

# Low-Rank Factorizations for Fast Inference in Structured Models

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# Structured Models

- ▶ Explicitly model output associations
  - ▶ Directly or through latent variables
- ▶ Focus on combinatorially large latent discrete structures
  - ▶ Complementary to continuous, distributed representations

# Scaling Structured Models

- ▶ Prior work demonstrated: Size  Performance 
  - ▶ Hidden Markov Models (HMM)<sup>1</sup>
  - ▶ Probabilistic Context-Free Grammars (PCFG)<sup>2</sup>
- ▶ This work: low-rank matrix factorizations
  - ▶ Generalize to hypergraph models

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<sup>1</sup>Dedieu et al., *Learning higher-order sequential structure with cloned HMMs*; Chiu and Rush, *Scaling Hidden Markov Language Models*.

<sup>2</sup>Yang, Zhao, and Tu, 'PCFGs Can Do Better: Inducing Probabilistic Context-Free Grammars with Many Symbols'.

# Inference as Matrix-Vector Products

- ▶ Inference: sequence of matrix-vector products
- ▶ Speed up via fast matvec methods
- ▶ Applies to a large family of structured models

# Fast Matrix-Vector Products

- ▶ Matvecs take  $O(L^2)$  computation
- ▶ Various fast methods
  - ▶ Sparsity (nnz entries)
  - ▶ Fast Fourier Transform ( $L \log L$ )
  - ▶ Low-Rank factorization ( $LR$ )
- ▶ Connected to efficient attention and kernel approximations<sup>3</sup>

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<sup>3</sup>Choromanski et al., *Rethinking Attention with Performers*; Peng et al., *Random Feature Attention*; Blanc and Rendle, *Adaptive Sampled Softmax with Kernel Based Sampling*.

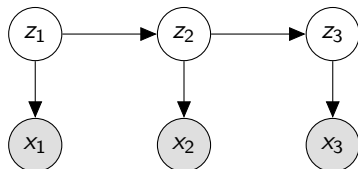
# Roadmap

- ▶ Inference in HMMs and PCFGs as matvecs
- ▶ Low-rank matvec inference
- ▶ Generalization to hypergraph inference
- ▶ Experiments

## Inference as Matvecs

# Model 1: Hidden Markov Models (HMMs)

For times  $t$ , model states  $z_t \in [L] = \mathcal{L}$ , and tokens  $x_t \in [X] = \mathcal{X}$ ,



with joint distribution

$$p(x, z) = \prod_t p(x_t \mid z_t) p(z_t \mid z_{t-1})$$



# Inference in HMMs

Given observed  $x = (x_1, \dots, x_T)$ , we wish to maximize

$$p(x) = \sum_{z_1} \cdots \sum_{z_T} p(x, z) = \mathbf{1}^\top \psi_1 \psi_2 \cdots \psi_T \mathbf{1},$$

where

$$\begin{aligned} [\psi_t]_{z_t, z_{t+1}} &= p(z_{t+1}, x_t \mid z_t) \\ [\psi_1]_{z_1, z_2} &= p(z_2, x_1 \mid z_1) p(z_1) \end{aligned}$$

# Matvec Inference in HMMs

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**Algorithm 1** HMM Inference

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**for**  $t \leftarrow (t + 1)$  in right-to-left order **do**

$$\beta_t \stackrel{+}{\leftarrow} \Psi_t \beta_{t+1}$$

**return**  $\beta_0^\top \mathbf{1}$

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## Model 2: Probabilistic Context-Free Grammars (PCFG)

A context-free grammar  $\mathcal{G} = (\mathcal{L}, \mathcal{R})$  where

$\mathcal{L}$  : Node labels;       $\mathcal{X}$  : Tokens;       $\mathcal{R}$  : Rules,

where rules take the form

$$\begin{array}{ll} A \rightarrow B C, & A, B, C \in \mathcal{L} \\ P \rightarrow x, & P \in \mathcal{L}, x \in \mathcal{X} \end{array}$$

In a PCFG, each rule has probability mass

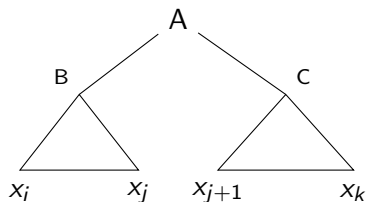
$$p(r) = p(B, C \mid A)$$

The joint distribution over rules in a tree  $t$

$$p(t) = \prod_{r \in t} p(r)$$

# Inference in PCFGs

- ▶ For a given observation  $x$ , compute  $p(x) = \sum_{t: \text{yield}(t)=x} p(t)$  via dynamic programming
- ▶ For each span  $(i, k)$ , sum over split points  $j \in (i, k)$ :



# Matvec Inference in PCFGs

For each rule define

$$[\Psi]_{z_u, (z_1, z_2)} = p(B = z_1, C = z_2 \mid A = z_u),$$

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**Algorithm 2** PCFG Inference

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for  $(i, k) \leftarrow (i, j), (j, k)$  in span-size order do  
  for  $z_1, z_2 \in \mathcal{L}_{i,j} \times \mathcal{L}_{j,k}$  do  
     $[\beta_{i,j,k}]_{(z_1, z_2)} = [\alpha_{i,j}]_{z_1} [\alpha_{j,k}]_{z_2}$   
     $\alpha_{i,k} \stackrel{+}{\leftarrow} \Psi_{i,j,k} \beta_{i,j,k}$   
return  $\alpha_{1,T}^\top \mathbf{1}$ 
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# Speeding Up Inference

# Low-Rank Factorization

- ▶ Factor matrices  $\Psi = UV^T$ ,  $U \in \mathbb{R}^{L \times R}$ ,  $V \in \mathbb{R}^{L' \times R}$

$$\boxed{\Psi} \times \boxed{\beta} = \boxed{U} \times \left( \boxed{V^T} \times \boxed{\beta} \right)$$

- ▶ Two matrix-vector products of cost  $O(LR)$  and  $O(L'R)$

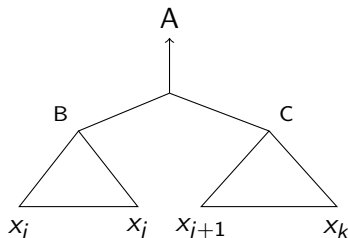
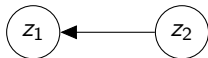
# Expressiveness and Generality

- ▶ Rank constraints limit expressivity
- ▶ Only apply to a subset of parameters
  - ▶ Transition matrix for HMMs
  - ▶ Subset of the transition matrix for PCFGs
- ▶ Is it more expressive than a smaller model?
  - ▶ An  $L$ -state HMM with rank  $R$  ( $< L$ ) is more expressive than an  $R$ -state HMM
- ▶ What other models does this work for?
  - ▶ Hypergraph models!



# Hypergraph Marginalization

- ▶ Models where exact inference is a directed acyclic hypergraph
- ▶ Hypergraph contains a node set and hyperedge set
  - ▶ Nodes have label set  $\mathcal{L}$
  - ▶ Hyperedges join a single head node  $u$  and a list of tail nodes  $v$



Hyperedge representations for HMMs and PCFGs

# Hypergraph Marginalization Algorithms

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**Algorithm 3** Hypergraph marginalization

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**for**  $u \leftarrow v$  hyperedge  $e$  topologically **do**

$$\beta_v \leftarrow \alpha_{v_1} \alpha_{v_2}^\top \quad \triangleright O(L^{|e|})$$

$$\alpha_u \stackrel{+}{\leftarrow} \Psi_e \beta_v \quad \triangleright O(L^{|e|+1})$$

**return**  $\alpha_S^\top \mathbf{1}$

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**Algorithm 4** Low-rank marginalization

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**for**  $u \leftarrow v_1, v_2$  hyperedge  $e$  topologically **do**

$$\beta_v \leftarrow \alpha_{v_1} \alpha_{v_2}^\top \quad \triangleright O(L^{|e|})$$

$$\gamma \leftarrow V_e^\top \beta_v \quad \triangleright O(L^{|e|}R)$$

$$\alpha_u \stackrel{+}{\leftarrow} U_e \gamma \quad \triangleright O(LR)$$

**return**  $\alpha_S^\top \mathbf{1}$

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# Experiments

# Experiments

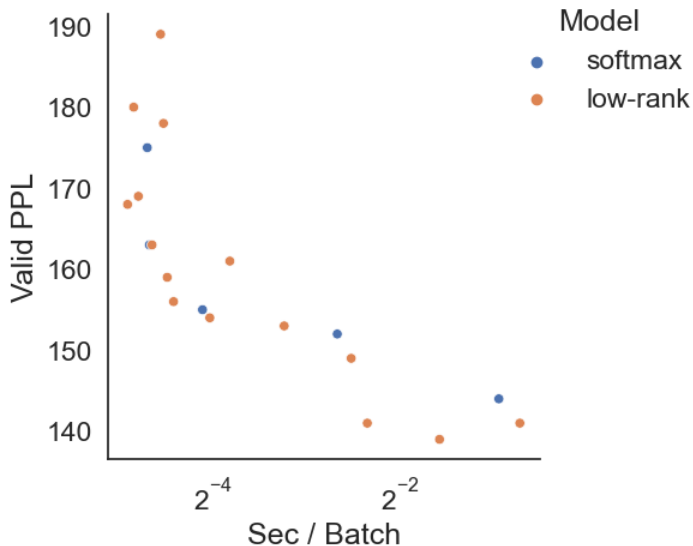
- ▶ Language modeling on PENN TREEBANK<sup>4</sup>
  - ▶ Compare speed vs accuracy frontier
  - ▶ Softmax HMM and PCFG vs low-rank versions (LHMM, LPCFG)
  - ▶ Evaluate accuracy with perplexity, a function of likelihood
- ▶ Video modeling on CROSSTASK<sup>5</sup>
  - ▶ Scale state size with fixed computation budget
  - ▶ Softmax HSMM vs low-rank HSMM
  - ▶ Evaluate accuracy with negative log-likelihood

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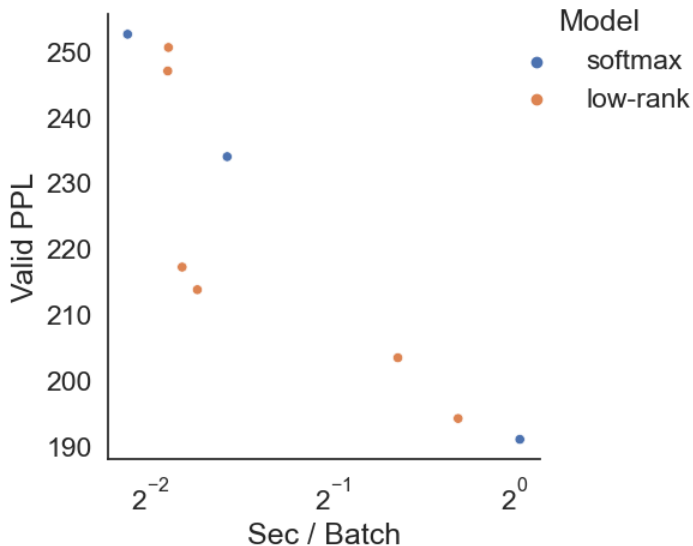
<sup>4</sup>Marcus, Santorini, and Marcinkiewicz, 'Building a Large Annotated Corpus of English: The Penn Treebank'.

<sup>5</sup>Zhukov et al., 'Cross-task weakly supervised learning from instructional videos'.

# HMM Speed vs Accuracy



# PCFG Speed vs Accuracy



# HSMM Results

Model	$L$	$N$	NLL	Sec/Batch
HSMM	$2^6$	-	1.428e5	0.78
HSMM	$2^7$	-	1.427e5	2.22
HSMM	$2^8$	-	1.426e5	7.69
LHSMM	$2^7$	$2^7$	1.427e5	4.17
LHSMM	$2^8$	$2^6$	1.426e5	5.00
LHSMM	$2^9$	$2^5$	1.424e5	5.56
LHSMM	$2^{10}$	$2^4$	1.423e5	10.00

# Conclusion

- ▶ Introduce a low-rank factorization to speed up inference
- ▶ Applies to models with hypergraph inference
- ▶ Most effective with large models



# Citations I

-  Blanc, Guy and Steffen Rendle. *Adaptive Sampled Softmax with Kernel Based Sampling*. 2018. arXiv: 1712.00527 [cs.LG].
-  Chiu, Justin T. and Alexander M. Rush. *Scaling Hidden Markov Language Models*. 2020. arXiv: 2011.04640 [cs.CL].
-  Choromanski, Krzysztof et al. *Rethinking Attention with Performers*. 2021. arXiv: 2009.14794 [cs.LG].
-  Dedieu, Antoine et al. *Learning higher-order sequential structure with cloned HMMs*. 2019. arXiv: 1905.00507 [stat.ML].
-  Marcus, Mitchell P., Beatrice Santorini, and Mary Ann Marcinkiewicz. 'Building a Large Annotated Corpus of English: The Penn Treebank'. In: *Computational Linguistics* 19.2 (1993), pp. 313–330. URL: <https://www.aclweb.org/anthology/J93-2004>.
-  Peng, Hao et al. *Random Feature Attention*. 2021. arXiv: 2103.02143 [cs.CL].

# Citations II



Yang, Songlin, Yanpeng Zhao, and Kewei Tu. 'PCFGs Can Do Better: Inducing Probabilistic Context-Free Grammars with Many Symbols'. In: *CoRR* abs/2104.13727 (2021). arXiv: 2104.13727. URL: <https://arxiv.org/abs/2104.13727>.



Zhukov, Dimitri et al. 'Cross-task weakly supervised learning from instructional videos'. In: *Proceedings of the IEEE/CVF Conference on Computer Vision and Pattern Recognition*. 2019, pp. 3537–3545.

# HMM Music Results

Model	Nott	Piano	Muse	JSB
RNN-NADE	2.31	<b>7.05</b>	<b>5.6</b>	5.19
R-Transformer	<b>2.24</b>	7.44	7.00	8.26
LSTM	3.43	7.77	7.23	8.17
LV-RNN	2.72	7.61	6.89	<b>3.99</b>
SRNN	2.94	8.20	6.28	4.74
TSBN	3.67	7.89	6.81	7.48
HMM	2.43	8.51	7.34	5.74
LHMM	2.60	8.89	7.60	5.80

## PCFG Results

$ \mathcal{N} $	$ \mathcal{P} $	Model	$N$	PPL	Sec/Batch
30	60	PCFG	-	252.60	0.29
		LPCFG	8	247.02	0.27
		LPCFG	16	250.59	0.27
60	120	PCFG	-	234.01	0.33
		LPCFG	16	217.24	0.28
		LPCFG	32	213.81	0.30
100	200	PCFG	-	191.08	1.02
		LPCFG	32	203.47	0.64
		LPCFG	64	194.25	0.81

# HSMM Speed vs Accuracy

