Justin Ciocoi

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CSCI 360 Textbook Notes

Chapter 7: The RSA Cryptosystem

7.1: Introduction

- The RSA Crypto scheme is currently the most widely used asymmetric cryptographic scheme
- There are many different applications for RSA, but in practice it is most often used for
    - Encryption of *small* pieces of data, especially when dealing with *key transport*
    - Digital signatures, which are used for digital certificates on the internet
- It should be noted however, that RSA is not intended to replace symmetric cryptography due to the fact that RSA is several times slower as ciphers such as AES
    - This is because of the many computations which are involved in the RSA algorithm
- The underlying one-way function of RSA is the integer factorization problem:
    - Multiplying two large primes is computationally easy
    - Factoring the resulting problem is computationally hard

7.2: Encryption and Decryption

- **RSA Encryption**
    - Given the public key $(n, e) = k_{pub}$ and the plaintext, x, the encryption function is:

$$y = e_{k_{pub}}(x) \equiv x^e \ mod \ n$$

where $x, y \in \mathbb{Z}_n$

- **RSA Decryption**

- Given the private key $d = k_{pr}$ and the ciphertext y, the decryption function is:

$$x = d_{k_{pr}}(y) \equiv y^d \ mod \ n$$

where $x, y \in \mathbb{Z}_n$

- In practice, the variables, x, y, n, d, are very long numbers, usually 1024 bits long or more
- e is often referred to as the *encryption exponent* or *public exponent*, and d is often referred to as the *decryption exponent* or *private exponent*
- Let us say user A wants to send a message to user B
  - User A must first have user B's public key, (n, e)
  - User B will decrypt the message using their private key d
- We can now state a few requirements for the RSA cryptosystem
  - Since an attacker has access to the public key, it must be computationally infeasible to calculate d given public key values e and n
  - Since x is only unique up to the size of the modulus n, more than l bits cannot be encrypted with one RSA encryption, where l is the bit length of n
  - It should be relatively easy to calculate $x^e \ mod \ n$ (encryption) and $y^d \ mod \ n$ (decryption)
    - This means we need a method for fast exponentiation with very long numbers
  - For a given n, there should be many private/public key pairs, otherwise an attacker might be able to perform a brute force attack

7.3: Key Generation and Proof of Correctness

- A distinctive feature of all asymmetric cryptosystems is that there is a set-up phase during which the public and private keys are computed
- Depending on the scheme, key generation can be quite complex, which is usually not an issue for block or stream ciphers
- **RSA Key Generation**
  - **Output:** public key: $k_{pub} = (n, e)$ and private key $k_{pr} = (d)$

1. Choose two large primes, p, and q
2. Compute $n = p \cdot q$
3. Compute $\Phi(n) = (p-1)(q-1)$
4. Select the public exponent $e \in [1, 2, ..., \Phi(n) - 1]$ such that

$$gcd(e, \Phi(n)) = 1$$

5. Compute the private key d such that

$$d \cdot e \equiv mod\ \Phi(n)$$

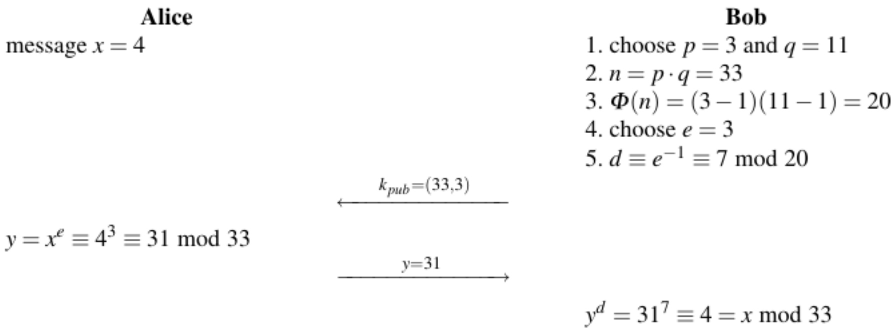
- The condition for public exponent selection will ensure that the inverse of e exists in modulo $\Phi(n)$, such that there will always be a private key d
- The computation of d and e can be done at once using the extended Euclidean algorithm
  - In practice, one first selects a public parameter e in the range $0 < e < \Phi(n)$, where the value of e satisfies the relationship $gcd(e, \Phi(n)) = 1$
  - We can apply the extended Euclidean algorithm with the input parameters n and e and obtain the following relationship

$$gcd(\Phi(n), e) = s \cdot \Phi(n) + t \cdot e$$

  - Where we know that e is a valid public key if $gcd(e, \Phi(n)) = 1$
  - We also know that the parameter t calculated using the extended Euclidean algorithm is the inverse of e and thus

$$d = t\ mod\ \Phi(n)$$

  - In the case that e does not satisfy the above condition, we simply choose a new value for e and repeat the process
- Here we can see an example



- Expressed as an equation, we can consider the process as:

$$d_{k_{pr}}(y) = d_{k_{pr}}(e_{pub}(x)) \equiv (x^e)^d \equiv x^{de} \equiv x \; mod \; n$$

7.4: Encryption and Decryption: Fast Exponentiation

- Since asymmetric algorithms are based on arithmetic with very long numbers, we need a method by which to quickly exponentiate large integers such that asymmetric schemes do not use too much system overhead in encryption and decryption

- Recall the RSA encryption and decryption functions

$$y = e_{k_{pub}}(x) \equiv x^e \; mod \; n$$
$$x = d_{k_{pr}}(y) \equiv y^d \; mod \; n$$

- A straightforward exponentiation method looks as follows

$$\boxed{x} \xrightarrow{\text{square}} \boxed{x\hat{}2} \xrightarrow{\text{multiply}} \boxed{x\hat{}3} \xrightarrow{\text{multiply}} \boxed{x\hat{}4}$$

- For example, to compute the value of x^8 would require one squaring and six multiplications

- Instead, we could do the following

$$\boxed{x} \xrightarrow{\text{square}} \boxed{x\hat{}2} \xrightarrow{\text{square}} \boxed{x\hat{}4} \xrightarrow{\text{square}} \boxed{x\hat{}8}$$

- This requires only 3 operations as opposed to the seven from the naive approach

- However, for a specific given exponent, we do not know the exact order of the square and multiply operations, and we thus have an algorithm which provides a systematic way to check the order of these operations

- **Square-and-Multiply Algorithm for Modular Exponentiation**

- *Input:*
  - Base element: x
  - Exponent: $H = \sum_{i=0}^{t} h_i 2^i$ with $h_i \in [0, 1]$ and $h_t = 1$ and modulus n
- *Output:* $x^H \ mod \ n$
- *Initialization:* $r = x$
- *Algorithm:*

```
for(i=t−1; i!=0; i++)
{
    r = (r*r) mod n

    if(h_i == 1)
        r = (r*x) mod n
}
return r
```

7.5: Speed-up Techniques for RSA

- **7.5.1: Fast Encryption with Short Public Exponents**
  - When concerned with the public exponent, e, which is used for encryption, we can choose a very small value for e
    - In practice, the values 3, 17, and $2^{16}+1$ are of particular importance for RSA
    - Below we can see a table of the complexities of the three important public exponents

Public key e	e as binary string	#MUL + #SQ
3	11_2	3
17	10001_2	5
$2^{16}+1$	$1\,0000\,0000\,0000\,0001_2$	17

  - RSA is still secure when using these particularly short public exponents
  - This scheme is useful for speeding up the encryption portion of RSA, but when the public key, d is involved, i.e. for decryption or signature generation, it is not possible to speed RSA up with this technique
- **7.5.2: Fast Decryption with the Chinese Remainder Theorem**

- In RSA, a short private key cannot be chosen without compromising the security of encryption
- The goal here is to perform $x^d \ mod \ n$ efficiently
  - The party who possesses the private key also possesses the primes p and q
  - The basic idea of the Chinese Remainder Theorem is that rather than doing arithmetic with one long modulus, n we can perform two individual exponentiations modulus the shorter primes p and q
  - We have to transform *into* the CRT domain, perform computations within the CRT domain, and inverse transform to find the result
- First we reduce x to obtain what is known as the *modular* representation of x

$$x_p \equiv x \ mod \ p$$
$$x_q \equiv x \ mod \ q$$

- Now, we will perform the following exponentiations

$$y_p = x_p^{d_p} \ mod \ p$$
$$y_q = x_q^{d_q} \ mod \ q$$

- Where the exponents d_i are given by

$$d_p \equiv d \ mod \ (p-1)$$
$$d_q \equiv d \ mod \ (q-1)$$

- To perform the inverse transformation we must first compute two coefficients given by

$$c_p \equiv q^{-1} \ mod \ p$$
$$c_q \equiv p^{-1} \ mod \ q$$

- And then assemble the final result, y

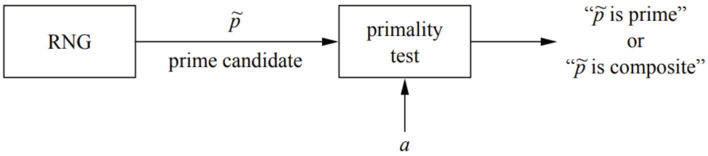
$$y = [q * c_p]y_p + [p * c_q]y_q \ mod \ n$$

- The total speed-up obtained by this implementation of the Chinese Remainder Theorem is a factor of 4, and there are hardly any drawbacks, so it is very frequently used in real-world cryptographic applications

7.6: Finding Large Primes

- One thing that hasn't been discussed is how the two primes p and q are generated

- Since their product is the RSA modulus, $n = p \cdot q$, each of the two primes should have about half the bit length of n



- The practicality of this approach is put into question by two factors
    - The likelihood of finding a prime number using an RNG
    - The speed at which we can check whether the randomly generated number is prime or composite
- **Fermat Primality Test**
    - One primality test which is commonly used is based on *Fermat's Little Theorem*
    - *Input:* prime candidate $\tilde{p}$ and security parameter s
    - *Output:* statement "$\tilde{p}$ is composite" or "$\tilde{p}$ is likely prime"
    - *Algorithm:*

    ```
    for(i=1; i<s; i++)
    {
        choose random a in {2, 3, 4, ..., p-2}
        if a^{p-1} != 1
            return p is composite
    }
    return p is likely prime
    ```

    - Since this algorithm is still capable of producing incorrect results via Carmichael numbers, or *Fermat liars*
- **Miller-Rabin Primality Test**
    - For a given odd integer, $n > 2$, we can write $n - 1$ as $2^s d$ where s is a positive integer and d is an odd positive integer
    - We will then consider an integer a which we can call the *base*, which is co-prime to n
    - Then, n is said to be a *strong probable prime* to base a if one of these congruence relations holds

$$a^d \equiv 1 \; (mod \; n)$$
$$a^{2^r d} \equiv -1 \; (mod \; n)$$

where $0 \leq r < s$

7.7: RSA in Practice: Padding

- In practice, RSA has to be used with a *padding scheme* which is extremely important and can lead to insecurity if they are not implemented properly
- This is due to the problematic properties of RSA encryption listed below
    - RSA encryption is deterministic, i.e., for a specific key, a particular plaintext is always mapped to a particular ciphertext, creating vectors for a potential statistical attack
    - Plaintext values $x = 0, x = 1, x = -1$ produce ciphertexts equal to $0, 1, -1$
    - Small public exponents e and small plaintexts x might be subject to attacks if no padding, or weak padding, is used
- Another undesirable property of RSA is that it is *malleable*
    - This means that the attacker is capable of transforming the ciphertext into another ciphertext which leads to a known transformation of the plaintext
    - The attacker cannot decrypt the ciphertext here, but they are capable of manipulating the plaintext in a predictable manner
- One possible solution to all of the problems mentioned above is the use of padding
- Modern techniques, such as the *Optimal Asymmetric Encryption Padding (OAEP)* for padding RSA messages are specified and standardized in *Public Key Cryptography Standard #1 (PKCS#1)*
- Let M be the message which will be padded, let k be the length of modulus n in bytes, let $|H|$ be the length of the hash function output in bytes, and let $|M|$ be the length of the message in bytes
- Let L be an optional label which is left as an empty string if not used
- According to the most recent standard, padding a message within the RSA Encryption scheme is done in the following way:
    1. Generate a string, PS of length $k - |M| - 2|H| - 2$ of zeroed byte

- The length of PS may be 0

2. Concatenate $hash(L), PS$, a single byte with ha hexadecimal value of $0x01$, and the message M, to form a data block, DB of length $k - |H| - 1$ bytes
   - $$DB = Hash(L)||PS||0x01||M$$

3- Generate a random byte string *seed* of length $|H|$

4- Let $dbMask = MGF(seed, k - |H| - 1)$, where MGF is the mask generation function which, in practice, is often a hash function such as SHA-1

5- Let $maskedDB = DB \oplus dbMask$

6- Let $seedMask = MGF(maskeedDB, |H|)$

7- Let $maskedSeed = seed \oplus seedMask$

8- Concatenate a single byte with hexadecimal value $0x00$, $maskedSeed$, and $maskedDB$ to form an encoded message, EM, of length k bytes

- $$EM = 0x00||maskedSeed||maskedDB$$

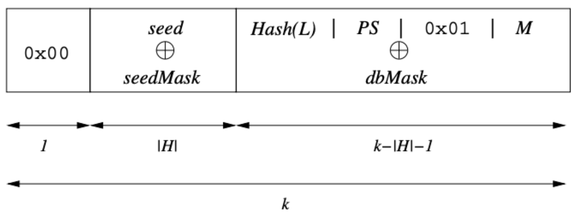


Fig. 7.3 RSA encryption of a message *M* with Optimal Asymmetric Encryption Padding (OAEP)