On background dependent formalisms for general relativity

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Background dependent formalisms

- General relativity is background independent
    - g_{ab} is defined without reference to an underlying geometric structure
- Why introduce a background?
    - Can choose background with symmetries---potentially useful feature
    - Can build new solutions out of existing ones
    - Uncover new properties of the Einstein field equations (EFE)

Examples (ordered in increasing generality)

- Kerr-schild perturbation:[*] $g_{ab} = \bar{g}_{ab} + \psi k_a k_b, \quad k \text{ null}$
  - Includes Kerr soln, can linearize vac EFE under some conds.[⊛]
- Extended Kerr-schild:[†] $g_{ab} = \varphi \bar{g}_{ab} + \psi k_{(a} l_{b)}, \quad k, l \text{ null wrt } \bar{g}_{ab}$
  - Quite general, can reduce nonlinearity in EFE to fifth order.[‡]
- Matrix deformations:[*†] $g_{ab} = \bar{g}_{mn} {\tau_a}^m {\tau_b}^n$
  - Generalization of KS, extended KS, and tetrad formalism
  - Can write: ${\tau_a}^m = \tau_a^{(1)\,n} \tau_n^{(2)\,p} \ldots \tau_q^{(N)\,m}$ (*"Turtles all the way down"*)

[*]R. P. Kerr, A. Schild, Gen. Rel. Grav. 41 (10): 2485–2499 (2009).
[†]J. Llosa and D. Soler, CQG 22, 893 (2005); J. Llosa and J. Carot, CQG 26, 055013 (2009)
[*]S. Capozziello, C. Stornaiolo, Int.J.Geom.Meth.Mod.Phys. 5 (2008) 185-195
[⊛]B. C. Xanthopoulos, J. Math. Phys 19, 1607 (1978)
[‡]A. I. Harte, Phys. Rev. Lett. 113, 261103 (2014)
[†]JCF, S. Carloni, Phys. Rev. D 101, 064002 (2020)$

Reference connections

- Introduce reference connection $\bar{\nabla}$, then construct: $W^{\sigma}{}_{\mu\nu} = \Gamma^{\sigma}{}_{\mu\nu} - \bar{\Gamma}^{\sigma}{}_{\mu\nu}$
  - Can recover covariant counterpart of $\Gamma\Gamma$ action:[*,⊛]

$$\bar{S}_{EH} = \int_U d^4x\sqrt{-g}\left[g^{ab}W^{c}{}_{ad}W^{d}{}_{cb} - W^{a}{}_{ab}W^{b}{}_{cd}g^{cd} + g^{ab}\bar{R}_{ab} + \nabla_a B^a\right]$$

$$B^a := g^{cd}W^{a}{}_{cd} - g^{ad}W^{c}{}_{cd}$$

  - Explored as early as 1940s[†], studied by Katz, Bičák, and Lynden-Bell (KBL)[†]
  - Formalism rediscovered at least twice in the past ten years[‡]

[*] D. Lynden-Bell, J. Katz, and J. Bičák, MNRAS 272,150 (1995); J. Katz, J. Bičák, and D. Lynden-Bell, Phys. Rev. D 55, 5957 (1997)
[⊛] JCF, S. Chakraborty, Gen. Rel. Grav. 54, 67 (2022)
[†] N. Rosen, Phys. Rev. 57, 147–150 (1940); A. Papapetrou, Proc. Roy. Irish Acad. A 52, 11–23 (1948).
[‡] J. Harada, Phys. Rev. D 101, 024053 (2020), arXiv:2001.06990 [gr-qc]; E. T. Tomboulis, JHEP 09, 145 (2017), arXiv:1708.03977 [hep-th]

Freedom to choose background: Advantages

- Can choose background with symmetries to get conserved quantities
  - Conserved canonical "energy-momentum" tensor!

    For flat background $\bar{g}_{\mu\nu} = \eta_{\mu\nu}$ and $\bar{\nabla}_\mu \xi^\nu = 0$:*

    $$\Pi^\sigma{}_\tau := \sqrt{g/\bar{g}}\,(2G^\mu{}_\nu - \Theta^\mu{}_\nu), \qquad \bar{\nabla}_\sigma(\Pi^\sigma{}_\tau \xi^\tau) = \xi^\tau \bar{\nabla}_\sigma \Pi^\sigma{}_\tau = 0$$

    $$\Theta^\alpha{}_\beta := W^{\alpha\sigma\tau}\,(W_{\sigma\beta\tau} + W_{\tau\beta\sigma}) - W^\sigma{}_{\sigma\tau}\,(W^\alpha{}_\beta{}^\tau + W^\tau{}_\beta{}^\alpha) - W^\sigma{}_{\beta\sigma} B^\alpha - \delta^\alpha{}_\beta(R - \nabla_\sigma B^\sigma)$$
- Can we really have our cake and eat it too?
  - Maybe not: Schwarzschild (or PG) with static flat BG:† $\Pi^\sigma{}_\tau = 0$

*D. Lynden-Bell, J. Katz, and J. Bičák, MNRAS 272,150 (1995); J. Katz, J. Bičák, and D. Lynden-Bell, Phys. Rev. D 55, 5957 (1997)

†JCF, S. Chakraborty, Gen. Rel. Grav. 54, 67 (2022)

‡M. Krššák, Phys.Rev.D 110 (2024) 10, 104061

Freedom is not free: Tradeoffs

- Can choose coordinates for $\bar{g}_{\mu\nu}$ independently of $g_{\mu\nu}$
  - Pseudotensorial ambiguities $\Rightarrow$ extra coordinate freedom for $\bar{g}_{\mu\nu}$
  - Schwarzschild in harmonic coords: "energy" obtained from ${\Pi^\sigma}_\tau$ can either be nice,[†] or negative and divergent[‡]
  - Can appropriate coordinate condition for $\bar{g}_{\mu\nu}$ fix this?
- Pseudotensor pathologies still present, but tensors nicer than pseudotensors.

The cake is not edible, but looks appetizing

[†] N. Nakanishi Prog.Theor.Phys. 75 (1986) 1351,

[‡] JCF, S. Chakraborty, Gen. Rel. Grav. 54, 67 (2022)

Energy of Schwarzschild in harmonic coords.

- Transform Schwarzschild to Harmonic coordinates:[*][†]

$$\tilde{r}(r) = c_1(r-m) + c_2\left\{(r-m)\,\ln\left[1-\frac{2m}{r}\right] + 2m\right\}$$

- Integrate $e = \Pi^{\sigma}{}_{\tau}\xi^{\tau}n_{\sigma}$ from horizon to r to get:

$$E_I = \frac{m^2}{2(r-m)} - c_2 m^3(\epsilon_1+\epsilon_2)$$

$$\frac{1}{\epsilon_1} := \; 2c_2 m(m-r) + r(2m-r)\left\{c_1 + c_2\ln\left[1-\frac{2m}{r}\right]\right\}$$

$$\frac{1}{\epsilon_2} := \; (m-r)^2\left(c_1 - \frac{2c_2 m}{m-r} + c_2\ln\left(1-\frac{2m}{r}\right)\right).$$

- Can have meaningful result if $c_2 = 0$, but unbounded in general

[*]JCF, S. Chakraborty, Gen. Rel. Grav. 54, 67 (2022)

[†]See also J. Bičák, J. Katz, Czech.J.Phys. 55 (2005) 105-118

Weiss variation

Action: $S[q] := \int_{t_1}^{t_2} L(q, \dot{q}, t)\, dt$

Weiss variation:

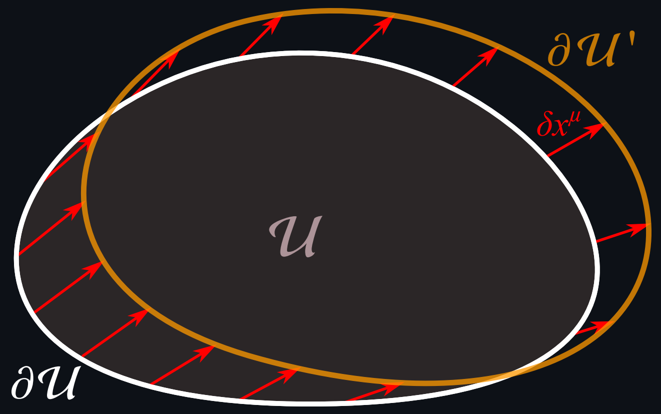
$$\boxed{\Delta S = \int_{t_1}^{t_2} \mathcal{E}[q] \cdot \delta q \, dt + (p \cdot \Delta q - H\, \Delta t)\Big|_{t_1}^{t_2}}$$

Endpoint change: $(\Delta q := \delta q + \dot{q} \Delta t)$

For field on U, move bdy ∂U $(\delta x^\mu = \xi^\mu \Delta t)$:

$$\Delta \varphi^a = [\delta \varphi^a + \pounds_\xi \varphi^a \Delta t]_{\partial \mathcal{U}}$$

Action: $S[\varphi] = \int_{\mathcal{U}} d\underline{\mu}\, \mathscr{L}(\varphi, \partial\varphi)$



$$\Delta S = \int_{\mathcal{U}} d\underline{\mu}\, \mathcal{E}_a[\varphi]\, \delta\varphi^a$$

$$+ \int_{\partial\mathcal{U}} d\underline{\Sigma}_\mu \left[\pi_a{}^\mu \Delta\varphi^a - (\Theta^\mu{}_\nu \xi^\nu + S^\mu)\Delta t\right]$$

$$\Theta^\mu{}_\nu := \pi^\mu_a\, \partial_\nu \varphi^a - \delta^\mu{}_\nu\, \mathscr{L}$$

$$S^\mu := \pi^\mu_a [\pounds_\xi \varphi^a - \xi^\nu \partial_\nu \varphi^a]$$

[1] M. P. Weiss, Proc. R. Soc. Lond. A 156, 192 (1936)
[2] E.C.G. Sudarshan and N. Mukunda, Classical Dynamics: A Modern Perspective (1983); R.A. Matzner and L.C. Shepley, Classical Mechanics (1991)
[4] JCF and R.A. Matzner, Gen. Rel. Grav. 50, 99 (2018); JCF, S. Chakraborty, Gen. Rel. Grav. 54, 67 (2022)