ECEN 2703: Discrete Mathematics
Computer Project #3
Optimal Coin Denominations

The third computer project for this class will focus on deriving the most efficient coin denomination. The first part of this project will focus on the optimal coin dispersion problem which involves providing the fewest number of coins possible to achieve a desired change amount. The second part of this project will then focus on deriving the most efficient coin denominations. Efficiency will be measured in two ways: The first measure of efficiency will focus on a worst-case analysis. That is, what is the largest number of coins that must be used to achieve any change amount in $\{1, 2, \ldots, 98, 99\}$. The second measure of efficiency will focus on an average-case analysis. That is, what is the expected number of coins that must be used when all change amounts $\{1, 2, \ldots, 98, 99\}$ are equally likely.

Your report should include the following analysis:

1. Model the optimal coin dispersion problem as a shortest path routing problem. Here, the "shortest path" associated with your defined routing problem should correspond to the optimal solution to the coin dispersion problem. Describe the details of your model. What do the nodes represent? What do the stages represent? What do the stage costs represent? Describe what a path and path cost represent. What does the optimal cost-to-go represent? How would you implement Dynamic Programming to find the shortest path?

2. Next, write a Matlab function to solve for the optimal coin dispersion problem using Dynamic Programming. This function should have the format

   ```
   [number_coins] = optimal_coin_dispersion(change,coin_denominations)
   ```

   with the following notation:

   - `change`: This variable should be an integer ≥ 1 that represents the desired change amount.
   - `coin_denominations`: This variable should be a vector of integers of length $m \geq 1$ that represents the different coin denominations. For example, our current coin denominations would be represented by $[1, 5, 10, 25]$. Note that the integer 1 must always be included in any coin denomination.
   - `number_coins`: This variable should be a vector of integers of length $m \geq 1$ that represents the number of of each coin type that is needed to achieve the desired change. Note that if the coin denominations are $[x_1, x_2, \ldots, x_m]$, the coin numbers are $[n_1, n_2, \ldots, n_m]$, and the desired change is $z \geq 1$, then we have

     $$z = (n_1 \times x_1) + (n_2 \times x_2) + \cdots + (n_{m-1} \times x_{m-1}) + (n_m \times x_m).$$

     Clearly, $n_i \geq 0$ for all i.

   Verify that your code gives you the correct answers for the following test cases.

   - `change`=11, `coin_denominations`=[1,5,8]
   - `change`=13, `coin_denominations`=[1,5,8]
   - `change`=47, `coin_denominations`=[1,5,10,25]

3. Now focus on our current coin denominations, i.e., $[1, 5, 10, 25]$. Construct a plot where the x-axis is all desired change amounts $\{1, 2, \ldots, 99\}$ and the y-axis represents the number of coins necessary to achieve each desired change amount. What is the largest number of coins that could be needed? What is the expected number of coins needed assuming that all desired change amounts $\{1, 2, \ldots, 99\}$ are equally likely?

4. Pick an alternative denomination of coins and compute the same analysis done in the previous question. Compare your answers.

5. Suppose you can have four different denomination of coins, i.e., $[1, x, y, z]$. What are the values of x, y, and z that minimize the maximum number of coins needed to achieve any desired change amount in $\{1, 2, \ldots, 99\}$? Include the same plot as done in the previous questions.

6. Suppose you can have four different denomination of coins, i.e., $[1, x, y, z]$. What are the values of x, y, and z that minimize the expected number of coins need to achieve a desired change amount when all desired change amounts $\{1, 2, \ldots, 99\}$ are equally likely? Include the same plot as done in the previous questions.

7. Is our current coin denominations $[1, 5, 10, 25]$ efficient? Explain your answer.

Discussions are allowed between students. However, every student must submit their own individual program and report. There should be no collaborations on the writing of your MATLAB function or report. Your responsibilities for this project include the following:

- Submitting one MATLAB function with the exact syntax above. Submission instructions will be posted on the course website.
- Submitting a typed report. This report should be on the order of 5-8 pages and should include an analysis of your algorithm as highlighted above.
- Please do not print out your code.