Justin Bornais - 110037759 **2310 Assignment 2**

1. $(\forall x)(P(x) \Rightarrow (\exists x)(\forall y)(Q(y,x) \wedge P(x)))$

$U = \{1,2\}$

x	$P(x)$
1	T
2	F

x	y	$Q(x,y)$
1	1	F
1	2	T
2	1	F
2	2	T

Let $x=1$ in $(\forall x)(P(x) \Rightarrow (\exists x)(\forall y)(Q(y,x) \wedge P(x)))$. We obtain $P(1) \Rightarrow (\exists x)(\forall y)(Q(y,x) \wedge P(x))$
Since $P(1) \equiv T$, we have to determine the value of $(\exists x)(\forall y)(Q(y,x) \wedge P(x))$

i) Let $x=1$ in $(\exists x)(\forall y)(Q(y,x) \wedge P(x))$. We obtain $(\forall y)(Q(y,1) \wedge P(1))$
Since $P(1) \equiv T$ we obtain $(\forall y)(Q(y,1) \wedge T) \equiv (\forall y)Q(y,1)$
Let $y=2$ in $(\forall y)Q(y,1)$. We obtain $Q(2,1) \equiv F$.
Hence $(\forall y)Q(y,1)$ is evaluated to false.
Hence $(\forall y)(Q(y,1) \wedge P(1))$ is evaluated to false.

ii) Let $x=2$ in $(\exists x)(\forall y)(Q(y,x) \wedge P(x))$ We obtain $(\forall y)(Q(y,2) \wedge P(2))$
Since $P(2) = F$, we obtain $(\forall y)(Q(y,2) \wedge F) \equiv F$
Hence $(\forall y)(Q(y,2) \wedge P(2))$ is evaluated to false.

Since $(\forall y)(Q(y,x) \wedge P(x))$ is evaluated to false for all values of x, thus $(\exists x)(\forall y)(Q(y,x) \wedge P(x))$ is evaluated to false.

Hence $P(1) \Rightarrow (\exists x)(\forall y)(Q(y,x) \wedge P(x))$ is evaluated to false.
Hence $(\forall x)(P(x) \Rightarrow (\exists x)(\forall y)(Q(y,x) \wedge P(x)))$ is evaluated to false. $\square$

2.2 e) $\vdash (\forall x)(\alpha(x) \Rightarrow \beta) \Longleftrightarrow ((\exists x)\alpha(x) \Rightarrow \beta)$
where x is not free in β.
(Biconditional proof)

1) Prove $\vdash (\forall x)(\alpha(x) \Rightarrow \beta) \Longrightarrow (\exists x)(\alpha(x) \Rightarrow \beta)$
where x is not free in β.
(Direct Proof)

1. $(\forall x)(\alpha(x) \Rightarrow \beta)$ Premise
2. $(\forall x)(\sim\alpha(x) \vee \beta)$ 1, E18
3. $(\forall x)\sim\alpha(x) \vee \beta$ 2, FE3
4. $\sim(\exists x)\alpha(x) \vee \beta$ 3 FE8
5. $(\exists x)\alpha(x) \Rightarrow \beta$ 4, E18

Hence $\vdash (\forall x)(\alpha(x) \Rightarrow \beta) \Longrightarrow ((\exists x)\alpha(x) \Rightarrow \beta)$
where x is not free in β.

2) Prove $\vdash ((\exists x)\alpha(x) \Rightarrow \beta) \Longrightarrow (\forall x)(\alpha(x) \Rightarrow \beta)$
where x is not free in β.
(Direct Proof)

1. $(\exists x)\ \alpha(x) \Rightarrow \beta$ Premise
2. $\sim(\exists x)\alpha(x) \vee \beta$ 1, E18
3. $(\forall x)\sim\alpha(x) \vee \beta$ 2, FE8
4. $(\forall x)(\sim\alpha(x) \vee \beta)$ 3, FE3
5. $(\forall x)(\alpha(x) \Rightarrow \beta)$ 4, E18

Hence $\vdash ((\exists x)\alpha(x) \Rightarrow \beta) \Rightarrow (\forall x)(\alpha(x) \Rightarrow \beta$
where x is not free in β.
Hence $\vdash (\forall x)(\alpha(x) \Rightarrow \beta) \Longleftrightarrow ((\exists x)\alpha(x) \Rightarrow \beta)$
where x is not free in β. $\square$

2.2 j) Prove that

$(\forall x)(\alpha(x) \Leftrightarrow (\exists y)\beta(y,x)), (\forall x)(\forall y)(\beta(x,y) \Rightarrow \gamma(x)\wedge\gamma(y))$
$\vdash (\forall x)(\alpha(x) \Rightarrow \gamma(x))$.

1. $(\forall x)(\alpha(x) \Longleftrightarrow (\exists y)\beta(y,x))$ from Γ
2. $(\forall x)(\forall y)(\beta(x,y) \Rightarrow \gamma(x)\wedge\gamma(y))$ from Γ
3. $\alpha(x) \Longleftrightarrow (\exists y)\beta(y,x)$ 1, UI
4. $(\alpha(x) \Rightarrow (\exists y)\beta(y,x)) \wedge ((\exists y)\beta(y,x) \Rightarrow \alpha(x))$ 3, E20
5. $\alpha(x) \Rightarrow (\exists y)\beta(y,x)$ 4, I2
6. $(\forall x)(\forall z)(\beta(x,z) \Rightarrow \gamma(x)\wedge\gamma(z))$ 2, FE1
7. $(\forall y)(\forall z)(\beta(y,z) \Rightarrow \gamma(y)\wedge\gamma(z))$ 6, FE1
8. $(\forall y)(\forall x)(\beta(y,x) \Rightarrow \gamma(y)\wedge\gamma(x))$ 7, FE1
9. $(\forall x)(\beta(y,x) \Rightarrow \gamma(y)\wedge\gamma(x))$ 8, UI
10. $\beta(y,x) \Rightarrow \gamma(y)\wedge\gamma(x)$ 9, UI
11. $\sim\beta(y,x) \vee (\gamma(y)\wedge\gamma(x))$ 10, E18
12. $(\sim\beta(y,x) \vee \gamma(y)) \wedge (\sim\beta(y,x) \vee \gamma(x))$ 11, E14
13. $(\sim\beta(y,x) \vee \gamma(x)) \wedge (\sim\beta(y,x) \vee \gamma(y))$ 12, E9
14. $\sim\beta(y,x) \vee \gamma(x)$ 13, I2
15. $(\forall y)(\sim\beta(y,x) \vee \gamma(x))$ 14, Gen
16. $(\forall y)\sim\beta(y,x) \vee \gamma(x)$ 15, FE3
17. $\sim(\exists y)\beta(y,x) \vee \gamma(x)$ 16, FE8
18. $(\exists y)\beta(y,x) \Rightarrow \gamma(x)$ 17, E18
19. $\alpha(x) \Rightarrow \gamma(x)$ 5, 18, I5
20. $(\forall x)(\alpha(x) \Rightarrow \gamma(x))$ 19, Gen

Hence, $(\forall x)(\alpha(x) \Longleftrightarrow (\exists y)\beta(y,x)), (\forall x)(\forall y)(\beta(x,y) \Rightarrow (\gamma(x)\wedge\gamma(y))$
$\vdash (\forall x)(\alpha(x) \Rightarrow \gamma(x))$. $\square$

2.3 c) Prove that P1, P2, P3 $\vdash$ C

P1: $(\forall x)(\sim Q(x) \Rightarrow P(x))$

P2: $\sim(\exists x)(R(x) \wedge Q(x))$

P3: $\sim(\exists x)(S(x) \wedge P(x))$

C: $\sim(\exists x)(R(x) \wedge S(x))$

1. $(\forall x)(\sim Q(x) \Rightarrow P(x))$ from I
2. $\sim(\exists x)(R(x) \wedge Q(x))$ from I
3. $\sim(\exists x)(S(x) \wedge P(x))$ from I
4. $(\forall x) \sim(R(x) \wedge Q(x))$ 2, FE8
5. $(\forall x) \sim(S(x) \wedge P(x))$ 3, FE8
6. $\sim(R(x) \wedge Q(x))$ 4, UI
7. $\sim(S(x) \wedge P(x))$ 5, UI
8. $\sim R(x) \vee \sim Q(x)$ 6, E16
9. $R(x) \Rightarrow \sim Q(x)$ 8, E18
10. $\sim S(x) \vee \sim P(x)$ 7, E16
11. $S(x) \Rightarrow \sim P(x)$ 10, E18
12. $\sim\sim P(x) \Rightarrow \sim S(x)$ 11, E19
13. $P(x) \Rightarrow \sim S(x)$ 12, E15
14. $\sim Q(x) \Rightarrow P(x)$ 1, UI
15. $\sim Q(x) \Rightarrow \sim S(x)$ 14, 13, IS
16. $R(x) \Rightarrow \sim S(x)$ 9, 15, IS
17. $\sim R(x) \vee \sim S(x)$ 16, E18
18. $\sim(R(x) \wedge S(x))$ 17, E16
19. $(\forall x) \sim(R(x) \wedge S(x))$ 18, Gen
20. $\sim(\exists x)(R(x) \wedge S(x))$ 19, FE8

Hence $(\forall x)(\sim Q(x) \Rightarrow P(x)), \sim(\exists x)(R(x) \wedge Q(x)), \sim(\exists x)(S(x) \wedge P(x)) \vdash \sim(\exists x)(R(x) \wedge S(x))$ $\square$

2.36) P1: $(\forall x)(A(x) \Rightarrow (\forall y)(E(x,y) \Rightarrow H(y)))$

P2: $(\forall x)(T(x) \Rightarrow ((\exists y)(M(y) \wedge E(x,y)) \wedge (\exists y)(B(y) \wedge E(x,y))))$

P3: $(\forall x)(M(x) \Rightarrow H(x)) \wedge (\forall x)(B(x) \Rightarrow \sim H(x))$

C: $\sim(\exists x)(T(x) \wedge A(x))$

(Proof by Contradiction)

1.	$\sim\sim(\exists x)(T(x) \wedge A(x))$	Hypothesis
2.	$(\forall x)(A(x) \Rightarrow (\forall y)(E(x,y) \Rightarrow H(y)))$	from I
3.	$(\forall x)(T(x) \Rightarrow ((\exists y)(M(y) \wedge E(x,y)) \wedge (\exists y)(B(y) \wedge E(x,y))))$	from I
4.	$(\forall x)(M(x) \Rightarrow H(x)) \wedge (\forall x)(B(x) \Rightarrow \sim H(x))$	from I
5.	$(\exists x)(T(x) \wedge A(x))$	1, E15
6.	$T(z) \wedge A(z)$	5, EI, z is a new constant
7.	$T(z)$	6, I2
8.	$A(z) \wedge T(z)$	7, E9
9.	$A(z)$	8, I2
10.	$T(z) \Rightarrow ((\exists y)(M(y) \wedge E(z,y)) \wedge (\exists y)(B(y) \wedge E(z,y)))$	3, UI
11.	$((\exists y)(M(y) \wedge E(z,y)) \wedge (\exists y)(B(y) \wedge E(z,y)))$	7, 10, I3
12.	$(\exists y)(B(y) \wedge E(z,y)) \wedge (\exists y)(M(y) \wedge E(z,y))$	11, E9
13.	$(\exists y)(B(y) \wedge E(z,y))$	12, I2
14.	$B(a) \wedge E(z,a)$	13, EI, a is a new constant
15.	$A(z) \Rightarrow (\forall y)(E(z,y) \Rightarrow H(y))$	2, UI
16.	$(\forall y)(E(z,y) \Rightarrow H(y))$	9, 15, I3
17.	$E(z,a) \Rightarrow H(a)$	16, UI
18.	$(\forall x)(B(x) \Rightarrow \sim H(x)) \wedge (\forall x)(M(x) \Rightarrow H(x))$	4, E9
19.	$(\forall x)(B(x) \Rightarrow \sim H(x))$	18, I2
20.	$B(a) \Rightarrow \sim H(a)$	19, UI
21.	$\sim\sim H(a) \Rightarrow B(a)$	20, E17
22.	$H(a) \Rightarrow B(a)$	21, E15
23.	$E(z,a) \Rightarrow B(a)$	17, 22, I5
24.	$\sim E(z,a) \vee B(a)$	23, E18
25.	$(B(a) \wedge E(z,a)) \wedge (\sim E(z,a) \vee B(a))$	14, 24, I6
26.	$((B(a) \wedge E(z,a)) \wedge \sim E(z,a)) \vee B(a)$	25, E11
27.	$B(a) \wedge (E(z,a) \wedge \sim E(z,a)) \vee B(a)$	26, E11
28.	$B(a) \vee B(a) \wedge (E(z,a) \wedge \sim E(z,a))$	27, E10
29.	$B(a) \wedge (E(z,a) \wedge \sim E(z,a))$	28, E4
30.	$B(a) \wedge \text{false}$	29, E1
31.	false	30, E7

Hence $(\forall x)(A(x) \Rightarrow (\forall y)(E(x,y) \Rightarrow H(y)))$, $(\forall x)(T(x) \Rightarrow ((\exists y)(M(y) \wedge E(x,y)) \wedge (\exists y)(B(y) \wedge E(x,y))))$, $(\forall x)(M(x) \Rightarrow H(x)) \wedge (\forall x)(B(x) \Rightarrow \sim H(x))$

$\vdash \sim(\exists x)(T(x) \wedge A(x))$ $\square$

2.4h) Prove that the following conclusion follows from the given premises.

Animals that do not kick are always unexitable;
Donkeys have no horns;
A buffalo can always toss one over a gate;
No animals that kick are easy to swallow.
No hornless animal can toss one over a gate;
All animals are exitable, except buffalos.
<u>Conclusion</u>: Donkeys are not easy to swallow.

Let U (the Universe of discourse) be the set of animals;
$K(x)$ denote animal x can kick;
$H(x)$ denote animal x has horns;
$T(x)$ denote animal x can toss one over a gate;
$E(x)$ denote animal x is exitable;
$S(x)$ denote animal x is easy to swallow;
$D(x)$ denote animal x is a donkey;
$B(x)$ denote animal x is a buffalo.

Then, the corresponding FOL-wffs are as follows:

P1: $(\forall x)(\sim K(x) \Rightarrow \sim E(x))$
P2: $(\forall x)(D(x) \Rightarrow \sim H(x))$
P3: $(\forall x)(B(x) \Rightarrow T(x))$
P4: $\sim(\exists x)(K(x) \Rightarrow S(x))$
P5: $\sim(\exists x)(\sim H(x) \Rightarrow T(x))$
P6: $(\forall x)(\sim B(x) \Leftrightarrow E(x))$
C: $(\forall x)(D(x) \Rightarrow \sim S(x))$

1. $(\forall x)(\sim K(x) \Rightarrow \sim E(x))$ from Γ
2. $(\forall x)(D(x) \Rightarrow \sim H(x))$ from Γ
3. $(\forall x)(B(x) \Rightarrow T(x))$ from Γ
4. $\sim(\exists x)(K(x) \Rightarrow S(x))$ from Γ
5. $\sim(\exists x)(\sim H(x) \Rightarrow T(x))$ from Γ
6. $(\forall x)(\sim B(x) \Leftrightarrow E(x))$ from Γ
7. $\sim K(x) \Rightarrow \sim E(x)$ 1, UI
8. $D(x) \Rightarrow \sim H(x)$ 2, UI
9. $B(x) \Rightarrow T(x)$ 3, UI
10. $(\forall x) \sim(K(x) \Rightarrow S(x))$ 4, FE8
11. $\sim(K(x) \Rightarrow S(x))$ 10, UI
12. $(\forall x) \sim(\sim H(x) \Rightarrow T(x))$ 5, FE8
13. $\sim(\sim H(x) \Rightarrow T(x))$ 12, UI
14. $\sim B(x) \Leftrightarrow E(x)$ 6, UI
15. $\sim(\sim K(x) \vee S(x))$ 11, E18
16. $\sim\sim K(x) \wedge \sim S(x)$ 15, E17
17. $\sim S(x) \wedge \sim\sim K(x)$ 16, E9
18. $\sim S(x)$ 17, I2
19. $\sim(\sim H(x) \vee T(x))$ 13, E18
20. $\sim\sim H(x) \wedge \sim T(x)$ 19, E17
21. $\sim\sim H(x)$ 20, I2
22. $\sim D(x)$ 21, 8, I4
23. $\sim D(x) \vee \sim S(x)$ 22, I1
24. $D(x) \Rightarrow \sim S(x)$ 23, E18
25. $(\forall x)(D(x) \Rightarrow \sim S(x))$ 24, Gen

Hence $(\forall x)(\sim K(x) \Rightarrow \sim E(x))$, $(\forall x)(D(x) \Rightarrow \sim H(x))$, $(\forall x)(B(x) \Rightarrow T(x))$, $\sim(\exists x)(K(x) \Rightarrow S(x))$, $\sim(\exists x)(\sim H(x) \Rightarrow T(x))$, $(\forall x)(\sim B(x) \Leftrightarrow E(x))$ $\vdash (\forall x)(D(x) \Rightarrow \sim S(x))$ □