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# COMP 2310 Assignment 4

5.8.36)

i) Prove or disprove R and S are antisymmetric $\Rightarrow R \cup S$ is an antisymmetric relation

(disproof by counterexample)
Let $R = \leq$ and $S = \geq$ ...(0)
Since $\leq$ and $\geq$ are partial orders
$\Rightarrow \leq$ and $\geq$ are antisymmetric (definition of partial order)
$\Rightarrow$ R and S are antisymmetric ((0), sub=)

Since $0 \leq 1$, we have $(0,1) \in R$ (definition of R)
$\Rightarrow (0,1) \in R \vee (0,1) \in S$ (I1)
$\Rightarrow (0,1) \in R \cup S$ (definition of $\cup$)...(A)
Since $1 \geq 0$, we have $(1,0) \in S$ (definition of R)
$\Rightarrow (1,0) \in S \vee (1,0) \in R$ (I1)
$\Rightarrow (1,0) \in R \vee (1,0) \in S$ (E10)
$\Rightarrow (1,0) \in R \cup S$ (definition of $\cup$)...(B)

Now we have $(0,1) \in R \cup S \wedge (1,0) \in R \cup S$ ((A), (B), I6)
Since $0 \neq 1$, then $(0,1) \in R \cup S \wedge (1,0) \in R \cup S \Rightarrow 0=1$ is evaluated to false
$\Rightarrow R \cup S$ is antisymmetric is evaluated to false. (definition of antisymmetric)
Thus $R \cup S$ is not antisymmetric.
□

ii) R and S are antisymmetric $\Rightarrow R \cap S$ is antisymmetric.
(direct proof)
Suppose R and S are antisymmetric (hypothesis)
Then $(a,b) \in R \wedge (b,a) \in R \Rightarrow a=b$ (R is antisymmetric) ... (A)
Also $(a,b) \in S \wedge (b,a) \in S \Rightarrow a=b$ (S is antisymmetric) ... (B)

We are to prove $(a,b) \in R \cap S \wedge (b,a) \in R \cap S \Rightarrow a=b$
(direct proof)
Suppose $(a,b) \in R \cap S \wedge (b,a) \in R \cap S$ (hypothesis)
$\Rightarrow (a,b) \in R \wedge (a,b) \in S \wedge (b,a) \in R \wedge (b,a) \in S$ (definition of $\cap$)
$\Rightarrow (a,b) \in R \wedge (b,a) \in R \wedge (a,b) \in S \wedge (b,a) \in S$ (E9)
$\Rightarrow (a,b) \in R \wedge (b,a) \in R$ (I2)
$\Rightarrow a=b$ ((P 5.3(c))

Thus, $R \cap S$ is antisymmetric
□

5.8.17. $R \circ Y_= = R$ and $X_= \circ R = R$ $\quad (\exists z)(z \in Y \wedge ((x,z) \in R \wedge (z,y) \in Y_=))$

a) We are to prove $(x,y) \in R \circ Y_= \Longleftrightarrow (x,y) \in R$
(Bidirectional proof)
$\Rightarrow$) Prove $(x,y) \in R \circ Y_= \Longrightarrow (x,y) \in R$
(direct proof)
Suppose $(x,y) \in R \circ Y_=$ (hypothesis)
$\Rightarrow (\exists z)(z \in Y \wedge ((x,z) \in R \wedge (z,y) \in Y_=))$ (definition of $\circ$)
$\Rightarrow a \in Y \wedge ((x,a) \in R \wedge (a,y) \in Y_=)$ (EI)
$\Rightarrow a \in Y \wedge ((x,a) \in R \wedge a = y)$ (definition of $Y_=$)
$\Rightarrow (x,a) \in R \wedge a = y$ (I2)
$\Rightarrow \begin{cases} (x,a) \in R & \text{(E9, I2)} \quad \ldots(\mathrm{I}) \\ a = y & \quad \ldots(\mathrm{II}) \end{cases}$
Then $(x,y) \in R$ ((I), (II), sub$_=$)

($\Leftarrow$) We are to prove $(x,y) \in R \Longrightarrow (x,y) \in R \circ Y_=$
(direct proof)
Suppose $(x,y) \in R$ (hypothesis) (III)
Let $a = y$ ...(X)
$\Rightarrow (a,y) \in Y_=$ (definition of $Y_=$)...(IV)
$\Rightarrow (a,y) \in Y \times Y$ (definition of $Y_=$)
$\Rightarrow a \in Y \wedge y \in Y$ (definition of $\times$)
$\Rightarrow a \in Y$ (I2)
$\Rightarrow a \in Y \wedge (x,y) \in R \wedge (a,y) \in Y_=$ ((III), (IV), I6)
$\Rightarrow a \in Y \wedge (x,a) \in R \wedge (a,y) \in Y_=$ ((X), sub$_=$)
$\Rightarrow (\exists z)(z \in Y \wedge (x,z) \in R \wedge (z,y) \in Y_=)$ (EQ)
$\Rightarrow (x,y) \in R \circ Y_=$ (definition of $\circ$)
Thus $(x,y) \in R \Longrightarrow (x,y) \in R \circ Y_=$
Thus $(x,y) \in R \circ Y_= \Longleftrightarrow (x,y) \in R$
$\Rightarrow R \circ Y_= = R$ (principle of extension)

b) We are to prove $(x,y) \in X_= \circ R \Longleftrightarrow (x,y) \in R$
(Bidirectional proof)
$\Rightarrow$) We are to prove $(x,y) \in X_= \circ R \Longrightarrow (x,y) \in R$
(direct proof)
Suppose $(x,y) \in X_= \circ R$ (hypothesis)
$\Rightarrow (\exists z)(z \in X \wedge ((x,z) \in X_= \wedge (z,y) \in R))$ (definition of $\circ$)
$\Rightarrow b \in X \wedge (x,b) \in X_= \wedge (b,y) \in R$ (EI)
$\Rightarrow b \in X \wedge x = b \wedge (b,y) \in R$ (definition of $X_=$)
$\Rightarrow x = b \wedge (b,y) \in R$ (I2)
$\Rightarrow \begin{cases} x = b & \text{(E9, I2)} \quad \ldots(\mathrm{V}) \\ (b,y) \in R & \quad \ldots(\mathrm{VI}) \end{cases}$
Then $(x,y) \in R$ ((VI), (V), sub$_=$)

($\Leftarrow$) We are to prove $(x,y) \in R \Longrightarrow (x,y) \in X_= \circ R$
(direct proof)
Suppose $(x,y) \in R$ (hypothesis) ...(VII)
Let $x = b$ ...(IX)
$\Rightarrow (x,b) \in X_=$ (definition of $X_=$) ...(VIII)
$\Rightarrow (x,b) \in X \times X$ (definition of $X_=$)
$\Rightarrow x \in X \wedge b \in X$ (definition of $\times$)
$\Rightarrow b \in X$ (E9, I2)
$\Rightarrow b \in X \wedge (x,b) \in X_= \wedge (x,y) \in R$ ((VIII), (VII), I6)
$\Rightarrow b \in X \wedge (x,b) \in X_= \wedge (b,y) \in R$ ((IX), sub$_=$)

$\Rightarrow (\exists z)(z \in X \wedge (x,z) \in X_= \wedge (z,y) \in R)$ (EQ)
$\Rightarrow (x,y) \in X_= \circ R$ (definition of $\circ$)

Thus $(x,y) \in R \Rightarrow (x,y) \in X_= \circ R$
Thus $(x,y) \in X_= \circ R \Leftrightarrow (x,y) \in R$
$\Rightarrow X_= \circ R = R$ (principle of extension)

6.6.14. Let $f : X \to Y$

a) $(\exists g,h)(g : Y \to X \wedge h : Y \to X \wedge f \circ g = I_X \wedge h \circ f = I_Y) \Rightarrow f$ is a one-to-one correspondence.
(direct proof)
Suppose $(\exists g,h)(g : Y \to X \wedge h : Y \to X \wedge f \circ g = I_X \wedge h \circ f = I_Y)$
$\Rightarrow c : Y \to X \wedge d : Y \to X \wedge f \circ c = I_X \wedge d \circ f = I_Y$ (EI)
$\Rightarrow \begin{cases} c : Y \to X \\ d : Y \to X \\ f \circ c = I_X & \ldots (F) \\ d \circ f = I_Y & \ldots (G) \end{cases}$ (E9, I2)

i) We are to prove $f$ is one-to-one.
$\Leftrightarrow (x,z) \in f \wedge (y,z) \in f \Rightarrow x = y$ (definition of one-to-one)
(direct proof)
Suppose $(x,z) \in f \wedge (y,z) \in f$ (A)
$\Rightarrow z = f(x) \wedge z = f(y)$ (equivalent notation)
$\Rightarrow \begin{cases} z = f(x) & \ldots (\mathrm{I}) \\ z = f(y) & \ldots (\mathrm{II}) \end{cases}$ (E9, I2)
Now, $z = f(y)$ ((II))
$\Rightarrow f(x) = f(y)$ ((I), sub=) ... (III)

Also $(x,z) \in f$ ((A), I2) ... (X)
$\Rightarrow (x,z) \in X \times Y$ (definition of $f$)
$\Rightarrow x \in X \wedge z \in Y$ (definition of $\times$)
$\Rightarrow z \in Y$ (E9, I2) ... (Z)

Similarly, $I_X(y) = y$ (definition of $I_X$)
$\Rightarrow (y,y) \in I_X$ (equivalent notation)
$\Rightarrow (y,y) \in f \circ c$ ((F), sub=)
$\Rightarrow (\exists z \in Y)((y,z) \in f \wedge (z,y) \in c)$ (definition of $\circ$)
$\Rightarrow a \in Y \wedge ((y,a) \in f \wedge (a,y) \in c)$ (EI)
$\Rightarrow (y,a) \in f \wedge (a,y) \in c$ (I2)
$\Rightarrow a = f(y) \wedge (a,y) \in c$ (equivalent notation)
$\Rightarrow \begin{cases} a = f(y) & \ldots (B) \\ (a,y) \in c & \ldots (C) \end{cases}$ (E9, I2)
Then $(f(y), y) \in c$ ((C), (B), sub=)
$\Rightarrow (z, y) \in c$ ((II), sub=)
$\Rightarrow (x,z) \in f \wedge (z,y) \in c$ ((X), I6, E9)
$\Rightarrow z \in Y \wedge ((x,z) \in f \wedge (z,y) \in c)$ ((Z), I6, E9)
$\Rightarrow (\exists z \in Y)((x,z) \in f \wedge (z,y) \in c)$ (EQ)
$\Rightarrow (x,y) \in f \circ c$ (definition of $\circ$)
$\Rightarrow (x,y) \in I_X$ ((F), sub=)
$\Rightarrow y = I_X(x)$ (equivalent notation)
$\Rightarrow y = x$ (definition of $I_X$)
$\Rightarrow x = y$ (= is reflexive)

Thus $f$ is one-to-one.

ii) We are to prove $f$ is onto

$\Rightarrow (\forall y)(y\in Y \Rightarrow ((\exists x) x\in X \wedge y=f(x)))$ (definition of onto)

$\Rightarrow y\in Y \Rightarrow ((\exists x) x\in X \wedge y=f(x))$ (UI)

(direct proof)

Suppose $y\in Y$ (hypothesis)

Let $x\in X$ such that $x=d(y)$ (definition of $d$) ... (1)

Now, $y = I_Y(y)$ (definition of $I_Y(y)$)

$= d\circ f(y)$ ((G), sub$_=$)

$= f(d(y))$ (equivalent notation)

$= f(x)$ ((1), sub$_=$) ... (2)

Thus $x\in X \wedge y=f(x)$ (definition of $x$, (2), I6)

$\Rightarrow (\exists x)(x\in X \wedge y=f(x))$ (EQ)

Thus, $y\in Y \Rightarrow (\exists x)(x\in X \wedge y=f(x))$

$\Rightarrow (\forall y)(y\in Y \Rightarrow (\exists x)(x\in X \wedge y=f(x)))$ (Gen)

$\Rightarrow f$ is onto (definition of onto)

Since $f$ is one-to-one and onto, thus $f$ is one-to-one correspondence. □

b) $(\exists g,h)(g: Y\to X \wedge h: Y\to X \wedge f\circ g = I_X \wedge h\circ f = I_Y) \Longrightarrow g=h=f^{-1}$

(direct proof)

Suppose $(\exists g,h)(g: Y\to X \wedge h: Y\to X \wedge f\circ g = I_X \wedge h\circ f = I_Y)$

$\Rightarrow c: Y\to X \wedge d: Y\to X \wedge f\circ c = I_X \wedge d\circ f = I_Y$ (EI)

$$\Rightarrow \begin{cases} c: Y\to X & \dots (A) \\ d: Y\to X \quad (\in 9, I2) & \dots (B) \\ f\circ c = I_X & \dots (C) \\ d\circ f = I_Y & \dots (D) \end{cases}$$

i) We are to show $c = f^{-1}$

$\Rightarrow (\forall y\in Y)(c(y)=f^{-1}(y))$ (lemma 6.1.1)

(direct proof)

Let $y\in Y$ ... (1)

Then $x=c(y)$ for some $x\in X$. (definition of $c$) ... (X)

Then $z=f^{-1}(y)$ for some $z\in X$ (definition of $f^{-1}$) ... (2)

$\Rightarrow y=f(z)$. (definition of inverse) ... (Y)

Also $z = I_X(z)$ (definition of $I_X$)

$= f\circ c(z)$ ((C), sub$_=$)

$= c(f(z))$ (equivalent notation)

$= c(y)$ ((Y), sub$_=$)

$= x$ ((X), sub$_=$)

Thus $z=x \Rightarrow x=z$ (= is reflexive)

$\Rightarrow x = f^{-1}(y)$ ((2), sub$_=$)

$\Rightarrow c(y) = f^{-1}(y)$ ((X), sub$_=$)

$\Rightarrow y\in Y \wedge c(y)=f^{-1}(y)$ ((1), I6, $\in$9)

$\Rightarrow (\forall y\in Y)(c(y)=f^{-1}(y))$ (Gen)

$\Rightarrow c=f^{-1}$ (lemma 6.1.1)

ii) We are to show $d=f^{-1} \Rightarrow (\forall y\in Y)(d(y)=f^{-1}(y))$ (lemma 6.1.1)

(direct proof)

Let $y\in Y$

Then $x=d(y)$ for some $x\in X$. (definition of $d$) ... (2)

$y = I_Y(y)$ (definition of $I_Y$)

$= d\circ f(y)$ ((D), sub$_=$)

$= f(d(y))$ (equivalent notation)

Thus $y = f(d(y))$

$\Rightarrow d(y) = f^{-1}(y)$ (definition of $f^{-1}$)

$\Rightarrow (\forall y \in Y)(d(y) = f^{-1}(y))$ (Gen)

Hence $(\forall y \in Y)(d(y) = f^{-1}(y)) \Rightarrow d = f^{-1}$ (lemma 6.1.1)

Hence, $c = d = f^{-1}$ $\square$

7.8.3. Let $A$ and $B$ be any two sets. Prove that $A \times B \sim B \times A$

We are to prove $(\exists f) f: A \times B \xrightarrow[\text{onto}]{1-1} B \times A$

Let $f: A \times B \to B \times A$ such that $((a,b),(b,a)) \in f$ (or equivalently, $f((a,b)) = (b,a)$)

i) We are to prove that $f((a,b)) = f((c,d)) \Rightarrow (a,b) = (c,d)$

(direct proof)

Suppose $f((a,b)) = f((c,d))$ (hypothesis)

$\Rightarrow (b,a) = (d,c)$ (definition of $f$)

$\Rightarrow b = d \wedge a = c$ (lemma 4.4.8)

$\Rightarrow a = c \wedge b = d$ (Eq)

$\Rightarrow (a,b) = (c,d)$ (lemma 4.4.8)

Hence $f$ is one-to-one.

ii) We are to prove $(\forall c,d)((c,d) \in B \times A \Rightarrow (\exists a,b)((a,b) \in A \times B \wedge (c,d) = f((a,b)))$

$\Rightarrow (c,d) \in B \times A \Rightarrow (\exists a,b)((a,b) \in A \times B \wedge (c,d) = f((a,b)))$ (UI)

(direct proof)

Suppose $(c,d) \in B \times A$

Now, $(c,d) = f((d,c))$ (definition of $f$) ... (A)

$\Rightarrow ((d,c),(c,d)) \in f$ (equivalent notation)

$\Rightarrow ((d,c),(c,d)) \in (A \times B) \times (B \times A)$ (definition of $f$)

$\Rightarrow (d,c) \in A \times B \wedge (c,d) \in B \times A$ (definition of $\times$)

$\Rightarrow (d,c) \in A \times B$ (I2)

$\Rightarrow (d,c) \in A \times B \wedge (c,d) = f((d,c))$ ((A), IC)

Thus, $(\exists a,b)((a,b) \in A \times B \wedge (c,d) = f((a,b)))$

Hence, $f$ is onto.

Thus, $f$ is bijective.

Thus, $A \times B \sim B \times A$ (definition of $\sim$)

7.8.12. $g = f^{-1}$

a) $g(i) =$ "program h(input, output);

```
var x: integer;
begin
    read(x);
    write(2x);
end."
```

$$F_i(x) = \begin{cases} 2x & \text{if } x \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

Thus $d(i) = F_i(i) + 1$

$$= \begin{cases} 2i + 1 & \text{if } i \in \mathbb{N} \\ 2 & \text{otherwise} \end{cases} \quad \text{(definition of } F_i\text{)}$$

b) $g(i) =$ "program h(input, output);

```
var x: integer;
begin
  read(x);
  repeat x:=x until (false)
end."
```

By inspection, $g(i)$ has no output (it has no write, which dictates the output).

Thus, $F_i(x) = 1$

Hence $d(i) = F_i(i) + 1$

$= 1 + 1$ (definition of $F_i(i)$)

$= 2$ (h.s.a.)

c) $g(i) =$ "program h(input, output);

```
var x: integer;
begin
  anjgishgroegjskklsjjbumijk
end."
```

By inspection, $g(i)$ is an invalid program since there's no input or output.

Thus, $F_i(x) = 1$

Hence $d(i) = F_i(i) + 1$

$= 1 + 1$ (definition of $F_i(i)$)

$= 2$ (h.s.a.)

d) $g(i) =$ "program h(input, output);

```
var x: integer;
begin
  read(x);
  if (x<50) then write(x-30)
  elseif (x<1000) then write(x/2)
  else write("too large")
end."
```

By inspection, we can derive $F_i(x)$ and $d(i)$ as follows:

$$F_i(x) = \begin{cases} x-30 & \text{if } 30 < x < 50 \\ x/2 & \text{if } 50 \le x < 1000 \text{ and } x/2 \in \mathbb{N} \\ 1 & \text{otherwise} \end{cases}$$

$$d(i) = F_i(i) + 1$$

$$= \begin{cases} i-29 & \text{if } 30 < i < 50 \\ i/2+1 & \text{if } 50 \le i < 1000 \text{ and } i/2 \in \mathbb{N} \\ 2 & \text{otherwise} \end{cases} \quad \text{(definition of } F_i(i))$$