MATH 3940 Numerical Analysis for Computer Scientists
Midterm Solutions Fall 2021

1. (9 marks) Consider the following system

$$\begin{array}{rcrcrcr} x_1 & + & 2x_2 & - & x_3 & = & 6 \\ 4x_1 & + & 4x_2 & + & x_3 & = & 9 \\ & - & 4x_2 & - & 7x_3 & = & 21 \end{array}$$

(a) (6.5 marks) Solve the above system using Gaussian elimination method with partial pivoting.

(b) (2.5 marks) Do you expect that the iterations of Jacobi method for the above system will converge? Justify your answer using the condition of convergence.

Solution. (a) We will perform Gaussian elimination on the augmented matrix.

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 6 \\ 4 & 4 & 1 & 9 \\ 0 & -4 & -7 & 21 \end{array}\right] \quad R_2 \leftrightarrow R_1 \quad \left[\begin{array}{rrr|r} 4 & 4 & 1 & 9 \\ 1 & 2 & -1 & 6 \\ 0 & -4 & -7 & 21 \end{array}\right]$$

$$R_2 - \frac{1}{4}R_1 \to R_2 \quad \left[\begin{array}{rrr|r} 4 & 4 & 1 & 9 \\ 0 & 1 & -\frac{5}{4} & \frac{15}{4} \\ 0 & -4 & -7 & 21 \end{array}\right]$$

$$R_2 \leftrightarrow R_3 \quad \left[\begin{array}{rrr|r} 4 & 4 & 1 & 9 \\ 0 & -4 & -7 & 21 \\ 0 & 1 & -\frac{5}{4} & \frac{15}{4} \end{array}\right] \quad R_3 + \frac{1}{4}R_2 \to R_3 \quad \left[\begin{array}{rrr|r} 4 & 4 & 1 & 9 \\ 0 & -4 & -7 & 21 \\ 0 & 0 & -3 & 9 \end{array}\right]$$

Now we will use back substitution to find the solution to the following system.

$$\begin{array}{rcrcrcr} 4x_1 & + & 4x_2 & + & x_3 & = & 9 \\ & - & 4x_2 & - & 7x_3 & = & 21 \\ & & & - & 3x_3 & = & 9 \end{array}$$

The third equation gives $-3x_3 = 9 \ \Rightarrow \ x_3 = -3$. Putting $x_3 = -3$ into the second equation we obtain $-4x_2 - 7(-3) = 21 \Rightarrow -4x_2 = 0 \Rightarrow x_2 = 0$.
Finally, substituting the value of x_2 and x_3 into the first equation we obtain
$4x_1 + 4(0) + (-3) = 9 \ \Rightarrow \ 4x_1 = 12 \ \Rightarrow \ x_1 = 3$.
Thus the solution is $(x_1, x_2, x_3) = (3, 0, -3)$.

(b) If A is strictly diagonally dominant, then Jacobi method will converge.

Here $|1| > |2| + |-1|$ is not true for the first row so A is not strictly diagonally dominant, Jacobi method may or may not converge. $\square$

2. (6 marks) Find Cholesky decomposition of the matrix $A = \begin{bmatrix} 4 & 0 & 6 \\ 0 & 9 & -3 \\ 6 & -3 & 27 \end{bmatrix}$

Solution. Let $A = LL^T$ where L is a lower triangular matrix. Then we have

$$\begin{bmatrix} 4 & 0 & 6 \\ 0 & 9 & -3 \\ 6 & -3 & 27 \end{bmatrix} = \begin{bmatrix} l_{11} & 0 & 0 \\ l_{21} & l_{22} & 0 \\ l_{31} & l_{32} & l_{33} \end{bmatrix} \begin{bmatrix} l_{11} & l_{21} & l_{31} \\ 0 & l_{22} & l_{32} \\ 0 & 0 & l_{33} \end{bmatrix}$$
$$= \begin{bmatrix} l_{11}^2 & l_{11}l_{21} & l_{11}l_{31} \\ l_{11}l_{21} & l_{21}^2 + l_{22}^2 & l_{21}l_{31} + l_{22}l_{32} \\ l_{11}l_{31} & l_{21}l_{31} + l_{22}l_{32} & l_{31}^2 + l_{32}^2 + l_{33}^2 \end{bmatrix}$$

Equating the corresponding entries of the matrices, we obtain following equations.

$$\begin{aligned}
&l_{11}^2 = 4 \;\Rightarrow\; l_{11} = 2 \\
&l_{11}l_{21} = 0 \;\Rightarrow\; l_{21} = 0 \\
&l_{11}l_{31} = 6 \;\Rightarrow\; (2)l_{31} = 6 \;\Rightarrow l_{31} = 3 \\
&l_{21}^2 + l_{22}^2 = 9 \;\Rightarrow\; (0)^2 + l_{22}^2 = 9 \;\Rightarrow\; l_{22}^2 = 9 \;\Rightarrow\; l_{22} = 3 \\
&l_{21}l_{31} + l_{22}l_{32} = -3 \;\Rightarrow\; 0 + 3(l_{32}) = -3 \;\Rightarrow\; l_{32} = -1 \\
&l_{31}^2 + l_{32}^2 + l_{33}^2 = 27 \;\Rightarrow\; (3)^3 + (-1)^2 + l_{33}^2 = 27 \Rightarrow\; l_{33}^2 = 17 \Rightarrow\; l_{33} = \sqrt{17}
\end{aligned}$$

Thus we have

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 3 & -1 & \sqrt{17} \end{bmatrix} \begin{bmatrix} 2 & 0 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & \sqrt{17} \end{bmatrix}$$

□

3. (8 marks) Consider the matrix $A = \begin{bmatrix} -5 & 1 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix}$ and $X_0 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

(a) (5 marks) Find all the eigenvalues and eigenvectors of matrix A.
(b) (3 marks) Perform one iteration of the power method starting with X_0.

Solution. (a) Note that A is an upper triangular matrix, so the eigenvalues are -5 and 2, the entries on the main diagonal.

Alternatively, To find the eigenvalues, we have to solve $|A - \lambda I| = 0$.

$$\begin{vmatrix} -5-\lambda & 1 & 4 \\ 0 & 2-\lambda & -1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$$

Expanding along the first column, we obtain $(-5-\lambda)(2-\lambda)(2-\lambda) = 0$. Thus the eigenvalues are -5 and 2.

Now we will find the eigenvectors.

For $\lambda_1 = -5$, we have

$$\left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 7 & -1 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] R_2{-}7R_1 \to R_2 \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 0 & -29 & 0 \\ 0 & 0 & 7 & 0 \end{array}\right] R_3{+}(7/29)R_2 \to R_3 \left[\begin{array}{ccc|c} 0 & 1 & 4 & 0 \\ 0 & 0 & -29 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

So we have

$$\begin{array}{rcrcl} x_2 & + & 4x_3 & = & 0 \\ & - & 29x_3 & = & 0 \end{array}$$

and x_1 is a free variable. The second equation gives $x_3 = 0$, substituting $x_3 = 0$ into the first equation, we get $x_2 = 0$. Taking $x_1 = 1$, the eigenvector is $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$.

For $\lambda_2 = 2$, we have

$$\left[\begin{array}{rrr|r} -7 & 1 & 4 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{array}{rcrcrcl} -7x_1 & + & x_2 & + & 4x_3 & = & 0 \\ & & & - & x_3 & = & 0 \end{array}$$

and x_2 is a free variable. The first equation gives $x_1 = \frac{1}{7}x_2$. If we take $x_2 = 1$, then $x_1 = 1/7$, and the eigenvector is $\mathbf{v} = \begin{bmatrix} 1/7 \\ 1 \\ 0 \end{bmatrix}$

(b) The initial approximation is $X_0 = [2\ 1\ 1]'$. Using the power method

$$Y_0 = AX_0 = \begin{bmatrix} -5 & 1 & 4 \\ 0 & 2 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -10+1+4 \\ 0+2-1 \\ 0+0+2 \end{bmatrix} = \begin{bmatrix} -5 \\ 1 \\ 2 \end{bmatrix}$$

The element of largest magnitude is -5 so $c_1 = -5$ and

$$X_1 = \frac{1}{c_1} Y_0 = \begin{bmatrix} 1 \\ -1/5 \\ -2/5 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ -0.2 \\ -0.4 \end{bmatrix}$$

□

4. (6 marks) Let $g(x) = 2 + \dfrac{8}{x}$

(a) (2.5 marks) Solve the equation $x = g(x)$.
(b) (3.5 marks) Do you expect fixed point method to converge starting with an initial approximation $p_0 = 3.2$? Justify your answer using the conditions of convergence.

Solution. (a) $x = g(x) \;\Rightarrow\; x = 2 + \dfrac{8}{x} \;\Rightarrow\; x^2 = 2x + 8 \Rightarrow\; x^2 - 2x - 8 = 0$
$\Rightarrow\; (x-4)(x+2) = 0 \;\Rightarrow\; x = 4, -2.$
Thus -2 and 4 are solutions.

(b) Here $g'(x) = -\dfrac{8}{x^2}$

Since $|g'(-2)| = \left|-\dfrac{8}{(-2)^2}\right| = 2 > 1$ the fixed point method will not converge to -2.

Now $|g'(4)| = \left|-\dfrac{8}{(4)^2}\right| = \dfrac{1}{2} < 1$

We can choose interval $[3, 5]$. Now g and g' are continuous on $[3, 5]$, both p_0 and the solution 4 are in $[3, 5]$. Also $|g'(4)| < 1$, so the fixed point method will converge to 4.

□

5. (5 marks) Consider the equation: $x - \sin x = 0$.

(a) (3 marks) Given that $x = 0$ is a solution of the equation, find the order of the root $x = 0$.
(b) (2 marks) What is the convergence rate if secant method is used to find the root $x = 0$?

Solution. (a) $f(x) = x - \sin x$ and $f(0) = 0$

$f'(x) = 1 - \cos x$ and $f'(0) = 0$

$f''(x) = \sin x$ and $f''(0) = 0$

$f'''(x) = \cos x$ and $f'''(0) = 1 \neq 0$

So the order of the root $x = 0$ is 3.

(b) The convergence will be linear because $x = 0$ is a multiple root. □

6. (a) (2 marks) Can we use regula falsi method to solve the equation $x^3 - \dfrac{1}{x+1} = 0$ starting with the interval $[-2, 2]$? Justify your answer using the conditions of convergence.

(b) (5 marks) Consider the equation: $x^2 + \cos x = 2$
Perform one iteration of Newton's method to solve the above equation starting with the initial approximation $p_0 = -1$.

Solution. (a) Here $f(x) = x^3 - \dfrac{1}{x+1}$ which is not continuous at $x = -1$.

We cannot use regula falsi method starting with the interval$[-2, 2]$ because the function is not continuous on the interval $[-2, 2]$. ,

(b) $x^2 + \cos x = 2 \;\Rightarrow\; x^2 + \cos x - 2 = 0$.

Let $f(x) = x^2 + \cos x - 2$, then $f'(x) = 2x - \sin x$

The Newton's iterations are $p_k = p_{k-1} - \dfrac{f(p_{k-1})}{f'(p_{k-1})}$

$p_0 = -1$, thus

$$p_1 = p_0 - \frac{(p_0)^2 + \cos p_0 - 2}{2p_0 - \sin p_0} = -1 - \frac{(-1)^2 + \cos(-1) - 2}{2(-1) - \sin(-1)}$$

Using calculator in radian mode we obtain

$$p_1 = -1 - \frac{1 + 0.540 - 2}{-2 - (-0.841)} = -1 - 0.397 = -1.397$$ □

7. (9 marks) Consider the system of nonlinear equations

$$\begin{array}{rcrcl} 2x & + & y^3 & = & 3 \\ -2x & + & 9.84y & = & 1 \end{array}$$

(a) (2.5 marks) Perform one iteration of Gauss-Seidel method starting with $x_0 = 1$ and $y_0 = 0$.
(b) (6.5 marks) Do you expect Gauss-Seidel method to converge to the solution $(1.468, 0.4)$ starting with $x_0 = 1$ and $y_0 = 0$? Justify your answer using the conditions of convergence.

Solution. (a) Solving the first equation for x and the second equation for y, we have Gauss-Seidel iterations as

$$x_{k+1} = \frac{3 - y_k^3}{2} \quad \text{and} \quad y_{k+1} = \frac{1 + 2x_{k+1}}{9.84}$$

Using $x_0 = 1$ and $y_0 = 0$, we have

$$x_{k+1} = \frac{3-0}{2} = \frac{3}{2} \text{ or } 1.5$$

$$y_{k+1} = \frac{1 + 2(3/2)}{9.84} = \frac{4}{9.84} = 0.407 \text{ or } \frac{50}{123}$$

(b) Here

$$g_1(x, y) = \frac{3 - y^3}{2} \quad \text{and} \quad g_2(x, y) = \frac{1 + 2x}{9.84}$$

$$\frac{\partial g_1}{\partial x} = 0, \quad \frac{\partial g_1}{\partial y} = -\frac{3y^2}{2}, \quad \frac{\partial g_2}{\partial x} = \frac{2}{9.84} = 0.203, \quad \frac{\partial g_2}{\partial y} = 0$$

The functions g_1, g_2 and their first order partial derivatives are continuous on $\mathbb{R}^2$. So we can take region $R = \{(x, y) | 0 \le x \le 2, -1 \le y \le 1\}$. The intial guess $(1, 0)$ and the solution $(1.468, 0.4)$ are in R.

$$\left|\frac{\partial g_1}{\partial x}\right| + \left|\frac{\partial g_1}{\partial y}\right| = |0| + \left|-\frac{3y^2}{2}\right| = \frac{3y^2}{2}$$

$$\left|\frac{\partial g_2}{\partial x}\right| + \left|\frac{\partial g_2}{\partial y}\right| = |0.203| + |0| = 0.203 < 1 \text{ for all } x, y$$

For the solution $(1.468, 0.4)$, the value of $\frac{3y^2}{2} = \frac{3(0.4)^2}{2} = 0.24 < 1$. The sufficient condition for the convergence is satisfied and the iterations will converge to the solution $(1.468, 0.4)$. $\square$