

Differential Equations

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1 Introduction to Differential Equations

Differential Equation Any equation that is differentiable. An example is $3y'' + xy' - x^2 = e^x$
The **order** of a DE is the highest order of derivative.

$$\text{Linear DE} \left\{ \begin{array}{l} (i)y, y', y'', \dots, \quad \text{Cannot have more than one power} \\ (ii) \text{We can have } x^n, e^x, \sin x, \text{ etc. but not } y^n, e^y, \cos y, \text{ etc.} \end{array} \right\} \quad (1)$$

Example of nonlinear DE: $3x^3y'' + yy' = e^x$

$y = f(x)$ is a **solution** of a DE on an interval I (interval of existence of solution):

- y satisfies the DE.
- y, y', y'', \dots are continuous on I.

1.1 Initial Value Problems (IVP)

The first-order DE $y' = f(x, y)$ subject to $y(x_0) = y_0 \rightarrow$ is an IVP.

The second-order DE $y'' = f(x, y, y')$ subject to $(y(x_0) = y_0, y'(x_0) = y_1)$ is an IVP, while $(y(x_0) = y_0, y(x_1) = y_1)$ is a **Boundary Value Problem (BVP)**.

- Note: After y_1 both cases can be used.

1.1.1 Example 1

$y = c_1 \cos x + c_2 \sin x$ is a solution of the DE $y'' + y = 0$. Find the solution subject to the conditions $y(\pi) = 1, y'(\pi) = -2$

Solution: $y(\pi) = 1 \Rightarrow 1 = c_1 \cos \pi + c_2 \sin \pi$
 $\Rightarrow 1 = -c_1 \Rightarrow c_1 = -1$

Then: $y = c_1 \cos x + c_2 \sin x \Rightarrow y' = -c_1 \sin x + c_2 \cos x$

Using $y'(\pi) = -2 \Rightarrow -2 = -c_1 \sin \pi + c_2 \cos \pi \Rightarrow -2 = -c_2 \Rightarrow c_2 = 2$

The solution is $y = -\cos x + 2 \sin x$

1.1.2 Example 2

$y = \frac{1}{x^2+c}$ is the one parameter solution of the DE $y' + 2xy^2 = 0$. Find a solution of the IVP:
 $y' + 2xy^2 = 0, y(-3) = \frac{1}{5}$. Give the largest interval over which the solution is defined.

Solution: Using $y(-3) = \frac{1}{5}$, $y = \frac{1}{x^2+c} \Rightarrow \frac{1}{5} = \frac{1}{(-3)^2+c}$
 $\Rightarrow \frac{1}{5} = \frac{1}{9+c}$
 $\Rightarrow 9+c = 5$
 $\Rightarrow c = -4$

Thus $y = \frac{1}{x^2-4}$ is a solution of the IVP.

y is continuous when $x^2 - 4 \neq 0 \Rightarrow x^2 \neq 4 \Rightarrow x \neq \pm 2$

$$y = \frac{1}{x^2-4} = (x^2 - 4)^{-1}$$

$$y' = -1(x^2 - 4)^{-2}(2x) = \frac{-2x}{(x^2-4)^2} \text{ is continuous when } x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$$

The longest interval is $(-\infty, -2)$ on which the solution is defined.

1.1.3 Not every DE is solvable

Consider the first-order IVP $xy' = 2y, y(0) = 0$. $y = 0$ is a solution $y = 0, y' = 0 \Rightarrow x(0) = 2(0) \Rightarrow 0 = 0$

• **Note:** a solution must be valid for all values of x .

$y = x^2$ is also a solution $\Rightarrow y' = 2x \Rightarrow x(2x) = 2x^2 \Rightarrow 2x^2 = 2x^2 \Rightarrow y(0) = 0$

1.2 Existence Theorem

Theorem 1.1 (1.2.1) Let R be a rectangular region $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$ that contains (x_0, y_0) in its interior. If f and $\frac{df}{dy}$ are continuous on R , then there exists an interval $(x_0 - h, x_0 + h)$ in R on which the IVP $y' = f(x, y), y(x_0) = y_0$ has a unique solution.

• **Note:** We need to have the form $y' = f(x, y)$ to decide f .

Examples:

• $f(x) = x^3 + \cos x \Rightarrow f'(x) = 3x^2 - \sin x$

• $f(x, y) = x^3 \cos y + e^y - x^7$

For a partial derivative of f with respect to $x \rightarrow \frac{df}{dx}$, we treat y as a constant.

For a partial derivative of f with respect to $y \rightarrow \frac{df}{dy}$, we treat x as a constant.

An example: $f(x, y) = x^3 \cos y + e^y - x^7$

• $\frac{df}{dx} = (\cos y)(3x^2) + 0 - 7x^6$

• $\frac{df}{dy} = x^3(-\sin y) + e^y + 0$

1.2.1 Example 1

Determine whether the existence theorem guarantees that the IVP $xy' = 2y, y(0) = 0$ has a unique solution.

Solution: $y' = \frac{2y}{x} \rightarrow f(x, y) = \frac{2y}{x}$ is continuous when $x \neq 0$

Conditions of existence theorem are not satisfied. So there is no guarantee of a unique solution.

1.2.2 Example 2

Determine a region R of the xy -plane for which the DE $(1 + y^3)y' = x^2$ would have a unique solution in an interval around $(0, 2)$ (a unique solution passing through $(0, 2)$).

Solution: $(1 + y^3)y' = x^2 \Rightarrow y' = \frac{x^2}{1+y^3} \Rightarrow f(x, y) = \frac{x^2}{1+y^3} = x^2(1 + y^3)^{-1}$ is continuous when

$$1 + y^3 \neq 0, y^3 \neq -1, y \neq -1$$

$$\frac{df}{dy} = x^2 [-(1 + y^3)^{-2}(3y^2)] = \frac{-3x^2 y^2}{(1 + y^3)^2}$$

$\frac{df}{dy}$ is continuous when $1 + y^3 \neq 0 \Rightarrow y \neq -1$

Define $R = \{(x, y) | -10 \leq x \leq 10, 0 \leq y \leq 9\}$

• **Note:** the boundaries of x can be anything since there are no restrictions on x , so long as it contains $x = 0$.

• The boundaries of y must contain $y = 2$ and must not cross $y = -1$.

1.2.3 Example 3

Find a region in the xy-plane on which the IVF $(1 + y^3)y' = x^2, y(1) = -3$ would have a unique solution.

Solution: From example 2, $f, \frac{df}{dy}$ are continuous when $y \neq -1$

$R = \{(x, y) | -3 \leq x \leq 5, -6 \leq y \leq -2\}$ where the x boundaries contain $x = 1$ and the y boundaries contain $y = -3$.

2 Solutions of First Order Differential Equations

In last class, 2.2, separable equation $\frac{dy}{dx} = g(x)h(y)$ which can turn into $\frac{dy}{h(y)} = g(x)dx$ and this can be integrated. When integrating, it must occur with the same variable.

- $\int x^2 dx = \frac{x^3}{3}$
- $\int x^2 du$ cannot integrate.

There are singular and implicit solutions.

2.1 Linear Equations

From section 1.1, nth order linear DE is represented by $\frac{d^n y}{dx^n}$ or $y^{(n)}$

The equation is $a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)y^{(n-1)} + \dots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$

First order is only concerned with $a_1(x)y' + a_0(x)y = g(x)$

Linear equations are easy to solve and get explicit solutions easily. However there are **no** singular solutions of linear DEs.

2.1.1 Solving Linear 1st Order DEs

Step 1 Wrote the DE in the standard form $y' + P(x)y = f(x)$

$a_1(x)y' + a_0(x)y = g(x)$ $y' + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)} \Rightarrow y' + P(x)y = f(x)$ (standard form of 1st order linear DE)

Step 2 Find the integrating factor (I.F.)

I.F. = $e^{\int P(x)dx}$ (do not write $+C$ in this integration since that adds a wasted simplification step)

Step 3 Multiply every term of the standard form equation (from **Step 1**) by the I.F.

$$e^{\int P(x)dx}y' + e^{\int P(x)dx}P(x)y = f(x)e^{\int P(x)dx}$$

The LHS will automatically be $\frac{d}{dx}((I.F.)y)$ i.e. $\frac{d}{dx}(e^{\int P(x)dx}y)$

$$= e^{\int P(x)dx}y' + ye^{\int P(x)dx}(P(x))$$

Step 4 Integrate both sides w.r.t. x .

$$(I.F.)y = \int f(x)e^{\int P(x)dx}dx$$

The explicit solution will be $y = \frac{\int f(x)e^{\int P(x)dx}dx}{I.F.}$

We get $y' + P(x)y = f(x)$. The interval of existence of solution is the interval over which $P(x)$ **and** $f(x)$ **are continuous**.

As a reminder of section 1.2, for any 1st order DE $y' = f(x, y); y(x_0) = y_0$, if f & $\frac{df}{dy}$ is continuous on \mathbb{R} , then a unique solution exists.

For linear equations, $y' = f(x) - P(x)y$, $\frac{df}{dy} = 0 - P(x)(1)$, where $f(x) - P(x)y = f(x, y)$

2.2 Example 1

Solve the differential equation $y' - 2xy = x$. Give the largest interval I over which the solution is defined.

Solution $y' - 2xy = x$ is a linear equation where $P(x) = -2x$, $f(x) = x$

Since $P(x)$ and $f(x)$ are continuous on \mathbb{R} , then the largest interval of existence of the solution is $I = (-\infty, \infty)$

$$\int P(x)dx = \int -2x dx = \frac{-2x^2}{2} = -x^2 \text{ (remember, don't write } +C)$$

$$\text{Thus } I.F. = e^{\int P(x)dx} = e^{-x^2}$$

Next, multiple the DE by the I.F.

$$e^{-x^2}y' - 2xe^{-x^2}y = e^{-x^2}x, \frac{d}{dx}(e^{-x^2}y) = e^{-x^2}x$$

Integrating w.r.t. x : $e^{-x^2}y = \int e^{-x^2}x dx$, let $u = -x^2$, $du = -2x dx$

$$\int e^u \frac{du}{-2} = \frac{-1}{2} \int e^u du = \frac{-1}{2} e^u + C = \frac{-e^{-x^2}}{2} + C$$

$$\text{Thus } e^{-x^2}y = \frac{-1}{2}e^{-x^2} + C \Rightarrow y = \frac{\frac{-1}{2}e^{-x^2}}{e^{-x^2}} + \frac{C}{e^{-x^2}} \Rightarrow y = \frac{-1}{2} + Ce^{x^2}$$

Question: Is the equation $y' - 2xy = x$ separable?

Answer: If you can separate x from y , then it is separable. So $y' - 2xy = x \Rightarrow y' = 2xy + x \Rightarrow y' = x(2y + 1)$
Thus, the equation is separable.

2.3 Example 2

Solve the IVP: $xy' + 2y = 12x^4$; $y(1) = 4$. Give the largest interval over which the solution is defined.

Solution $xy' + 2y = 12x^4 \Rightarrow y' + \frac{2}{x}y = 12x^3$ is standard form. $P(x) = \frac{2}{x}$ & $f(x) = 12x^3$ are continuous on \mathbb{R} except at $x = 0$. Thus the longest interval of existence of the solution is $(0, \infty)$.

$$I.F. = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2$$

Next we multiply the standard form function by the I.F. to get $x^2y' + x^2\left(\frac{2}{x}\right)y = x^2(12x^3) \Rightarrow \frac{d}{dx}(x^2y) = 12x^5$

Now we integrate w.r.t. x : $x^2y = \int 12x^5dx = 2x^6 + C$

$$\text{Thus } y = \frac{2x^6}{x^2} + \frac{C}{x^2} \Rightarrow y = 2x^4 + \frac{C}{x^2}$$

For the IVP, $y(1) = 4$, $4 = 2(1) + \frac{C}{1} \Rightarrow C = 2$

Thus the solution of the IVP is $y = 2x^4 + 2$

Now, find the transient term in the solution: $\lim_{x \rightarrow \infty} \frac{2}{x^2} = 0 \Rightarrow \frac{2}{x^2}$ is the transient term.

- **Note:** a **transient term** is a term which approaches 0 as x approaches ∞ .

2.4 Example 3

Find the general solution (a linear DE) of $xy' + (1+x)y = e^{-x} \sin^2 x$ (or, solve the DE).

Solution $y' + \frac{1+x}{x}y = \frac{e^{-x} \sin^2 x}{x}$ is standard form, $P(x) = \frac{1+x}{x}$

$$\int P(x)dx = \int \frac{1+x}{x}dx = \int \left(\frac{1}{x} + 1\right)dx = \ln|x| + x \text{ (no } +C)$$

Thus $I.F. = e^{\int P(x)dx} = e^{\int \ln|x| + xdx} = e^{\ln|x|} \cdot e^x = |x|e^x \Rightarrow I.F. = xe^x$

$$xe^x y' + xe^x(1+x)y = xe^x \frac{e^{-x} \sin^2 x}{x} \Rightarrow \frac{d}{dx}(xe^x y) = \sin^2 x$$

Integrating w.r.t. x : $xe^x y = \int \sin^2 x dx = \frac{1}{2} \int (1 - \cos 2x)dx = \frac{1}{2} \left(x - \frac{\sin 2x}{2}\right)$

$$\text{Thus } xe^x y = \frac{1}{2} \left(x - \frac{\sin 2x}{2}\right) + C \Rightarrow y = \frac{\frac{x}{2} - \frac{\sin 2x}{4}}{xe^x} + \frac{C}{xe^x}$$

2.5 (2.4) Exact Equations

First order DEs:

$$\left\{ \begin{array}{ll} 2.2 \rightarrow \text{Separable} & \frac{dy}{dx} = g(x)h(y) \\ 2.3 \rightarrow \text{Linear equation} & y' + P(x)y = f(x) \end{array} \right\} \quad (2)$$

Differential: $\delta x = dx$

- $y = f(x) \rightarrow$ differential is $dy = f' dx$
- $z = f(x, y) \rightarrow$ differential is $dz = \frac{df}{dx} + \frac{df}{dy} dy$
- $\frac{df}{dx} \rightarrow$ treat y as a constant.
- $\frac{df}{dy} \rightarrow$ treat x as a constant.

Consider $f(x, y) = x^3 + x^2 y \Rightarrow \frac{df}{dx} = 3x^2 + (2x)y$ and $\frac{df}{dy} = 0 + x^2(1)$
 $x^3 + x^2 y = 7$ [1], take derivatives to get: $(3x^2 + 2xy)dx + x^2 dy = 0$ [2]. [1] is a solution of the DE [2].

A DE of the form $\frac{df}{dx} dx + \frac{df}{dy} dy = 0$ is called an **exact DE**.

How to decide if a DE is exact A DE of the form $M(x, y)dx + N(x, y)dy = 0$ is an exact equation iff $\frac{dM}{dy} = \frac{dN}{dx}$ or $M_y = N_x$

How to solve the exact equation We have two equations $\frac{df}{dx} = M$ [1] and $\frac{df}{dy} = N$ [2]

- The partial integration $\int \frac{df}{dx} dx = f(x, y)$
- The partial integration $\int \frac{df}{dy} dy = f(x, y)$

Doing the partial integration of [1]: $f(x, y) = \int M dx + g(y)$, where $g(y)$ is the constant of integration.

Find $\frac{df}{dy}$ and substitute in [2] to find $g(y)$.

$f(x, y) = \dots$ (the solution is $f(x, y) = C$)

2.5.1 Example 1

Determine whether the DE is exact or not. If it is exact, then solve it. $y' = \frac{-2xy}{1+x^2}$

Solution $y' = \frac{-2xy}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{-2xy}{1+x^2} \Rightarrow (1+x^2)dy = -2xydx \Rightarrow 2xydx + (1+x^2)dy = 0$
 $M_y = \frac{dM}{dy} = 2x(1)$ and $N_x = \frac{dN}{dx} = 2x$, $M_y = N_x \Rightarrow$ exact.

To solve, $\frac{df}{dx} = M$ and $\frac{df}{dy} = N \Rightarrow \frac{df}{dx} = 2xy$ [1] and $\frac{df}{dy} = 1 + x^2$ [2]

Perform partial integration of [1] w.r.t. x : $f(x, y) = 2y \frac{x^2}{2} + g(y)$ [3] $\Rightarrow \frac{df}{dy} = (1)x^2 + g'(y)$

Next substitute equation [2]: $x^2 + g'(y) = 1 + x^2 \Rightarrow g'(y) = 1$ (cannot have any x terms)

Integrating w.r.t. y : $g(y) = \int 1 dy \Rightarrow g(y) = y$ or $g(y) = y + C$ (preferred without the $+C$)

Substitute $g(y)$ in [3] to get $f(x, y) = x^2 y + y$ or $f(x, y) = x^2 y + y + C$

The solution of the DE is $f(x, y) = C \Rightarrow x^2 y + y = C$ or $f(x, y) = 0 \Rightarrow x^2 y + y + C = 0$

2.5.2 Example 2

Determine whether the DE is exact or not. If it is exact, then solve it. $(xy + y^2)dx + (x^2 + xy)dy = 0$

Solution $M_y = x(1) + 2y$ and $N_x = 2x + (1)y \Rightarrow M_x \neq N_y \Rightarrow$ not exact.

2.5.3 Example 3

Determine whether the DE is exact or not. If it is exact, then solve it. $(x + \sin y)dx = (e^y = x \cos y)dy$

Solution $(x + \sin y)dx - (e^y - x \cos y)dy = 0$

Here, $M = x + \sin y$ and $N = -(e^y - x \cos y) = -e^y + x \cos y$

$M_y = \cos y$ and $N_x = (1) \cos y \Rightarrow M_y = N_x \Rightarrow$ Exact.

To solve, $\frac{df}{dx} = M \Rightarrow \frac{df}{dx} = x + \sin y$ [1]

$\frac{df}{dy} = N \Rightarrow \frac{df}{dy} = -e^y + x \cos y$ [2]

Partial integration of [2] w.r.t. y : $f(x, y) = -e^y + x \sin y + g(x)$ [3]

$\frac{df}{dx} = 0 + (1) \sin y + g'(x)$

Substitute in [1]: $\sin y + g'(x) = x + \sin y \Rightarrow g'(x) = x$ (cannot have y) $\Rightarrow g(x) = \frac{x^2}{2}$

Substitute $g(x)$ in [3]: $f(x, y) = -e^y + x \sin y + \frac{x^2}{2}$

The solution is $f(x, y) = C \Rightarrow -e^y + x \sin y + \frac{x^2}{2} = C$

2.5.4 Example 4

Solve the IVP: $(ey - xe^{xy})y' = 2 + ye^{xy}$; $y(0) = 1$

Solution $(2y - xe^{xy})dy = (2 + ye^{xy})dx \Rightarrow (2 + ye^{xy})dx - (2y - xe^{xy}) = 0$

$M = 2 + ye^{xy} \Rightarrow \frac{dM}{dy} = 0 + (1)e^{xy} + ye^{xy}(x)$

$N = -2y + xe^{xy} \Rightarrow \frac{dN}{dx} = 0 + (1)e^{xy} + xe^{xy}(y)$

Since $M_y = N_x$, this is an exact equation.

$\frac{df}{dx} = M \Rightarrow \frac{df}{dx} = 2 + ye^{xy}$ [1]

$\frac{df}{dy} = N \Rightarrow \frac{df}{dy} = -2y + xe^{xy}$ [2]

Partial integration of [1] w.r.t. x : $f(x, y) = \int (2 + ye^{xy})dx = 2x + y \frac{e^{xy}}{y} + g(y)$ [3]

$\frac{df}{dy} = 0 + e^{xy}(x) + g'(y)$

Substitute in [2]: $xe^{xy} + g'(y) = -2y + xe^{xy} \Rightarrow g'(y) = -2y$

$g(y) = \int -2y dy = -2 \frac{y^2}{2}$

Substituting in [3]: $f(x, y) = 2x + e^{xy} - y^2$

The solution of the DE is $f(x, y) = C \Rightarrow 2x + e^{xy} - y^2 = C$

$y(0) = 1 \Rightarrow$ Substitute $x = 0$ and $y = 1$: $2(0) + e^{(0)(1)} - (1)^2 = C \Rightarrow 1 - 1 = C \Rightarrow C = 0$

The solution of the IVP is $ex + e^{xy} - y^2 = 0$

2.6 More on Exact Equations

Sometimes, we can multiply the DE by an integrating factor and the DE becomes an exact DE. We solve by the method of exact equations:

1. If $\frac{M_y - N_x}{N}$ is a function of x only (no y terms), then $I.F. = \mu = e^{\int \frac{M_y - N_x}{N} dx}$
2. If $\frac{N_x - M_y}{M}$ is a function of y only (no x terms), then $I.F. = \mu = e^{\int \frac{N_x - M_y}{M} dy}$

2.6.1 Example 1

Find an I.F. to make this DE exact and then solve it: $(y^2 - y)dx + xdy = 0$

Solution $M = y^2 - y$, $M_y = 2y - 1$, $N = x$, $N_x = 1$ $M_y \neq N_x \Rightarrow$ not exact.

$\frac{M_y - N_x}{N} = \frac{(2y-1)-1}{x} = \frac{2y-2}{x}$ (not in terms of x only)

$\frac{N_x - M_y}{M} = \frac{1-(2y-1)}{y^2-y} = \frac{1-2y+1}{y^2-y} = \frac{2-2y}{y^2-y} = \frac{-2(1-y)}{y(y-1)} = \frac{-2}{y}$ (good!)

$I.F. = \mu = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int \frac{-2}{y} dy} = e^{-2 \ln |y|} = e^{\ln |y|^{-2}} = |y|^{-2} = \frac{1}{y^2}$

Multiply the given DE by $\frac{1}{y^2}$: $\frac{1}{y^2}(y^2 - y)dx + \frac{1}{y^2}xdy = 0 \Rightarrow (1 - \frac{1}{y})dx + \frac{x}{y^2}dy = 0$

Now, $M = 1 - \frac{1}{y}$ and $N = \frac{x}{y^2} \Rightarrow M_y = -(-1y^{-2}) = \frac{1}{y^2}$, $N_x = \frac{1}{y^2}(1) = \frac{1}{y^2}$ (unnecessary to check)

To solve: $\frac{df}{dx} = M \Rightarrow \frac{df}{dx} = 1 - \frac{1}{y}$ [1] $\frac{df}{dy} = N \Rightarrow \frac{df}{dy} = \frac{x}{y^2}$ [2]

Partial integration of [1] w.r.t. $x \Rightarrow f(x, y) = x - \frac{1}{y}(x) + g(y)$ [3]

$\frac{df}{dy} = 0 + \frac{x}{y^2} + g'(y)$ (substitute this into [2])

$\frac{x}{y^2} + g'(y) = \frac{x}{y^2} \Rightarrow g'(y) = 0 \Rightarrow g(y) = 0$ or $g(y) = C$

So [3] gives $f(x, y) = x - \frac{x}{y}$ or $f(x, y) = x - \frac{x}{y} + C$

Thus the solution is $f(x, y) = C \Rightarrow x - \frac{x}{y} = C$ or $f(x, y) = 0 \Rightarrow x - \frac{x}{y} + C = 0$

Now, this is only the case if $y \neq 0$ (since the I.F. involves y being in the denominator).

So we can check if $y = 0$ is a solution: $dy = y'dx = 0dx = 0$

$(0^2 - 0)dx + xdy = 0 \Rightarrow (0^2 - 0)dx + x(0) = 0 \Rightarrow 0 + 0 = 0 \Rightarrow y = 0$ is a solution.

2.6.2 Example 2

Find an I.F. to make this DE exact: $y(x + y + 1)dx + (x + 2y)dy = 0$ (do not solve the equation)

Solution $M = xy + y^2 + y$ and $N = x + 2y$ $M_y = x + 2y + 1$ and $N_x = 1$

$\frac{M_y - N_x}{N} = \frac{(x+2y+1)-1}{x+2y} = \frac{x+2y}{x+2y} = 1 \Rightarrow I.F. = \mu = e^{\int 1 dx} = e^x$

2.7 Solutions by Substitution

We do 3 kinds of substitutions which can be used to solve 1st order DEs.

2.8 Homogeneous Equations

A function $f(x, y)$ is called a homogeneous function of degree a if $f(tx, ty) = t^a f(x, y)$

To determine if the equation $f(x, y) = x^2 + xy$ is homogeneous, calculate $f(tx, ty)$.

$f(tx, ty) = (tx)^2 + (tx)(ty) = t^2x^2 + t^2xy = t^2(x^2 + xy) = t^2f(x, y) \Rightarrow$ Thus f is a homogeneous function of degree 2.

For more functions:

- $f(x, y) = x + 3$, $f(tx, ty) = tx + 3 \Rightarrow$ not homogeneous.
- $f(x, y) = \sin x - \sin y$ $f(tx, ty) = \sin tx - \sin ty \Rightarrow$ not homogeneous. (Usually anything with $\sin x$ or $\sin y$ is not homogeneous)
- $f(x, y) = \sin \frac{x}{y}$ $f(tx, ty) = \sin \frac{tx}{ty} = \sin \frac{x}{y} = t^0 \sin \frac{x}{y} \Rightarrow$ homogeneous. (this is an exception to the above)
- $f(x, y) = \ln x - \ln y$ $f(tx, ty) = \ln tx - \ln ty = \ln \frac{tx}{ty} = \ln \frac{x}{y} = t^0(\ln x - \ln y) \Rightarrow$ homogeneous of degree 0.

A differential equation $Mdx + Ndy = 0$ is homogeneous if both M and N are homogeneous functions of the same degree.

Example 1 For $(x^2 + xy)dx + xy^2dy = 0$, M is homogeneous of degree 2 and N is homogeneous of degree 3. Thus the equation is not homogeneous.

Example 2 For $(x + 3)dx + ydy = 0$, it is not homogeneous because M is not homogeneous.

Example 3 For $(x^2 + xy)dx + (xy - y^2)dy = 0$, since both M and N are homogeneous of degree 2, thus the equation is homogeneous. (you can often check the power of M and N to check if it's homogeneous)

2.8.1 Solving a Homogeneous Equation

Case 1: Let $y = ux$ and $dy = udx + xdu \rightarrow$ the homogeneous equation will become a separable equation in u and x . Solve it and then replace $u = \frac{y}{x}$ in the solution.

Case 2: Let $x = vy$ and $dx = vdy + ydv \rightarrow$ the homogeneous equation will become a separable equation in v and y . Solve it and then replace $v = \frac{x}{y}$ in the solution.

Example Solve the DE by using an appropriate substitution: $ydx - 2(x + y)dy = 0$

Solution Let $y = ux$ and $dy = udx + xdu$ **or** let $x = vy$ and $dx = vdy + ydv$
 $uxdx - (2x + 2ux)(udx + xdu) = 0$ **or** $y(vdy + ydv) - (2vy + 2y)dy = 0$
This is harder to solve **or** $vydy + y^2dv - 2vydy - 2ydy = 0$

$$y^2dv - vydy - 2ydy = 0$$

$$y^2dv = vydy + 2ydy \Rightarrow y^2dv = ydy(v + 2) \Rightarrow \frac{dv}{v+2} = \frac{ydy}{y^2} \Rightarrow \frac{dv}{v+2} = \frac{1}{y}dy \rightarrow \text{a separable equation.}$$

If we went the u way: $\int \frac{-dx}{x} = \int \frac{2+2u}{u+2u^2} du \dots$ needs partial fraction.

$$\text{Instead, } \int \frac{dv}{v+2} = \int \frac{1}{y} dy \Rightarrow \ln|v+2| = \ln|y| + C$$

$$\text{Replace } v \text{ by } \frac{x}{y} \Rightarrow \ln\left|\frac{x}{y} + 2\right| = \ln|y| + C$$

Is $y = 0$ also a solution?

Well, $y = 0 \Rightarrow dy = 0$. Substitute into original equation: $(0)dx - 2(x + 0)(0) = 0 \Rightarrow 0 - 0 = 0$

$\Rightarrow y = 0$ is also a solution.

$$v + 2 = 0 \Rightarrow \frac{x}{y} + 2 = 0 \Rightarrow \frac{x}{y} = -2 \Rightarrow x = -2y \Rightarrow y = \frac{-x}{2}$$

Is this a solution? Well, $y = \frac{-x}{2} \Rightarrow dy = y'dx = \frac{-1}{2}dx$

$$\text{Substitute into original equation: } \frac{-x}{2}dx - 2\left(x - \frac{x}{2}\right)\left(\frac{-1}{2}dx\right) = 0 \Rightarrow \frac{-x}{2}dx + \frac{x}{2}dx = 0 \Rightarrow 0 = 0$$

$\Rightarrow y = \frac{-x}{2}$ is also a solution.

2.9 Bernoulli's Equation

2.10 Linear Substitution