

SUBMANIFOLDS WITH PARALLEL MEAN CURVATURE VECTOR IN A SPHERE

MEI-JIAO WANG AND SHI-JIE LI

Abstract

In this paper, we study submanifolds in a unit sphere with parallel mean curvature vector, a formula of Simons' type is obtained and a corresponding pinching theorem is proved.

1. Introduction

Let M be a closed n -dimensional Riemannian manifold immersed in the unit sphere S^{n+p} of dimension $n+p$. Denote by S the square of the length of the second fundamental form and by H the mean curvature of M . When M is minimal, J. Simons [9] obtained a pinching constant $n/(2-p^{-1})$ of S and Chern, do Carmo and Kobayashi [5] proved that if $S \leq n/(2-p^{-1})$ on M , then either M is totally geodesic, or the equality holds and M is either a Clifford hypersurface or a Veronese surface in S^4 . Then A. M. Li and J. M. Li [7] obtained a better pinching constant $(2/3)n$ of S and proved that if $S \leq (2/3)n$ on M , then M is either totally geodesic or a Veronese surface in S^4 . When M has parallel mean curvature vector, Z. H. Hou [6] obtained a pinching constant $2\sqrt{n-1}$ of S for the case of $p=1$, i.e., M is a hypersurface of constant mean curvature immersed in the unit sphere, and characterized all such hypersurfaces with $S \leq 2\sqrt{n-1}$. On the other hand G. Chen and X. Zou [4] discussed the case of $p > 2$ and proved that if $2 \leq n \leq 7$ and $S \leq (2/3)n$ on M , then M is totally umbilical.

In this paper, we prove the following:

THEOREM 1. *Let M be a closed n -dimensional Riemannian manifold immersed in the unit sphere S^{n+p} of dimension $n+p$, $p \geq 2$. If the mean curvature vector of M is non-zero parallel, then*

$$(1.1) \quad \int_M (aS - n)(S - nH^2) * 1 \geq 0,$$

where $a = \max\{3/2, n/2\sqrt{n-1}\}$ and $*1$ denotes the volume element of M .

The authors are supported by PNSF (Project 960179) of Guangong Province, China, and NSFC (Project 19771039).

Received February 10, 1998; revised March 23, 1998.

THEOREM 2. *Let M be a closed n -dimensional Riemannian manifold immersed in the unit sphere S^{n+p} of dimension $n+p$, $p \geq 2$, with non-zero parallel mean curvature vector. If $S \leq n/a$ on M , where $a = \max\{3/2, n/2\sqrt{n-1}\}$, then M is one of the following:*

- (1) *M is totally umbilical, and is a small sphere in S^{n+1} with constant curvature $(1 + |H|^2)$.*
- (2) *M is a hypersurface in S^{n+1} of S^{n+p} , and is either $S^n(r_0)$ or $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = n/(n + 2\sqrt{n-1})$, $r^2 = 1/(1 + \sqrt{n-1})$ and $s^2 = \sqrt{n-1}/(1 + \sqrt{n-1})$.*

2. Preliminaries

Let M be a closed n -dimensional Riemannian manifold immersed in the unit sphere S^{n+p} of dimension $n+p$. Denote by A the Weingarten map of M . Choose a local orthonormal frame $\{e_1, \dots, e_{n+p}\}$ in S^{n+p} such that, restricted to M , $\{e_1, \dots, e_n\}$ are tangent to M and e_{n+1} is in the direction of the mean curvature vector of M in S^{n+p} , i.e., the normalized mean curvature vector. Then we have that $\text{tr } A_{n+1} = nH$, $\text{tr } A_\alpha = 0$, for $n+2 \leq \alpha \leq n+p$, and $S = |A|^2 = \sum_{\alpha=n+1}^{n+p} |A_\alpha|^2 = \sum_{\alpha=n+1}^{n+p} \text{tr } A_\alpha^2$, where $A_\alpha = A_{e_\alpha}$ denotes the Weingarten map with respect to e_α .

As Alencar and do Carmo in [1] and Santos in [8], we define a bilinear map $\Phi : TM \times TM \rightarrow T^\perp M$ by

$$\Phi(X, Y) = \sum_{\alpha=n+1}^{n+p} \langle \Phi_\alpha X, Y \rangle e_\alpha,$$

where Φ_α is given by

$$\begin{cases} \Phi_{n+1} = H \text{id} - A_{n+1}, \\ \Phi_\alpha = -A_\alpha, \quad n+2 \leq \alpha \leq n+p. \end{cases}$$

Then we have

$$|\Phi|^2 := \sum_{\alpha=n+1}^{n+p} \text{tr } \Phi_\alpha^2 = S - nH^2.$$

We need the following results of W. Santos [8], and A. M. Li and J. M. Li [7].

LEMMA 1 [8]. *Let B_1 and B_2 be symmetric $(n \times n)$ -matrices such that $[B_1, B_2] = 0$ and $\text{tr } B_1 = \text{tr } B_2 = 0$. Then*

$$(2.1) \quad \text{tr } B_1^2 B_2 \leq \frac{n-2}{\sqrt{n(n-1)}} (\text{tr } B_1^2) \sqrt{\text{tr } B_2^2}.$$

We rewrite Theorem 1 of [7] as follows.

LEMMA 2 [7]. Let B_1, \dots, B_p , $p \geq 2$, be symmitric $(n \times n)$ -matrices. Then

$$(2.2) \quad \sum_{\alpha, \beta=1}^p \{ \operatorname{tr} [B_\alpha, B_\beta]^2 - (\operatorname{tr} B_\alpha B_\beta)^2 \} \geq -\frac{3}{2} \left(\sum_{\alpha=1}^p \operatorname{tr} B_\alpha^2 \right)^2.$$

We also need the following results of B. Y. Chen and K. Yano [3] and Z. H. Hou [6].

PROPOSITION 1 [3]. Let M^n be a non-minimal pseudo-umbilical submanifold of S^{n+p} . If the mean curvature vector of M is parallel, then M is a minimal submanifold of a hypersphere of S^{n+p} .

PROPOSITION 2 [6]. Let M^n be a closed hypersurface of constant mean curvature in S^{n+1} . Then

- (1) If $S < 2\sqrt{n-1}$, M is a small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n+S)}$.
- (2) If $S = 2\sqrt{n-1}$, M is either $S^n(r_0)$ or $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = n/(n+2\sqrt{n-1})$, $r^2 = 1/(1+\sqrt{n-1})$ and $s^2 = \sqrt{n-1}/(1+\sqrt{n-1})$.

3. Proof of theorems

First, we estimate $\Delta|\Phi|^2$. Santos [8] calculated the Laplacian $\Delta|\Phi|$ of Φ by using a formula of J. J. Erbacher for $\Delta|A|^2$ and obtained that if the mean curvature vector of M is parallel, with the above notation, we have

$$(3.1) \quad \begin{aligned} \frac{1}{2} \Delta|\Phi|^2 &= \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_\alpha|^2 + n(1+H^2)|\Phi|^2 - nH \sum_{\alpha=n+1}^{n+p} \operatorname{tr} \Phi_{n+1} \Phi_\alpha^2 \\ &\quad - \sum_{\alpha, \beta=n+1}^{n+p} (\operatorname{tr} \Phi_\alpha \Phi_\beta)^2 + \sum_{\alpha, \beta > n+1}^{n+p} \operatorname{tr}([\Phi_\alpha, \Phi_\beta])^2, \end{aligned}$$

where $[\Phi_\alpha, \Phi_\beta] = \Phi_\alpha \Phi_\beta - \Phi_\beta \Phi_\alpha$.

Rewrite (3.1) as the following:

$$(3.2) \quad \begin{aligned} \frac{1}{2} \Delta|\Phi|^2 &= \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_\alpha|^2 + n(1+H^2)|\Phi|^2 - (\operatorname{tr} \Phi_{n+1}^2)^2 \\ &\quad - nH \sum_{\alpha=n+1}^{n+p} \operatorname{tr} \Phi_{n+1} \Phi_\alpha^2 + \sum_{\alpha, \beta > n+1}^{n+p} \{ \operatorname{tr}([\Phi_\alpha, \Phi_\beta])^2 - (\operatorname{tr} \Phi_\alpha \Phi_\beta)^2 \} \\ &\quad - 2 \sum_{\alpha > n+1}^{n+p} (\operatorname{tr} \Phi_{n+1} \Phi_\alpha)^2. \end{aligned}$$

Since $\operatorname{tr} \Phi_\alpha = 0$, $\alpha = n+1, \dots, n+p$, and $[\Phi_{n+1}, \Phi_\alpha] = [A_{n+1}, A_\alpha] = 0$, $\alpha = n+2, \dots, n+p$, we may apply Lemma 1 to the fourth term of (3.2) and

have

$$(3.3) \quad \begin{aligned} \sum_{\alpha=n+1}^{n+p} \operatorname{tr} \Phi_{n+1} \Phi_{\alpha}^2 &\leq \frac{n-2}{\sqrt{n(n-1)}} \left(\sum_{\alpha=n+1}^{n+p} \operatorname{tr} \Phi_{\alpha}^2 \right) \sqrt{\operatorname{tr} \Phi_{n+1}^2} \\ &= \frac{n-2}{\sqrt{n(n-1)}} |\Phi|^2 |\Phi_{n+1}|. \end{aligned}$$

Applying Lemma 2 to the fifth term of (3.2), we may have for $p \geq 3$

$$(3.4) \quad \begin{aligned} \sum_{\alpha, \beta > n+1}^{n+p} \{ \operatorname{tr}([\Phi_{\alpha}, \Phi_{\beta}])^2 - (\operatorname{tr} \Phi_{\alpha} \Phi_{\beta})^2 \} &\geq -\frac{3}{2} \left(\sum_{\alpha > n+1}^{n+p} |\Phi_{\alpha}|^2 \right)^2 \\ &= -\frac{3}{2} (|\Phi|^2 - |\Phi_{n+1}|^2)^2. \end{aligned}$$

Since when $p = 2$, (3.4) becomes

$$(3.5) \quad -(\operatorname{tr} \Phi_{n+2} \Phi_{n+2})^2 \geq -\frac{3}{2} (\operatorname{tr} \Phi_{n+2}^2)^2,$$

which holds of course and we really have (3.4) for $p \geq 2$.

For the last term of (3.2), Cauchy-Schwarz's inequality gives the following for any α .

$$(\operatorname{tr} \Phi_{n+1} \Phi_{\alpha})^2 \leq |\Phi_{n+1}|^2 |\Phi_{\alpha}|^2.$$

Therefore, we have

$$(3.6) \quad -2 \sum_{\alpha > n+1}^{n+p} (\operatorname{tr} \Phi_{n+1} \Phi_{\alpha})^2 \geq -2 |\Phi_{n+1}|^2 (|\Phi|^2 - |\Phi_{n+1}|^2).$$

Apply (3.3)–(3.6) to (3.2), we may have for $p \geq 2$

$$(3.7) \quad \begin{aligned} \frac{1}{2} \Delta |\Phi|^2 &\geq \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_{\alpha}|^2 + n(1+H^2) |\Phi|^2 - \frac{n-2}{\sqrt{n(n-1)}} nH |\Phi_{n+1}| |\Phi|^2 \\ &\quad - \frac{3}{2} (|\Phi|^2 - |\Phi_{n+1}|^2)^2 - |\Phi_{n+1}|^4 - 2 |\Phi_{n+1}|^2 (|\Phi|^2 - |\Phi_{n+1}|^2) \\ &= \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_{\alpha}|^2 + \frac{1}{2} |\Phi_{n+1}|^2 (|\Phi|^2 - |\Phi_{n+1}|^2) \\ &\quad + |\Phi|^2 \left\{ n(1+H^2) - \frac{(n-2)\sqrt{n}}{\sqrt{(n-1)}} H |\Phi_{n+1}| \right. \\ &\quad \left. - \frac{3}{2} (|\Phi|^2 - |\Phi_{n+1}|^2) - |\Phi_{n+1}|^2 \right\} \end{aligned}$$

$$\begin{aligned}
&= \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_\alpha|^2 + \frac{1}{2} |\Phi_{n+1}|^2 (|\Phi|^2 - |\Phi_{n+1}|^2) \\
&\quad + |\Phi|^2 \left\{ n - \frac{3}{2} (|\Phi|^2 - |\Phi_{n+1}|^2) \right. \\
&\quad \left. + (nH^2 - \frac{(n-2)\sqrt{n}}{\sqrt{(n-1)}} H |\Phi_{n+1}| - |\Phi_{n+1}|^2) \right\}.
\end{aligned}$$

In the last term of (3.7), we need to estimate the following

$$(3.8) \quad z(H, |\Phi_{n+1}|) = nH^2 - \frac{(n-2)\sqrt{n}}{\sqrt{(n-1)}} H |\Phi_{n+1}| - |\Phi_{n+1}|^2.$$

Let $x = \sqrt{n}H$ and $y = |\Phi_{n+1}|$, then (3.8) becomes

$$z(H, |\Phi_{n+1}|) = x^2 - \frac{n-2}{\sqrt{n-1}} xy - y^2, \quad y \geq 0,$$

which could be consider as a quadratic surface in E^3 . By the following coordinate transformation:

$$(3.9) \quad \begin{cases} u = \frac{1}{\sqrt{2n}} \{ (1 + \sqrt{n-1})x + (1 - \sqrt{n-1})y \}, \\ v = \frac{1}{\sqrt{2n}} \{ -(1 - \sqrt{n-1})x + (1 + \sqrt{n-1})y \}, \end{cases}$$

we may find that

$$\begin{aligned}
(3.10) \quad z(H, |\Phi_{n+1}|) &= \frac{n}{2\sqrt{n-1}} (u^2 - v^2) \\
&= -\frac{n}{2\sqrt{n-1}} (v^2 + u^2 - 2u^2) \\
&\geq -\frac{n}{2\sqrt{n-1}} (v^2 + u^2) \\
&= -\frac{n}{2\sqrt{n-1}} (x^2 + y^2) \\
&= -\frac{n}{2\sqrt{n-1}} (nH^2 + |\Phi_{n+1}|^2).
\end{aligned}$$

Substituting (3.10) into (3.7), we have

$$\begin{aligned}
(3.11) \quad \frac{1}{2} \Delta |\Phi|^2 &\geq \sum_{\alpha=n+1}^{n+p} |\nabla \Phi_\alpha|^2 + \frac{1}{2} |\Phi_{n+1}|^2 (|\Phi|^2 - |\Phi_{n+1}|^2) \\
&\quad + |\Phi|^2 \left\{ n - \frac{3}{2} (|\Phi|^2 - |\Phi_{n+1}|^2) - \frac{n}{2\sqrt{n-1}} (nH^2 + |\Phi_{n+1}|^2) \right\}.
\end{aligned}$$

And taking $a = \max\{3/2, n/(2\sqrt{n-1})\}$, we may finally obtain the following, since $|\Phi|^2 = S - nH^2$,

$$(3.12) \quad \frac{1}{2}\Delta|\Phi|^2 \geq \sum_{\alpha=n+1}^{n+p} |\nabla\Phi_\alpha|^2 + \frac{1}{2}|\Phi_{n+1}|^2(|\Phi|^2 - |\Phi_{n+1}|^2) + |\Phi|^2(n - aS).$$

Proof of Theorem 1. Integrating (3.12) on M , we get (1.1). q.e.d.

Proof of Theorem 2. If $S \leq n/a$ on M , then either $S \equiv nH^2$ or $S \equiv n/a$ by Theorem 1. In the first case, M is totally umbilical and thus contained in S^{n+1} with constant curvature $(1 + |H|^2)$. (cf. [2, pp. 49–50]).

For the later, we may see from (3.12) that both $|\Phi_{n+1}|^2(|\Phi|^2 - |\Phi_{n+1}|^2)$ and $\sum_{\alpha=n+1}^{n+p} |\nabla\Phi_\alpha|^2$ must vanish.

(1) If $|\Phi_{n+1}| = 0$, which implies that $|A_{n+1}|^2 = nH^2$, M is a pseudo-umbilical submanifold of S^{n+p} . Applying Proposition 1 of Chen and Yano, we find that M is minimal in a hypersphere S^{n+p-1} of S^{n+p} with constant scalar curvature. But, since the equality in (3.10) holds when $s = n/a$, we may find that u must vanish. Consequently, from (3.9) the mean curvature H of M would be vanish which contradicts the hypothesis of the theorem.

(2) If $|\Phi|^2 - |\Phi_{n+1}|^2 = 0$, and plus that $\nabla\Phi_\alpha = 0$, for $\alpha = n+1, \dots, n+p$, we may find that M is a hypersurface of S^{n+1} in S^{n+p} . Then by Proposition 2 of Hou, we obtain that

(a) For $2 \leq n \leq 7$, we have $a = 3/2$, and then $S = 2n/3 < 2\sqrt{n-1}$, thus M is a small hypersphere $S^n(r)$ of radius $r = \sqrt{n/(n+S)}$.

(b) For $n \geq 8$, we have $a = n/2\sqrt{n-1}$, and then $S = 2\sqrt{n-1}$, thus M is either $S^n(r_0)$ or $S^1(r) \times S^{n-1}(s)$, where $r_0^2 = n/(n+2\sqrt{n-1})$, $r^2 = 1/(1+\sqrt{n-1})$ and $s^2 = \sqrt{n-1}/(1+\sqrt{n-1})$. q.e.d.

Remark. When the dimension n of M is in $2 \leq n \leq 7$, then $a = 3/2$ and our Theorem 1 is a generalization of Theorem 2 of A. M. Li and J. M. Li [7].

Theorem 2 generalizes Theorem 2 of [4].

REFERENCES

- [1] H. ALENCAR AND M. DO CARMO, Hypersurfaces with constant mean curvature in spheres, Proc. Amer. Math. Soc., **120** (1994), 1223–1229.
- [2] B. Y. CHEN, Geometry of Submanifolds, M. Dekker, New York, 1973.
- [3] B. Y. CHEN AND K. YANO, Minimal submanifolds of a higher dimensional sphere, Tensor (N. S.), **22** (1971), 369–373.
- [4] G. CHEN AND X. ZOU, Rigidity of compact submanifolds in a unit sphere, Kodai Math. J., **18** (1995), 75–85.
- [5] S. S. CHERN, M. DO CARMO AND S. KOBAYASHI, Minimal submanifolds of a sphere with second fundamental form of constant length, Functional Analysis and Related Fields (Proc. Conf. for Stone), Springer-Verlag, New York, 1970, 59–75.

- [6] Z. H. HOU, Hypersurfaces in a sphere with constant mean curvature, Proc. Amer. Math. Soc., **125** (1997), 1193–1196.
- [7] A. M. LI AND J. M. LI, An intrinsic rigidity theorem for minimal submanifolds in a sphere, Arch. Math. (Basel), **58** (1992), 582–594.
- [8] W. SANTOS, Submanifolds with parallel mean curvature vector in spheres, Tôhoku Math. J., **46** (1994), 403–415.
- [9] J. SIMONS, Minimal varieties in Riemannian manifolds, Ann. of Math., **88** (1968), 62–105.

DEPARTMENT OF MATHEMATICS
GUANGZHOU EDUCATIONAL COLLEGE
GUANGZHOU 510030
CHINA

DEPARTMENT OF MATHEMATICS
SOUTH CHINA NORMAL UNIVERSITY
GUANGZHOU 510631
CHINA
E-mail: lisj@scnu.edu.cn