MATH2780 - Tutorials

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1 (13.3) Arc Length & Curvature

Reminder of common formulas:

$$K = \frac{|\vec{T}\,'(t)|}{|\vec{r}\,'(t)|} \qquad K = \frac{|\vec{r}\,' \times \vec{r}\,''|}{|\vec{r}\,'|^3} \qquad \vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$$

1.1 Unit Normal Vector

The unit normal vectors tell us the direction in which the curve is turning. The vector $\vec{N}$ points towards the inside of the curve. The formula for the unit normal vector is as follows:

$$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)}$$

The normal vector $\vec{N}$ is orthogonal to $\vec{T}$. Additionally, $\vec{T}$ and $\vec{T}\,'$ are orthogonal (or perpendicular). That means $\vec{N} \cdot \vec{T} = 0$ and $\vec{T} \cdot \vec{T}\,' = 0$.

- Also note that $\vec{a} \cdot \vec{a} = |\vec{a}|^2$ or $|\vec{a}|\,|\vec{a}| \cos 0 = |\vec{a}|^2$

1.1.1 Example 1

Find the unit tangent vector and unit normal vector of the following curve: $\vec{r}(t) =< t, t^2 >$

Solution $\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{<1,2t>}{\sqrt{1+4t^2}} =< \frac{1}{\sqrt{1+4t^2}}, \frac{2t}{\sqrt{1+4t^2}} >$
$\vec{T}\,'(t) =< \frac{1}{2}(1+4t^2)^{\frac{-3}{2}}(8t), \frac{2}{1+4t^2} >$

- Side math: $\frac{d}{dx}\left(\frac{2t}{\sqrt{1+4t^2}}\right) = \frac{(2)\sqrt{1+4t^2} - 2t \cdot \frac{1}{2}(1+4t^2)^{\frac{-1}{2}}(8t)}{\left(\sqrt{1+4t^2}\right)^2} = \frac{2\sqrt{1+4t^2} - \frac{8t^2}{\sqrt{1+4t^2}}}{1+4t^2} = \frac{\frac{2(1+4t^2)-8t^2}{\sqrt{1+4t^2}}}{1+4t^2} = \frac{2+8t^2-8t^2}{(1+4t^2)^{\frac{3}{2}}}$

$|\vec{T}\,'(t)| = \sqrt{\frac{16t^2}{(1+4t^2)^3} + \frac{4}{(1+4t^2)^3}} = \sqrt{\frac{16t^2+4}{(1+4t^2)^3}} = \sqrt{\frac{4(4t^2+1)}{(1+4t^2)^3}} = \sqrt{\frac{4}{(1+4t^2)^2}} = \frac{2}{1+4t^2}$
$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \frac{< \frac{-4t}{(1+4t^2)^{\frac{3}{2}}}, \frac{2}{(1+4t^2)^{\frac{3}{2}}} >}{\frac{2}{1+4t^2}} =< \frac{-2t}{\sqrt{1+4t^2}}, \frac{1}{\sqrt{1+4t^2}} >$

1.1.2 Example 2

Find the unit tangent vector and unit normal vector of the following curve: $\vec{r}(t) =< 3\cos t, 3\sin t, 2t >$

Solution $\vec{r}\,'(t) =< -3\sin t, 3\cos t, 2 >$
$|\vec{r}\,'(t)| = \sqrt{9\sin^2 t + 9\cos^2 t + 4} = \sqrt{13}$ (note that $9\sin^2 t + 9\cos^2 t = 9$)
$\vec{T}(t) = \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} = \frac{<-3\sin t, 3\cos t, 2>}{\sqrt{13}} =< \frac{-3}{\sqrt{13}}\sin t, \frac{3}{\sqrt{13}}\cos t, \frac{2}{\sqrt{13}} = \frac{1}{\sqrt{13}} < -3\sin t, 3\cos t, 2 >$
$\vec{T}\,'(t) = \frac{1}{\sqrt{13}} < -3\cos t, -3\sin t, 0 >$
$|\vec{T}\,'(t)| = \sqrt{\frac{1}{13}\left(9\cos^2 t + 9\sin^2 t\right)} = \sqrt{\frac{1}{13}(9)} = \frac{\sqrt{9}}{\sqrt{13}} = \frac{3}{\sqrt{13}}$
$\vec{N}(t) = \frac{\vec{T}\,'(t)}{|\vec{T}\,'(t)|} = \frac{\frac{1}{13}<-3\cos t, -3\sin t, 0>}{\frac{3}{\sqrt{13}}} = \frac{<-3\cos t, -3\sin t, 0>}{3} =< -\cos t, -\sin t, 0 >$

2 (14.2) Limits

In last class, we discussed limits. Namely, for $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ if we get an answer, we get the limit. If we get $\frac{0}{0}$ then we need to do something:

- Factorization.
- Rationalization.
- Polar coordinates $\left(x^2+y^2=r^2,\ x=r\cos\theta,\ y=r\sin\theta,\ r\to 0,\ \theta\in[0,2\pi]\right)$.
- Show two paths with different answers, so the limit DNE.

2.0.1 Example 1

Evaluate the limit if it exists, or show that the limit DNE: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^4+y^2}{5x^2+5y^2}$

Solution Use polar coordinates $x=r\cos\theta,\ y=r\sin\theta$
$\lim\limits_{(x,y)\to(0,0)} \frac{x^2+y^4+y^2}{5x^2+5y^2} = \lim\limits_{r\to 0} \frac{r^2\cos^2\theta+r^4\sin^4\theta+r^2\sin^2\theta}{5r^2\cos^2\theta+5r^2\sin^2\theta} = \lim\limits_{r\to 0} \frac{r^2+r^4\sin^4\theta}{5r^2(\cos^3\theta+\sin^2\theta)} = \lim\limits_{r\to 0} \frac{r^2(1+r^2\sin^4\theta)}{5r^2} = \lim\limits_{r\to 0} \frac{1+r^2\sin^4\theta}{5}$
$= \frac{1+0}{5} = \frac{1}{5}$

2.0.2 Example 2

Evaluate the limit if it exists, or show that the limit DNE: $\lim\limits_{(x,y)\to(0,0)} \frac{xy}{x^2+y^2}$

Solution Along $x=0 \Rightarrow \lim\limits_{y\to 0} \frac{0}{y^2} = \lim\limits_{y\to 0} 0 = 0$
However, along $y=x \Rightarrow \lim\limits_{x\to 0} \frac{x(x)}{x^2+x^2} = \lim\limits_{x\to 0} \frac{x^2}{2x^2} = \frac{1}{2}$ Thus the limit DNE.

Alternate Solution $x=r\cos\theta,\ y=r\sin\theta \Rightarrow = \lim\limits_{r\to 0} \frac{(r\cos\theta)(r\sin\theta)}{r^2} = \lim r\to 0\cos\theta\sin\theta = \cos\theta\sin\theta$
If $\theta=0 \Rightarrow$ the limit is 0, If $\theta=\frac{\pi}{4} \Rightarrow$ the limit is $\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$

2.0.3 Example 3

Evaluate the limit if it exists, or show that the limit DNE: $\lim\limits_{(x,y)\to(0,0)} \frac{x^2\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{3x^2+3y^2}$

Solution Using the same polar coordinates: $= \lim\limits_{r\to 0} \frac{r^2\cos^2\theta\cos^{-1}(\cos\theta)}{3r^2} = \frac{\cos^2\theta(\theta)}{3}$
Note: $\frac{x}{\sqrt{x^2+y^2}} = \frac{r\cos\theta}{\sqrt{r^2\cos^2\theta+r^2\sin^2\theta}} = \frac{r\cos\theta}{\sqrt{r^2(\cos^2\theta+\sin^2\theta)}} = \frac{r\cos\theta}{\sqrt{r^2(1)}} = \frac{r\cos\theta}{r} = \cos\theta$
If $\theta=0 \Rightarrow$ the limit is $\frac{0}{3}=0$, but if $\theta=\pi \Rightarrow$ the limit is $\frac{(\cos^2\pi)(\pi)}{3} = \frac{\pi}{3}$ Thus the limit DNE.

2.0.4 Example 4

Evaluate the limit if it exists, or show that the limit DNE: $\lim\limits_{(x,y)\to(0,0)} \frac{x^3\cos^{-1}\left(\frac{x}{\sqrt{x^2+y^2}}\right)}{3x^2+3y^2}$

Solution Using the same polar coordinates: $= \lim\limits_{r\to 0} \frac{r^3\cos^3\theta\cos^{-1}(\cos\theta)}{3r^2} = 0$

2.0.5 Example 5

Evaluate the limit if it exists, or show that the limit DNE: $\lim\limits_{(x,y,z)\to(4,0,0)} \frac{10x^2+y^2+z^2}{x-yz}$

Solution $\lim\limits_{(x,y,z)\to(4,0,0)} \frac{10x^2+y^2+z^2}{x-yz} = \frac{10(4)^2+0+0}{4-0} = \frac{160}{4} = 40$

2.0.6 Example 6

Evaluate the limit if it exists, or show that the limit DNE: $\lim\limits_{(x,y,z)\to(1,0,2)} \frac{4x-2z}{2xy-zy+2x-z}$

Solution $\lim\limits_{(x,y,z)\to(1,0,2)} \frac{4x-2z}{2xy-zy+2x-z} = \frac{4(1)-2(2)}{2(1)(0)-2(0)-2(1)-2} = \frac{0}{0}$
Instead: $= \lim\limits_{(x,y,z)\to(1,0,2)} \frac{2(2x-z)}{(2x-z)(y+1)} = \frac{2}{0+1} = 2$

2.0.7 Example 7

Evaluate the limit if it exists, or show that the limit DNE: $\lim\limits_{(x,y,z)\to(0,0,0)} \frac{6xz^2}{x^3+y^2+z^3}$

Solution Along the x-axis (set $y=0$ and $z=0$) $= \lim\limits_{x\to 0} \frac{6x(0)^2}{x^3+0+0} = \lim\limits_{x\to 0} \frac{0}{x^3} = \lim\limits_{x\to 0} 0 = 0$
Similarly, along the y and z axes respectively, you will get the same value of 0.
However, along $y=0$, $z=x \Rightarrow \lim\limits_{x\to 0} \frac{6xx^2}{x^3+0+x^3} = \lim\limits_{x\to 0} \frac{6x^3}{2x^3} = 3$
Since 2 paths give different limits, the limit DNE.

3 (14.7) Maximum and Minimum Values

In last class, (a,b) is a critical point of f if $f_x(a,b)=0$ and $f_y(a,b)=0$ or if one of f_x and f_y does not exist. A critical point is a candidate for a maximum or minimum value. To check, we can use the **Second Derivative Test:** Check if (a,b) is a critical point by calculating $D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix}$

- If $D(a,b) > 0$ and $f_{xx}(a,b) < 0$ then there is a local maximum at (a,b).
- If $D(a,b) > 0$ and $f_{xx}(a,b) > 0$ then there is a local minimum at (a,b).
- If $D(a,b) < 0$ then (a,b) is a saddle point.

3.0.1 Example 1

Find the critical points of the function $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2$. Classify them as a local max, local min, or a saddle point.
The question could also be: Find the local max and min values of the function.

Solution The critical points occur where $f_x = 0$ and $f_y = 0$
$f_x = 6x^2 + y^2 + 10x = 0$ [1]
$f_y = 2xy + 2y = $ [2] $\Rightarrow 2y(x+1) = 0 \Rightarrow y = 0$ or $x+1=0 \Rightarrow x=-1$

If $y=0$ then equation [1] becomes $6x^2+10x=0 \Rightarrow 2x(3x+5)=0 \Rightarrow x=0$ or $3x+5=0 \Rightarrow x=\frac{-5}{3}$
Thus we have the critical points $(0,0)$ and $(\frac{-5}{3}, 0)$

If $x=-1$ then equation [1] becomes $6(-1)^2+y^2+10(-1)=0 \Rightarrow y^2=4 \Rightarrow y=\pm 2$
Thus we also have the critical points $(-1,2)$ and $(-1,-2)$

$f_{xx} = 6(2x) + 0 + 10(1) = 12x + 10$
$f_{xy} = 0 + 2y + 0 = 2y$
$f_{yy} = 2x(1) + 2(1) = 2x + 2$
$D = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{bmatrix} = \begin{bmatrix} 12x+10 & 2y \\ 2y & 2x+2 \end{bmatrix} = (12x+10)(2x+2) - 4y^2$
$D(0,0) = \begin{bmatrix} 0+10 & 0 \\ 0 & 0+2 \end{bmatrix} = 20 - 0 = 20 > 0 \Rightarrow f_{xx}(0,0) = 10 > 0 \Rightarrow$ Local minimum occurs at $(0,0)$.
The local minim value is $f(0,0) = 0$
$D(\frac{-5}{3}, 0) = \begin{bmatrix} 12\frac{-5}{3}+10 & 0 \\ 0 & 2\frac{-5}{3}+2 \end{bmatrix} = \begin{bmatrix} -10 & 0 \\ 0 & \frac{-4}{3} \end{bmatrix} = (-10)(\frac{-4}{3}) = \frac{40}{3} > 0 \Rightarrow f_{xx}(\frac{-5}{3}, 0) = -10 < 0$
Thus we have a local max at $(\frac{-5}{3}, 0)$. The local max value is:
$f(\frac{-5}{3}, 0) = 2(\frac{-5}{3})^3 + (\frac{-5}{3})(0)^2 + 5(\frac{-5}{3})^2 + 0^2 = (\frac{-5}{3})^2(\frac{-10}{3}+5) = \frac{25}{9}(\frac{-10+15}{3}) = \frac{25}{9} \cdot \frac{5}{3} = \frac{125}{27}$

$$D(-1,2) = \begin{bmatrix} 12(-1)+10 & 2(2) \\ 2(2) & 2(-1)+2 \end{bmatrix} = \begin{bmatrix} -2 & 4 \\ 4 & 0 \end{bmatrix} = 0 - 16 = -16 < 0 \Rightarrow (-1,2) \text{ is a saddle point.}$$

$D(-1,-2) = \begin{bmatrix} -2 & -4 \\ -4 & 0 \end{bmatrix} = 0 - 16 = -16 < 0 \Rightarrow (-1,-2)$ is a saddle point.

This will be discussed in Chapter 14.7, which is optimization (word problems).