MATH2790 - Tutorials

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1 (2.3) Linear 1st Order DE

Reminder of standard form: $y' + P(x)y = f(x)$ where $P(x)$ and $f(x)$ are continuous. In many real life applications, $P(x)$ or $f(x)$ may be *piecewise continuous*.

Piecewise Linear DEs $P(x)$ or $f(x)$ is piecewise continuous on I.

Example Solve the IVP: $\frac{dy}{dx} + y = f(x),\ y(0) = 1$ where $f(x) = \begin{cases} 1 & 0 \leq x \leq 1 \\ -1 & x > 1 \end{cases}$

Solution For $0 \leq x \leq 1$, the DE is $\frac{dy}{dx} + y = 1$
$I.F. = e^{\int P(x)dx} = e^{\int 1dx} = e^x$
Multiply the equation by I.F: $e^x y' + e^x y = e^x \Rightarrow \frac{d}{dx}(e^x y) = e^x$
$\int \frac{d}{dx}(e^x y)dx = \int e^x dx$
$e^x y = e^x + C_1 \Rightarrow y = \frac{e^x}{e^x} + \frac{C_1}{e^x} \Rightarrow y = 1 + C_1 e^{-x},\ 0 \leq x \leq 1$

For $x > 1$, the DE is $\frac{dy}{dx} + y = -1$
$I.F. = e^{\int P(x)dx} = e^x$ (it's the same as the above case since $P(x)$ remains unchanged)
Multiply the equation by I.F: $e^x y' + e^x y = -e^x \Rightarrow \frac{d}{dx}(e^x y) = -e^x$
$\int \frac{d}{dx}(e^x y)dx = \int -e^x dx$
$e^x y = -e^x + C_2 \Rightarrow y = \frac{-e^x}{e^x} + \frac{C_2}{e^x} \Rightarrow y = -1 + \frac{C_2}{e^x}, x > 1$

Thus $y = \begin{cases} 1 + C_1 e^{-x} & 0 \leq x \leq 1 \\ -1 + C_2 e^{-x} & x > 1 \end{cases}$ $y(0) = 1 \Rightarrow 1 = 1 + C_1 e^{-0} \Rightarrow 1 = 1 + C_1 \Rightarrow C_1 = 0$

So $y = \begin{cases} 1 & 0 \leq x \leq 1 \\ -1 + C_2 e^{-x} & x > 1 \end{cases}$ For y to be the solution, we need y to be continuous at $x = 1$.
$y(1) = \lim_{x \to 1^-} y = \lim +x \to 1^+$
$1 = \lim_{x \to 1^-} 1 = \lim_{x \to 1^+} (-1 + C_2 e^{-x}) \Rightarrow 1 = 1 = -1 + C_2 e^{-1} \Rightarrow 2 = C_2 e^{-1} \Rightarrow C_2 = 2e$
Thus the solution of the IVP is $y = \begin{cases} 1 & 0 \leq x \leq 1 \\ -1 + 2e^{1-x} & x > 1 \end{cases}$

1.0.1 Example 1.1 #19

Verify that $\ln \frac{2x-1}{x-1} = t$ is an implicit solution of the DE $\frac{dx}{dt} = (x-1)(2x-1)$

Solution Differentiate each term w.r.t. t.
$\frac{1}{\frac{2x-1}{x-1}} \cdot \frac{(2x')(x-1) - (2x-1)x'}{(x-1)^2} = 1 \Rightarrow \frac{1}{2x-1} \cdot \frac{(2x')(x-1) - (2x-1)(x')}{x-1} = 1$
$x'[2x - 2 - 2x + 1] = (2x-1)(x-1) \Rightarrow x' = (x-1)(1-2x) \to DE$
Now we will find one explicit solution: $\ln \frac{2x-1}{x-1} = t \Rightarrow \frac{2x-1}{x-1} = e^t \Rightarrow 2x - 1 = e^t(x-1) \Rightarrow 2x - 1 = xe^t - e^t \Rightarrow$
$2x - xe^t = -e^t + 1 \Rightarrow x = \frac{-e^t + 1}{2 - e^t}$

2 Solutions by Homogeneous Substitutions

Reminder: $Mdx + Ndy = 0$ is a homogeneous equation if both M and N are homogeneous functions of the same degree (that is, $f(tx,ty) = t^a f(x,y)$ with a degree of a). To solve a homogeneous equation, choose one of the following:

- Let $y = ux$ and $dy = udx + xdu \rightarrow$ separable equation in x and $u \rightarrow$ solve and replace $u = \frac{y}{x}$.
- Let $x = vy$ and $dx = vdy + ydv \rightarrow$ separable equation in v and $y \rightarrow$ solve and replace $v = \frac{x}{y}$.

Homoegeneous equations can have a singular solution.

2.0.1 Example

Solve the DE using appropriate substitution: $(x^4 + 2y^4)dx - xy^3dy = 0$

Solution Both M and N are homogeneous of degree 4, thus the given DE is homogeneous.
Let $y = ux$ and $dy = udx + xdu \Rightarrow (x^4 + 2u^4x^4)dx - xu^3x^3(udx + xdu) = 0$
$x^4dx + 2u^4x^4dx - u^4x^4dx - u^3x^4du = 0$
$x^4dx + u^4x^4dx - u^3x^5du = 0$
$x^4dx + u^4x^4dx = u^3x^4du$
$x^4dx(1 + u^4) = u^3x^5du$
$\frac{x^4}{x^5}dx = \frac{u^3}{1+u^4}du$
$\int \frac{1}{x}dx = \int \frac{u^3}{1+u^4}du$ $\rightarrow$separable if $x \neq 0$ and if $1 + u^4 \neq 0$.
$\ln|x| = \frac{1}{4}\ln|1 + u^4| + C$
Replace $u = \frac{y}{x}$: $\ln|x| = \frac{1}{4}\ln|1 + \frac{y^4}{x^4}| + C$
$\ln|x| = \frac{1}{4}\ln\left|\frac{x^4+y^4}{x^4}\right| + C$
$\ln|x| = \frac{1}{4}\ln|x^4 + y^4| - \frac{1}{4}\ln x^4 + C$
$\ln|x| = \frac{1}{4}\ln|x^4 + y^4| - \ln|x^4|^{\frac{1}{4}} + C$
$2\ln|x| = \ln|x^4 + y^4|^{\frac{1}{4}} + C$
$\ln|x|^2 - \ln|x^4 + y^4|^{\frac{1}{4}} = C$
$\ln\left(\frac{x^2}{|x^4+y^4|^{\frac{1}{4}}}\right) = C$

3 Midterm Review

The midterm is on June 15th (11:30 AM - 12:50 PM) in the Education Building, room 1101. It covers:

- Chapter 1 (1.1, 1.2)
  - Find interval of existence of solution, or
  - Find a region where an IVP has a unique solution (no solving the equation). Take $y' = f(x, y)$ and check that f and $\frac{\partial f}{\partial y}$ are continuous on the region containing (x_0, y_0)
    * An example: take $\sqrt{x^2 - 4}y' = x^5y^2 \Rightarrow y' = \frac{x^5y^2}{x^2-4} \Rightarrow y' = f$ is continuous when $x^2 + 4 > 0 \Rightarrow x > 2$ or $x < -2$
    * Also $\frac{\partial f}{\partial y} = \frac{x^5(2y)}{\sqrt{x^2-4}}$, also continuous when $x > 2$ or $x < -2$
    * Can use region $\{(x, y)| - 5 \leq x \leq -2, -3 \leq y \leq 3\}$
- Chapter 2 (2.2-2.5)
  - You may be given separable equations: $\frac{dy}{dx} = g(x) \cdot h(y)$
  - Might be given a linear equation: $y' + P(x)y = f(x)$
  - If a question provides an implicit solution and says to find an explicit solution, then simplify this implicit solution to get explicit.
  - **Singular solutions** $\rightarrow$ need to check for singular solutions, in the form $\int \frac{d(\text{dependent})}{\ldots(=0)}$
    * $\frac{dy}{dx} = \frac{y-1}{x} \Rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x}$ since $x \neq 0$ then x is not a solution.
    * If $y = 1$ then substitue in original equation and check if it satisfies or not.
- Chapter 3 (3.1) (Models)
  - Think about growth/decay, cooling/warming, mixture and circuits.
- Chapter 4 (4.1 and 4.2)
  - Think about $W = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix}$. If $W \neq 0$ for all x in $I \Rightarrow$ L.I. Otherwise, L.D.
  - Think about reduction of order: $y_2 = uy_1$ and homogeneous linear DEs $\Rightarrow u = \int \frac{e^{\int -P(x)dx}}{y_1^2}dx$
  - Exact equations $Mdx + Ndy = 0$ If $M_y = N_x$ then it is exact.
    * To solve, $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ and the solution is $f(x, y) = C$
    * The I.F. to make it exact is either $\mu = e^{\int \frac{M_y - N_x}{N}dx}$ with no y or $\mu = e^{\int \frac{N_x - M_y}{M}dy}$ with no x.
  - Solve a DE by using appropriate substitution:
    * Homogeneous DE: $Mdx + Ndy = 0$ (M and N are homogeneous of the same degree) $\Rightarrow$ Let $y = ux, dy = udx + xdu$ or let $x = vy, dx = vdy + ydv$, and make them separable.
    * Bernoulli's Equation: $y' + f(x)y = f(x)y^n$, $n \in R$, $n \neq 0$ and $n \neq 1 \Rightarrow$ Let $u = y^{1-n}$, $u' + (1 - n)P(x)u = (1 - n)f(x) \Rightarrow$ a linear equation.
    * Linear Substitution: $y' = f(Ax + By + C) \Rightarrow$ Let $u = Ax + By + C \Rightarrow$ separable equation.

4 (4.4) Method of Undetermined Coefficients

$g(x)$	y_p
3	A
$x^3 - 4$	$Ax^3 + Bx^2 + Cx + D$
e^{2x}	Ae^{2x}
$\cos 3x$	$A\cos 3x + B\sin 3x$
$\sin 3x$	$A\cos 3x + B\sin 3x$
$3 + e^{2x}$	$A + Be^{2x}$
xe^{3x}	$(Ax + B)e^{3x}$
$x\sin 5x$	$(Ax + B)\cos 5x + (Cx + D)\sin 5x$

To solve a nonhomogeneous equation:

1. Solve associated homoegenous equation and get y_c (4.3).
2. Find y_p.
   - You have $a_n, a_{n-1}, \ldots, a_1, a_0$ as constants.
   - $g(x)$ is a constant, polynomial, e^{kx}, $\sin kx$, $\cos kx$, their sum, difference or product.
   - $g(x)$ cannot be $\ln x$, $\tan x$, etc.
3. The solution of nonhomogeneous equation is $y = y_c + y_p$

4.0.1 Example 2 (Example 1 in Lecture Notes)

Solve the DE: $y'' - 9y = 2e^{3x}$

Solution The homogeneous DE is $y'' - 9y = 0$. The auxiliary equation is $m^2 - 9 = 0 \Rightarrow m^2 = 9 \Rightarrow m = \pm 3$
Then $y_c = c_1e^{3x} + c_2e^{-x}$
Let $y_p = Ae^{3x}$, $y_p' = 3Ae^{3x}$, $y_p'' = 9Ae^{3x} \Rightarrow 9Ae^{3x} - 9Ae^{3x} = 2e^{3x} \Rightarrow 0 = 2e^{3x}$

Case 1: There is no duplication of y_p with any term of y_c. The y_p will be a linear combination of the linearly independent functions generated by repeated differentiation of $g(x)$.
Case 2: If any term in y_{p_i} duplicates with any term of y_c, then we multiply y_{p_i} by x^n where n is the smallest power that removes the duplication.

Redo Example 2 $y_c = c_1e^{3x} + c_2e^{-3x}$
Let $y_p = (Ae^{3x})x = Axe^{3x}$
$y_p' = Ae^{3x} + Ax3e^{3x} = Ae^{3x} + 3Axe^{3x}$ $\quad$ $y_p'' = 3Ae^{3x} + 3Ae^{3x} + 3Ax3e^{3x} = 6Ae^{3x} + 9Axe^{3x}$
The solution is $y = y_c = +y_p - c_1e^{3x} + c_2e^{-3x} + \frac{1}{3}xe^{3x}$ (notice no c terms for y_p)

4.0.2 Example 3

Solve $y''' + 2y'' = 2x + 5 - e^{-2x}$

Solution The associated homogeneous equation is $y''' + 2y'' = 0$
The auxiliary equation is $m^3 + 2m^2 = 0 \Rightarrow m^2(m + 2) = 0 \Rightarrow m = 0, 0, -2$
So $y_c = c_1e^{0x} + c_2xe^{0x} + c^3e^{-2x} \Rightarrow y_c = c_1 + c_2x + c_3e^{-2x}$

Let $y_p = Ax^3 + Bx^2 + Cxe^{-2x}$
$y_p' = 3Ax^2 + 2Bx + Ce^{-2x} - 2Cxe^{-2x}$
$y_p'' = 6Ax - 2B - 2Ce^{-2x} - 2Ce^{-2x} - 2Cx(-2e^{-2x}) = 6Ax + 2B - 4Ce^{-2x} + 4Cxe^{-2x}$
$y_p''' = 6A - 4Ce^{-2x}(-2) + 4Ce^{-2x} + 4Cx(-2e^{-2x}) = 6A + 12Ce^{-2x} - 8Cxe^{-2x}$

Substitute into original DE: $6A + 12Ce^{-2x} - 8Cxe^{-2x} + 12Ax + 4B - 8Ce^{-2x} + 8Cxe^{-2x} = 2x + 5 - e^{-2x}$
$\Rightarrow 6A + 4Ce^{-2x} + 12Ax + 4B = 2x + 5 - e^{-2x}$

Complete coefficients of x: $12A = 2 \Rightarrow A = \frac{2}{12} = \frac{1}{6}$

Constants: $6A+4B=5 \Rightarrow 6(\frac{1}{6})+4B=5 \Rightarrow 4B=4 \Rightarrow B=1$
$e^{-2x}:\ 12C-8C=-1 \Rightarrow 4C=-1 \Rightarrow C=\frac{-1}{4}$ So $y_p=\frac{1}{6}x^3+x^2-\frac{1}{4}e^{-2x}$

Thus the solution is $y=y_c+y_p=c_1+c_2x+c_3e^{-2x}+\frac{1}{6}x^3+x^2-\frac{1}{4}e^{-2x}$

4.0.3 Example 4

Suppose the method of undetermined coefficients is to be used to solve the DE: $y''-10y'+25y=2e^{-5x}+xe^{5x}$ and $y_c=c_1e^{5x}+c_2e^{5x}$. Write an appropriate form of y_p (do not solve for coefficients).

Solution $y_p=Ae^{-5x}+(Bx+C)e^{5x}x^2$ is proper form.
Note: Bx^3+Cx^2 was the original form, but x^2 was factored out.

4.0.4 Example 5

Suppose the DE $y^{(5)}+49y'''=x^2\cos 7x+4x^2$ has to be solved by the method of undetermined coefficeints. Write the appropriateform of y_p (do not solve for coefficients).

Solution $y_p=\left[(Ax^2+Bx+C)\cos 7x+(Dx^2+Ex+F)\sin 7x\right]x+\left(\cos x^2+Hx+I\right)x^3$
Note: $y_{p_1}=(Ax^2+Bx+C)\cos 7x+(Dx^2+Ex+F)\sin 7x$ and $y_{p_2}=\cos x^2+Hx+I$

Now, for y_c: the auxiliary equation is $m^5+49m^3=0 \Rightarrow m^3(m^2+49)=0$
Thus $m=0,0,0,\pm 7i$
Thus $y_c=c_1+c_2x+c_3x^2+c_4\cos 7x+c_5\sin 7x$ (for c_4, $e^{0x}=1$).

Thus, $y_p=(Ax^3+Bx^2+Cx)\cos 7x+(Dx^3+Ex^2+Fx)\sin 7x+Gx^5+Hx^4+Ix^3$