

## SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR IN A SPHERE

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### Abstract

In this paper, we study submanifolds in a unit sphere with parallel mean curvature vector, a formula of Simons' type is obtained and a corresponding pinching theorem is proved.

### 1. Introduction

Let  $M$  be a closed  $n$ -dimensional Riemannian manifold immersed in the unit sphere  $S^{n+p}$  of dimension  $n+p$ . Denote by  $S$  the square of the length of the second fundamental form and by  $H$  the mean curvature of  $M$ . When  $M$  is minimal, J. Simons [9] obtained a pinching constant  $n/(2-p^{-1})$  of  $S$  and Chern, do Carmo and Kobayashi [5] proved that if  $S \leq n/(2-p^{-1})$  on  $M$ , then either  $M$  is totally geodesic, or the equality holds and  $M$  is either a Clifford hypersurface or a Veronese surface in  $S^4$ . Then A. M. Li and J. M. Li [7] obtained a better pinching constant  $(2/3)n$  of  $S$  and proved that if  $S \leq (2/3)n$  on  $M$ , then  $M$  is either totally geodesic or a Veronese surface in  $S^4$ . When  $M$  has parallel mean curvature vector, Z. H. Hou [6] obtained a pinching constant  $2\sqrt{n-1}$  of  $S$  for the case of  $p=1$ , i.e.,  $M$  is a hypersurface of constant mean curvature immersed in the unit sphere, and characterized all such hypersurfaces with  $S \leq 2\sqrt{n-1}$ . On the other hand G. Chen and X. Zou [4] discussed the case of  $p > 2$  and proved that if  $2 \leq n \leq 7$  and  $S \leq (2/3)n$  on  $M$ , then  $M$  is totally umbilical.

In this paper, we prove the following:

**THEOREM 1.** *Let  $M$  be a closed  $n$ -dimensional Riemannian manifold immersed in the unit sphere  $S^{n+p}$  of dimension  $n+p$ ,  $p \geq 2$ . If the mean curvature vector of  $M$  is non-zero parallel, then*

$$(1.1) \quad \int_M (aS - n)(S - nH^2) * 1 \geq 0,$$

where  $a = \max\{3/2, n/2\sqrt{n-1}\}$  and  $*1$  denotes the volume element of  $M$ .

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**THEOREM 2.** *Let  $M$  be a closed  $n$ -dimensional Riemannian manifold immersed in the unit sphere  $S^{n+p}$  of dimension  $n+p$ ,  $p \geq 2$ , with non-zero parallel mean curvature vector. If  $S \leq n/a$  on  $M$ , where  $a = \max\{3/2, n/2\sqrt{n-1}\}$ , then  $M$  is one of the following:*

- (1)  *$M$  is totally umbilical, and is a small sphere in  $S^{n+1}$  with constant curvature  $(1+|H|^2)$ .*
- (2)  *$M$  is a hypersurface in  $S^{n+1}$  of  $S^{n+p}$ , and is either  $S^n(r_0)$  or  $S^1(r) \times S^{n-1}(s)$ , where  $r_0^2 = n/(n+2\sqrt{n-1})$ ,  $r^2 = 1/(1+\sqrt{n-1})$  and  $s^2 = \sqrt{n-1}/(1+\sqrt{n-1})$ .*

## 2. Preliminaries

Let  $M$  be a closed  $n$ -dimensional Riemannian manifold immersed in the unit sphere  $S^{n+p}$  of dimension  $n+p$ . Denote by  $A$  the Weingarten map of  $M$ . Choose a local orthonormal frame  $\{e_1, \dots, e_{n+p}\}$  in  $S^{n+p}$  such that, restricted to  $M$ ,  $\{e_1, \dots, e_n\}$  are tangent to  $M$  and  $e_{n+1}$  is in the direction of the mean curvature vector of  $M$  in  $S^{n+p}$ , i.e., the normalized mean curvature vector. Then we have that  $\text{tr } A_{n+1} = nH$ ,  $\text{tr } A_\alpha = 0$ , for  $n+2 \leq \alpha \leq n+p$ , and  $S = |A|^2 = \sum_{\alpha=n+1}^{n+p} |A_\alpha|^2 = \sum_{\alpha=n+1}^{n+p} \text{tr } A_\alpha^2$ , where  $A_\alpha = A_{e_\alpha}$  denotes the Weingarten map with respect to  $e_\alpha$ .

As Alencar and do Carmo in [1] and Santos in [8], we define a bilinear map  $\Phi : TM \times TM \rightarrow T^\perp M$  by

$$\Phi(X, Y) = \sum_{\alpha=n+1}^{n+p} \langle \Phi_\alpha X, Y \rangle e_\alpha,$$

where  $\Phi_\alpha$  is given by

$$\begin{cases} \Phi_{n+1} = H \text{id} - A_{n+1}, \\ \Phi_\alpha = -A_\alpha, \quad n+2 \leq \alpha \leq n+p. \end{cases}$$

Then we have

$$|\Phi|^2 := \sum_{\alpha=n+1}^{n+p} \text{tr } \Phi_\alpha^2 = S - nH^2.$$

We need the following results of W. Santos [8], and A. M. Li and J. M. Li [7].

**LEMMA 1** [8]. *Let  $B_1$  and  $B_2$  be symmetric  $(n \times n)$ -matrices such that  $[B_1, B_2] = 0$  and  $\text{tr } B_1 = \text{tr } B_2 = 0$ . Then*

$$(2.1) \quad \text{tr } B_1^2 B_2 \leq \frac{n-2}{\sqrt{n(n-1)}} (\text{tr } B_1^2) \sqrt{\text{tr } B_2^2}.$$

We rewrite Theorem 1 of [7] as follows.

LEMMA 2 [7]. *Let  $B_1, \dots, B_p$ ,  $p \geq 2$ , be symmetric  $(n \times n)$ -matrices. Then*

$$(2.2) \quad \sum_{\alpha, \beta=1}^p \{ \text{tr}[B_\alpha, B_\beta]^2 - (\text{tr } B_\alpha B_\beta)^2 \} \geq -\frac{3}{2} \left( \sum_{\alpha=1}^p \text{tr } B_\alpha^2 \right)^2.$$

We also need the following results of B. Y. Chen and K. Yano [3] and Z. H. Hou [6].

PROPOSITION 1 [3]. *Let  $M^n$  be a non-minimal pseudo-umbilical submanifold of  $S^{n+p}$ . If the mean curvature vector of  $M$  is parallel, then  $M$  is a minimal submanifold of a hypersphere of  $S^{n+p}$ .*

PROPOSITION 2 [6]. *Let  $M^n$  be a closed hypersurface of constant mean curvature in  $S^{n+1}$ . Then*

- (1) *If  $S < 2\sqrt{n-1}$ ,  $M$  is a small hypersphere  $S^n(r)$  of radius  $r = \sqrt{n/(n+S)}$ .*
- (2) *If  $S = 2\sqrt{n-1}$ ,  $M$  is either  $S^n(r_0)$  or  $S^1(r) \times S^{n-1}(s)$ , where  $r_0^2 = n/(n+2\sqrt{n-1})$ ,  $r^2 = 1/(1+\sqrt{n-1})$  and  $s^2 = \sqrt{n-1}/(1+\sqrt{n-1})$ .*

### 3. Proof of theorems

First, we estimate  $\Delta|\Phi|^2$ . Santos [8] calculated the Laplacian  $\Delta|\Phi|$  of  $\Phi$  by using a formula of J. J. Erbacher for  $\Delta|A|^2$  and obtained that if the mean curvature vector of  $M$  is parallel, with the above notation, we have

$$(3.1) \quad \begin{aligned} \frac{1}{2} \Delta |\Phi|^2 &= \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_\alpha|^2 + n(1+H^2)|\Phi|^2 - nH \sum_{\alpha=n+1}^{n+p} \text{tr } \Phi_{n+1} \Phi_\alpha^2 \\ &\quad - \sum_{\alpha, \beta=n+1}^{n+p} (\text{tr } \Phi_\alpha \Phi_\beta)^2 + \sum_{\alpha, \beta>n+1}^{n+p} \text{tr}([\Phi_\alpha, \Phi_\beta])^2, \end{aligned}$$

where  $[\Phi_\alpha, \Phi_\beta] = \Phi_\alpha \Phi_\beta - \Phi_\beta \Phi_\alpha$ .

Rewrite (3.1) as the following:

$$(3.2) \quad \begin{aligned} \frac{1}{2} \Delta |\Phi|^2 &= \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_\alpha|^2 + n(1+H^2)|\Phi|^2 - (\text{tr } \Phi_{n+1}^2)^2 \\ &\quad - nH \sum_{\alpha=n+1}^{n+p} \text{tr } \Phi_{n+1} \Phi_\alpha^2 + \sum_{\alpha, \beta>n+1}^{n+p} \{ \text{tr}([\Phi_\alpha, \Phi_\beta])^2 - (\text{tr } \Phi_\alpha \Phi_\beta)^2 \} \\ &\quad - 2 \sum_{\alpha>n+1}^{n+p} (\text{tr } \Phi_{n+1} \Phi_\alpha)^2. \end{aligned}$$

Since  $\text{tr } \Phi_\alpha = 0$ ,  $\alpha = n+1, \dots, n+p$ , and  $[\Phi_{n+1}, \Phi_\alpha] = [A_{n+1}, A_\alpha] = 0$ ,  $\alpha = n+2, \dots, n+p$ , we may apply Lemma 1 to the fourth term of (3.2) and

have

$$(3.3) \quad \begin{aligned} \sum_{\alpha=n+1}^{n+p} \operatorname{tr} \Phi_{n+1} \Phi_{\alpha}^2 &\leq \frac{n-2}{\sqrt{n(n-1)}} \left( \sum_{\alpha=n+1}^{n+p} \operatorname{tr} \Phi_{\alpha}^2 \right) \sqrt{\operatorname{tr} \Phi_{n+1}^2} \\ &= \frac{n-2}{\sqrt{n(n-1)}} |\Phi|^2 |\Phi_{n+1}|. \end{aligned}$$

Applying Lemma 2 to the fifth term of (3.2), we may have for  $p \geq 3$

$$(3.4) \quad \begin{aligned} \sum_{\alpha, \beta > n+1}^{n+p} \{ \operatorname{tr}([\Phi_{\alpha}, \Phi_{\beta}])^2 - (\operatorname{tr} \Phi_{\alpha} \Phi_{\beta})^2 \} &\geq -\frac{3}{2} \left( \sum_{\alpha > n+1}^{n+p} |\Phi_{\alpha}|^2 \right)^2 \\ &= -\frac{3}{2} (|\Phi|^2 - |\Phi_{n+1}|^2)^2. \end{aligned}$$

Since when  $p = 2$ , (3.4) becomes

$$(3.5) \quad -(\operatorname{tr} \Phi_{n+2} \Phi_{n+2})^2 \geq -\frac{3}{2} (\operatorname{tr} \Phi_{n+2}^2)^2,$$

which holds of course and we really have (3.4) for  $p \geq 2$ .

For the last term of (3.2), Cauchy-Schwarz's inequality gives the following for any  $\alpha$ .

$$(\operatorname{tr} \Phi_{n+1} \Phi_{\alpha})^2 \leq |\Phi_{n+1}|^2 |\Phi_{\alpha}|^2.$$

Therefore, we have

$$(3.6) \quad -2 \sum_{\alpha > n+1}^{n+p} (\operatorname{tr} \Phi_{n+1} \Phi_{\alpha})^2 \geq -2 |\Phi_{n+1}|^2 (|\Phi|^2 - |\Phi_{n+1}|^2).$$

Apply (3.3)–(3.6) to (3.2), we may have for  $p \geq 2$

$$(3.7) \quad \begin{aligned} \frac{1}{2} \Delta |\Phi|^2 &\geq \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_{\alpha}|^2 + n(1+H^2) |\Phi|^2 - \frac{n-2}{\sqrt{n(n-1)}} n H |\Phi_{n+1}| |\Phi|^2 \\ &\quad - \frac{3}{2} (|\Phi|^2 - |\Phi_{n+1}|^2)^2 - |\Phi_{n+1}|^4 - 2 |\Phi_{n+1}|^2 (|\Phi|^2 - |\Phi_{n+1}|^2) \\ &= \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_{\alpha}|^2 + \frac{1}{2} |\Phi_{n+1}|^2 (|\Phi|^2 - |\Phi_{n+1}|^2) \\ &\quad + |\Phi|^2 \left\{ n(1+H^2) - \frac{(n-2)\sqrt{n}}{\sqrt{(n-1)}} H |\Phi_{n+1}| \right. \\ &\quad \left. - \frac{3}{2} (|\Phi|^2 - |\Phi_{n+1}|^2) - |\Phi_{n+1}|^2 \right\} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_\alpha|^2 + \frac{1}{2} |\Phi_{n+1}|^2 (|\Phi|^2 - |\Phi_{n+1}|^2) \\
&\quad + |\Phi|^2 \left\{ n - \frac{3}{2} (|\Phi|^2 - |\Phi_{n+1}|^2) \right. \\
&\quad \left. + (nH^2 - \frac{(n-2)\sqrt{n}}{\sqrt{(n-1)}} H |\Phi_{n+1}| - |\Phi_{n+1}|^2) \right\}.
\end{aligned}$$

In the last term of (3.7), we need to estimate the following

$$(3.8) \quad z(H, |\Phi_{n+1}|) = nH^2 - \frac{(n-2)\sqrt{n}}{\sqrt{(n-1)}} H |\Phi_{n+1}| - |\Phi_{n+1}|^2.$$

Let  $x = \sqrt{n}H$  and  $y = |\Phi_{n+1}|$ , then (3.8) becomes

$$z(H, |\Phi_{n+1}|) = x^2 - \frac{n-2}{\sqrt{n-1}} xy - y^2, \quad y \geq 0,$$

which could be consider as a quadratic surface in  $E^3$ . By the following coordinate transformation:

$$(3.9) \quad \begin{cases} u = \frac{1}{\sqrt{2n}} \{(1 + \sqrt{n-1})x + (1 - \sqrt{n-1})y\}, \\ v = \frac{1}{\sqrt{2n}} \{-(1 - \sqrt{n-1})x + (1 + \sqrt{n-1})y\}, \end{cases}$$

we may find that

$$\begin{aligned}
(3.10) \quad z(H, |\Phi_{n+1}|) &= \frac{n}{2\sqrt{n-1}} (u^2 - v^2) \\
&= -\frac{n}{2\sqrt{n-1}} (v^2 + u^2 - 2u^2) \\
&\geq -\frac{n}{2\sqrt{n-1}} (v^2 + u^2) \\
&= -\frac{n}{2\sqrt{n-1}} (x^2 + y^2) \\
&= -\frac{n}{2\sqrt{n-1}} (nH^2 + |\Phi_{n+1}|^2).
\end{aligned}$$

Substituting (3.10) into (3.7), we have

$$\begin{aligned}
(3.11) \quad \frac{1}{2} \Delta |\Phi|^2 &\geq \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_\alpha|^2 + \frac{1}{2} |\Phi_{n+1}|^2 (|\Phi|^2 - |\Phi_{n+1}|^2) \\
&\quad + |\Phi|^2 \left\{ n - \frac{3}{2} (|\Phi|^2 - |\Phi_{n+1}|^2) - \frac{n}{2\sqrt{n-1}} (nH^2 + |\Phi_{n+1}|^2) \right\}.
\end{aligned}$$

And taking  $a = \max\{3/2, n/(2\sqrt{n-1})\}$ , we may finally obtain the following, since  $|\Phi|^2 = S - nH^2$ ,

$$(3.12) \quad \frac{1}{2}\Delta|\Phi|^2 \geq \sum_{\alpha=n+1}^{n+p} |\nabla\Phi_\alpha|^2 + \frac{1}{2}|\Phi_{n+1}|^2(|\Phi|^2 - |\Phi_{n+1}|^2) + |\Phi|^2(n - aS).$$

*Proof of Theorem 1.* Integrating (3.12) on  $M$ , we get (1.1). q.e.d.

*Proof of Theorem 2.* If  $S \leq n/a$  on  $M$ , then either  $S \equiv nH^2$  or  $S \equiv n/a$  by Theorem 1. In the first case,  $M$  is totally umbilical and thus contained in  $S^{n+1}$  with constant curvature  $(1 + |H|^2)$ . (cf. [2, pp. 49–50]).

For the later, we may see from (3.12) that both  $|\Phi_{n+1}|^2(|\Phi|^2 - |\Phi_{n+1}|^2)$  and  $\sum_{\alpha=n+1}^{n+p} |\nabla\Phi_\alpha|^2$  must vanish.

(1) If  $|\Phi_{n+1}| = 0$ , which implies that  $|A_{n+1}|^2 = nH^2$ ,  $M$  is a pseudo-umbilical submanifold of  $S^{n+p}$ . Applying Proposition 1 of Chen and Yano, we find that  $M$  is minimal in a hypersphere  $S^{n+p-1}$  of  $S^{n+p}$  with constant scalar curvature. But, since the equality in (3.10) holds when  $s = n/a$ , we may find that  $u$  must vanish. Consequently, from (3.9) the mean curvature  $H$  of  $M$  would be vanish which contradicts the hypothesis of the theorem.

(2) If  $|\Phi|^2 - |\Phi_{n+1}|^2 = 0$ , and plus that  $\nabla\Phi_\alpha = 0$ , for  $\alpha = n+1, \dots, n+p$ , we may find that  $M$  is a hypersurface of  $S^{n+1}$  in  $S^{n+p}$ . Then by Proposition 2 of Hou, we obtain that

(a) For  $2 \leq n \leq 7$ , we have  $a = 3/2$ , and then  $S = 2n/3 < 2\sqrt{n-1}$ , thus  $M$  is a small hypersphere  $S^n(r)$  of radius  $r = \sqrt{n/(n+S)}$ .

(b) For  $n \geq 8$ , we have  $a = n/2\sqrt{n-1}$ , and then  $S = 2\sqrt{n-1}$ , thus  $M$  is either  $S^n(r_0)$  or  $S^1(r) \times S^{n-1}(s)$ , where  $r_0^2 = n/(n+2\sqrt{n-1})$ ,  $r^2 = 1/(1+\sqrt{n-1})$  and  $s^2 = \sqrt{n-1}/(1+\sqrt{n-1})$ . q.e.d.

*Remark.* When the dimension  $n$  of  $M$  is in  $2 \leq n \leq 7$ , then  $a = 3/2$  and our Theorem 1 is a generalization of Theorem 2 of A. M. Li and J. M. Li [7].

Theorem 2 generalizes Theorem 2 of [4].

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