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Midterm 2

3. Let $\mathbb{Z}$ be the set of all integers.
Let R be a relation in $\mathbb{Z}\times\mathbb{Z}$ such that $((a,b),(c,d)) \in R \Leftrightarrow a+d = c+b$
Prove R is an equivalence relation.

To prove R is an equivalence relation, we are to prove R is reflexive, symmetric and transitive.

i) Prove R is reflexive.
We are to prove $(\forall a)(\forall b)((a,b) \in \mathbb{Z}\times\mathbb{Z} \Rightarrow ((a,b),(a,b)) \in R)$ (definition of reflexive)
$\Rightarrow ((a,b) \in \mathbb{Z}\times\mathbb{Z} \Rightarrow ((a,b),(a,b)) \in R)$ (UI)

(direct proof)
Suppose $(a,b) \in \mathbb{Z}\times\mathbb{Z}$ (hypothesis)
$\Rightarrow a \in \mathbb{Z} \wedge b \in \mathbb{Z}$ (definition of $\mathbb{Z}\times\mathbb{Z}$)
Let $c = a+b$ ... (I)
We have $c=c$ (= is reflexive)
$\Rightarrow a+b = a+b$ ((I), $sub_=$)
$\Rightarrow ((a,b),(a,b)) \in R$ (definition of R)
Thus, $(a,b) \in \mathbb{Z}\times\mathbb{Z} \Rightarrow ((a,b),(a,b)) \in R$
Hence, R is reflexive.

ii) Prove R is symmetric.
We are to prove $((a,b),(c,d)) \in R \Rightarrow ((c,d),(a,b)) \in R$ (definition of symmetric)
(direct proof)
Suppose $((a,b),(c,d)) \in R$ (hypothesis)
$\Rightarrow a+d = c+b$ (definition of R)
$\Rightarrow c+b = a+d$ (= is symmetric)
$\Rightarrow ((c,d),(a,b)) \in R$ (definition of R)
Thus, $((a,b),(c,d)) \in R \Rightarrow ((c,d),(a,b)) \in R$
Hence R is symmetric.

iii) Prove R is transitive.
(proof by definition)

We have $((a,b),(a,b)) \in R$ (R is reflexive)
Let $((a,b),(c,d)) \notin R$ ... (II)
$\Rightarrow ((a,b),(c,d)) \notin R \vee ((c,d),(a,b)) \notin R$ (I1)
$\Rightarrow \sim(((a,b),(c,d)) \in R \wedge ((c,d),(a,b)) \in R)$ (E16)
$\Rightarrow \sim(((a,b),(c,d)) \in R \wedge ((c,d),(a,b)) \in R) \vee ((a,b),(a,b)) \in R$ (I1)
$\Rightarrow (((a,b),(c,d)) \in R \wedge ((c,d),(a,b)) \in R) \Rightarrow ((a,b),(a,b)) \in R$ (E18)
Thus, R is transitive.

We have shown that R is reflexive, symmetric and transitive.
Hence, R is an equivalence relation.
$\square$

1. Prove or disprove $\vdash (A \in B) \wedge (B \in C) \Rightarrow A \in C$

(disproof by counterexample)

Let $A = \{1\}$

Let $B = \{\{1\}\}$

Let $C = \{\{\{1\}\}\}$

By inspection, we see $A \in B$ and $B \in C$.

However, by inspection, $A \notin C$.

$\therefore$ The conclusion is false.

Hence, if $(A \in B) \wedge (B \in C)$

2. Prove $B - (B - A) = A \Leftrightarrow A \subseteq B$

(Biconditional proof)

$$
\begin{aligned}
B - (B - A) = A &\Leftrightarrow B \cap A = A \text{ (theorem)} \\
&\Leftrightarrow A \cap B = A \quad \text{(theorem)} \\
&\Leftrightarrow A \subseteq B \quad \text{(theorem)}
\end{aligned}
$$

Hence, $B - (B - A) = A \Leftrightarrow A \subseteq B$

$\square$

(question 4 is on the next page)

4. Let A, B, C, D be sets such that $A \neq \emptyset$ and $B \neq \emptyset$.

Prove if $(A \times B) \cup (B \times A) = C \times C$, then $A = C$.

We are to prove $(A \times B) \cup (B \times A) = C \times C \Rightarrow A = C$

(direct proof)

Suppose $(A \times B) \cup (B \times A) = C \times C$

$\Rightarrow (\forall x)(x \in (A \times B) \cup (B \times A) \Leftrightarrow x \in C \times C)$ (principle of extension)

$\Rightarrow x \in (A \times B) \cup (B \times A) \Leftrightarrow x \in C \times C$ (UI)

$\Rightarrow (x \in A \times B \vee x \in B \times A) \Longleftrightarrow x \in C \times C$ (definition of $\cup$)

$\Rightarrow (x \in A \times B \vee x \in B \times A) \Longleftrightarrow x \in C \wedge x \in C$ (definition of $C \times C$)

$\Rightarrow ((x \in A \wedge x \in B) \vee (x \in B \wedge x \in A)) \Longleftrightarrow x \in C \wedge x \in C$ (definition of cartesian product)

$\Rightarrow ((x \in A \wedge x \in B) \vee (x \in A \wedge x \in B)) \Longleftrightarrow x \in C$ (E9, E3)

$\Rightarrow (x \in A \wedge x \in B) \Longleftrightarrow x \in C$ (E4)

$\Rightarrow ((x \in A \wedge x \in B) \Rightarrow x \in C) \wedge (x \in C \Rightarrow (x \in A \wedge x \in B))$ (E20)

$\Rightarrow (\sim(x \in A \wedge x \in B) \vee x \in C) \wedge (x \notin C \vee (x \in A \wedge x \in B))$ (E18)

$\Rightarrow ((x \notin A \vee x \notin B) \vee x \in C) \wedge ((x \notin C \vee x \in A) \wedge (x \notin C \vee x \in B))$ (E16, E14)

$\Rightarrow (x \notin A \vee (x \notin B \vee x \in C)) \wedge ((x \in C \Rightarrow x \in A) \wedge (x \notin C \vee x \in B))$ (E12, E18)

$\Rightarrow (x \notin A \vee (x \in C \vee x \notin B)) \wedge ((x \in C \Rightarrow x \in A) \wedge (x \notin C \vee x \in B))$ (E10)

$\Rightarrow ((x \notin A \vee x \in C) \vee x \notin B) \wedge ((x \in C \Rightarrow x \in A) \wedge (x \notin C \vee x \in B))$ (E12)

$\Rightarrow ((x \in A \Rightarrow x \in C) \vee x \notin B) \wedge ((x \in C \Rightarrow x \in A) \wedge (x \notin C \vee x \in B))$ (E18)

$\Rightarrow (x \notin B \vee (x \in A \Rightarrow x \in C)) \wedge ((x \in C \Rightarrow x \in A) \wedge (x \in C \Rightarrow x \in B))$ (E10, E18)

$\Rightarrow (x \in B \Rightarrow (x \in A \Rightarrow x \in C)) \wedge ((x \in C \Rightarrow x \in A) \wedge (x \in C \Rightarrow x \in B))$ (E18) ... (I)

$\Rightarrow x \in B \Rightarrow (x \in A \Rightarrow x \in C)$ (I2) ... (II)

$\Rightarrow (x \in C \Rightarrow x \in A) \wedge (x \in C \Rightarrow x \in B)$ ((I), E9, I2) ... (III)

$\Rightarrow x \in C \Rightarrow x \in A$ (I2) ... (IV)

$\Rightarrow x \in C \Rightarrow x \in B$ ((III), E9, I2)

$\Rightarrow x \in C \Rightarrow (x \in A \Rightarrow x \in C)$ ((II), I5)

We have $(x \in C \Rightarrow x \in A) \wedge (x \in B \Rightarrow (x \in A \Rightarrow x \in C))$ ((IV), (II), I6)

$\Rightarrow (x \in C \Rightarrow x \in A) \wedge (x \notin B \vee (x \in A \Rightarrow x \in C))$ (E18)

$\Rightarrow ((x \in C \Rightarrow x \in A) \wedge x \notin B) \vee ((x \in C \Rightarrow x \in A) \wedge (x \in A \Rightarrow x \in C))$ (E13)

(question unfinished) :(