

The Weiss Variation in Gravitation Theory

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Part I: The Weiss variation in mechanics

What is the Weiss variation?

- Weiss variation is a variation that includes boundary displacements^{1–4}
- Provides unified formalism for deriving wide array of results in mechanics:
 - Hamiltonian mechanics
 - Noether theorem
 - Hamilton-Jacobi theory
 - Path Integral \Rightarrow Schrödinger eq.
- Useful for constructing Hamiltonians in field theory without explicit 3+1 split

¹M. P. Weiss, Proc. R. Soc. Lond. A 156, 192 (1936)

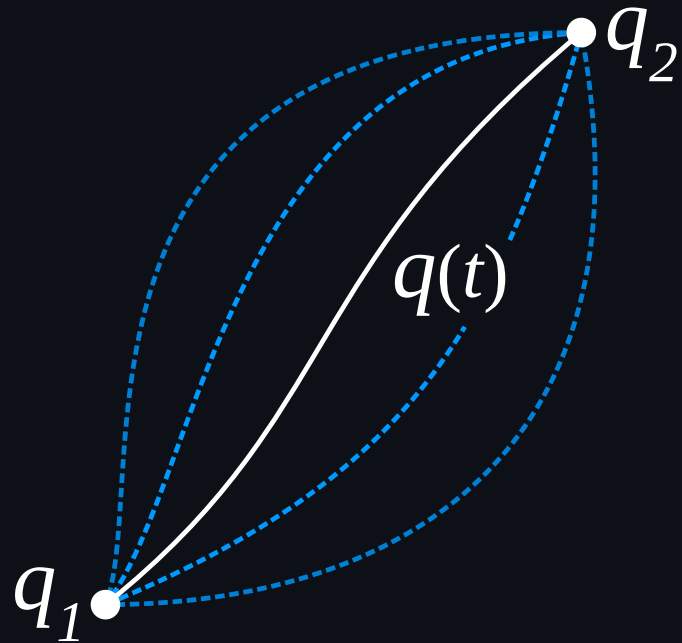
²E.C.G. Sudarshan and N. Mukunda, Classical Dynamics: A Modern Perspective (R.E. Krieger, 1983)

³R.A. Matzner and L.C. Shepley, Classical Mechanics (Prentice Hall, 1991)

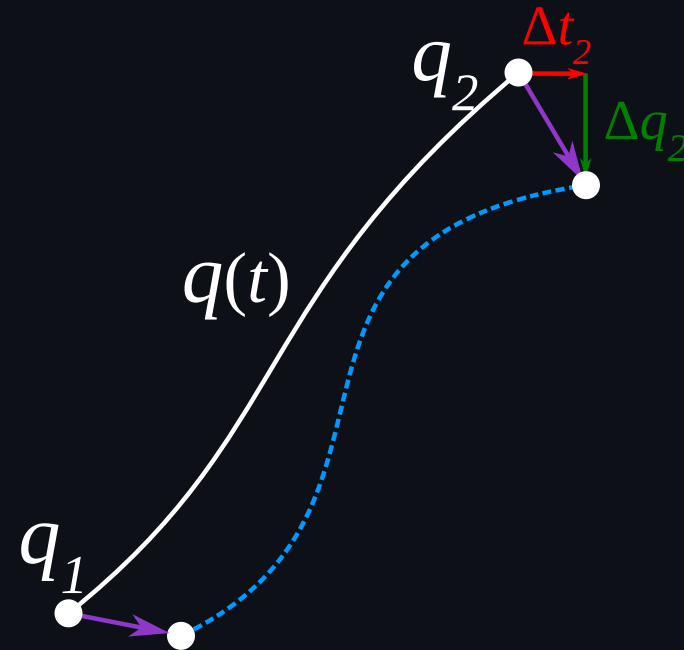
⁴JCF and R.A. Matzner, Gen. Rel. Grav. 50, 99 (2018), arXiv:1708.04489 [gr-qc]

What's different?

``Standard'' variation



Weiss variation



The game: rewrite endpoint contributions in terms of total changes: Δq , Δt

Weiss variation

Action:

$$S[q] := \int_{t_1}^{t_2} L(q, \dot{q}, t) dt$$

Variation takes the form:

$$\Delta S = \int_{t_1}^{t_2} \mathcal{E}[q] \cdot \delta q dt + (p \cdot \delta q + L \Delta t) \Big|_{t_1}^{t_2}$$

$$\mathcal{E}[q] := \frac{\partial L}{\partial q} - \frac{dp}{dt} \qquad p := \frac{\partial L}{\partial \dot{q}}$$

At endpoint:

$$\Delta q = \delta q + \dot{q} \Delta t$$

Variation of action rewritten as:

$$\Delta S = \int_{t_1}^{t_2} \mathcal{E}[q] \cdot \delta q dt + (p \cdot \Delta q - H \Delta t) \Big|_{t_1}^{t_2}$$

- Term in front of Δt is Hamiltonian:

$$H := p \cdot \dot{q} - L$$

- When $\mathcal{E}[q] = 0$, ΔS is endpoint terms.
- Canonical vars. emerge naturally; diffs. dL , dH yield Hamilton eqs.

Hamilton-Jacobi equation (w/o canonical transformations)

Weiss variation:

$$\Delta S = \int_{t_1}^{t_2} \mathcal{E}[q] \cdot \delta q dt + (p \cdot \Delta q - H \Delta t) \Big|_{t_1}^{t_2}$$

On physical paths ($\mathcal{E}[q] = 0$), ΔS consists of boundary terms for arbitrary variations

Compare with differential of classical action $S_c = S_c(q_2, t_2)$ (hold q_1 and t_1 fixed) to obtain Hamilton-Jacobi Eqs:

$$dS_c = \frac{\partial S_c}{\partial q_2} \cdot dq_2 + \frac{\partial S_c}{\partial t_2} dt_2 \quad \Rightarrow \quad \frac{\partial S_c}{\partial q_2} = p_2, \quad \frac{\partial S_c}{\partial t_2} = -H_2$$

Noether theorem

Let action S be invariant under transformation

$$q(t) \rightarrow q(t) + \varepsilon^A \eta_A(t) \quad t \rightarrow t + \varepsilon^A \tau_A(t)$$

where ε^A are infinitesimal parameters with generators $\eta_A(t)$ and $\tau_A(t)$.

Setting $\Delta q = \varepsilon^A \eta_A(t)$, $\Delta t = \varepsilon^A \tau_A(t)$, the Weiss variation on solutions ($\mathcal{E}[q] = 0$) is:

$$\Delta S = \varepsilon^A (Q_A|_{t_2} - Q_A|_{t_1}).$$

where:

$$Q_A := p \cdot \eta_A - H \tau_A$$

If action is invariant, $\Delta S = 0$, which implies conservation: $Q_A|_{t_2} = Q_A|_{t_1}$.

Schrödinger equation from path integral

Probability amplitude ($S = S[q](t_1, t_2)$):

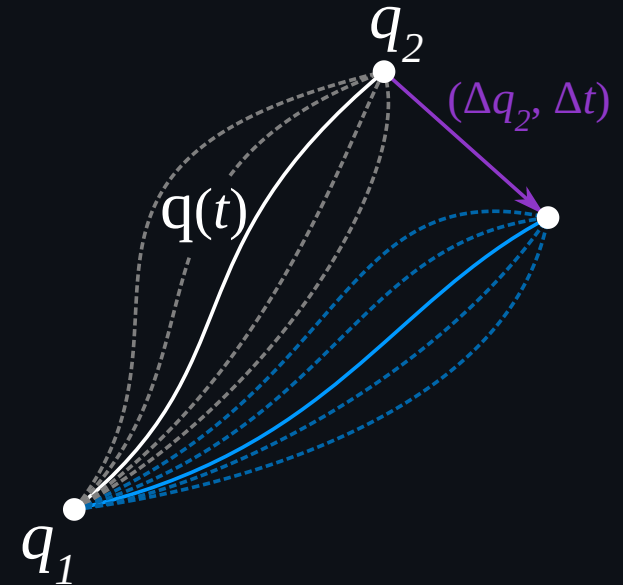
$$\langle q_1, t_1 | q_2, t_2 \rangle = \int \mathcal{D}q e^{iS/\hbar} \Rightarrow \Psi(q, t) := \int dq_1 \langle q_1, t_1 | q, t \rangle$$

Using Ehrenfest theorem⁶ $\int \mathcal{D}q \mathcal{E}[q] e^{iS/\hbar} = 0$,

$$\Delta \Psi = \frac{i}{\hbar} \int \mathcal{D}q [p \cdot \Delta q - H \Delta t] e^{iS/\hbar} = \frac{i}{\hbar} \left[\int \mathcal{D}q p e^{iS/\hbar} \right] \cdot \Delta q - \frac{i}{\hbar} \left[\int \mathcal{D}q H e^{iS/\hbar} \right] \Delta t$$

Compare with $d\Psi = (\partial_q \Psi) \cdot dq + \partial_t \Psi dt$ to obtain :

$$\hat{p}\Psi := -i\hbar \frac{\partial \Psi}{\partial q} = \int \mathcal{D}q p e^{iS/\hbar} \quad \hat{H}\Psi := i\hbar \frac{\partial \Psi}{\partial t} = \int \mathcal{D}q H e^{iS/\hbar}$$



⁴ M. Blau, <http://www.blau.itp.unibe.ch/lecturesPI.pdf>; H. Murayama, <http://hitoshi.berkeley.edu/221a/pathintegral.pdf>.

⁵ The Ehrenfest theorem follows from invariance of $\int \mathcal{D}q$ under redefinitions $q \rightarrow q + \delta q$

Part II: Classical fields & Gravity

JCF and R.A. Matzner, Gen. Rel. Grav. 50, 99 (2018), [arXiv:1708.04489]

JCF, S. Chakraborty, Gen. Rel. Grav. 54, 67 (2022), [arXiv:2111.06897]

Full variation of classical field action

Field theory action (defining $d\underline{\mu} := d^4x$):

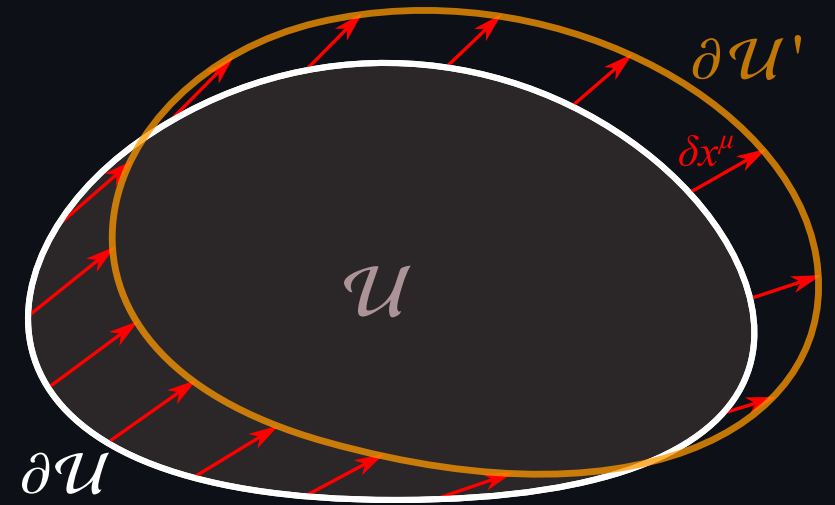
$$S[\varphi] = \int_{\mathcal{U}} d\underline{\mu} \mathcal{L}(\varphi, \partial\varphi)$$

Under a general variation

$$\Delta S = \int_{\mathcal{U}} d\underline{\mu} \mathcal{E}_a[\varphi] \delta\varphi^a + \int_{\partial\mathcal{U}} d\underline{\Sigma}_\mu [\pi_a{}^\mu \delta\varphi^a + \mathcal{L} \delta x^\mu]$$

$$\pi_a{}^\mu := \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi^a)} \quad \mathcal{E}_a := \frac{\partial \mathcal{L}}{\partial \varphi^a} - \partial_\mu \pi_a{}^\mu$$

$$d\underline{\Sigma}_\mu := \frac{1}{3!} \epsilon_{\mu\alpha\beta\gamma} dx^\alpha \wedge dx^\beta \wedge dx^\gamma$$



$$\Delta S = \int_{t_1}^{t_2} (\mathcal{E}[q] \cdot \delta q) dt + (p \cdot \delta q + L \Delta t)|_{t_1}^{t_2}$$

Weiss form of the variation

Defining $\delta x^\mu := \xi^\mu \Delta t$, total change in field is:

$$\Delta \varphi^a := (\varphi^a + \delta \varphi^a)|_{\partial \mathcal{U}'} - \varphi^a|_{\partial \mathcal{U}} = [\delta \varphi^a + \mathcal{L}_\xi \varphi^a \Delta t]_{\partial \mathcal{U}}$$

Weiss variation is:

$$\Delta S = \int_{\mathcal{U}} d\underline{\mu} \mathcal{E}_a[\varphi] \delta \varphi^a + \int_{\partial \mathcal{U}} d\underline{\Sigma}_\mu [\pi_a^\mu \Delta \varphi^a - (\Theta^\mu{}_\nu \xi^\nu + S^\mu) \Delta t]$$

Canonical energy-momentum

$$\Theta^\mu{}_\nu := \pi_a^\mu \partial_\nu \varphi^a - \delta^\mu{}_\nu \mathcal{L}$$

"Belinfante-Rosenfeld" vector

$$S^\mu := \pi_a^\mu [\mathcal{L}_\xi \varphi^a - \xi^\nu \partial_\nu \varphi^a]$$

Einstein-Hilbert action and variation⁶

Einstein-Hilbert action for general relativity:

$$S_{EH}[g^{\cdot\cdot}] = \frac{1}{2\kappa} \int_U d\mu R \quad d\mu := d\underline{\mu} \sqrt{|g|} = d^4x \sqrt{|g|}$$

Full variation takes the form ($d\Sigma_\mu := d\underline{\Sigma}_\mu \sqrt{|g|}$):

$$\Delta S_{EH} = \frac{1}{2\kappa} \left\{ \int_U d\mu G_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial U} d\Sigma_\mu \left[\delta^{\mu\alpha}_{\nu\sigma} g^{\sigma\beta} \Delta \Gamma^\nu_{\alpha\beta} - [2G^\mu_\sigma \xi^\sigma - j^\mu] \Delta t \right] \right\}.$$

where $j^\mu := \nabla_\sigma (\nabla^\mu \xi^\sigma - \nabla^\sigma \xi^\mu)$ is Noether-Komar current, $\delta^{\mu\alpha}_{\nu\sigma} := \delta^\mu_\nu \delta^\alpha_\sigma - \delta^\mu_\sigma \delta^\alpha_\nu$.

⁶For the non-null boundary result, see: JCF and R.A. Matzner, Gen. Rel. Grav. 50, 99 (2018), arXiv:1708.04489 [gr-qc]

Usual conventions: $\kappa = 8\pi G$, R is Ricci scalar, $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is Einstein tensor, $g = \det(g_{..})$, $\Gamma^\alpha_{\mu\nu}$ are Christoffel symbols

Weiss variation of gravitational action for non-null boundaries

Problem: need to rewrite boundary term with $\Delta g^{\mu\nu}$ (don't want $\Delta \Gamma^\alpha_{\mu\nu}$)

Can do this for non-null boundary by adding the Gibbons-Hawking York term⁶ to action ($d\Sigma := d\Sigma_\nu n^\nu$):

$$S_{GHY}[g^{\cdot\cdot}] = \frac{1}{2\kappa} \int_U d\mu R + \frac{1}{\kappa} \int_{\partial U} d\Sigma K$$

with variation⁷ ($p_{\mu\nu} := K_{\mu\nu} - \gamma_{\mu\nu} K$, $\gamma_{\mu\nu} := g_{\mu\nu} - \varepsilon n_\mu n_\nu$)

$$\Delta S_{GHY} = \frac{1}{2\kappa} \left\{ \int_U d\mu G_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial U} d\Sigma \left[p_{\mu\nu} \Delta \gamma^{\mu\nu} - \xi^\mu \left[2D_\alpha p^{\alpha\beta} \gamma_{\mu\beta} - n_\mu \left({}^3R - \varepsilon (K^2 - K_{\alpha\beta} K^{\alpha\beta}) \right) \right] \Delta t \right] \right\}.$$

Hamiltonian proportional to constraints, which vanish on solutions of Einstein eqs.

⁶For null boundary GHY term, see: K. Parattu, S. Chakraborty, B. R. Majhi, and T. Padmanabhan, Gen. Rel. Grav. 48, 94 (2016), 1501.01053

⁷JCF and R.A. Matzner, Gen. Rel. Grav. 50, 99 (2018), arXiv:1708.04489 [gr-qc]

Weiss variation with a different boundary term⁸

Can add a different boundary term, with $\bar{\Gamma}^\lambda_{\mu\nu}$ being nondynamical:^{9,10}

$$S_{gW} = \frac{1}{2\kappa} \int_U d\mu [R - \nabla_\mu W^\mu], \quad W^\mu := W^{\mu\nu}{}_\nu - W_\nu{}^{\nu\mu}, \quad W^\lambda{}_{\mu\nu} := \Gamma^\lambda{}_{\mu\nu} - \bar{\Gamma}^\lambda{}_{\mu\nu}$$

Variation is

$$\Delta S_{gW} = \frac{1}{2\kappa} \left\{ \int_U d\mu G_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial U} d\Sigma_\mu \left[P^\mu{}_{\sigma\beta} \Delta g^{\sigma\beta} + \delta \bar{\Gamma}^\mu - \left[({}_E\Theta^\mu{}_\beta - \sigma^\mu{}_\beta) \xi^\beta - \Sigma^{\mu\alpha}{}_\beta \bar{\nabla}_\alpha \xi^\beta \right] \Delta t \right] \right\}$$

$$P^\sigma{}_{\mu\nu} := -\delta^{\sigma\alpha}{}_{\rho(\mu} W^\rho{}_{\alpha|\nu)} + \frac{1}{2} W^\sigma g_{\mu\nu},$$

$$\delta \bar{\Gamma}^\mu := \delta^{\mu\alpha}{}_{\nu\sigma} \delta \bar{\Gamma}^\nu{}_{\alpha\beta}, \quad \sigma^\mu{}_\beta \propto \bar{\Gamma}^\sigma{}_{[\mu\nu]}, \quad \Sigma^{\mu\alpha}{}_\beta = \Sigma^\mu{}_\beta{}^\alpha \propto W^\sigma{}_{\mu\nu}$$

The tensor ${}_E\Theta^\alpha{}_\beta$ generalizes Einstein pseudotensor:⁹

$${}_E\Theta^\alpha{}_\beta = (W_\tau{}^{\tau\alpha} - W^{\alpha\tau}{}_\tau) W^\sigma{}_{\beta\sigma} - W^\sigma{}_{\sigma\tau} (W^\alpha{}_\beta{}^\tau + W^\tau{}_\beta{}^\alpha) + W^{\alpha\sigma\tau} (W_{\sigma\beta\tau} + W_{\tau\beta\sigma}) - \delta^\alpha{}_\beta (R - \nabla_\sigma W^\sigma)$$

⁸JCF, S. Chakraborty, Gen. Rel. Grav. 54, 67 (2022) arXiv:2111.06897 [gr-qc]

⁹D. Lynden-Bell, J. Katz, and J. Bicak, MNRAS 272,150 (1995); J. Katz, J. Bičák, and D. Lynden-Bell, Phys. Rev. D 55, 5957 (1997)

¹⁰J Harada Phys. Rev. D 101, 024053 (2020) arXiv:2001.06990

Symmetric reference

For a flat, Levi-Civita connection $(\bar{\Gamma}^\lambda_{\mu\nu}, \bar{\nabla})$ for metric $\bar{g}_{\mu\nu}$, and a covariantly constant vector ξ^μ satisfying $\bar{\nabla}_\nu \xi^\mu = 0$:

$$\Delta S_{gW} = \frac{1}{2\kappa} \left\{ \int_U d\mu G_{\mu\nu} \delta g^{\mu\nu} + \int_{\partial U} d\Sigma_\mu \left[P^\mu_{\alpha\beta} \Delta g^{\alpha\beta} - {}_E \Theta^\mu_{\beta} \xi^\beta \Delta t \right] \right\}.$$

Using (Katz-Ori) identity,^{8,9} can obtain tensor satisfying conservation law:

$$\Pi^\sigma_{\tau} := \sqrt{g/\bar{g}} (2G^\mu_{\nu} - {}_E \Theta^\mu_{\nu}), \quad \bar{\nabla}_\sigma (\Pi^\sigma_{\tau} \xi^\tau) = \xi^\tau \bar{\nabla}_\sigma \Pi^\sigma_{\tau} = 0$$

Has two features that make it potentially useful:

1. Using $G_{\mu\nu} = \kappa T_{\mu\nu}$, can separate into matter and gravitational parts
2. Satisfies an exact conservation law.

⁸D. Lynden-Bell, J. Katz, and J. Bicak, MNRAS 272,150 (1995)

⁹J. Katz, J. Bičák, and D. Lynden-Bell, Phys. Rev. D 55, 5957 (1997)

Does this define local gravitational energy? (Not really)

- Many attempts to define energy in GR¹¹ but no reasonable local defn. found
- Regarding $\Pi^\sigma{}_\tau$:
 - Pseudotensor ambiguities \Rightarrow ambiguity of coordinates for $\bar{\Gamma}^\lambda{}_{\mu\nu}$
- Studied energy density $e = \Pi^\sigma{}_\tau \xi^\tau n_\sigma$ for various coordinate choices for Schwarzschild metric:
 - Get trivial or pathological energy density in many cases, get sensible result for one choice of harmonic coordinates, but not others
- Can obtain a quasilocal energy from superpotential and reference choice.^{12,13}

¹¹L. B. Szabados, Liv. Rev. Relativ. 12:4 (2009); S. De Haro, arxiv:2103.17160; S. Aoki, T. Onogi, S. Yokoyama, Int.J.Mod.Phys.A 36 (2021) 29, 2150201.

¹²C.-M. Chen, J.-L. Liu, J. M. Nester, Int. J. Mod.Phys. D27, 1847017 (2018) arxiv:1805.07692

¹³C.-M. Chen, J.-L. Liu, J. M. Nester, Gen. Rel. Grav. 50, 158 (2018) arxiv:1811.05640

Einstein-Schrödinger equation

Formal path integral for quantum GR (QGR):

$$\mathcal{A}[\dot{g}^{\cdot\cdot}] = \int \mathcal{D}g^{\cdot\cdot} e^{(i/\hbar)S_{gW}[g^{\cdot\cdot}]}$$

With Ehrenfest theorem, variation is:

$$\Delta \mathcal{A} = \frac{i}{2\kappa\hbar} \left\{ \int_{\partial U} d\underline{\Sigma} \left[\hat{P}_{\alpha\beta} \Delta g^{\alpha\beta} - \hat{\mathcal{H}}_{\nu} \xi^{\nu} \Delta t \right] \right\}$$

$$\hat{P}_{\alpha\beta} \mathcal{A} = \int \mathcal{D}g^{\cdot\cdot} \sqrt{-\dot{g}} n_{\mu} P^{\mu}_{\alpha\beta} e^{(i/\hbar)S_{gW}[g^{\cdot\cdot}]} \quad \hat{\mathcal{H}}_{\nu} \mathcal{A} = \int \mathcal{D}g^{\cdot\cdot} \sqrt{-\dot{g}} n_{\mu} E \Theta^{\mu}_{\nu} e^{(i/\hbar)S_{gW}[g^{\cdot\cdot}]}$$

Similarly to the case in mechanics, one obtains a Schrödinger equation:

$$i\hbar \frac{\partial \mathcal{A}}{\partial t} = \hat{H} \mathcal{A} \quad \hat{H} \mathcal{A} := \frac{1}{2\kappa} \int_{\partial U} d\underline{\Sigma} \xi^{\nu} \hat{\mathcal{H}}_{\nu} \mathcal{A}$$

Does this solve problem of time in QGR? (Not really)

- Problem of time in quantum GR (QGR):
 - In usual 3+1 GR with Gibbons-Hawking-York term, Ham. density \mathcal{H} vanishes---one obtains the Wheeler-DeWitt equation¹² $\hat{\mathcal{H}}\Psi[\gamma^\cdot] = 0$
- Einstein-Schrödinger Eq. naively provides time evolution, but...
 - Hamiltonian \hat{H} is ambiguous---depends on coords. for reference
 - Bad coordinate choices for reference geometry can lead to pathologies
- Might have use as a formal tool in certain contexts (e.g. perturbation theory)

¹³JCF and R. A. Matzner, Phys. Rev. D96, 106005(2017) arXiv:1708.07001.

Summary and Prospects

- Weiss var. unifies derivation of many results in classical mechanics
- Can obtain suggestive results in GR, QGR, and field theory
 - Canonical EMT, currents, Schrödinger/WDW Eqs.
 - Can get Hamiltonian without explicit 3+1 split (useful in mod. grav.)
- May lead to better understanding of energy in GR
 - Sharpens understanding of pseudotensorial ambiguities, at least
 - Can we find a definition that can be cleanly exchanged and satisfies strong conservation laws?

¹³H. Friedrich, Classical and Quantum Gravity, 20(1):101–117, Dec 2002

¹⁴JCF, S. Mukohyama, S. Carloni, Phys. Rev. D 105, 104036 (2022)

Belinfante-Rosenfeld Correction

Consider the covariant Weiss variation of some matter action $S_M = \int d\mu \mathcal{L}(\varphi, d\varphi)$:

$$\Delta S_M = \int_U d\mu \mathcal{E}_I \delta\varphi^I + \int_{\partial U} d\Sigma_\sigma [\mathcal{L} \delta x^\sigma + \pi_I^\sigma \delta\varphi^I]$$

Define:

$$\Delta\varphi^I := \delta\varphi^I + \mathcal{L}_{\Delta\lambda\xi}\varphi^I, \quad \mathcal{L}_\xi\varphi^I := \xi^\sigma \nabla_\sigma \varphi^I + w_J^I \varphi^J, \quad \Theta^\mu{}_\nu := \pi_I^\mu \nabla_\nu \varphi^I - \delta^\mu{}_\nu \mathcal{L}$$

$$B_{\mu\nu}{}^\sigma \quad \text{such that} \quad P_I^\sigma w_J^I \varphi^J = \Delta\lambda \xi^\mu \nabla^\nu B_{\mu\nu}{}^\sigma - \nabla^\nu (\Delta\lambda \xi^\mu B_{\mu\nu}{}^\sigma),$$

One has ($\Theta^\mu{}_\nu := \pi_I^\mu \nabla_\nu \varphi^I - \delta^\mu{}_\nu \mathcal{L}$ being the canonical energy-momentum tensor):

$$\Delta S_M = \int_U d\mu \mathcal{E}_I \delta\varphi^I + \int_{\partial U} d\Sigma_\sigma [\pi_I^\sigma \Delta\varphi^I + \Delta\lambda \xi^\mu (\Theta^\sigma{}_\mu + \nabla^\nu B_{\mu\nu}{}^\sigma) + \nabla_\nu (\Delta\lambda \xi^\mu B_\mu{}^{\nu\sigma})]$$

If $B_\mu{}^{\nu\sigma} = -B_\mu{}^{\sigma\nu}$, last term is bdy. term; get Belinfante-Rosenfeld correction to $\Theta^\sigma{}_\mu$:

$$\bar{T}^\sigma{}_\mu := \Theta^\sigma{}_\mu + \nabla^\nu B_{\mu\nu}{}^\sigma$$

Higher derivative actions

Action:

$$S[q, \lambda] = \int_{t_1}^{t_2} L \left(q, \frac{dq}{dt}, \dots, \frac{d^n q}{dt^n}, \lambda, t \right) dt$$

Variation is:

$$\Delta S = \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial q} + \sum_{m=1}^n (-1)^m \frac{d^m \tilde{P}^m}{dt^m} \right) \cdot \delta q(t) dt + \left[\sum_{m=1}^n P^m \cdot \Delta Q_m - H \Delta t \right]_{t_1}^{t_2}$$

$$Q_m := \frac{d^{m-1} q}{dt^{m-1}} \quad P^a := \sum_{m=a}^n (-1)^{m-a} \frac{d^{m-a} \tilde{P}^m}{dt^{m-a}} \quad \tilde{P}^m := \frac{\partial L}{\partial (d^m q / dt^m)}$$

Ostrogradsky Hamiltonian: $H := (\sum_m P^m \cdot \dot{Q}_m) - L$.

Modified gravity: Higher curvature example

Consider a metric theory defined by the Lagrangian $\mathcal{L}_g(R \dots)$. Boundary terms are:

$$\begin{aligned} \int_{\partial U} d\Sigma_\mu \left(P^\mu_{\alpha\beta} \Delta g^{\alpha\beta} + P^\mu_{\nu}{}^{\alpha\beta} \Delta \Gamma^\nu_{\alpha\beta} - \Delta \lambda \left\{ 2P^\mu_{\alpha\beta} \nabla^{(\alpha} \xi^{\beta)} + P^\mu_{\nu}{}^{\alpha\beta} (R_{\alpha\sigma\beta}{}^\nu \xi^\sigma + \nabla_\alpha \nabla_\beta \xi^\nu) - \mathcal{L}_g \xi^\mu \right\} \right) \\ = \int_{\partial U} d\Sigma_\mu \left[P^\mu_{\alpha\beta} \Delta g^{\alpha\beta} + P^\mu_{\nu}{}^{\alpha\beta} \Delta \Gamma^\nu_{\alpha\beta} - \Delta \lambda \{ \Theta^\mu_{\nu} \xi^\nu + \nabla_\nu \omega^{\mu\nu} \} \right]. \end{aligned}$$

If $\nabla_\nu \omega^{(\mu\nu)} = 0$, you're done---otherwise, have three options:

1. Introduce flat auxiliary connection $\bar{\nabla}$, choose ξ^μ to satisfy $\bar{\nabla}_\mu \xi^\nu = 0$, then:

$$\nabla^{(\alpha} \xi^{\beta)} \propto \xi^\sigma, \quad \nabla_\alpha \nabla_\beta \xi^\nu \propto \xi^\sigma \quad (\text{lose some freedom to choose } \xi^\sigma)$$

2. Do 3+1 decomposition of $\nabla_\nu \omega^{(\mu\nu)}$ wrt to bdy surface (lose 4D covariance).

3. Add boundary terms to cancel $\nabla_\nu \omega^{(\mu\nu)}$ (depends on integrability)