Justin Bornais
bornais3 - 110037759

COMP 2310 Assignment 3

3.8.4. Prove that the set of all sets does not exist

We shall prove that $\vdash \sim(\exists X)(X = \{A \mid A \text{ is a set}\})$
(Proof by contradiction)

Suppose $(\exists X)(X = \{A \mid A \text{ is a set}\})$
Then $Y = \{A \mid A \text{ is a set}\}$ (EI) Y is a set ... (I)
Then $(\exists B)(B = \{x \in Y \mid x \notin x\})$ (Principle of Specification)
$\Rightarrow Z = \{x \in Y \mid x \notin x\}$ (EI, Z is a constant)
Then $Z \in Y$ ((I), UI)...(II)

It follows that we have either $Z \in Z$ or $Z \notin Z$.
i) Suppose $Z \in Z$ (III)
Then $Z \in Y \wedge Z \notin Z$ (Definition of Z)
$\Rightarrow Z \notin Z$ (E9, I2) ... (IV)
$\Rightarrow Z \in Z \wedge Z \notin Z$ ((III), I6)
$\Rightarrow$ false (E1)

ii) Suppose $Z \notin Z$ ... (V)
Then $Z \in Y \wedge Z \notin Z$ ((V), (II), I6)
$\Rightarrow Z \in Z$ (Definition of Z)
$\Rightarrow Z \in Z \wedge Z \notin Z$ ((V), I6)
$\Rightarrow$ false (E1)

$\because$ in both cases, the result is a contradiction,
$\therefore \vdash \sim(\exists X)(X = \{A \mid A \text{ is a set}\})$
Hence, the set of all sets does not exist.
□

4.9.5b) $(\bar{A} \cap B \cap C) \cup (\bar{A} \cap C \cap A) \cup (\bar{A} \cap B \cap C \cap \overline{D \cap E}) = \bar{A} \cap B \cap C$

Suppose $(\bar{A} \cap B \cap C) \cup (\bar{A} \cap C \cap A) \cup (\bar{A} \cap B \cap C \cap \overline{D \cap E})$
$= (\bar{A} \cap B \cap C) \cup (A \cap \bar{A} \cap C) \cup (\bar{A} \cap B \cap C \cap \overline{D \cap E})$ (Theorem 4.2.2(iii))
$= (\bar{A} \cap B \cap C) \cup (\emptyset \cap C) \cup (\bar{A} \cap B \cap C \cap \overline{D \cap E})$ (Theorem 4.3.2(iii))
$= (\bar{A} \cap B \cap C) \cup \emptyset \cup (\bar{A} \cap B \cap C \cap \overline{D \cap E})$ (Theorem 4.2.2(iii), Theorem 4.2.2(i))
$= (\bar{A} \cap B \cap C) \cup (\bar{A} \cap B \cap C \cap \overline{D \cap E})$ (Theorem 4.1.1(i))
$= (\bar{A} \cap B \cap C \cap U) \cup (\bar{A} \cap B \cap C \cap \overline{D \cap E})$ (Theorem 4.2.2(v) where $(\bar{A} \cap B \cap C) \subseteq U$)
$= (\bar{A} \cap B \cap C) \cap (U \cup \overline{D \cap E})$ (Theorem 4.2.3(ii))
$= (\bar{A} \cap B \cap C) \cap U$ (Theorem 4.1.1(iii), Theorem 4.1.1(v) where $\overline{D \cap E} \subseteq U$)
$= \bar{A} \cap B \cap C$ (Theorem 4.2.2(v))

Hence $(\bar{A} \cap B \cap C) \cup (\bar{A} \cap C \cap A) \cup (\bar{A} \cap B \cap C \cap \overline{D \cap E}) = \bar{A} \cap B \cap C$
□

4.9. 8a) $A \cap (B-A) = \emptyset$

Suppose $A \cap (B-A)$
$= A \cap (B \cap \bar{A})$ (Theorem 4.3.7 (v))
$= (A \cap \bar{A}) \cap B$ (Theorem 4.2.2(iii), Theorem 4.2.2 (iv))
$= \emptyset \cap B$ (Theorem 4.3.7(iii))
$= B \cap \emptyset$ (Theorem 4.2.2(iii))
$= \emptyset$ (Theorem 4.2.2(i))

Hence $A \cap (B-A) = \emptyset$
□

4.9. 11a) $(A \cup C) \times (B \cup D) \neq (A \times B) \cup (C \times D)$ (as per his posted Assignment 3.pdf)

(Proof by Counterexample)

Let $A = \{a\}$, $B = \{b\}$, $C = \{c\}$, $D = \{d\}$

Then $(A \times B) \cup (C \times D) = \{(a,b)\} \cup \{(c,d)\}$ (Definition of $\times$)
$= \{(a,b), (c,d)\}$ (Definition of $\cup$)

Then $(A \cup C) \times (B \cup D) = \{a, c\} \times \{b, d\}$ (Definition of $\cup$)
$= \{(a,b), (a,d), (c,b), (c,d)\}$ (Definition of $\times$)

Notice: $(a,d) \in ((A \cup C) \times (B \cup D))$ but $(a,d) \notin ((A \times B) \cup (C \times D))$

Hence $\sim(\forall x)(x \in ((A \cup C) \times (B \cup D)) \Longleftrightarrow ((A \times B) \cup (C \times D)))$

By the Principle of Extension $(A \cup C) \times (B \cup D) \neq (A \times B) \cup (C \times D)$
Hence, $(A \cup C) \times (B \cup D) \neq (A \times B) \cup (C \times D)$
□

4.9.14b) Prove that $\bigcup_{X \in \{A,B\}} X = A \cup B$

We shall prove that $(\forall x)(x \in \bigcup_{X \in \{A,B\}} X \Leftrightarrow x \in A \cup B)$

$\Rightarrow x \in \bigcup_{X \in \{A,B\}} X \Leftrightarrow x \in A \cup B$ (UI)

(Biconditional Proof)

$\Rightarrow$) Prove that $x \in \bigcup_{X \in \{A,B\}} X \Rightarrow x \in A \cup B$

(Direct Proof)

Suppose $x \in \bigcup_{X \in \{A,B\}} X$

Then $(\exists X)(X \in \{A,B\} \wedge x \in X)$ (Definition of $\bigcup_{X \in C} X$)

$\Rightarrow M \in \{A,B\} \wedge x \in M$ (EI, M is a constant) ... (I)

$\Rightarrow M \in \{A,B\}$ (I2) ... (IX)

Then $x \in M$ ((I), E9, I2) ... (II)

This means we have either $M = A$ or $M = B$ (IX)

We are now trying to prove $x \in M \vdash (M = A \vee M = B) \Rightarrow x \in A \cup B$

(Proof by cases)

i) Suppose $M = A$

$\Rightarrow x \in A$ ((II), sub$_=$)

$\Rightarrow x \in A \vee x \in B$ (I1)

$\Rightarrow x \in A \cup B$ (definition of $\cup$)

ii) Suppose $M = B$

$\Rightarrow x \in B$ ((II), sub$_=$)

$\Rightarrow x \in A \vee x \in B$ (I1, E10)

$\Rightarrow x \in A \cup B$ (definition of $\cup$)

We have now shown that for both cases, $x \in M \vdash (M = A \vee M = B) \Rightarrow x \in A \cup B$

Thus, $\bigcup_{X = \{A,B\}} X \Rightarrow x \in A \cup B$

$\Leftarrow$) Now we are to prove $x \in A \cup B \Rightarrow x \in \bigcup_{X = \{A,B\}} X$ ... (X)

So we are to show $x \in A \vee x \in B \Rightarrow x \in \bigcup_{X = \{A,B\}} X$ ((X), Definition of $\cup$)

(Proof by cases)

i) Suppose $x \in A$ ... (III)

Let Y be a set where $Y = A$ ... (IV)

$\Rightarrow x \in Y$ ((III), sub$_=$) ... (V)

$Y = A \vee Y = B$ ((IV), I1)

$\Rightarrow Y \in \{A, B\}$ (equivalent notation)

$\Rightarrow x \in Y \wedge Y \in \{A,B\}$ ((V), I6)

$\Rightarrow x \in \bigcup_{X \in \{A,B\}} X$ (Definition of $\bigcup_{X \in C} X$)

Thus $x \in A \Rightarrow x \in \bigcup_{X \in \{A,B\}} X$

ii) Suppose $x \in B$ ... (VI)

Let Y be a set where $Y = B$ ... (VII)

$\Rightarrow x \in Y$ ((VI), Sub$_=$) ... (VIII)

$Y = B \vee Y = A$ ((VII), I1)

$\Rightarrow Y = A \vee Y = B$ (E10)

$\Rightarrow Y \in \{A, B\}$ (equivalent notation)

$\Rightarrow x \in Y \wedge Y \in \{A, B\}$ ((VIII), I6)

$\Rightarrow x \in \bigcup_{X \in \{A,B\}} X$ (definition of $\bigcup_{X \in C} X$)

Thus $x \in B \Rightarrow x \in \bigcup_{X \in \{A,B\}} X$

We have now shown that for both cases, $x \in A \vee x \in B \vdash x \in \bigcup_{X \in \{A,B\}} X$

Thus $x \in A \cup B \Rightarrow x \in \bigcup_{X = \{A,B\}} X$

Hence $x \in A \cup B \Rightarrow x \in \bigcup_{X = \{A,B\}} X$ □

5.8.7. If for every $a \in A$, there exists $b \in A$ such that $(a,b) \in R$, then R is an equivalence relation.

R is defined as being transitive and symmetric. If we want to show that R is an equivalence relation, we shall prove that R is reflexive.

We shall prove $(\forall a \in A)(\exists b \in A)(a,b) \in R \Rightarrow (\forall x \in A)(x,x) \in R$
(Direct Proof)

Suppose $(\forall a \in A)(\exists b \in A)(a,b) \in R$

$\Rightarrow a \in A \wedge (\exists b \in A)(a,b) \in R$ (UI)

$\Rightarrow (\exists b \in A)(a,b) \in R$ (E9, I2)

$\Rightarrow d \in A \wedge (a,d) \in R$ (EI)

$\Rightarrow (a,d) \in R$ (E9, I2) ... (I)

$\Rightarrow (d,a) \in R$ (def of symmetric) ... (II)

$\Rightarrow (a,d) \in R \wedge (d,a) \in R$ ((I), (II), I6)

$\Rightarrow (a,a) \in R$ (def of transitive)

$\Rightarrow (\forall x)(x,x) \in R$ (Gen)

Hence $(\forall a \in A)(\exists b \in A)(a,b) \in R \Rightarrow (\forall x \in A)(x,x) \in R$

Hence R is reflexive

Thus, R is an equivalence relation (definition of equivalence relation)

□

Part 2: Show that the conclusion is false if symmetricity is replaced by reflexivity.

R is defined as being reflexive and transitive. To disprove R is an equivalence relation, we shall prove R is not symmetric.

(Proof by Counterexample)

Let $A = \{x, y, z\}$ and R is a relation in set A.

Let $R = \{(x,x), (y,y), (z,z), (x,y), (y,z), (x,z)\}$

By inspection, R is reflexive and transitive

If we assume R is symmetric, then since $(x,y) \in R$, we would have $(y,x) \in R$

However, $(y,x) \notin R$

Thus R is not symmetric.

Thus, R is not an equivalence relation. □

5.8.9. Let N be the set of all positive integers
Let R be a relation in N where $R = \{(a,b) \in N \times N \mid$ the sum of the decimal digits in a = the sum of the decimal digits in $b\}$
Prove R is an equivalence relation.

i) To prove that R is reflexive, we shall prove that $(\forall a \in N), (a,a) \in R$

Let $a \in N$.

We have $(\forall x)\ x = x$ (A1)
$\Rightarrow$ the sum of the decimal digits in a = the sum of the decimal digits in a (UI)
Hence, the sum of the decimal digits in a = the sum of the decimal digits in a
$\Rightarrow (a,a) \in R$ (Definition of R)
Hence, R is reflexive.

ii) To prove R is symmetric, we shall prove that $(a,b) \in R \Rightarrow (b,a) \in R$.
(Direct Proof)

Assume $(a,b) \in R$
This means the sum of the decimal digits in a = the sum of the decimal digits in b ... (I)

Let c = The sum of the decimal digits in a, and d = the sum of the decimal digits in b ... (II)
$\Rightarrow c = d$ ((I), sub$_=$) ... (III)
We also have $(\forall x)(\forall y)(x = y \Rightarrow y = x)$ (A2)
$\Rightarrow c = d \Rightarrow d = c$ (UI)
$\Rightarrow d = c$ ((III), I3)
$\Rightarrow$ The sum of the decimal digits in b = the sum of the decimal digits in a ((II), sub$_=$)
$\Rightarrow (b,a) \in R$ (definition of R)

Thus $(a,b) \in R \Rightarrow (b,a) \in R$

Hence, R is symmetric.

iii) To prove R is transitive, we shall prove $((a,b) \in R \wedge (b,c) \in R) \Rightarrow (a,c) \in R$
(Direct proof)

Suppose $(a,b) \in R \wedge (b,c) \in R$ ... (IV)
Let g = the sum of the decimal digits of a
Let h = the sum of the decimal digits of b
Let j = the sum of the decimal digits of c
$(a,b) \in R$ ((IV), I2)
$\Rightarrow$ the sum of the decimal digits in a = the sum of the decimal digits in b (definition of R)
$\Rightarrow g = h$ (the definition of g, the definition of h) ... (V)

$(b,c) \in R$ ((IV), E9, I2)
$\Rightarrow$ the sum of the decimal digits in b = the sum of the decimal digits in c (definition of R)
$\Rightarrow h = j$ (the definition of h, the definition of j) ... (VI)

We have $(\forall x)(\forall y)(\forall z)((x = y) \wedge (y = z) \Rightarrow x = z)$ (A3)
$\Rightarrow (g = h) \wedge (h = j) \Rightarrow g = j$ (UI) ... (VII)
We also have $(g = h) \wedge (h = j)$ ((V), (VI), I6)
$\Rightarrow g = j$ ((VII), I3)
$\Rightarrow$ the sum of the decimal digits in a = the sum of the decimal digits in c (definition of g, definition of j)
$\Rightarrow (a,c) \in R$ (definition of R)
Thus $((a,b) \in R \wedge (b,c) \in R) \Rightarrow (a,c) \in R$
Hence, R is transitive.

Since we've shown R is reflexive, symmetric and transitive, hence R is an equivalence relation. $\square$

Part 2: describe the equivalence class $[98]/R$.

$[98]/R = \{x \in \mathbb{N} \mid (98, x) \in R\}$

$\Rightarrow \{x \in \mathbb{N} \mid$ the sum of the decimal digits of $98 =$ The sum of the decimal digits of $x\}$ (definition of R)

$\Rightarrow \{x \in \mathbb{N} \mid 17 =$ The sum of the decimal digits of $x\}$ ($\because$ the sum of the decimal digits of $98 = 9+8 = 17$)

There are no integers in $[98]/R$ that fall within the range of 0 to 50, inclusive.
Reasoning: every integer in this range has a smaller digit sum than 98.

The number in this range with the largest digit sum is 49.
4 is the largest possible tens digit that can pair with 9 (the largest digit) in this range.
While 5 is greater than 4, the range only goes up to 50, who's digit sum $= 5+0 = 5$.
Hence, 49 has the largest possible digit sum in this range.

The digit sum of $49 = 4+9 = 13$.
Since 13 is less than 17 and 49 has the largest digit sum, we conclude that there is no number in this range that can reach 17 (the digit sum of 98).

Hence, in the range 0 to 50, inclusive, $[98]/R$ is empty. $\square$

Lemma A: $y \notin X \Leftrightarrow (y,y) \notin X_=$

(Biconditional proof)

$\Rightarrow$) Prove $y \notin X \Rightarrow (y,y) \notin X_=$

(Direct proof)

Suppose $y \notin X$ ... (I)

Then $y \notin X \wedge y \notin X$ ((I), E3)

$\Rightarrow (y,y) \notin X \times X$ (Definition of $X \times X$) ... (II)

We also have $(\forall x)\, x = x$ (A1)

$\Rightarrow y = y$ (UI) ... (III)

Then we have $(y,y) \notin X \times X \vee \sim(y = y)$ ((II), I1)

$\Rightarrow \sim((y,y) \in X \times X \wedge y = y)$ (E16)

$\Rightarrow (y,y) \notin X_=$ (Definition of $X_=$)

Thus $y \notin X \Rightarrow (y,y) \notin X_=$

$\Leftarrow$) Prove $(y,y) \notin X_= \Rightarrow y \notin X$

(Direct proof)

Suppose $(y,y) \notin X_=$

Then $\sim((y,y) \in X \times X \wedge y = y)$ (Definition of $X_=$)

$\Rightarrow \sim(y,y) \in X \times X \vee \sim(y = y)$ (E16) ... (IV)

We also have $(\forall x)\, x = x$ (A1)

$\Rightarrow y = y$ (UI) ... (V)

Then we have $y = y \wedge (\sim(y,y) \in X \times X \vee \sim(y = y))$ ((V), (IV), I6)

$\Rightarrow (y = y \wedge \sim(y,y) \in X \times X) \vee (y = y \wedge \sim(y = y))$ (E13)

$\Rightarrow (\sim(y,y) \in X \times X \wedge y = y) \vee \text{false}$ (E9, E1)

$\Rightarrow (y,y) \notin X \times X \wedge y = y$ (E6)

$\Rightarrow (y,y) \notin X \times X$ (I2)

$\Rightarrow y \notin X \wedge y \notin X$ (Definition of Cartesian Product)

$\Rightarrow y \notin X$ (E3)

Hence $(y,y) \notin X_= \Rightarrow y \notin X$

Hence $y \notin X \Leftrightarrow (y,y) \notin X_=$ $\square$

5.8.18a) R is reflexive iff $X_= \subseteq R$

We shall prove R is reflexive $\Leftrightarrow X_= \subseteq R$
(Biconditional proof)

$$
\begin{aligned}
R \text{ is reflexive} &\Longleftrightarrow (\forall x)(x \in X \Rightarrow (x,x) \in R) && \text{(Def. of Reflexivity)}\\
&\Longleftrightarrow x \in X \Rightarrow (x,x) \in R && (\Rightarrow)\text{UI}, (\Leftarrow)\text{Gen})\\
&\Longleftrightarrow x \notin X \vee (x,x) \in R && \text{(E18)}\\
&\Longleftrightarrow (x,x) \notin X_= \vee (x,x) \in R && \text{(Lemma A)}\\
&\Longleftrightarrow (x,x) \in X_= \Rightarrow (x,x) \in R && \text{(E18)}\\
&\Longleftrightarrow (\forall x)(x \in X_= \Rightarrow x \in R) && ((\Rightarrow)\text{Gen}, (\Leftarrow)\text{UI})\\
&\Longleftrightarrow X_= \subseteq R && \text{(Def. of subset)}
\end{aligned}
$$

Hence, R is reflexive iff $X_= \subseteq R$ □