Low-Rank Factorizations for Fast Inference in Structured Models

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Structured Models

- Explicitly model output associations
  - Directly or through latent variables
- Focus on combinatorially large latent discrete structures
  - Complementary to continuous, deterministic representations
- More difficult to scale than alternative representations
  - Bottlenecked by time and space complexity of inference

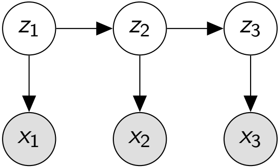
Scaling Structured Models

- Target hypergraph models
- Impose a low-rank model constraint
  - Trades off model expressivity for cheaper inference
- Only constrain parameters used in key steps of inference

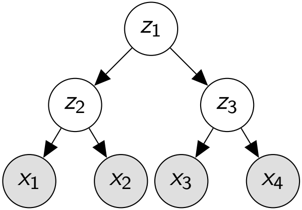
Problem Setup

- Model an observation $x = (x_1, \ldots, x_T)$ via latent structure z
- Perform training and evaluation via marginalization

$$p(x) = \sum_z p(x, z)$$



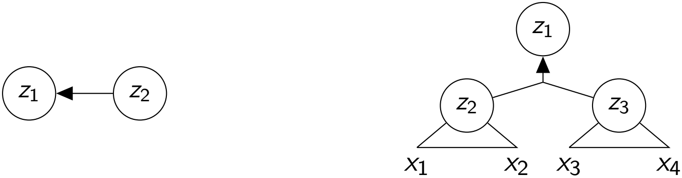
Hidden Markov models



Probabilistic context-free grammars

Hypergraphs for Inference

- Represent dynamic programs for inference as hypergraphs
- Propagate marginals along hyperedges
  - Hyperedge consists of a head nodes and tail nodes
  - Aggregate marginals from tails to head



Hyperedge representations for HMMs and PCFGs

Hypergraph Marginalization

- Combine tail marginals α_1, α_2 into joint tail marginal β_v
- Apply score matrix Ψ_e and aggregate in head marginal α_u
  - Multiple hyperedges may have the same head node

Algorithm 1 Hypergraph marginalization

for $u \leftarrow v$ hyperedge e topologically **do**

$\quad \beta_v \leftarrow \alpha_{v_1} \alpha_{v_2}^\top$ $\qquad \triangleright\ O(L^{|e|})$

$\quad \alpha_u \xleftarrow{+} \Psi_e \beta_v$ $\qquad \triangleright\ O(L^{|e|+1})$

return $\alpha_S^\top \mathbf{1}$

Low-rank Constraints

- Factor matrices $\Psi = UV^\top$, $U \in \mathbb{R}^{L \times R}$, $V \in \mathbb{R}^{L' \times R}$

$$\boxed{\Psi} \times \boxed{\beta} = \boxed{U} \times \left(\boxed{V^\top} \times \boxed{\beta} \right)$$

- Two matrix-vector products of cost $O(LR)$ and $O(L'R)$

Low-rank Hypergraph Marginalization

Algorithm 2 Low-rank marginalization

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for $u \leftarrow v_1, v_2$ hyperedge e topologically **do**

  $\beta_v \leftarrow \alpha_{v_1} \alpha_{v_2}^\top$ $\triangleright$ $O(L^{|e|})$

  $\gamma \leftarrow V_e^\top \beta_v$ $\triangleright$ $O(L^{|e|} R)$

  $\alpha_u \xleftarrow{+} U_e \gamma$ $\triangleright$ $O(LR)$

return $\alpha_S^\top \mathbf{1}$

Expressivity of Rank-constrained Models

- Rank constraints limit expressivity
- Only apply to a subset of parameters
  - Transition matrix for HMMs
  - Subset of the transition matrix for PCFGs
- Is it more expressive than a smaller model?
  - An L-state HMM with rank R ($< L$) is more expressive than an R-state HMM

Experiments

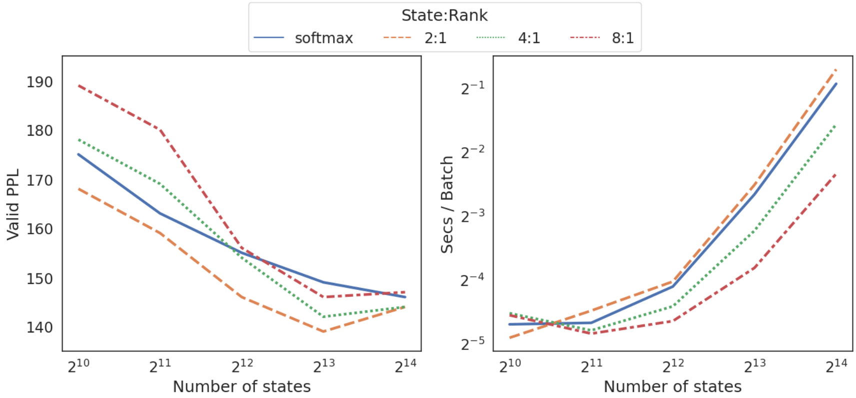
- Language modeling on Penn Treebank[1]
  - Compare size vs speed and accuracy
  - Softmax HMM and PCFG vs low-rank versions (LHMM, LPCFG)
  - Evaluate accuracy with perplexity, a function of likelihood

- Video modeling on CrossTask[2]
  - Scale state with fixed computation budget vs accuracy
  - Softmax HSMM vs low-rank HSMM
  - Evaluate accuracy with negative log-likelihood

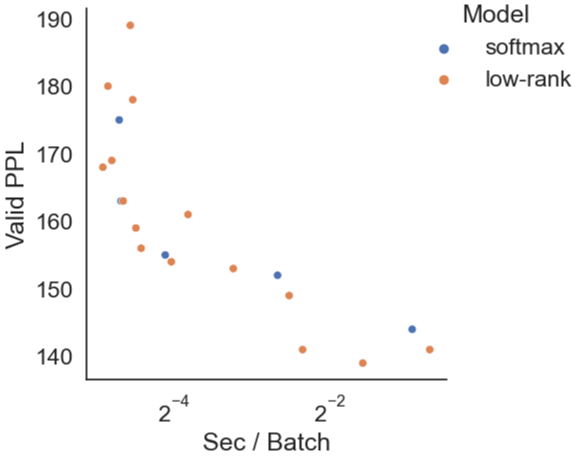
[1] Marcus, Santorini, and Marcinkiewicz, 'Building a Large Annotated Corpus of English: The Penn Treebank'.

[2] Zhukov et al., 'Cross-task weakly supervised learning from instructional videos'.

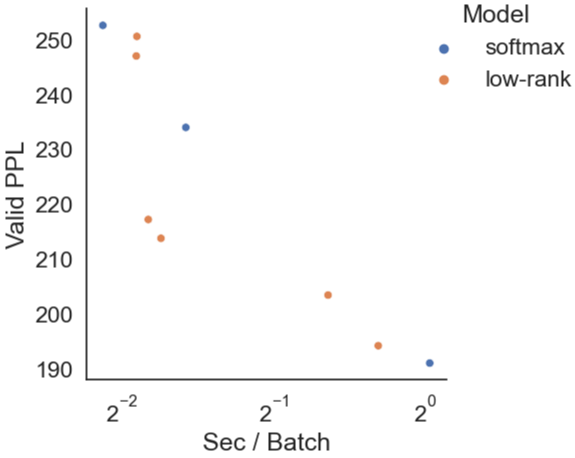
HMM Results


State:Rank
softmax
2:1
4:1
8:1
Valid PPL
190
180
170
160
150
140
2^{10}
2^{11}
2^{12}
2^{13}
2^{14}
Number of states
Secs / Batch
2^{-1}
2^{-2}
2^{-3}
2^{-4}
2^{-5}
2^{10}
2^{11}
2^{12}
2^{13}
2^{14}
Number of states

HMM Speed vs Accuracy Frontier



PCFG Speed vs Accuracy Frontier



HSMM Results

Model	L	N	NLL	Sec/Batch
HSMM	2^6	-	$1.428e5$	0.78
HSMM	2^7	-	$1.427e5$	2.22
HSMM	2^8	-	$1.426e5$	7.69
LHSMM	2^7	2^7	$1.427e5$	4.17
LHSMM	2^8	2^6	$1.426e5$	5.00
LHSMM	2^9	2^5	$1.424e5$	5.56
LHSMM	2^{10}	2^4	$1.423e5$	10.00

Conclusion

- Introduce a low-rank factorization to speed up inference
- Applies to models with hypergraph inference
- Most effective with large models

Citations I

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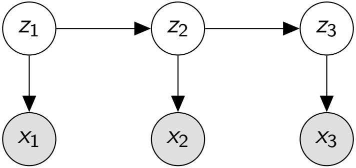
Inference as Matrix-Vector Products

- Inference: sequence of matrix-vector products
- Speed up via fast matvec methods
- Applies to a large family of structured models

Model 1: Hidden Markov Models (HMMs)

For times t, model states $z_t \in [L] = \mathcal{L}$, and tokens $x_t \in [X] = \mathcal{X}$,



We wish to maximize

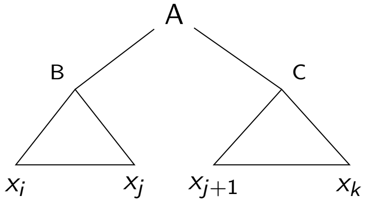
$$p(x) = \sum_{z_1} \cdots \sum_{z_T} p(x, z) = \mathbf{1}^\top \Psi_1 \Psi_2 \cdots \Psi_T \mathbf{1},$$

where

$$\begin{aligned} [\Psi_t]_{z_t, z_{t+1}} &= p(z_{t+1}, x_t \mid z_t) \\ [\Psi_1]_{z_1, z_2} &= p(z_2, x_1 \mid z_1) p(z_1) \end{aligned}$$

Model 2: Probabilistic Context-Free Grammars (PCFG)

Assign mass to each rule in a rewrite system



We wish to maximize

$$p(x) = \sum_{\text{tree:yield(tree)}=x} p(\text{tree})$$

Matvec Inference in PCFGs

For each rule define

$$[\Psi]_{z_u,(z_1,z_2)} = p(B = z_1, C = z_2 \mid A = z_u),$$

Algorithm 3 PCFG Inference

for $(i,k) \leftarrow (i,j),(j,k)$ in span-size order **do**
  for $z_1, z_2 \in \mathcal{L}_{i,j} \times \mathcal{L}_{j,k}$ **do**
    $[\beta_{i,j,k}]_{(z_1,z_2)} = [\alpha_{i,j}]_{z_1}[\alpha_{j,k}]_{z_2}$
  $\alpha_{i,k} \overset{+}{\leftarrow} \Psi_{i,j,k}\beta_{i,j,k}$
return $\alpha_{1,T}^{\top}\mathbf{1}$

Low-Rank Factorization

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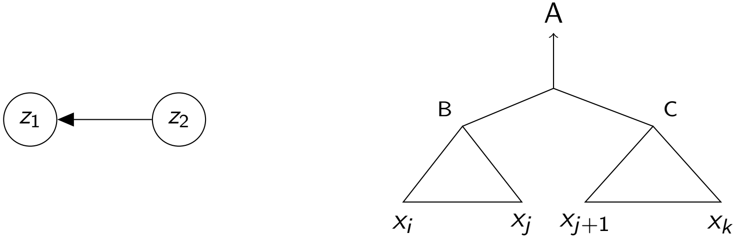
Hypergraph Marginalization

- Models where exact inference is a directed acyclic hypergraph
- Hypergraph contains a node set and hyperedge set
  - Nodes have label set $\mathcal{L}$
  - Hyperedges join a single head node u and a list of tail nodes v



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Hypergraph Marginalization Algorithms

Algorithm 4 Hypergraph marginalization

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Expressiveness and Generality

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- Only apply to a subset of parameters
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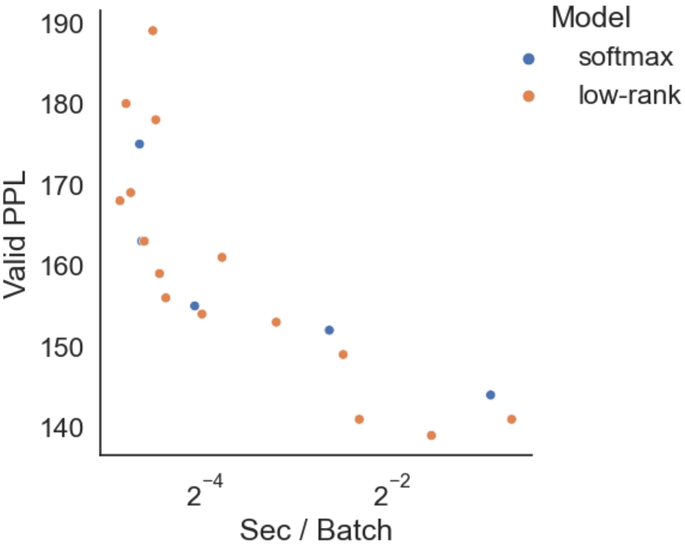
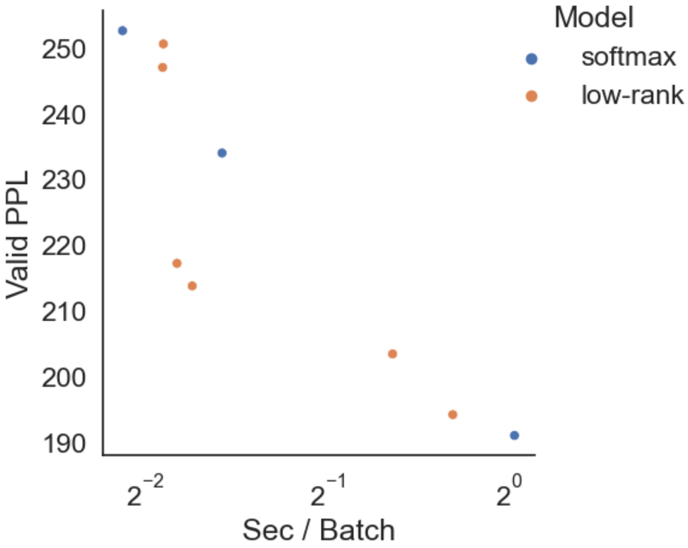


PCFG Speed vs Accuracy



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HMM Music Results

Model	Nott	Piano	Muse	JSB
RNN-NADE	2.31	**7.05**	**5.6**	5.19
R-Transformer	**2.24**	7.44	7.00	8.26
LSTM	3.43	7.77	7.23	8.17
LV-RNN	2.72	7.61	6.89	**3.99**
SRNN	2.94	8.20	6.28	4.74
TSBN	3.67	7.89	6.81	7.48
HMM	2.43	8.51	7.34	5.74
LHMM	2.60	8.89	7.60	5.80

PCFG Results

$\lvert\mathcal{N}\rvert$	$\lvert\mathcal{P}\rvert$	Model	N	PPL	Sec/Batch
30	60	PCFG	-	252.60	0.29
		LPCFG	8	247.02	0.27
		LPCFG	16	250.59	0.27
60	120	PCFG	-	234.01	0.33
		LPCFG	16	217.24	0.28
		LPCFG	32	213.81	0.30
100	200	PCFG	-	191.08	1.02
		LPCFG	32	203.47	0.64
		LPCFG	64	194.25	0.81

HSMM Speed vs Accuracy