# Low-Rank Constraints for Fast Inference in Structured Models

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# Structured Models

- Explicitly model output associations
  - Directly or through latent variables
- Focus on combinatorially large latent discrete structures
  - Complementary to continuous, distributed representations

# Scaling Structured Models

- Prior work demonstrated: Size ⇧ Performance ⇧
  - Hidden Markov Models (HMM)
  - Probabilistic Context-Free Grammars (PCFG)

- Prior work scaled via
  - Sparsity for HMMs[1]
  - Low-rank tensor decompositions for PCFGs[2]

- This work: low-rank matrix constraints
  - More general
  - Less speedup

[1] Chiu and Rush, *Scaling Hidden Markov Language Models*.

[2] Yang, Zhao, and Tu, 'PCFGs Can Do Better: Inducing Probabilistic Context-Free Grammars with Many Symbols'.

# Inference as Matrix-Vector Products

- Inference: sequence of matrix-vector products
- Speed up via fast matvec methods
- Applies to a large family of structured models

# Fast Matrix-Vector Products

- Matvecs take $O(L^2)$ computation
- Various fast methods
  - Sparsity (nnz entries)
  - Fast Fourier Transform ($L \log L$)
  - Low-Rank factorization ($LR$)
- Connected to efficient attention and kernel approximations[3]

[3] Choromanski et al., *Rethinking Attention with Performers*; Peng et al., *Random Feature Attention*; Blanc and Rendle, *Adaptive Sampled Softmax with Kernel Based Sampling*.

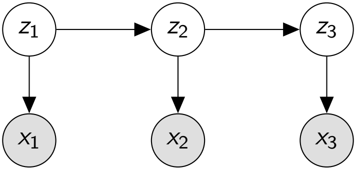
# Roadmap

- Inference in HMMs and PCFGs as matvecs
- Low-rank matvec inference
- Generalization to hypergraph inference
- Experiments

# Inference as Matvecs

# Hidden Markov Models (HMMs)

For times $t$, model states $z_t \in [L] = \mathcal{L}$, and tokens $x_t \in [X] = \mathcal{X}$,




with joint distribution

$$p(x, z) = \prod_t p(x_t \mid z_t) p(z_t \mid z_{t-1})$$

# Inference in HMMs

Given observed $x = (x_1, \ldots, x_T)$, we wish to maximize

$$p(x) = \sum_{z_1} \cdots \sum_{z_T} p(x, z) = \mathbf{1}^\top \Psi_1 \Psi_2 \cdots \Psi_T \mathbf{1},$$

where

$$\begin{aligned} [\Psi_t]_{z_t, z_{t+1}} &= p(z_{t+1}, x_t \mid z_t) \\ [\Psi_1]_{z_1, z_2} &= p(z_2, x_1 \mid z_1) p(z_1) \end{aligned}$$

# Matvec Inference in HMMs

**Algorithm** HMM Inference

**for** $t \leftarrow (t+1)$ in right-to-left order **do**

$\beta_t \stackrel{+}{\leftarrow} \boldsymbol{\Psi}_t \beta_{t+1}$

**return** $\beta_0^\top \mathbf{1}$

# Probabilistic Context-Free Grammars (PCFG)

A context-free grammar $\mathcal{G} = (\mathcal{L}, \mathcal{R})$ where

$$\mathcal{L}: \text{ Label symbols;} \qquad \mathcal{X}: \text{ Tokens;} \qquad \mathcal{R}: \text{ Rules,}$$

where rules take the form

$$\begin{aligned} A &\to B\ C, & A, B, C &\in \mathcal{L} \\ P &\to x, & P \in \mathcal{L}, x &\in \mathcal{X} \end{aligned}$$

In a PCFG, each rule has probability mass

$$p(r) = p(B, C \mid A)$$

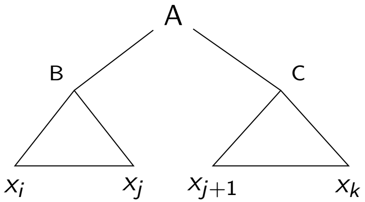
The joint distribution over rules in a tree $t$

$$p(t) = \prod_{r \in t} p(r)$$

# Inference in PCFGs

- For a given observation $x$, compute $p(x) = \sum_{t:\text{yield}(t)=x} p(t)$ via dynamic programming

- For each span $(i, k)$, sum over split point $j \in (i, k)$:




- Similar to HMMs, define

$$[\Psi]_{z_u,(z_1,z_2)} = p(B = z_1, C = z_2 \mid A = z_u),$$

for each rule

# Matvec Inference in PCFGs

**Algorithm** PCFG Inference

**for** $(i,k) \leftarrow (i,j),(j,k)$ in span-size order **do**
  **for** $z_1, z_2 \in \mathcal{L}_{i,j} \times \mathcal{L}_{j,k}$ **do**
    $[\beta_{i,j,k}]_{(z_1,z_2)} = [\alpha_{i,j}]_{z_1}[\alpha_{j,k}]_{z_2}$
  $\alpha_{i,k} \xleftarrow{+} \Psi_{i,j,k}\beta_{i,j,k}$
**return** $\alpha_{1,T}^{\top}\mathbf{1}$

# Speeding Up Inference

# Low-Rank Factorization

- Factor matrices $\Psi \in \mathbb{R}^{L \times L}$ into product of $U, V \in \mathbb{R}^{L \times R}$

$$\boxed{\Psi} \times \boxed{\beta} = \boxed{U} \times \left( \boxed{V^\top} \times \boxed{\beta} \right)$$

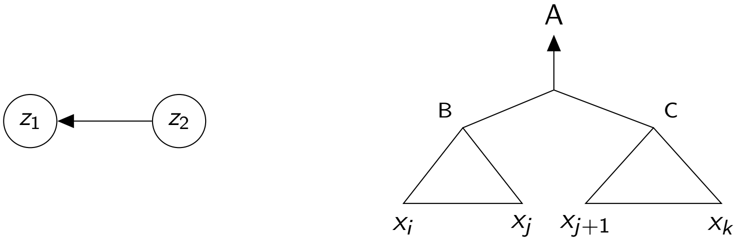
- Two matrix-vector products of cost $O(LR)$ each
- Also holds for rectangular $\Psi$

# Expressiveness and Generality

- Rank constraints limit expressivity
- Replace $\Psi = UV^\top$ for a subset of parameter matrices
  - Transition matrix for HMMs
  - Subset of the transition matrix for PCFGs
- An $L$-state HMM with rank $R$ ($< L$) is more expressive than an $R$-state HMM
- What other models does this work for?

# Hypergraph Marginalization

- Models where exact inference is a directed acyclic hypergraph
- Hypergraph contains a node set and hyperedge set
  - Nodes have label set $\mathcal{L}$
  - Hyperedges join a single head node $u$ and a list of tail nodes $v$




Hyperedge representations for HMMs and PCFGs

# Hypergraph Marginalization Algorithms

**Algorithm** Hypergraph marginalization

**for** $u \leftarrow v$ hyperedge $e$ topologically **do**

  $\beta_v \leftarrow \alpha_{v_1} \alpha_{v_2}^\top$  $\triangleright\ O(L^{|e|})$

  $\alpha_u \xleftarrow{+} \Psi_e \beta_v$  $\triangleright\ O(L^{|e|+1})$

**return** $\alpha_S^\top \mathbf{1}$

**Algorithm** Low-rank marginalization

**for** $u \leftarrow v_1, v_2$ hyperedge $e$ topologically **do**

  $\beta_v \leftarrow \alpha_{v_1} \alpha_{v_2}^\top$  $\triangleright\ O(L^{|e|})$

  $\gamma \leftarrow V_e^\top \beta_v$  $\triangleright\ O(L^{|e|} R)$

  $\alpha_u \xleftarrow{+} U_e \gamma$  $\triangleright\ O(LR)$

**return** $\alpha_S^\top \mathbf{1}$

# Experiments

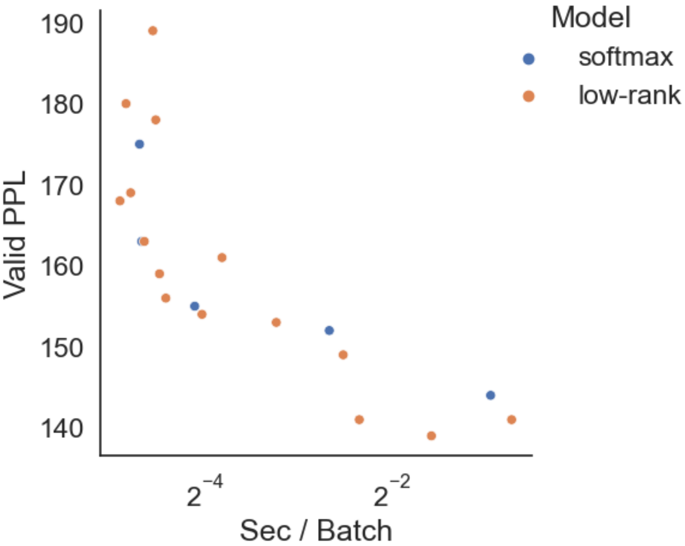
# Experiments

- Compare speed vs accuracy frontier
- Language modeling on PENN TREEBANK[4]
  - Softmax HMM and PCFG vs low-rank versions (LHMM, LPCFG)
  - Evaluate accuracy with perplexity, a function of likelihood
- Video modeling on CROSSTASK[5]
  - Softmax HSMM vs low-rank HSMM
  - Evaluate accuracy with negative log-likelihood

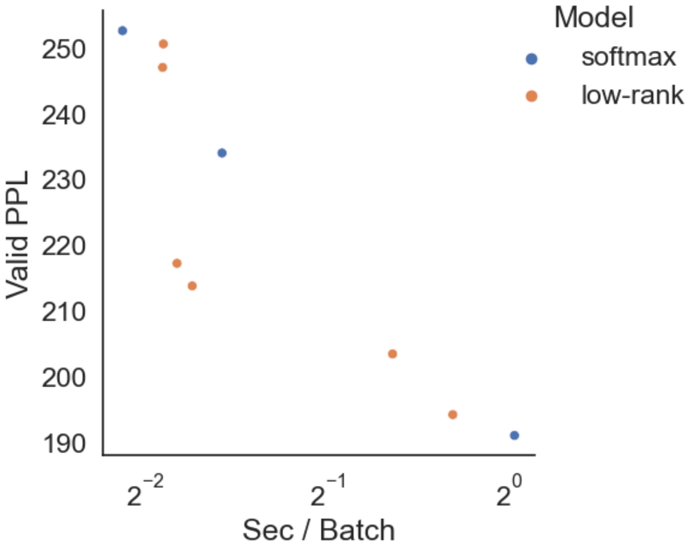
[4] Marcus, Santorini, and Marcinkiewicz, 'Building a Large Annotated Corpus of English: The Penn Treebank'.

[5] Zhukov et al., 'Cross-task weakly supervised learning from instructional videos'.

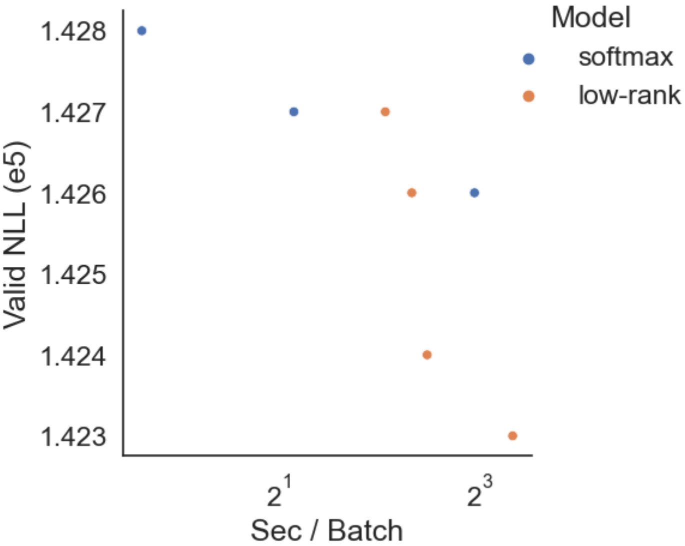
# HMM Speed vs Accuracy



# PCFG Speed vs Accuracy



# HSMM Speed vs Accuracy



# Conclusion

- Introduce a low-rank constraint to speed up inference
- Applies to models with hypergraph inference
- Most effective with large models

# Citations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# HMM Music Results

| Model | Nott | Piano | Muse | JSB |
|---|---|---|---|---|
| RNN-NADE | 2.31 | **7.05** | **5.6** | 5.19 |
| R-Transformer | **2.24** | 7.44 | 7.00 | 8.26 |
| LSTM | 3.43 | 7.77 | 7.23 | 8.17 |
| LV-RNN | 2.72 | 7.61 | 6.89 | **3.99** |
| SRNN | 2.94 | 8.20 | 6.28 | 4.74 |
| TSBN | 3.67 | 7.89 | 6.81 | 7.48 |
| HMM | 2.43 | 8.51 | 7.34 | 5.74 |
| LHMM | 2.60 | 8.89 | 7.60 | 5.80 |