Vector Calculus

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1 Construction of a Line

The equation of a line is $y = mx + b$, where y is the dependent variable and x is the dependent variable. Now, the equation of a line passing through (x_0, y_0, z_0) (or $\vec{r_0}$) and is parallel to the vector $\vec{v}$ is $\vec{r} = \vec{r_0} + t\vec{v}, t \in R$

2 Planes

The equation of the plane is $ax + by + cz = d$ where $\vec{n} =< a, b, c >$ is the normal vector of the plane.

2.1 Cylinders

A **cylinder** is a surface that consists of all lines parallel to a line and passing through a.

Examples: Identify and sketch the surface

1. $x^2 + y^2 = 4 \rightarrow$ **a circle in 2 dimensions** $x^2 + y^2 = 4$ is a cylinder in 3 dimensions (a surface where one variable is missing is a cylinder, the missing variable is the axis).

2. $y^2 + z^2 = 9$ A cyliinder with x-axis as the axis.

3. $z = x^2$ A cylinder with y-axis as the axis.

3 Quadrics

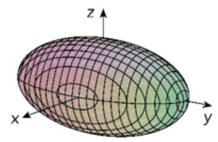
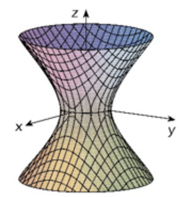
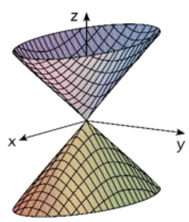
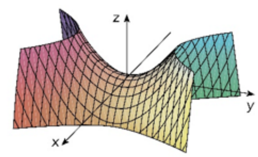
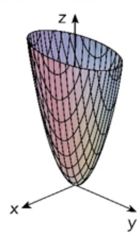
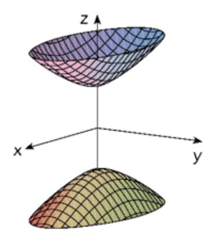
A quadric in two dimensions is:

1. A parabola $y = x^2$ or $x = y^2$
2. An ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
3. A hyperbola: $\frac{x^2}{a^2} - y^2 b^2 = 1$

A trace is the curve of the intersection of the surface with the coordinate plane $\rightarrow$ 3 traces.

3.1 Quadric Surface

$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fzy + Gx + Hy + Iz + J = 0$
In this course, we need to know 6 quadric surfaces.

Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ "A bunch of ellipses stacked together" Special case: If $a = b = c$, we have a sphere	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are hyperbolas. *Whichever variable is negative corresponds to the axis of symmetry
Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are hyperbolas, except when $x = 0$ or $y = 0$, then the traces are pairs of lines	Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ In the xy plane, the traces are hyperbolas. In the xz or yz plane, the traces are parabolas.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are parabolas.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ In the xy plane, the traces are ellipses if $z > c$ or $z < -c$ In the xz or yz planes, the traces are hyperbolas.

We can actually know some patterns for the rest of the quadric surfaces:

- **Ellipsoid** $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
- **Hyperboloid of One Sheet** $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
- **Hyperboloid of Two Sheets** $\Rightarrow -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
- **Cone** $\Rightarrow \frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$
- **Elliptic Paraboloid** $\Rightarrow \frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$
- **Hyperbolic Paraboloid** $\Rightarrow \frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$

Notice the patterns with the equations?

3.1.1 Elliptic Paraboloids

The equation of an elliptic paraboloid is $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ where z is the axis, xy traces are ellipses $(x = 0)$ and both $\frac{x^2}{a^2}$ and $\frac{y^2}{b^2}$ have the same signs.

2 traces is a parabola, 1 trace is an ellipse.

Effectively, the variable with a power of 1 is the axis.

$z = 0 \quad \Rightarrow 0 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow x = 0, y = 0, z = 0$

$z = k \quad \Rightarrow \frac{k}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow$ ellipses for $k > 0$

3.1.2 Hyperboloid of One Sheet

The variable with the negative sign is the axis. The **xy-trace** is where $z = 0$. The **xz-trace** and **yz-trace** is a hyperbola.

3.1.3 Hyperboloid of Two Sheets

The variable with the positive sign is the axis. The **xy-trace** is an ellipses if $|z| > |c|$

3.1.4 Cone

The variable with the negative sign is the axis. Two traces are hyperbolas, one trace is an ellipse for $k \neq 0$.

3.1.5 Examples: Identify and sketch the surfaces

1. $x^2 + 4y^2 + z^2 = 4 \quad \Rightarrow \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{4} = 1$

3. $z^2 = x^2 + 4y^2 + 64 \quad \Rightarrow -x^2 - 4y^2 + z^2 = 64 \Rightarrow -\frac{x^2}{64} - \frac{y^2}{16} + \frac{z^2}{64} = 1$ (hyperboloid on 2 sheets, axis is z-axis with $c = 8$)

4 Vector Functions

These are chapters 13.1 and 13.2 from last class.

A vector function is $\vec{r}(t) =< f(t), g(t), h(t) >$ such that $f(t) = x$, $g(t) = y$ and $h(t) = z$.

- For 2 dimensions: $\vec{r}(t) =< f(t), g(t) >$
- $\vec{r}\,'(t) =< f'(t), g'(t) >$ or $\vec{r}\,'(t) =< f'(t), g'(t), h'(t) >$

4.1 Arc Length

The equation for length is $|\vec{r}\,'(t)| = \sqrt{(f'(t))^2 + (g'(t))^2}$ for 2 dimensions, or $|\vec{r}\,'(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$ for 3 dimensions.

Now, the formula is effectively $\sqrt{(\delta x)^2 + (\delta y)^2}$, where $x = f(t)$ and $\delta x = f'(t)dt$ and similarly for y. Thus the formula is $\sqrt{(f'(t)dt)^2 + (g'(t)dt)^2} = \sqrt{(f'(t))^2 + (g'(t))^2 (dt)^2} = \sqrt{(f'(t))^2 + (g'(t))^2}dt$

Arc Length is: $L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}dt$

Thus, the arc length of a curvee $\vec{r}(t)$ for $a \leq t \leq b$ is $\int_a^b |\vec{r}\,'(t)|dt$

4.2 Example 1

Find the length of the curve $\vec{r}(t) =< 2\sin^3 t, 2cos^3 t >, 0 \leq t \leq \frac{\pi}{4}$

Solution: $\vec{r}\,'(t) =< 2(3\sin^2 t)(\cos t), 2(3\cos^2 t)(-\sin t) >$
$|\vec{r}\,'(t)| = \sqrt{(6sin^2 t\cos t)^2 + (-6\cos^2 t\sin t)^2}$
$= \sqrt{36\sin^4 t\cos^2 t + 36\cos^4 t\sin^2 t}$
$= \sqrt{36\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)}$
$= \sqrt{(36\sin^2 t\cos^2 t(1))}$
$= 6\sin t\cos t$

The arc length $L = \int_a^b |\vec{r}\,'(t)|dt = \int_0^{\frac{\pi}{4}} 6\sin t\cos t dt$
Let $u = \sin t$ so $du = \cos t dt$

Then we have $L = \int_0^{\frac{\pi}{4}} 6u du = \frac{6u^2}{2} = 3u^2$
$= 3\sin^2 t|_{t=0}^{\frac{\pi}{4}} = 3\sin^2 \frac{\pi}{4} - 3\sin^2 0 = 3(\frac{1}{2}) = \frac{3}{2}$

4.3 Example 2

Find the length of the curve $\vec{r}(t) =< t^2, 2t, \ln t >, 1 \leq t \leq e$

Solution: $\vec{r}\,'(t) =< 2t, 2, \frac{1}{t} >$
$|\vec{r}\,'(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\frac{4t^4+4t^2+1}{t^2}} = \frac{\sqrt{(2t^2+1)^2}}{\sqrt{t^2}} = \frac{2t^2+1}{t}$
$L = \int_a^b |\vec{r}\,'(t)|dt = \int_1^e \frac{2t^2+1}{t}dt = \int_1^e \left(\frac{2t^2}{t} + \frac{1}{t}\right) dt$
$L = \int_1^e \left(2t + \frac{1}{t}\right) dt = t^2 + \ln|t| \;|_1^e$
$L = e^2 + \ln|e| - 1^2 - \ln 1 = e^2$

4.4 Curvature

In the last class, the unit tangent vector $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$. Curvature is the magnitude of the rate of change of the unit tangent vector w.r.t. the arc length.

Arc Length Function $S = \int_a^t |\vec{r}\,'(u)|\, du$
Curvature Function $K = \left|\frac{d\vec{T}}{ds}\right|$

- Note: $\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt} \rightarrow$ chain rule.
  $\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}|\vec{r}\,'(t)| \Rightarrow \left|\frac{d\vec{T}}{ds}\right| = \frac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{r}\,'(t)|} = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$
- Thus $K = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$
- Another formula for $K = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

4.5 Example 1

Find the curvature of the following curve: $\vec{r}(t) =< 5\sin t, 3t, 5\cos t >$

Solution: $\vec{r}\,'(t) =< 5\cos t, 3, -5\sin t >$, $|\vec{r}\,'(t) = \sqrt{25\cos^2 t + 9 + 25\sin^2 t} = \sqrt{25(\cos^2 t + \sin^2 t) + 9} = \sqrt{34}$
$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)} = \frac{<5\cos t, 3, -5\sin t>}{\sqrt{34}}$
$\vec{T}\,'(t) = \frac{1}{\sqrt{34}} < -5\sin t, 0, -5\cos t >$
$\left|\vec{T}\,'(t)\right| = \sqrt{\frac{1}{34}\left(25\sin^2 t + 25\cos^2 t\right)} = \sqrt{\frac{25}{34}} = \frac{\frac{5}{\sqrt{34}}}{\sqrt{34}}$ (divide by $\sqrt{34}$ one more time since we're trying to find K)

4.6 Example 2

Find the curvature of the following curve: $\vec{r}\,'(t) =< t, t, 1 + t^2 >$

Solution: $\vec{r}\,'(t) =< 1, 1, 2t >$, $|\vec{r}\,'(t)| = \sqrt{1^2 + 1^2 + 4t^2} = \sqrt{2 + 4t^2}$
$\vec{T}\,'(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)} = \frac{<1,1,2t>}{\sqrt{2+4t^2}} = \left\langle \frac{1}{\sqrt{2+4t^2}}, \frac{1}{\sqrt{2+4t^2}}, \frac{2t}{\sqrt{2+4t^2}} \right\rangle$
$\vec{T}\,'(t) = ...$ (hard to calculate)
Instead, another formula for K: $K = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$
$\vec{r}\,''(t) =< 0, 0, 2 >$
$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2t \\ 0 & 0 & 2 \end{bmatrix} = \hat{i}(2-0) - \hat{j}(2-0) + \hat{k}(0-0) = 2\hat{i} - 2\hat{j} + 0\hat{k}\ or\ < 2, -2, 0 >$ (remember cross product in MATH1250)
$K = \frac{|\vec{r}\,'(t) \times \vec{r}\,''(t)|}{|\vec{r}^3|} = \frac{\sqrt{8}}{\left(\sqrt{2+4t^2}\right)^3}$

5 (13.4) Motion in Space

We learned that $\vec{r}(t) =< f(t), g(t), h(t) >$ where $f(t) = x$, $g(t) = y$ and $h(t) = z$.

- If the position vector/function of an object is $\vec{r}(t) =< f(t), g(t), h(t) >$ then the velocity of the object is $\vec{v}(t) = \vec{r}\,'(t)$ and it will be in the direction of the tangent vector $\vec{r}\,'(t)$.
- The speed is $|\vec{v}(t)| = |\vec{r}\,'(t)|$
- The acceleration of the object is $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

5.0.1 Example 1

Find the velocity, acceleration and speed of a particle with position vector (function) $\vec{r}(t) =< e^t, t^4, e^{-t} >$

Solution $\vec{v}(t) = \vec{r}\,'(t) =< e^t, 4t^3, -e^{-t} > \rightarrow$ velocity
Acceleration is $\vec{a}(t) = \vec{v}\,'(t) =< e^t, 12t^2, e^{-t} >$
Speed is $|\vec{v}(t)| = \sqrt{(e^t)^2 + (4t^3)^2 + (e^{-t})} = \sqrt{e^{2t} + 16t^6 + e^{-2t}}$

5.0.2 Example 2

Find the velocity, acceleration and speed of a particle with position vector $\vec{r}(t) =< e^t, t^4, e^{-t} >$ at $t = 0$.

Solution From example 1, $\vec{v}(t) =< e^t, 4t^3, -e^{-t} >$
At $t = 0$, the velocity is $\vec{v}(0) =< e^0, 4(0)^3, -e^{-0} >=< 1, 0, -1 >$
The speed at $t = 0$ is $\sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$
$\vec{a}(t) =< e^t, 12t^2, e^{-t} >$
At $t = 0$, $\vec{a}(0) =< e^0, 12(0)^2, e^{-0} >=< 1, 0, 1 >$

5.0.3 Example 3

Given the acceleration vector $\vec{a}(t) =< 2t, 1, t^2 >$ amd the initial velocity is $\vec{v}(0) =< 0, 1, 1 >$ and the initial position vector is $\vec{r}(0) =< 2, 0, -1 >$, find the velocity and position vectors of the particle.

Solution The velocity is $\vec{v}(t) = \int \vec{a}(t)dt = \int < 2t, 1, t^2 > dt$
$=< \frac{2t^2}{2} + C_1, t + C_2, \frac{t^3}{3} + C_3 >$ or $< t^2, t, \frac{t^3}{3} > + < C_1, C_2, C_3 > (\vec{C})$
$\vec{v}(0) =< 0, 1, 1 >\Rightarrow< 0, 1, 1 >=< 0 + C_1, 0 + C_2, 0 + C_3 >\Rightarrow C_1 = 0, C_2 = 1, C_3 = 1$
Thus the velocity is $\vec{v}(t) =< t^2, t + 1, \frac{t^3}{3} + 1 >$

The position vector is $\vec{r}(t) = \int \vec{v}(t)dt = \int < t^2, t + 1, \frac{t^3}{3} + 1 > dt$
$\vec{r}(t) =< \frac{t^3}{3} + d_1, \frac{t^2}{2} + t + d_2, \frac{t^4}{12} + t + d_3 >$
$\vec{r}(0) =< 2, 0, -1 >\Rightarrow< 2, 0, -1 >=< 0 + d_1, 0 + d_2, 0 + d_3 >\Rightarrow d_1 = 2, d_2 = 0, d_3 = -1$
So $\vec{r}(t) =< \frac{t^3}{3} + 2, \frac{t^2}{2} + t, \frac{t^4}{12} + t - 1 >$

6 Functions of Several Variables (or Multivariable Functions)

Multivariable Functions are functions with at least 2 independent variables. In **chapters 14 and 15**, we cover domains, limits, continuity, derivatives and applications, integration and applications.

- One prominent example is the volume functions $V = xyz$ and $V = \pi r^2 h$

6.1 Functions of 2 Variables

x and y are independent variables, and the domain will be in R^2: $D = \{(x, y) \mid \text{properties}\}$
A function in 2 variables, $z = f(x, y)$ is a rule that assigns to each $(x, y) \in D$ a unique value z in R.

- The domain D is R^2 (inputs).
- The range is a subset of R (outputs).
- $z = f(x, y) \rightarrow$ explicit.
- $f(x, y, z) = 0 \rightarrow$ implicit.

The vertical line test (VLT) can be used to check if a single-variable relation is a function. This is similar for 2-variable functions, the VLT will instead draw lines parallel to the z-axis. The relation will be a function if every vertical line crosses the surface of a function only once.

6.1.1 Example 1

Find the domain of $f(x, y) = \ln(x + y)$

Solution We need $x + y > 0 \Rightarrow y > -x$
Thus the domain of $f = \{(x, y) \mid y > -x\}$

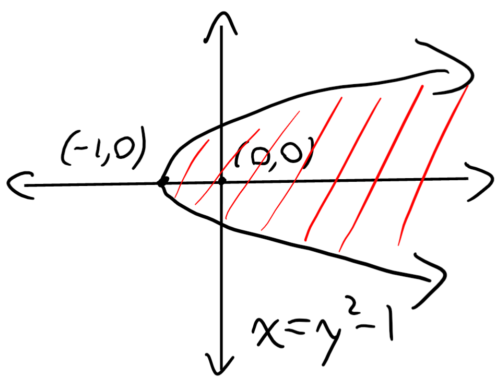
6.1.2 Example 2

Find the domain of $f(x, y) = \sqrt{1 + x - y^2}$

Solution We need $1 + x - y^2 \geq 0 \Rightarrow x \geq y^2 - 1$
Thus the domain of $f = \{(x, y) \mid x \geq y^2 - 1\}$

6.2 Sketch the Domains

Draw $x = y^2 - 1$



Choose a test point, (0,0)
$x \geq y^2 - 1 \Rightarrow 0 \geq 0 - 1 \Rightarrow 0 \geq -1$ True $\Rightarrow$ Shade the side that has (0,0)
Graphing the surface $z = f(x, y)$ will be in 3 dimensions.

6.2.1 Example 1

Sketch the graph of the following function: $f(x, y) = 8 - x - 2y$

Solution $z = 8 - x - 2y \Rightarrow x + 2y + z = 8 \Rightarrow$ a plane
x-intercept: $y = 0, z = 0 \Rightarrow x = 8 \Rightarrow (8, 0, 0)$
y-intercept: $x = 0, z = 0 \Rightarrow 2y = 8 \Rightarrow y = 4 \Rightarrow (0, 4, 0)$
z-intercept: $x = 0, y = 0 \Rightarrow z = 8 \Rightarrow (0, 0, 8)$

6.2.2 Example 2

Sketch the graph of the following function: $f(x, y) = \sqrt{2x^2 + y^2}$

Solution $z = \sqrt{2x^2 + y^2} \Rightarrow z^2 = 2x^2 + y^2 \Rightarrow 2x^2 + y^2 - z^2 = 0 \Rightarrow$ a cone

6.2.3 Example 3

Sketch the graph of the following function: $f(x, y) = -\sqrt{2x^2 + y^2}$

6.3 (14.1) Functions

In last class, we did functions of 2 variables, such as $z = f(x, y)$ where $(x, y) \in D$ where D is the domain in R^2. However, this idea can be extended to more than 2 variables. A 3-variable function would have $w = f(x, y, z)$ (explicit form) or $f(x, y, z, w) = 0$ (implicit form).

6.3.1 Example

Find the domain of $f(x, y, z) = \frac{1}{\sqrt{16-x^2-y^2-z^2}}$ and sketch it.

Solution For the domain, we need $16 - x^2 - y^2 - z^2 > 0$.
$D = \{(x, y, z) \mid 16 - x^2 - y^2 - z^2 > 0\}$ (or $x^2 + y^2 + z^2 < 16$)

To sketch, draw $x^2 + y^2 + z^2 = 16$ (a sphere with a radius of $\sqrt{16} = 4$). $(0, 0, 0) \Rightarrow 0 + 0 + 0 < 16 \Rightarrow 0 < 16$

6.4 Limits and Continuity

$\lim_{x\to a} f(x) = L$ if $f(x) \to L$ as $x \to a$ and the righthand/lefthand limits are equal to L.
$\lim_{(x,y)\to(a,b)} f(x, y) = L$ if $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every possible path.
If two paths give different answers, then this proves the limit DNE. To find the limit, when it exists, we substitute $x = a$ and $y = b$. If we get an answer, that is the limit. If we get $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$,$\infty - \infty$, 0^∞, ∞^0, 1^∞ or any other indeterminate form, then do something:

$$\begin{cases} \text{To show the limit DNE, show two paths with different limits. This can be done with polar coordinates} \\ \quad x = \gamma\cos\theta, y = \gamma\sin\theta \\ \text{Factorization.} \\ \text{Rationalization.} \end{cases}$$

Note: for the indeterminate forms $0 \times \infty$,$\infty - \infty$, 0^∞, ∞^0 and 1^∞, try changing them to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and use L'Hopital's rule. **However:** there is no L'Hopital's rule for 2 variables.

6.4.1 Example 1

Evaluate the limit if it exists, or show that the limit DNE. $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{2+xy}$

Solution $\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{2+xy} = \frac{0+0}{2+0} = \frac{0}{2} = 0$

6.4.2 Example 2

Evaluate the limit if it exists, or show that the limit DNE. $\lim_{(x,y)\to(4,1)} \frac{x^2-6xy+8y^2}{2x-8y}$

Solution $\lim_{(x,y)\to(4,1)} \frac{x^2-6xy+8y^2}{2x-8y} = \frac{4^2-6(4)(1)+8(1)^2}{2(4)-8(1)} = \frac{16-24+8}{8-8} = \frac{0}{0}$
This did not work, so we can try factorization: $\lim_{(x,y)\to(4,1)} \frac{(x-4y)(x-2y)}{2(x-4y)} = \lim_{(x,y)\to(4,1)} \frac{x-2y}{2} = \frac{4-2(1)}{2} = \frac{2}{2} = 1$

6.4.3 Example 3

Evaluate the limit if it exists, or show that the limit DNE. $\lim_{(x,y)\to(0,0)} \frac{x+2y}{\sqrt{x+2y+4}-2}$

Solution $\lim_{(x,y)\to(0,0)} \frac{x+2y}{\sqrt{x+2y+4}-2} = \frac{0+0}{\sqrt{4}-2} = \frac{0}{0}$
This did not work, so let's try rationalization: $\lim_{(x,y)\to(0,0)} \frac{x+2y}{\sqrt{x+2y+4}-2} \cdot \frac{\sqrt{x+2y+4}+2}{\sqrt{x+2y+4}+2} = \lim_{(x,y)\to(0,0)} \frac{(x+2y)(\sqrt{x+2y+4}+2)}{x+2y+4-4}$
$= \lim_{(x,y)\to(0,0)} \frac{(x+2y)(\sqrt{x+2y+4}+2)}{x+2y} = \lim_{(x,y)\to(0,0)} \sqrt{x+2y+4} + 2 = \sqrt{0+0+4} + 2 = 2 + 2 = 4$

6.4.4 Example 4

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$

Solution $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2} = \frac{0+\sin^2 0}{0+0} = \frac{0}{0}$
This did not work, so let's do some tricks:
Along the x-axis (or along $y = 0$)$\Rightarrow \lim\limits_{x\to 0} \frac{x^2+\sin^2 0}{2x^2+0^2} = \lim\limits_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$
Along the y-axis (or along $x = 0$)$\Rightarrow \lim\limits_{y\to 0} \frac{0+\sin^2 y}{0+y^2} = \lim\limits_{y\to 0} \frac{\sin^2 y}{y^2} = \frac{0}{0}$
This still doesn't work, but since we only have one variable, we can use L'Hopital's rule:
$\lim\limits_{y\to 0} \frac{\sin^2 y}{y^2} = \lim\limits_{y\to 0} \frac{2\sin y\cos y}{2y} = \lim\limits_{\sin y(-\sin y)+\cos y\cos y} 1 = 0 + 1 = 1$
Or, we can break up the limit like this: $\lim\limits_{y\to 0} \frac{\sin^2 y}{y^2} = \lim\limits_{y\to 0} \frac{\sin y}{y} \cdot \frac{\sin y}{y} = 1 \cdot 1 = 1$

6.4.5 Example 5

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+5y^4}$

Solution $\lim\limits_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+5y^4} \to \frac{0}{0}$
Along $x = 0 \Rightarrow \lim\limits_{y\to 0} \frac{0}{5y^4} = \lim\limits_{y\to 0} 0 = 0$ (similarly, along $y = 0 \Rightarrow \lim\limits_{x\to 0} \frac{0}{2x^4} = \lim\limits_{x\to 0} 0 = 0$)
Along $y = x \Rightarrow \lim\limits_{x\to 0} \frac{6x^3x}{2x^4+5x^4} = \lim\limits_{x\to 0} \frac{6x^4}{7x^4} = \frac{6}{7}$
Since two paths have different answers, the limit DNE.
Alternate answer: Along $y = mx \Rightarrow \lim\limits_{x\to 0} \frac{6x^3mx}{2x^4+6m^4x^4} = \lim\limits_{x\to 0} \frac{6mx^4}{x^4(2+5m^4)} = \frac{6m}{2+5m^4}$
Notice the answer depends on m. Thus the limit DNE.

6.4.6 Example 6

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y)\to(0,0)} \frac{3xy^3}{x^2+y^2}$

Solution $\lim\limits_{(x,y)\to(0,0)} \frac{3xy^3}{x^2+y^2} \to \frac{0}{0}$
Use polar coordinates: $x = r\cos\theta, y = r\sin\theta, r \to 0as(x,y) \to (0,0)$
Thus $x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$
So $\lim\limits_{(x,y)\to(0,0)} \frac{3xy^3}{x^2+y^2} = \lim\limits_{r\to 0} \frac{3(r\cos\theta)(r^3\sin^3\theta)}{r^2}$
• If the answer does not depend on θ, then we get the limit. However if it depends on θ, then the limit DNE.
$\lim\limits_{r\to 0} \frac{3(r\cos\theta)(r^3\sin^3\theta)}{r^2} = \lim\limits_{r\to 0} \frac{3r^4\cos\theta\sin^3\theta}{r^2} = \lim\limits_{r\to 0} 3r^2\cos\theta\sin^3\theta = 3(0)^2\cos\theta\sin^3\theta = 0$
Thus the limit is 0.

6.4.7 Example 7

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+2y^4}$

Solution $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+2y^4} \to \frac{0}{0}$
Along $y = mx \to \lim\limits_{x\to 0} \frac{xm^2x^2}{x^2+2m^4x^4} = \lim\limits_{x\to 0} \frac{m^2x^3}{x^2(1+2m^4x^2)} = \frac{m^2(0)}{1+2m^4(0)} = \frac{0}{1} = 0$
Along $x = y^2$ (or more general, $x = cy^2$)$\to \lim\limits_{y\to 0} \frac{y^2y^2}{(y^2)^2+2y^4} = \lim\limits_{y\to 0} \frac{y^4}{3y^4} = \frac{1}{3} \neq 0$
Thus the limit DNE.

6.5 (14.2) Limits and Continuity

A function $f(x, y)$ is continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y) = f(a, b)$. Similarly, for functions with 3 variables, $f(x, y, z)$ is continuous at (a, b, c) if $\lim_{(x,y,z)\to(a,b,c)} f(x, y, z) = f(a, b, c)$
This basically states that a value exists, limit exists and both of them are equal. Every function is continuous on its domain. Additionally, polynomials and rational functions are continuous on their domain.

6.5.1 Example 1

Find the points where f is continuous: $f(x, y) = \frac{x+y}{\sqrt{x-1}}$

Solution The domain of f is $x - 1 > 0 \Rightarrow x > 1$
Thus f is continuous on R^2 except when $x \leq 1$ or f is continous on $\{(x, y)|x > 1\}$

6.5.2 Example 2

Find the points where f is continuous: $f(x, y) = \frac{e^y+3}{x^2+y^2}$

Solution The domain of f is when $x^2 + y^2 \neq 0$
Thus f is continuous on R^2 except at $(0, 0)$ (or when $x = 0$ and $y = 0$)
f is continuous on $\{(x, y)(x, y) \neq (0, 0)\}$

6.5.3 Example 3

Find the points where f is continuous: $f(x, y) = \begin{cases} \frac{2x^2+y^2}{x^2+y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases}$

Solution f is continuous when $(x, y) \neq (0, 0)$
Now we check if f is continuous at $(0, 0)$ by checking if $\lim_{(x,y)\to(0,0)} f(x, y) = f(0, 0)$
First, we know $f(0, 0) = 0$ exists as a value.
Now, $\lim_{(x,y)\to(0,0)} f(x, y) = \lim_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2} \Rightarrow \frac{0}{0}$
Along $x = 0 \Rightarrow \lim_{y\to 0} \frac{0+y^2}{0+y^2} = \lim_{y\to 0} \frac{y^2}{y^2} = \lim_{y\to 0} 1 = 1$
Along $y = 0 \Rightarrow \lim_{x\to 0} \frac{2x^2+0}{x^2+0} = \lim_{x\to 0} \frac{2x^2}{x^2} = \lim_{x\to 0} 2 = 2$
Thus the limit DNE $\Rightarrow f$ is not continuous at (0,0)
f is continuous on R^2 except at $(0, 0)$

6.6 (14.3) Partial Derivatives

Suppose y is fixed, say $y = b$. Let $g(x) = f(x, b)$ where b is some constant. Then $g'(x) = \lim_{h\to 0} \frac{g(x+h)-g(x)}{h}$ and $\frac{df}{dx} = \lim_{h\to 0} \frac{f(x+h,b)-f(x,b)}{h}$
The 1st order partial derivatives of $f(x, y)$ are:

- $f_x = \frac{\partial f}{\partial x} \Rightarrow$ treat y as a constant.
- $f_y = \frac{\partial f}{\partial y} \Rightarrow$ treat x as a constant.

6.6.1 Example 1

Find the 1st partial derivative of $f(x, y) = y^2 \ln x + x \sin^2 y + y^3$

Solution $f_x = \frac{df}{dx} = y^2(\frac{1}{x}) + (1)\sin^2 y + 0 = \frac{y^2}{x} + \sin^2 y$
$f_y = \frac{df}{dy} = 2y \ln x + x \cdot 2 \sin y \cos y + 3y^2$ (note that $\frac{d}{dy} \sin^2 y = 2 \sin y \cos y$)

6.6.2 Example 2

Find the 1st partial derivative of $f(r,t) = t^2e^r + \frac{r^2}{t}$

Solution $f_r = \frac{df}{dr} = t^2e^r + \frac{2}{t}r$
$f_t = \frac{df}{dt} = 2te^r + r^2(-\frac{1}{t^2})$

6.6.3 Example 3

Find the 1st partial derivative of $f(x,y) = \frac{x^2+y}{x+1}$

Solution $f_x = \frac{\partial f}{\partial x} = \frac{(2x)(x+1)-(x^2+y)(1)}{(x+1)^2}$ or $\frac{2x^2+2x-x^2-y}{(x+y)^2}$ (both answers are acceptable)
$f_y = \frac{\partial f}{\partial y} = \frac{1}{x+1}(0+1) = \frac{1}{x+1}$

6.6.4 Example 4

Find the 1st partial derivative of $f(x,y,z) = xyz + x^2\ln(2y-z)$

Solution $f_x = (1)yz + 2x\ln(2y-z)$
$f_y = \frac{\partial f}{\partial y} = x(1)z + x^2\frac{1}{2y-z}(2)$
$f_z = \frac{\partial f}{\partial z} = xy(1) + x^2\frac{1}{2y-z}(-1)$

6.7 Higher Order Derivatives

Higher order derivatives can be $\frac{\partial}{\partial y}(f_x)$ and $\frac{\partial}{\partial x}(f_x)$, or $\frac{\partial}{\partial x}(f_y)$ and $\frac{\partial}{\partial y}(f_y)$

- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$
- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$
- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

6.7.1 Example 1

Find all four 2nd order partial derivatives of $f(x,y) = 3x^2y^3 + 5x^4y$

Solution $f_x = \frac{\partial f}{\partial x} = 3(2x)y^3 + 5(4x^3)y = 6xy^3 + 20x^3y$
$f_y = 3x^2(3y^2) + 5x^4(1) = 9x^2y^2 + 5x^4$
$f_{xx} = 6(1)y^3 + 20(3x^2)y$
$f_{yx} = 9(2x)y^2 + 20x^3$
$f_{xy} = 6x(3y^2) + 20x^3(1)$ (also the same as f_{yx})
$f_{yy} = 9x62(2y) + 0$

6.8 Clairaut's Theorem

Theorem 6.1 *Suppose f is defined on a disk that contains (a,b) and f_{xy} and f_{yx} are continuous on D. Then* $f_{xy}(a,b) = f_{yx}(a,b)$

6.8.1 More Examples of Partial Derivatives

Let $f(x,y) = y\tan 2x$, find f_{xx} and f_{yx}

Solution $f_x = y\sec^2 2x(2) = 2y\sec^2 x$
$f_y = (1)\tan 2x$
$f_{xx} = 2y(2\sec 2x)(\sec 2x\tan 2x)(2)$ $\quad f_{yx} = \sec^2 2x(2) = f_{xy}$ (since $f_{xy} = f_{yx}$)

6.8.2 Example 2

Given that $f(x,y,z) = 3x^3 + 7xy\cos z + x^2y^3$, find $f_{xy}(-1,2,0)$ and $f_{xyz}(-1,2,0)$

Solution $f_x = 9x^2 + 7(1)y\cos z + 2xy^3 z$
$f_{xy} = 0 + 7(1)\cos z + 2x(3y^2)z = 7\cos z + 6xy^2 z$
$f_{xy}(-1,2,0) = 7\cos 0 + 6(-1)(2)^2(0) = 7(1) + 0 = 7$
$f_{xyz} = 7(-\sin z) + 6xy^2(1)$
$f_{xyz}(-1,2,0) = 7(-\sin 0) + 6(-1)(2)^2 = -24$

6.9 (14.5) Chain Rule

You have seen that if we have to find y' where $y = (x^2+1)^3$, we use the chain rule like so:
Let $u = x^2 + 1$, then $y = u^3 \Rightarrow \frac{du}{dx} = 2x$, $\frac{dy}{du} = 3u^2$
We need $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = 3u^2(2x) = 3(x^2+1)^2(2x)$

Now we can expand on this by examining the different cases of the chain rule:

1. Let $z = f(x,y)$ be a differentiable function in x and y, and $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Then, $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$
2. Let $z = f(x,y)$ be a differentiable function of x and y and let $x = g(s,t)$ and $y = h(s,t)$ be differentiable functions of s and t. Then, $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$
3. The general version of the chain rule is as follows: $w = f(x,y,z)$, $x = g(s,t,u,r)$, $y = h(s,t,u,r)$ $z = h(s,t,u,r)$. Then, $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$
4. This pattern continues.

6.9.1 Example 1

Find $\frac{dz}{dt}$ where $z = \sqrt{x^2+y}$, $x = e^{2t}$, $y = \sin t$

Solution $\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt} = \frac{x}{\sqrt{x^2+y}}(e^{2t}(2)) + \frac{1}{2\sqrt{x^2+y}}(\cos t)$
More details: $z = (x^2+y)^{\frac{1}{2}}$,
$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2+y)^{-\frac{1}{2}} \cdot (2x)$,
$\frac{\partial z}{\partial y} = \frac{1}{2}(x^2+y)^{-\frac{1}{2}}(1)$

6.9.2 Example 2

Find $\frac{dw}{dt}$ where $w = x^2y + y^3\cos z$, $x = t^2$, $y = t+1$, $z = t^3$

Solution $\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt} = (2x)y(2t) + (x^2 + 3y^2\cos z)(1) + y^3(-\sin z)(3t^2)$

6.9.3 Example 3

Let $z = \frac{x}{y}$, $x = re^t$, $y = 4re^{-t}$. Find z_r and z_t. (Note: $z_r = \frac{\partial z}{\partial r}$ and $z_t = \frac{\partial z}{\partial t}$)

Solution $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r} = \frac{1}{y}(1e^t) + \frac{-x}{y^2}(1e^{-t})$
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = \frac{1}{y}(re^t) + \frac{-x}{y^2}(re^{-t}(-1))$

6.10 (14.5) Implicit Differentiation

If you have an explicit definition of y, such as $y = f(x)$, but y is used impllicitly: $F(x, y) = 0$, then $\frac{dF}{dy} \cdot \frac{dy}{dx} = \frac{-\partial F}{\partial x}$ and $\frac{dy}{dx} = \frac{\frac{-\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$ or $\frac{dy}{dx} = \frac{-F_x}{F_y}$

Similarly with 3 variables: If we have an explicit definition $z = f(x, y)$ but z is used impllicitly: $F(x, y, z) = 0$. Then: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

6.10.1 Example 1

Find $\frac{dy}{dx}$ where $x^2y + e^{xy} = 9$ using partial derivatives.

Solution $\frac{dy}{dx} = \frac{-F_x}{F_y} = -\frac{2xy+e^{xy}(y)}{x^2+e^{xy}(x)}$

6.10.2 Example 2

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where $yz = \ln(2x + 3z)$

Solution The equation becomes $yz - \ln(2x + 3z) = 0$ (move everything to one side)
We can see $F = yz - \ln(2x + 3z)$
$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-\frac{1}{2x+3z}\cdot(2)}{y-\frac{1}{2x+3z}\cdot(3)}$ or $-\frac{-\frac{2}{2x+3z}}{\frac{y(2x+3z)-3}{2x+3z}} = \frac{2}{2xy+3yz-3}$
$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(1)z}{y-\frac{3}{2x+3z}}$

6.11 (14.4) Tangent Planes

A tangent plane to a surface is a plane that contains all of its tangent lines. A tangent plane to the surface $z = f(x, y)$ at the point (x_0, y_0) is $z - z_0 = \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0)$
Then $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

6.11.1 Example 1

Find an equation of the tangent plane to the surface $z = x\cos y + x^2$ at the point $(1, 0, 2)$

Solution Here, $f(x, y) = x\cos y + x^2 \qquad f_x = (1)\cos y + 2x \qquad f_y = x(-siny)$
$f_x(1, 0, 2) = \cos 0 + 2(1) = 3 \qquad f_y(1, 0, 2) = (1)(-\sin 0) = 0$
The equation of the tangent plane at the point (1,0,2) is $z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$
$\Rightarrow z - 2 = 3(x - 1) + 0(y - 0) \Rightarrow x - 2 = 3x - 3 \Rightarrow 3x - z - 1 = 0$ or $3x - z = 1$
Note: the equation of a tangent plane is always $ax + by + cz = d$ or $ax + by + cz + d = 0$

6.11.2 Linear Approximations

For $z = f(x, y)$, the linear approximation of $f(x, y)$ near (x_0, y_0) is:
$L(x, y) \approx f(x_0, y_0) + \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x - x_0) + \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y - y_0)$
This is effectively $f(x, y) \approx z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

6.11.3 Example

find the linear approximation of the function $f(x,y) = \ln(2x-5y)$ at the point $(3,1)$. Use linear approximation to estimate the value of $f(2.98, 1.01)$

Solution $f(3,1) = \ln(2(3)-5(1)) = \ln 1 = 0$
$f_x = \frac{1}{2x-5y} \cdot (2) = \frac{2}{2x-5y}$, $f_x|_{(3,1)} = \frac{2}{2(3)-5(1)} = \frac{2}{1} = 2$
$f_y = \frac{1}{2x-5y} \cdot (-5) = \frac{-5}{2x-5y}$, $f_y|_{(3,1)} = \frac{-5}{2(3)-5(1)} = \frac{-5}{1} = -5$

The linear approximation at $(3,1)$ is: $L(x,y) \approx f(x_0,y_0) + \frac{\partial f}{\partial x}|_{(x_0,y_0)}(x-x_0) + \frac{\partial f}{\partial y}|_{(x_0,y_0)}(y-y_0)$
$\approx f(3,1) + f_x|_{(3,1)}(x-3) + f_y|_{(3,1)}(y-1)$
$= 0 + 2(x-3) + (-5)(y-1)$
$= 2x - 6 - 5y + 5 \Rightarrow = 2x - 5y - 1$
Thus the linear approximation is $f(x,y) \approx 2x - 5y - 1$
$f(2.98, 1.09) \approx 2(2.98) - 5(1.01) - 1 = 5.96 - 5.05 - 1 = -0.09$

6.11.4 Differentiable?

If f_x and f_y are defined at (x_0,y_0) and are continuous near (x_0,y_0), then f is differentiable at (x_0,y_0).

6.12 (14.6) Directional Derivatives and Gradient Vector

If f is a differentiable function of x and y, then the gradient of f is defined as $\vec{\bigtriangledown} f = < f_x, f_y >$ or $f_x\hat{i} + f_y\hat{j}$
If f is a differentiable function of x, y, z, then the gradient vector is $\vec{\bigtriangledown} f = < f_x, f_y, f_z >$

The directional derivative of $f(x,y)$ is the direction of a **unit vector** $\vec{u} = < a, b >$ is $D_u f = \vec{\bigtriangledown} f \cdot \vec{u}$
Note: $\vec{u} = < 1, 0 > = \hat{i} \Rightarrow D_u f = < f_x, f_y > \cdot < 1, 0 > = f_x$

6.12.1 Example 1

Find the gradient of $f(x,y) = \ln(x^2+y^2)$

Solution $\vec{\bigtriangledown} f = < f_x, f_y > = \left\langle \frac{1}{x^2+y^2}(2x), \frac{1}{x^2+y^2} 2y \right\rangle$

6.12.2 Example 2

Find the gradient of $f(x,y,z) = ye^x + zx^2$ at the point $(0,1,-1)$

Solution $f_x = ye^x + z(2x)$, $f_x(0,1,-1) = 1e^0 + (-1)(2(0)) = 1$
$f_y = (1)e^x$, $f_y(0,1,-1) = e^0 = 1$
$f_z = 0 + (1)(x^2)$, $f_z(0,1,-1) = 0^2 = 0$

The gradient of f is $\vec{\bigtriangledown} f(0,1,-1) = < 1, 1, 0 >$

6.12.3 Example 3

Find the directional derivative of $f(x,y) = \ln(x^2+y^2)$ at $P(2,1)$ in the direction of the vector $< -1, 3 >$.

Solution $\vec{\bigtriangledown} f(2,1) = \left\langle \frac{1}{2^2+1^2}(2(2)), \frac{1}{2^2+1^2}(2(1)) \right\rangle = \left\langle \frac{4}{5}, \frac{2}{5} \right\rangle$
The directional derivative is $D_u f = \vec{\bigtriangledown} f \cdot \vec{u} = \left\langle \frac{4}{5}, \frac{2}{5} \right\rangle \cdot \left\langle \frac{-1}{\sqrt{10}}, \frac{3}{\sqrt{10}} \right\rangle = \frac{4}{5} \cdot \frac{-1}{\sqrt{10}} + \frac{2}{5} \cdot \frac{3}{\sqrt{10}} = \frac{2}{5\sqrt{10}}$
Note: We divided $\vec{u}$ by $|\vec{u}|$ and used that instead.

In which direction do we have the maximum/minimum rate of change?
$D_u f = \vec{\bigtriangledown} f \cdot \vec{u} = |\vec{\bigtriangledown} f||\vec{u}| \cos\theta = |\vec{\bigtriangledown} f| \cos\theta$ (since $|\vec{u}| = 1$)
The value of $\cos\theta$ is 1 when $\theta = 0$.
Thus the maximum rate of change is $|\vec{\bigtriangledown} f|$ and it occurs in the direction of $\vec{u}$.

6.13 (14,6) Tangent Plane for Implicit Functions

If a surfacae is given by $F(x(t), y(t), z(t)) = K$ where K is some constant, then
$\frac{dF}{dt} = \vec{\bigtriangledown} F \cdot r'(t) = \frac{dF}{dx} \cdot \frac{dx}{dt} + \frac{dF}{dy} \cdot \frac{dy}{dt} + \frac{dF}{dz} \cdot \frac{dz}{dt} = 0$ where $\vec{\bigtriangledown} F$ is the normal to the tangent plane (so we define the tangent plane using $\vec{\bigtriangledown} F$ and some point P (x_0, y_0, z_0) or $(x(0), y(0), z(0))$).

Also, $F_x(x_0, y_0, z_0)(x - x_0) + F_y(x_0, y_0, z_0)(y - y_0) + F_z(x_0, y_0, z_0)(z - z_0) = 0$

6.14 (14.7) Maximum and Minimum

- A function $f(x, y)$ has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all nearby values of x and y.
- A function $f(x, y)$ has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all nearby values of x and y.
- The point (a, b) is a critical minimum if $f_x(a, b) = 0$ and $f_y(a, b) = 0$, or if f_x and f_y do not exist.

6.14.1 Example 1

Determine the critical values of $f(x, y)$: $f(x, y) = x^2 + y^2$

Solution $f_x = 2x$ and $f_y = 2y$. $\Rightarrow f_x(0, 0) = 2(0) = 0 = f_y(0, 0)$
Thus $(0, 0)$ is a critical minimum.

6.14.2 Example 2

Determine the critical values of $f(x, y)$: $f(x, y) = y^2 - x^2$

Solution $f(x, y)$ has a saddle point, but is it a local minimum or maximum?
To answer this, we will perform the second derivative test.
$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = f_{xx} \cdot f_{yy} - (f_{xy})^2$

- If $D(a, b)$ is positive and $f_{xx}(a, b)$ is negative, then (a, b) is a local minimum and concave down.
- If $D(a, b)$ is positive and $f_{xx}(a, b)$ is positive, then (a, b) is a local minimum.
- If $D(a, b)$ is negative, then $f(a, b)$ is a saddle point.
- If $D(a, b) = 0$ then it is inconclusive.

6.14.3 Example 3

Find the critical points of the function $f(x, y) = x^3 - 12xy + 8y^3$. Classify them as local min, max, or saddle point.

Solution $f_x = 3x^2 - 12y = 0$ [1] and $f_y = -12x + 24y^2 = 0$ [2]
Equation [1] gives us $12y = 3x^2 \Rightarrow y = \frac{x^2}{4}$.
Substituting these results into [2]: $-12x + 24(\frac{x^2}{4})^2 = 0 \Rightarrow -12x + 24\frac{x^4}{16} = 0$
$-12x + \frac{3x^4}{2} = 0 \Rightarrow -24x + 3x^4 = 0 \Rightarrow 3x(-8 + x^3) = 0 \Rightarrow x = 0$ and $x^3 = 8 \Rightarrow x = 2$
When $x = 0, y = \frac{0}{4} = 0 \Rightarrow (0, 0)$
When $x = 2, y = \frac{2^2}{4} = 1 \Rightarrow (2, 1)$
$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 6x & -12 \\ -12 & 48y \end{bmatrix}$
$D(0, 0) = \begin{bmatrix} 0 & -12 \\ -12 & 0 \end{bmatrix} = 0 - (-12)(-12) = -144 < 0 \Rightarrow (0, 0)$ is a saddle point.
$D(2, 1) = \begin{bmatrix} 12 & -12 \\ -12 & 48 \end{bmatrix} = (12)(14) - (-12)(-12) = 12(48 - 12) = 12(36) = 432 > 0$
$f_{xx}(2, 1) = 12 > 0$, so we have a local minimum at $(2, 1)$.

Note: we could have also isolated values of y, and we would've found $y = 0$ and $y = 1$. Using those values, we would get $x = 0$ and $x = 2$. Be aware that you can do it either order, whichever is mathematically easier to calculate.

6.15 (14.7) Optimization Problems

Optimization problems describe real-life problems. Here, we need to find absolute minimums and maximums. For optimization problems, **read carefully**. Assign symbols to different quantities, decide the objective function (the function that has to be maximized or minimized).

If the objective function has 3 variables, then we eliminate one of the variables (primarily z) using the **constraint** (any condition). Find the domain of the object function, then find critical points and decide between a max or a min, based on what is asked.

6.15.1 Example

Find the points on the plane $x - y + z = 4$ that is closest to the point $(1, 2, 3)$.

Solution Let (x, y, z) be any point on the plane $x - y + z = 4$.
The distance from (x, y, z) to $(1, 2, 3)$ is $d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$
To make calculations easier, we will minimize d^2 :
$f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$
Substitute $z = 4 - x + y$: $f(x, y) = (x-1)^2 + (y-2)^2 + (4 - x - y - 3)^2$
Minimizing: $f(x, y) = (x-1)^2 + (y-2)^2 + (1 - x - y)^2$ where $(x, y) \in R$

Critical points: $f_x = 0 \Rightarrow 2(x-1)(1) + 2(1-x-y)(-1) = 0 \Rightarrow 2(x - 1 - 1 + x - y) = 0$
$f_y = 0 \Rightarrow 2(y-2)(1) + 2(1-x+y)(1) = 0 \Rightarrow 2(y - 2 + 1 - x + y) = 0$
$2x - y - 2 = 0 \Rightarrow 2x - y = 2$ [1]
$2y - x - 1 = 0 \Rightarrow -x + 2y = 1$ [2]
Multiply [2] by 2 and adding to 1:
$2x - 1 = 2$
$-2x + 4y = 2$
$\Rightarrow 3y = 4 \Rightarrow y = \frac{4}{3}$
Substitute into [1] $\Rightarrow 2x - \frac{4}{3} = 2 \Rightarrow 2x = 2 + \frac{4}{3} = \frac{10}{3} \Rightarrow x = \frac{5}{3}$
And $z = 4 - x + y = 4 - \frac{5}{3} + \frac{4}{3} = \frac{12-5+4}{3} = \frac{11}{3}$
Thus we have the point $(\frac{5}{3}, \frac{4}{3}, \frac{11}{3})$
By the geometry of the problem, the maximum distance will be infinite, so $(\frac{5}{3}, \frac{4}{3}, \frac{11}{3})$ is the closest point.

Note: $D = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = 4 - 1 = 3 > 0$ and $f_{xx} = 2 > 0 \Rightarrow$ the point is also a local minimum.
Since there is only one critical point and it is a local minimum, it will be an absolute minimum.

6.16 Lagrange Multipliers

To find the absolute max and min values of $f(x, y)$ or $f(x, y, z)$ subject to a constraint $g(x, y) = k$ or $g(x, y, z) = k$, the normal vectors are $\vec{\triangledown} f$ and $\vec{\triangledown} g$.
At the max and min values, the normal vectors would be parallel: $\vec{\triangledown} f = \lambda \vec{\triangledown} g$, where λ is a scalar called a **Lagrange multiplier**. Follow these steps:

1. Solve the equations:
   - For two variable functions: $f_x = \lambda g_x$, $f_y = \lambda g_y$, $g(x, y) = k$
   - For three variable functions: $f_x = \lambda g_x$, $f_y = \lambda g_y$, $f_z = \lambda g_z$, $g(x, y, z) = k$
2. Evaluate f at all points obtained in step 1. The largest value of f is the absolute max and the smallest value will be the absolute min.

6.16.1 Example 1

Find the maximum and minimum values of the function: $f(x, y) = 2x^2 + 3y^2 - 4x - 5$ subject to $x^2 + y^2 = 16$ (or $g(x, y) = x^2 + y^2$).

Solution $f_x = \lambda g_x \Rightarrow 4x - 4 = \lambda(2x)$ [1]
$f_y = \lambda g_y \Rightarrow 6y = \lambda(2y)$ [2]
$g(x, y) = k \Rightarrow x^2 + y^2 = 16$ [3]

From equation 2: $6y = 2\lambda y \Rightarrow (6y - 2\lambda y) = 0$
$\Rightarrow 2y(3 - \lambda) = 0 \Rightarrow y = 0$ or $\lambda = 3$
When $y = 0$, equation 3 becomes: $x^2 + 0 = 16 \Rightarrow x^2 = 16 \Rightarrow x = \pm 4 \Rightarrow (4, 0)$ and $(-4, 0)$.

When $\lambda = 3$, equation 1 becomes: $4x - 4 = 3(2x) \Rightarrow -4 = 2x \Rightarrow x = -2$
For $x = -2$, equation 3 becomes: $(-2)^2 + y^2 = 16 \Rightarrow y^2 = 12 \Rightarrow y = \pm\sqrt{12} \Rightarrow (-2, \sqrt{12})$ and $(-2, -\sqrt{12})$.

Now plug and play:
$f(4, 0) = 2(4)^2 + 3(0)^2 - 4(4) - 5 = 32 - 16 - 5 = 11$
$f(-4, 0) = 2(-4)^2 + 3(0)^2 - 4(-4) - 5 = 32 + 16 - 5 = 43$
$f(-2, \sqrt{12}) = 2(-2)^2 + 3(\sqrt{12})^2 - 4(-2) - 5 = 8 + 36 + 8 - 5 = 47$
$f(-2, -\sqrt{12}) = 2(-2)^2 + 3(\sqrt{12})^2 - 4(-2) - 5 = 8 + 36 + 8 - 5 = 47$
From this, the minimum is 11 and the absolute max is 47.

6.16.2 Example 2

Use Lagrange multipliers to find the point(s) o the plane $x - y + z = 4$ which is closest to the point $(1, 2, 3)$.

Solution In the last class, we did (let (x, y, z) be any point on the plane, then we had to minimize $d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$, then we made calculations easier by creating the objective function $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$).

Now, with Lagrange multipliers, we have to find the minimum of $f(x, y, z) = (x-1)^2 + (y-2)^2 + (z-3)^2$ subject to $x - y + z = 4$ (or $g(x, y, z) = x - y + z$).
$f_x = \lambda g_x \Rightarrow 2(x-1) = \lambda(1)$ [1] $\Rightarrow x - 1 = \frac{\lambda}{2} \Rightarrow x = \frac{\lambda}{2} + 1$
$f_y = \lambda g_y \Rightarrow 2(y-2) = \lambda(-1)$ [2] $\Rightarrow y - 2 = \frac{-\lambda}{2} \Rightarrow y = \frac{-\lambda}{2} + 2$
$f_z = \lambda g_z \Rightarrow 3(z-3) = \lambda(1)$ [3] $\Rightarrow z - 3 = \frac{\lambda}{2} \Rightarrow z = \frac{\lambda}{2} + 3$
$g(x, y, z) = k \Rightarrow x - y + z = 4$ [4]

Subbing the values of x, y and z into equation 4 yields: $(\frac{\lambda}{2} + 1) - (\frac{-\lambda}{2} + 2) + (\frac{\lambda}{2} + 3) = 4$
$\frac{\lambda}{2} + 1 + \frac{\lambda}{2} - 2 + \frac{\lambda}{2} + 3 = 4$
$\Rightarrow \frac{3\lambda}{2} = 4 - 2 = 2 \Rightarrow \lambda = \frac{4}{3}$
For $\lambda = \frac{4}{3}$, we have:

$$\begin{aligned}
x &= \tfrac{\lambda}{2} + 1 = \tfrac{2}{3} + 1 = \tfrac{5}{3} \\
y &= \tfrac{-\lambda}{2} + 2 = \tfrac{-2}{3} + 2 = \tfrac{4}{3} \\
z &= \tfrac{\lambda}{2} + 3 = \tfrac{2}{3} + 3 = \tfrac{11}{3}
\end{aligned}$$

By the geometry of the problem, the farthest point will be infinite, so $(\frac{5}{3}, \frac{4}{3}, \frac{11}{3})$ is the closest point.

6.16.3 Example 3

Find the dimensions of the rectangular box with the largest volume if the total surface area is given as 64cm^2. (If 64cm^2 of material is available to make a rectangular box, find the largest possible volume of the box.)

Solution Let x, y, z be the length, width and height of the box respectively. We have to minimize the volume: $V = xyz,\ x \geq 0,\ y \geq 0,\ z \geq 0$ subject to the surface area $SA = 64 \Rightarrow xy + xy + yz + yz + xz + xz = 64$
$\Rightarrow 2xy + 2yz + 2xz = 64 \Rightarrow xy + yz + xz = 32$

For the Lagrange multipliers, we have to maximize $f(x, y, z) = xyz$ subject to $xy + yz + xz = 32$, where $f(x, y, z) = V$ and $g(x, y, z) = xy + yz + xz$.
$f_x = \lambda g_x \Rightarrow yz = \lambda(y + z)$ [1]
$f_y = \lambda g_y \Rightarrow xz = \lambda(x + z)$ [2]
$f_z = \lambda g_z \Rightarrow xy = \lambda(y + x)$ [3]
$g(x, y, z) = k \Rightarrow xy + yz + xz = 32$ [4]

Multiply [1] by $x \Rightarrow xyz = \lambda(xy + xz)$
Multiply [2] by $y \Rightarrow xyz = \lambda(xy + zy)$
Equating them: $\lambda(xy + xz) = \lambda(xy + zy)$
$\lambda(xy + xz - xy - zy) = 0$
$\lambda(xz - zy) = 0$
$\lambda z(x - y) = 0 \Rightarrow \lambda = 0,\ z = 0,\ x = y$
When $\lambda = 0$, equations [1], [2] and [3] give $x = 0$ or $y = 0$ or $z = 0 \Rightarrow V = 0$
When $z = 0$, then $V = 0$.

Alternate path to these results:
Equation [1] becomes $\lambda = \frac{yz}{y+z}$ if $y + z \neq 0$, $y + z = 0$ if $y = 0$ and $z = 0$.
Equation [2] becomes $\lambda = \frac{xz}{x+z}$ if $x + z \neq 0$, $x + z = 0$ if $x = 0$ and $z = 0$.
Equate: $\frac{yz}{y+z} = \frac{xz}{x+z} \Rightarrow xyz + xz^2 = xyz + yz^2 \Rightarrow xz^2 + yz^2 = 0 \Rightarrow x^2(x - y) = 0$
Also gives $z = 0$ and $x = y$.

Obviously $z \neq 0$ because that would have the opposite effect we're trying to have! We want to **maximize** the volume.
Multiply [3] by $z \Rightarrow xyz = \lambda(yz + xz)$
Equating [2] and the above: $\lambda(xy + zy) = \lambda(yz + xz) \Rightarrow \lambda(xy - xz) = 0 \Rightarrow \lambda x(y - z) = 0$
$\lambda = 0$ (already done), $x = 0$, $z = y$.
When $x = 0$, then $V = 0$.

However, sub in $x = y$ and $z = y$ to get $yy + yy + yy = 32 \Rightarrow 3y^2 = 32 \Rightarrow y^2 = \frac{32}{3} \Rightarrow y = \sqrt{\frac{32}{3}}$
Note: y cannot be negative.
We thus have $x = y = z = \sqrt{\frac{32}{3}}$
Thus $V = \sqrt{\frac{32}{3}}\sqrt{\frac{32}{3}}\sqrt{\frac{32}{3}} = (\sqrt{\frac{32}{3}})^3\text{cm}^3$.

7 (Chapter 15) Multiple Integrals

A reminder about integrals: an integral is only with respect to one variable. So $\int_a^b f(x)dx$ is valid, but $\int_a^b f(x)dy$ is not.

7.1 (15.1) Double Integrals Over Rectangles

We have $\int\int_R f(x,y)dA = \lim_{m,n\to\infty} \sum_{i=1}^n \sum_{j=1}^m f(x_i,y_j)\Delta x\Delta y$ where:
$R = \{(x,y)|a \le x \le b,\ c \le y \le d\}$

7.1.1 Iterated Integral

Again we have $R = \{(x,y)|a \le x \le b,\ c \le y \le d\}$ where a,b,c,d are constants.
$\int\int_R f(x,y)dA = \int_{x=a}^b \left(\int_{y=c}^d f(x,y)dy\right) dx$ where $dA = dxdy$ or $dA = dydx$ and $\left(\int_{y=c}^d f(x,y)dy\right)$ is the partial integration w.r.t. y (treat x as a constant).

Alternate: $\int\int_R f(x,y)dA = \int_{y=c}^d \int_{x=a}^b f(x,y)dx\ dy$, where $\int_{x=a}^b f(x,y)dx$ is the partial integration w.r.t. x (treat y as a constant).

Clairaut's theorem states $f_{xy} = f_{yx}$
Similarly, we have **Fubinacci's theorem** stating: $\int_{y=c}^d \int_{x=a}^b f(x,y)dx\ dy = \int_{x=a}^b \int_{y=c}^d f(x,y)dy\ dx$
We can decide the order to make our calculations easier.

7.1.2 Example 1

Evaluate $\int_0^1 \int_1^2 \frac{xe^x}{y} dy\ dx$

Solution $\int_0^1 \int_1^2 \frac{xe^x}{y} dy\ dx = \left(\int_0^1 xe^x dx\right)\left(\int_1^2 \frac{1}{y} dy\right)$ (treated the x terms as a constant in the inner integral)
$= \left(xe^x - e^x \Big|_0^1\right)\left(\ln|y| \Big|_1^2\right)$
$= \left[(e-e) - (0-e^0)\right][\ln 2 - \ln 1]$
$= \ln 2$

Alternate $= \int_0^1 xe^x \ln|y| \Big|_{y=1}^2 dx$
$= \int_0^1 xe^x(\ln|2| - \ln|1|)dx$
$= \int_0^1 xe^x \ln|2| dx$
$= \ln|2| \int_0^1 xe^x dx$ Let $u = x$, $du = dx$, $v = e^x$, $dv = e^x dx$, $xe^x - \int e^x dx = xe^x - e^x$
$= \ln|2| \left[xe^x - e^x \Big|_{x=0}^1\right] = \ln|2| \left[(1e^1 - e^1) - (0e^0 - e^0)\right] = \ln 2(1) = \ln 2$

7.1.3 Example 2

Evaluate $\int\int_R x\cos(xy)dA$ where $R = \{(x,y)|0 \le x \le \pi,\ 1 \le y \le 2\}$

Solution $\int\int_R x\cos(xy)dA = \int_{x=0}^\pi y = 12x\cos(xy)dy\ dx$
$= \int_0^\pi x \frac{\sin(xy)}{x} \Big|_{y=1}^2 dx$
$= \int_0^\pi (\sin(2x) - \sin x)dx = \frac{-\cos 2x}{2} - (-\cos x) \Big|_{x=0}^\pi$
$= \left(\frac{-\cos 2\pi}{2} + \cos\pi\right) - \left(\frac{-\cos 0}{2} + \cos 0\right)$
$= \left(\frac{-1}{2} + (-1)\right) - \left(\frac{-1}{2} + 1\right)$
$= \frac{-3}{2} - \frac{1}{2} = \frac{-4}{2} = -2$

Could also try $\int_1^2 \int_0^\pi x\cos(xy)dx\ dy$: let $u = x$, $du = dx$, $v = \frac{\sin(xy)}{y}$, $dv = \cos(xy)dx$. Try it yourself.

7.1.4 Area and Volume

We have equations for area and volume. For area, only two dimensions are considered. So we have $A = \int_a^b f(x)dx$ (if $f(x) \geq 0$ on $[a, b]$).
For volume, we have three dimensions, so we have $V = \int\int_R f(x, y)dA$ (if $f(x, y) \geq 0$ on R).

7.1.5 Example 3

Find the volume of the solid bounded by the elliptic paraboloid $z = 1 + (x-1)^2 + 4y^2$, the planes $x = 3$, $y = 2$ and the coordinate planes.
(Note: $z = f(x, y)$)

Solution $V = \int\int_R f(x, y)dA = \int_0^3 \int_0^2 (1 + (x-1)^2 + 4y^2)dy\ dx$

$V = \int_0^3 y + (x-1)^2 y + \frac{4y^3}{3}\Big|_0^2 dx = \int_0^3 2 + (x-1)^2 2 + \frac{4(2)^3}{3} - (0 + 0 + 0)dx$

$= \int_0^3 \left[\frac{38}{3} + 2(x-1)^2\right] dx$

$= \frac{38}{3}x + \frac{2(x-1)^3}{3}\Big|_0^3$

$= \left(\frac{38}{3}(3) + \frac{2(3-1)^3}{3}\right) - \left(0 + \frac{2(-1)^3}{3}\right)$

$= \frac{38(3)}{3} + \frac{16}{3} + \frac{2}{3} = 38 + \frac{18}{3} = 44$

7.2 (15.2) Double Integrals Over General Regions

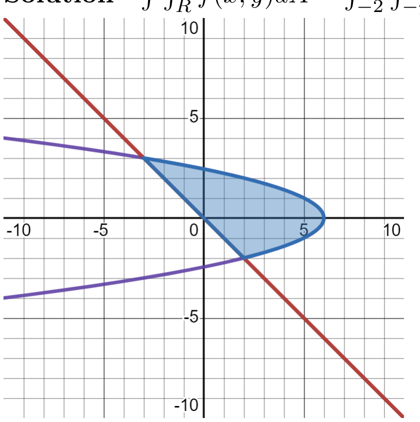
For some region where $y = h(x)$ is the upper portion and $y = g(x)$ is some lower portion, then use:
$\int\int_R f(x, y)dA = \int_a^b \int_{g(x)}^{h(y)} f(x, y)dy\ dx$

If you have some sideways function where $x = h(y)$ is the "upper" portion and $x = g(y)$ is the "lower", then use:
$\int\int_R f(x, y)dA = \int_c^d \int_{g(y)}^{h(y)} f(x, y)dx\ dy$
The outer integral **always** has constant boundaries.

7.2.1 Example 1

Sketch the region and write an iterated integral of $\int\int_R f(x, y)dA$ where R is the region bounded by $x = 6 - y^2$ and $y = -x$.

Solution $\int\int_R f(x, y)dA = \int_{-2}^3 \int_{-y}^{6-y^2} f(x, y)dx\ dy$. The function can be illustrated as such:



7.2.2 Example 2

Evaluate $\int\int_D (x+y)dA$ where D is the region bounded by $y=\sqrt{x}$ and $y=x^2$

Solution $\int\int_D (x+y)dA = \int_0^1 \int_{x^2}^{\sqrt{x}} (x+y)dydx$
$= \int_0^1 xy + \frac{y^2}{2}\Big|_{y=x^2}^{\sqrt{x}} dx$
$= \int_0^1 \left(x\sqrt{x} + \frac{\sqrt{x}^2}{2}\right) - \left(xx^2 + \frac{(x^2)^2}{2}\right) dx$
$= \int_0^1 \left(x^{\frac{3}{2}} + \frac{x}{2} - x^3 - \frac{x^4}{2}\right) dx$
$= \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{x^2}{4} - \frac{x^4}{4} - \frac{x^5}{2(5)}\Big|_0^1$
$= \frac{2}{5} + \frac{1}{4} - \frac{1}{4} - \frac{1}{10} = \frac{4-1}{10} = \frac{3}{10}$

7.2.3 Example 3

Evaluate $\int\int_D 2xydA$, where D is the triangular region with vertices $(0,0)$, $(1,2)$ and $(0,3)$.

Solution Use the equation $m = \frac{y_2-y_1}{x_2-x_1}$ for the points $(0,0)$ and $(1,2)$ to get $y - y_1 = m(x - x_1)$ to get $y = 2x$, then do the same with $(0,3)$ and $(1,2)$ to get the equation $y = -x+3$
We then get: $\int\int_D 2xydA = \int_0^1 \int_{2x}^{-x+3} 2xydydx$
$= \int_0^1 2x\frac{y^2}{2}\Big|_{2x}^{-x+3} dx$
$= \int_0^1 \left[x(-x+3)^2 - x(2x)^2\right] dx$
$= \int_0^1 \left[x(x^2-6x+9) - x(4x^2)\right] dx$
$= \int_0^1 (x^3 - 6x^2 + 9x - 4x^3)dx$
$= \int_0^1 (-3x^3 - 6x^2 + 9x)dx$
$= \frac{-3x^4}{4} - \frac{6x^3}{3} + \frac{9x^2}{2}\Big|_0^1$
$= \left(\frac{-3}{x} - 2 + \frac{9}{2}\right) - (0)$
$= \frac{-3-8+18}{4} = \frac{7}{4}$

7.2.4 Example 4

Evaluate $\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)dydx$

Solution We cannot integrate $\int siny^3dy$. Therefore, we must reverse the order of integration.
$\int_0^1 \int_{x^2 \to y=x^2}^{1 \to y=1} x^3 \sin(y^3)dydx = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)dxdy$
$= \int_0^1 \frac{x^4}{4}\sin(y^3)\Big|_0^{\sqrt{y}} dy = \int_0^1 \frac{(\sqrt{y})^4}{4}\sin(y^3) - 0dy$
$= \int_0^1 \frac{y^2}{4}\sin(y^3)dy$, let $u = y^3$, $du = 3y^2dy$
$= \int \frac{1}{4}\sin udu = \frac{-1}{12}\cos(y^3)\Big|_0^1 = \frac{-1}{12}(\cos 1 - \cos 0) = \frac{-1}{12}(\cos 1 - 1)$

7.3 Applications of Double Integral

Area $= \int\int_D dA$
Volume $= \int\int_D^(z_{upper} - z_{lower})dA$

7.3.1 Example 5

Use double integrals to compute the area of the region bounded by $y = x^2$ and $y = 2 - x$

Solution Calculate the POIs of the equations: $x^2 = 2 - x \Rightarrow x^2 + x - 2 = 0 \Rightarrow (x+2)(x-1) = 0 \Rightarrow x = -2, 1$

Now, Area $= \int_{-2}^{1} \int_{x^2}^{2-x} dydx = \int_{-2}^{1} y \Big|_{x^2}^{2-x} dx$

$= \int_{-2}^{1} (2 - x - x^2)dx = 2x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-2}^{1}$

$= (2 - \frac{1}{2} - \frac{1}{3}) - (2(-2) - \frac{(-2)^2}{2} - \frac{(-2)^3}{3}) = \frac{7}{6} - \frac{-10}{3} = \frac{7+20}{6} = \frac{9}{2}$

7.4 Midterm on July 4th, CHN G133

Covers sections 12.5, 12.6, 13.1-13.4, 14.1-14.8

- Sction 12.5 (1 or 2 questions):
  - Equation of a line: $\vec{r} = \vec{r_0} + t\vec{v}$, in the case of tangent line: $\vec{V} = \vec{r}(t_0)$ where $\vec{V}$ is the direction vector.
  - Equation of a plane: $a(x - x_0) + b(y - y_0) + c(z - z_0) = 0$, in the case of a tangent plane: $\vec{n} = \vec{\bigtriangledown} f$ (sections 14.4 and 14.6)
- Section 13.1 (2 or 3 questions):
  - 13.1: Domain $\vec{r}(t) =< f(t), g(t), h(t) >$ and limits $\lim \vec{r}(t) =< \lim f, \lim g, \lim h >$
  - 13.2: Derivatives $\vec{r}'(t) =< f'(t), g'(t), h'(t) >$ and integrals $\int \vec{r}(t)dt =< \int fdt, \int gdt, \int hdt >$ (covered indirectly).
  - Unit tangent vector $\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$ and unit normal vector $\vec{N} = \frac{\vec{T}'(t)}{|\vec{T}'(t)|}$
- Section 13.3 (1 question): Arc length $= \int_a^b |\vec{r}'(t)|dt$
- 1 question: Curvature $K = \frac{|\vec{T}'(t)|}{|\vec{r}'(t)|}$ or $K = \frac{|\vec{r}' \times \vec{r}''|}{|\vec{r}'|^3}$
- Chapter 3 (section 14.2) (4,5 or 6 questions): limits and continuity:
  - Sub in $x = a$ and if we get an answer, we get the limit.
  - If it is $\frac{0}{0}$, then we need to do any of the following: factorization, rationalization, polar coordinates, etc.
  - We can show two paths with different answers to show that the limit does not exist.
  - L'Hopital's rule only works with one variable.
  - You can use polar coordinates to potentially show if the limit depends on θ, then the limit DNE.
- Section 14.7: local maximum/minimum, saddle point:
  - If $f_x = 0$ and $f_y = 0$, then you have a critical point.
  - $D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$. If $D < 0 \Rightarrow$ saddle point, otherwise it is a local minimm or maximum.
  - If $D > 0$ and $f_{xx} > 0$, then it is a local minimum. If $f_{xx} < 0$ then it is a local maximum.
- Optimization problem (1 question):
  - You can eliminate z and use section 14.7.
  - If not, use Lagrange multipliers: find the maximum or minimum values of $f(x \dots) = \dots$ subject to *dots*.
- Section 12.6: Quadric surfaces (no direct questions).
- Section 14.3: partial derivatives (covered indirectly).
- Section 14.4: tangent plane and linear approximation: $f(x, y) \approx f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$
- Section 14.6: directional derivative: $D_u f = \vec{\bigtriangledown} f \cdot \vec{u}$ where $\vec{u}$ is the unit vector.