

# Network Optimization

Justin Chiu

Cornell Tech

jtc257@cornell.edu

September 17, 2021

## Abstract

None

## 1 Network Traffic Problem

We consider the problem of maximizing network traffic. A network is a graph with  $V$  vertices  $v$  and  $E$  edges  $e = (i, j)$ . Each edge  $e$  has an associated capacity limit  $c_e$ . We get a set of requests, with each request  $r$  represented as source and target pairs  $(s, t)$ ,<sup>1</sup> plus a traffic demand  $d_r$ . There are  $R$  requests. We would like to fulfill the demand for each request as much as possible, by splitting traffic across  $P_r$  valid paths. Each path associated with a request is a sequence of edges given by  $p = ((s, a), (a, b), \dots, (z, t))$  from  $s$  to  $t$ . Our first constraint, constraint [1], ensures that each request has the constraint that the total traffic across associated paths  $\sum_{p \in r} x_p \leq d_r$  does not exceed the request's demand. We say that  $p \in r$  if  $p$  is a valid path from  $s \rightarrow t$ . We denote the total number of paths  $P$ . Additionally, we must ensure that edge traffic constraints hold. Multiple paths, as well as multiple requests, may result in overlapping traffic across particular edges, resulting in constraint [2]:  $\sum_{p \in \pi(e)} x_p \leq c_e$ , where  $\pi(e) = \{p \mid e \in p\}$ .

This yields the following optimization problem:

$$\begin{aligned} \text{minimize} \quad & - \sum_r \sum_{p \in r} x_p \\ \text{subject to} \quad & \sum_{p \in r} x_p \leq d_r, \forall r \quad [1] \\ & \sum_{p \in \pi(e)} x_p \leq c_e, \forall e \quad [2] \\ & x_p \geq 0, \forall p. \end{aligned} \tag{1}$$

While this form is compact, we would like the problem to separate over paths. This is not possible in Eqn. 1 because constraint [2] couples the traffic in paths across requests. In order to

---

<sup>1</sup>We assume there is only one unique request for each source and target pair.

20 decouple this constraint, we will transform the problem by adding dummy variables. Afterwards,  
 21 we will introduce slack variables to turn each of the resulting inequality constraints into equality  
 22 constraints, which makes it more straightforward to apply the ADMM algorithm.

### 23 1.1 Dummy variables for edge traffic

We add one dummy variable  $z_{pe}$  for each combination of path and edge.<sup>2</sup> We then replace the  $x_p$  variables constraint [2] with dummy variables  $z_{pe}$ , resulting in constraint [3]. We also for  $x_p = z_{pe}$  as constraint [4]. This results in the following problem:

$$\begin{aligned}
 & \text{minimize} && - \sum_r \sum_{p \in r} x_p \\
 & \text{subject to} && \sum_{p \in r} x_p \leq d_r, \forall r & [1] \\
 & && \sum_{p \in \pi(e)} z_{pe} \leq c_e, \forall e & [3] \\
 & && x_p - z_{pe} = 0, \forall e, p & [4] \\
 & && x_p \geq 0, \forall p \\
 & && z_{pe} \geq 0, \forall e, p.
 \end{aligned} \tag{2}$$

### 24 1.2 Slack variables for equality constraints

Finally, we add slack variables  $s_{1r}, s_{3e}$  to turn the inequality constraints into equality constraints.

$$\begin{aligned}
 & \text{minimize} && - \sum_r \sum_{p \in r} x_p \\
 & \text{subject to} && d_r - \sum_{p \in r} x_p - s_{1r} = 0, \forall r & [1] \\
 & && c_e - \sum_{p \in \pi(e)} z_{pe} - s_{3e} = 0, \forall e & [3] \\
 & && x_p - z_{pe} = 0, \forall e, p & [4] \\
 & && x_p \geq 0, z_{pe} \geq 0, s_{1r} \geq 0, s_{3e} \geq 0.
 \end{aligned} \tag{3}$$

In order to write the Lagrangian for this problem, we introduce Lagrange multipliers  $\lambda = (\lambda_1, \lambda_3, \lambda_4)$  for constraints [1], [3], and [4] respectively, where  $\lambda_1 \in \mathbb{R}^{|R|}$ ,  $\lambda_3 \in \mathbb{R}^{|E|}$ ,  $\lambda_4 \in \mathbb{R}^{|P||E|}$ . The Lagrangian is then given by

$$\mathcal{L}_\rho(x, z, s, \lambda) = - \sum_r \sum_{p \in r} x_p + \lambda^\top F(x, z, s), \tag{4}$$

---

<sup>2</sup>There are  $P_r$  paths for each request  $r$ , resulting in  $E \sum_r \sum_{P_r}$  dummy variables.

where  $F(x, z, s) = (F_1, F_3, F_4)^\top$  and

$$[F_1]_r = d_r - \sum_{p \in r} x_p - s_{1r}, \quad [F_3]_e = c_e - \sum_{p \in \pi(e)} z_{pe} - s_{3e}, \quad [F_4]_{pe} = x_p - z_{pe}.$$

The augmented Lagrangian for this problem is

$$\mathcal{L}_\rho(x, z, s, \lambda) = - \sum_r \sum_{p \in r} x_p + \lambda^\top F(x, z, s) + (\rho/2) \|F(x, z, s)\|_2^2, \quad (5)$$

where we have added a quadratic term based on  $F$  and the hyperparameter  $\rho \in \mathbb{R}_+$ . This quadratic term is key to ADMM, as it must be 0 at the solution, resulting in a feasible solution for the original problem.

## 2 ADMM Updates

Given the augmented Lagrangian, the ADMM updates are as follows:

$$\begin{aligned} x^{k+1} &:= \operatorname{argmin}_x \mathcal{L}_\rho(x, z^k, s^k, \lambda^k) \\ z^{k+1} &:= \operatorname{argmin}_z \mathcal{L}_\rho(x^{k+1}, z, s^k, \lambda^k) \\ s^{k+1} &:= \operatorname{argmin}_s \mathcal{L}_\rho(x^{k+1}, z^{k+1}, s, \lambda^k) \\ \lambda^{k+1} &:= \lambda^k + \rho(F(x^{k+1}, z^{k+1}, s^{k+1})). \end{aligned} \quad (6)$$

We solve for each update below.

### 2.1 Solving for $x$

We can compute  $\operatorname{argmin}_x \mathcal{L}_\rho(x, z^k, s^k, \lambda^k)$  by restricting our attention to terms of  $\mathcal{L}_\rho$  involving each  $x_p$  (for each path  $p \in P$ ) and setting the derivative equal to 0:

$$\begin{aligned} 0 &= \nabla_{x_p} (-x_p + \lambda_{1r}(d_r - \sum_{p \in r} x_p - s_{1r}) + \sum_{e \in p} \lambda_{4pe}(x_p - z_{pe}) \\ &\quad + (\rho/2)((d_r - \sum_{p \in r} x_p - s_{1r})^2 + \sum_{e \in p} (x_p - z_{pe})^2)) \\ &= -1 - \lambda_{1r} + \sum_{e \in p} \lambda_{4pe} + \rho(-d_r + \sum_{p \in r} x_p + s_{1r}) + \rho \sum_{e \in p} (x_p - z_{pe}). \end{aligned}$$

This gives us a system of equations for each request:  $0 = A_r x_r + b_r$ , where  $x_r = (x_p)_{p \in r}$ . Let  $P_r$  be the number of paths in request  $r$ . We can then solve for  $x_r = -A_r^{-1} b_r$ , where

$$\begin{aligned} A_r &= \rho(\mathbf{1}_{P_r \times P_r} + \operatorname{diag}(|p|)_{p \in r}) \\ [b_r]_p &= -1 - \lambda_{1r} + \sum_{e \in p} \lambda_{4pe} + \rho(-d_r + s_{1r}) - \rho \sum_{e \in p} z_{pe}. \end{aligned}$$

## 31 2.2 Solving for $z$

We perform a similar computation for  $\operatorname{argmin}_z \mathcal{L}_\rho(x^{k+1}, z, s^k, \lambda^k)$ :

$$\begin{aligned} 0 &= \nabla_{z_{pe}} \lambda_{3e} (c_e - \sum_{p' \in \pi(e)} z_{p'e} - s_{3e}) + \lambda_{4pe} (x_p - z_{pe}) \\ &\quad + (\rho/2) ((c_e - \sum_{p' \in \pi(e)} z_{p'e} - s_e)^2 + (x_p - z_{pe})^2) \\ &= -\lambda_{3e} - \lambda_{4pe} + \rho(-c_e + \sum_{p' \in \pi(e)} z_{p'e} + s_e + z_{pe} - x_p). \end{aligned}$$

This gives us a system of equations for each edge:  $0 = A_e z_e + b_e$  allowing us to solve for  $z_e = -A_e^{-1} b_e$ . Let  $P_e = |\pi(e)|$  be the number of paths that pass through edge  $e$ . We then have

$$\begin{aligned} A_e &= \mathbf{1}_{P_e \times P_e} + I_{P_e \times P_e} \\ [b_e]_p &= -\lambda_{3e} - \lambda_{4pe} + \rho(-c_e + s_e - x_p). \end{aligned}$$

## 32 2.3 Solving for $s$

Then, for  $\operatorname{argmin}_{s_1} \mathcal{L}_\rho(x^{k+1}, z^{k+1}, s, y^k)$ :

$$\begin{aligned} 0 &= \nabla_{s_{1r}} \lambda_{1r} (d_r - \sum_{p' \in r} x_{p'} - s_{1p}) + (\rho/2) ((d_r - \sum_{p' \in r} x_{p'} - s_{1p})^2) \\ &= -\lambda_{1r} + \rho(-d_r + \sum_{p' \in r} x_{p'} + s_r) \\ s_{1r} &= \frac{\lambda_{1r} + \rho(d_r - \sum_{p' \in r} x_{p'})}{\rho}, \end{aligned}$$

and for  $\operatorname{argmin}_{s_3} \mathcal{L}_\rho(x^{k+1}, z^{k+1}, s, y^k)$ :

$$\begin{aligned} 0 &= \nabla_{s_{3e}} \lambda_{3e} (c_e - \sum_{p \in \pi(e)} z_{pe} - s_{3e}) + (\rho/2) ((c_e - \sum_{p \in \pi(e)} z_{pe} - s_{3e})^2) \\ &= -\lambda_{3e} + \rho(-c_e + \sum_{p \in \pi(e)} z_{pe} + s_{3e}) \\ s_{3e} &= \frac{\lambda_{3e} + \rho(c_e - \sum_{p \in \pi(e)} z_{pe})}{\rho}. \end{aligned}$$