

# The Weiss Variation in Gravitation Theory

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# **Part I: The Weiss variation in mechanics**

# What is the Weiss variation?

- Weiss variation is a variation that includes boundary displacements<sup>1–4</sup>
- Provides unified formalism for deriving wide array of results in mechanics:
  - Hamiltonian mechanics
  - Hamilton-Jacobi theory
  - Noether theorem
  - Path Integral  $\Rightarrow$  Schrödinger eq.
- Useful for constructing Hamiltonians in field theory without explicit 3+1 split

<sup>1</sup>M. P. Weiss, Proc. R. Soc. Lond. A 156, 192 (1936)

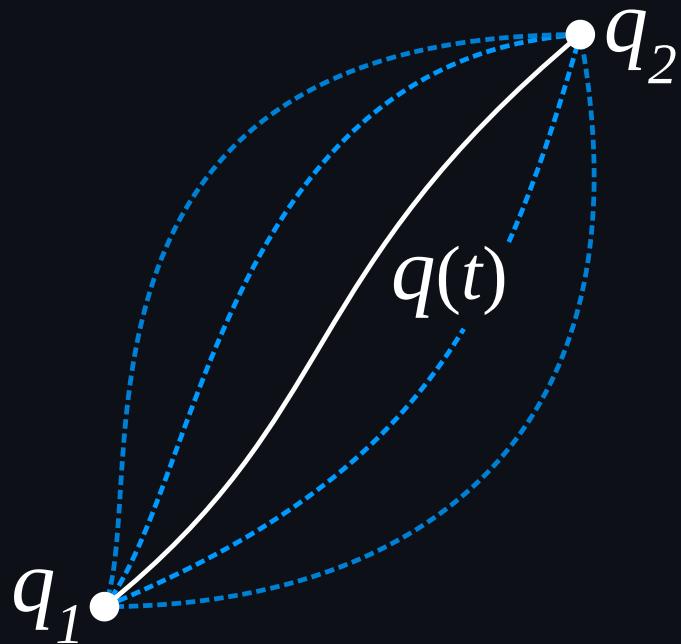
<sup>2</sup>E.C.G. Sudarshan and N. Mukunda, Classical Dynamics: A Modern Perspective (R.E. Krieger, 1983)

<sup>3</sup>R.A. Matzner and L.C. Shepley, Classical Mechanics (Prentice Hall, 1991)

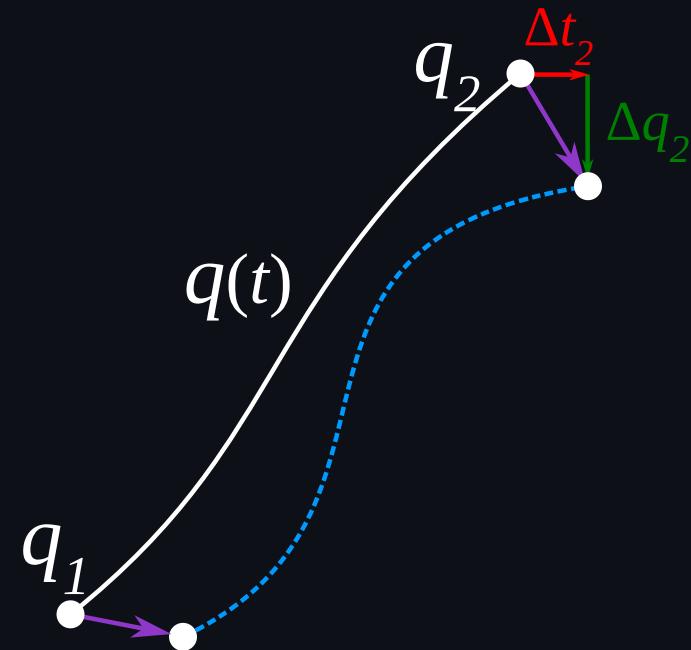
<sup>4</sup>JCF and R.A. Matzner, Gen. Rel. Grav. 50, 99 (2018), arXiv:1708.04489 [gr-qc]

# What's different?

``Standard'' variation



Weiss variation



The game: rewrite endpoint contributions in terms of total changes:  $\Delta q$ ,  $\Delta t$

## Weiss variation

Action:

$$S[q] := \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

Variation takes the form:

$$\Delta S = \int_{t_1}^{t_2} \mathcal{E}[q] \cdot \delta q dt + (p \cdot \delta q + L \Delta t)|_{t_1}^{t_2}$$

$$\mathcal{E}[q] := \frac{\partial L}{\partial q} - \frac{dp}{dt} \quad p := \frac{\partial L}{\partial \dot{q}}$$

At endpoint:

$$\Delta q = \delta q + \dot{q} \Delta t$$

Variation of action rewritten as:

$$\boxed{\Delta S = \int_{t_1}^{t_2} \mathcal{E}[q] \cdot \delta q dt + (p \cdot \Delta q - H \Delta t)|_{t_1}^{t_2}}$$

- Term in front of  $\Delta t$  is Hamiltonian:  
$$H := p \cdot \dot{q} - L$$
- When  $\mathcal{E}[q] = 0$ ,  $\Delta S$  is endpoint terms.
- Canonical vars. emerge naturally;  
diffs.  $dL, dH$  yield Hamilton eqs.

## Hamilton-Jacobi equation (w/o canonical transformations)

Weiss variation:

$$\Delta S = \int_{t_1}^{t_2} \mathcal{E}[q] \cdot \delta q \, dt + (p \cdot \Delta q - H \Delta t) \Big|_{t_1}^{t_2}$$

On physical paths ( $\mathcal{E}[q] = 0$ ),  $\Delta S$  consists of boundary terms for arbitrary variations

Compare with differential of classical action  $S_c = S_c(q_2, t_2)$  (hold  $q_1$  and  $t_1$  fixed) to obtain Hamilton-Jacobi Eqs:

$$dS_c = \frac{\partial S_c}{\partial q_2} \cdot dq_2 + \frac{\partial S_c}{\partial t_2} dt_2 \quad \Rightarrow \quad \frac{\partial S_c}{\partial q_2} = p_2, \quad \frac{\partial S_c}{\partial t_2} = -H_2$$

## Noether theorem

Let action  $S$  be invariant under transformation

$$q(t) \rightarrow q(t) + \varepsilon^A \eta_A(t) \quad t \rightarrow t + \varepsilon^A \tau_A(t)$$

where  $\varepsilon^A$  are infinitesimal parameters with generators  $\eta_A(t)$  and  $\tau_A(t)$ .

Setting  $\Delta q = \varepsilon^A \eta_A(t)$ ,  $\Delta t = \varepsilon^A \tau_A(t)$ , the Weiss variation on solutions ( $\mathcal{E}[q] = 0$ ) is:

$$\Delta S = \varepsilon^A (Q_A|_{t_2} - Q_A|_{t_1}).$$

where:

$$Q_A := p \cdot \eta_A - H \tau_A$$

If action is invariant,  $\Delta S = 0$ , which implies conservation:  $Q_A|_{t_2} = Q_A|_{t_1}$ .

# Schrödinger equation from path integral

Probability amplitude ( $S = S[\mathbf{q}](t_1, t_2)$ ):

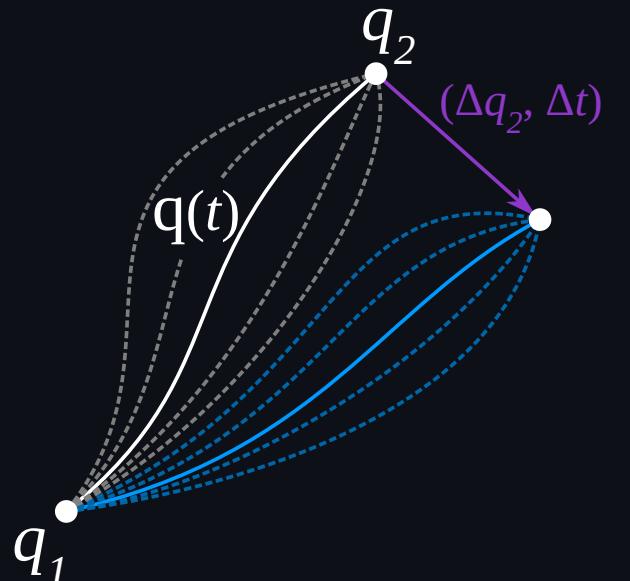
$$\langle q_1, t_1 | q_2, t_2 \rangle = \int \mathcal{D}\mathbf{q} e^{iS/\hbar} \quad \Rightarrow \quad \Psi(\mathbf{q}, t) := \int dq_1 \langle q_1, t_1 | q, t \rangle$$

Using Ehrenfest theorem<sup>6</sup>  $\int \mathcal{D}\mathbf{q} \mathcal{E}[\mathbf{q}] e^{iS/\hbar} = 0$ ,

$$\Delta\Psi = \frac{i}{\hbar} \int \mathcal{D}\mathbf{q} [p \cdot \Delta q - H \Delta t] e^{iS/\hbar} = \frac{i}{\hbar} \left[ \int \mathcal{D}\mathbf{q} p e^{iS/\hbar} \right] \cdot \Delta q - \frac{i}{\hbar} \left[ \int \mathcal{D}\mathbf{q} H e^{iS/\hbar} \right] \Delta t$$

Compare with  $d\Psi = (\partial_q \Psi) \cdot dq + \partial_t \Psi dt$  to obtain :

$$\hat{p}\Psi := -i\hbar \frac{\partial\Psi}{\partial q} = \int \mathcal{D}\mathbf{q} p e^{iS/\hbar} \quad \hat{H}\Psi := i\hbar \frac{\partial\Psi}{\partial t} = \int \mathcal{D}\mathbf{q} H e^{iS/\hbar}$$



<sup>4</sup> M. Blau, <http://www.blau.itp.unibe.ch/lecturesPI.pdf>; H. Murayama, <http://hitoshi.berkeley.edu/221a/pathintegral.pdf>.

<sup>5</sup> The Ehrenfest theorem follows from invariance of  $\int \mathcal{D}\mathbf{q}$  under redefinitions  $\mathbf{q} \rightarrow \mathbf{q} + \delta\mathbf{q}$

## Part II: Classical fields & Gravity

JCF and R.A. Matzner, Gen. Rel. Grav. 50, 99 (2018), [arXiv:1708.04489]

JCF, S. Chakraborty, Gen. Rel. Grav. 54, 67 (2022), [arXiv:2111.06897]

# Full variation of classical field action

Field theory action (defining  $d\underline{\mu} := d^4x$ ):

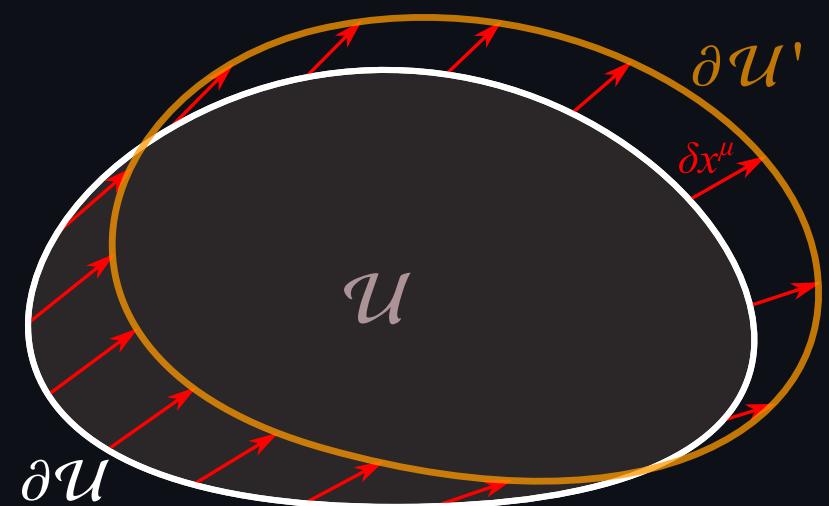
$$S[\varphi] = \int_{\mathcal{U}} d\underline{\mu} \mathcal{L}(\varphi, \partial\varphi)$$

Under a general variation

$$\Delta S = \int_{\mathcal{U}} d\underline{\mu} \mathcal{E}_a[\varphi] \delta\varphi^a + \int_{\partial\mathcal{U}} d\underline{\Sigma}_{\mu} [\pi_a{}^{\mu} \delta\varphi^a + \mathcal{L} \delta x^{\mu}]$$

$$\pi_a{}^{\mu} := \frac{\partial \mathcal{L}}{\partial(\partial_{\mu}\varphi^a)} \quad \mathcal{E}_a := \frac{\partial \mathcal{L}}{\partial \varphi^a} - \partial_{\mu}\pi_a{}^{\mu}$$

$$d\underline{\Sigma}_{\mu} := \frac{1}{3!} \epsilon_{\mu\alpha\beta\gamma} dx^{\alpha} \wedge dx^{\beta} \wedge dx^{\gamma}$$



Compare with mechanical expression:  $\Delta S = \int_{t_1}^{t_2} (\mathcal{E}[q] \cdot \delta q) dt + (p \cdot \delta q + L \Delta t)|_{t_1}^{t_2}$

## Weiss form of the variation

Defining  $\delta x^\mu := \xi^\mu \Delta t$ , total change in field is:

$$\Delta\varphi^a := (\varphi^a + \delta\varphi^a)|_{\partial\mathcal{U}} - \varphi^a|_{\partial\mathcal{U}} = [\delta\varphi^a + \mathcal{L}_\xi\varphi^a \Delta t]_{\partial\mathcal{U}}$$

Weiss variation is:

$$\Delta S = \int_{\mathcal{U}} d\underline{\mu} \mathcal{E}_a[\varphi] \delta\varphi^a + \int_{\partial\mathcal{U}} d\underline{\Sigma}_\mu [\pi_a{}^\mu \Delta\varphi^a - (\Theta^\mu{}_\nu \xi^\nu + S^\mu) \Delta t]$$

Canonical energy-momentum

$$\Theta^\mu{}_\nu := \pi_a^\mu \partial_\nu \varphi^a - \delta^\mu{}_\nu \mathcal{L}$$

"Belinfante-Rosenfeld" vector

$$S^\mu := \pi_a^\mu [\mathcal{L}_\xi \varphi^a - \xi^\nu \partial_\nu \varphi^a]$$

## Einstein-Hilbert action and variation<sup>6</sup>

Einstein-Hilbert action for general relativity:

$$S_{EH}[g^{\cdot\cdot}] = \frac{1}{2\kappa} \int_U d\mu R \quad d\mu := \underline{d\mu} \sqrt{|g|} = d^4x \sqrt{|g|}$$

Full variation takes the form ( $d\Sigma_\mu := d\underline{\Sigma}_\mu \sqrt{|g|}$ ):

$$\Delta S_{EH} = \frac{1}{2\kappa} \left\{ \int_U d\mu G_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial U} d\Sigma_\mu \left[ \delta^{\mu\alpha}{}_{\nu\sigma} g^{\sigma\beta} \Delta \Gamma^\nu{}_{\alpha\beta} - [2G^\mu{}_\sigma \xi^\sigma - j^\mu] \Delta t \right] \right\}.$$

where  $j^\mu := \nabla_\sigma (\nabla^\mu \xi^\sigma - \nabla^\sigma \xi^\mu)$  is Noether-Komar current,  $\delta^{\mu\alpha}{}_{\nu\sigma} := \delta^\mu{}_\nu \delta^\alpha{}_\sigma - \delta^\mu{}_\sigma \delta^\alpha{}_\nu$ .

<sup>6</sup>For the non-null boundary result, see: JCF and R.A. Matzner, Gen. Rel. Grav. 50, 99 (2018), arXiv:1708.04489 [gr-qc]

Usual conventions:  $\kappa = 8\pi G$ ,  $R$  is Ricci scalar,  $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$  is Einstein tensor,  $g = \det(g_{\mu\nu})$ ,  $\Gamma^\alpha{}_{\mu\nu}$  are Christoffel symbols

# Weiss variation of gravitational action for non-null boundaries

Problem: need to rewrite boundary term with  $\Delta g^{\mu\nu}$  (don't want  $\Delta\Gamma^\alpha_{\mu\nu}$ )

Can do this for non-null boundary by adding the Gibbons-Hawking York term<sup>6</sup> to action ( $d\Sigma := d\Sigma_\nu n^\nu$ ):

$$S_{GHY}[g^{\cdot\cdot}] = \frac{1}{2\kappa} \int_U d\mu R + \frac{1}{\kappa} \int_{\partial U} d\Sigma K$$

with variation<sup>7</sup> (  $p_{\mu\nu} := K_{\mu\nu} - \gamma_{\mu\nu}K$  ,  $\gamma_{\mu\nu} := g_{\mu\nu} - \varepsilon n_\mu n_\nu$  )

$$\Delta S_{GHY} = \frac{1}{2\kappa} \left\{ \int_U d\mu G_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial U} d\Sigma \left[ p_{\mu\nu} \Delta \gamma^{\mu\nu} - \xi^\mu \left[ 2D_\alpha p^{\alpha\beta} \gamma_{\mu\beta} - n_\mu \left( {}^3R - \varepsilon(K^2 - K_{\alpha\beta} K^{\alpha\beta}) \right) \right] \Delta t \right] \right\}.$$

Hamiltonian proportional to constraints, which vanish on solutions of Einstein eqs.

<sup>6</sup>For null boundary GHY term, see: K. Parattu, S. Chakraborty, B. R. Majhi, and T. Padmanabhan, Gen. Rel. Grav. 48, 94 (2016), 1501.01053

<sup>7</sup>JCF and R.A. Matzner, Gen. Rel. Grav. 50, 99 (2018), arXiv:1708.04489 [gr-qc]

# Weiss variation with a different boundary term<sup>8</sup>

Can add a different boundary term, with  $\bar{\Gamma}^\lambda_{\mu\nu}$  being nondynamical:<sup>9,10</sup>

$$S_{gW} = \frac{1}{2\kappa} \int_U d\mu [R - \nabla_\mu W^\mu], \quad W^\mu := W^{\mu\nu}{}_\nu - W_\nu{}^{\nu\mu}, \quad W^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \bar{\Gamma}^\lambda{}_{\mu\nu}$$

Variation is

$$\boxed{\Delta S_{gW} = \frac{1}{2\kappa} \left\{ \int_U d\mu G_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial U} d\Sigma_\mu \left[ P^\mu{}_{\sigma\beta} \Delta g^{\sigma\beta} + \delta \bar{\Gamma}^\mu - \left[ ({}_E\Theta^\mu{}_\beta - \sigma^\mu{}_\beta) \xi^\beta - \Sigma^{\mu\alpha}{}_\beta \bar{\nabla}_\alpha \xi^\beta \right] \Delta t \right] \right\}}$$

$$P^\sigma{}_{\mu\nu} := -\delta^{\sigma\alpha}{}_{\rho(\mu} W^\rho{}_{\alpha|\nu)} + \tfrac{1}{2} W^\sigma g_{\mu\nu},$$

$$\delta \bar{\Gamma}^\mu := \delta^{\mu\alpha}{}_{\nu\sigma} \delta \bar{\Gamma}^\nu{}_{\alpha\beta}, \quad \sigma^\mu{}_\beta \propto \bar{\Gamma}^\sigma{}_{[\mu\nu]}, \quad \Sigma^{\mu\alpha}{}_\beta = \Sigma^\mu{}_\beta{}^\alpha \propto W^\sigma{}_{\mu\nu}$$

The tensor  ${}_E\Theta^\alpha{}_\beta$  generalizes Einstein pseudotensor:<sup>9</sup>

$${}_E\Theta^\alpha{}_\beta = (W_\tau{}^{\tau\alpha} - W^{\alpha\tau}{}_\tau) W^\sigma{}_{\beta\sigma} - W^\sigma{}_{\sigma\tau} (W^\alpha{}_{\beta\tau} + W^\tau{}_{\beta\alpha}) + W^{\alpha\sigma\tau} (W_{\sigma\beta\tau} + W_{\tau\beta\sigma}) - \delta^\alpha{}_\beta (R - \nabla_\sigma W^\sigma)$$

<sup>8</sup>JCF, S. Chakraborty, Gen. Rel. Grav. 54, 67 (2022) arXiv:2111.06897 [gr-qc]

<sup>9</sup>D. Lynden-Bell, J. Katz, and J. Bičák, MNRAS 272, 150 (1995); J. Katz, J. Bičák, and D. Lynden-Bell, Phys. Rev. D 55, 5957 (1997)

<sup>10</sup>J Harada Phys. Rev. D 101, 024053 (2020) arXiv:2001.06990

## Symmetric reference

For a flat, Levi-Civita connection  $(\bar{\Gamma}^\lambda_{\mu\nu}, \bar{\nabla})$  for metric  $\bar{g}_{\mu\nu}$ , and a covariantly constant vector  $\xi^\mu$  satisfying  $\bar{\nabla}_\nu \xi^\mu = 0$ :

$$\Delta S_{gW} = \frac{1}{2\kappa} \left\{ \int_U d\mu G_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial U} d\Sigma_\mu \left[ P^\mu{}_{\alpha\beta} \Delta g^{\alpha\beta} - {}_E\Theta^\mu{}_\beta \xi^\beta \Delta t \right] \right\}.$$

Using (Katz-Ori) identity,<sup>8,9</sup> can obtain tensor satisfying conservation law:

$$\Pi^\sigma{}_\tau := \sqrt{g/\bar{g}} (2G^\mu{}_\nu - {}_E\Theta^\mu{}_\nu), \quad \bar{\nabla}_\sigma (\Pi^\sigma{}_\tau \xi^\tau) = \xi^\tau \bar{\nabla}_\sigma \Pi^\sigma{}_\tau = 0$$

Has two features that make it potentially useful:

1. Using  $G_{\mu\nu} = \kappa T_{\mu\nu}$ , can separate into matter and gravitational parts
2. Satisfies an exact conservation law.

<sup>8</sup>D. Lynden-Bell, J. Katz, and J. Bičák, MNRAS 272, 150 (1995)

<sup>9</sup>J. Katz, J. Bičák, and D. Lynden-Bell, Phys. Rev. D 55, 5957 (1997)

# Does this define local gravitational energy? (Not really)

- Many attempts to define energy in GR<sup>11</sup> but no reasonable local defn. found
- Regarding  $\Pi^{\sigma}_{\tau}$ :
  - Pseudotensor ambiguities  $\Rightarrow$  ambiguity of coordinates for  $\bar{\Gamma}^{\lambda}_{\mu\nu}$
- Studied energy density  $e = \Pi^{\sigma}_{\tau} \xi^{\tau} n_{\sigma}$  for various coordinate choices for Schwarzschild metric:
  - Get trivial or pathological energy density in many cases, get sensible result for one choice of harmonic coordinates, but not others
- Can obtain a quasilocal energy from superpotential and reference choice.<sup>12,13</sup>

<sup>11</sup>L. B. Szabados, Liv. Rev. Relativ. 12:4 (2009); S. De Haro, arxiv:2103.17160; S. Aoki, T. Onogi, S. Yokoyama, Int.J.Mod.Phys.A 36 (2021) 29, 2150201.

<sup>12</sup>C.-M. Chen, J.-L. Liu, J. M. Nester, Int. J. Mod.Phys. D27, 1847017 (2018) arxiv:1805.07692

<sup>13</sup>C.-M. Chen, J.-L. Liu, J. M. Nester, Gen. Rel. Grav. 50, 158 (2018) arxiv:1811.05640

# Einstein-Schrödinger equation

Formal path integral for quantum GR (QGR):

$$\mathcal{A}[\dot{g}^{\cdot\cdot}] = \int \mathcal{D}g^{\cdot\cdot} e^{(i/\hbar)S_{gW}[g^{\cdot\cdot}]}$$

With Ehrenfest theorem, variation is:

$$\Delta\mathcal{A} = \frac{i}{2\kappa\hbar} \left\{ \int_{\partial U} d\Sigma \left[ \hat{P}_{\alpha\beta} \Delta g^{\alpha\beta} - \hat{\mathcal{H}}_\nu \xi^\nu \Delta t \right] \right\}$$

$$\hat{P}_{\alpha\beta}\mathcal{A} = \int \mathcal{D}g^{\cdot\cdot} \sqrt{-\dot{g}} n_\mu P^\mu{}_{\alpha\beta} e^{(i/\hbar)S_{gW}[g^{\cdot\cdot}]} \quad \hat{\mathcal{H}}_\nu\mathcal{A} = \int \mathcal{D}g^{\cdot\cdot} \sqrt{-\dot{g}} n_\mu {}_E\Theta^\mu{}_\nu e^{(i/\hbar)S_{gW}[g^{\cdot\cdot}]}$$

Similarly to the case in mechanics, one obtains a Schrödinger equation:

$$i\hbar \frac{\partial \mathcal{A}}{\partial t} = \hat{H}\mathcal{A} \quad \hat{H}\mathcal{A} := \frac{1}{2\kappa} \int_{\partial U} d\Sigma \xi^\nu \hat{\mathcal{H}}_\nu \mathcal{A}$$

## Does this solve problem of time in QGR? (Not really)

- Problem of time in quantum GR (QGR):
  - In usual 3+1 GR with Gibbons-Hawking-York term, Ham. density  $\mathcal{H}$  vanishes---one obtains the Wheeler-DeWitt equation<sup>12</sup>  $\hat{\mathcal{H}}\Psi[\gamma^{\cdot\cdot}] = 0$
- Einstein-Schrödinger Eq. naively provides time evolution, but...
  - Hamiltonian  $\hat{H}$  is ambiguous---depends on coords. for reference
  - Bad coordinate choices for reference geometry can lead to pathologies
- Might have use as a formal tool in certain contexts (e.g. perturbation theory)

<sup>13</sup>JCF and R. A. Matzner, Phys. Rev. D96, 106005(2017) arXiv:1708.07001.

# Summary and Prospects

- Weiss var. unifies derivation of many results in classical mechanics
- Can obtain suggestive results in GR, QGR, and field theory
  - Canonical EMT, currents, Schrödinger/WDW Eqs.
  - Can get Hamiltonian without explicit 3+1 split (useful in mod. grav.)
- May lead to better understanding of energy in GR
  - Sharpens understanding of pseudotensorial ambiguities, at least
  - Can we find a definition that can be cleanly exchanged and satisfies strong conservation laws?

<sup>13</sup>H. Friedrich, Classical and Quantum Gravity, 20(1):101–117, Dec 2002

<sup>14</sup>JCF, S. Mukohyama, S. Carloni, Phys. Rev. D 105, 104036 (2022)



## Belinfante-Rosenfeld Correction

Consider the covariant Weiss variation of some matter action  $S_M = \int d\mu \mathcal{L}(\varphi, d\varphi)$ :

$$\Delta S_M = \int_U d\mu \mathcal{E}_I \delta\varphi^I + \int_{\partial U} d\Sigma_\sigma [\mathcal{L} \delta x^\sigma + \pi_I{}^\sigma \delta\varphi^I]$$

Define:

$$\begin{aligned} \Delta\varphi^I &:= \delta\varphi^I + \mathcal{L}_{\Delta\lambda\xi}\varphi^I, & \mathcal{L}_\xi\varphi^I &:= \xi^\sigma \nabla_\sigma \varphi^I + w_J{}^I \varphi^J, & \Theta^\mu{}_\nu &:= \pi_I^\mu \nabla_\nu \varphi^I - \delta^\mu{}_\nu \mathcal{L} \\ B_{\mu\nu}{}^\sigma &\quad \text{such that} \quad P_I{}^\sigma w_J{}^I \varphi^J = \Delta\lambda \xi^\mu \nabla^\nu B_{\mu\nu}{}^\sigma - \nabla^\nu (\Delta\lambda \xi^\mu B_{\mu\nu}{}^\sigma), \end{aligned}$$

One has ( $\Theta^\mu{}_\nu := \pi_I^\mu \nabla_\nu \varphi^I - \delta^\mu{}_\nu \mathcal{L}$  being the canonical energy-momentum tensor):

$$\Delta S_M = \int_U d\mu \mathcal{E}_I \delta\varphi^I + \int_{\partial U} d\Sigma_\sigma [\pi_I{}^\sigma \Delta\varphi^I + \Delta\lambda \xi^\mu (\Theta^\sigma{}_\mu + \nabla^\nu B_{\mu\nu}{}^\sigma) + \nabla_\nu (\Delta\lambda \xi^\mu B_{\mu}{}^{\nu\sigma})]$$

If  $B_\mu{}^{\nu\sigma} = -B_\mu{}^{\sigma\nu}$ , last term is bdy. term; get Belinfante-Rosenfeld correction to  $\Theta^\sigma{}_\mu$ :

$$\bar{T}^\sigma{}_\mu := \Theta^\sigma{}_\mu + \nabla^\nu B_{\mu\nu}{}^\sigma$$

# Higher derivative actions

Action:

$$S[q, \lambda] = \int_{t_1}^{t_2} L \left( q, \frac{dq}{dt}, \dots, \frac{d^n q}{dt^n}, \lambda, t \right) dt$$

Variation is:

$$\Delta S = \int_{t_1}^{t_2} \left( \frac{\partial L}{\partial q} + \sum_{m=1}^n (-1)^m \frac{d^m \tilde{P}^m}{dt^m} \right) \cdot \delta q(t) dt + \left[ \sum_{m=1}^n P^m \cdot \Delta Q_m - H \Delta t \right]_{t_1}^{t_2}$$

$$Q_m := \frac{d^{m-1} q}{dt^{m-1}} \quad P^a := \sum_{m=a}^n (-1)^{m-a} \frac{d^{m-a} \tilde{P}^m}{dt^{m-a}} \quad \tilde{P}^m := \frac{\partial L}{\partial (d^m q / dt^m)}$$

Ostrogradsky Hamiltonian:  $H := (\sum_m P^m \cdot \dot{Q}_m) - L$ .

## Modified gravity: Higher curvature example

Consider a metric theory defined by the Lagrangian  $\mathcal{L}_g(R \dots)$ . Boundary terms are:

$$\begin{aligned} & \int_{\partial U} d\Sigma_\mu \left( P^\mu{}_{\alpha\beta} \Delta g^{\alpha\beta} + P^\mu{}_\nu{}^{\alpha\beta} \Delta \Gamma^\nu{}_{\alpha\beta} - \Delta \lambda \left\{ 2P^\mu{}_{\alpha\beta} \nabla^{(\alpha} \xi^{\beta)} + P^\mu{}_\nu{}^{\alpha\beta} (R_{\alpha\sigma\beta}{}^\nu \xi^\sigma + \nabla_\alpha \nabla_\beta \xi^\nu) - \mathcal{L}_g \xi^\mu \right\} \right) \\ &= \int_{\partial U} d\Sigma_\mu [P^\mu{}_{\alpha\beta} \Delta g^{\alpha\beta} + P^\mu{}_\nu{}^{\alpha\beta} \Delta \Gamma^\nu{}_{\alpha\beta} - \Delta \lambda \{ \Theta^\mu{}_\nu \xi^\nu + \nabla_\nu \omega^{\mu\nu} \}]. \end{aligned}$$

If  $\nabla_\nu \omega^{(\mu\nu)} = 0$ , you're done---otherwise, have three options:

1. Introduce flat auxiliary connection  $\bar{\nabla}_.$ , choose  $\xi^\mu$  to satisfy  $\bar{\nabla}_\mu \xi^\nu = 0$ , then:

$$\nabla^{(\alpha} \xi^{\beta)} \propto \xi^\sigma, \quad \nabla_\alpha \nabla_\beta \xi^\nu \propto \xi^\sigma \quad (\text{lose some freedom to choose } \xi^\sigma)$$

2. Do 3+1 decomposition of  $\nabla_\nu \omega^{(\mu\nu)}$  wrt to bdy surface (lose 4D covariance).
3. Add boundary terms to cancel  $\nabla_\nu \omega^{(\mu\nu)}$  (depends on integrability)