Differential Equations

Justin Bornais

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1 Introduction to Differential Equations

Differential Equation Any equation that is differentiable. An example is $3y'' + xy' - x^2 = e^x$
The **order** of a DE is the highest order of derivative.

$$\textbf{Linear DE}\left\{\begin{array}{ll} (i) y, y', y'', ..., & \text{Cannot have more than one power} \\ (ii) \text{We can have ,} & x^n, e^x, \sin x, \text{ etc. but not } y^n, e^y, \cos y, \text{ etc.} \end{array}\right\} \tag{1}$$

Example of nonlinear DE: $3x^3y'' + yy' = e^x$

$y = f(x)$ is a **solution** of a DE on an interval I (interval of existence of solution):

- y satisfies the DE.
- y, y', y'', ... are continuous on I.

1.1 Initial Value Problems (IVP)

The first-order DE $y' = f(x, y)$ subject to $y(x_0) = y_0 \rightarrow$ is an IVP.
The second-order DE $y'' = f(x, y, y')$ subject to $(y(x_0) = y_0, y'(x_0) = y_1)$ is an IVP, while $(y(x_0) = y_0, y(x_1) = y_1)$ is a **Boundary Value Problem (BVP)**.

- Note: After y_1 both cases can be used.

1.1.1 Example 1

$y = c_1 \cos x + c_2 \sin x$ is a soluation of the DE $y'' + y = 0$. Find the solution subject to the conditions $y(\pi) = 1, y'(\pi) = -2$

Solution: $y(\pi) = 1 \Rightarrow 1 = c_1 \cos \pi + c_2 \sin \pi$
$\Rightarrow 1 = -c_1 \Rightarrow c_1 = -1$

Then: $y = c_1 \cos x + c_2 \sin x \Rightarrow y' = -c_1 \sin x + c_2 \cos x$
Using $y'(\pi) = -2 \Rightarrow -2 = -c_1 \sin pi + c_2 \cos pi \Rightarrow -2 = -c_2 \Rightarrow c_2 = 2$

The solution is $y = -\cos x + 3 \sin x$

1.1.2 Example 2

$y = \frac{1}{x^2+c}$ is the one parameter solution of the DE $y' + 2xy^2 = 0$. Find a solution of the IVP:
$y' + 2xy^2 = 0, y(-3) = \frac{1}{5}$. Give the longest interval over which the solution is defined.

Solution: Using $y(-3) = \frac{1}{5}$, $y = \frac{1}{x^2+c} \Rightarrow \frac{1}{5} = \frac{1}{(-3)^2+c}$
$\Rightarrow \frac{1}{5} = \frac{1}{9+c}$
$\Rightarrow 9 + c = 5$
$\Rightarrow c = -4$
Thus $y = \frac{1}{x^2-4}$ is a solution of the IVP.

y is continuous when $x^2 - 4 \neq 0 \Rightarrow x^2 \neq 4 \Rightarrow x \neq \pm 2$
$y = \frac{1}{x^2-4} = \left(x^2 - 4\right)^{-1}$
$y' = -1\left(x^2 - 4\right)^{-2}(2x) = \frac{-2x}{(x^2-4)^2}$ is continuous when $x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$

The longest interval is $(-\infty, -2)$ on which the solution is defined.

1.1.3 Not every DE is solvable

Consider the first-order IVP $xy' = 2y, y(0) = 0$. $y = 0$ is a solution $y = 0, y' = 0 \Rightarrow x(0) = 2(0) \Rightarrow 0 = 0$

- **Note:** a solution must be valid for all values of x.

$y = x^2$ is also a solution $\Rightarrow y' = 2x \Rightarrow x(2x) = 2x^2 \Rightarrow 2x^2 = 2x^2 \Rightarrow y(0) = 0$

1.2 Existence Theorem

Theorem 1.1 (1.2.1) *Let R be a rectangular region $R = \{(x, y)|a \leq x \leq b, c \leq y \leq d\}$ that contains (x_0, y_0) in its interior. If f and $\frac{df}{dy}$ are continous on R, then there exists an interval $(x_0 - h, x_0 + h)$ in R on which the IVP $y' = f(x, y), y(x_0) = y_0$ has a unique solution.*

- **Note:** We need to have the form $y' = f(x, y)$ to decide f.

Examples:

- $f(x) = x^3 + \cos x \Rightarrow f'(x) = 3x^2 - \sin x$
- $f(x, y) = x^3 \cos y + e^y - x^7$

For a partial derivative of f with respect to $x \rightarrow \frac{df}{dx}$, we treat y as a constant.
For a partial derivative of f with respect to $y \rightarrow \frac{df}{dy}$, we treat x as a constant.
An example: $f(x, y) = x^3 \cos y + e^y - x^7$

- $\frac{df}{dx} = (\cos y)(3x^2) + 0 - 7x^6$
- $\frac{df}{dy} = x^3(-\sin y) + e^y + 0$

1.2.1 Example 1

Determine whether the existence theorem guarantees that the IVP $xy' = 2y, y(0) = 0$ has a unique solution.

Solution: $y' = \frac{2y}{x} \rightarrow f(x, y) = \frac{2y}{x}$ is continuous when $x \neq 0$
Conditions of existence theorem are not satisfied. So there is no guarantee of a unique solution.

1.2.2 Example 2

Determine a region R of the xy-plane for which the DE $\left(1 + y^3\right) y' = x^2$ would have a unique solution in an interval around (0,2) (a unique solution passing through (0,2)).

Solution: $(1 + y^3)y' = x^2 \Rightarrow y' = \frac{x^2}{1+y^3} \Rightarrow f(x, y) = \frac{x^2}{1+y^3} = x^2(1 + y^3)^{-1}$ is continous when $1 + y^3 \neq 0, y^3 \neq 1, y \neq -1$

$\frac{df}{dy} = x^2\left[-(1 + y^3)^{-2}(3y^2)\right] = \frac{-3x^2y^2}{(1+y^3)^2}$
$\frac{df}{dy}$ is continuous when $1 + y^3 \neq 0 \Rightarrow y \neq -1$
Define $R = \{(x, y)| - 10 \leq x \leq 10, 0 \leq y \leq 9\}$

- **Note:** the boundaries of x can be anything since there are no restrictions on x, so long as it contains $x = 0$.
- The boundaries of y must contain $y = 2$ and must not cross $y = -1$.

1.2.3 Example 3

Find a region in the xy-plane on which the IVF $(1+y^3)y' = x^2, y(1) = -3$ would have a unique solution.

Solution: From example 2, $f, \frac{df}{dy}$ are continous when $y \neq -1$

$R = \{(x,y)| -3 \leq x \leq 5, -6 \leq y \leq -2\}$ where the x boundaries contain $x = 1$ and the y boundaries contain $y = -3$.

2 Solutions of First Order Differential Equations

In last class, 2.2, separable equation $\frac{dy}{dx} = g(x)h(y)$ which can turn into $\frac{dy}{h(y)} = g(x)dx$ and this can be integrated. When integrating, it must occur with the same variable.

- $\int x^2 dx = \frac{x^3}{3}$
- $\int x^2 du$ cannot integrate.

There are singular and implicit solutions.

2.1 Linear Equations

From section 1.1, nth order linear DE is represented by $\frac{d^n y}{dx^n}$ or $y^{(n)}$
The equation is $a_n(x)\frac{d^n y}{dx^n} + a_{n-1}y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$
First order is only concerned with $a_1(x)y' + a_0(x)y = g(x)$
Linear equations are easy to solve and get explicit solutions easily. However there are **no** singular solutions of linear DEs.

2.1.1 Solving Linear 1st Order DEs

Step 1 Wrote tje DE in the standard form $y' + P(x)y = f(x)$
$a_1(x)y' + a_0(x)y = g(x)$ $y' + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)} \Rightarrow y' + P(x)y = f(x)$ (standard form of 1st order linear DE)

Step 2 Find the integrating factor (I.F.)
I.F.$= e^{\int P(x)dx}$ (do not write $+C$ in this integration since that adds a wasted simplification step)

Step 3 Multiply every term of the standard form equation (from **Step 1**) by the I.F.
$e^{\int P(x)dx}y' + e^{\int P(x)dx}P(x)y = f(x)e^{\int P(x)dx}$
The LHS will automatically be $\frac{d}{dx}((I.F.)y)$ i.e. $\frac{d}{dx}\left(e^{\int P(x)dx}y\right)$
$= e^{\int P(x)dx}y' + ye^{\int P(x)dx}(P(x))$

Step 4 Integrate both sides w.r.t. x.
$(I.F.)y = \int f(x)e^{\int P(x)dx}dx$
The explicit solution will be $y = \dfrac{\int f(x)e^{\int P(x)dx}dx}{I.F.}$

We get $y' + P(x)y = f(x)$. The interval of existence of solution is the interval over which $P(x)$ **and** $f(x)$ **are continuous**.
As a reminder of section 1.2, for any 1st order DE $y' = f(x, y); y(x_0) = y_0$, if $f \& \frac{df}{dy}$ is continuous on R, then a unique solution exists.
For linear equations, $y' = f(x) - P(x)y$, $\frac{df}{dy} = 0 - P(x)(1)$, where $f(x) - P(x)y = f(x, y)$

2.1.2 Example 1

Solve the differential equation $y' - 2xy = x$. Give the largest interval I over which the solution is defined.

Solution $y' - 2xy = x$ is a linear equation where $P(x) = -2x$, $f(x) = x$
Since $P(x)$ and $f(x)$ are continuous on R, then the largest interval of existence of the solutio is $I = (-\infty, \infty)$

$\int P(x)dx = \int -2xdx = \frac{-2x^2}{2} = -x^2$ (remember, don't write $+C$)
Thus $I.F. = e^{\int P(x)dx} = e^{-x^2}$

Next, multiple the DE by the I.F.
$e^{-x^2}y' - 2xe^{-x^2}y = e^{-x^2}x$, $\frac{d}{dx}\left(e^{-x^2}y\right) = e^{-x^2}x$
Integrating w.r.t. x: $e^{-x^2}y = \int e^{-x^2}xdx$, let $u = -x^2$, $du = -2xdx$
$\int e^u \frac{du}{-2} = \frac{-1}{2}\int e^u du = \frac{-1}{2}e^u + C = \frac{-e^{-x^2}}{2} + C$
Thus $e^{-x^2}y = \frac{-1}{2}e^{-x^2} + C \Rightarrow y = \frac{\frac{-1}{2}e^{-x^2}}{e^{-x^2}} + \frac{C}{e^{-x^2}} \Rightarrow y = \frac{-1}{2} + Ce^{x^2}$

Question: Is the equation $y' - 2xy = x$ separable?
Answer: If you can separate x from y, then it is separable. So $y' - 2xy = x \Rightarrow y' = 2xy + x \Rightarrow y' = x(2y+1)$
Thus, the equation is separable.

2.1.3 Example 2

Solve the IVP: $xy' + 2y = 12x^4$; $y(1) = 4$. Give the largest interval over which the solution is defined.

Solution $xy' + 2y = 12x^4 \Rightarrow y' + \frac{2}{x}y = 12x^3$ is standard form. $P(x) = \frac{2}{x}$ & $f(x) = 12x^3$ are continuous on R except at $x = 0$. Thus the longest interval of existence of the solution is $(0, \infty)$.
$I.F. = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2$

Next we multiply the standard form function by the I.F. to get $x^2y' + x^2\left(\frac{2}{x}\right)y = x^2(12x^3) \Rightarrow \frac{d}{dx}(x^2y) = 12x^5$
Now we integrate w.r.t. x: $x^2y = \int 12x^5dx = 2x^6 + C$
Thus $y = \frac{2x^6}{x^2} + \frac{C}{x^2} \Rightarrow y = 2x^4 + \frac{C}{x^2}$

For the IVP, $y(1) = 4$, $4 = 2(1) + \frac{C}{1} \Rightarrow C = 2$
Thus the solution of the IVP is $y = 2x^4 + 2$
Now, find the transient term in the solution: $\lim_{x\to\infty} \frac{2}{x^2} = 0 \Rightarrow \frac{2}{x^2}$ is the transient term.

- **Note:** a **transient term** is a term which approaches 0 as x approaches ∞.

2.1.4 Example 3

Find the general solution (a linear DE) of $xy' + (1+x)y = e^{-x}\sin^2 x$ (or, solve the DE).

Solution $y' + \frac{1+x}{x}y = \frac{e^{-x}\sin^2 x}{x}$ is standard form, $P(x) = \frac{1+x}{x}$
$\int P(x)dx = \int \frac{1+x}{x}dx = \int\left(\frac{1}{x} + 1\right)dx = \ln|x| + x$ (no $+C$)
Thus $I.F. = e^{\int P(x)dx} = e^{\int \ln|x| + xdx} = e^{\ln|x|} \cdot e^x = |x|e^x \Rightarrow I.F. = xe^x$
$xe^xy' + xe^x(1+x)y = xe^x\frac{e^{-x}\sin^2 x}{x} \Rightarrow \frac{d}{dx}(xe^xy) = \sin^2 x$

Integrating w.r.t. x: $xe^xy = \int \sin^2 xdx = \frac{1}{2}\int(1-\cos 2x)dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)$
Thus $xe^xy = \frac{1}{2}\left(x - \frac{sin2x}{2}\right) + C \Rightarrow y = \frac{\frac{x}{2} - \frac{\sin 2x}{4}}{xe^x} + \frac{C}{xe^x}$

2.2 (2.4) Exact Equations

First order DEs:

$$\left\{ \begin{array}{ll} 2.2 \rightarrow \text{ Separable} & \frac{dy}{dx} = g(x)h(y) \\ 2.3 \rightarrow \text{ Linear equation} & y' + P(x)y = f(x) \end{array} \right\} \tag{2}$$

Differential: $\delta x = dx$

- $y = f(x) \rightarrow$ differential is $dy = f'dx$
- $z = f(x,y) \rightarrow$ differential is $dz = \frac{df}{dx} + \frac{df}{dy}dy$
- $\frac{df}{dx} \rightarrow$ treat y as a constant.
- $\frac{df}{dx} \rightarrow$ treat x as a constant.

Consider $f(x,y) = x^3 + x^2y \Rightarrow \frac{df}{dx} = 3x^2 + (2x)y$ and $\frac{df}{dy} = 0 + x^2(1)$
$x^3 + x^2y = 7$ [1], take derivatives to get: $(3x^2 + 2xy)dx + x^2dy = 0$ [2]. [1] is a solution of the DE [2].

A DE of the form $\frac{df}{dx}dx + \frac{df}{dy}dy = 0$ is called an **exact DE**.

How to decide if a DE is exact A DE of the form $M(x,y)dx + N(x,y)dy = 0$ is an exact equation iff $\frac{dM}{dy} = \frac{dN}{dx}$ or $M_y = N_x$

How to solve the exact equation We have two equations $\frac{df}{dx} = M$ [1] and $\frac{df}{dy} = N$ [2]

- The partial integration $\int \frac{df}{dx}dx = f(x,y)$
- The partial integration $\int \frac{df}{dy}dy = f(x,y)$

Doing the partial integration of [1]: $f(x,y) = \int Mdx + g(y)$, where $g(y)$ is the constant of integration.
Find $\frac{df}{dy}$ and substitute in [2] to find $g(y)$.
$f(x,y) = ...$ (the solution is $f(x,y) = C$)

2.2.1 Exampole 1

Determine whether the DE is exact or not. If it is exact, then solve it. $y' = \frac{-2xy}{1+x^2}$

Solution $y' = \frac{-2xy}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{-2xy}{1+x^2} \Rightarrow (1+x^2)dy = -2xydx \Rightarrow 2xydx + (1+x^2)dy = 0$
$M_y = \frac{dM}{dy} = 2x(1)$ and $N_x = \frac{dN}{dx} = 2x$, $M_y = N_x \Rightarrow$ exact.

To solve, $\frac{df}{dx} = M$ and $\frac{df}{dy} = N \Rightarrow \frac{df}{dx} = 2xy$ [1] and $\frac{df}{dy} = 1 + x^2$ [2]
Perform partial integration of [1] w.r.t. x: $f(x,y) = 2y\frac{x^2}{2} + g(y)$ [3] $\Rightarrow \frac{df}{dy} = (1)x^2 + g'(y)$
Next substitute equation [2]: $x^2 + g'(y) = 1 + x^2 \Rightarrow g'(y) = 1$ (cannot have any x terms)
Integrating w.r.t. y: $g(y) = \int 1dy \Rightarrow g(y) = y$ or $g(y) = y + C$ (preferred without the $+C$)
Substitute $g(y)$ in [3] to get $f(x,y) = x^2y + y$ or $f(x,y) = x^2y + y + C$

The solution of the DE is $f(x,y) = C \Rightarrow x^2y + y = C$ or $f(x,y) = 0 \Rightarrow x^2y + y + C = 0$

2.2.2 Example 2

Determine whether the DE is exact or not. If it is exact, then solve it. $(xy + y^2)dx + (x^2 + xy)dy = 0$

Solution $M_y = x(1) + 2y$ and $N_x = 2x + (1)y \Rightarrow M_x \neq N_y \Rightarrow$ not exact.

2.2.3 Example 3

Determine whether the DE is exact or not. If it is exact, then solve it. $(x + \sin y)dx = (e^y = x\cos y)dy$

Solution $(x + \sin y)dx - (e^y - x\cos y)dy = 0$
Here, $M = x + \sin y$ and $N = -(e^y - x\cos y) = -e^y + x\cos y$
$M_y = \cos y$ and $N_x = (1)\cos y \Rightarrow M_y = N_x \Rightarrow$ Exact.

To solve, $\frac{df}{dx} = M \Rightarrow \frac{df}{dx} = x + \sin y$ [1]
$\frac{df}{dy} = N \Rightarrow \frac{df}{dy} = -e^y + x\cos y$ [2]
Partial integration of [2] w.r.t. y: $f(x, y) = -e^y + x\sin y + g(x)$ [3]
$\frac{df}{dx} = 0 + (1)\sin y + g'(x)$
Substitute in [1]: $\sin y + g'(x) = x + \sin y \Rightarrow g'(x) = x$ (cannot have y) $\Rightarrow g(x) = \frac{x^2}{2}$
Substitute $g(x)$ in [3]: $f(x, y) = -e^y + x\sin y + \frac{x^2}{2}$
The solution is $f(x, y) = C \Rightarrow -e^y + x\sin y + \frac{x^2}{2} = C$

2.2.4 Example 4

Solve the IVP: $(ey - xe^{xy})y' = 2 + ye^{xy}\ ;\ y(0) = 1$

Solution $(2y - xe^{xy})dy = (2 + ye^{xy})dx \Rightarrow (2 + ye^{xy})dx - (2y - xe^{xy}) = 0$
$M = 2 + ye^{xy} \Rightarrow \frac{dM}{dy} = 0 + (1)e^{xy} + ye^{xy}(x)$
$N = -2y + xe^{xy} \Rightarrow \frac{dN}{dx} = 0 + (1)e^{xy} + xe^{xy}(y)$
Since $M_y = N_x$, this is an exact equation.

$\frac{df}{dx} = M \Rightarrow \frac{df}{dx} = 2 + ye^{xy}$ [1]
$\frac{df}{dy} = N \Rightarrow \frac{df}{dy} = -2y + xe^{xy}$ [2]
Partial integration of [1] w.r.t. x: $f(x, y) = \int(2 + ye^{xy})dx = 2x + y\frac{e^{xy}}{y} + g(y)$ [3]
$\frac{df}{dy} = 0 + e^{xy}(x) + g'(y)$
Substitute in [2]: $xe^{xy} + g'(y) = -2y + xe^{xy} \Rightarrow g'(y) = -2y$
$g(y) = \int -2y dy = -2\frac{y^2}{2}$
Substituting in [3]: $f(x, y) = 2x + e^{xy} - y^2$
The solution of the DE is $f(x, y) = C \Rightarrow 2x + e^{xy} - y^2 = C$

$y(0) = 1 \Rightarrow$ Substitute $x = 0$ and $y = 1$: $2(0) + e^{(0)(1)} - (1)^2 = C \Rightarrow 1 - 1 = C \Rightarrow C = 0$
The solution of the IVP is $ex + e^{xy} - y^2 = 0$

2.3 More on Exact Equations

Sometimes, we can multiply the DE by an integrating factor and the DE becomes an exact DE. We solve by the mothod of exact equations:

1. If $\frac{M_y - N_x}{N}$ is a function of x only (no y terms), then $I.F. = \mu = e^{\int \frac{M_y - N_x}{N} dx}$
2. If $\frac{N_x - M_y}{M}$ is a function of y only (no x terms), then $I.F. = \mu = e^{\int \frac{N_x - M_y}{M} dy}$

2.3.1 Example 1

Find an I.F. to make this DE exact and then solve it: $(y^2 - y)dx + xdy = 0$

Solution $M = y^2 - y,\ M_y = 2y - 1,\ N = x,\ N_x = 1 \qquad M_y \neq N_x \Rightarrow$ not exact.
$\frac{M_y - N_x}{N} = \frac{(2y-1)-1}{x}$ (not in terms of x only)
$\frac{N_x - M_y}{M} = \frac{1-(2y-1)}{y^2-y} = \frac{1-2y+1}{y^2-y} = \frac{2-2y}{y^2-y} = \frac{-2(1-y)}{y(y-1)} = \frac{-2}{y}$ (good!)
$I.F. = \mu = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int \frac{-2}{y} dy} = e^{-2\ln|y|} = e^{\ln|y|^{-2}} = |y|^{-2} = \frac{1}{y^2}$

Multiply the given DE by $\frac{1}{y^2}$: $\frac{1}{y^2}(y^2 - y)dx + \frac{1}{y^2}xdy = 0 \Rightarrow (1 - \frac{1}{y})dx + \frac{x}{y^2}dy = 0$
Now, $M = 1 - \frac{1}{y}$ and $N = \frac{x}{y^2} \Rightarrow M_y = -(-1y^{-2}) = \frac{1}{y^2},\ \ N_x = \frac{1}{y^2}(1) = \frac{1}{y^2}$ (unnecessary to check)

To solve: $\frac{df}{dx} = M \Rightarrow \frac{df}{dx} = 1 - \frac{1}{y}$ [1] $\quad \frac{df}{dy} = N \Rightarrow \frac{df}{dy} = \frac{x}{y^2}$ [2]
Partial integration of [1] w.r.t. $x \Rightarrow f(x,y) = x - \frac{1}{y}(x) + g(y)$ [3]
$\frac{df}{dy} = 0 + \frac{x}{y^2} + g'(y)$ (substitute this into [2])
$\frac{x}{y^2} + g'(y) = \frac{x}{y^2} \Rightarrow g'(y) = 0 \Rightarrow g(y) = 0$ or $g(y) = C$
So [3] gives $f(x,y) = x - \frac{x}{y}$ or $f(x,y) = x - \frac{x}{y} + C$
Thus the solution is $f(x,y) = C \Rightarrow x - \frac{x}{y} = C$ or $f(x,y) = 0 \Rightarrow x - \frac{x}{y} + C = 0$

Now, this is only the case if $y \neq 0$ (since the I.F. involves y being in the denominator).
So we can check if $y = 0$ is a solution: $dy = y'dx = 0dx = 0$
$(0^2 - 0)dx + xdy = 0 \Rightarrow (0^2 - 0)dx + x(0) = 0 \Rightarrow 0 + 0 = 0 \Rightarrow y = 0$ is a solution.

2.3.2 Example 2

Find an I.F. to make this DE exact: $y(x + y + 1)dx + (x + 2y)dy = 0$ (do not solve the equation)

Solution $M = xy + y^2 + y$ and $N = x + 2y \quad M_y = x + 2y + 1$ and $N_x = 1$
$\frac{M_y - N_x}{N} = \frac{(x+2y+1)-1}{x+2y} = \frac{x+2y}{x+2y} = 1 \Rightarrow I.F. = \mu = e^{\int 1dx} = e^x$

2.4 Solutions by Substitution

We do 3 kinds of substitutions which can be used to solve 1st order DEs.

2.5 Homogeneous Equations

A function $f(x,y)$ is called a homogeneous function of degree a if $f(tx,ty) = t^a f(x,y)$

To determine if the equation $f(x,y) = x^2 + xy$ is homogeneous, calculate $f(tx,ty)$.
$f(tx,ty) = (tx)^2 + (tx)(ty) = t^2x^2 + t^2xy = t^2(x^2 + xy) = t^2 f(x,y) \Rightarrow$ Thus f is a homogeneous function of degree 2.
For more functions:

- $f(x,y) = x + 3$, $f(tx,ty) = tx + 3 \Rightarrow$ not homogeneous.
- $f(x,y) = \sin x - \sin y \quad f(tx,ty) = \sin tx - \sin ty \Rightarrow$ not homogeneous. (Usually anything with $\sin x$ or $\sin y$ is not homogeneous)
- $f(x,y) = \sin \frac{x}{y} \quad f(tx,ty) = \sin \frac{tx}{ty} = \sin \frac{x}{y} = t^0 \sin \frac{x}{y} \Rightarrow$ homogeneous. (this is an exception to the above)
- $f(x,y) = \ln x - \ln y \quad f(tx,ty) = \ln tx - \ln ty = \ln \frac{tx}{ty} = \ln \frac{x}{y} = t^0(\ln x - \ln y) \Rightarrow$ homogeneous of degree 0.

A differential equation $Mdx + Ndy = 0$ is homogeneous if both M and N are homogeneous functions of the same degree.

Example 1 For $(x^2 + xy)dx + xy^2dy = 0$, M is homogeneous of degree 2 and N is homogeneous of degree 3. Thus the equation is not homogeneous.

Example 2 For $(x + 3)dx + ydy = 0$, it is not homogeneous because M is not homogeneous.

Example 3 For $(x^2 + xy)dx + (xy - y^2)dy = 0$, since both M and N are homogeneous of degree 2, thus the equation is homogeneous. (you can often check the power of M and N to check if it's homogeneous)

2.5.1 Solving a Homogeneous Equation

Case 1: Let $y = ux$ and $dy = udx + udu \rightarrow$ the homogeneous equation will become a separable equation in u and x. Solve it and then replace $u = \frac{y}{x}$ in the solution.
Case 2: Let $x = vy$ and $dx = vdy + ydv \rightarrow$ the homogeneous equation will become a separable equation in v and y. Solve it and then replace $v = \frac{x}{y}$ in the solution.

Example Solve the DE by using an approprite substitution: $ydx - 2(x+y)dy = 0$

Solution Let $y = ux$ and $dy = udx + xdu$ **or** let $x = vy$ and $dx = vdy + ydv$

$uxdx - (2x + 2ux)(udx + xdu) = 0$ **or** $y(vdy + ydv) - (2vy + 2y)dy = 0$

This is harder to solve **or** $vydy + y^2dv - 2vydy - 2ydy = 0$

$y^2dv - vydy - 2ydy = 0$

$y^2dv = vydy + 2ydy \Rightarrow y^2dv = ydy(v+2) \Rightarrow \frac{dv}{v+2} = \frac{ydy}{y^2} \Rightarrow \frac{dv}{v+2} = \frac{1}{y}dy \rightarrow$ a separable equation.

If we went the u way: $\int \frac{-dx}{x} = \int \frac{2+2u}{u+2u^2}du \ldots$ needs partial fraction.
Instead, $\int \frac{dv}{v+2} = \int \frac{1}{y}dy \Rightarrow \ln|v+2| = \ln|y| + C$
Replace v by $\frac{x}{y} \Rightarrow \ln|\frac{x}{y} + 2| = \ln|y| + C$

Is $y = 0$ also a solution?
Well, $y = 0 \Rightarrow dy = 0$. Substitute into original equation: $(0)dx - 2(x+0)(0) = 0 \Rightarrow 0 - 0 = 0$
$\Rightarrow y = 0$ is also a solution.
$v + 2 = 0 \Rightarrow \frac{x}{y} + 2 = 0 \Rightarrow \frac{x}{y} = -2 \Rightarrow x = -2y \Rightarrow y = \frac{-x}{2}$
Is this a solution? Well, $y = \frac{-x}{2} \Rightarrow dy = y'dx = \frac{-1}{2}dx$
Substitute into original equation: $\frac{-x}{2}dx - 2(x - \frac{x}{2})(\frac{-1}{2}dx) = 0 \Rightarrow \frac{-x}{2}dx + \frac{x}{2}dx = 0 \Rightarrow 0 = 0$
$\Rightarrow y = \frac{-x}{2}$ is also a solution.

2.6 Bernoulli's Equation

$\frac{dy}{dx} + P(x)y = f(x)y^n$ where $n \in R, n \neq 0$ and $n \neq 1$
If $n = 0 \Rightarrow \frac{dy}{dx} + P(x)y = f(x)(1) \rightarrow$ linear.
If $n = 1 \Rightarrow \frac{dy}{dx} + P(x)y = f(x)y \Rightarrow \frac{dy}{dx} + (P(x) - f(x))y = 0 \rightarrow$ linear equation.
If $n \neq 0$ and $n \neq 1$ then we use the substitution $u = y^{1-n}$ and the Bernoulli's equation changes to a linear equation $\rightarrow$ solve linear equation $\rightarrow$ replace u.

Another way: for the equation $\frac{dy}{dx} + P(x)y = f(x)y^n$, let $u = y^{1-n}$ and the equation turns into:
$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x) \Rightarrow$ linear equation in u.

2.6.1 Example 1

Solve the DEs using an appropriate substitution: $\frac{dy}{dx} - y = e^x y^2$

Solution This is Bernoulli's equation with $n = 2$.
Let $u = y^{1-n} = y^{1-2} = y^{-1} = \frac{1}{y}$ if $y \neq 0$
$\Rightarrow y = \frac{1}{u} = u^{-1}$ and $\frac{dy}{dx} = -1u^{-2}\frac{du}{dx} = \frac{-1}{u^2}\frac{du}{dx}$
Substituting in the original equation: $\frac{-1}{u^2}\frac{du}{dx} - \frac{1}{u} = e^x\frac{1}{u^2}$
Multiplying by $-u^2 \Rightarrow \frac{du}{dx} + u = -e^x$ [1]

An alternate route: Let $u = y^{1-n}$. Then Bernoulli's equation becomes $\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x)$
$\Rightarrow \frac{du}{dx} - (-1)u = (-1)e^x$
$\frac{du}{dx} + u = -e^x$ (linear in u)

$I.F. = e^{\int P(x)dx} = e^{\int 1dx} = e^x$
Multiplying [1] by $e^x \Rightarrow e^x\frac{du}{dx} + e^x u = -e^x e^x$
$\frac{d}{dx}(e^x u) = -e^{2x}$
Integrating w.r.t. $x \Rightarrow e^x u = \int -e^{2x}dx \Rightarrow e^x u = \frac{-e^{2x}}{2} + C \Rightarrow u = \frac{-e^{2x}}{2e^x} + \frac{C}{e^x}$
Replace $u = \frac{1}{y} \Rightarrow \frac{1}{y} = -\frac{e^x}{2} + Ce^{-x}$

2.6.2 Example 2

Solve the DEs using an appropriate substitution: $\frac{dy}{dx} - y = e^x y^2,\ y(0) = 1$ (note: $y = 0$ does not satisfy $y(0) = 1$)

Solution From example 1: $\frac{1}{y} = \frac{-e^x}{2} + Ce^{-x}$
Set $x = 0$ and $y = 1 \Rightarrow \frac{1}{1} = \frac{-e^0}{2} + Ce^{-0} \Rightarrow 1 = \frac{-1}{2} + C \Rightarrow C = \frac{3}{2}$
The solution is $\frac{1}{y} = \frac{-e^x}{2} + \frac{3}{2}e^{-x}$

2.6.3 Example 3

Solve the DEs using an appropriate substitution: $xy' - x^5 y^{\frac{1}{3}} = 3y$

Solution $\Rightarrow y' - \frac{x^5}{x} y^{\frac{1}{3}} = \frac{3}{x} y$
$\Rightarrow y' - \frac{3}{x} y = x^4 y^{\frac{1}{3}}$ [$*$] This is Bernoulli's equation with $n = \frac{1}{3}$
Let $u = y^{1-n} = y^{1-\frac{1}{3}} = y^{\frac{2}{3}}$
$u' - (\frac{2}{3}) \frac{3}{x} u = \frac{2}{3} x^4$
$\frac{du}{dx} - \frac{2}{x} u = \frac{2}{3} x^4$ [2] linear equation in u

$I.F. = e^{\int \frac{-2}{x} dx} = e^{-2 \ln |x|} = e^{\ln |x|^{-2}} = |x|^{-2} = \frac{1}{x^2}$
Multiply [2] by $\frac{1}{x^2} \Rightarrow \frac{1}{x^2} \frac{du}{dx} - \frac{2}{x} \frac{1}{x^2} u = \frac{2}{3} \frac{1}{x^2}$
$\int \frac{d}{dx} (\frac{1}{x^2} u) dx = \int \frac{2}{3} x^2 dx$
$\frac{1}{x^2} u = \frac{2}{3} \frac{x^3}{3} + C \Rightarrow u = \frac{2}{9} x^3 x^2 + C x^2$
Replace $u = y^{\frac{2}{3}} \Rightarrow y^{\frac{2}{3}} = \frac{2}{9} x^5 + C x^2$

2.7 Linear Substition

A DE of the form $\frac{dy}{dx} = f(Ax + By + C)$ where $B \neq 0$ can be converted to a separable equation by using the substitution $u = Ax + By + C$, then solve the separable equation and replace u.

2.7.1 Example 1

Solve the DE by using an appropriate substitution: $\frac{dy}{dx} = 2 + e^{y-2x+6}$

Solution Let $u = y - 2x + 6 \Rightarrow y = u + 2x - 6$ and $\frac{dy}{dx} = \frac{du}{dx} + 2$
Sub into original equation: $\frac{du}{dx} + 2 = 2 + e^u \Rightarrow \frac{du}{dx} = e^u \Rightarrow \int \frac{du}{e^u} = \int dx \Rightarrow$ separable equation.
$\frac{e^{-u}}{-1} = x + C$, replace $u = y + -2x + 6 \Rightarrow -e^{-(y-2x+6)} = x + C$

2.7.2 Example 2

Solve the DE by using an appropriate substitution: $\frac{dy}{dx} = \frac{1-x-y}{x+y}$

Solution $\Rightarrow \frac{dy}{dx} = \frac{1-(x+y)}{x+y}$, we can have a linear substitution $f(x+y)$
Let $u = x + y \Rightarrow y = u - x$ and $\frac{dy}{dx} = \frac{du}{dx} - 1$
Sub into original equation: $\frac{du}{dx} - 1 = \frac{1-u}{u} \Rightarrow \frac{du}{dx} = 1 + \frac{1-u}{u} = \frac{u+1-u}{u} = \frac{1}{u}$
$udu = dx$ (cross multiplication of $\frac{du}{dx} = \frac{1}{u}$), so $\int udu = \int dx \Rightarrow$ separable equation
$\frac{u^2}{2} = x + C$
Replace $u = x + y \Rightarrow \frac{(x+y)^2}{2} + x + C$

Now, back to the beginning: $\frac{Dy}{dx} = \frac{1-x-y}{x+y}$ is not separable, ont linear, not Bernoulli's equation.
We can check homogeneous and exact$\Rightarrow Mdx + Ndy = 0 \Rightarrow (x+y)dy = (1-x-y)dx$
$\Rightarrow (1-x-y)dx - (x+y)dy = 0 \Rightarrow$ not homogeneous ($1 - x - y$ is not homogeneous)
$M = 1 - x - y$ and $N = -(x+y) \Rightarrow M_y = -1$ and $N_x = -1 \Rightarrow$ exact equation.
You can solve by exact equation to get $x - \frac{x^2}{2} - xy - \frac{y^2}{2} = C$

3 Real Life Problems

Real life problems can be modelled as differential equations (especially first-order DEs). We will discuss this further:

3.1 Linear Models

Real life applications can be written as 1st order linear DEs (section 2.1 in this document). Let's look at some examples:

3.1.1 Growth and Decay

If P is a population at time t, then the rate of change of that population is proportional to $P \Rightarrow \frac{dP}{dt} = kP$ where $k > 0$