# Lecture 23

Online secretary problem

Problem

- Hire a secretary from n candidates we interview in a random order

- when a candidate interviews, we need to make an irrevocable accept / reject decision

Objective: Maximize probability of selecting the best

n: total number of candidates

Threshold policy: Sample x candidates & then after that pick the first candidate who beats the maximum among x candidates

Exploration   Exploitation

1   2   3 x n
✗   ✗   ✗   ✗   ✗   ✗   ✗   ✓
reject ..   reject   accept
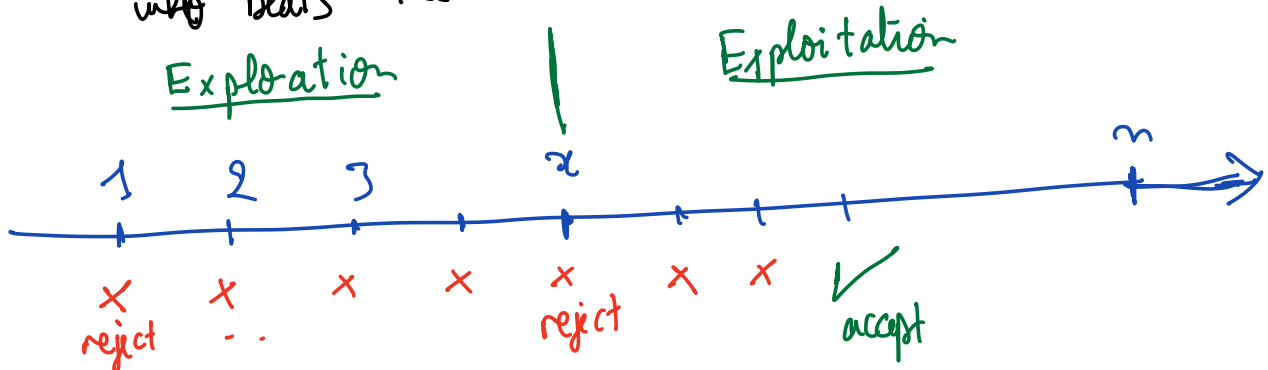
$$\mathbb{P}(\text{selecting the best}) = \sum_{j=x+1}^{n} \mathbb{P}\left(\begin{array}{l} j \text{ is the best } \& \\ \text{policy selects } j \end{array}\right)$$

$$= \sum_{j=x+1}^{n} \mathbb{P}\left(\text{policy selects } j \mid j \text{ is the best}\right) \times \underbrace{\mathbb{P}\left(\begin{array}{c} j \text{ is the} \\ \text{best} \end{array}\right)}_{\dfrac{1}{n}}$$
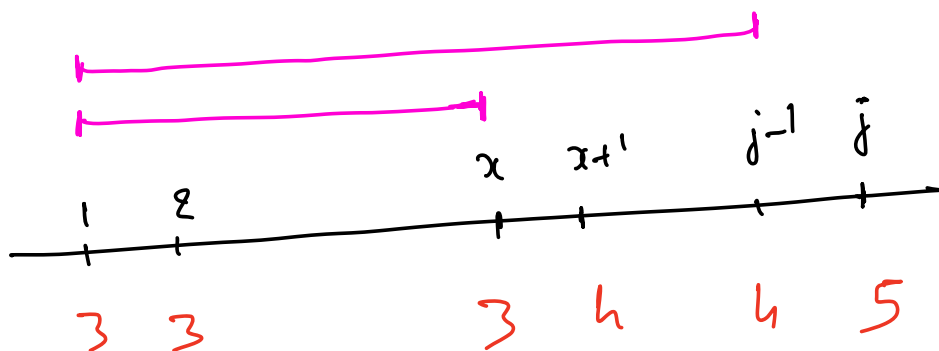
$$= \frac{1}{n} \sum_{j=x+1}^{x} \mathbb{P}\left(\begin{array}{c} \text{best among } \{1, \dots j-1\} \\ \text{is in } \{1, \dots x\} \end{array}\right)$$

$$= \frac{1}{n} \sum_{j=x+1}^{n} \frac{x}{j-1}$$

$$= \frac{x}{n}\left(\sum_{j=1}^{n-1} \frac{1}{j} - \sum_{i=1}^{x-1} \frac{1}{j}\right)$$

$$\cong \frac{x}{n}\left[\ln(n-1) - \ln(x-1)\right] \underset{n\to\infty}{\cong} \frac{x}{n} \ln\left(\frac{n}{x}\right)$$

$$f(x) = x \; h\left(n/x\right)$$

$$f'(x) = h\left(n/x\right) + x\left(-\frac{1}{x}\right)$$

$$= h(n) - h(x) - 1 = 0$$

$$\Rightarrow h(x) = h\left(\frac{n}{e}\right)$$

$$\Rightarrow \boxed{x^* = \frac{n}{e}}$$

$$x^* \simeq 37\% \; n$$