Network Optimization

Justin Chiu
Cornell Tech
jtc257@cornell.edu

September 6, 2021

Abstract

None

1 Network Traffic Problem

In this section, we consider the problem of optimizing network traffic. A network is a graph $G = (V, E)$ with vertices $v_i \in V$ and edges $e_{ij} \in E = V \times V$. We would like to maximize the total traffic through a series of K paths across the network. A path is a sequence of edges $p^k_{st} = (e_{sv_1}, e_{v_1v_2}, \dots, e_{v_mt})$ from source vertex s to target vertex t. A path contributes a constant amount of traffic x_p to each included edge $e \in p$. We denote the set of paths that pass through a particular edge by $\pi(e) = \{p \mid e \in p\}$. We additionally have the following constraints: traffic must be nonnegative $x_p \geq 0$, and each edge has a capacity constraint so that the total traffic on that edge cannot exceed $c_e \geq \sum_{p\in\pi(e)} x_p$.

This yields the following optimization problem:

$$\begin{aligned} \text{maximize} \quad & \sum_p x_p \text{ or } \sum_e \sum_{p\in\pi(e)} x_p \\ \text{subject to} \quad & \sum_{p\in\pi(e)} x_p \leq c_e, \forall e \\ & x_p \geq 0, \forall p. \end{aligned} \tag{1}$$

The first objective assigns equal weight to each path, while the second objective weights paths based on length. We focus on the first objective, since the goal is to maximize the amount of traffic each customer receives (represented as a path) rather than the congestion of the network.

While this form is compact, we would like to apply the ADMM algorithm, which applies to problems of the form

$$\begin{aligned} \text{minimize} \quad & f(x) + g(z) \\ \text{subject to} \quad & Ax + Bz = c. \end{aligned} \tag{2}$$

To massage the problem in Eqn. 1 into the correct form, we will introduce two sets of new variables: slack variables s_e for each edge[1] and decoupled edge weights z_{pe} for each edge and path.

The new problem is given by

$$\begin{array}{rl} \text{minimize} & f(x) + g(z) + h(s) \\ \text{subject to} & c_e - \displaystyle\sum_{p \in \pi(e)} z_{pe} - s_e = 0, \forall e \\ & x_p - z_{pe} = 0, \forall e, \forall p \in \pi(e) \\ & x, s, z \succeq 0, \end{array} \tag{3}$$

where $f(x) = -\sum_p x_p$, $g(z) = \sum_{pe} \bar{\delta}(z_{pe} \geq 0)$, $h(s) = \sum_e \bar{\delta}(s_e \geq 0)$. We use the delta notation to indicate a function

$$\bar{\delta}(b) = \begin{cases} \infty & \text{if condition } b \text{ does not hold,} \\ 0 & \text{o.w.} \end{cases}$$

This problem is decomposable along each path (for x) or edge (for z and s) as follows:

$$\begin{array}{rl} \text{minimize} & -\displaystyle\sum_p x_p + \sum_e \sum_{p \in \pi(e)} \bar{\delta}(z_{pe}) + \sum_e \bar{\delta}(s_e) \\ \text{subject to} & c_e - \displaystyle\sum_{p \in \pi(e)} z_{pe} - s_e = 0, \forall e \\ & x_p - z_{pe} = 0, \forall e, \forall p \in \pi(e) \\ & x, s, z \succeq 0. \end{array} \tag{4}$$

The augmented Lagrangian for this problem is

$$\mathcal{L}_\rho(x, z, s, \lambda) = -\sum_p x_p + \sum_e \sum_{p \in \pi(e)} \bar{\delta}(z_{pe}) + \sum_e \bar{\delta}(s_e) + \lambda^\top F(x, z, s) + (\rho/2) \|F(x, z, s)\|_2^2, \tag{5}$$

where

$$F(x, z, s) = \begin{bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \end{bmatrix}$$

and

$$[F_1]_e = c_e - \sum_{p \in \pi(e)} z_{pe} - s_e, [F_2]_{pe} = x_p - z_{pe}, [F_3]_p = x_p, [F_4]_e = s_e, [F_5]_{pe} = z_{pe}.$$

The elements of λ corresponding to F_1 are $\lambda_{1,e}$, F_2 are $\lambda_{2,pe}$, and so forth.

[1] The slack variable replaces the inequality constraints $c_e - \sum_p x_p \geq 0$ with equality constraints $c_e - \sum_p x_p - s_e = 0$.

The ADMM updates are as follows:

$$\begin{aligned} x^{k+1} &:= \underset{x}{\operatorname{argmin}}\, \mathcal{L}_\rho(x, z^k, s^k, \lambda^k) \\ z^{k+1} &:= \underset{z}{\operatorname{argmin}}\, \mathcal{L}_\rho(x^{k+1}, z, s^k, \lambda^k) \\ s^{k+1} &:= \underset{s}{\operatorname{argmin}}\, \mathcal{L}_\rho(x^{k+1}, z^{k+1}, s, \lambda^k) \\ \lambda^{k+1} &:= \lambda^k + \rho(F(x^{k+1}, z^{k+1}, s^{k+1})). \end{aligned} \tag{6}$$

We can compute $\operatorname{argmin}_x \mathcal{L}_\rho(x, z^k, s^k, \lambda^k)$ by restricting our attention to terms of $\mathcal{L}_\rho$ involving x and setting the derivative equal to 0:

$$\begin{aligned} 0 &= \nabla_{x_p}(-x_p + \sum_{e \in p} \lambda_{2,pe}(x_p - z_{pe}) + \lambda_{3,p} x_p + (\rho/2)(x_p^2 + \sum_{e \in p}(x_p - z_{pe})^2)) \\ &= -1 + \sum_{e \in p} \lambda_{2,pe} + \lambda_{3,p} + \rho x_p + \rho \sum_{e \in p}(x_p - z_{pe}) \\ x_p &= \max(0, \frac{1 - \sum_{e \in p} \lambda_{2,pe} - \lambda_{3,p} + \rho \sum_{e \in p} z_{pe}}{(1 + |p|)\rho}). \end{aligned}$$

We perform a similar computation for $\operatorname{argmin}_z \mathcal{L}_\rho(x^{k+1}, z, s^k, \lambda^k)$:

$$\begin{aligned} 0 &= \nabla_{z_{pe}} \bar{\delta}(z_{pe}) + \lambda_{1,e}(c_e - \sum_{p \in \pi(e)} z_{pe} - s_e) + \lambda_{2,pe}(x_p - z_{pe}) + \lambda_{5,pe} z_{pe} \\ &\quad + (\rho/2)((c_e - \sum_{p \in \pi(e)} z_{pe} - s_e)^2 + (x_p - z_{pe})^2 + z_{pe}^2) \\ &= -\lambda_{1,e} - \lambda_{2,pe} + \lambda_{5,pe} + \rho(-c_e + \sum_{p \in \pi(e)} z_{pe} + s_e + 2z_{pe} - x_p). \end{aligned}$$

This gives us a system of equations for each edge $0 = A_e z_e + b_e$ allowing us to solve for $z_e = -A_e^{-1} b_e$. Let $P_e = |\pi(e)|$. We then have

$$\begin{aligned} A_e &= \mathbf{1}_{P_e \times P_e} + 2I_{P_e \times P_e} \\ [b_e]_p &= (-\lambda_{1,e} - \lambda_{2,pe} + \lambda_{5,pe} + \rho(-c_e + s_e - x_p). \end{aligned}$$

Then, for $\operatorname{argmin}_s \mathcal{L}_\rho(x^{k+1}, z^{k+1}, s, y^k)$:

$$\begin{aligned} 0 &= \nabla_{s_e} \bar{\delta}(s_e) + \lambda_{1,e}(c_e - \sum_{p \in \pi(e)} z_{pe} - s_e) + \lambda_{4,e} s_e + (\rho/2)((c_e - \sum_{p \in \pi(e)} z_{pe} - s_e)^2 + s_e^2) \\ &= -\lambda_{1,e} + \lambda_{4,e} + \rho(-c_e + \sum_{p \in \pi(e)} z_{pe} + 2s_e) \\ s_e &= \max(0, \frac{\lambda_{1,e} - \lambda_{4,e} + \rho(c_e - \sum_{p \in \pi(e)} z_{pe})}{2\rho}). \end{aligned}$$