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COMP 2310 Final Exam

1. Prove $P1, P2, P3 \vdash C$

$P1: (A \wedge B) \Rightarrow C$ $\qquad \sim A \vee \sim B \vee C$

$P2: (A \Rightarrow C) \Rightarrow D$ $\qquad B \Rightarrow E$

$P3: \sim B \vee E$ $\qquad E$

$C: B \Rightarrow (D \wedge E)$

(Direct proof)

1. B Premise
2. $(A \wedge B) \Rightarrow C$ from I
3. $(A \Rightarrow C) \Rightarrow D$ from I
4. $\sim B \vee E$ from I
5. $B \Rightarrow E$ 4, E18
6. E 1, 5, I3
7. $\sim(A \wedge B) \vee C$ 2, E18
8. $(\sim A \vee \sim B) \vee C$ 7, E16
9. $\sim A \vee (\sim B \vee C)$ 8, E12
10. $\sim A \vee (C \vee \sim B)$ 9, E10
11. $(\sim A \vee C) \vee \sim B$ 10, E12
12. $(A \Rightarrow C) \vee \sim B$ 11, E18
13. $\sim B \vee (A \Rightarrow C)$ 12, E10
14. $B \Rightarrow (A \Rightarrow C)$ 13, E18
15. $B \Rightarrow D$ 14, 3, I5
16. D 1, 15, I3
17. $D \wedge E$ 16, 6, I6

Thus $P1, P2, P3 \vdash C$.

$\square$

2. Prove $P1, P2, P3 \vdash C$

$P1: (\forall x)(\forall y)((P(x) \land Q(y)) \Rightarrow R(x,y))$

$P2: (\exists x)(\forall y)((P(x) \land S(x,y)) \Rightarrow Q(y))$

$P3: (\forall x)(\exists y)(P(x) \land \sim R(x,y) \land T(x,y))$

$C: (\exists x)(\exists y)(P(x) \land \sim S(x,y) \land T(x,y))$

(proof by definition)

1. $(\forall x)(\forall y)((P(x) \land Q(y)) \Rightarrow R(x,y))$ from I
2. $(\exists x)(\forall y)((P(x) \land S(x,y)) \Rightarrow Q(y))$ from I
3. $(\forall y)((P(a) \land S(a,y)) \Rightarrow Q(y))$ 2, EI, a is a constant
4. $(\forall x)(\exists y)(P(x) \land \sim R(x,y) \land T(x,y))$ from I
5. $(\exists y)(P(a) \land \sim R(a,y) \land T(a,y))$ 4, UI
6. $(P(a) \land \sim R(a,b)) \land T(a,b)$ 5, EI, b is a constant
7. $P(a) \land \sim R(a,b)$ 6, I2
8. $T(a,b) \land (P(a) \land \sim R(a,b))$ 6, E9
9. $T(a,b)$ 8, I2
10. $P(a)$ 7, I2
11. $\sim R(a,b) \land P(a)$ 7, E9
12. $\sim R(a,b)$ 11, I2
13. $(\forall y)((P(a) \land Q(y)) \Rightarrow R(a,y))$ 1, UI
14. $(P(a) \land Q(b)) \Rightarrow R(a,b)$ 13, UI
15. $\sim (P(a) \land Q(b))$ 12, 14, I4
16. $\sim P(a) \lor \sim Q(b)$ 15, E16
17. $\sim Q(b) \lor \sim P(a)$ 16, E10
18. $Q(b) \Rightarrow \sim P(a)$ 17, E18
19. $(P(a) \land S(a,b)) \Rightarrow Q(b)$ 3, UI
20. $(P(a) \land S(a,b)) \Rightarrow \sim P(a)$ 19, 18, I5
21. $\sim\sim P(a)$ 10, E15
22. $\sim (P(a) \land S(a,b))$ 21, 20, I4
23. $\sim P(a) \lor \sim S(a,b)$ 22, E16
24. $P(a) \land (\sim P(a) \lor \sim S(a,b))$ 10, 23, I6
25. $(P(a) \land \sim P(a)) \lor (P(a) \land \sim S(a,b))$ 24, E13
26. $\text{false} \lor (P(a) \land \sim S(a,b))$ 25, E1
27. $(P(a) \land \sim S(a,b)) \lor \text{false}$ 26, E10
28. $P(a) \land \sim S(a,b)$ 27, E6
29. $(P(a) \land \sim S(a,b)) \land T(a,b)$ 28, 9, I6
30. $(\exists y)(P(a) \land \sim S(a,y) \land T(a,y))$ 29, EQ
31. $(\exists x)(\exists y)(P(x) \land \sim S(x,y) \land T(x,y))$ 30, EQ

Hence $P1, P2, P3 \vdash C$

$\square$

3. Prove that $B - A = B \iff B \cap A = \emptyset$
(biconditional proof)

$$
\begin{aligned}
B - A = B &\iff B \cap \bar{A} = B \quad \text{(theorem)} \\
&\iff (\forall x)(x \in B \cap \bar{A} \iff x \in B) \quad \text{(Principle of extension)} \\
&\iff (\forall x)((x \in B \land x \in \bar{A}) \iff x \in B) \quad \text{(definition of } \cap) \\
&\iff (\forall x)((x \in B \land x \notin A) \iff x \in B) \quad \text{(theorem)} \\
&\iff (\forall x)(((x \in B \land x \notin A) \implies x \in B) \land (x \in B \implies (x \in B \land x \notin A))) \quad (E20) \\
&\iff (\forall x)((\sim(x \in B \land x \notin A) \lor x \in B) \land (\sim x \in B \lor (x \in B \land x \notin A))) \quad (E18) \\
&\iff (\forall x)(((x \notin B \lor x \in A) \lor x \in B) \land (\sim x \in B \lor (x \in B \land x \notin A))) \quad (E16, E15) \\
&\iff (\forall x)((x \notin B \lor (x \in B \lor x \in A)) \land ((\sim x \in B \lor x \in B) \land (\sim x \in B \lor x \notin A))) \quad (E12, E10, E14) \\
&\iff (\forall x)(((x \notin B \lor x \in B) \lor x \in A) \land ((x \in B \lor \sim x \in B) \land (\sim x \in B \lor \sim x \in A))) \quad (E12, E10) \\
&\iff (\forall x)(((x \in B \lor \sim x \in B) \lor x \in A) \land (\text{true} \land (\sim x \in B \lor \sim x \in A))) \quad (E10, E2) \\
&\iff (\forall x)((\text{true} \lor x \in A) \land ((\sim x \in B \lor \sim x \in A) \land \text{true}) \quad (E2, E9) \\
&\iff (\forall x)((x \in A \lor \text{true}) \land (\sim x \in B \lor \sim x \in A)) \quad (E10, E5) \\
&\iff (\forall x)(\text{true} \land (\sim x \in B \lor \sim x \in A)) \quad (E8) \\
&\iff (\forall x)(\sim(x \in B \land x \in A) \land \text{true}) \quad (E16, E9) \\
&\iff (\forall x) \sim(x \in B \land x \in A) \quad (E5) \\
&\iff (\forall x)(\sim(x \in B \land x \in A) \lor \text{false}) \quad (E4) \\
&\iff (\forall x)(\sim(x \in B \land x \in A) \lor x \in \emptyset) \quad \text{(theorem)} \\
&\iff (\forall x)(\sim(x \in B \cap A) \lor x \in \emptyset) \quad \text{(definition of } \cap) \\
&\iff (\forall x)(x \in B \cap A \implies x \in \emptyset) \quad (E18) \\
&\iff B \cap A = \emptyset \quad \text{(principle of extension)}
\end{aligned}
$$

Thus $B - A = B \iff B \cap A = \emptyset$.

$\square$

4. Let N be the set of all positive integers.

Let R be a relation in $N \times N$ such that $((a,b),(x,y)) \in R \iff \frac{2a+1}{2^b} \le \frac{2x+1}{2^y}$

Prove R is a partial order.

A partial order is reflexive, antisymmetric and transitive. Thus we break the proof into 3 steps:

a) R is reflexive.

We shall show that $(\forall (a,b) \in N \times N)((a,b),(a,b)) \in R$ (definition of reflexive)

$\Rightarrow (a,b) \in N \times N \Rightarrow ((a,b),(a,b)) \in R$ (UI)

(Direct proof)

Suppose $(a,b) \in N \times N$ (hypothesis)

$\Rightarrow a \in N \wedge b \in N$ (definition of $\times$)

We have $\frac{2a+1}{2^b} = \frac{2a+1}{2^b}$ (= is reflexive)

$\Rightarrow \frac{2a+1}{2^b} \le \frac{2a+1}{2^b}$ (definition of $\le$)

$\Rightarrow ((a,b),(a,b)) \in R$ (definition of R)

Thus R is reflexive (definition of reflexive)

b) R is antisymmetric

We shall prove $((a,b),(x,y)) \in R \wedge ((x,y),(a,b)) \in R \Rightarrow (a,b) = (x,y)$

(direct proof)

Suppose $((a,b),(x,y)) \in R \wedge ((x,y),(a,b)) \in R$ (hypothesis)

$$\Rightarrow \begin{cases} ((a,b),(x,y)) \in R \\ ((x,y),(a,b)) \in R \end{cases} \quad (E9, I2)$$

$$\Rightarrow \begin{cases} \frac{2a+1}{2^b} \le \frac{2x+1}{2^y} & \dots (1) \\ \frac{2x+1}{2^y} \le \frac{2a+1}{2^b} & \dots (2) \end{cases} \quad \text{(definition of } R)$$

We then have $\frac{2a+1}{2^b} \le \frac{2x+1}{2^y} \wedge \frac{2x+1}{2^y} \le \frac{2a+1}{2^b}$ ((1), (2), I6)

$\Rightarrow \left(\frac{2a+1}{2^b} < \frac{2x+1}{2^y} \vee \frac{2a+1}{2^b} = \frac{2x+1}{2^y}\right) \wedge \left(\frac{2x+1}{2^y} < \frac{2a+1}{2^b} \vee \frac{2x+1}{2^y} = \frac{2a+1}{2^b}\right)$ (definition of $\le$)

$\Rightarrow \left(\frac{2a+1}{2^b} = \frac{2x+1}{2^y} \vee \frac{2a+1}{2^b} < \frac{2x+1}{2^y}\right) \wedge \left(\frac{2a+1}{2^b} = \frac{2x+1}{2^y} \vee \frac{2x+1}{2^y} < \frac{2a+1}{2^b}\right)$ (E10, = is reflexive)

$\Rightarrow \frac{2a+1}{2^b} = \frac{2x+1}{2^y} \vee \left(\frac{2a+1}{2^b} < \frac{2x+1}{2^y} \wedge \frac{2x+1}{2^y} < \frac{2a+1}{2^b}\right)$ (E14)

$\Rightarrow \frac{2a+1}{2^b} = \frac{2x+1}{2^y} \vee$ false ($\because \frac{2a+1}{2^b}$ can't be $<$ and $> \frac{2x+1}{2^y}$)

$\Rightarrow a = x \wedge b = y$ (hsa)

$\Rightarrow (a,b) = (x,y)$ (theorem)

Thus R is antisymmetric.

c) R is transitive

We shall prove $((a,b),(c,d)) \in R \wedge ((c,d),(e,f)) \in R \Rightarrow ((a,b),(e,f)) \in R$ (definition of transitive)

(direct proof)

Suppose $((a,b),(c,d)) \in R \wedge ((c,d),(e,f)) \in R$ (hypothesis)

$$\Rightarrow \begin{cases} ((a,b),(c,d)) \in R \\ ((c,d),(e,f)) \in R \end{cases} \quad (Eq, I2)$$

$$\Rightarrow \begin{cases} \frac{2a+1}{2^b} \le \frac{2c+1}{2^d} & \text{(definition of } R) \quad \ldots(3) \\ \frac{2c+1}{2^d} \le \frac{2e+1}{2^f} & \ldots(4) \end{cases}$$

We then have $\frac{2a+1}{2^b} \le \frac{2c+1}{2^d} \wedge \frac{2c+1}{2^d} \le \frac{2e+1}{2^f}$ ((3),(4), I6)

$\Rightarrow \frac{2a+1}{2^b} \le \frac{2e+1}{2^f}$ ($\le$ is transitive $\because$ $\le$ is a partial order)

$\Rightarrow ((a,b),(e,f)) \in R$ (definition of R)

Thus R is transitive.

We have shown that R is reflexive, antisymmetric and transitive.

Thus, R is a partial order.

$\square$

5. Let $f: A \to B$.

Prove $(\exists g)(g: B \to A \land g \circ f = I_B) \Rightarrow f$ is onto

$\Rightarrow (\exists g)(g: B \to A \land g \circ f = I_B) \Rightarrow (\forall y \in B)(\exists x \in A) y = f(x)$ (definition of onto)

(direct proof)

Suppose $(\exists g)(g: B \to A \land g \circ f = I_B)$ (hypothesis)

$\Rightarrow h: B \to A \land h \circ f = I_B$ (EI)

$\Rightarrow h \circ f = I_B$ (E9, I2) ... (1)

We have $x = I_B(x)$ (definition of I_B)

$\Rightarrow x = h \circ f(x)$ ((1), sub=)

$\Rightarrow (\exists z \in B)(z = h(x) \land x = f(z))$ (definition of $\circ$)

$\Rightarrow a \in B \land (a = h(x) \land x = f(a))$ (EI)

$\Rightarrow a \in B \land (x = f(a) \land a = h(x))$ (E9)

$\Rightarrow (a \in B \land x = f(a)) \land a = h(x)$ (E11)

$\Rightarrow a \in B \land x = f(a)$ (I2)

$\Rightarrow (\exists a \in B) x = f(a)$ (EQ)

Hence, f is onto.

$\square$

6. Prove there does not exist a simple graph of order 4 and size 7.

Equivalently, we shall prove that there does not exist a simple graph where $|V|=4$ and $|E|=7$.

(proof by contradiction)

Suppose there exists a graph G where $|V|=4$ and $|E|=7$.

We know the maximum degree sequence of G is a complete graph, since a complete graph has all vertices paired with an edge

or more formally, $(\forall v_1, v_2 \in V)\ \{v_1, v_2\} \in E \wedge v_1 \neq v_2$

The maximum degree sequence would be 3,3,3,3.

This is because every vertex has 3 other vertices to pair with, in an edge.

A vertex with a degree >3 implies $|V|>4$ since G is a simple graph.

However, $|V|=4$ in graph G.

Thus the largest degree sequence is 3,3,3,3.

If we add up all the degrees in the sequence, we get

$$\sum_{v\in V} \deg(v) = 3+3+3+3 = 12.$$

However,
$$\sum_{v\in V} \deg(v) = 2|E| \quad \text{(theorem)}$$
$$\Rightarrow \sum_{v\in V} \deg(v) = 2(7) \quad (|E|=7)$$
$$\Rightarrow \sum_{v\in V} \deg(v) = 14.$$

We know $12<14$ (hsa)

This implies there are some self-loops or parallel edges in graph G.

However, G is simple, so there are no self-loops or parallel edges in G.

Hence, we have a contradiction!

Thus, there does not exist a simple graph G of order 4 and size 7.

□