Differential Equations

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1 Introduction to Differential Equations

Differential Equation Any equation that is differentiable. An example is $3y'' + xy' - x^2 = e^x$
The **order** of a DE is the highest order of derivative.

$$\textbf{Linear DE} \left\{ \begin{array}{ll} (i) y, y', y'', ..., & \text{Cannot have more than one power} \\ (ii)\text{We can have },\ & x^n, e^x, \sin x, \text{ etc. but not } y^n, e^y, \cos y, \text{ etc.} \end{array} \right\} \tag{1}$$

Example of nonlinear DE: $3x^3y'' + yy' = e^x$

$y = f(x)$ is a **solution** of a DE on an interval I (interval of existence of solution):

- y satisfies the DE.
- y, y', y'', ... are continuous on I.

1.1 Initial Value Problems (IVP)

The first-order DE $y' = f(x, y)$ subject to $y(x_0) = y_0 \to$ is an IVP.
The second-order DE $y'' = f(x, y, y')$ subject to $(y(x_0) = y_0, y'(x_0) = y_1)$ is an IVP, while $(y(x_0) = y_0, y(x_1) = y_1)$ is a **Boundary Value Problem (BVP)**.

- Note: After y_1 both cases can be used.

1.1.1 Example 1

$y = c_1 \cos x + c_2 \sin x$ is a soluation of the DE $y'' + y = 0$. Find the solution subject to the conditions $y(\pi) = 1, y'(\pi) = -2$

Solution: $y(\pi) = 1 \Rightarrow 1 = c_1 \cos \pi + c_2 \sin \pi$
$\Rightarrow 1 = -c_1 \Rightarrow c_1 = -1$

Then: $y = c_1 \cos x + c_2 \sin x \Rightarrow y' = -c_1 \sin x + c_2 \cos x$
Using $y'(\pi) = -2 \Rightarrow -2 = -c_1 \sin pi + c_2 \cos pi \Rightarrow -2 = -c_2 \Rightarrow c_2 = 2$

The solution is $y = -\cos x + 3 \sin x$

1.1.2 Example 2

$y = \frac{1}{x^2+c}$ is the one parameter solution of the DE $y' + 2xy^2 = 0$. Find a solution of the IVP:
$y' + 2xy^2 = 0, y(-3) = \frac{1}{5}$. Give the longest interval over which the solution is defined.

Solution: Using $y(-3) = \frac{1}{5}$, $y = \frac{1}{x^2+c} \Rightarrow \frac{1}{5} = \frac{1}{(-3)^2+c}$
$\Rightarrow \frac{1}{5} = \frac{1}{9+c}$
$\Rightarrow 9 + c = 5$
$\Rightarrow c = -4$
Thus $y = \frac{1}{x^2-4}$ is a solution of the IVP.

y is continuous when $x^2 - 4 \neq 0 \Rightarrow x^2 \neq 4 \Rightarrow x \neq \pm 2$
$y = \frac{1}{x^2-4} = \left(x^2-4\right)^{-1}$
$y' = -1\left(x^2-4\right)^{-2}(2x) = \frac{-2x}{(x^2-4)^2}$ is continuous when $x^2 - 4 \neq 0 \Rightarrow x \neq \pm 2$

The longest interval is $(-\infty, -2)$ on which the solution is defined.

1.1.3 Not every DE is solvable

Consider the first-order IVP $xy' = 2y, y(0) = 0$. $y = 0$ is a solution $y = 0, y' = 0 \Rightarrow x(0) = 2(0) \Rightarrow 0 = 0$

- **Note:** a solution must be valid for all values of x.

$y = x^2$ is also a solution $\Rightarrow y' = 2x \Rightarrow x(2x) = 2x^2 \Rightarrow 2x^2 = 2x^2 \Rightarrow y(0) = 0$

1.2 Existence Theorem

Theorem 1.1 (1.2.1) *Let R be a rectangular region $R = \{(x,y)|a \leq x \leq b, c \leq y \leq d\}$ that contains (x_0, y_0) in its interior. If f and $\frac{df}{dy}$ are continous on R, then there exists an interval $(x_0 - h, x_0 + h)$ in R on which the IVP $y' = f(x,y), y(x_0) = y_0$ has a unique solution.*

- **Note:** We need to have the form $y' = f(x,y)$ to decide f.

Examples:

- $f(x) = x^3 + \cos x \Rightarrow f'(x) = 3x^2 - \sin x$
- $f(x,y) = x^3 \cos y + e^y - x^7$

For a partial derivative of f with respect to $x \to \frac{df}{dx}$, we treat y as a constant.
For a partial derivative of f with respect to $y \to \frac{df}{dy}$, we treat x as a constant.
An example: $f(x,y) = x^3 \cos y + e^y - x^7$

- $\frac{df}{dx} = (\cos y)(3x^2) + 0 - 7x^6$
- $\frac{df}{dy} = x^3(-\sin y) + e^y + 0$

1.2.1 Example 1

Determine whether the existence theorem guarantees that the IVP $xy' = 2y, y(0) = 0$ has a unique solution.

Solution: $y' = \frac{2y}{x} \to f(x,y) = \frac{2y}{x}$ is continuous when $x \neq 0$
Conditions of existence theorem are not satisfied. So there is no guarantee of a unique solution.

1.2.2 Example 2

Determine a region R of the xy-plane for which the DE $\left(1+y^3\right)y' = x^2$ would have a unique solution in an interval around (0,2) (a unique solution passing through (0,2)).

Solution: $(1+y^3)y' = x^2 \Rightarrow y' = \frac{x^2}{1+y^3} \Rightarrow f(x,y) = \frac{x^2}{1+y^3} = x^2(1+y^3)^{-1}$ is continous when
$1 + y^3 \neq 0, y^3 \neq 1, y \neq -1$

$\frac{df}{dy} = x^2\left[-(1+y^3)^{-2}(3y^2)\right] = \frac{-3x^2y^2}{(1+y^3)^2}$
$\frac{df}{dy}$ is continuous when $1 + y^3 \neq 0 \Rightarrow y \neq -1$
Define $R = \{(x,y)| -10 \leq x \leq 10, 0 \leq y \leq 9\}$

- **Note:** the boundaries of x can be anything since there are no restrictions on x, so long as it contains $x = 0$.
- The boundaries of y must contain $y = 2$ and must not cross $y = -1$.

1.2.3 Example 3

Find a region in the xy-plane on which the IVF $(1 + y^3)y' = x^2, y(1) = -3$ would have a unique solution.

Solution: From example 2, $f, \frac{df}{dy}$ are continous when $y \neq -1$

$R = \{(x, y)| -3 \leq x \leq 5, -6 \leq y \leq -2\}$ where the x boundaries contain $x = 1$ and the y boundaries contain $y = -3$.

2 Solutions of First Order Differential Equations

In last class, 2.2, separable equation $\frac{dy}{dx} = g(x)h(y)$ which can turn into $\frac{dy}{h(y)} = g(x)dx$ and this can be integrated. When integrating, it must occur with the same variable.

- $\int x^2 dx = \frac{x^3}{3}$
- $\int x^2 du$ cannot integrate.

There are singular and implicit solutions.

2.1 Linear Equations

From section 1.1, nth order linear DE is represented by $\frac{d^n y}{dx^n}$ or $y^{(n)}$
The equation is $a_n(x)\frac{d^n y}{dx^n} + a_{n-1}y^{(n-1)} + \cdots + a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x)$
First order is only concerned with $a_1(x)y' + a_0(x)y = g(x)$
Linear equations are easy to solve and get explicit solutions easily. However there are **no** singular solutions of linear DEs.

2.1.1 Solving Linear 1st Order DEs

Step 1 Wrote tje DE in the standard form $y' + P(x)y = f(x)$
$a_1(x)y' + a_0(x)y = g(x)$ $y' + \frac{a_0(x)}{a_1(x)}y = \frac{g(x)}{a_1(x)} \Rightarrow y' + P(x)y = f(x)$ (standard form of 1st order linear DE)

Step 2 Find the integrating factor (I.F.)
I.F.$= e^{\int P(x)dx}$ (do not write $+C$ in this integration since that adds a wasted simplification step)

Step 3 Multiply every term of the standard form equation (from **Step 1**) by the I.F.
$e^{\int P(x)dx}y' + e^{\int P(x)dx}P(x)y = f(x)e^{\int P(x)dx}$
The LHS will automatically be $\frac{d}{dx}((I.F.)y)$ i.e. $\frac{d}{dx}\left(e^{\int P(x)dx}y\right)$
$= e^{\int P(x)dx}y' + ye^{\int P(x)dx}(P(x))$

Step 4 Integrate both sides w.r.t. x.
$(I.F.)y = \int f(x)e^{\int P(x)dx}dx$
The explicit solution will be $y = \dfrac{\int f(x)e^{\int P(x)dx}dx}{I.F.}$

We get $y' + P(x)y = f(x)$. The interval of existence of solution is the interval over which $P(x)$ **and** $f(x)$ **are continuous**.
As a reminder of section 1.2, for any 1st order DE $y' = f(x, y); y(x_0) = y_0$, if $f \& \frac{df}{dy}$ is continuous on R, then a unique solution exists.
For linear equations, $y' = f(x) - P(x)y$, $\frac{df}{dy} = 0 - P(x)(1)$, where $f(x) - P(x)y = f(x, y)$

2.1.2 Example 1

Solve the differential equation $y' - 2xy = x$. Give the largest interval I over which the solution is defined.

Solution $y' - 2xy = x$ is a linear equation where $P(x) = -2x$, $f(x) = x$
Since $P(x)$ and $f(x)$ are continuous on R, then the largest interval of existence of the solutio is $I = (-\infty, \infty)$

$\int P(x)dx = \int -2xdx = \frac{-2x^2}{2} = -x^2$ (remember, don't write $+C$)
Thus $I.F. = e^{\int P(x)dx} = e^{-x^2}$

Next, multiple the DE by the I.F.
$e^{-x^2}y' - 2xe^{-x^2}y = e^{-x^2}x$, $\frac{d}{dx}\left(e^{-x^2}y\right) = e^{-x^2}x$
Integrating w.r.t. x: $e^{-x^2}y = \int e^{-x^2}xdx$, let $u = -x^2$, $du = -2xdx$
$\int e^u \frac{du}{-2} = \frac{-1}{2}\int e^u du = \frac{-1}{2}e^u + C = \frac{-e^{-x^2}}{2} + C$
Thus $e^{-x^2}y = \frac{-1}{2}e^{-x^2} + C \Rightarrow y = \frac{\frac{-1}{2}e^{-x^2}}{e^{-x^2}} + \frac{C}{e^{-x^2}} \Rightarrow y = \frac{-1}{2} + Ce^{x^2}$

Question: Is the equation $y' - 2xy = x$ separable?
Answer: If you can separate x from y, then it is separable. So $y' - 2xy = x \Rightarrow y' = 2xy + x \Rightarrow y' = x(2y+1)$
Thus, the equation is separable.

2.1.3 Example 2

Solve the IVP: $xy' + 2y = 12x^4$; $y(1) = 4$. Give the largest interval over which the solution is defined.

Solution $xy' + 2y = 12x^4 \Rightarrow y' + \frac{2}{x}y = 12x^3$ is standard form. $P(x) = \frac{2}{x}$ & $f(x) = 12x^3$ are continuous on R except at $x = 0$. Thus the longest interval of existence of the solution is $(0, \infty)$.
$I.F. = e^{\int P(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\ln|x|} = e^{\ln|x|^2} = |x|^2$

Next we multiply the standard form function by the I.F. to get $x^2y' + x^2\left(\frac{2}{x}\right)y = x^2(12x^3) \Rightarrow \frac{d}{dx}(x^2y) = 12x^5$
Now we integrate w.r.t. x: $x^2y = \int 12x^5dx = 2x^6 + C$
Thus $y = \frac{2x^6}{x^2} + \frac{C}{x^2} \Rightarrow y = 2x^4 + \frac{C}{x^2}$

For the IVP, $y(1) = 4$, $4 = 2(1) + \frac{C}{1} \Rightarrow C = 2$
Thus the solution of the IVP is $y = 2x^4 + 2$
Now, find the transient term in the solution: $\lim_{x\to\infty} \frac{2}{x^2} = 0 \Rightarrow \frac{2}{x^2}$ is the transient term.

- **Note:** a **transient term** is a term which approaches 0 as x approaches ∞.

2.1.4 Example 3

Find the general solution (a linear DE) of $xy' + (1+x)y = e^{-x}\sin^2 x$ (or, solve the DE).

Solution $y' + \frac{1+x}{x}y = \frac{e^{-x}\sin^2 x}{x}$ is standard form, $P(x) = \frac{1+x}{x}$
$\int P(x)dx = \int \frac{1+x}{x}dx = \int \left(\frac{1}{x} + 1\right)dx = \ln|x| + x$ (no $+C$)
Thus $I.F. = e^{\int P(x)dx} = e^{\int \ln|x| + xdx} = e^{\ln|x|} \cdot e^x = |x|e^x \Rightarrow I.F. = xe^x$
$xe^xy' + xe^x(1+x)y = xe^x\frac{e^{-x}\sin^2 x}{x} \Rightarrow \frac{d}{dx}(xe^xy) = \sin^2 x$

Integrating w.r.t. x: $xe^xy = \int \sin^2 xdx = \frac{1}{2}\int(1 - \cos 2x)dx = \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right)$
Thus $xe^xy = \frac{1}{2}\left(x - \frac{sin2x}{2}\right) + C \Rightarrow y = \frac{\frac{x}{2} - \frac{\sin 2x}{4}}{xe^x} + \frac{C}{xe^x}$

2.2 (2.4) Exact Equations

First order DEs:

$$\left\{ \begin{array}{ll} 2.2 \rightarrow \text{ Separable} & \frac{dy}{dx} = g(x)h(y) \\ 2.3 \rightarrow \text{ Linear equation} & y' + P(x)y = f(x) \end{array} \right\} \tag{2}$$

Differential: $\delta x = dx$

- $y = f(x) \rightarrow$ differential is $dy = f' dx$
- $z = f(x, y) \rightarrow$ differential is $dz = \frac{df}{dx} + \frac{df}{dy} dy$
- $\frac{df}{dx} \rightarrow$ treat y as a constant.
- $\frac{df}{dx} \rightarrow$ treat x as a constant.

Consider $f(x, y) = x^3 + x^2 y \Rightarrow \frac{df}{dx} = 3x^2 + (2x)y$ and $\frac{df}{dy} = 0 + x^2(1)$
$x^3 + x^2 y = 7$ [1], take derivatives to get: $(3x^2 + 2xy)dx + x^2 dy = 0$ [2]. [1] is a solution of the DE [2].

A DE of the form $\frac{df}{dx} dx + \frac{df}{dy} dy = 0$ is called an **exact DE**.

How to decide if a DE is exact A DE of the form $M(x, y)dx + N(x, y)dy = 0$ is an exact equation iff $\frac{dM}{dy} = \frac{dN}{dx}$ or $M_y = N_x$

How to solve the exact equation We have two equations $\frac{df}{dx} = M$ [1] and $\frac{df}{dy} = N$ [2]

- The partial integration $\int \frac{df}{dx} dx = f(x, y)$
- The partial integration $\int \frac{df}{dy} dy = f(x, y)$

Doing the partial integration of [1]: $f(x, y) = \int M dx + g(y)$, where $g(y)$ is the constant of integration.
Find $\frac{df}{dy}$ and substitute in [2] to find $g(y)$.
$f(x, y) = ...$ (the solution is $f(x, y) = C$)

2.2.1 Exampole 1

Determine whether the DE is exact or not. If it is exact, then solve it. $y' = \frac{-2xy}{1+x^2}$

Solution $y' = \frac{-2xy}{1+x^2} \Rightarrow \frac{dy}{dx} = \frac{-2xy}{1+x^2} \Rightarrow (1 + x^2)dy = -2xydx \Rightarrow 2xydx + (1 + x^2)dy = 0$
$M_y = \frac{dM}{dy} = 2x(1)$ and $N_x = \frac{dN}{dx} = 2x$, $M_y = N_x \Rightarrow$ exact.

To solve, $\frac{df}{dx} = M$ and $\frac{df}{dy} = N \Rightarrow \frac{df}{dx} = 2xy$ [1] and $\frac{df}{dy} = 1 + x^2$ [2]
Perform partial integration of [1] w.r.t. x: $f(x, y) = 2y\frac{x^2}{2} + g(y)$ [3] $\Rightarrow \frac{df}{dy} = (1)x^2 + g'(y)$
Next substitute equation [2]: $x^2 + g'(y) = 1 + x^2 \Rightarrow g'(y) = 1$ (cannot have any x terms)
Integrating w.r.t. y: $g(y) = \int 1dy \Rightarrow g(y) = y$ or $g(y) = y + C$ (preferred without the $+C$)
Substitute $g(y)$ in [3] to get $f(x, y) = x^2 y + y$ or $f(x, y) = x^2 y + y + C$

The solution of the DE is $f(x, y) = C \Rightarrow x^2 y + y = C$ or $f(x, y) = 0 \Rightarrow x^2 y + y + C = 0$

2.2.2 Example 2

Determine whether the DE is exact or not. If it is exact, then solve it. $(xy + y^2)dx + (x^2 + xy)dy = 0$

Solution $M_y = x(1) + 2y$ and $N_x = 2x + (1)y \Rightarrow M_x \neq N_y \Rightarrow$ not exact.

2.2.3 Example 3

Determine whether the DE is exact or not. If it is exact, then solve it. $(x + \sin y)dx = (e^y = x \cos y)dy$

Solution $(x+\sin y)dx - (e^y - x\cos y)dy = 0$
Here, $M = x+\sin y$ and $N = -(e^y - x\cos y) = -e^y + x\cos y$
$M_y = \cos y$ and $N_x = (1)\cos y \Rightarrow M_y = N_x \Rightarrow$ Exact.

To solve, $\frac{df}{dx} = M \Rightarrow \frac{df}{dx} = x + \sin y$ [1]
$\frac{df}{dy} = N \Rightarrow \frac{df}{dy} = -e^y + x\cos y$ [2]
Partial integration of [2] w.r.t. y: $f(x,y) = -e^y + x\sin y + g(x)$ [3]
$\frac{df}{dx} = 0 + (1)\sin y + g'(x)$
Substitute in [1]: $\sin y + g'(x) = x + \sin y \Rightarrow g'(x) = x$ (cannot have y) $\Rightarrow g(x) = \frac{x^2}{2}$
Substitute $g(x)$ in [3]: $f(x,y) = -e^y + x\sin y + \frac{x^2}{2}$
The solution is $f(x,y) = C \Rightarrow -e^y + x\sin y + \frac{x^2}{2} = C$

2.2.4 Example 4

Solve the IVP: $(ey - xe^{xy})y' = 2 + ye^{xy}\ ;\ y(0) = 1$

Solution $(2y - xe^{xy})dy = (2 + ye^{xy})dx \Rightarrow (2 + ye^{xy})dx - (2y - xe^{xy}) = 0$
$M = 2 + ye^{xy} \Rightarrow \frac{dM}{dy} = 0 + (1)e^{xy} + ye^{xy}(x)$
$N = -2y + xe^{xy} \Rightarrow \frac{dN}{dx} = 0 + (1)e^{xy} + xe^{xy}(y)$
Since $M_y = N_x$, this is an exact equation.

$\frac{df}{dx} = M \Rightarrow \frac{df}{dx} = 2 + ye^{xy}$ [1]
$\frac{df}{dy} = N \Rightarrow \frac{df}{dy} = -2y + xe^{xy}$ [2]
Partial integration of [1] w.r.t. x: $f(x,y) = \int(2 + ye^{xy})dx = 2x + y\frac{e^{xy}}{y} + g(y)$ [3]
$\frac{df}{dy} = 0 + e^{xy}(x) + g'(y)$
Substitute in [2]: $xe^{xy} + g'(y) = -2y + xe^{xy} \Rightarrow g'(y) = -2y$
$g(y) = \int -2y dy = -2\frac{y^2}{2}$
Substituting in [3]: $f(x,y) = 2x + e^{xy} - y^2$
The solution of the DE is $f(x,y) = C \Rightarrow 2x + e^{xy} - y^2 = C$

$y(0) = 1 \Rightarrow$ Substitute $x = 0$ and $y = 1$: $2(0) + e^{(0)(1)} - (1)^2 = C \Rightarrow 1 - 1 = C \Rightarrow C = 0$
The solution of the IVP is $ex + e^{xy} - y^2 = 0$

2.3 More on Exact Equations

Sometimes, we can multiply the DE by an integrating factor and the DE becomes an exact DE. We solve by the mothod of exact equations:

1. If $\frac{M_y - N_x}{N}$ is a function of x only (no y terms), then $I.F. = \mu = e^{\int \frac{M_y - N_x}{N} dx}$
2. If $\frac{N_x - M_y}{M}$ is a function of y only (no x terms), then $I.F. = \mu = e^{\int \frac{N_x - M_y}{M} dy}$

2.3.1 Example 1

Find an I.F. to make this DE exact and then solve it: $(y^2 - y)dx + xdy = 0$

Solution $M = y^2 - y,\ M_y = 2y - 1,\ N = x,\ N_x = 1 \qquad M_y \neq N_x \Rightarrow$ not exact.
$\frac{M_y - N_x}{N} = \frac{(2y-1)-1}{x}$ (not in terms of x only)
$\frac{N_x - M_y}{M} = \frac{1-(2y-1)}{y^2-y} = \frac{1-2y+1}{y^2-y} = \frac{2-2y}{y^2-y} = \frac{-2(1-y)}{y(y-1)} = \frac{-2}{y}$ (good!)
$I.F. = \mu = e^{\int \frac{N_x - M_y}{M} dy} = e^{\int \frac{-2}{y} dy} = e^{-2\ln|y|} = e^{\ln|y|^{-2}} = |y|^{-2} = \frac{1}{y^2}$

Multiply the given DE by $\frac{1}{y^2}$: $\frac{1}{y^2}(y^2 - y)dx + \frac{1}{y^2}xdy = 0 \Rightarrow (1 - \frac{1}{y})dx + \frac{x}{y^2}dy = 0$
Now, $M = 1 - \frac{1}{y}$ and $N = \frac{x}{y^2} \Rightarrow M_y = -(-1y^{-2}) = \frac{1}{y^2},\ \ N_x = \frac{1}{y^2}(1) = \frac{1}{y^2}$ (unnecessary to check)

To solve: $\frac{df}{dx} = M \Rightarrow \frac{df}{dx} = 1 - \frac{1}{y}$ [1] $\quad \frac{df}{dy} = N \Rightarrow \frac{df}{dy} = \frac{x}{y^2}$ [2]
Partial integration of [1] w.r.t. $x \Rightarrow f(x,y) = x - \frac{1}{y}(x) + g(y)$ [3]
$\frac{df}{dy} = 0 + \frac{x}{y^2} + g'(y)$ (substitute this into [2])
$\frac{x}{y^2} + g'(y) = \frac{x}{y^2} \Rightarrow g'(y) = 0 \Rightarrow g(y) = 0$ or $g(y) = C$
So [3] gives $f(x,y) = x - \frac{x}{y}$ or $f(x,y) = x - \frac{x}{y} + C$
Thus the solution is $f(x,y) = C \Rightarrow x - \frac{x}{y} = C$ or $f(x,y) = 0 \Rightarrow x - \frac{x}{y} + C = 0$

Now, this is only the case if $y \neq 0$ (since the I.F. involves y being in the denominator).
So we can check if $y = 0$ is a solution: $dy = y'dx = 0dx = 0$
$(0^2 - 0)dx + xdy = 0 \Rightarrow (0^2 - 0)dx + x(0) = 0 \Rightarrow 0 + 0 = 0 \Rightarrow y = 0$ is a solution.

2.3.2 Example 2

Find an I.F. to make this DE exact: $y(x + y + 1)dx + (x + 2y)dy = 0$ (do not solve the equation)

Solution $M = xy + y^2 + y$ and $N = x + 2y \quad M_y = x + 2y + 1$ and $N_x = 1$
$\frac{M_y - N_x}{N} = \frac{(x+2y+1)-1}{x+2y} = \frac{x+2y}{x+2y} = 1 \Rightarrow I.F. = \mu = e^{\int 1dx} = e^x$

2.4 Solutions by Substitution

We do 3 kinds of substitutions which can be used to solve 1st order DEs.

2.5 Homogeneous Equations

A function $f(x,y)$ is called a homogeneous function of degree a if $f(tx,ty) = t^a f(x,y)$

To determine if the equation $f(x,y) = x^2 + xy$ is homogeneous, calculate $f(tx,ty)$.
$f(tx,ty) = (tx)^2 + (tx)(ty) = t^2x^2 + t^2xy = t^2(x^2 + xy) = t^2 f(x,y) \Rightarrow$ Thus f is a homogeneous function of degree 2.
For more functions:

- $f(x,y) = x + 3$, $f(tx,ty) = tx + 3 \Rightarrow$ not homogeneous.
- $f(x,y) = \sin x - \sin y \quad f(tx,ty) = \sin tx - \sin ty \Rightarrow$ not homogeneous. (Usually anything with $\sin x$ or $\sin y$ is not homogeneous)
- $f(x,y) = \sin\frac{x}{y} \quad f(tx,ty) = \sin\frac{tx}{ty} = \sin\frac{x}{y} = t^0 \sin\frac{x}{y} \Rightarrow$ homogeneous. (this is an exception to the above)
- $f(x,y) = \ln x - \ln y \quad f(tx,ty) = \ln tx - \ln ty = \ln\frac{tx}{ty} = \ln\frac{x}{y} = t^0(\ln x - \ln y) \Rightarrow$ homogeneous of degree 0.

A differential equation $Mdx + Ndy = 0$ is homogeneous if both M and N are homogeneous functions of the same degree.

Example 1 For $(x^2 + xy)dx + xy^2dy = 0$, M is homogeneous of degree 2 and N is homogeneous of degree 3. Thus the equation is not homogeneous.

Example 2 For $(x + 3)dx + ydy = 0$, it is not homogeneous because M is not homogeneous.

Example 3 For $(x^2 + xy)dx + (xy - y^2)dy = 0$, since both M and N are homogeneous of degree 2, thus the equation is homogeneous. (you can often check the power of M and N to check if it's homogeneous)

2.5.1 Solving a Homogeneous Equation

Case 1: Let $y = ux$ and $dy = udx + udu \rightarrow$ the homogeneous equation will become a separable equation in u and x. Solve it and then replace $u = \frac{y}{x}$ in the solution.
Case 2: Let $x = vy$ and $dx = vdy + ydv \rightarrow$ the homogeneous equation will become a separable equation in v and y. Solve it and then replace $v = \frac{x}{y}$ in the solution.

Example Solve the DE by using an approprite substitution: $ydx - 2(x+y)dy = 0$

Solution Let $y = ux$ and $dy = udx + xdu$ **or** let $x = vy$ and $dx = vdy + ydv$

$uxdx - (2x + 2ux)(udx + xdu) = 0$ **or** $y(vdy + ydv) - (2vy + 2y)dy = 0$

This is harder to solve **or** $vydy + y^2dv - 2vydy - 2ydy = 0$

$y^2dv - vydy - 2ydy = 0$

$y^2dv = vydy + 2ydy \Rightarrow y^2dv = ydy(v+2) \Rightarrow \frac{dv}{v+2} = \frac{ydy}{y^2} \Rightarrow \frac{dv}{v+2} = \frac{1}{y}dy \rightarrow$ a separable equation.

If we went the u way: $\int \frac{-dx}{x} = \int \frac{2+2u}{u+2u^2}du \ldots$ needs partial fraction.
Instead, $\int \frac{dv}{v+2} = \int \frac{1}{y}dy \Rightarrow \ln|v+2| = \ln|y| + C$
Replace v by $\frac{x}{y} \Rightarrow \ln|\frac{x}{y} + 2| = \ln|y| + C$

Is $y = 0$ also a solution?
Well, $y = 0 \Rightarrow dy = 0$. Substitute into original equation: $(0)dx - 2(x+0)(0) = 0 \Rightarrow 0 - 0 = 0$
$\Rightarrow y = 0$ is also a solution.
$v + 2 = 0 \Rightarrow \frac{x}{y} + 2 = 0 \Rightarrow \frac{x}{y} = -2 \Rightarrow x = -2y \Rightarrow y = \frac{-x}{2}$
Is this a solution? Well, $y = \frac{-x}{2} \Rightarrow dy = y'dx = \frac{-1}{2}dx$
Substitute into original equation: $\frac{-x}{2}dx - 2(x - \frac{x}{2})(\frac{-1}{2}dx) = 0 \Rightarrow \frac{-x}{2}dx + \frac{x}{2}dx = 0 \Rightarrow 0 = 0$
$\Rightarrow y = \frac{-x}{2}$ is also a solution.

2.6 Bernoulli's Equation

$\frac{dy}{dx} + P(x)y = f(x)y^n$ where $n \in R, n \neq 0$ and $n \neq 1$
If $n = 0 \Rightarrow \frac{dy}{dx} + P(x)y = f(x)(1) \rightarrow$ linear.
If $n = 1 \Rightarrow \frac{dy}{dx} + P(x)y = f(x)y \Rightarrow \frac{dy}{dx} + (P(x) - f(x))y = 0 \rightarrow$ linear equation.
If $n \neq 0$ and $n \neq 1$ then we use the substitution $u = y^{1-n}$ and the Bernoulli's equation changes to a linear equation $\rightarrow$ solve linear equation $\rightarrow$ replace u.

Another way: for the equation $\frac{dy}{dx} + P(x)y = f(x)y^n$, let $u = y^{1-n}$ and the equation turns into:
$\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x) \Rightarrow$ linear equation in u.

2.6.1 Example 1

Solve the DEs using an appropriate substitution: $\frac{dy}{dx} - y = e^x y^2$

Solution This is Bernoulli's equation with $n = 2$.
Let $u = y^{1-n} = y^{1-2} = y^{-1} = \frac{1}{y}$ if $y \neq 0$
$\Rightarrow y = \frac{1}{u} = u^{-1}$ and $\frac{dy}{dx} = -1u^{-2}\frac{du}{dx} = \frac{-1}{u^2}\frac{du}{dx}$
Substituting in the original equation: $\frac{-1}{u^2}\frac{du}{dx} - \frac{1}{u} = e^x \frac{1}{u^2}$
Multiplying by $-u^2 \Rightarrow \frac{du}{dx} + u = -e^x$ [1]

An alternate route: Let $u = y^{1-n}$. Then Bernoulli's equation becomes $\frac{du}{dx} + (1-n)P(x)u = (1-n)f(x)$
$\Rightarrow \frac{du}{dx} - (-1)u = (-1)e^x$
$\frac{du}{dx} + u = -e^x$ (linear in u)

$I.F. = e^{\int P(x)dx} = e^{\int 1dx} = e^x$
Multiplying [1] by $e^x \Rightarrow e^x\frac{du}{dx} + e^x u = -e^x e^x$
$\frac{d}{dx}(e^x u) = -e^{2x}$
Integrating w.r.t. $x \Rightarrow e^x u = \int -e^{2x}dx \Rightarrow e^x u = \frac{-e^{2x}}{2} + C \Rightarrow u = \frac{-e^{2x}}{2e^x} + \frac{C}{e^x}$
Replace $u = \frac{1}{y} \Rightarrow \frac{1}{y} = -\frac{e^x}{2} + Ce^{-x}$

2.6.2 Example 2

Solve the DEs using an appropriate substitution: $\frac{dy}{dx} - y = e^x y^2$, $y(0) = 1$ (note: $y = 0$ does not satisfy $y(0) = 1$)

Solution From example 1: $\frac{1}{y} = \frac{-e^x}{2} + Ce^{-x}$
Set $x = 0$ and $y = 1 \Rightarrow \frac{1}{1} = \frac{-e^0}{2} + Ce^{-0} \Rightarrow 1 = \frac{-1}{2} + C \Rightarrow C = \frac{3}{2}$
The solution is $\frac{1}{y} = \frac{-e^x}{2} + \frac{3}{2}e^{-x}$

2.6.3 Example 3

Solve the DEs using an appropriate substitution: $xy' - x^5 y^{\frac{1}{3}} = 3y$

Solution $\Rightarrow y' - \frac{x^5}{x} y^{\frac{1}{3}} = \frac{3}{x} y$
$\Rightarrow y' - \frac{3}{x} y = x^4 y^{\frac{1}{3}}$ [$*$] This is Bernoulli's equation with $n = \frac{1}{3}$
Let $u = y^{1-n} = y^{1-\frac{1}{3}} = y^{\frac{2}{3}}$
$u' - (\frac{2}{3}) \frac{3}{x} u = \frac{2}{3} x^4$
$\frac{du}{dx} - \frac{2}{x} u = \frac{2}{3} x^4$ [2] linear equation in u

$I.F. = e^{\int \frac{-2}{x} dx} = e^{-2 \ln |x|} = e^{\ln |x|^{-2}} = |x|^{-2} = \frac{1}{x^2}$
Multiply [2] by $\frac{1}{x^2} \Rightarrow \frac{1}{x^2} \frac{du}{dx} - \frac{2}{x} \frac{1}{x^2} u = \frac{2}{3} \frac{1}{x^2}$
$\int \frac{d}{dx} (\frac{1}{x^2} u) dx = \int \frac{2}{3} x^2 dx$
$\frac{1}{x^2} u = \frac{2}{3} \frac{x^3}{3} + C \Rightarrow u = \frac{2}{9} x^3 x^2 + C x^2$
Replace $u = y^{\frac{2}{3}} \Rightarrow y^{\frac{2}{3}} = \frac{2}{9} x^5 + C x^2$

2.7 Linear Substition

A DE of the form $\frac{dy}{dx} = f(Ax + By + C)$ where $B \neq 0$ can be converted to a separable equation by using the substitution $u = Ax + By + C$, then solve the separable equation and replace u.

2.7.1 Example 1

Solve the DE by using an appropriate substitution: $\frac{dy}{dx} = 2 + e^{y-2x+6}$

Solution Let $u = y - 2x + 6 \Rightarrow y = u + 2x - 6$ and $\frac{dy}{dx} = \frac{du}{dx} + 2$
Sub into original equation: $\frac{du}{dx} + 2 = 2 + e^u \Rightarrow \frac{du}{dx} = e^u \Rightarrow \int \frac{du}{e^u} = \int dx \Rightarrow$ separable equation.
$\frac{e^{-u}}{-1} = x + C$, replace $u = y + -2x + 6 \Rightarrow -e^{-(y-2x+6)} = x + C$

2.7.2 Example 2

Solve the DE by using an appropriate substitution: $\frac{dy}{dx} = \frac{1-x-y}{x+y}$

Solution $\Rightarrow \frac{dy}{dx} = \frac{1-(x+y)}{x+y}$, we can have a linear substitution $f(x+y)$
Let $u = x + y \Rightarrow y = u - x$ and $\frac{dy}{dx} = \frac{du}{dx} - 1$
Sub into original equation: $\frac{du}{dx} - 1 = \frac{1-u}{u} \Rightarrow \frac{du}{dx} = 1 + \frac{1-u}{u} = \frac{u+1-u}{u} = \frac{1}{u}$
$udu = dx$ (cross multiplication of $\frac{du}{dx} = \frac{1}{u}$), so $\int udu = \int dx \Rightarrow$ separable equation
$\frac{u^2}{2} = x + C$
Replace $u = x + y \Rightarrow \frac{(x+y)^2}{2} + x + C$

Now, back to the beginning: $\frac{Dy}{dx} = \frac{1-x-y}{x+y}$ is not separable, ont linear, not Bernoulli's equation.
We can check homogeneous and exact$\Rightarrow Mdx + Ndy = 0 \Rightarrow (x+y)dy = (1-x-y)dx$
$\Rightarrow (1-x-y)dx - (x+y)dy = 0 \Rightarrow$ not homogeneous ($1-x-y$ is not homogeneous)
$M = 1 - x - y$ and $N = -(x+y) \Rightarrow M_y = -1$ and $N_x = -1 \Rightarrow$ exact equation.
You can solve by exact equation to get $x - \frac{x^2}{2} - xy - \frac{y^2}{2} = C$

3 Real Life Problems

Real life problems can be modelled as differential equations (especially first-order DEs). We will discuss this further:

3.1 Linear Models

Real life applications can be written as 1st order linear DEs (section 2.1 in this document). Let's look at some examples:

3.1.1 Growth and Decay

If P is a population at time t, then the rate of change of that population is proportional to $P \Rightarrow \frac{dP}{dt} \propto P \Rightarrow \frac{dP}{dt} = kP$ where $k > 0$
Radioactive elements decay with time. The rate of change of A, where A represents the amount of decay, is proportional to $A \Rightarrow \frac{dA}{dt} \propto A \Rightarrow \frac{dA}{dt} = kA$ where $k < 0$

3.1.2 Example 1

The population of bacteria in a culture grows at a rate proportional to the number of bacteria present at time t. After 3 hours, it is observed that 10,000 bacteria are present. If the initial population is 2000 bacteria, when will the bacteria population reach 17,500?

Solution The DE is $\frac{dP}{dt} = kP$ (where P is the population) and the given conditions are $P(0) = 2000$ and $P(3) = 10,000$
To find t where $P(t) = 17,500$: $\frac{dP}{dt} = kP \Rightarrow \frac{dP}{dt} - kP = 0$ and we now have a linear equation.
$I.F. = e^{\int -kdt} = e^{-kt}$
$e^{-kt}\frac{dP}{dt} - e^{-kt}kP = e^{-kt}(0) \Rightarrow \int \frac{d}{dt}(e^{-kt}P)dt = \int 0dt$

Alternate solution: Make separable: $\int \frac{dP}{P} = \int kdt$ if $P \neq 0$
$\ln|P| = kt + C \Rightarrow |P| = e^{kt+C} \Rightarrow P = e^{kt} \cdot e^{C} \Rightarrow P = C_1 e^{kt}$

Anyways, $e^{-kt}P = C \Rightarrow P = \frac{C}{e^{-kt}}$
$P(0) = 2000 \Rightarrow 2000 = Ce^{k(0)} \Rightarrow 2000 = C \Rightarrow P = 2000e^{kt}$
$P(3) = 10,000 \Rightarrow 10000 = 2000e^{k3} \Rightarrow e^{3k} = \frac{10,000}{2000} \Rightarrow \ln e^{3k} = \ln 5 \Rightarrow 3k = \ln 5 \Rightarrow k = \frac{\ln 5}{3} \approx 0.536$
So, $P = 2000e^{0.536t}$
When $P = 17,500 \Rightarrow 17,500 = 2000e^{0.536t}$
$e^{0.536t} = \frac{17,500}{2000} \Rightarrow \ln e^{0.536t} = \ln \frac{175}{20} \Rightarrow 0.536t = \ln \frac{175}{20} \Rightarrow t = \frac{\ln \frac{175}{20}}{0.536} \approx 4.044$
Therefore, the population will reach 17,500 bacteria a little after 4 hours.

3.1.3 Example 2

A sample of bismuth-210 decays at a rate proportional to the amount present at time t. If 67% of its original amount has decayed after 8 days, find the half life of this sample.

Solution The DE is $\frac{dA}{dt} = kA$, where A is the amount, given the conditions are $A(8) = \frac{33}{100}A_0$ where A_0 is the initial amount.
Reminder: Half-life refers to how long it takes for the substance to reduce to 50% of its original amount.
From example 1: $A = Ce^{kt}$.
$A_0 = Ce^{k(0)} = C(1) \Rightarrow C = A_0$
So $A = A_0e^{kt}$
Now, $A(8) = \frac{33}{100}A_0 \Rightarrow \frac{33}{100}A_0 = A_0e^{k(8)}$.
Take $\ln \Rightarrow \ln \frac{33}{100} = 8k \Rightarrow k = \frac{\ln \frac{33}{100}}{8} \approx -0.139$
We thus have $A = A_0e^{-0.139t}$
When $A = \frac{1}{2}A_0 \Rightarrow \frac{1}{2}A_0 = A_0e^{-0.139t} \Rightarrow \ln \frac{1}{2} = -0.139t \Rightarrow t = \frac{\ln \frac{1}{2}}{-0.139} \approx 4.985$

3.1.4 Cooling and Warming

The DE is $\frac{dT}{dt} = k(T - T_m)$ where T is the temperature of the object, t is time and T_m is the temperature of the surroundings. Here, $k < 0$.

- When $T > T_m \Rightarrow \frac{dT}{dt} < 0$ since $T - T_m > 0$
- When $T < T_m \Rightarrow \frac{dT}{dt} > 0$ since $T - T_m < 0$

3.1.5 Example 3 (from 3.1 #14)

A thermometer is taken from an inside room to the outside, where the air temperature is 5°F. After 1 minute, the termperature reads 55°F and after 5 minutes, it reads 30°F. What is the initial temperature of the room?

Solution The DE is $\frac{dT}{dt} = k(T - T_m)$ given $T(1) = 55$ and $T(5) = 30$
Since the air temperature is 5°F, then $T_m = 5 \Rightarrow \frac{dT}{dt} = kT - 5k$
$\frac{dT}{dt} - kT = -5k$
$I.F. = e^{\int -kdt} = e^{-kt}$
$e^{-kt}\frac{dT}{dt} - kTe^{-kt} = -5ke^{-kt}$
$\frac{d}{dt}(e^{-kt}T) = -5ke^{-kt}$
Integrating: $e^{-kt}T = \int -5ke^{-kt}dt = -5k\frac{e^{-kt}}{-k} + C = 5e^{-kt} + C$
$T = \frac{5e^{-kt}}{e^{-kt}} + \frac{C}{e^{-kt}} \Rightarrow T = 5 + Ce^{kt}$

$T(1) = 55 \Rightarrow 55 = 5 + Ce^{k(1)} \Rightarrow Ce^{k} = 50$ [1]
$T(5) = 30 \Rightarrow 30 = 5 + Ce^{k(5)} \Rightarrow Ce^{5k} = 25$ [2]
$\frac{Ce^{5k}}{Ce^{k}} = \frac{25}{50} \Rightarrow e^{4k} = \frac{1}{2} \Rightarrow 4k = \ln\frac{1}{2} \Rightarrow k = \frac{\ln\frac{1}{2}}{4} \approx -0.173287$
Thus equation 1 becomes: $C = 50e^{-k} = 50e^{-(-0.173287)} \approx 59.46$
So, $T = 5 + 59.46e^{-0.173287t}$
At $t = 0 \Rightarrow T = 5 + 59.46e^{0} \approx 64.46$
Thus, the temperature of the room is 64.46°F.

3.1.6 Mixtures

IF $A(t)$ is the amount of salt in a tank, then the DE is $\frac{dA}{dt} = R_{\text{in}} - R_{\text{out}}$ where R_{in} is the input rate of salt and R_{out} is the output rate of salt.
$R_{\text{in}} \Rightarrow$ (the input rate of flow) $\times$ (concentration of salt in solution).
$R_{\text{out}} \Rightarrow$ (the output rate of flow) $\times$ (concentration of salt).

3.1.7 (3.1) Series Circuits (Section 1.3)

We previously covered growth/decay, cooling/warming and mixtures. Now we're covering circuits:

- For LR series, the DE is $L\frac{di}{dt} + Ri = E$, where L is the inductance, i is the current and R is the resistance.
- For RC series, the DE is $R\frac{dq}{dt} + \frac{1}{c}q = F$, where R is the resistance, q is the charge and c is the capacitance.

3.1.8 Example

A 200 volt electromotive force is applied to an RC series circuit in which the resistance is 100 ohms and the capacitance is 5×10^{-6} Farad. Find the charge q on the capacitor if $i(0) = 0.4$. Determine the charge as $t \to \infty$. Find current at $t = 0.005$ seconds.

Solution The DE is $R\frac{dq}{dt} + \frac{1}{c}q = E \Rightarrow 1000\frac{dq}{dt} + \frac{1}{5\times10^{-6}}q = 200$
$\frac{dq}{dt} + \frac{10^6}{1000(5)}q = \frac{200}{1000}$
$\frac{dq}{dt} + 200q = \frac{1}{5}$
$I.F. = e^{\int 200dt} = e^{200t}$

$e^{200t}\frac{dq}{dt} + 200e^{200t}q = \frac{1}{5}e^{200t}$
$\frac{d}{dt}(e^{200t}q) = \frac{1}{5}e^{200t}$

Integrating w.r.t. $t \Rightarrow e^{200t}q = \frac{1}{5}\frac{e^{200t}}{200} + C \Rightarrow q = \frac{e^{200t}}{1000e^{200t}} + \frac{C}{e^{200t}}$
$q = \frac{1}{1000} + Ce^{-200t}$
Given the initial condition $i(0) = 0.4$, but $i = \frac{dq}{dt} = Ce^{-200t}(-200)$
$i(0) = 0.4 \Rightarrow 0.4 = Ce^{-200(0)}(-200) \Rightarrow C = \frac{0.4}{-200} = \frac{-1}{500}$
Note: the 0 in $i(0)$ is the value of t and the 0.4 is the value of i.
The charge is $q = \frac{1}{1000} - \frac{1}{500}e^{-200t}$

$\lim_{t\to\infty}\left(\frac{1}{1000} - \frac{1}{500}e^{-200t}\right) = \frac{1}{1000}$
When $t \to \infty$, then the charge is $\frac{1}{1000}$ coulomb.
The current is $i = \frac{dq}{dt} = \frac{-1}{500}e^{-200t}(-200)$
$i = \frac{2}{5}e^{-200t}$

When $t = 0.005$, $i = \frac{2}{5}e^{-200(0.005)} = \frac{2}{5}e^{-1} \approx 0.1472$ amperes.

4 (Ch. 4) Higher Order Differential Equations

4.1 Linear Differential Equations

An nth order linear DE is $a_n(x)\frac{d^ny}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_1(x)\frac{dy}{dx} + a_0(x)y = g(x)$
For IVPs: $y(x_0) = y_0, y'(x_0) = y_1, \ldots, y^{(n-1)}(x_0) = y_{n-1} \Rightarrow$ you need n conditions for an nth order DE.

4.1.1 Example 1

Given that $y = c_1 + c_2\cos x + c_3\sin x$ is a solution of the DE $y''' + y' = 0$ on the interval $(-\infty, \infty)$, find the solution subject to the conditions: $y(\pi) = 0$, $y'(\pi) = 2$ and $y''(\pi) = -1$

Solution $y(\pi) = 0 \Rightarrow 0 = c_1 + c_2\cos\pi + c_3\sin\pi \Rightarrow c_1 - c_2 = 0$ [1]
$y' = c_2(-\sin x) + c_3\cos x$
$y'(\pi) = 2 \Rightarrow 2 = c_2(-\sin\pi) + c_3\cos\pi \Rightarrow 2 = -c_3 \Rightarrow c_3 = -2$
$y'' = c_2(-cosx) + c_3(-\sin x)$
$y''(\pi) = -1 \Rightarrow -1 = c_2(-\cos\pi) + c_3(-\sin\pi) \Rightarrow -1 = c_2$
From [1], $c_1 = c_2 = -1$
Thus the solution is $y = -1 - \cos x - 2\sin x$

4.2 Existence and Uniqueness Theorem for IVPs

Consider the IVP: $a_n(x)y^{(n)} + a_{n-1}(x)y^{(n-1)} + \cdots + a_1(x)y' + a_0(x)y = g(x)$ subject to $y(x_0) = y_0, \ldots, y^{(n-1)}(x_0) = y_{n-1}$. If $a_0(x)$, $a_1(x)$, $\ldots$, $a_n(x)$ and $g(x)$ are continuous on an interval containing x_0 and $a_n(x) \neq 0$ for any x in the interval, then the IVP has a unique solution.

Back in section 2.3 of the textbook, we covered first-order DEs had to be written in standard form: $a_1y' + a_0y = g(x) \Rightarrow y' + \frac{a_0}{a_1}y = \frac{g(x)}{a_1}$ where $\frac{a_0}{a_1} = P(x)$ and $\frac{g(x)}{a_1} = f(x)$, where $P(x)$ and $f(x)$ had to be continuous. This is a similar situation but for higher order DEs.

4.2.1 Example

Find an interval centered about $x = 0$ for which the IVP has a unique solution: $(x-1)y'' + (\sec x)y = e^x;\ y(0) = 3,\ y'(0) = 1$

Solution $a_2 = (x-1)$ and e^x are continuous on R, while $a_0 = \sec x = \frac{1}{\cos x}$ is continuous on $(\frac{-\pi}{2}, \frac{\pi}{2})$
Thus the interval where the DE is continuous is $(\frac{-\pi}{2}, \frac{\pi}{2})$
$a_2 \neq 0 \Rightarrow x - 1 \neq 0 \Rightarrow x \neq 1$
Because of that, the largest interval is $(-1, 1)$

4.2.2 Linear Dependence/Independence

A set of functions $f_1, f_2, \ldots, f_n$ are **linearly dependent** on I if there exists constants $c_1, c_2, \ldots, c_n$ where not all of them are 0 and $c_1 f_1 + c_2 f_2 + \cdots + c_n f_n = 0$

The **wronskian** of $f_1, f_2, \ldots, f_n$ is $W = \begin{bmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ f_1'' & f_2'' & \cdots & f_n'' \\ \vdots & \ddots & & \\ f_1^{(n-1)} & \cdots & & f_n^{(n-1)} \end{bmatrix}$
$f_1, f_2, \ldots, f_n$ are linearly independent on an interval iff $W \neq 0$ for all x in I.

4.2.3 Example 1

Determine wheter the function is linearly independent: $f_1(x) = e^x, f_2(x) = e^{-x}$ on the interval $(-\infty, \infty)$

Solution $W = \begin{bmatrix} f_1 & f_2 \\ f_1' & f_2' \end{bmatrix} = \begin{bmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{bmatrix} = -e^x e^{-x} - e^{-x} e^x = -1 - 1 = -2 \neq 0$
Thus the functions are linearly independent.

4.3 Fundamental Set of Solutions

$y_1, y_2, \ldots, y_n$ are the fundamental set of solutions of an nth order DE on I if:

1. $y_1, y_2, \ldots, y_n$ satisfy the DE on I, and
2. $y_1, y_2, \ldots, y_n$ are linearly independent (l.i.) on I.

4.3.1 Example

Show that $y_1 = \cos 5x$ and $y_2 = \sin 5x$ are fundamental set of solutions of the DE $y'' + 25y = 0$

Solution $W = \begin{bmatrix} \cos 5x & \sin 5x \\ (-\sin 5x)(5) & 5\cos 5x \end{bmatrix} = 5\cos^2 5x + 5\sin^2 5x = 5(\cos^2 5x + \sin^2 5x) = 5 \neq 0 \Rightarrow$ l.i.
$y = \cos 5x, y' = -5\sin 5x, y'' = -25\cos 5x \Rightarrow -25\cos 5x + 25\cos 5x = 0 \Rightarrow 0 = 0$
$y = \sin 5x, y' = 5\cos 5x, y'' = -25\sin 5x \Rightarrow -25\sin 5x + 25\sin 5x = 0 \Rightarrow 0 = 0$
So y_1, y_2 are fundamental set of solutions.

4.4 (4.1) Linear Equations

The equation $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$ is an nth order linear DE.
The equation $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = 0$ is a homogeneous equation.

- The equation $3xy'' + 2y' - 6y = 7x$ is linear.
- The equation $3xy'' + 2y' - 6y = 0$ is homogeneous.

If $y_1, y_2, \ldots, y_n$ are the fundamental solutions of an nth order homogeneous DE, then $y = c_1 y_1 + c_2 y_2 + \cdots + c_n y_n$ will be the general solution of the DE.

For non-homogeneous DEs, the solution is $y = y_c + y_p$ where:

- y_c is the complementary function (solution of associated homogeneous function); and
- y_p is the solution of nonhomogeneous equation, has no constants.

4.5 (4.2) Reduction of Order

Consider the second-order linear DE: $a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \Rightarrow y'' + \frac{a_1(x)}{a_2(x)}y' + \frac{a_0(x)}{a_2(x)}y = \frac{g(x)}{a_2(x)}$.
Notice we have the equation in standard form: $y'' + P(x)y' + Q(x)y = f(x)$ where $P(x) = \frac{a_1(x)}{a_2(x)}$, $Q(x) = \frac{a_0(x)}{a_2(x)}$ and $f(x) = \frac{g(x)}{a_2(x)}$

If we know that y_1 is a solution of the above equation, then if we can find a second solution y_2 which is linearly independent of y_1, then we can have the solution of the DE $y = y_c + y_p$, where $y_c = 2c_1y_1 + c_2y_2$ and y_p has no constant c.
We say that $y_2 = uy_1$ where $u(x)$ is a function of x. The challenge is finding u.

4.5.1 Example 1

Given that $y_1 = x^7$ is a solution of the DE $x^2y'' - 13xy' + 49y = 0$, find the second solution.

Solution Let $y_2 = uy_1 = ux^7$
Let $y = ux^7$. This should satisfy the equation.
$y' = u'x^7 + u7x^6$
$y'' = u''x^7 + u'7x^6 + u'7x^6 + 42ux^5$

4.6 (4.3) Homogeneous Equations with Constant Coefficients

2nd order linear homogeneous equations with a constant coefficient would be $ay'' + by' + cy = 0$.
The auxiliary equation is $am^2 + bm + c = 0 \Rightarrow m = \frac{-b\pm\sqrt{b^2-4ac}}{2a}$
We must examine the following cases:

1. When $b^2 - 4ac > 0$, we have two distinct reql roots m_1 and m_2. The solution of the DE is: $y = C_1e^{m_1x} + C_2e^{m_2x}$ (or $C_1y_1 + C_2y_2$, where $y_1 = e^{m_1x}$ and $y_2 = e^{m_2x}$)
2. When $b^2 - 4ac = 0$, we have one real root, which means it will appear twice in the solution. The solution of the DE is $y = C_1e^{mx} + C_2e^{mx}$ where $y_1 = y_2 = e^{mx}$
3. When $b^2 - 4ac < 0$, we have two imaginary/complex roots: $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$. The solution of the DE is $y = e^{\alpha x}(C_1 \cos \beta x + C_2 \sin \beta x)$.
   A useful formula is Euler's formula: $e^{i\theta} = \cos\theta + i\sin\theta$

4.6.1 Example 1

Solve the DE (find the general solution of the DE) $y'' - y' - 2y = 0$

Solution The auxiliary equation is $m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0 \Rightarrow m = 2, -1$
The solution is $y = c_1e^{2x} + c_2e^{-x}$

4.6.2 Example 2

Solve the DE (find the general solution of the DE) $y'' - 18y' + 81y = 0$

Solution The auxiliary equation is $m^2 - 18m + 81 = 0 \Rightarrow (m-9)(m-9) = 0 \Rightarrow m = 9, 9$
The solution of the DE is $y = c_1e^{9x} + c_2e^{9x}$

The idea can be extended to linear equations with any order.
Suppose we had a 7th order equation where $m = 0, 1, 1, 1, 2, 2+i, 2+i$.
Th solution of that DE is $yc_1e^{0x} + c_2e^x + c_3e^x + c_4e^x + c_5e^{2x} + e^{2x}(c_6 \cos x + c_7 \sin x)$

4.6.3 Example 3

Solve the DE (find the general solution of the DE) $y^{(4)} - 7y'' - 18y = 0$

Solution The auxiliary equation is $m^4 - 7m^2 - 18 = 0$. Let $m^2 = t$.
Then $t^2 - 7t - 18 = 0 \Rightarrow (t-9)(t+2) = 0 \Rightarrow t = 9, -2$
Thus $m^2 = 9 \Rightarrow m = \pm\sqrt{9} = \pm 3$
or $m^2 = -2 \Rightarrow m = \pm\sqrt{-2} = \pm i\sqrt{2}$
The solution is $y = c_1 e^{3x} + c_2 e^{-3x} + e^{0x}(c_3 \cos\sqrt{2}x + c_4 \sin\sqrt{2}x)$

4.6.4 Example 4

Solve the DE (find the general solution of the DE) $y^{(5)} - 8y^{(4)} + 18y''' = 0$

Solution The auxiliary equation is $m^5 - 8m^4 + 18m^3 = 0 \Rightarrow m^3(m^2 - 8m + 18) = 0$
From this, we already have 3 out of 5 solutions: $m = 0, 0, 0$.
$m = \frac{-(-8)\pm\sqrt{(-8)^2-4(1)(18)}}{2(1)} = \frac{8\pm\sqrt{-8}}{2} = \frac{8\pm\sqrt{-4(2)}}{2} = \frac{8\pm 2i\sqrt{2}}{2} = \frac{2(4\pm i\sqrt{2})}{2} = 4 \pm i\sqrt{2}$

The solution of the DE is $y = c_1 e^{0x} + c_2 x e^{0x} + c_3 x^2 e^{0x} + e^{4x}(c_4 \cos\sqrt{2}x + c_5 \sin\sqrt{2}x)$

For nonhomogeneous linear DEs $a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(x)$, we have the following steps:

1. Find a solution of the associated homogeneous linear equation (i.e. $g(x) = 0$). The solution will be $y_c = c_1 y_1 + c_2 y_2 + ... + c_n y_n \Rightarrow$ complementary function.
2. Solve the nonhomogeneous equation and this will give $y_p \Rightarrow$ particular solution (remember: y_p has no c_1, c_2, etc.). Learn this from sections 4.4 and 4.6.
3. The solution is $y = y_c + y_p$.

4.7 (4.4) Method of Undetermined Coefficients

This method finds y_p for nonhomogeneous linear DEs if:

1. The coefficients $a_n, a_{n-1}, \ldots, a_2, a_1, a_0$ are constants.
2. $g(x)$ can be a constant, a polynomial $p(x)$, e^{kx}, $\sin kx$, $\cos kx$, or a sum/difference and product of these functions.
   $g(x) = e^{3x}$, $g(x) = x^2 + 2 - \cos 5x$, $g(x) = (\sin 3x)e^x$ are fine. However, $g(x)$ cannot be $\ln x$, e^{3x^2}, $\sec x$, $\tan x$, $\sqrt{x+3}$, etc.

For the method of undetermined coefficient, we assume a form of y_p based on $g(x)$ and y_c. We will find the coefficients A, B, etc.

$g(x)$	y_p
3	A
$x^3 - 4$	$Ax^3 + Bx^2 + Cx + D$
e^{2x}	Ae^{2x}
$\cos 3x$	$A\cos 3x + B\sin 3x$
$\sin 3x$	$A\cos 3x + B\sin 3x$
$3 + e^{2x}$	$A + Be^{2x}$
xe^{3x}	$(Ax+B)e^{3x}$
$x\sin 5x$	$(Ax+B)\cos 5x + (Cx+D)\sin 5x$

Note that xe^{3x} turns into $y_p = (Ax+B)e^{3x}$ because the x turns into $(Ax+B)$ and the e^{3x} turns into $y_p = Ce^{3x}$. Then, distributive property turns it into $ACxe^{3x} + BCe^{3x}$, but we could just say $A = AC$ and $B = BC$ to make it $Axe^{3x} + Be^{3x}$.

4.7.1 Example

Solve the DE: $y'' - y' - 2y = 4x^2$

Solution **Step 1:** The associated homogeneous linear equation is $y'' - y' - 2y = 0$
The auxiliary equation is $m^2 - m - 2 = 0 \Rightarrow (m-2)(m+1) = 0 \Rightarrow m = 2, -1$
$y_c = c_1 e^{2x} + c_2 e^{-x}$

Step 2: To find y_p: Let $y_p = Ax^2 + Bx + C$, $y_p' = 2Ax + B$, $y_p'' = 2A$

Substituting into the original equation: $2A - (2Ax + B) - 2(Ax^2 + Bx + C) = 4x^2$
$2A - 2Ax - B - 2Ax^2 - 2Bx - 2C = 4x^2$
Comparing coefficients of x^2: $-2A = 4 \Rightarrow A = -2$
$x:\ -2A - 2B = 0 \Rightarrow -2(-2) - 2B = 0 \Rightarrow 2B = 4 \Rightarrow B = 2$
Constants: $-2A - B - 2C = 0 \Rightarrow -2(-2) - 2 - 2C = 0 \Rightarrow -6 = 2C \Rightarrow C = -3$
So $y_p = -2x^2 + 2x - 3$.

Step 3: The solution is $y = y_c + y_p = c_1e^{2x} + c_2e^{-x} - 2x^2 + 2x - 3$

4.8 (4.6) Variations of Parameters

Sometimes, the method of undetermined coefficients (section 4.4) doesn't work. In this case, we can use variations of parameters. In all cases, this method works but sometimes we cannot integrate. Choose this method if the method of undetermined coefficients does not work.

To find y_p for 2nd order equations: $a_2y'' + a_1y' + a_0y = g(x)$, use section 4.3 to solve $y_c = c_1y_1 + c_2y_2$ where c_1 and c_2 are constant parameters.
Then let $y_p = u_1y_1 + u_2y_2$, where u_1 and u_2 are variable parameters.
$a_2y'' + a_1y' + a_0y = g(x) \Rightarrow y'' + \frac{a_1}{a_2}y' + \frac{a_0}{a_2}y = \frac{g(x)}{a_2} \Rightarrow y'' + P(x)y' + Q(x)y = f(x)$
To derive the formula, we need the system:

- $y_1u_1' + y_2u_2' = 0$
- $y_1'u_1' + y_2'u_2' = f(x)$

$u_1' = \dfrac{\begin{bmatrix} 0 & y_2 \\ f(x) & y_2' \end{bmatrix}}{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \frac{W_1}{W}$ and $u_2' = \dfrac{\begin{bmatrix} y_1 & 0 \\ y_1' & f(x) \end{bmatrix}}{\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}} = \frac{W_2}{W}$. Also: $W = W(y_1, y_2)$, $u_1 = \int \frac{W_1}{W}dx$ and $u_2 = \int \frac{W_2}{W}dx$.
This is Cramer's Rule.

4.8.1 Example 1

Find the general solution of the equation $y'' - 2y' + y = \frac{e^x}{x}$.

Solution We cannot use the method of undetermined coefficients, since x^{-1} is not a polynomial.
The associated homogeneous equation is $y'' - 2y' + y = 0$ (section 4.3)
The auxiliary equation is $m^2 - 2m + 1 = 0 \Rightarrow (m-1)(m-1) = 0 \Rightarrow m = 1, 1$
So $y_c = c_1e^x + c_2xe^x$

Now, $y'' - 2y' + y = xe^x \Rightarrow$ can use undetermined coefficients.
However, $y'' - 2y' + xy = xe^x \Rightarrow$ cannot use undetermined coefficients.
$y'' - 2y' + y = \frac{x}{e^x} \Rightarrow y'' - 2y' + y = xe^{-x} \Rightarrow$ can use undetermined coefficients.
$y'' - 2y' + y = \frac{e^x}{x^2+1} = e^x \cdot \frac{1}{x^2+1}$

To find y_p: $W = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} e^x & xe^x \\ e^x & (1)e^x + xe^x \end{bmatrix} = e^xe^x + xe^xe^x - xe^xe^x = e^{2x}$
$W_1 = \begin{bmatrix} 0 & y_2 \\ f(x) & y_2' \end{bmatrix} = \begin{bmatrix} 0 & xe^x \\ \frac{e^x}{x} & e^x + xe^x \end{bmatrix} = 0 - xe^x\frac{e^x}{x} = -e^{2x}$
$W_2 = \begin{bmatrix} y_1 & 0 \\ y_1' & f(x) \end{bmatrix} = \begin{bmatrix} e^x & 0 \\ e^x & \frac{e^x}{x} \end{bmatrix} = \frac{e^xe^x}{x} - 0 = \frac{e^{2x}}{x}$
$u_1' = \frac{W_1}{W} = \frac{-e^{2x}}{e^{2x}} = -1 \Rightarrow u_1 = \int -1dx = -x$
$u_2' = \frac{W_2}{W} = \frac{\frac{e^{2x}}{x}}{e^{2x}} = \frac{1}{x} \Rightarrow u_2 = \int \frac{1}{x}dx = \ln|x|$
(Note: Do not write the $+C$ for u_1 or u_2)
Thus $y_p = u_1y_1 + u_2y_2 = -xe^x + \ln|x|xe^x$

The solution is $y = y_c + y_p = c_1e^x + c_2xe^x - xe^x + x\ln|x|e^x$

4.8.2 Example 2

Solve the DE: $5y''' - 5y' = \sec x \Rightarrow$ cannot use undetermined coefficients.

Solution The associated homogeneous equation is $5y''' + 5y' = 0$
The auxiliary equation is $5m^3 + 5m = 0 \Rightarrow 5m(m^2+1) = 0 \Rightarrow m = 0$ and $m^2 = -1 \Rightarrow m = \pm i$ $(\alpha = 0,\ \beta = 1)$
So $y_c = c_1 e^{0x} + e^{0x}(c_2 \cos x + c_3 \sin x) = c_1 + c_2 \cos x + c_3 \sin x$

Now, $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3$
$W = \begin{bmatrix} y_1 & y_2 & y_3 \\ y_1' & y_2' & y_3' \\ y_1'' & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} 1 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ 0 & -\cos x & -\sin x \end{bmatrix} = 1 \begin{bmatrix} -\sin x & \cos x \\ -\cos x & -\sin x \end{bmatrix} - 0 \begin{bmatrix} \cos x & \sin x \\ -\cos x & -\sin x \end{bmatrix} + 0 \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix}$
$= 1(\sin^2 x + \cos^2 x) = 1$
$W_1 = \begin{bmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{bmatrix} = \begin{bmatrix} 0 & \cos x & \sin x \\ 0 & -\sin x & \cos x \\ \frac{\sec x}{5} & -\cos x & -\sin x \end{bmatrix} = 0 \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix} - 0 \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix} + \frac{\sec x}{5} \begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$
$= \frac{\sec x}{5}(\cos^2 x + \sin^2 x) = \frac{\sec x}{5}$
$W_2 = \begin{bmatrix} 1 & 0 & \sin x \\ 0 & 0 & \cos x \\ 0 & \frac{\sec x}{5}(f(x)) & -\sin x \end{bmatrix} = 1 \begin{bmatrix} 0 & \cos x \\ \frac{\sec x}{5} & -\sin x \end{bmatrix} - 0 \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix} + 0 \begin{bmatrix} \dots & \dots \\ \dots & \dots \end{bmatrix} = 1(0 - \cos x \cdot \frac{\sec x}{5}) = \frac{-1}{5}$
$W_3 = \begin{bmatrix} 1 & \cos x & 0 \\ 0 & -\sin x & 0 \\ 0 & -\cos x & \frac{\sec x}{5} \end{bmatrix} = 1 \begin{bmatrix} -\sin x & 0 \\ -\cos x & \frac{\sec x}{5} \end{bmatrix} = -\sin x \cdot \frac{\sec x}{5} - 0 = \frac{-\sin x}{5 \cos x}$
$u_1' = \frac{W_1}{W} = \frac{\frac{\sec x}{5}}{1} = \frac{\sec x}{5} \Rightarrow u_1 = \int \frac{\sec x}{5} dx = \frac{1}{5} \ln|\sec x + \tan x|$
$u_2' = \frac{W_2}{W} = \frac{\frac{-1}{5}}{1} = \frac{-1}{5} \Rightarrow u_2 = \int \frac{-1}{5} dx = \frac{-1}{5} x$
$u_3' = \frac{W_3}{W} = \frac{\frac{-\sin x}{5\cos x}}{1} = \frac{-\sin x}{5 \cos x}$
$u_3 = \int \frac{-\sin x}{5 \cos x} dx$ let $u = \cos x$, $du = -\sin x dx \Rightarrow u_3 = \frac{1}{5} \int \frac{du}{u} = \frac{1}{5} \ln|u| = \frac{1}{5} \ln|\cos x|$
Thus $y_p = u_1 y_1 + u_2 y_2 + u_3 y_3 = \frac{1}{5} \ln|\sec x + \tan x|(1) - \frac{x}{5} \cos x + \frac{1}{5} \ln|\cos x| \sin x$

Thus the solution is $y = y_c + y_p = c_1 + c_2 \cos x + c_3 \sin x + \frac{1}{5} \ln|\sec x + \tan x| - \frac{x}{5} \cos x + \frac{1}{5} \ln|\cos x| \sin x$

For future reference:

- $W_1 = \begin{bmatrix} 0 & y_2 & y_3 \\ 0 & y_2' & y_3' \\ f(x) & y_2'' & y_3'' \end{bmatrix}$
- $W_2 = \begin{bmatrix} y_1 & 0 & y_3 \\ y_1' & 0 & y_3' \\ y_1'' & f(x) & y_3'' \end{bmatrix}$
- $W_2 = \begin{bmatrix} y_1 & y_2 & 0 \\ y_1' & y_2' & 0 \\ y_1'' & y_2'' & f(x) \end{bmatrix}$

4.8.3 Example 3

Solve the IVP: $2y'' + y' - y = x;\ y(0) = 1, y'(0) = 0$

Solution The auxiliary homogeneous equation is $2y'' + y' - y = 0$
The auxiliary equation is $2m^2 + m - 1 = 0 \Rightarrow (2m+2)(2m-1) = 0 \Rightarrow (m+1)(2m-1) = 0 \Rightarrow m = -1, m = \frac{1}{2}$
Thus $y_c = c_1 e^{-x} + c_2 e^{\frac{x}{2}}$ or $y_c = c_1 e^{\frac{x}{2}} + c_2 e^{-x}$

To find y_p:
$W = \begin{bmatrix} e^{\frac{x}{2}} & e^{-x} \\ \frac{1}{2} e^{\frac{x}{2}} & -e^{-x} \end{bmatrix} = -e^{-x} e^{\frac{x}{2}} - \frac{1}{2} e^{-x} e^{\frac{x}{2}} = -e^{\frac{-x}{2}} - \frac{1}{2} e^{\frac{-x}{2}} = -\frac{3}{2} e^{\frac{-x}{2}}$

$W_1 = \begin{bmatrix} 0 & e^{-x} \\ \frac{x}{2} & -e^{-x} \end{bmatrix} = \frac{-x}{2}e^{-x} \Rightarrow u_1' = \frac{W_1}{W} = \frac{\frac{-x}{2}e^{-x}}{\frac{-3}{2}e^{\frac{-x}{2}}} = \frac{x}{3}e^{\frac{-x}{2}}$
$u_1 = \int \frac{x}{3}e^{\frac{-x}{2}}dx \ldots$ Solve yourself.
Continue with W_2.

The easier method for this problem is undetermined coefficients: Let $y_p = Ax + B,\ y_p' = A, y_p'' = 0$
$\Rightarrow 2(0) + A - Ax - B = x \Rightarrow x:\ -A = 1 \Rightarrow A = -1$
Constants: $A - B = 0 \Rightarrow B = A = -1 \Rightarrow y_p = -x - 1$

The solution is $y = y_c + y_p = c_1e^{\frac{x}{2}} + c_2e^{-x} - x - 1$
$y(0) = 1 \Rightarrow c_1e^0 + c_2e^{-0} - 0 - 1 \Rightarrow 1 = c_1 + c_2 - 1 \Rightarrow c_1 + c_2 = 2$
Then $y'(0) = 0 \Rightarrow y' = \frac{1}{2}c_1e^{\frac{x}{2}} - c_2e^{-x} - 1$
$y'(0) = 0 \Rightarrow 0 = \frac{1}{2}c_1e^0 - c_2e^0 - 1 \Rightarrow \frac{1}{2}c_1 - c_2 = 1$
By elimination, $\frac{3}{2}c_1 = 3 \Rightarrow c_1 = 3\frac{2}{3} = 2$
Then $c_1 + c_2 = 2 \Rightarrow c_2 = 0$
The solution of the IVP is $y = 2e^{\frac{x}{2}} - x - 1$

4.9 (4.7) Cauchy-Euler Equations

$3y''' - 2y'' = \cos x$ has constant coefficients: y_c using section 4.3.
If the coefficients are not constant (i.e. $3x^2y''' + 3y' = 7e^x$), normally infinite series are used to solve it.
We will learn to solve a special case of DE: $a_nx^ny^{(n)} + a_{n-1}x^{n-1}y^{(n-1)} + \cdots + a_2x^2y'' + a_1xy' + a_0y = g(x)$
The power of x and the order of derivative is the same in every term.

$x^3y''' - 2xy' + y = \sin x \Rightarrow$ Cauchy Euler equation (CE)
However, $xy'' + 3y' = e^x$ is not a CE.
Also, $xy'' + 3y' = e^x$ is a CE as multiplying by x yields: $x^2y'' + 2xy' = xe^x$

How to solve:

1. Find y_c the solution of the associated homogeneous equation. You **cannot use 4.3 for this**!
2. Find y_p by variation of parameter method.
3. Solution is $y = y_c + y_p$.

To find y_c, we need a new method:

4.9.1 New Method to Find y_c

Start with a 2nd order CE equation: $ax^2y'' + bxy' + cy = 0$
Let $y = x^m$ be a solution: $y' = mx^{m-1}$ and $y'' = m(m-1)x^{m-2}$
Substitute this into the original equation: $ax^2m(m-1)x^{m-2} + bxmx^{m-1} + cx^m = 0$
$a(m^2 - m)x^m + bmx^m + cx^m = 0 \Rightarrow x^m(am^2 - am + bm + c) = 0$, where $x \neq 0$

We get the auxiliary equation $am^2 - (b - a)m + c = 0$, and the following cases:

1. We have distinct real roots m_1, m_2: $y_c = c_1x^{m_1} + c_2x^{m_2}$
2. Repeated roots m, m: $y_c = c_1x^m + c_2x^m \ln x$
3. Complex roots $m = \alpha \pm i\beta$: $y_c = x^\alpha(c_1 \cos(\beta \ln x) + c_2 \sin(\beta \ln x))$

4.9.2 Example 1

Solve the DE: $x^2y'' + 2xy' - 20y = 0$ (this is a CE).

Solution We have $a = 1,\ b = 2,\ c = 20$
The auxiliary equation is $am^2 + (b - a)m + c = 0 \Rightarrow m^2 + (2 - 1)m - 20 = 0 \Rightarrow m^2 + m - 20 = 0$
$\Rightarrow (m + 5)(m - 4) = 0 \Rightarrow m = -5, 4$
So, $y = c_1x^{-5} + c_2x^4$

4.9.3 Example 2

Solve the DE: $x^2y'' - 7xy' + 41y = 0$

Solution We have a CE with $a = 1,\ b = -7,\ c = 41$
The auxiliary equation is $m^2 + (-7-1)m + 41 = 0 \Rightarrow m^2 - 8m + 41 = 0$
$m = \frac{-(-8)\pm\sqrt{(-8)^2-4(1)(41)}}{2(1)} = \frac{8\pm\sqrt{64-164}}{2} = \frac{8\pm i10}{2} = 4 \pm 5i$
The solution is $y = x^4(c_1\cos(5\ln x) + c_2\sin(5\ln x))$

4.9.4 Example 3

Solve the DE: $4x^2y'' + y = 0$

Solution We have a CE with $a = 4, b = 0, c = 1$
The auxiliary equation is $4m^2 + (0-4)m + 1 = 0 \Rightarrow 4m^2 - 4m + 1 = 0 \Rightarrow (2m-1)(2m-1) = 0 \Rightarrow m = \frac{1}{2}, \frac{1}{2}$
The solution is: $y = c_1x^{\frac{1}{2}} + c_2x^{\frac{1}{2}}\ln x$

4.9.5 Example 4

Solve $x^2y'' - 4xy' + 6y = \ln x^2$

Solution From this, we see that $f(x) = \frac{\ln x^2}{x^2}$
The associated homogeneous equation is $x^2y'' - 4xy' + 6y = 0$
This is a CE with $a = 1,\ b = -4,\ c = 6$
The auxiliary equation is $m^2 - 5m + 6 = 0 \Rightarrow (m-2)(m-3) = 0 \Rightarrow m = 2, 3$
Thus $y_c = c_1x^2 + c_2x^3$

Let $y_p = u_1y_1 + u_2y_2$ (variation of parameters)
$W = \begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} = \begin{bmatrix} x^2 & x^3 \\ 2x & 3x^2 \end{bmatrix} = 3x^4 - 2x^4 = x^4$
$W_1 = \begin{bmatrix} 0 & y_2 \\ f(x) & y_2' \end{bmatrix} = \begin{bmatrix} 0 & x^3 \\ \frac{\ln x^2}{x^2} & 3x^2 \end{bmatrix} = 0 - x^3\frac{\ln x^2}{x^2} = -x\ln x^2 = -2x\ln x$
$W_2 = \begin{bmatrix} y_1 & 0 \\ y_1' & f(x) \end{bmatrix} = \begin{bmatrix} x^2 & 0 \\ 2x & \frac{\ln x^2}{x^2} \end{bmatrix} = x^2\frac{\ln x^2}{x^2} = \ln x^2 = 2\ln x$
$u_1' = \frac{W_1}{W} = \frac{-2x\ln x}{x^4} = \frac{-2\ln x}{x^3}$
$u_1 = -2\int\frac{\ln x}{x^3}dx$, let $u = \ln x$, $du = \frac{1}{x}dx$, $v = \frac{x^{-2}}{-2}, dv = x^{-3}dx$
$u_1 = -2\left[\ln x\frac{x^{-2}}{-2} - \int\frac{1}{x}\frac{x^{-2}}{-2}dx\right]$
$u_1 = \frac{\ln x}{x^2} - \frac{x^{-2}}{-2} = \frac{\ln x}{x^2} + \frac{1}{2x^2}$
$u_2' = \frac{W_2}{W} = \frac{2\ln x}{x^4} \Rightarrow u_2 = 2\int\frac{\ln x}{x^4}dx$
Let $u = \ln x$, $du = \frac{1}{x}dx$, $v = \frac{x^{-3}}{-3}$, $dv = x^{-4}dx$
$u_2 = 2\left[\ln x(\frac{x^{-3}}{-3}) - \int\frac{1}{x}\frac{x^{-3}}{-3}dx\right]$
$= \frac{-2}{3}\frac{\ln x}{x^3} + \frac{2}{3}\int x^{-4}dx = \frac{-2}{3}\frac{\ln x}{x^3} + \frac{2}{3}\frac{x^{-3}}{-3} = \frac{-2}{3}\frac{\ln x}{x^3} - \frac{2}{9x^3}$
$y_p = u_1y_1 + u_2y_2 = (\frac{\ln x}{x^2} + \frac{1}{2x^2})x^2 + (\frac{-2}{3}\frac{\ln x}{x^3} - \frac{2}{9x^3})x^3$
The solution is $y = y_c + y_p = c_1x^2 + c_2x^3 + \ln x + \frac{1}{2} - \frac{2}{3}\ln x - \frac{2}{9}$

Side note: Put $t = \ln x \Rightarrow x = e^t$
$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx} = \frac{dy}{dt}\cdot\frac{1}{x}$