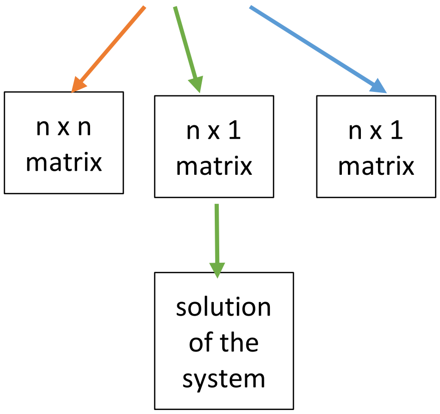
Chapter 3 Solution of Linear Systems

In linear algebra, system of linear equations can be written as

$$Ax = B$$



$$\text{N x N matrix} \Rightarrow A = a_{ij} \Rightarrow \begin{bmatrix} a_{1\,1} & a_{1\,2} & a_{1\,3} & a_{1\,N} \\ a_{2\,1} & a_{2\,2} & a_{2\,3} & a_{2\,N} \\ \dots & \dots & \dots & \dots \\ a_{N\,1} & a_{N\,2} & a_{N\,3} & a_{N\,N} \end{bmatrix}$$

3.3 Upper Triangular Systems

An N x N matrix $A = [a_{ij}]$ is called upper triangular matrix provided that $a_{i\,j} = 0$ whenever

$$i > j \Rightarrow A = \begin{bmatrix} a_{1\,1} & a_{1\,2} & a_{1\,3} & a_{1\,N} \\ 0 & a_{2\,2} & a_{2\,3} & a_{2\,N} \\ 0 & 0 & a_{3\,3} & a_{3\,N} \\ 0 & 0 & 0 & a_{N\,N} \end{bmatrix}$$

An N x N matrix $A = [a_{ij}]$ is called lower triangular matrix provided that $a_{i\,j} = 0$ whenever

$$i < j \Rightarrow A = \begin{bmatrix} a_{1\,1} & 0 & 0 & 0 \\ a_{2\,1} & a_{2\,2} & 0 & 0 \\ a_{3\,1} & a_{3\,2} & a_{3\,3} & 0 \\ a_{N\,1} & a_{N\,2} & a_{N\,3} & a_{N\,N} \end{bmatrix}$$

The diagonal elements are $a_{1\,1}, a_{2\,2}, \dots, a_{N\,N}$ $\quad (a_{k\,k}\,, k = 1, \dots, N)$

If A is an upper triangular matrix, then the system $Ax = B$ is said to be an upper triangular system of linear equations.

$$\left.\begin{aligned} a_{1\,1}\,x_1 + a_{1\,2}\,x_2 + a_{1\,3}\,x_3 + \cdots + a_{1\,N-1}\,x_{N-1} + a_{1\,N}\,x_N &= b_1 \\ a_{2\,2}\,x_2 + a_{2\,3}\,x_3 + \cdots + a_{2\,N-1}\,x_{N-1} + a_{2\,N}\,x_N &= b_2 \\ \cdots \\ \cdots \\ a_{N-1\,N-1}\,x_{N-1} + a_{N-1\,N}\,x_N &= b_{N-1} \\ a_{N\,N}\,x_N &= b_N \end{aligned}\right\} \boxed{\text{upper triangular system}}$$

$$\boxed{x_N = \frac{b_N}{a_{N\,N}} \text{ , sub. } X_n \text{ in } eq\ (N-1) \text{ we will get } X_{N-1}}$$

We can **<u>solve</u>** <u>an</u> **<u>upper triangular system</u>** by <u>using back substitution</u>.

Solve the following system of equations.

Example 1:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 1 \\ x_2 - 0x_3 &= 0 \\ x_3 &= 5 \end{aligned} \rightarrow \underset{\boldsymbol{A}}{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \underset{\boldsymbol{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\boldsymbol{B}}{\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}}$$

Solution: upper triangular system.

Last eq $\Rightarrow x_3 = 5$

2^{nd} eq $\Rightarrow x_2 - 0(5) = 0 \Rightarrow x_2 = 0$

1^{st} eq $\Rightarrow x_1 - 2(0) + 3(5) = 1 \Rightarrow x_1 = -14$

The solution is $(x_1, x_2, x_3) = (-14, 0, 5)$ → **UNIQUE** solution (here $a_{k\,k} \neq 0$ for all $k = 1,2,3$)

Example 2:

$$\begin{aligned} x_1 + 0x_2 + 2x_3 &= 0 \\ x_2 + 0x_3 &= 0 \\ 0x_3 &= 0 \end{aligned} \rightarrow \underset{\boldsymbol{A}}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}} \underset{\boldsymbol{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}} = \underset{\boldsymbol{B}}{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}$$

$a_{3\,3} = 0$

Solution:

Last eq. is satisfied for any real number x_3, say $x_3 = t$

2nd eq $\Rightarrow x_2 = 0$

1st eq $\Rightarrow x_1 - 2t = 0 \Rightarrow x_1 = -2t$

The solution is $(x_1, x_2, x_3) = (-2t, 0, t)$, where $t \in \mathbb{R} \Rightarrow$ **INFINITE** solutions. ($a_{k\,k} \neq 0$ for all $k = 3$)

Example 3:

$$\begin{aligned} x_1 - x_2 + x_3 + x_4 &= 1 \\ 0x_2 + x_3 + x_4 &= 3 \\ x_3 - x_4 &= 0 \\ x_4 &= 1 \end{aligned} \rightarrow \underset{\boldsymbol{A}}{\begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix}} \underset{\boldsymbol{x}}{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}} = \underset{\boldsymbol{B}}{\begin{bmatrix} 1 \\ 3 \\ 0 \\ 1 \end{bmatrix}}$$

Solution:

Last eq. $\Rightarrow x_4 = 1$

3rd eq. $\Rightarrow x_3 - 1 = 0 \Rightarrow x_3 = 1$

2nd eq $\Rightarrow 0x_2 + 1 + 1 = 3$

0 + 2 = 3 not true for any value of x_2

The system has **NO** solutions. ($a_{2\,2} = 0$)

0 + 2 = 3 not true for any value of x_2, so the system has no solution ($a_{2\,2} = 0$)

The **upper triangular system** has **<u>a unique solution</u>** if $a_{k\,k} \neq 0$ for all $k = 1, 2, \ldots, N$

In this course, we will focus on the solutions of system for which we can find a **unique solution.**

Textbook: Appendix: Introduction to MATLAB. Book-marked.
Most of the programs are in the textbook.
Upper triangular system, the book has a program for back substitution on page 123.

name of the file ⇒ save as backsub.m

output

input separated by commas.

$function \quad x = backsub(A, B)$
$n = length(B)$
$x = zeros(n, 1)$

For output

[x k] or [x, k]

no comma or comma

$x(n) = {B(n)}/{A(n,n)}$;

means it will not show (display) output for $x(n)$ at this time

$B \to [\vdots]_{N\times 1}$

$for\ k = n-1: -1: 1$
$x(k) = (B(k) - A(k, k+1: n) * x(k+1: n))/A_{k,k}$;
end

$for\ k = 1 : 5$

$k = 1, 2, 3, 4, 5$

$$\left.\begin{aligned}
a_{1\,1}x_1 + a_{1\,2}x_2 + a_{1\,3}x_3 + \cdots + a_{1\,N-1}x_{N-1} + a_{1\,N}x_N &= b_1 - (1)\\
a_{2\,2}x_2 + a_{2\,3}x_3 + \cdots + a_{2\,N-1}x_{N-1} + a_{2\,N}x_N &= b_2 - (2)\\
\vdots \qquad\qquad \vdots \qquad\qquad \vdots &\\
a_{N-1\,N-1}x_{N-1} + a_{N-1\,N}x_N &= b_{N-1} - (N-1)\\
a_{N\,N}x_N &= b_N - (N)
\end{aligned}\right\}$$

upper triangular system

$x_N = \frac{b_N}{a_{N\,N}}$, sub. x_N in eq $(N-1)$ we will get x_{N-1}

In back substitution, we do N dimensions (1 at each step)

multiplications: $0 + 1 + 2 + \cdots + (N-1) = \frac{(N-1)N}{2} = \frac{N^2-N}{2}$

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

subtractions / additions: $0 + 1 + 2 + \cdots + (N-1) = \frac{(N-1)N}{2} = \frac{N^2-N}{2}$

Total operations are $N + \frac{N^2-N}{2} + \frac{N^2-N}{2} = \frac{2N+N^2-N+N^2-N}{2} = \frac{2N^2}{2} = N^2$

Computational complexity of back substitution is of $O(N^2)$.