Network Optimization

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Abstract

None

1 Network Traffic Problem

We consider the problem of maximizing network traffic. A network is a graph $G = (V, E)$ with vertices $v \in V$ and edges $e = (i, j) \in E = V \times V$. Each edge e has an associated capacity limit c_e. We get a set of requests, with each request r represented as source and target pairs (s, t),[1] plus a traffic demand d_r. We would like to fulfill the demand for each request as much as possible, by splitting traffic across the K_r valid paths $p_{rk} = ((s, a), (a, b), \ldots, (z, t))$ from s to t: $\sum_{p \in r} x_p \leq d_r$, where we should not exceed the request's demand (constraint [1]). We will say that $p \in r$ if p is a valid path from $s \to t$. Additionally, we must ensure that edge traffic constraints hold. Multiple paths, as well as multiple requests, may result in overlapping traffic across particular edges, resulting in constraint [2]: $\sum_{p \in \pi(e)} x_p \leq c_e$, where $\pi(e) = \{p \mid e \in p\}$.

This yields the following optimization problem:

$$\begin{aligned}
\text{minimize} \quad & -\sum_r \sum_{p \in r} x_p \\
\text{subject to} \quad & \sum_{p \in r} x_p \leq d_r, \forall r \qquad [1] \\
& \sum_{p \in \pi(e)} x_p \leq c_e, \forall e \quad [2] \\
& x_p \geq 0, \forall p.
\end{aligned} \tag{1}$$

While this form is compact, we would like the problem to separate over paths. This is not possible in Eqn. 1 because constraint [2] couples the traffic in paths. In order to decouple this constraint, we will transform the problem by adding dummy variables. Afterwards, we will introduce slack

[1] We assume there is only one unique request for each source and target pair.

variables to turn each of the resulting inequality constraints into equality constraints, which makes it more straightforward to apply the ADMM algorithm.

We add one dummy variable z_{pe} for each combination of path and edge.[2] We then replace the x_p variables constraint [2] with dummy variables z_{pe}, resulting in constraint [3]. We also for $x_p = z_{pe}$ as constraint [4]. This results in the following problem:

$$
\begin{array}{ll}
\text{minimize} & -\sum_r \sum_{p \in r} x_p \\
\text{subject to} & \sum_{p \in r} x_p \leq d_r, \forall r \qquad [1] \\
& \sum_{p \in \pi(e)} z_{pe} \leq c_e, \forall e \qquad [3] \\
& x_p - z_{pe} = 0, \forall e, p \qquad [4] \\
& x_p \geq 0, \forall p \\
& z_{pe} \geq 0, \forall e, p.
\end{array}
\tag{2}
$$

Finally, we add slack variables s_{1r}, s_{3e} to turn the inequality constraints into equality constraints.

$$
\begin{array}{ll}
\text{minimize} & -\sum_r \sum_{p \in r} x_p \\
\text{subject to} & d_r - \sum_{p \in r} x_p - s_{1r} = 0, \forall r \qquad [1] \\
& c_e - \sum_{p \in \pi(e)} z_{pe} - s_{3e} = 0, \forall e \qquad [3] \\
& x_p - z_{pe} = 0, \forall e, p \qquad [4] \\
& x_p \geq 0, z_{pe} \geq 0, s_{1r} \geq 0, s_{2e} \geq 0.
\end{array}
\tag{3}
$$

In order to write the Lagrangian for this problem, we introduce Lagrange multipliers $\lambda = (\lambda_1, \lambda_3, \lambda_4)$ for constraints [1], [3], and [4] respectively, where $\lambda_1 \in \mathbb{R}^{|R|}$, $\lambda_3 \in \mathbb{R}^{|E|}$, $\lambda_4 \in \mathbb{R}^{|P||E|}$. The Lagrangian is then given by

$$\mathcal{L}_\rho(x, z, s, \lambda) = -\sum_r \sum_{p \in r} x_p + \lambda^\top F(x, z, s), \tag{4}$$

where $F(x, z, s) = (F_1, F_2, F_3)^\top$ and

$$[F_1]_r = d_r - \sum_{p \in r} x_p - s_{1r}, \quad [F_3]_e = c_e - \sum_{p \in \pi(e)} z_{pe} - s_{3e}, \quad [F_4]_{pe} = x_p - z_{pe}.$$

The augmented Lagrangian for this problem is

$$\mathcal{L}_\rho(x, z, s, \lambda) = -\sum_r \sum_{p \in r} x_p + \lambda^\top F(x, z, s) + (\rho/2)\|F(x, z, s)\|_2^2, \tag{5}$$

[2] There are K_r paths for each request r, resulting in $|E| \sum_r \sum_{K_r}$ dummy variables.

where we have added a quadratic term based on F and the hyperparameter $rho \in \mathbb{R}_+$. This quadratic term is key to ADMM.

Given the augmented Lagrangian, the ADMM updates are as follows:

$$\begin{aligned}
x^{k+1} &:= \operatorname*{argmin}_x \mathcal{L}_\rho(x, z^k, s^k, \lambda^k) \\
z^{k+1} &:= \operatorname*{argmin}_z \mathcal{L}_\rho(x^{k+1}, z, s^k, \lambda^k) \\
s^{k+1} &:= \operatorname*{argmin}_s \mathcal{L}_\rho(x^{k+1}, z^{k+1}, s, \lambda^k) \\
\lambda^{k+1} &:= \lambda^k + \rho(F(x^{k+1}, z^{k+1}, s^{k+1})).
\end{aligned} \tag{6}$$

We can compute $\operatorname{argmin}_x \mathcal{L}_\rho(x, z^k, s^k, \lambda^k)$ by restricting our attention to terms of $\mathcal{L}_\rho$ involving x and setting the derivative equal to 0:

$$\begin{aligned}
0 &= \nabla_{x_p}(-x_p + \lambda_{1r}(d_r - x_p - s_{1r}) + \sum_{e\in p} \lambda_{4pe}(x_p - z_{pe}) + (\rho/2)((d_r - x_p - s_{1r})^2 + \sum_{e\in p}(x_p - z_{pe})^2)) \\
&= -1 - \lambda_{1r} + \sum_{e\in p} \lambda_{4pe} + \rho(-d_r + x_p + s_{1r}) + \rho \sum_{e\in p}(x_p - z_{pe}) \\
x_p &= \max(0, \frac{1 + \lambda_{2,p} - \sum_{e\in p} \lambda_{3,pe} + \rho(d_p - s_p + \sum_{e\in p} z_{pe})}{(1 + |p|)\rho}).
\end{aligned}$$

We perform a similar computation for $\operatorname{argmin}_z \mathcal{L}_\rho(x^{k+1}, z, s^k, \lambda^k)$:

$$\begin{aligned}
0 &= \nabla_{z_{pe}} \bar{\delta}(z_{pe}) + \lambda_{1,e}(c_e - \sum_{p'\in\pi(e)} z_{p'e} - s_e) + \lambda_{3,pe}(x_p - z_{pe}) \\
&\quad + (\rho/2)((c_e - \sum_{p'\in\pi(e)} z_{p'e} - s_e)^2 + (x_p - z_{pe})^2) \\
&= -\lambda_{1,e} - \lambda_{3,pe} + \rho(-c_e + \sum_{p'\in\pi(e)} z_{p'e} + s_e + z_{pe} - x_p).
\end{aligned}$$

This gives us a system of equations for each edge $0 = A_e z_e + b_e$ allowing us to solve for $z_e = -A_e^{-1} b_e$. Let $P_e = |\pi(e)|$. We then have

$$\begin{aligned}
A_e &= \mathbf{1}_{P_e \times P_e} + I_{P_e \times P_e} \\
[b_e]_p &= -\lambda_{1,e} - \lambda_{3,pe} + \rho(-c_e + s_e - x_p).
\end{aligned}$$

Then, for $\operatorname{argmin}_s \mathcal{L}_\rho(x^{k+1}, z^{k+1}, s, y^k)$:

$$\begin{aligned}
0 &= \nabla_{s_e} \bar{\delta}(s_e) + \lambda_{1,e}(c_e - \sum_{p\in\pi(e)} z_{pe} - s_e) + (\rho/2)((c_e - \sum_{p\in\pi(e)} z_{pe} - s_e)^2) \\
&= -\lambda_{1,e} + \rho(-c_e + \sum_{p\in\pi(e)} z_{pe} + s_e) \\
s_e &= \max(0, \frac{\lambda_{1,e} + \rho(c_e - \sum_{p\in\pi(e)} z_{pe})}{\rho}),
\end{aligned}$$

and

$$
\begin{aligned}
0 &= \nabla_{s_p}\bar{\delta}(s_p) + \lambda_{2,p}(d_p - x_p - s_p) + (\rho/2)((d_p - x_p - s_p)^2) \\
&= -\lambda_{2,p} + \rho(-d_p + x_p + s_p) \\
s_p &= \max(0, \frac{\lambda_{2,p} + \rho(d_p - x_p)}{\rho}).
\end{aligned}
$$