Gaussian Elimination

Gaussian Elimination: A system of N linear equations in N unknowns.

$$Ax = B$$

$$\begin{bmatrix} a_{1\,1} & a_{1\,2} & \cdots & a_{1\,N} \\ a_{2\,1} & a_{2\,2} & \cdots & a_{2\,N} \\ \cdots & \cdots & \cdots & \cdots \\ a_{N\,1} & a_{N\,2} & a_{N\,3} & a_{N\,N} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_N \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_N \end{bmatrix}$$

N x N

Unknowns

We will use Gaussian elimination method to change it to an upper triangular system and we can then use back substitution to find the solution.

$$\begin{bmatrix} a_{1\,1} & a_{1\,2} & \cdots & a_{1\,N} \\ 0 & a_{2\,2} & \cdots & a_{2\,N} \\ \vdots & \cdots & \cdots & \vdots \\ \vdots & \cdots & \cdots & \vdots \\ 0 & 0 & 0 & a_{N\,N} \end{bmatrix}$$

In **<u>Gaussian elimination method</u>** we can **<u>use three elementary operations to reduce</u>** the augmented matrix $[A|B]$ <u>to an upper triangular matrix.</u>

(1) We can interchange any two rows of the matrix.
(2) we can multiply any row by a nonzero number.
(3) We can multiply a row by a nonzero number and add to another row.

multipliers are $m_{i\,j} = \frac{a_{i\,j}}{a_{j\,j}}$

Example 1:

Solve the system using Gaussian elimination with no pivoting.

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 4x_3 &= 4 \\ x_1 - 3x_2 - 2x_3 &= 5 \end{aligned}$$

Solution:

$a_{1\,1}$ *is pivot*

$$\left[\begin{array}{ccc|c} \boxed{1} & 4 & 3 & 1 \\ 2 & 5 & 4 & 4 \\ 1 & -3 & -2 & 5 \end{array}\right] \longrightarrow \text{pivot row}$$

$$\begin{array}{cccc|c} R_2 & 2 & 5 & 4 & 4 \\ 2R_1 & 2 & 8 & 6 & 2 \\ & - & - & - & - \\ \hline & 0 & -3 & -2 & 2 \end{array}$$

The Multipliers are $m_{2\,1} = \frac{a_{2\,1}}{a_{1\,1}} = \frac{2}{1} = 2$ and $m_{3\,1} = \frac{a_{3\,1}}{a_{1\,1}} = \frac{1}{1} = 1$

$R_2 - m_{2\,1}R_1 \to R_2 \Rightarrow R_2 - 2R_1 \to R_2$

$R_3 - m_{3\,1}R_1 \to R_3 \Rightarrow R_3 - R_1 \to R_3$

$a_{2\,2}$ ***is pivot***

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & \boxed{-3} & -2 & 2 \\ 0 & -7 & -5 & 4 \end{array}\right] \longrightarrow \textit{pivot row is } R_2$$

$$\begin{array}{cccc|c} R_3 \to & 0 & -7 & -5 & 4 \\ \frac{7}{3}R_2 \to & 0 & -7 & -\frac{14}{3} & \frac{14}{3} \\ & - & + & + & - \\ \hline & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{array}$$

$$R_3 - \frac{7}{3}R_2 \to R_3 \quad \left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 0 & -3 & -2 & 2 \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{array}\right]$$

This is upper triangular system

$$x_1 + 4x_2 + 3x_3 = 1 - (1)$$

$$-3x_2 - 2x_3 = 2 - (2)$$

$$-\frac{1}{3}x_3 = -\frac{2}{3} - (3) \Rightarrow x_3 = \frac{\frac{-2}{3}}{-\frac{1}{3}} = 2$$

Substitute $x_3 = 2$ in the eq (2)

$$-3x_2 - 2(2) = 2 \Rightarrow -3x_2 = 6 \Rightarrow x_2 = \frac{6}{-3} = -2$$

Substitute $x_2 = -2$ & $x_3 = 2$ into eq (1)

$$x_1 + 4(-2) + 3(2) = 1 \Rightarrow x_1 = 1 + 8 - 6 = 3$$

The solution is $(x_1, x_2, x_3) = (3, -2, 2)$ ans

Note: If $a_{k\,k} = 0$ then kth row will be interchanged with a row below the diagonal to obtain a nonzero pivot element. If this can not be done, then A is a singular matrix and in this case the system does not have a unique solution.

The Gaussian elimination is used to solve a system whose coefficient matrix A is non-singular (i.e., A^{-1} exists).

Partial Pivoting: Since compute uses fixed precision (rounding)
If operations are done in computer, then a small error is introduced at every arithmetic operation.

$\frac{1}{3} \approx 0.333333\ldots$
Computer's do rounding.

Example from textbook on page 317. (pg. 132, Ex. 3.18)
Consider the system

$$\begin{aligned} 1.133x_1 + 5.281x_2 &= 6.414 \\ 24.14x_1 - 1.210x_2 &= 22.93 \end{aligned}$$

Note that $x_1 = 1, x_2 = 1$ are solutions.

Solving it using Gaussian elimination & 4 digits calculations.

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & -1.210 & 22.93 \end{array}\right] R_2 - 21.31R_1 \to R_2 \left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 0 & -113.7 & 113.8 \end{array}\right]$$

$$\boxed{m_{2\,1} = \frac{24.14}{1.133} = 21.31}$$

$1.133x_1 + 5.281x_2 = 6.414$ – (1)

$-113.7x_2 = -113.8 \Rightarrow x_2 = -\frac{113.8}{-113.7} = 0.9956$ Substitute in eq (1) $\Rightarrow x_1 = 1.001$

We have a small error in the solution x_1& x_2.
Note: The error is due to the value of $m_{2\,1}$ being large, we can reduce the error by having a small multiplier. For this we will interchange the rows.

Example from textbook on page 318.

$$\left[\begin{array}{cc|c} 1.133 & 5.281 & 6.414 \\ 24.14 & -1.210 & 22.93 \end{array}\right] R_2 \leftrightarrow R_1 \left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 1.133 & 5.281 & 6.414 \end{array}\right]$$

how $M_{2\,1} = \frac{1.133}{24.14} = 0.04693$ $R_2 - 0.04693R1 \to R_2 \left[\begin{array}{cc|c} 24.14 & -1.210 & 22.93 \\ 0 & 5.338 & 5.338 \end{array}\right]$

$$\left.\begin{array}{l} 24.14x_1 - 1.210x_2 = 22.93 - (*) \\ 5.338x_2 = 5.338 \Rightarrow x_2 = 1 \\ \text{Substitute in } (*) \text{ and get } x_1 = 1 \end{array}\right\}$$ no error. *Note:* that here $|m_{2\,1}| < 1$

<u>To reduce the error</u>, we check the magnitude of all elements in the column that lie on or below the main diagonal and then the element with largest magnitude needs to be used as a pivot, so we do now interchange to have pivot in the correct place.

In this way since the magnitude of the pivot is larger or equal to the elements in the column below the pivot row, the multipliers will have magnitude less then or equal to 1. ($|m_{i\,j}| \leq 1$) this strategy is called PARTIAL PIVOTING.

$$\begin{bmatrix} 1 & \dots & \dots \\ 2 & \dots & \dots \\ -3 & \dots & \dots \\ 0 & \dots & \dots \end{bmatrix} R_3 \leftrightarrow R_1$$

$|-3| > 1$

$|-3| > 2$

-3 should be pivot

Example 2:

Solve the system using Gaussian elimination with partial pivoting.

$$\begin{aligned} x_1 + 4x_2 + 3x_3 &= 1 \\ 2x_1 + 5x_2 + 4x_3 &= 4 \\ x_1 - 3x_2 - 2x_3 &= 5 \end{aligned}$$

Solution:

$$\left[\begin{array}{ccc|c} 1 & 4 & 3 & 1 \\ 2 & 5 & 4 & 4 \\ 1 & -3 & -2 & 5 \end{array}\right] \qquad |2| > |1|\ R_2 \leftrightarrow R_1 \left[\begin{array}{ccc|c} 2 & 5 & 4 & 4 \\ 1 & 4 & 3 & 1 \\ 1 & -3 & -2 & 5 \end{array}\right]$$

$$\begin{array}{ccccc} R_2 & 1 & 4 & 3 & 1 \\ \frac{1}{2}R_1 & 1 & \frac{5}{2} & 2 & 2 \\ \hline & 0 & \frac{3}{2} & -1 & -1 \end{array}$$

$$\left.\begin{aligned} m_{2\,1} &= \tfrac{1}{2} \\ m_{3\,1} &= \tfrac{1}{2} \end{aligned}\right\} \quad \begin{aligned} |m_{2\,1}| &\leq 1 \\ |m_{3\,1}| &\leq 1 \end{aligned} \quad \begin{aligned} R_2 - \tfrac{1}{2}R_1 &\to R_2 \\ R_3 - \tfrac{1}{2}R_1 &\to R_3 \end{aligned} \quad \left[\begin{array}{ccc|c} 2 & 5 & 4 & 4 \\ 0 & \frac{3}{2} & 1 & -1 \\ 0 & -\frac{11}{2} & -4 & 3 \end{array}\right]$$

$$\left|-\tfrac{11}{2}\right| > \left|\tfrac{3}{2}\right| \quad R_3 \leftrightarrow R_2 \left[\begin{array}{ccc|c} 2 & 5 & 4 & 4 \\ 0 & -\frac{11}{2} & -4 & 3 \\ 0 & \frac{3}{2} & 1 & -1 \end{array}\right]$$

$$\begin{array}{ccccc} R_3 & 0 & \frac{3}{2} & 1 & -1 \\ -\frac{3}{11}R_2 & 0 & \frac{3}{2} & \frac{12}{11} & -\frac{9}{11} \\ \hline & 0 & 0 & -\frac{1}{11} & -\frac{2}{11} \end{array}$$

$m_{3\,2} = \dfrac{\frac{3}{2}}{-\frac{11}{2}} = -\dfrac{3}{11}$

$|m_{3\,2}| \leq 1$

$R_3 - \dfrac{3}{11}R_2 \to R_3$

or $R_3 + \frac{3}{11}R_2 \to R_3$

$$\left[\begin{array}{ccc|c} 2 & 5 & 4 & 4 \\ 0 & -\frac{11}{2} & -4 & 3 \\ 0 & 0 & -\frac{1}{11} & -\frac{2}{11} \end{array}\right]$$

$2x_1 + 5x_2 + 4x_3 = 4 \to 2x_1 + 5(-2) + 4(2) = 4 \Rightarrow 2x_1 = 4 + 10 - 8 = 6 \Rightarrow x_1 = 3$

$-\dfrac{11}{2}x_2 - 4x_3 = -3 \to -\dfrac{11}{2}x_2 - 4(2) = 3 \Rightarrow -\dfrac{11}{2}x_2 = 11 \Rightarrow x_2 = -2$

$-\dfrac{1}{11}x_3 = -\dfrac{2}{11} \Rightarrow x_3 = 2$

The solution is $(x_1, x_2, x_3) = (3, -2, 2)$ ans

$Ax = B$, linear algebra that solution is $x = A^{-1}B$ if A^{-1}exists.

In MATLAB:

```
>> A = [   ] ;
>> B = [   ]';
> > X = inv(A)*B
```

solution

MATLAB also has a built-in command to solve system of linear equations Ax=B

```
>> A = [ - - - ];
>> B = [     ]';
>> X = A \ B
```

$\longrightarrow$ [ ]

$\longrightarrow$ will give solution of the system

$B = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

```
>>B=[ 1 2 3 ]';
or
>>B=[1;2;3]
```

$>> 2/3 \Rightarrow \frac{2}{3}$