Network Optimization

Justin Chiu
Cornell Tech
jtc257@cornell.edu

September 10, 2021

Abstract

None

1 Network Traffic Problem

We consider the problem of maximizing network traffic. A network is a graph $G = (V, E)$ with vertices $v \in V$ and edges $e = (i, j) \in E = V \times V$. Each edge e has an associated capacity limit c_e. We get a set of requests, with each request r represented as source and target pairs (s, t), plus a traffic demand d_r. We would like to fulfill the demand for each request as much as possible, by splitting traffic across the K_{st} valid paths $p_{stk} = ((s, a), (a, b), \ldots, (z, t))$ from s to t: $\sum_{p \in r} x_p \leq d_r$, where we should not exceed the request's demand. We will say that $p \in r$ if p is a valid path from $s \to t$. Additionally, we must ensure that edge traffic constraints hold. Multiple paths, as well as multiple requests, may result in overlapping traffic across particular edges, resulting in constraint (2): $\sum_{p \in \pi(e)} x_p \leq c_e$, where $\pi(e) = \{p \mid e \in p\}$.

This yields the following optimization problem:

$$\begin{aligned}
\text{maximize} \quad & \sum_r \sum_{p \in r} x_p \\
\text{subject to} \quad & \sum_{p \in r} x_p \leq d_r, \forall r \\
& \sum_{p \in \pi(e)} x_p \leq c_e, \forall e \\
& x_p \geq 0, \forall p.
\end{aligned} \tag{1}$$

While this form is compact, we would like the problem to separate over paths. This is not possible in Eqn. 1 because constraint 2 couples the traffic in paths. In order to decouple this constraint, we will transform the problem by adding dummy variables z_{stek} for the traffic contributed to each edge from each path p_{stk}. Afterwards, we will introduce slack variables for each of the resulting inequality constraints in order to apply the ADMM algorithm.

The new problem is given by

$$
\begin{array}{ll}
\text{minimize} & f(x) + g(z) + h(s) \\
\text{subject to} & c_e - \displaystyle\sum_{p \in \pi(e)} z_{pe} - s_e = 0, \forall e \\
& d_p - x_p - s_p = 0, \forall p \\
& x_p - z_{pe} = 0, \forall e, \forall p \in \pi(e) \\
& x, s, z \succeq 0,
\end{array}
\tag{2}
$$

where $f(x) = -\sum_p x_p$, $g(z) = \sum_{pe} \bar{\delta}(z_{pe} \geq 0)$, $h(s) = \sum_e \bar{\delta}(s_e \geq 0) + \sum_p \bar{\delta}(s_p \geq 0)$. We use the delta notation to indicate a function

$$
\bar{\delta}(b) = \begin{cases} \infty & \text{if condition } b \text{ does not hold,} \\ 0 & \text{o.w.} \end{cases}
$$

This problem is decomposable along each path (for x) or edge (for z) as follows:

$$
\begin{array}{ll}
\text{minimize} & -\displaystyle\sum_p x_p + \sum_e \sum_{p \in \pi(e)} \bar{\delta}(z_{pe}) + \sum_e \bar{\delta}(s_e) + \sum_p \bar{\delta}(s_p) \\
\text{subject to} & c_e - \displaystyle\sum_{p \in \pi(e)} z_{pe} - s_e = 0, \forall e \\
& d_p - x_p - s_p = 0, \forall p \\
& x_p - z_{pe} = 0, \forall e, \forall p \in \pi(e) \\
& x, s, z \succeq 0.
\end{array}
\tag{3}
$$

The augmented Lagrangian for this problem is

$$
\mathcal{L}_\rho(x, z, s, \lambda) = -\sum_p x_p + \sum_e \sum_{p \in \pi(e)} \bar{\delta}(z_{pe}) + \sum_e \bar{\delta}(s_e) + \sum_p \bar{\delta}(s_p) + \lambda^\top F(x, z, s) + (\rho/2)\|F(x, z, s)\|_2^2,
\tag{4}
$$

where

$$
F(x, z, s) = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix}
$$

with $F_1 \in \mathbb{R}^{|E|}$, $F_2 \in \mathbb{R}^{K|E|}$ ($|E|$ the number of edges and K the number of paths). Each subvector is given by

$$
[F_1]_e = c_e - \sum_{p \in \pi(e)} z_{pe} - s_e, [F_2]_p = d_p - x_p - s_p, [F_3]_{pe} = x_p - z_{pe},
$$

The elements of λ corresponding to F_1 are $\lambda_{1,e}$, F_2 are $\lambda_{2,p}$, and F_3 are $\lambda_{3,pe}$.

The ADMM updates are as follows:

$$
\begin{aligned}
x^{k+1} &:= \operatorname*{argmin}_x \mathcal{L}_\rho(x, z^k, s^k, \lambda^k) \\
z^{k+1} &:= \operatorname*{argmin}_z \mathcal{L}_\rho(x^{k+1}, z, s^k, \lambda^k) \\
s^{k+1} &:= \operatorname*{argmin}_s \mathcal{L}_\rho(x^{k+1}, z^{k+1}, s, \lambda^k) \\
\lambda^{k+1} &:= \lambda^k + \rho(F(x^{k+1}, z^{k+1}, s^{k+1})).
\end{aligned} \tag{5}
$$

We can compute $\operatorname{argmin}_x \mathcal{L}_\rho(x, z^k, s^k, \lambda^k)$ by restricting our attention to terms of $\mathcal{L}_\rho$ involving x and setting the derivative equal to 0:

$$
\begin{aligned}
0 &= \nabla_{x_p}(-x_p + \lambda_{2,p}(d_p - x_p - s_p) + \sum_{e\in p} \lambda_{3,pe}(x_p - z_{pe}) + (\rho/2)((d_p - x_p - s_p)^2 + \sum_{e\in p}(x_p - z_{pe})^2)) \\
&= -1 - \lambda_{2,p} + \sum_{e\in p} \lambda_{3,pe} + \rho(-d_p + x_p + s_p) + \rho \sum_{e\in p}(x_p - z_{pe}) \\
x_p &= \max(0, \frac{1 + \lambda_{2,p} - \sum_{e\in p} \lambda_{3,pe} + \rho(d_p - s_p + \sum_{e\in p} z_{pe})}{(1 + |p|)\rho}).
\end{aligned}
$$

We perform a similar computation for $\operatorname{argmin}_z \mathcal{L}_\rho(x^{k+1}, z, s^k, \lambda^k)$:

$$
\begin{aligned}
0 &= \nabla_{z_{pe}} \bar{\delta}(z_{pe}) + \lambda_{1,e}(c_e - \sum_{p'\in\pi(e)} z_{p'e} - s_e) + \lambda_{3,pe}(x_p - z_{pe}) \\
&\quad + (\rho/2)((c_e - \sum_{p'\in\pi(e)} z_{p'e} - s_e)^2 + (x_p - z_{pe})^2) \\
&= -\lambda_{1,e} - \lambda_{3,pe} + \rho(-c_e + \sum_{p'\in\pi(e)} z_{p'e} + s_e + z_{pe} - x_p).
\end{aligned}
$$

This gives us a system of equations for each edge $0 = A_e z_e + b_e$ allowing us to solve for $z_e = -A_e^{-1} b_e$. Let $P_e = |\pi(e)|$. We then have

$$
\begin{aligned}
A_e &= \mathbf{1}_{P_e \times P_e} + I_{P_e \times P_e} \\
[b_e]_p &= -\lambda_{1,e} - \lambda_{3,pe} + \rho(-c_e + s_e - x_p).
\end{aligned}
$$

Then, for $\operatorname{argmin}_s \mathcal{L}_\rho(x^{k+1}, z^{k+1}, s, y^k)$:

$$
\begin{aligned}
0 &= \nabla_{s_e} \bar{\delta}(s_e) + \lambda_{1,e}(c_e - \sum_{p\in\pi(e)} z_{pe} - s_e) + (\rho/2)((c_e - \sum_{p\in\pi(e)} z_{pe} - s_e)^2) \\
&= -\lambda_{1,e} + \rho(-c_e + \sum_{p\in\pi(e)} z_{pe} + s_e) \\
s_e &= \max(0, \frac{\lambda_{1,e} + \rho(c_e - \sum_{p\in\pi(e)} z_{pe})}{\rho}),
\end{aligned}
$$

and

$$
\begin{aligned}
0 &= \nabla_{s_p}\bar{\delta}(s_p) + \lambda_{2,p}(d_p - x_p - s_p) + (\rho/2)((d_p - x_p - s_p)^2) \\
&= -\lambda_{2,p} + \rho(-d_p + x_p + s_p) \\
s_p &= \max(0, \frac{\lambda_{2,p} + \rho(d_p - x_p)}{\rho}).
\end{aligned}
$$