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MATHEMATICS

Lesson 1: Functions

Learning Outcome(s): At the end of the lesson, the learner is able to represent real-life situations using functions, including piecewise functions.

Lesson Outline:

1. Functions and Relations
2. Vertical Line Test
3. Representing real-life situations using functions, including piecewise functions.

Definition: A relation is a rule that relates values from a set of values (called the domain) to a second set of values (called the range).
A relation is a set of ordered pairs (x,y).

Definition: A function is a relation where each element in the domain is related to only one value in the range by some rule.

A **function** is a set of ordered pairs (x,y) such that no two ordered pairs have the same x-value but different y-values. Using functional notation, we can write f(x) = y, read as "f of x is equal to y."
In particular, if (1, 2) is an ordered pair associated with the function f,
then we say that $f(2) = 1$.

Example. Which of the following relations are functions?

$f = \{(1,2),(2,3),(3,5),(4,7)\}$
$g = \{(1,3),(1,4),(2,5),(2,6),(3,7)\}$
$h = \{(1,3),(2,6),(3,9),\ldots,(n,3n),\ldots\}$

Solution.
The relations f and h are functions because no two ordered pairs have the same x-value but different y-values. Meanwhile, g is not a function because (1,3) and (1,4) are ordered pairs with the same x-value but different y-values.

Relations and **functions** can be represented by mapping diagrams where the elements of the domain are mapped to the elements of the range using arrows. In this case, the relation or function is represented by the set of all the connections represented by the arrows.

The Vertical Line Test
A graph represents a function if and only if each vertical line intersects the graph at most once.

Important Concepts.

- **Relations** are rules that relate two values, one from a set of inputs and the second from the set of outputs.
- **Functions** are rules that relate only one value from the set of outputs to a value from the set of inputs.

Definition: The domain of a relation is the set of all possible values that the variable x can take.

Functions as representations of real-life situations.

Functions can often be used to model real situations. Identifying an appropriate functional model will lead to a better understanding of various phenomena.

Example. Give a function *C* that can represent the cost of buying x meals, if one meal costs P40.

Solution. Since each meal costs P40, then the cost function is $C(x) = 40x$.

Piecewise functions.
Some situations can only be described by more than one formula, depending on the value of the independent variable.

Lesson 2: Evaluating Functions

Learning Outcome(s): At the end of the lesson, the learner is able to evaluate functions and solve problems involving functions.

Lesson Outline:

1. Evaluating functions

Evaluating a function means replacing the variable in the function, in this case x, with a value from the function's domain and computing for the result. To denote that we are evaluating *f* at *a* for some *a* in the domain of *f*, we write *f*(*a*).

Example. Evaluate the following functions at $x = 1.5$:
(g) $f(x) = 2x + 1$
(h) $q(x) = x^2 - 2x + 2$
(i) $g(x) = \sqrt{x+1}$
(j) $r(x) = (2x+1)/(x-1)$
(k) $f(x) = [x] + 1$, where $[x]$ is the greatest integer function.

Solution. Substituting 1.5 for x in the functions above, we have
(a) $f(1.5) = 2(1.5) + 1 = 4$
(b) $q(1.5) = (1.5)^2 - 2(1.5) + 2 = 2.253 - 3 + 2 = 1.25$
(c) $g(1.5) = \sqrt{1.5 + 1} = \sqrt{2.5}$
(d) $r(1.5) = [2(1.5) + 1] / (1.5 - 1) = [(3 + 1) / 0.5] = 8$
(e) $f(1.5) = [1.5] + 1 = 1 + 1 = 2$

Lesson 3: Operations on Functions

Learning Outcome(s): At the end of the lesson, the learner is able to perform addition, subtraction, multiplication, division, composition of functions, and solve problems involving functions.

Lesson Outline:

1. Review: Operations on algebraic expressions
2. Addition, subtraction, multiplication, and division of functions
3. Function composition

Addition and Subtraction:
(a) Find the least common denominator (LCD) of both fractions.
(b) Rewrite the fractions as equivalent fractions with the same LCD.
(c) The LCD is the denominator of the resulting fraction.
(d) The sum or difference of the numerators is the numerator of the resulting fraction.

Example. Find the sum of 1/3 and 2/5

Solution. The LCD of the two fractions is 15.

$$1/3 + 2/5 = 5/15 + 6/15 = 5 + 6/15 = 11/15$$

Multiplication:
(a) Rewrite the numerator and denominator in terms of its prime factors.
(b) Common factors in the numerator and denominator can be simplified as "1" (this is often called "cancelling").
(c) Multiply the numerators together to get the new numerator.
(d) Multiply the denominators together to get the new denominator.

Example. Find the product of 10/21 and 15/8. Use cancellation of factors when convenient.

Solution. Express the numerators and denominators of the two fractions into their prime factors. Multiply and cancel out common factors in the numerator and the denominator to reduce the final answer to lowest terms.

$$10/21 \times 15/8 = 10 \times 15 / 21 \times 8 = (2 \times 5 / 3 \times 7) \times (3 \times 5 / 2 \times 2 \times 2)$$
$$= (\cancel{2} \times 5 \times \cancel{3} \times 5 / \cancel{3} \times 7 \times \cancel{2} \times 2 \times 2) = 25/28$$

Division:
To divide two fractions or rational expressions, multiply the dividend with the reciprocal of the divisor.

Example. Divide $[(2x^2 + x - 6) / (2x^2 + 7x + 5)]$ by $[(x^2 - 2x - 8) / (2x^2 - 3x - 20)]$

Solution. $[(2x^2 + x - 6) / (2x^2 + 7x + 5)] / [(x^2 - 2x - 8) / (2x^2 - 3x - 20)]$

$[(2x^2 + x - 6) / (2x^2 + 7x + 5)] \times [(2x^2 - 3x - 20) / (x^2 - 2x - 8)]$

$[(2x - 3)(x + 2) / (2x+5)(x + 1)] \times [(x - 4)(2x + 5) / (x + 2)(x - 4)]$

$[(2x - 3)\cancel{(x + 2)(x - 4)(2x + 5)}] / [\cancel{(2x+5)}(x + 1)\cancel{(x + 2)(x - 4)}]$

$2x - 3 / x + 1$

Definition. Let f and g be functions.
1. Their sum, denoted by $f + g$, is the function denoted by
   $(f+g)(x) = f(x) + g(x)$.
2. Their difference, denoted by $f - g$, is the function denoted by
   $(f-g)(x) = f(x) - g(x)$.
3. Their product, denoted by $f \times g$, is the function denoted by
   $(f \times g)(x) = f(x) \times g(x)$.
4. Their quotient, denoted by f/g, is the function denoted by
   $(f/g)(x) = f(x)/g(x)$, excluding the values of x where $g(x) = 0$.

Lesson 4: Rational Functions, Equations, and Inequalities
Learning Outcome(s): At the end of the lesson, the learner is able to distinguish among rational functions, rational equations, and rational inequalities

Lesson Outline:
1. Rational functions, rational equations, and rational inequalities

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Definition:
A **rational expression** is an expression that can be written as a ratio of two polynomials.

Some examples of rational expressions are $2/x$, $5/x-3$, and $5x + 1/ 4x^2$

A **rational equation or inequality** can be solved for all x values that satisfy the equation or inequality. A rational function expresses a relationship between two variables (such as x and y), and can be represented by a table of values or a graph.

A **rational equation** is an equation involving rational expressions.

Example.

$$(2/x) – (3/2x) = 1/5$$

A **rational inequality** is an equation involving rational inequality.

Example.

$$(5/ x – 3) > 2x$$

A **rational function** is a function of the form f(x) = p(x) q(x) where p(x) and q(x) are polynomial functions and q(x) is not the zero function (i.e., q(x) ≠ 0).

Example.

$$f(x) = x^2 + 2x + 3/ x+1$$

Lesson 5: Solving Rational Equations and Inequalities
Learning Outcome(s): At the end of the lesson, the learner is able to solve rational equation and inequalities, and solve problems involving rational equations and inequalities.
Lesson Outline:

1. Solving rational equations.
2. Solving rational inequalities.
3. Solving word problems involving rational equations or inequalities.

To solve a rational equation:
(a) Eliminate denominators by multiplying each term of the equation by the least common denominator.
(b) Note that eliminating denominators may introduce extraneous solutions. Check the solutions of the transformed equations with the original equation.

Example. Solve for x: 2/x – 3/2x = 1/5

Solution. The LCD of all the denominators is 10x.
Multiply both sides of the equation by 10x and solve the resolving equation.

$$10x(2/x) – 10(3/2x) = 10x(1/5)$$
$$20 – 15 = 2x$$
$$5 = 2x$$
$$5/2 = x$$

To solve rational inequalities:

(a) Rewrite the inequality as a single rational expression on one side of the inequality symbol and 0 on the other side.

(b) Determine over what intervals the rational expression takes on positive and negative values.

i. Locate the x values for which the rational expression is zero or undefined (factoring the numerator and denominator is a useful strategy).

ii. Mark the numbers found in (i) on a number line. Use a shaded circle to indicate that the value is included in the solution set, and a hollow circle to indicate that the value is excluded. These numbers partition the number line into intervals.

iii. Select a test point within the interior of each interval in (ii). The sign of the rational expression at this test point is also the sign of the rational expression at each interior point in the aforementioned interval.

iv. Summarize the intervals containing the solutions.

Warning! Multiplying both sides of an inequality by a number requires that the sign (positive or negative) of the number is known. Since the sign of a variable is unknown, it is not valid to multiply both sides of an inequality by a variable.

Example. Solve the inequality $2x/(x+1) \geq 1$.

Solution.

(a) Rewrite the inequality as a single rational expression.

$$2x/(x+1) \geq 1$$
$$[2x/(x+1)] - 1 \geq 0$$
$$2x - (x+1) / (x+1) \geq 0$$
$$x - 1 / x + 1 \geq 0$$
$$x = 1$$

(b) The rational expression will be zero for $x = 1$ and undefined for $x = -1$.The value $x = 1$ is included while $x = -1$ is not.

(c) Since we are looking for the intervals where the rational expression is positive or zero, we determine the solution to be the set **$\{x \in R \mid x < -1 \text{ or } x \geq 1\}$.**

Example.

In an inter-barangay basketball league, the team from Barangay Culiat has won 12 out of 25 games, a winning percentage of 48%. How many games should they win in a row to improve their win percentage to 60%.

Solution.

Let **x** represent the number of games that they need to win to raise their percentage to 60%. The team has already won 12 out of their 25 games. If they win x games in a row to increase their percentage to 60%, then they would have played 12+x games out of their 25+x games.

The equation is (12 + x)/(25 + x) = 0.60. Multiply 25 + x to both sides of the equation and solve the resulting equation.

$$(12 + x) / (25 + x) = 0.60$$
$$12 + x = (25 + x)(0.6)$$
$$12 + x = 0.6(25) + 0.6(x)$$
$$x - 0.6x = 15 - 12$$
$$0.4x = 3$$
$$x = 7.5$$

Therefore, Barangay Culiat needs to win 8 games in a row to raise their winning percentage to 60%.