Example 1:

Approximate $\int_0^2 x^4\,dx$ using Trapezoidal rule, Simpson's rule, and Simpson's 3/8 rule. Find the exact error and relative error.

Solution:

Trapezoidal: $h = b - a = 2 - 0 = 2$

$$x_0 = 0 \rightarrow f_0 = f(0) = (0)^4 = 0$$
$$x_1 = 2 \rightarrow f_1 = f(2) = (2)^4 = 16$$

$$\int_a^b f(x)\,dx = \frac{h}{2}[f_0 + f_1]$$
$$= \frac{2}{2}[0 + 16]$$
$$= 16$$

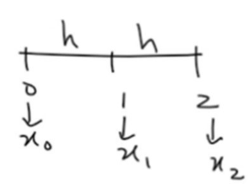
Exact value:

$$\int_0^2 x^4\,dx = \frac{x^5}{5}\bigg|_0^2 = \frac{(2)^5}{5} = \frac{32}{5} = 6.4$$

Exact error is $|16 - 6.4| = 9.6$

Relative error is $\frac{9.6}{6.4} = 1.5$

Simpson's rule: $h = \frac{b-a}{2} = \frac{2-0}{2} = 1$



$$\int_a^b f(x)\,dx = \frac{h}{3}[f_0 + 4f_1 + f_2]$$

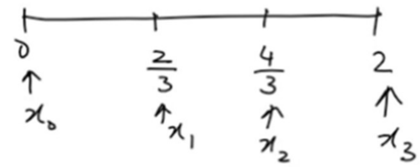
$$\boxed{\begin{aligned} f_0 &= f(0) = (0)^4 = 0 \\ f_1 &= f(1) = (1)^4 = 1 \\ f_2 &= f(2) = (2)^4 = 16 \end{aligned}}$$

$$\int_0^2 x^4\,dx = \frac{1}{3}[0 + 4(1) + 16]$$
$$= \frac{20}{3} = 6.6667$$

Exact error is $|6.6667 - 6.4| = 0.2667$

Relative error is $\frac{0.2667}{6.4} = 0.041667$

Simpson's $\frac{3}{8}$ rule: $h = \frac{b-a}{3} = \frac{2-0}{3} = \frac{2}{3}$



$$\int_a^b f(x)\,dx = \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3]$$

$$\int_0^2 x^4\,dx = \frac{3\left(\frac{2}{3}\right)}{8}\left[0 + 3\left(\frac{16}{81}\right) + 3\left(\frac{256}{81}\right)_2 + 16\right]$$

$$= \frac{1}{4}\left[\frac{704}{27}\right]$$

$$= 6.518518$$

$f_0 = f(0) = (0)^4 = 0$
$f_1 = f\left(\frac{2}{3}\right) = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$
$f_2 = f\left(\frac{4}{3}\right) = \left(\frac{4}{3}\right)^4 = \frac{256}{81}$
$f_3 = f(2) = (2)^4 = 16$

Exact error = $|6.518518 - 6.4| = 0.118518$

Relative error is $\frac{0.118518}{6.4} = 0.018518$

Note: If you were asked to find error bound for the above example.

Trapezoidal rule → error is $-\frac{h^3}{12}f''(c)$ where $c \in [0, 2]$

$$|E| = \left|\frac{-(2)^3}{12}(12c^2)\right| \le |(8)(4)| = 32$$

The max value of c^2 on $[0, 2]$ is $(2)^2 = 4$

$f(x) = x^4$
$f'(x) = 4x^3$
$f''(x) = 12x^2$
$f'''(x) = 24x$
$f''''(x) = 24$

Simpson's Rule → error is $-\frac{h^2}{90}f^{(4)}(c)$ where $c \in [0,2]$

$|E| = \left|\frac{-(1)^5}{90}(24)\right| \le 0.26667$ → This was exact error

Simpson's $\frac{3}{8}$ Rule → error is $-\frac{3h^5}{80}f^{(4)}(c)$ where $c \in [0,2]$

$|E| = \left|\frac{-3\left(\frac{2}{3}\right)^5}{80}(24)\right| \le 0.118518$ → This was exact error

Example 2:

Consider the integral $\int_0^1 e^{x^2}\,dx$

Approximate the integral using Trapezoidal, Simpson's & Simpson's $\frac{3}{8}$ rules.

Solution:

Trapezoidal : $h = b - a = 1 - 0 = 1$

The nodes are $x_0 = 0$ & $x_1 = 1$

$$\boxed{\int_a^b f(x)\,dx = \frac{h}{2}[f_0 + f_1]}$$

$f_0 = f(0) = e^0 = 1$ & $f_1 = f(1) = e^1 = 2.7183$

$$\int_0^1 e^{x^2}\,dx = \frac{1}{2}[1 + 2.7183] = 1.85915$$

Simpson's rule: $h = \frac{b-a}{2} = \frac{1-0}{2} = \frac{1}{2}$

The nodes are $x_0 = 0, x_1 = \frac{1}{2}, x_2 = 1$

$$f_0 = f(0) = e^0 = 1, f_1 = f\left(\frac{1}{2}\right) = e^{\left(\frac{1}{2}\right)^2} = e^{\frac{1}{4}} = 1.2840 \ \& \ f_2 = e^{(1)^2} = e^1 = 2.7183$$

$$\int_a^b f(x)dx = \frac{h}{3}[f_0 + 4f_1 + f_2]$$

$$\int_0^1 e^{x^2}dx = \frac{\frac{1}{2}}{3}[1 + 4(1.2840) + 2.7183] = 1.4757$$

Simpson's $\frac{3}{8}$ rule: $h = \frac{b-a}{3} = \frac{1-0}{3} = \frac{1}{3}$

The nodes are $x_0 = 0, x_1 = \frac{1}{3}, x_2 = \frac{2}{3}, x_1 = 1$

$$f_0 = f(0) = e^0 = 1$$

$$f_1 = f\left(\frac{1}{3}\right) = e^{\frac{1}{9}} = 1.1175$$

$$f_2 = f\left(\frac{2}{3}\right) = e^{\frac{4}{9}} = 1.5596$$

$$f_3 = f(1) = e^{\frac{1}{9}} = 2.7183$$

$$\int_a^b f(x)dx = \frac{3h}{8}[f_0 + 3f_1 + 3f_2 + f_3]$$

$$\int_0^1 e^{x^2} dx = \frac{3\left(\frac{1}{3}\right)}{8}[1 + 3(1.1175) + 3(1.5596) + 2.7183]$$

$$= 1.4687$$

The Degree of Precision

The standard deviation of quadrature error formulas is based on determining the class? of polynomials for which these formulas produce exact results. The degree of accuracy or precision of quadrature formula is the largest positive integer n such that the formula is exact for x^k , for each $k = 0,1, \dots, n$

Example:
Determine the degree of precision of the trapezoidal rule.

Solution:
It is sufficient to apply trapezoidal rule over $\underbrace{[0\,,1]}_{h=b-a=1}$ with $1, x, x^2, \dots, x^n$

Simpson's use → [0,2]

$$h = \frac{b-a}{2} = \frac{2}{2} = 1$$

Simpson's $\frac{3}{8}$ use → [0,3]

$$h = \frac{b-a}{3} = \frac{3}{3} = 1$$

Trapezoidal rule is

$$\int_a^b f(x)\,dx = \frac{h}{2}[f_0 + f_1]$$

Exact value

$$\int_0^1 1\,dx = x \Big|_0^1 = 1 - 0 = 1$$

<u>For $f(x) = 1$</u> on $[0,1] \Rightarrow h = 1$

$$f_0 = f(0) = 1\,,\ f_1 = f(1) = 1$$

$$\int_0^1 1\,dx = \frac{1}{2}[1+1] = \frac{1}{2}$$ ✓

$$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = \frac{1-0}{2} = \frac{1}{2}$$

<u>For $f(x) = x$</u> on $[0,1] \Rightarrow h = 1$

$$f_0 = f(0) = 0\,,\ f_1 = f(1) = 1$$

$$\int_0^1 1\,dx = \frac{1}{2}[0+1] = \frac{1}{2}$$ ✓

For $f(x) = x^2$ on $[0,1] \Rightarrow h = 1 \rightarrow f_0 = f(0) = 0$ & $f_1 = f(1) = (1)^2 = 1$

$$\int_0^1 x^2\,dx = \frac{1}{2}[0+1] = \frac{1}{2} \neq \frac{1}{3}$$

Exact value

$$\int_0^1 x^2\,dx = \frac{x^3}{3}\bigg|_0^1 = \frac{1-0}{3} = \frac{1}{3}$$

The degree of precision of Trapezoidal rule is one.

The degree of precision of Simpson's and Simpson's $\frac{3}{8}$ is 3