Low-Rank Factorizations for Fast Inference in Structured Models

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Structured Models

- Explicitly model output associations
  - Directly or through latent variables
- Focus on combinatorially large latent discrete structures
  - Complementary to continuous, distributed representations

Scaling Structured Models

- Prior work demonstrated: Size ⇧ Performance ⇧
  - Hidden Markov Models (HMM)[1]
  - Probabilistic Context-Free Grammars (PCFG)[2]
- This work: low-rank matrix factorizations
  - Generalize to hypergraph models

[1] Dedieu et al., *Learning higher-order sequential structure with cloned HMMs*; Chiu and Rush, *Scaling Hidden Markov Language Models*.

[2] Yang, Zhao, and Tu, 'PCFGs Can Do Better: Inducing Probabilistic Context-Free Grammars with Many Symbols'.

Inference as Matrix-Vector Products

- Inference: sequence of matrix-vector products
- Speed up via fast matvec methods
- Applies to a large family of structured models

Fast Matrix-Vector Products

- Matvecs take $O(L^2)$ computation
- Various fast methods
  - Sparsity (nnz entries)
  - Fast Fourier Transform ($L \log L$)
  - Low-Rank factorization (LR)
- Connected to efficient attention and kernel approximations[3]

[3] Choromanski et al., *Rethinking Attention with Performers*; Peng et al., *Random Feature Attention*; Blanc and Rendle, *Adaptive Sampled Softmax with Kernel Based Sampling*.

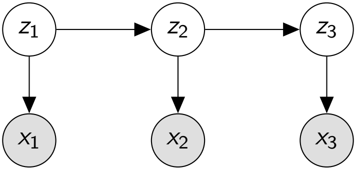
Roadmap

- Inference in HMMs and PCFGs as matvecs
- Low-rank matvec inference
- Generalization to hypergraph inference
- Experiments

Inference as Matvecs

Hidden Markov Models (HMMs)

For times t, model states $z_t \in [L] = \mathcal{L}$, and tokens $x_t \in [X] = \mathcal{X}$,



with joint distribution

$$p(x, z) = \prod_t p(x_t \mid z_t) p(z_t \mid z_{t-1})$$

Inference in HMMs

Given observed $x = (x_1, \dots, x_T)$, we wish to maximize

$$p(x) = \sum_{z_1} \cdots \sum_{z_T} p(x, z) = \mathbf{1}^\top \Psi_1 \Psi_2 \cdots \Psi_T \mathbf{1},$$

where

$$\begin{aligned} [\Psi_t]_{z_t, z_{t+1}} &= p(z_{t+1}, x_t \mid z_t) \\ [\Psi_1]_{z_1, z_2} &= p(z_2, x_1 \mid z_1) p(z_1) \end{aligned}$$

Matvec Inference in HMMs

Algorithm HMM Inference

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for $t \leftarrow (t+1)$ in right-to-left order **do**

$\beta_t \xleftarrow{+} \mathbf{\Psi}_t \beta_{t+1}$

return $\beta_0^\top \mathbf{1}$

Probabilistic Context-Free Grammars (PCFG)

A context-free grammar $\mathcal{G} = (\mathcal{L}, \mathcal{R})$ where

$$\mathcal{L}: \text{ Node labels;} \qquad \mathcal{X}: \text{ Tokens;} \qquad \mathcal{R}: \text{ Rules,}$$

where rules take the form

$$\begin{aligned} &A \to B\ C, && A, B, C \in \mathcal{L} \\ &P \to x, && P \in \mathcal{L}, x \in \mathcal{X} \end{aligned}$$

In a PCFG, each rule has probability mass

$$p(r) = p(B, C \mid A)$$

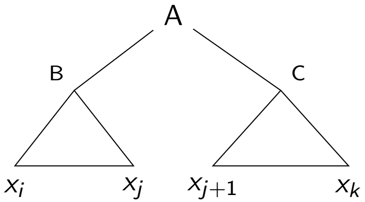
The joint distribution over rules in a tree t

$$p(t) = \prod_{r \in t} p(r)$$

Inference in PCFGs

- For a given observation x, compute $p(x) = \sum_{t:\text{yield}(t)=x} p(t)$ via dynamic programming

- For each span (i, k), sum over split point $j \in (i, k)$:



- Similar to HMMs, define

$$[\Psi]_{z_u,(z_1,z_2)} = p(B = z_1, C = z_2 \mid A = z_u),$$

for each rule

Matvec Inference in PCFGs

Algorithm PCFG Inference

for $(i, k) \leftarrow (i, j), (j, k)$ in span-size order **do**
  for $z_1, z_2 \in \mathcal{L}_{i,j} \times \mathcal{L}_{j,k}$ **do**
    $[\beta_{i,j,k}]_{(z_1,z_2)} = [\alpha_{i,j}]_{z_1}[\alpha_{j,k}]_{z_2}$
  $\alpha_{i,k} \xleftarrow{+} \Psi_{i,j,k}\beta_{i,j,k}$
return $\alpha_{1,T}^{\top}\mathbf{1}$

Speeding Up Inference

Low-Rank Factorization

- Factor matrices $\Psi = UV^\top$, $U \in \mathbb{R}^{L \times R}$, $V \in \mathbb{R}^{L' \times R}$

$$\boxed{\quad \Psi \quad} \times \boxed{\beta} = \boxed{U} \times \left(\boxed{\quad V^\top \quad} \times \boxed{\beta} \right)$$

- Two matrix-vector products of cost $O(LR)$ and $O(L'R)$

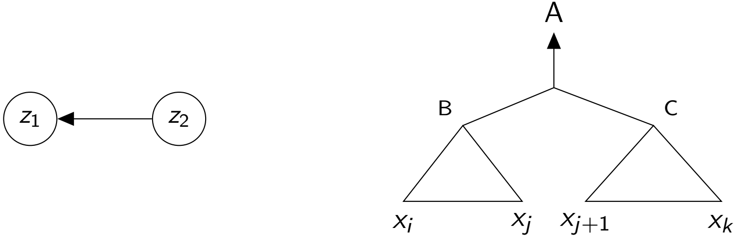
Expressiveness and Generality

- Rank constraints limit expressivity
- Replace $\Psi = UV^\top$ for a subset of parameter matrices
  - Transition matrix for HMMs
  - Subset of the transition matrix for PCFGs
- An L-state HMM with rank R ($< L$) is more expressive than an R-state HMM
- What other models does this work for?

Hypergraph Marginalization

- Models where exact inference is a directed acyclic hypergraph

- Hypergraph contains a node set and hyperedge set
  - Nodes have label set $\mathcal{L}$
  - Hyperedges join a single head node u and a list of tail nodes v



Hyperedge representations for HMMs and PCFGs

Hypergraph Marginalization Algorithms

Algorithm Hypergraph marginalization

for $u \leftarrow v$ hyperedge e topologically **do**

$\beta_v \leftarrow \alpha_{v_1}\alpha_{v_2}^\top$ $\qquad \triangleright\ O(L^{|e|})$

$\alpha_u \xleftarrow{+} \Psi_e \beta_v$ $\qquad \triangleright\ O(L^{|e|+1})$

return $\alpha_S^\top \mathbf{1}$

Algorithm Low-rank marginalization

for $u \leftarrow v_1, v_2$ hyperedge e topologically **do**

$\beta_v \leftarrow \alpha_{v_1}\alpha_{v_2}^\top$ $\qquad \triangleright\ O(L^{|e|})$

$\gamma \leftarrow V_e^\top \beta_v$ $\qquad \triangleright\ O(L^{|e|}R)$

$\alpha_u \xleftarrow{+} U_e \gamma$ $\qquad \triangleright\ O(LR)$

return $\alpha_S^\top \mathbf{1}$

Experiments

Experiments

- Compare speed vs accuracy frontier

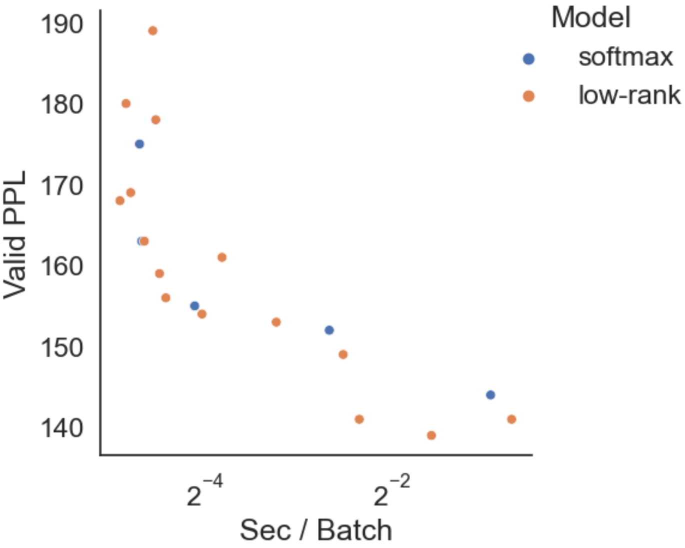
- Language modeling on PENN TREEBANK[4]
  - Softmax HMM and PCFG vs low-rank versions (LHMM, LPCFG)
  - Evaluate accuracy with perplexity, a function of likelihood

- Video modeling on CROSSTASK[5]
  - Softmax HSMM vs low-rank HSMM
  - Evaluate accuracy with negative log-likelihood

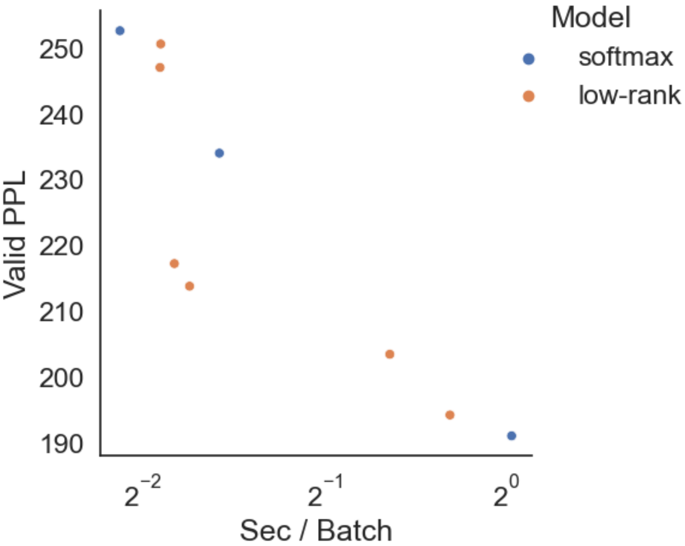
[4] Marcus, Santorini, and Marcinkiewicz, 'Building a Large Annotated Corpus of English: The Penn Treebank'.

[5] Zhukov et al., 'Cross-task weakly supervised learning from instructional videos'.

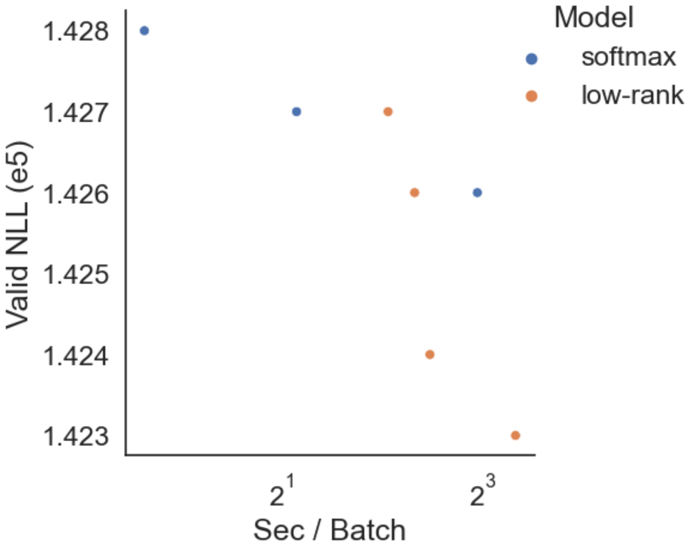
HMM Speed vs Accuracy



PCFG Speed vs Accuracy



HSMM Speed vs Accuracy



Conclusion

- Introduce a low-rank factorization to speed up inference
- Applies to models with hypergraph inference
- Most effective with large models

Citations

- Blanc, Guy and Steffen Rendle. *Adaptive Sampled Softmax with Kernel Based Sampling*. 2018. arXiv: 1712.00527 [cs.LG].
- Chiu, Justin T. and Alexander M. Rush. *Scaling Hidden Markov Language Models*. 2020. arXiv: 2011.04640 [cs.CL].
- Choromanski, Krzysztof et al. *Rethinking Attention with Performers*. 2021. arXiv: 2009.14794 [cs.LG].
- Dedieu, Antoine et al. *Learning higher-order sequential structure with cloned HMMs*. 2019. arXiv: 1905.00507 [stat.ML].
- Marcus, Mitchell P., Beatrice Santorini, and Mary Ann Marcinkiewicz. 'Building a Large Annotated Corpus of English: The Penn Treebank'. In: *Computational Linguistics* 19.2 (1993), pp. 313–330. URL: https://www.aclweb.org/anthology/J93-2004.
- Peng, Hao et al. *Random Feature Attention*. 2021. arXiv: 2103.02143 [cs.CL].
- Yang, Songlin, Yanpeng Zhao, and Kewei Tu. 'PCFGs Can Do

HMM Music Results

Model	Nott	Piano	Muse	JSB
RNN-NADE	2.31	**7.05**	**5.6**	5.19
R-Transformer	**2.24**	7.44	7.00	8.26
LSTM	3.43	7.77	7.23	8.17
LV-RNN	2.72	7.61	6.89	**3.99**
SRNN	2.94	8.20	6.28	4.74
TSBN	3.67	7.89	6.81	7.48
HMM	2.43	8.51	7.34	5.74
LHMM	2.60	8.89	7.60	5.80

PCFG Results

$\lvert\mathcal{N}\rvert$	$\lvert\mathcal{P}\rvert$	Model	N	PPL	Batch/s
30	60	PCFG	-	252.60	4.37
		LPCFG	8	247.02	3.75
		LPCFG	16	250.59	3.74
60	120	PCFG	-	234.01	2.99
		LPCFG	16	217.24	3.55
		LPCFG	32	213.81	3.35
100	200	PCFG	-	191.08	0.98
		LPCFG	32	203.47	1.56
		LPCFG	64	194.25	1.24

HSMM Results

Model	L	N	NLL	Batch/s
HSMM	2^6	-	$1.428e5$	1.28
HSMM	2^7	-	$1.427e5$	0.45
HSMM	2^8	-	$1.426e5$	0.13
LHSMM	2^7	2^7	$1.427e5$	0.24
LHSMM	2^8	2^6	$1.426e5$	0.20
LHSMM	2^9	2^5	$1.424e5$	0.18
LHSMM	2^{10}	2^4	$1.423e5$	0.10