MATH2790 - Tutorials

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1 (2.3) Linear 1st Order DE

Reminder of standard form: $y' + P(x)y = f(x)$ where $P(x)$ and $f(x)$ are continuous. In many real life applications, $P(x)$ or $f(x)$ may be *piecewise continuous*.

Piecewise Linear DEs $P(x)$ or $f(x)$ is piecewise continuous on I.

Example Solve the IVP: $\frac{dy}{dx} + y = f(x),\ y(0) = 1$ where $f(x) = \begin{cases} 1 & 0 \le x \le 1 \\ -1 & x > 1 \end{cases}$

Solution For $0 \le x \le 1$, the DE is $\frac{dy}{dx} + y = 1$
$I.F. = e^{\int P(x)dx} = e^{\int 1dx} = e^x$
Multiply the equation by I.F: $e^x y' + e^x y = e^x \Rightarrow \frac{d}{dx}(e^x y) = e^x$
$\int \frac{d}{dx}(e^x y)dx = \int e^x dx$
$e^x y = e^x + C_1 \Rightarrow y = \frac{e^x}{e^x} + \frac{C_1}{e^x} \Rightarrow y = 1 + C_1 e^{-x},\ 0 \le x \le 1$

For $x > 1$, the DE is $\frac{dy}{dx} + y = -1$
$I.F. = e^{\int P(x)dx} = e^x$ (it's the same as the above case since $P(x)$ remains unchanged)
Multiply the equation by I.F: $e^x y' + e^x y = -e^x \Rightarrow \frac{d}{dx}(e^x y) = -e^x$
$\int \frac{d}{dx}(e^x y)dx = \int -e^x dx$
$e^x y = -e^x + C_2 \Rightarrow y = \frac{-e^x}{e^x} + \frac{C_2}{e^x} \Rightarrow y = -1 + \frac{C_2}{e^x}, x > 1$

Thus $y = \begin{cases} 1 + C_1 e^{-x} & 0 \le x \le 1 \\ -1 + C_2 e^{-x} & x > 1 \end{cases}$ $\quad y(0) = 1 \Rightarrow 1 = 1 + C_1 e^{-0} \Rightarrow 1 = 1 + C_1 \Rightarrow C_1 = 0$

So $y = \begin{cases} 1 & 0 \le x \le 1 \\ -1 + C_2 e^{-x} & x > 1 \end{cases}$ For y to be the solution, we need y to be continuous at $x = 1$.
$y(1) = \lim_{x\to 1^-} y = \lim +x \to 1^+$
$1 = \lim_{x\to 1^-} 1 = \lim_{x\to 1^+} (-1 + C_2 e^{-x}) \Rightarrow 1 = 1 = -1 + C_2 e^{-1} \Rightarrow 2 = C_2 e^{-1} \Rightarrow C_2 = 2e$
Thus the solution of the IVP is $y = \begin{cases} 1 & 0 \le x \le 1 \\ -1 + 2e^{1-x} & x > 1 \end{cases}$

1.0.1 Example 1.1 #19

Verify that $\ln \frac{2x-1}{x-1} = t$ is an implicit solution of the DE $\frac{dx}{dt} = (x-1)(2x-1)$

Solution Differentiate each term w.r.t. t.
$\frac{1}{\frac{2x-1}{x-1}} \cdot \frac{(2x')(x-1)-(2x-1)x'}{(x-1)^2} = 1 \Rightarrow \frac{1}{2x-1} \cdot \frac{(2x')(x-1)-(2x-1)(x')}{x-1} = 1$
$x'[2x - 2 - 2x + 1] = (2x-1)(x-1) \Rightarrow x' = (x-1)(1-2x) \to DE$
Now we will find one explicit solution: $\ln \frac{2x-1}{x-1} = t \Rightarrow \frac{2x-1}{x-1} = e^t \Rightarrow 2x - 1 = e^t(x-1) \Rightarrow 2x - 1 = xe^t - e^t \Rightarrow$
$2x - xe^t = -e^t + 1 \Rightarrow x = \frac{-e^t+1}{2-e^t}$

2 Solutions by Homogeneous Substitutions

Reminder: $Mdx + Ndy = 0$ is a homogeneous equation if both M and N are homogeneous functions of the same degree (that is, $f(tx, ty) = t^a f(x, y)$ with a degree of a). To solve a homogeneous equation, choose one of the following:

- Let $y = ux$ and $dy = udx + xdu \rightarrow$ separable equation in x and $u \rightarrow$ solve and replace $u = \frac{y}{x}$.
- Let $x = vy$ and $dx = vdy + ydv \rightarrow$ separable equation in v and $y \rightarrow$ solve and replace $v = \frac{x}{y}$.

Homoegeneous equations can have a singular solution.

2.0.1 Example

Solve the DE using appropriate substitution: $(x^4 + 2y^4)dx - xy^3dy = 0$

Solution Both M and N are homogeneous of degree 4, thus the given DE is homogeneous.
Let $y = ux$ and $dy = udx + xdu \Rightarrow (x^4 + 2u^4x^4)dx - xu^3x^3(udx + xdu) = 0$
$x^4dx + 2u^4x^4dx - u^4x^4dx - u^3x^4du = 0$
$x^4dx + u^4x^4dx - u^3x^5du = 0$
$x^4dx + u^4x^4dx = u^3x^4du$
$x^4dx(1 + u^4) = u^3x^5du$
$\frac{x^4}{x^5}dx = \frac{u^3}{1+u^4}du$
$\int \frac{1}{x}dx = \int \frac{u^3}{1+u^4}du \rightarrow$separable if $x \neq 0$ and if $1 + u^4 \neq 0$.
$\ln|x| = \frac{1}{4}\ln|1 + u^4| + C$
Replace $u = \frac{y}{x}$: $\ln|x| = \frac{1}{4}\ln|1 + \frac{y^4}{x^4}| + C$
$\ln|x| = \frac{1}{4}\ln\left|\frac{x^4+y^4}{x^4}\right| + C$
$\ln|x| = \frac{1}{4}\ln|x^4 + y^4| - \frac{1}{4}\ln x^4 + C$
$\ln|x| = \frac{1}{4}\ln|x^4 + y^4| - \ln|x^4|^{\frac{1}{4}} + C$
$2\ln|x| = \ln|x^4 + y^4|^{\frac{1}{4}} + C$
$\ln|x|^2 - \ln|x^4 + y^4|^{\frac{1}{4}} = C$
$\ln\left(\frac{x^2}{|x^4+y^4|^{\frac{1}{4}}}\right) = C$

3 Midterm Review

The midterm is on June 15th (11:30 AM - 12:50 PM) in the Education Building, room 1101. It covers:

- Chapter 1 (1.1, 1.2)
  - Find interval of existence of solution, or
  - Find a region where an IVP has a unique solution (no solving the equation). Take $y' = f(x, y)$ and check that f and $\frac{\partial f}{\partial y}$ are continuous on the region containing (x_0, y_0)
    * An example: take $\sqrt{x^2 - 4}y' = x^5y^2 \Rightarrow y' = \frac{x^5y^2}{x^2-4} \Rightarrow y' = f$ is continuous when $x^2 + 4 > 0 \Rightarrow x > 2$ or $x < -2$
    * Also $\frac{\partial f}{\partial y} = \frac{x^5(2y)}{\sqrt{x^2-4}}$, also continuous when $x > 2$ or $x < -2$
    * Can use region $\{(x, y)| -5 \leq x \leq -2, -3 \leq y \leq 3\}$
- Chapter 2 (2.2-2.5)
  - You may be given separable equations: $\frac{dy}{dx} = g(x) \cdot h(y)$
  - Might be given a linear equation: $y' + P(x)y = f(x)$
  - If a question provides an implicit solution and says to find an explicit solution, then simplify this implicit solution to get explicit.
  - **Singular solutions** $\rightarrow$ need to check for singular solutions, in the form $\int \frac{d(\text{dependent})}{...(=0)}$
    * $\frac{dy}{dx} = \frac{y-1}{x} \Rightarrow \int \frac{dy}{y-1} = \int \frac{dx}{x}$ since $x \neq 0$ then x is not a solution.
    * If $y = 1$ then substitue in original equation and check if it satisfies or not.
- Chapter 3 (3.1) (Models)
  - Think about growth/decay, cooling/warming, mixture and circuits.
- Chapter 4 (4.1 and 4.2)
  - Think about $W = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix}$. If $W \neq 0$ for all x in $I \Rightarrow$ L.I. Otherwise, L.D.
  - Think about reduction of order: $y_2 = uy_1$ and homogeneous linear DEs $\Rightarrow u = \int \frac{e^{\int -P(x)dx}}{y_1^2}dx$
  - Exact equations $Mdx + Ndy = 0$ If $M_y = N_x$ then it is exact.
    * To solve, $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$ and the solution is $f(x, y) = C$
    * The I.F. to make it exact is either $\mu = e^{\int \frac{M_y - N_x}{N}dx}$ with no y or $\mu = e^{\int \frac{N_x - M_y}{M}dy}$ with no x.
  - Solve a DE by using appropriate substitution:
    * Homogeneous DE: $Mdx + Ndy = 0$ (M and N are homogeneous of the same degree) $\Rightarrow$ Let $y = ux, dy = udx + xdu$ or let $x = vy, dx = vdy + ydv$, and make them separable.
    * Bernoulli's Equation: $y' + f(x)y = f(x)y^n$, $n \in R$, $n \neq 0$ and $n \neq 1 \Rightarrow$ Let $u = y^{1-n}$, $u' + (1 - n)P(x)u = (1 - n)f(x) \Rightarrow$ a linear equation.
    * Linear Substitution: $y' = f(Ax + By + C) \Rightarrow$ Let $u = Ax + By + C \Rightarrow$ separable equation.