Abstract. We prove the following approximation of π

$$\pi = \frac{2M(1, \sqrt{2}/2)^2}{1 - \sum_{n=0}^{\infty} 2^n (a_n^2 - b_n^2)},$$

which is derived from the arithmetic-geometric mean of 1 and $\sqrt{2}/2$.

Note. Throughout this article tedious steps are often skipped to illustrate the main picture; in particular, computing integral substitutions and power series expansions.

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1. History

The methods used for approximating π span millennia and severely vary in complexity; we will describe a few of the popular methods. The reference here is [1]. Around 200 BC, Archimedes approximated the circumference C and radius r of a circle by inscribing it in a polygon with n sides. It is not hard to see that as n approaches infinity, we get π using the circumference formula $C = 2\pi r$. Then during the invention of calculus in the 1600s, Newton and others used integrals and power series expansions to calculate π. For instance, the identity

$$\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$$

evaluated at $x = 1$ gives us

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \frac{1}{7} - \cdots .$$

Furthermore, in the 1700s, Euler calculated the values of the Riemann zeta function. Famously, for $x = 2$, we get that

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

The underlying problem is this family of methods are slow, often taking hundreds of iterations to even yield a couple of digits.

In contrast, the Gauss-Legendre algorithm has quadratic convergence. Let us say we want to calculate 512 decimal places of π. Then the Gauss-Legendre algorithm only needs 9 iterations, while almost all of the older methods (normally) need at least 800 iterations, if not significantly more. As an aside, Ramanujan's equation for π

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!(1103+26390n)}{(n!)^4 396^{4n}}.$$

only need around 65 iterations to converge [3]. His equations are now used for large approximations of π due to computational complexity optimization and storage restrictions.

2. Elliptic integrals

Let $(a\cos\theta, b\sin\theta)$ be an ellipse parameterized by $\theta \in [0, 2\pi]$. Then it's arc-length is given by

$$\int_0^{2\pi} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta. \tag{1}$$

This integral is not easy to compute in itself, and it was generalized to the study of so-called elliptic integrals in the early 1700s. Now notice that in (1) all of the data concerning our specific ellipse's arc-length is contained within the interval $0 \le \theta \le \pi/2$. Hence, it would make sense to reduce our study to complete elliptic integrals, meaning those with amplitude $\pi/2$. Furthermore, we will restrict ourselves to complete elliptic integrals of the first and second kind. For the proof that every elliptic integral is of the first, second, or third kind, see [2].

Definition. Let

$$\begin{aligned}
F(k) &= \int_0^{\pi/2} \frac{1}{\sqrt{1-k^2\sin^2\theta}}\,d\theta,\\
E(k) &= \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta,\\
F_S(a,b) &= \int_0^{\pi/2} \frac{1}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}\,d\theta,\\
E_S(a,b) &= \int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta.
\end{aligned}$$

Then $F(k)$ and $E(k)$ are called *complete elliptic integrals of the first and second kinds*, respectively, with *symmetric forms* $F_S(a,b)$ and $E_S(a,b)$. We refer to k as the modulus of our integral.

From here on we will assume that by elliptic integral we mean a complete elliptic integral of the first or second kind. We see that the ordinary and symmetric forms of elliptic integrals are related by the following proposition.

Proposition 2.1. *Let $k^2 = 1 - b^2/a^2$. Then the identities*

$$\begin{aligned}
F(k) &= aF_S(a,b)\\
E(k) &= \frac{1}{a}E_S(a,b).
\end{aligned}$$

are true.

Proof. Let us prove the second equation with the first following similarly. Substituting, we get

$$
\begin{aligned}
E(k) &= \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\,d\theta \\
&= \int_0^{\pi/2} \sqrt{1-(1-b^2/a^2)\sin^2\theta}\,d\theta \\
&= \int_0^{\pi/2} \sqrt{\cos^2\theta + b^2/a^2\sin^2\theta}\,d\theta \\
&= \frac{1}{a}\int_0^{\pi/2} \sqrt{a^2\cos^2\theta + b^2\sin^2\theta}\,d\theta \\
&= \frac{1}{a}E_S(a,b),
\end{aligned}
$$

and hence our result. ∎

We also recall a result regarding the symmetry of these integrals originally discovered by Legendre.

Proposition 2.2. (Legendre's Identity). *Suppose $k_1^2 + k_2^2 = 1$. Then*

$$
F(k_1)E(k_2) + F(k_2)E(k_1) - F(k_1)F(k_2) = \frac{\pi}{2}
$$

holds.

Proof. We leave the details of this proof to the reader. Taking the derivative with respect to k_1 shows that the L.H.S. is constant. To see that this value is $\pi/2$, we take the limit as k_1 goes to 0. ∎

3. Main results

We now prove the Gauss-Legendre algorithm. We will not discuss error analysis, which is done in [4].

Definition. Let $a_0, b_0 \in \mathrm{R}$. Let $a_{n+1} = (a_n + b_n)/2$ and $b_{n+1} = (a_n b_n)^{1/2}$ be the arithmetic and geometric means, respectively, of the $ n $th terms. Then we call their common limit

$$
\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = M(a_0, b_0)
$$

the *arithmetic-geometric (AM-GM) mean* of a_0 and b_0.

Theorem (Gauss-Legendre). *Set $a_0 = 1$ and $b_0 = \sqrt{2}/2$. Then the series*

$$
\pi = \frac{2M(1,\sqrt{2}/2)^2}{1 - \sum_{n=0}^{\infty} 2^n (a_n^2 - b_n^2)}
$$

converges to π.

We need two lemmas in order to prove our result.

Lemma 3.1. *Let $a_0 = a$ and $b_0 = b$ as in Definition 3.1. Set

$$
S = a^2 - \sum_{n=0}^{\infty} 2^{n-1}\,(a_n^2 - b_n^2)\,.
$$

Then $E_S(a,b) = SF_S(a,b)$.*

Proof. This proof is taken from [5]. Consider the integral

$$L(a,b) = a^2 F_S(a,b) - E_S(a,b) \tag{2}$$

Explicitly, this expands to

$$L(a,b) = (a^2 - b^2)\int_0^{\pi/2} \frac{\sin^2\theta}{\sqrt{a^2\cos^2\theta + b^2\sin^2\theta}}d\theta.$$

Substituting $x^2 = a^2\cos^2\theta + b^2\sin^2\theta$,

$$L(a,b) = \int_b^a \sqrt{\frac{a^2 - x^2}{x^2 - b^2}}dx.$$

Now substituting $y = (x + ab/x)/2$ and considering the associated AM-GM sequence,

$$\begin{aligned} L(a,b) &= \frac{1}{2}\int_{b_1}^{a_1} \frac{(a^2 - b^2) + 4(a_1^2 - y^2)}{\sqrt{(a_1^2 - y^2)(y^2 - b_1^2)}}dy \\ &= \frac{1}{2}(a^2 - b^2)F_S(a,b) + 2L(a_1, b_1), \end{aligned}$$

and thus

$$\frac{L(a,b)}{F_S(a,b)} = \frac{1}{2}(a_0^2 - b_0^2) + 2\frac{L(a_1,b_1)}{F_S(a,b)}.$$

Since $2^n(a_n^2 - b_n^2) \to 0$, we get $2^n L(a_n, b_n) \to 0$, and hence repeatedly applying this identity we get

$$L(a,b) = \sum_{n=0}^{\infty} 2^{n-1}(a_n^2 - b_n^2)F_S(a,b),$$

or equivalently

$$L(a,b) = -SF_S(a,b) + a^2 F_S(a,b).$$

Combining this equation with (2) gives us the result. ■

Lemma 3.2. *We have*

$$F_S(a,b) = \frac{\pi}{2M(a,b)}$$

In particular,

$$F_S(1,\sqrt{2}) = \frac{\pi}{2M(1,\sqrt{2})}.$$

Proof. It is not hard to calculate the power series expansion

$$F(k) = \frac{\pi}{2M(1+k, 1-k)} = \sum_{n=0}^{\infty}\left(\frac{(2n-1)!!}{(2n)!!}k^n\right)^2.$$

(The "!!" means double factorial{:target="_blank"}.) Calculating the AM and GM means gives us

$$M(1+k, 1-k) = M(1, \sqrt{1-k^2}).$$

Hence, applying proposition 2.1{:target="_blank"} for $b/a = \sqrt{1-k^2}$, we get

$$aF_S(a,b) = \frac{\pi}{2M(1,b/a)}$$
$$F_S(a,b) = \frac{\pi}{2M(a,b)},$$

with the last equality following from the identity $M(ca,cb) = cM(a,b)$. ∎

We finally have the tools we need to prove the Gauss-Legendre Algorithm does indeed converge to π.

Proof (Gauss-Legendre). Set $k = \sqrt{2}/2$ to be our modulus. Then we notice $2k^2 = 1$, and hence we can apply proposition 2.2{:target="_blank"} to get

$$2F(k)E(k) - F(k)^2 = \frac{\pi}{2}.$$

Now let us evaluate these integrals by first converting them into symmetric form then applying our lemmas. Since $k^2 = 1 - 1/k^2$, we can set $a = 1$ and $b = k$ in proposition 2.1{:target="_blank"} to get

$$2F_S(1,k)E_S(1,k) - F_S(1,k)^2 = \frac{\pi}{2}.$$

Then applying lemma 3.1{:target="_blank"} gives us an equation only dependent on $F_S(1,k)$

$$(2S-1)F_S(1,k)^2 = \frac{\pi}{2}.$$

Finally, we apply lemma 3.2{:target="_blank"} to write our equation in terms of the AM-GM mean

$$(2S-1)\left(\frac{\pi}{2M(1,k)}\right)^2 = \frac{\pi}{2}$$
$$\pi = \frac{2M(1,k)^2}{2S-1},$$

with plugging in S gives us our result. ∎

4. References

1. David H. Bailey, Simon M. Plouffe, Peter B. Borwein, and Jonathan M. Borwein, *The quest for π*, The Mathematical Intelligencer **19** (1997), no. 1, 50–56.
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4. Eugene Salamin, *Computation of π using arithmetic-geometric mean*, Mathematics of Computation **30** (1976), no. 135, 565–570.
5. Paramanand Singh, *π and the AGM: Evaluating elliptic integrals*, 2009. Link{:target="_blank"}.