ORIE 6360: Optimization Under Uncertainty: Robust and Online Models Spring 2022

Lecture 23 (The Secretary Problem)

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In this lecture we introduce the secretary problem, a classical problem in optimal stopping.

1 Problem Setup

Imagine that we are an employer, interviewing candidates for a job opening. As we have both limited resources and a tight deadline, we can only interview at most n candidates (equivalent to drawing from a pool of n candidates without replacement). We assume the candidates are interviewed online in a random order.

After each candidate is interviewed, we make an irrevocable decision to either accept or reject them. Accepting a candidate means closing the job opening and terminating our search, while rejecting a candidate means we can continue the search by sampling a new candidate without replacement.

Objective: The goal is to maximize the probability of selecting the best candidate among the n total candidates. The main question we must answer is when should we accept a candidate and stop searching? Intuitively, we will break the search into two phases: explore and exploit. In the explore phase, we will gather information on candidate quality, then use that information in the exploit phase to make an informed selection. This leads naturally to the idea of a threshold policy.

Threshold policy: In the exploration phase, we sample x candidates and reject all of them. In the subsequent exploitation phase, you accept the first candidate who beats the maximum among the x candidates rejected during exploration, rejecting all others. We illustrate this process in Figure 1.

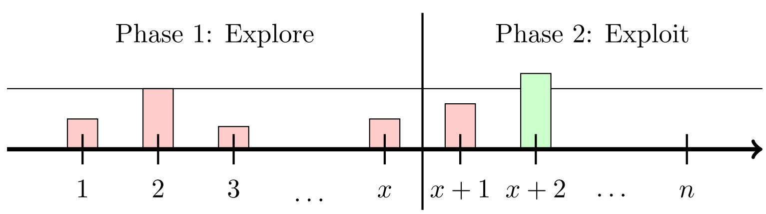

Figure 1: The scores of each candidate are given by the height of the corresponding rectangle. The threshold policy's exploration phase rejects the first x candidates (the rejected candidates are in red). Following this, we choose an acceptance threshold equal to the max among the first x candidates. In this example, the acceptance threshold is determined by candidate 2. For the exploitation phase, the first candidate that exceeds this threshold is accepted (the accepted candidate is in green).

The next question we must answer is how to determine the optimal explore-exploit tradeoff by setting x, the number of candidates in the explore phase. It turns out the optimal value of x is approximately n/e. To derive this, we will first determine the probability that the threshold policy selects the best candidate as a function of x, then optimize for the best x.

Derivation: The probability the threshold policy selects the best candidate (for a fixed x) is given by

$$\begin{aligned}
\mathsf{P}(\text{select the best}) &= \sum_{j=x+1}^{n} \mathsf{P}(j \text{ is the best} \wedge \text{policy selects } j) \\
&= \sum_{j=x+1}^{n} \mathsf{P}(\text{policy selects } j \mid j \text{ is the best}) \underbrace{\mathsf{P}(j \text{ is the best})}_{\frac{1}{n}} \\
&= \frac{1}{n} \sum_{j=x+1}^{n} \mathsf{P}(\text{best among } \{1, \ldots, j-1\} \text{ is in } \{1, \ldots, x\}) \\
&= \frac{1}{n} \sum_{j=x+1}^{n} \frac{x}{j-1} \\
&= \frac{x}{n} \left(\sum_{j=1}^{n-1} \frac{1}{j} - \sum_{j=1}^{x-1} \frac{1}{j} \right) \\
&\approx \frac{x}{n} \left[\log(n-1) - \log(x-1) \right] \underset{n\to\infty}{\approx} \frac{x}{n} \log \frac{n}{x}.
\end{aligned}$$

We can then maximize over x. Let

$$\begin{aligned}
f(x) &= x \log \frac{n}{x} \\
f'(x) &= \log \frac{n}{x} + x \left(-\frac{1}{x} \right) \\
&= \log n - \log x - 1 = 0 \\
&\Rightarrow \log x = \log \frac{n}{e} \\
&\Rightarrow x^* = \frac{n}{e} \approx .37n,
\end{aligned}$$

meaning you should explore 37% of the time then pick the first candidate that exceeds all the scores of the first 37% candidates.