Higher Order Derivatives

To find formulas of $f'', f''', etc.$

$$f(x+h) = f(x) + hf'(x) + \frac{h^2}{2!}f''(x) + \frac{h^3}{3!}f'''(c_1) + \frac{h^4}{4!}f^{(4)}(c_1)$$

$$+ \quad f(x-h) = f(x) - hf'(x) + \frac{h^2}{2!}f''(x) - \frac{h^3}{3!}f'''(c_2) + \frac{h^4}{4!}f^{(4)}(c_2)$$

$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + \frac{h^4}{4!}\left(f^{(4)}(c_1) + f^{(4)}(c_2)\right)$$

$$\frac{h^2 f''(x)}{h^2} = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} - \frac{O(h^4)}{h^2}$$

$$f''(x) = \frac{f(x+h) + f(x-h) - 2f(x)}{h^2} - O(h^2) \rightarrow Central\ Difference$$

Example 3:
Let $f(x) = x^4$ and $h = 0.1$. Find $f''(1)$ and find the exact error and relative error.

Solution:

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$f''(1) = \frac{f(1+0.1) - 2f(1) + f(1-0.1)}{(0.1)^2} = \frac{f(1.1) - 2f(1) + f(0.9)}{0.01}$$

$$= \frac{(1.1)^4 - 2(1)^4 + (0.9)^4}{0.01} = \frac{1.4641 - 2 + 0.6561}{0.01} = 12.02$$

exact value: $f(x) = x^4, f'(x) = 4x^3, f''(x) = 12x^2$

$f''(1) = 12(1)^2 = 12$

exact error is $|12.02 - 12| = 0.02$

relative error is $\frac{0.02}{12} = 0.001667$

NOTE: We have other formulas for $f''(x)$

$$f''(x) = \frac{-f(x+2h) + 16f(x+h) - 30f(x) + 16f(x-h) - f(x-2h)}{12h^2} + o(h^4)$$

$$f''(x) = \frac{-f(1+0.2) + 16f(1+0.1) - 30f(1) + 16f(1-0.1) - f(1-0.2)}{12h^2} + o(h^4)$$

$$= \frac{-(1.2)^4 + 16(1.1)^4 - 30(1)^4 + 16(0.9)^4 - (0.8)^4}{12(0.01)}$$

$$= \frac{1.44}{0.12}$$

$$= 12$$

error is zero.

$1.5 - 1.45 = 0.05 \to h$
$1.55 - 1.5 = 0.05 \to h$
$1.6 - 1.55 = 0.05 \to h$

Example 4: Consider the data

x_k	1.45	1.5	1.55	1.6
$f(x_k)$	4.263	4.481	4.711	4.953

Find the approximation to $f'(1.45), f'(1.5), f'(1.6)$ and $f''(1.5)$ Using difference formulas of order $O(h^2)$.

Solution:

$f'(1.45) \to$ forward difference formula $\quad f'(x) = \frac{-3f(x)+4f(x+h)-f(x+2h)}{2h}$

$$f'(1.45) = \frac{-3f(1.45) + 4f(\overbrace{1.45 + 0.05}^{1.5}) - f(\overbrace{1.45 + 2(0.05)}^{1.55})}{2h}$$

$$= \frac{-3f(1.45) + 4f(1.5) - f(1.55)}{2(0.05)}$$

$$= \frac{-3(4.263) + 4(4.481) - (4.711)}{2(0.05)}$$

$$= \frac{0.424}{0.1}$$

$$= 4.24$$

$f'(1.5) \to central\ difference$ OR

$$f'(1.5) = \frac{f(x+h) - f(x-h)}{2h}$$

$$f'(1.5) = \frac{f(1.5 + 0.05) - f(1.5 - 0.05)}{2(0.05)}$$

$$= \frac{f(1.55) - f(1.45)}{0.1}$$

$$= \frac{4.711 - 4.263}{0.1}$$

$$= 4.48$$

$f'(1.5) \to central\ difference$

$$f'(1.5) = \frac{-3f(1.5) + 4f(1.5 + 0.05)}{2(0.05)}$$

$$= \frac{-3f(1.5) + 4f(1.55) - f(1.6)}{2(0.05)}$$

$$= \frac{-3(4.481) + 4(4.711) - 4.953}{0.1}$$

$$= 4.48$$

$f'(1.6) \rightarrow backward\ difference$

$$f'(x) = \frac{3f(x) - 4f(x-h) + f(x-2h)}{2h}$$

$$f'(1.6) = \frac{3f(1.6) - 4f(1.6 - 0.05) + f(1.6 - 2(0.05))}{2(0.05)}$$

$$= \frac{3f(1.6) - 4f(1.55) + f(1.5)}{0.1}$$

$$= \frac{3(4.953) - 4(4.711) + 4.481}{0.1}$$

$$= \frac{0.496}{0.1}$$

$$= 4.96$$

$f''(x) \rightarrow central\ difference$

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

$$f''(1.5) = \frac{f(1.5 + 0.05) - 2f(1.5) + f(1.5 - 0.05)}{(0.05)^2}$$

$$= \frac{f(1.55) - 2f(1.5) + f(1.5 - 0.05)}{0.0025}$$

$$= \frac{4.711 - 2(4.481) + 4.263}{0.0025}$$

$$= \frac{0.012}{0.0025}$$

$$= 4.8$$

If we are having data from the experiment where the values are not recorded to many decimal digits, then the difference formulas will have more roundoff errors. So, if the data is available only to a few digits then using least squares techniques a curve is found for the data. And we differentiate the function in the curve to find the derivates.