Vector Calculus

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1 Construction of a Line

The equation of a line is $y = mx + b$, where y is the dependent variable and x is the dependent variable. Now, the equation of a line passing through (x_0, y_0, z_0) (or $\vec{r_0}$) and is parallel to the vector $\vec{v}$ is $\vec{r} = \vec{r_0} + t\vec{v}, t \in R$

2 Planes

The equation of the plane is $ax + by + cz = d$ where $\vec{n} =< a, b, c >$ is the normal vector of the plane.

2.1 Cylinders

A **cylinder** is a surface that consists of all lines parallel to a line and passing through a.

Examples: Identify and sketch the surface

1. $x^2 + y^2 = 4 \rightarrow$ **a circle in 2 dimensions** $x^2 + y^2 = 4$ is a cylinder in 3 dimensions (a surface where one variable is missing is a cylinder, the missing variable is the axis).

2. $y^2 + z^2 = 9$ A cyliinder with x-axis as the axis.

3. $z = x^2$ A cylinder with y-axis as the axis.

3 Quadrics

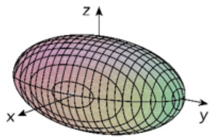
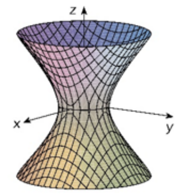
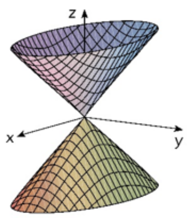
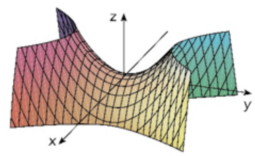
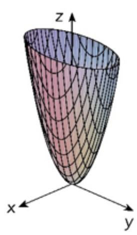
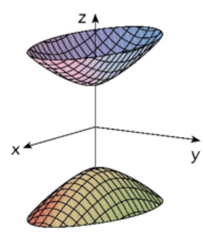
A quadric in two dimensions is:

1. A parabola $y = x^2$ or $x = y^2$
2. An ellipse: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$
3. A hyperbola: $\frac{x^2}{a^2} - y^2 b^2 = 1$

A trace is the curve of the intersection of the surface with the coordinate plane $\rightarrow$ 3 traces.

3.1 Quadric Surface

$Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fzy + Gx + Hy + Iz + J = 0$
In this course, we need to know 6 quadric surfaces.

Ellipsoid 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ "A bunch of ellipses stacked together" Special case: If $a=b=c$, we have a sphere	Hyperboloid of One Sheet 	$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are hyperbolas. *Whichever variable is negative corresponds to the axis of symmetry
Cone 	$\frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are hyperbolas, except when $x=0$ or $y=0$, then the traces are pairs of lines	Hyperbolic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$ In the xy plane, the traces are hyperbolas. In the xz or yz plane, the traces are parabolas.
Elliptic Paraboloid 	$\frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$ In the xy plane, the traces are ellipses. In the xz or yz planes, the traces are parabolas.	Hyperboloid of Two Sheets 	$-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ In the xy plane, the traces are ellipses if $z>c$ or $z<-c$ In the xz or yz planes, the traces are hyperbolas.

We can actually know some patterns for the rest of the quadric surfaces:

- **Ellipsoid** $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
- **Hyperboloid of One Sheet** $\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$
- **Hyperboloid of Two Sheets** $\Rightarrow -\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$
- **Cone** $\Rightarrow \frac{z^2}{c^2}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$
- **Elliptic Paraboloid** $\Rightarrow \frac{z}{c}=\frac{x^2}{a^2}+\frac{y^2}{b^2}$
- **Hyperbolic Paraboloid** $\Rightarrow \frac{z}{c}=\frac{x^2}{a^2}-\frac{y^2}{b^2}$

Notice the patterns with the equations?

3.1.1 Elliptic Paraboloids

The equation of an elliptic paraboloid is $\frac{z}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2}$ where z is the axis, xy traces are ellipses $(x = 0)$ and both $\frac{x^2}{a^2}$ and $\frac{y^2}{b^2}$ have the same signs.

2 traces is a parabola, 1 trace is an ellipse.

Effectively, the variable with a power of 1 is the axis.

$z = 0 \quad \Rightarrow 0 = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow x = 0, y = 0, z = 0$

$z = k \quad \Rightarrow \frac{k}{c} = \frac{x^2}{a^2} + \frac{y^2}{b^2} \Rightarrow$ ellipses for $k > 0$

3.1.2 Hyperboloid of One Sheet

The variable with the negative sign is the axis. The **xy-trace** is where $z = 0$. The **xz-trace** and **yz-trace** is a hyperbola.

3.1.3 Hyperboloid of Two Sheets

The variable with the positive sign is the axis. The **xy-trace** is an ellipses if $|z| > |c|$

3.1.4 Cone

The variable with the negative sign is the axis. Two traces are hyperbolas, one trace is an ellipse for $k \neq 0$.

3.1.5 Examples: Identify and sketch the surfaces

1. $x^2 + 4y^2 + z^2 = 4 \quad \Rightarrow \frac{x^2}{4} + \frac{y^2}{4} + \frac{z^2}{4} = 1$

3. $z^2 = x^2 + 4y^2 + 64 \quad \Rightarrow -x^2 - 4y^2 + z^2 = 64 \Rightarrow -\frac{x^2}{64} - \frac{y^2}{16} + \frac{z^2}{64} = 1$ (hyperboloid on 2 sheets, axis is z-axis with $c = 8$)

4 Vector Functions

These are chapters 13.1 and 13.2 from last class.

A vector function is $\vec{r}(t) =< f(t), g(t), h(t) >$ such that $f(t) = x$, $g(t) = y$ and $h(t) = z$.

- For 2 dimensions: $\vec{r}(t) =< f(t), g(t) >$
- $\vec{r}\,'(t) =< f'(t), g'(t) >$ or $\vec{r}\,'(t) =< f'(t), g'(t), h'(t) >$

4.1 Arc Length

The equation for length is $|\vec{r}\,'(t)| = \sqrt{(f'(t))^2 + (g'(t))^2}$ for 2 dimensions, or $|\vec{r}\,'(t)| = \sqrt{(f'(t))^2 + (g'(t))^2 + (h'(t))^2}$ for 3 dimensions.

Now, the formula is effectively $\sqrt{(\delta x)^2 + (\delta y)^2}$, where $x = f(t)$ and $\delta x = f'(t)dt$ and similarly for y. Thus the formula is $\sqrt{(f'(t)dt)^2 + (g'(t)dt)^2} = \sqrt{(f'(t))^2 + (g'(t))^2 (dt)^2} = \sqrt{(f'(t))^2 + (g'(t))^2}dt$

Arc Length is: $L = \int_a^b \sqrt{(f'(t))^2 + (g'(t))^2}dt$

Thus, the arc length of a curvee $\vec{r}(t)$ for $a \le t \le b$ is $\int_a^b |\vec{r}\,'(t)|dt$

4.2 Example 1

Find the length of the curve $\vec{r}(t) =< 2\sin^3 t, 2cos^3 t >, 0 \le t \le \frac{\pi}{4}$

Solution: $\vec{r}\,'(t) =< 2(3\sin^2 t)(\cos t), 2(3\cos^2 t)(-\sin t) >$

$|\vec{r}\,'(t)| = \sqrt{(6sin^2 t\cos t)^2 + (-6\cos^2 t\sin t)^2}$

$= \sqrt{36\sin^4 t\cos^2 t + 36\cos^4 t\sin^2 t}$

$= \sqrt{36\sin^2 t\cos^2 t(\sin^2 t + \cos^2 t)}$

$= \sqrt{(}36\sin^2 t\cos^2 t(1))$

$= 6\sin t\cos t$

The arc length $L = \int_a^b |\vec{r}\,'(t)|dt = \int_0^{\frac{\pi}{4}} 6\sin t\cos tdt$

Let $u = \sin t$ so $du = \cos tdt$

Then we have $L = \int_0^{\frac{\pi}{4}} 6udu = \frac{6u^2}{2} = 3u^2$

$= 3\sin^2 t|_{t=0}^{\frac{\pi}{4}} = 3\sin^2 \frac{\pi}{4} - 3\sin^2 0 = 3(\frac{1}{2}) = \frac{3}{2}$

4.3 Example 2

Find the length of the curve $\vec{r}(t) =< t^2, 2t, \ln t >, 1 \le t \le e$

Solution: $\vec{r}\,'(t) =< 2t, 2, \frac{1}{t} >$

$|\vec{r}\,'(t)| = \sqrt{4t^2 + 4 + \frac{1}{t^2}} = \sqrt{\frac{4t^4+4t^2+1}{t^2}} = \frac{\sqrt{(2t^2+1)^2}}{\sqrt{t^2}} = \frac{2t^2+1}{t}$

$L = \int_a^b |\vec{r}\,'(t)|dt = \int_1^e \frac{2t^2+1}{t}dt = \int_1^e \left(\frac{2t^2}{t} + \frac{1}{t}\right) dt$

$L = \int_1^e \left(2t + \frac{1}{t}\right) dt = t^2 + \ln|t| \;|_1^e$

$L = e^2 + \ln|e| - 1^2 - \ln 1 = e^2$

4.4 Curvature

In the last class, the unit tangent vector $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$. Curvature is the magnitude of the rate of change of the unit tangent vector w.r.t. the arc length.

Arc Length Function $S = \int_a^t |\vec{r}\,'(u)|\, du$
Curvature Function $K = \left|\frac{d\vec{T}}{ds}\right|$

- Note: $\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}\frac{ds}{dt} \rightarrow$ chain rule.
  $\frac{d\vec{T}}{dt} = \frac{d\vec{T}}{ds}|\vec{r}\,'(t)| \Rightarrow \left|\frac{d\vec{T}}{ds}\right| = \frac{\left|\frac{d\vec{T}}{dt}\right|}{|\vec{r}\,'(t)|} = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$
- Thus $K = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|}$
- Another formula for $K = \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$

4.5 Example 1

Find the curvature of the following curve: $\vec{r}(t) =< 5\sin t, 3t, 5\cos t >$

Solution: $\vec{r}\,'(t) =< 5\cos t, 3, -5\sin t >$, $|\vec{r}\,'(t) = \sqrt{25\cos^2 t + 9 + 25\sin^2 t} = \sqrt{25(\cos^2 t + \sin^2 t) + 9} = \sqrt{34}$
$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)} = \frac{<5\cos t, 3, -5\sin t>}{\sqrt{34}}$
$\vec{T}\,'(t) = \frac{1}{\sqrt{34}} < -5\sin t, 0, -5\cos t >$
$\left|\vec{T}\,'(t)\right| = \sqrt{\frac{1}{34}\left(25\sin^2 t + 25\cos^2 t\right)} = \sqrt{\frac{25}{34}} = \frac{\frac{5}{\sqrt{34}}}{\sqrt{34}}$ (divide by $\sqrt{34}$ one more time since we're trying to find K)

4.6 Example 2

Find the curvature of the following curve: $\vec{r}\,'(t) =< t, t, 1 + t^2 >$

Solution: $\vec{r}\,'(t) =< 1, 1, 2t >$, $|\vec{r}\,'(t)| = \sqrt{1^2 + 1^2 + 4t^2} = \sqrt{2 + 4t^2}$
$\vec{T}\,'(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)} = \frac{<1,1,2t>}{\sqrt{2+4t^2}} = \left\langle \frac{1}{\sqrt{2+4t^2}}, \frac{1}{\sqrt{2+4t^2}}, \frac{2t}{\sqrt{2+4t^2}} \right\rangle$
$\vec{T}\,'(t) = ...$ (hard to calculate)
Instead, another formula for K: $K = \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}\,'(t)|^3}$
$\vec{r}\,''(t) =< 0, 0, 2 >$
$\vec{r}\,'(t) \times \vec{r}\,''(t) = \begin{bmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 2t \\ 0 & 0 & 2 \end{bmatrix} = \hat{i}(2-0) - \hat{j}(2-0) + \hat{k}(0-0) = 2\hat{i} - 2\hat{j} + 0\hat{k}$ *or* $< 2, -2, 0 >$ (remember cross product in MATH1250)
$K = \frac{|\vec{r}\,'(t)\times\vec{r}\,''(t)|}{|\vec{r}^3|} = \frac{\sqrt{8}}{\left(\sqrt{2+4t^2}\right)^3}$

5 (13.4) Motion in Space

We learned that $\vec{r}(t) =< f(t), g(t), h(t) >$ where $f(t) = x$, $g(t) = y$ and $h(t) = z$.

- If the position vector/function of an object is $\vec{r}(t) =< f(t), g(t), h(t) >$ then the velocity of the object is $\vec{v}(t) = \vec{r}\,'(t)$ and it will be in the direction of the tangent vector $\vec{r}\,'(t)$.
- The speed is $|\vec{v}(t)| = |\vec{r}\,'(t)|$
- The acceleration of the object is $\vec{a}(t) = \vec{v}\,'(t) = \vec{r}\,''(t)$

5.0.1 Example 1

Find the velocity, acceleration and speed of a particle with position vector (function) $\vec{r}(t) =< e^t, t^4, e^{-t} >$

Solution $\vec{v}(t) = \vec{r}\,'(t) =< e^t, 4t^3, -e^{-t} > \rightarrow$ velocity
Acceleration is $\vec{a}(t) = \vec{v}\,'(t) =< e^t, 12t^2, e^{-t} >$
Speed is $|\vec{v}(t)| = \sqrt{(e^t)^2 + (4t^3)^2 + (e^{-t})} = \sqrt{e^{2t} + 16t^6 + e^{-2t}}$

5.0.2 Example 2

Find the velocity, acceleration and speed of a particle with position vector $\vec{r}(t) =< e^t, t^4, e^{-t} >$ at $t = 0$.

Solution From example 1, $\vec{v}(t) =< e^t, 4t^3, -e^{-t} >$
At $t = 0$, the velocity is $\vec{v}(0) =< e^0, 4(0)^3, -e^{-0} > =< 1, 0, -1 >$
The speed at $t = 0$ is $\sqrt{1^2 + 0^2 + (-1)^2} = \sqrt{2}$
$\vec{a}(t) =< e^t, 12t^2, e^{-t} >$
At $t = 0$, $\vec{a}(0) =< e^0, 12(0)^2, e^{-0} > =< 1, 0, 1 >$

5.0.3 Example 3

Given the acceleration vector $\vec{a}(t) =< 2t, 1, t^2 >$ amd the initial velocity is $\vec{v}(0) =< 0, 1, 1 >$ and the initial position vector is $\vec{r}(0) =< 2, 0, -1 >$, find the velocity and position vectors of the particle.

Solution The velocity is $\vec{v}(t) = \int \vec{a}(t)dt = \int < 2t, 1, t^2 > dt$
$=< \frac{2t^2}{2} + C_1, t + C_2, \frac{t^3}{3} + C_3 >$ or $< t^2, t, \frac{t^3}{3} > + < C_1, C_2, C_3 > (\vec{C})$
$\vec{v}(0) =< 0, 1, 1 > \Rightarrow < 0, 1, 1 > =< 0 + C_1, 0 + C_2, 0 + C_3 > \Rightarrow C_1 = 0, C_2 = 1, C_3 = 1$
Thus the velocity is $\vec{v}(t) =< t^2, t + 1, \frac{t^3}{3} + 1 >$

The position vector is $\vec{r}(t) = \int \vec{v}(t)dt = \int < t^2, t + 1, \frac{t^3}{3} + 1 > dt$
$\vec{r}(t) =< \frac{t^3}{3} + d_1, \frac{t^2}{2} + t + d_2, \frac{t^4}{12} + t + d_3 >$
$\vec{r}(0) =< 2, 0, -1 > \Rightarrow < 2, 0, -1 > =< 0 + d_1, 0 + d_2, 0 + d_3 > \Rightarrow d_1 = 2, d_2 = 0, d_3 = -1$
So $\vec{r}(t) =< \frac{t^3}{3} + 2, \frac{t^2}{2} + t, \frac{t^4}{12} + t - 1 >$

6 Functions of Several Variables (or Multivariable Functions)

Multivariable Functions are functions with at least 2 independent variables. In **chapters 14 and 15**, we cover domains, limits, continuity, derivatives and applications, integration and applications.

- One prominent example is the volume functions $V = xyz$ and $V = \pi r^2 h$

6.1 Functions of 2 Variables

x and y are independent variables, and the domain will be in R^2: $D = \{(x, y) \mid \text{properties}\}$
A function in 2 variables, $z = f(x, y)$ is a rule that assigns to each $(x, y) \in D$ a unique value z in R.

- The domain D is R^2 (inputs).
- The range is a subset of R (outputs).
- $z = f(x, y) \rightarrow$ explicit.
- $f(x, y, z) = 0 \rightarrow$ implicit.

The vertical line test (VLT) can be used to check if a single-variable relation is a function. This is similar for 2-variable functions, the VLT will instead draw lines parallel to the z-axis. The relation will be a function if every vertical line crosses the surface of a function only once.

6.1.1 Example 1

Find the domain of $f(x, y) = \ln(x + y)$

Solution We need $x + y > 0 \Rightarrow y > -x$
Thus the domain of $f = \{(x, y) \mid y > -x\}$

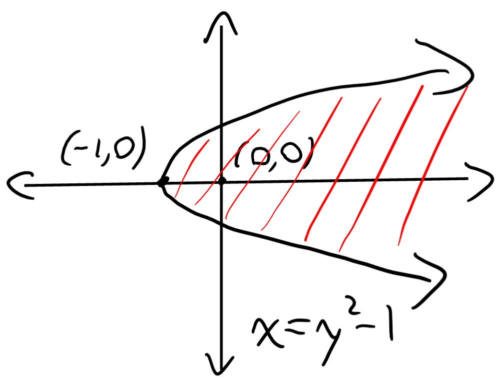
6.1.2 Example 2

Find the domain of $f(x, y) = \sqrt{1 + x - y^2}$

Solution We need $1 + x - y^2 \geq 0 \Rightarrow x \geq y^2 - 1$
Thus the domain of $f = \{(x, y) \mid x \geq y^2 - 1\}$

6.2 Sketch the Domains

Draw $x = y^2 - 1$



Choose a test point, (0,0)
$x \geq y^2 - 1 \Rightarrow 0 \geq 0 - 1 \Rightarrow 0 \geq -1$ True $\Rightarrow$ Shade the side that has (0,0)
Graphing the surface $z = f(x, y)$ will be in 3 dimensions.

6.2.1 Example 1

Sketch the graph of the following function: $f(x, y) = 8 - x - 2y$

Solution $z = 8 - x - 2y \Rightarrow x + 2y + z = 8 \Rightarrow$ a plane
x-intercept: $y = 0, z = 0 \Rightarrow x = 8 \Rightarrow (8, 0, 0)$
y-intercept: $x = 0, z = 0 \Rightarrow 2y = 8 \Rightarrow y = 4 \Rightarrow (0, 4, 0)$
z-intercept: $x = 0, y = 0 \Rightarrow z = 8 \Rightarrow (0, 0, 8)$

6.2.2 Example 2

Sketch the graph of the following function: $f(x, y) = \sqrt{2x^2 + y^2}$

Solution $z = \sqrt{2x^2 + y^2} \Rightarrow z^2 = 2x^2 + y^2 \Rightarrow 2x^2 + y^2 - z^2 = 0 \Rightarrow$ a cone

6.2.3 Example 3

Sketch the graph of the following function: $f(x, y) = -\sqrt{2x^2 + y^2}$

6.3 (14.1) Functions

In last class, we did functions of 2 variables, such as $z = f(x, y)$ where $(x, y) \in D$ where D is the domain in R^2. However, this idea can be extended to more than 2 variables. A 3-variable function would have $w = f(x, y, z)$ (explicit form) or $f(x, y, z, w) = 0$ (implicit form).

6.3.1 Example

Find the domain of $f(x, y, z) = \frac{1}{\sqrt{16-x^2-y^2-z^2}}$ and sketch it.

Solution For the domain, we need $16 - x^2 - y^2 - z^2 > 0$.
$D = \{(x, y, z) \mid 16 - x^2 - y^2 - z^2 > 0\}$ (or $x^2 + y^2 + z^2 < 16$)

To sketch, draw $x^2 + y^2 + z^2 = 16$ (a sphere with a radius of $\sqrt{16} = 4$). $(0, 0, 0) \Rightarrow 0 + 0 + 0 < 16 \Rightarrow 0 < 16$

6.4 Limits and Continuity

$\lim\limits_{x \to a} f(x) = L$ if $f(x) \to L$ as $x \to a$ and the righthand/lefthand limits are equal to L.
$\lim\limits_{(x,y) \to (a,b)} f(x, y) = L$ if $f(x, y) \to L$ as $(x, y) \to (a, b)$ along every possible path.
If two paths give different answers, then this proves the limit DNE. To find the limit, when it exists, we substitute $x = a$ and $y = b$. If we get an answer, that is the limit. If we get $\frac{0}{0}$, $\frac{\infty}{\infty}$, $0 \times \infty$,$\infty - \infty$, 0^∞, ∞^0, 1^∞ or any other indeterminate form, then do something:

$$\begin{cases} \text{To show the limit DNE, show two paths with different limits. This can be done with polar coordinates} \\ \quad x = \gamma \cos\theta, y = \gamma \sin\theta \\ \text{Factorization.} \\ \text{Rationalization.} \end{cases}$$

Note: for the indeterminate forms $0 \times \infty$,$\infty - \infty$, 0^∞, ∞^0 and 1^∞, try changing them to $\frac{0}{0}$ or $\frac{\infty}{\infty}$ and use L'Hopital's rule. **However:** there is no L'Hopital's rule for 2 variables.

6.4.1 Example 1

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y) \to (0,0)} \frac{x^2+y^2}{2+xy}$

Solution $\lim\limits_{(x,y) \to (0,0)} \frac{x^2+y^2}{2+xy} = \frac{0+0}{2+0} = \frac{0}{2} = 0$

6.4.2 Example 2

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y) \to (4,1)} \frac{x^2-6xy+8y^2}{2x-8y}$

Solution $\lim\limits_{(x,y) \to (4,1)} \frac{x^2-6xy+8y^2}{2x-8y} = \frac{4^2-6(4)(1)+8(1)^2}{2(4)-8(1)} = \frac{16-24+8}{8-8} = \frac{0}{0}$
This did not work, so we can try factorization: $\lim\limits_{(x,y) \to (4,1)} \frac{(x-4y)(x-2y)}{2(x-4y)} = \lim\limits_{(x,y) \to (4,1)} \frac{x-2y}{2} = \frac{4-2(1)}{2} = \frac{2}{2} = 1$

6.4.3 Example 3

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y) \to (0,0)} \frac{x+2y}{\sqrt{x+2y+4}-2}$

Solution $\lim\limits_{(x,y) \to (0,0)} \frac{x+2y}{\sqrt{x+2y+4}-2} = \frac{0+0}{\sqrt{4}-2} = \frac{0}{0}$
This did not work, so let's try rationalization: $\lim\limits_{(x,y) \to (0,0)} \frac{x+2y}{\sqrt{x+2y+4}-2} \cdot \frac{\sqrt{x+2y+4}+2}{\sqrt{x+2y+4}+2} = \lim\limits_{(x,y) \to (0,0)} \frac{(x+2y)(\sqrt{x+2y+4}+2)}{x+2y+4-4}$
$= \lim\limits_{(x,y) \to (0,0)} \frac{(x+2y)(\sqrt{x+2y+4}+2)}{x+2y} = \lim\limits_{(x,y) \to (0,0)} \sqrt{x+2y+4} + 2 = \sqrt{0+0+4} + 2 = 2 + 2 = 4$

6.4.4 Example 4

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2}$

Solution $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2} = \frac{0+\sin^2 0}{0+0} = \frac{0}{0}$
This did not work, so let's do some tricks:
Along the x-axis (or along $y=0$)$\Rightarrow \lim\limits_{x\to 0} \frac{x^2+\sin^2 0}{2x^2+0^2} = \lim\limits_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$
Along the y-axis (or along $x=0$)$\Rightarrow \lim\limits_{y\to 0} \frac{0+\sin^2 y}{0+y^2} = \lim\limits_{y\to 0} \frac{\sin^2 y}{y^2} = \frac{0}{0}$
This still doesn't work, but since we only have one variable, we can use L'Hopital's rule:
$\lim\limits_{y\to 0} \frac{\sin^2 y}{y^2} = \lim\limits_{y\to 0} \frac{2\sin y\cos y}{2y} = \lim\limits_{\sin y(-\sin y)+\cos y\cos y} 1 = 0+1 = 1$
Or, we can break up the limit like this: $\lim\limits_{y\to 0} \frac{\sin^2 y}{y^2} = \lim\limits_{y\to 0} \frac{\sin y}{y} \cdot \frac{\sin y}{y} = 1\cdot 1 = 1$

6.4.5 Example 5

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+5y^4}$

Solution $\lim\limits_{(x,y)\to(0,0)} \frac{6x^3y}{2x^4+5y^4} \to \frac{0}{0}$
Along $x=0 \Rightarrow \lim\limits_{y\to 0} \frac{0}{5y^4} = \lim\limits_{y\to 0} 0 = 0$ (similarly, along $y=0 \Rightarrow \lim\limits_{x\to 0} \frac{0}{2x^4} = \lim\limits_{x\to 0} 0 = 0$)
Along $y=x \Rightarrow \lim\limits_{x\to 0} \frac{6x^3x}{2x^4+5x^4} = \lim\limits_{x\to 0} \frac{6x^4}{7x^4} = \frac{6}{7}$
Since two paths have different answers, the limit DNE.
Alternate answer: Along $y=mx \Rightarrow \lim\limits_{x\to 0} \frac{6x^3mx}{2x^4+6m^4x^4} = \lim\limits_{x\to 0} \frac{6mx^4}{x^4(2+5m^4)} = \frac{6m}{2+5m^4}$
Notice the answer depends on m. Thus the limit DNE.

6.4.6 Example 6

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y)\to(0,0)} \frac{3xy^3}{x^2+y^2}$

Solution $\lim\limits_{(x,y)\to(0,0)} \frac{3xy^3}{x^2+y^2} \to \frac{0}{0}$
Use polar coordinates: $x = r\cos\theta, y = r\sin\theta, r\to 0 as (x,y)\to(0,0)$
Thus $x^2+y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta+\sin^2\theta) = r^2$
So $\lim\limits_{(x,y)\to(0,0)} \frac{3xy^3}{x^2+y^2} = \lim\limits_{r\to 0} \frac{3(r\cos\theta)(r^3\sin^3\theta)}{r^2}$
• If the answer does not depend on θ, then we get the limit. However if it depends on θ, then the limit DNE.
$\lim\limits_{r\to 0} \frac{3(r\cos\theta)(r^3\sin^3\theta)}{r^2} = \lim\limits_{r\to 0} \frac{3r^4\cos\theta\sin^3\theta}{r^2} = \lim\limits_{r\to 0} 3r^2\cos\theta\sin^3\theta = 3(0)^2\cos\theta\sin^3\theta = 0$
Thus the limit is 0.

6.4.7 Example 7

Evaluate the limit if it exists, or show that the limit DNE. $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+2y^4}$

Solution $\lim\limits_{(x,y)\to(0,0)} \frac{xy^2}{x^2+2y^4} \to \frac{0}{0}$
Along $y=mx \to \lim\limits_{x\to 0} \frac{xm^2x^2}{x^2+2m^4x^4} = \lim\limits_{x\to 0} \frac{m^2x^3}{x^2(1+2m^4x^2)} = \frac{m^2(0)}{1+2m^4(0)} = \frac{0}{1} = 0$
Along $x=y^2$ (or more general, $x=cy^2$)$\to \lim\limits_{y\to 0} \frac{y^2y^2}{(y^2)^2+2y^4} = \lim\limits_{y\to 0} \frac{y^4}{3y^4} = \frac{1}{3} \neq 0$
Thus the limit DNE.

6.5 (14.2) Limits and Continuity

A function $f(x,y)$ is continuous at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$. Similarly, for functions with 3 variables, $f(x,y,z)$ is continuous at (a,b,c) if $\lim\limits_{(x,y,z)\to(a,b,c)} f(x,y,z) = f(a,b,c)$
This basically states that a value exists, limit exists and both of them are equal. Every function is continuous on its domain. Additionally, polynomials and rational functions are continuous on their domain.

6.5.1 Example 1

Find the points where f is continuous: $f(x,y) = \frac{x+y}{\sqrt{x-1}}$

Solution The domain of f is $x - 1 > 0 \Rightarrow x > 1$
Thus f is continuous on R^2 except when $x \leq 1$ or f is continous on $\{(x,y)|x > 1\}$

6.5.2 Example 2

Find the points where f is continuous: $f(x,y) = \frac{e^y+3}{x^2+y^2}$

Solution The domain of f is when $x^2 + y^2 \neq 0$
Thus f is continuous on R^2 except at $(0,0)$ (or when $x = 0$ and $y = 0$)
f is continuous on $\{(x,y)(x,y) \neq (0,0)\}$

6.5.3 Example 3

Find the points where f is continuous: $f(x,y) = \begin{cases} \frac{2x^2+y^2}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$

Solution f is continuous when $(x,y) \neq (0,0)$
Now we check if f is continuous at $(0,0)$ by checking if $\lim\limits_{(x,y)\to(0,0)} f(x,y) = f(0,0)$
First, we know $f(0,0) = 0$ exists as a value.
Now, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \frac{2x^2+y^2}{x^2+y^2} \Rightarrow \frac{0}{0}$
Along $x = 0 \Rightarrow \lim\limits_{y\to 0} \frac{0+y^2}{0+y^2} = \lim\limits_{y\to 0} \frac{y^2}{y^2} = \lim\limits_{y\to 0} 1 = 1$
Along $y = 0 \Rightarrow \lim\limits_{x\to 0} \frac{2x^2+0}{x^2+0} = \lim\limits_{x\to 0} \frac{2x^2}{x^2} = \lim\limits_{x\to 0} 2 = 2$
Thus the limit DNE $\Rightarrow f$ is not continuous at (0,0)
f is continuous on R^2 except at $(0,0)$

6.6 (14.3) Partial Derivatives

Suppose y is fixed, say $y = b$. Let $g(x) = f(x,b)$ where b is some constant. Then $g'(x) = \lim\limits_{h\to 0} \frac{g(x+h)-g(x)}{h}$ and $\frac{df}{dx} = \lim\limits_{h\to 0} \frac{f(x+h,b)-f(x,b)}{h}$
The 1st order partial derivatives of $f(x,y)$ are:

- $f_x = \frac{\partial f}{\partial x} \Rightarrow$ treat y as a constant.
- $f_y = \frac{\partial f}{\partial y} \Rightarrow$ treat x as a constant.

6.6.1 Example 1

Find the 1st partial derivative of $f(x,y) = y^2 \ln x + x \sin^2 y + y^3$

Solution $f_x = \frac{df}{dx} = y^2(\frac{1}{x}) + (1)\sin^2 y + 0 = \frac{y^2}{x} + \sin^2 y$
$f_y = \frac{df}{dy} = 2y \ln x + x \cdot 2 \sin y \cos y + 3y^2$ (note that $\frac{d}{dy} \sin^2 y = 2 \sin y \cos y$)

6.6.2 Example 2

Find the 1st partial derivative of $f(r,t) = t^2e^r + \frac{r^2}{t}$

Solution $f_r = \frac{df}{dr} = t^2e^r + \frac{2}{t}r$
$f_t = \frac{df}{dt} = 2te^r + r^2(-\frac{1}{t^2})$

6.6.3 Example 3

Find the 1st partial derivative of $f(x,y) = \frac{x^2+y}{x+1}$

Solution $f_x = \frac{\partial f}{\partial x} = \frac{(2x)(x+1)-(x^2+y)(1)}{(x+1)^2}$ or $\frac{2x^2+2x-x^2-y}{(x+y)^2}$ (both answers are acceptable)
$f_y = \frac{\partial f}{\partial y} = \frac{1}{x+1}(0+1) = \frac{1}{x+1}$

6.6.4 Example 4

Find the 1st partial derivative of $f(x,y,z) = xyz + x^2\ln(2y-z)$

Solution $f_x = (1)yz + 2x\ln(2y-z)$
$f_y = \frac{\partial f}{\partial y} = x(1)z + x^2\frac{1}{2y-z}(2)$
$f_z = \frac{\partial f}{\partial z} = xy(1) + x^2\frac{1}{2y-z}(-1)$

6.7 Higher Order Derivatives

Higher order derivatives can be $\frac{\partial}{\partial y}(f_x)$ and $\frac{\partial}{\partial x}(f_x)$, or $\frac{\partial}{\partial x}(f_y)$ and $\frac{\partial}{\partial y}(f_y)$

- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}$
- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}$
- $\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}$
- $\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}$

6.7.1 Example 1

Find all four 2nd order partial derivatives of $f(x,y) = 3x^2y^3 + 5x^4y$

Solution $f_x = \frac{\partial f}{\partial x} = 3(2x)y^3 + 5(4x^3)y = 6xy^3 + 20x^3y$
$f_y = 3x^2(3y^2) + 5x^4(1) = 9x^2y^2 + 5x^4$
$f_{xx} = 6(1)y^3 + 20(3x^2)y$
$f_{yx} = 9(2x)y^2 + 20x^3$
$f_{xy} = 6x(3y^2) + 20x^3(1)$ (also the same as f_{yx})
$f_{yy} = 9x62(2y) + 0$

6.8 Clairaut's Theorem

Theorem 6.1 *Suppose f is defined on a disk that contains (a,b) and f_{xy} and f_{yx} are continuous on D. Then* $f_{xy}(a,b) = f_{yx}(a,b)$

6.8.1 More Examples of Partial Derivatives

Let $f(x,y) = y\tan 2x$, find f_{xx} and f_{yx}

Solution $f_x = y\sec^2 2x(2) = 2y\sec^2 x$
$f_y = (1)\tan 2x$
$f_{xx} = 2y(2\sec 2x)(\sec 2x\tan 2x)(2)$ $\quad$ $f_{yx} = \sec^2 2x(2) = f_{xy}$ (since $f_{xy} = f_{yx}$)

6.8.2 Example 2

Given that $f(x,y,z) = 3x^3 + 7xy\cos z + x^2y^3$, find $f_{xy}(-1,2,0)$ and $f_{xyz}(-1,2,0)$

Solution $f_x = 9x^2 + 7(1)y\cos z + 2xy^3z$
$f_{xy} = 0 + 7(1)\cos z + 2x(3y^2)z = 7\cos z + 6xy^2z$
$f_{xy}(-1,2,0) = 7\cos 0 + 6(-1)(2)^2(0) = 7(1) + 0 = 7$
$f_{xyz} = 7(-\sin z) + 6xy^2(1)$
$f_{xyz}(-1,2,0) = 7(-\sin 0) + 6(-1)(2)^2 = -24$

6.9 (14.5) Chain Rule

You have seen that if we have to find y' where $y = (x^2+1)^3$, we use the chain rule like so:
Let $u = x^2+1$, then $y = u^3 \Rightarrow \frac{du}{dx} = 2x$, $\frac{dy}{du} = 3u^2$
We need $\frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = 3u^2(2x) = 3(x^2+1)^2(2x)$

Now we can expand on this by examining the different cases of the chain rule:

1. Let $z = f(x,y)$ be a differentiable function in x and y, and $x = g(t)$ and $y = h(t)$ are differentiable functions of t. Then, $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$
2. Let $z = f(x,y)$ be a differentiable function of x and y and let $x = g(s,t)$ and $y = h(s,t)$ be differentiable functions of s and t. Then, $\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s}$ and $\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$
3. The general version of the chain rule is as follows: $w = f(x,y,z)$, $x = g(s,t,u,r)$, $y = h(s,t,u,r)$ $z = h(s,t,u,r)$. Then, $\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$
4. This pattern continues.

6.9.1 Example 1

Find $\frac{dz}{dt}$ where $z = \sqrt{x^2+y}$, $x = e^{2t}$, $y = \sin t$

Solution $\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = \frac{x}{\sqrt{x^2+y}}(e^{2t}(2)) + \frac{1}{2\sqrt{x^2+y}}(\cos t)$
More details: $z = (x^2+y)^{\frac{1}{2}}$,
$\frac{\partial z}{\partial x} = \frac{1}{2}(x^2+y)^{-\frac{1}{2}}\cdot(2x)$,
$\frac{\partial z}{\partial y} = \frac{1}{2}(x^2+y)^{-\frac{1}{2}}(1)$

6.9.2 Example 2

Find $\frac{dw}{dt}$ where $w = x^2y + y^3\cos z$, $x = t^2$, $y = t+1$, $z = t^3$

Solution $\frac{dw}{dt} = \frac{\partial w}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial w}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial w}{\partial z}\cdot\frac{dz}{dt} = (2x)y(2t) + (x^2+3y^2\cos z)(1) + y^3(-\sin z)(3t^2)$

6.9.3 Example 3

Let $z = \frac{x}{y}$, $x = re^t$, $y = 4re^{-t}$. Find z_r and z_t. (Note: $z_r = \frac{\partial z}{\partial r}$ and $z_t = \frac{\partial z}{\partial t}$)

Solution $\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial r} = \frac{1}{y}(1e^t) + \frac{-x}{y^2}(1e^{-t})$
$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t} = \frac{1}{y}(re^t) + \frac{-x}{y^2}(re^{-t}(-1))$

6.10 (14.5) Implicit Differentiation

If you have an explicit definition of y, such as $y = f(x)$, but y is used impllicitly: $F(x, y) = 0$, then $\frac{dF}{dy} \cdot \frac{dy}{dx} = \frac{-\partial F}{\partial x}$ and $\frac{dy}{dx} = \frac{\frac{-\partial F}{\partial x}}{\frac{\partial F}{\partial y}}$ or $\frac{dy}{dx} = \frac{-F_x}{F_y}$

Similarly with 3 variables: If we have an explicit definition $z = f(x, y)$ but z is used impllicitly: $F(x, y, z) = 0$. Then: $\frac{\partial z}{\partial x} = -\frac{F_x}{F_z}$ and $\frac{\partial z}{\partial y} = -\frac{F_y}{F_z}$

6.10.1 Example 1

Find $\frac{dy}{dx}$ where $x^2y + e^{xy} = 9$ using partial derivatives.

Solution $\frac{dy}{dx} = \frac{-F_x}{F_y} = -\frac{2xy + e^{xy}(y)}{x^2 + e^{xy}(x)}$

6.10.2 Example 2

Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$ where $yz = \ln(2x + 3z)$

Solution The equation becomes $yz - \ln(2x + 3z) = 0$ (move everything to one side)
We can see $F = yz - \ln(2x + 3z)$
$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-\frac{1}{2x+3z}\cdot(2)}{y - \frac{1}{2x+3z}\cdot(3)}$ or $-\frac{-\frac{2}{2x+3z}}{\frac{y(2x+3z)-3}{2x+3z}} = \frac{2}{2xy+3yz-3}$
$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{(1)z}{y - \frac{3}{2x+3z}}$