vdots \\ y_N \end{bmatrix}}_{\text{B}}$$

We have to solve the system $Ax = B$.

Matlab is used to solve this system.

Example: Consider the data

X	**2**	**3**	**4**	**5**	**6**
Y	4	3	5	4	7

(a) Find an interpolating polynomial of degree 4 by solving the system $Ax = B$ (Use Matlab to solve the system).

Let $P_4(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4$

$P_4(2) = 4 \Rightarrow a_0 + a_1(2) + a_2(2)^2 + a_3(2)^3 + a_4(2)^4 = 4$

$P_4(3) = 3 \Rightarrow a_0 + a_1(3) + a_2(3)^2 + a_3(3)^3 + a_4(3)^4 = 3$

$P_4(4) = 5 \Rightarrow a_0 + a_1(4) + a_2(4)^2 + a_3(4)^3 + a_4(4)^4 = 5$

$P_4(5) = 4 \Rightarrow a_0 + a_1(5) + a_2(5)^2 + a_3(5)^3 + a_4(5)^4 = 4$

$P_4(6) = 7 \Rightarrow a_0 + a_1(6) + a_2(6)^2 + a_3(6)^3 + a_4(6)^4 = 7$

$$\underbrace{\begin{bmatrix} 1 & 2 & 2^2 & 2^3 & 2^4 \\ 1 & 3 & 3^2 & 3^3 & 3^4 \\ 1 & 4 & 4^2 & 4^3 & 4^4 \\ 1 & 5 & 5^2 & 5^3 & 5^4 \\ 1 & 6 & 6^2 & 6^3 & 6^4 \end{bmatrix}}_{\text{A}} \underbrace{\begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix}}_{\text{x}} = \underbrace{\begin{bmatrix} 4 \\ 3 \\ 5 \\ 5 \\ 7 \end{bmatrix}}_{\text{=B}}$$

In Matlab, we can write

(rows, columns)

initial increment final value

\>>x5=[2:1:6];
\>>y5=[4 3 5 4 7];
\>>A=[ones(1,5); x5;x5.^2;x5.^3;x5.^4]';
\>>B=y5';
\>>X=A\B

X= 104.0000 $\rightarrow a_0$
-117.9167 $\rightarrow a_1$
48.9583 $\rightarrow a_2$
-8.5833 $\rightarrow a_3$
0.5417 $\rightarrow a_4$

x=[1 2 3]
x.^2 or x.*x = [1.1 2.2 3.3]= [1 4 3]
x^2 [1 2 3]·[1 2 3]
not possible

The polynomial is $P_4(x) = 104 - 117.9167x + 48.9583x^2 - 8.5833x^3 + 0.5417x^4$

(b) Use Matlab built in functions to find an interpolation polynomial of degree 4 and to approximate the value at x=4.5

\>>x5=[2:1:6];
\>>y5=[4 3 5 4 7];
\>>p=polyfit(x5, y5, 4)

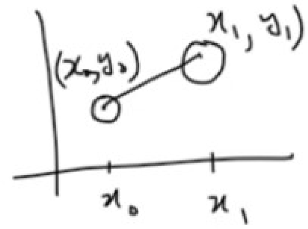
p=0.5417 -8.5833 48.9583 -117.9167 104.0000

a_4 a_3 a_2 a_1 a_0

\>>polyval(p, 4.5)

ans = 4.722

4.3 Lagrange Polynomials



$m = \frac{y_1 - y_0}{x_1 - x_0}$

The eq. of line is $y - y_0 = \overbrace{\left(\frac{y_1 - y_0}{x_1 - x_0}\right)}^{m}(x - x_0)$

$$\Rightarrow \underbrace{y}_{P_1(x)} = y_0 + \frac{y_1 - y_0}{x_1 - x_0}(x - x_0)$$

$$P_1(x) = y_0 \underbrace{\frac{x - x_1}{x_0 - x_1}}_{L_1, 0} + y_1 \underbrace{\frac{x - x_1}{x_1 - x_0}}_{L_1, 1}$$

Lagrange Coefficients

The generalization is the construction of a polynomial of degree at most N that passes through $N + 1$ points $(x_0, y_0)(x_1, y_1), \ldots, (x_N, y_N)$

$$P_N(x) = \sum_{k=0}^{N} y_k L_{N,k}$$

Lagrange coefficients & given by

$$L_{N,k} = \prod_{\substack{j=0 \\ j \neq k}}^{N} \frac{(x - x_j)}{(x_k - x_j)} \Rightarrow L_{N,k} = \frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_N)}{(x_k-x_0)(x_k-x_1)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_N)}$$

Example 1:

Consider the data

	x_0	x_1	x_2
x_k	**1**	**2**	**3**
y_k	3	3	3.5
	y_0	y_1	y_2

Find Lagrange polynomial $P_2(x)$ and use it to approximate the value at x=2.5

Solution:

$$P_2(x) = y_0 \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + y_1 \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + y_2 \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$P_2(x) = 3\frac{(x-2)(x-3)}{(1-2)(1-3)} + 3\frac{(x-1)(x-3)}{(2-1)(2-3)} + 3.5\frac{(x-1)(x-2)}{(3-1)(3-2)}$$

$$= 3\frac{x^2-5x+6}{(-1)(-2)} + 3\frac{x^2-4x+3}{(1)(-1)} + 3.5\frac{x^2-3x+2}{(2)(1)}$$

$$= \frac{3x^2-15x+18}{2} - (3x^2-12x+9) + \frac{3.5x^2-10.5x+7}{2}$$

$$= \left(\frac{3}{2} - 3 + \frac{3.5}{2}\right)x^2 + \left(\frac{-15}{2} + 12 - \frac{10.5}{2}\right)x + \left(\frac{18}{2} - 9 + \frac{7}{2}\right)$$

$$P_2(x) = \underbrace{0.25x^2}_{\frac{1}{4}x^2} - \underbrace{0.75x}_{\frac{-3}{4}x} + \underbrace{3.5}_{\frac{7}{2}}$$

$$P_2(2.5) = 0.25(2.5)^2 - 0.75(2.5) + 3.5$$
$$= 3.1875$$

Check: $P_2(1) = \frac{1}{4}(1) - \frac{3}{4}(1) + \frac{7}{2} = 3$ $\checkmark$

$P_2(2) = \frac{1}{4}(4) - \frac{3}{4}(2) + \frac{7}{2} = 3$ $\checkmark$ $\qquad$ $P_N(x_k) = y_k$ for all k

$P_2(3) = \frac{1}{4}(9) - \frac{3}{4}(3) + \frac{7}{2} = 3.5$ $\checkmark$