Solutions Manual

to Introduction to the Theory of Statistics (1974)

by A.M. Mood, F.A. Graybill, and D.C. Boes

Anonymous

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Contents

CHAPTER I

Chapter I PROBLEMS

1. (a) $\Omega = \{(B, W), (B, G), (G, W), (G, G)\}$. The sample space contains four outcomes; an outcome itself is a 2-tuple where the first component represents the result of drawing from urn one and the second component from urn two.

   (b) The event space is the collection of all subsets of the sample space. There are 16 such subsets.

   $$\mathscr{A} = \Big\{\phi, \Omega, \{(B,W)\}, \{(B,G)\}, \{(G,W)\}, \{(G,G)\},$$
   $$\{(B,W),(B,G)\}, \{(B,W),(G,W)\}, \{(B,W),(G,G)\}, \{(B,G),(G,W)\},$$
   $$\{(B,G),(G,G)\}, \{(G,W),(G,G)\}, \{(B,W),(B,G),(G,W)\},$$
   $$\{(B,W),(B,G),(G,G)\}, \{(B,W),(G,W),(G,G)\}, \{(B,G),(G,W),(G,G)\}\Big\}$$

   (c) 1/4

   (d) 0

2. (a) There are many ways to describe the outcomes of this experiment. For example, one could number the balls in urn one as 1, 2, 3 red; 4, 5 white; and 6 blue and those in urn two as 1 red, 2, 3 white; and 4, 5, 6 blue.

   i. Then $\Omega = \{(i_1, i_2) : i_1 = 1, \ldots, 6$ and $i_2 = 1, \ldots, 6$, where
   i_1 is the number on the ball drawn from urn 1 and
   i_2 is the number on the ball drawn from urn 2.$\}$
   Note that there are 36 outcomes in this experiment.

   ii. Let A denote the event both balls are red.
   B denote the event both balls are white, and
   C denote the vent both balls are blue.
   Then $P[\text{both balls are same color}] = P[A \cup B \cup C] = P[A] + P[B] + P[C] = \frac{3}{36} + \frac{4}{36} + \frac{3}{36}$.

   iii. $P[A] = \frac{3}{36} < \frac{4}{36} = P[B]$

   (b) (i) $\frac{12 \cdot 8 \cdot 4}{12^3}$ (ii) $\frac{12 \cdot 8 \cdot 4}{12 \cdot 11 \cdot 10}$

4. (a) $\Omega = \{(i_1, i_2) : i_1 = 1,\dots,5$ and $i_2 = 1,\dots,5$, where i_1 is the number on the first ball drawn and i_2 is the number on the second ball drawn$\}$.

$B_1 = \{(i_1, i_2) : i_1 = 1,2,3 \text{ and } i_2 = 1,\dots,5\}$

$B_2 = \{(i_1, i_2) : i_1 = 1,\dots,5 \text{ and } i_2 = 1,2,3\}$

$B_1B_2 = \{(i_1, i_2) : i_1 = 1,2,3 \text{ and } i_2 = 1,2,3\}$

(b) $P[B_1] = \dfrac{\text{size of } B_1}{\text{size of } \Omega} = \dfrac{3\cdot 5}{5\cdot 5}$.

(c) $\Omega = \{(i_1, i_2) : i_1 = 1,\dots,5 \text{ and } i_2 = 1,\dots,5 \text{ but } i_1 \neq i_2\}$.

$B_1 = \{(i_1, i_2) : i_1 = 1,2,3 \text{ and } i_2 = 1,\dots,5 \text{ but } i_1 \neq i_2\}$.

$P[B_1] = \dfrac{3\cdot 4}{5\cdot 4}$, etc.

7. Using H for hit, M for miss, R for right hand and L for left hand, the event the the participant is successful is

$\{(H,H,H),(H,H,M),(M,H,H)\} = A$, say.

Under strategy RLR, $P[A] = p_1p_2p_1 + p_1p_2(1-p_1) + (1-p_1)p_2p_1$ and

under strategy LRL, $P[A] = p_2p_1p_2 + p_2p_1(1-p_2) + (1-p_2)p_1p_2$.

8. (b) $P[A \text{ will beat B in three out of four}] = p^3 + 3p^3(1-p) = \binom{4}{3}p^3(1-p) + p^4$

$P[A \text{ will beat B in five out of seven}] = p^5 + 5p^5(1-p) + 15p^5(1-p)^2$

$$= \binom{7}{5}p^5(1-p)^2 + \binom{7}{6}p^6(1-p) + p^7$$

10. $A = B$ and $p = 1/2$ is a counterexample.

14. $P[AB] = P[A] + P[B] - P[A \cup B] \geq P[A] + P[B] - 1 = 1 - \alpha - \beta$.

18. (a) $(1/3)^4$

(b) $3(1/3)^4$

(c) $3(1/3)^4 + 4\cdot 3(1/3)^4 = 5/27$

19. (a) $P[\text{total of } 9] = 25/216$; $P[\text{total of } 10] = 27/216$

(b) $P[\text{at least one 6 in 4 tosses}] = 1 - (5/6)^4$

$P[\text{at least double 6 in 24 tosses}] = 1 - (35/36)^{24}$

(c) $P[\text{at least one 6 with 6 dice}] = 1 - (5/6)^6$

$P[\text{at least two 6's with 12 dice}] = 1 - (5/6)^{12} - (12)(1/6)(5/6)^{11}$

20. This is similar to Problem 27.

22. $(365)_{25}/(365)^{25}$.

23. $$\frac{\binom{5}{2}\binom{21}{11}}{\binom{26}{13}} + \frac{\binom{5}{3}\binom{21}{10}}{\binom{26}{13}}.$$

24. (a) $$\frac{\binom{r}{k}(n-1)^{r-k}}{n^r}$$

25. Consider that a single coin is tossed until the first head occurs.

$P[\text{first head occurs on toss } j] = (1/2)^j$.

$P[\text{Ace wins}] = (1/2) + (1/2)^5 + (1/2)^7 + \ldots = 4/7$.

$P[\text{Bones wins}] = (1/2)^2 + (1/2)^5 + (1/2)^8 + \ldots = 2/7$.

$P[\text{Clod wins}] = (1/2)^3 + (1/2)^6 + (1/2)^9 + \ldots = 1/7$.

26. $P[\text{single ring formed}] = (4/5)(2/3)$.

$P[\text{at least one ring formed}] = 1$.

27. You might test your intuition on this one and guess the answer before you proceed. Let A_1 = {Mr. Bandit does not get caught under strategy 1} where strategy 1 is to sell all twenty at once; strategy 2 is to put four stolen cattle in one set of ten; strategy 3 is to put three stolen cattle in one set of ten and one on the other; and strategy 4 is to put two stolen cattle in each set of ten.

$$P[A_1] = \frac{\binom{4}{0}\binom{16}{4}}{\binom{20}{4}}$$

$$P[A_2] = \frac{\binom{4}{0}\binom{6}{2}}{\binom{10}{2}} \cdot \frac{\binom{10}{2}}{\binom{10}{2}}$$

$$P[A_3] = \frac{\binom{3}{0}\binom{7}{1}}{\binom{10}{2}} \cdot \frac{\binom{1}{0}\binom{9}{1}}{\binom{10}{2}}$$

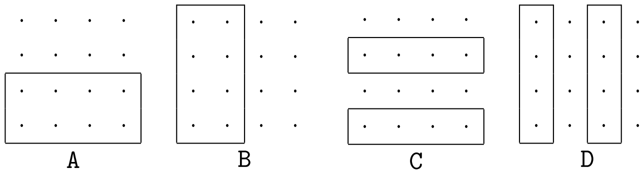
$$P[A_4] = \frac{\binom{2}{0}\binom{8}{2}}{\binom{10}{2}} \cdot \frac{\binom{2}{0}\binom{8}{2}}{\binom{10}{2}}$$

30. (b) Take $B = C$ and $P[A] > P[B]$ for a counterexample.

31. Use the corollary of Thorem 29.

32. This is similar to Problem 70. Use Bayes' Formula.

33. (c)



34. Use Theorem 29.

39. Use Bayes' Formula.

40. This problem is known as the "liars problem." It can be varied by changing the number of liars. In fact, the reader might want to try to solve it for only two or three liars before reading the solution. As is the case with most "story" problems some "modelling" is required. Let A_T = {statement that A makes is true}, and D_T = {D says that C says that B says that A is telling the truth}, then $P[A_T|D_T]$ is what is sought. Also, let

B_T = {B says that A is telling the truth}, and

C_T = {C says that B says that A is telling the truth}.

Note that $\overline{C}_T$ = {C says that B says that A is not telling the truth} and similarly for $\overline{B}_T$ and $\overline{D}_T$. Actually some "modelling" has been done in defining these events; for example, it has been assumed that B does say that A's statement is either true or false. Note that

$1/3 = P[A_T] = P[B_T|A_T] = P[C_T|B_T] + P[D_T|C_T] = P[C_T|B_TA_T] = P[D_T|C_TA_T]$, and

$2/3 = P[\overline{A}_T] = P[B_T|\overline{A}_T] = P[C_T|\overline{B}_T] = P[D_T|\overline{C}_T] = P[C_T|\overline{B}_TA_T] = P[D_T|\overline{C}_TA_T]$.

Implicitly, it has been assumed that not only does each liar lie with probability 2/3 in any given instance, but also the liars lie independently of each other. The solution given here includes the solution for the two and three liars problems.

$P[B_T] = P[B_T|A_T]P[A_T] + P[B_T|\overline{A}_T]P[\overline{A}_T] = (1/3)(1/3) + (2/3)(2/3) = 5/9$, so

$P[A_T|B_T] = \dfrac{P[B_T|A_T]P[A_T]}{P[B_T]} = 1/5$, the solution to the two liar problem.

Now $P[C_T] = (1/3)(5/9) + (3/2)(4/9) = 13/27$ and

$$P[C_T|A_T] = P[C_TB_T|A_T] + P[C_T\overline{B}_T|A_T]$$
$$= P[C_T|B_TA_T]P[B_T|A_T] + P[C_T|\overline{B}_TA_T]P[\overline{B}_T|A_T]$$
$$= 5/9, \text{ hence}$$

$P[A_T|C_T] = \dfrac{(5/9)(1/3)}{(13/27)}$, the solution to the three liar problem.

Similarly, $P[D_T] = P[D_T|C_T]P[C_T] + P[D_T|\overline{C}_T]P[\overline{C}_T]$

$$= (1/3)(13/27) + (2/3)(14/27) = 41/81, \text{ and}$$

$$P[D_T|A_T] = P[D_T|C_TA_T]P[C_T|A_T] + P[D_T|\overline{C}_TA_T]P[\overline{C}_T|A_T]$$
$$= (1/3)(5/9) + (2/3)(4/9) = 13/27, \text{ and}$$

finally,

$$P[A_T|D_T] = \frac{(13/27)(1/3)}{41/81} = \frac{13}{41}.$$

42. (a) 2/3

(b) 4/5

(c) 1

46. (b) A and B disjoint and $P[A] \neq P[B]$ gives a counterexample.

48. Let A_j = {exactly j seeds out of the fifty germinate}.

Model by assuming each seed germinates with probability 0.96. P[package will violate guarantee] =

$$\sum_{j=0}^{44} P[A_j] = 1 - \sum_{j=45}^{50} P[A_j] = 1 - \sum_{j=45}^{50} (.96)^j(.04)^{50-j}.$$

50. Intuition says the answer ought to be greater than 1/2.

Let A = {tested stone is real}

B = {son gets real diamonds} We want $P[B|A]$ and $P[B|\overline{A}]$. Symmetry suggests that these two conditional probabilities are equal.

Define C = {box with two real diamonds is selected for testing} and model by assuming $P[C] = 1/2$, $P[A|C] = 2/3$, and $P[A|\overline{C}] = 1/3$.

Then $P[A] = P[A|C]P[C] + P[A|\bar{C}]P[\bar{C}] = 2/3 \cdot 1/2 + 1/3 \cdot 1/2 = 1/2$.

$$\begin{aligned} P[B|A] &= \frac{P[AB]}{P[A]} = \frac{P[AB|C]P[C] + P[AB|\bar{C}]P[\bar{C}]}{P[A]} \\ &= \frac{(2/3)(1/2) + (0)(1/2)}{1/2} = 2/3. \text{ Similarly} \\ P[B|\bar{A}] &= \frac{(0)(1/2) + (2/3)(1/2)}{1/2} = 2/3. \end{aligned}$$

57. Let A = {player wins}. Let B_j = {total of j on first toss}.

$$P[A] = \sum_{j=2}^{12} P[A|B_j]P[B_j].$$

59. (a) $p^4 + 4p^3(1-p) + 4p^2(1-p)^2 = a$ (say)

(b) $p^4 + 4p^3(1-p) + 2p^2(1-p)^2 = b$ (say)

(c) $pa + (1-p)b$

62. Mark first in a corner. The random player must then mark in the center to keep you from winning. Next mark one of the two spaces adjacent to your first mark, etc. Your opponent's chance of forcing a tie under this strategy is (1/8)(1/6)(1/4)(2/2). No other strategy does better. Your chance of winning is 191/192. How does the problem change if you allow your opponent to mark first?

63. Apply Bayes' Formula.

67. 3/4; 1/3

68. (a) Outcomes are yellow-smooth (Y-S), yellow-wrinkled (Y-W), green-smooth (G-S), and green-wrinkled (G-W); they are equally likely.

(b)

Y - S	Y - W	G - S	G - W
3/8	1/8	3/8	1/8

(c)

Y - S	Y - W	G - S	G - W
9/16	3/16	3/16	1/16

70. (a) $P[B|A] = \dfrac{P[A|B]P[B]}{P[A|B]P[B] + P[A|\bar{B}]P[\bar{B}]} = \dfrac{(.95)(.05)}{(.95)(.05) + (.05)(.95)} = \dfrac{1}{2}$.

(b) $.9 = \dfrac{p(.05)}{p(.05) + (1-p)(.95)}$ implies $p = \dfrac{17.1}{17.2} \approx .9942$.

CHAPTER II

Chapter II PROBLEMS

Several of these problems requires showing that a given function is a p.d.f. This simply involves verifying the conditions of Definition 9.

1. (a) $f_1(\cdot)$ and $f_2(\cdot)$ are esily shown to be p.d.f.s. Also, the integral of f(x) is clearly unity. One can show that $f(x) \geq 0$.

   (b) You can disprove this by taking $\theta_1 = -1, \theta_2 = 2$, $f_1(x) = I_{(0,1)}(x)$ and $f_2(x) = I_{(1,2)}(x)$.

2. The median is α.

3. Need $K\int_{-K}^{K} x^2\,dx = 1$, which gives K = fourth root of 3/2.

4. (a) Since $F_X(x)$ can be written as a function of $(x-\alpha)/\beta$, let's do it. That is, write $F_X(x) = F\left(\frac{x-\alpha}{\beta}\right)$.

$$
\begin{aligned}
\text{Now } E[X] &= \int_0^\infty \left[1 - F\left(\frac{x-\alpha}{\beta}\right)\right]\,dx - \int_{-\infty}^0 F\left(\frac{x-\alpha}{\beta}\right)\,dx \\
&= \beta\int_{-\alpha/\beta}^\infty (1-F(y))\,dy - \beta\int_{-\infty}^{-\alpha/\beta} F(y)\,dy \\
&= \beta\left\{\int_0^\infty (1-F(y))\,dy - \int_{-\infty}^0 F(y)\,dy + \int_{-\alpha/\beta}^0 (1-F(y))\,dy + \int_{-\alpha/\beta}^0 F(y)\,dy\right\} \\
&= \beta\left\{\int_0^\infty (1-F(y))\,dy - \int_{-\infty}^0 F(y)\,dy\right\} + \alpha.
\end{aligned}
$$

   E[X] equals α plus a quantity that does not depend on α; hence if α is increased by $\Delta\alpha$ so is E[X].

5. (b) X is a discrete random variable taking on values 0, 1, 2, and $P[X = 2] = (1/4)^2$, $P[X = 1] = 2(1/4)(3/4)$, and $P[X = 0] = (3/4)^2$.

   (c) $E[X] = 1/2$ and $\mathrm{var}[X] = 3/8$.

7. (a) The game ends at the first trial if and only if A wins first match; the game ends at the second trial if and only if B wins the first two matches; the game ends at the third trial if and only if B wins the first match and A wins the next two; etc.

   $P[X = j] = (1/2)^j$, $j = 1, 2, \ldots$.

(b) $E[X] = \sum_{j=1}^{\infty} j(1/2)^j = (1/2)\sum_{j=1}^{\infty} j(1/2)^j = 2.$

$\text{var}[X] = E[X^2] - 4 = E[X(X-1)] + 2 - 4 =$

$\sum_{j=1}^{\infty} j(j-1)(1/2)^j - 2 = (1/4)\sum_{j=2}^{\infty} j(j-1)(1/2)^{j-2} = 2.$

(c) B wins the game if and only if the game ends on an even numbered trial; hence P[B wins the game] $= (1/2)^2 + (1/2)^4 + \ldots = 1/3$.

Also, let p_A = probability that A wins the game and p_b = probability that B wins the game. Note $p_B = 1 - p_A$. In order for B to win the game, B must win the first match, having done so B is then in the same position as A at the start of the game, hence $p_B = (1/2)p_A$ and $p_B = 1 - p_A$ imply $p_B = 1/3$.

Problems 8 and 9 are very similar. The density of 8 is "triangular" whereas that of 9 is "parabolic." Both densities are symmetric about α.

8. (c) $E[X] = \alpha$ and $\text{var}[X] = \beta^2/6$.

(b) For $\alpha < q < 1/2$, $\xi_q = \alpha - \beta + \beta\sqrt{2q}$

11. Write μ_θ and σ_θ^2 for the mean and variance of $f(\cdot;\theta)$ including $\theta = 0$ and $\theta = 1$.

(b) $\mu_\theta = \theta\mu_1 + (1-\theta)\mu_0$

$\sigma_\theta^2 = \theta\sigma_1^2 + (1-\theta)\sigma_0^2 + \theta(1-\theta)(\mu_1 - \mu_0)^2$

(c) $\theta m_1(t) + (1-\theta)m_0(t)$.

12. (a) 16/25

(b) Model the problem by assuming that the bombs fall independently of one another. Then if at least one of the three large bombs falls within 40 feet of the track, traffic will be disrupted. Answer is $1 - (9/25)^3$.

13. (a) $E[(X-b)^2] = E[(X-\mu)^2] + (\mu - b)^2$ which is minimized when $b = \mu$.

(b) The result follows from the hint by nothing that the integral on the right hand side of the equality is non-negative for all b and zero for b = m.

To prove the hint assume $m < b$ ($m > b$ is similar). Write $E[|X-b|] - E[|X-m|]$

$$= \int_{-\infty}^{b} (b-x)f(x)\,dx + \int_{b}^{m} (x-b)f(x)\,dx + \int_{m}^{\infty} (x-b)f(x)\,dx -$$
$$\left(\int_{-\infty}^{b} (m-x)f(x)\,dx + \int_{b}^{m} (m-x)f(x)\,dx + \int_{m}^{\infty} (x-m)f(x)\,dx\right)$$
$$= 2\int_{b}^{m} (x-b)f(x)\,dx + (b-m)[F(b) + F(m) - F(b) - 1 + F(m)]$$
$$= 2\int_{b}^{m} (x-b)f(x)\,dx$$

14. (a) 21/25

(b) $\mu_X = 0$ and $\sigma_X = 1/2$, hence

$P[|X - \mu_k| \geq k\sigma_X] = 1/4 = 1/k^2$.

(c) See problem 20.

15. $E[X] = 1$ and $var[X] = 1/2$.

17. No, by Chebyshev inequality.

20. $P[X \leq \mu t] \geq P[X < \mu t] = 1 - P[(X/\mu) \geq t] \geq 1 - E[(X/\mu)]/t = 1 - (1/t)$ by Chebyshev inequality.

24. (a) $f_X(x;\theta) \geq 0$ for $-1/2 \leq \theta \leq 1/2$.

(b) $E[X] = (2/3)\theta$; median $= \dfrac{-1 + (1+4\theta^2)^{1/2}}{2\theta}$ for $\theta \neq 0$.

(c) $\theta = 0$.

CHAPTER III

Chapter III PROBLEMS

1. (f) No, the variance of a negative binomial random variable cannot be smaller than its mean.

   (h) Rectangular, normal, logistic, and beta with $a = b$. Note that the binomial for $p = 1/2$ and n even does not work.

   (n) No.

   (o) Yes, if the distribution of X is symmetric about zero.

2. (b) If $r \leq 1$, the mode is zero. If $r > 1$, the mode is $(r - 1)/\lambda$.

4. (b) $2\Phi(-2)$

   (c) $P[X \leq 0] = \Phi(-\mu/\sqrt{h(\mu)}) = \Phi(-1/\sqrt{a})$ for $h(\mu) = a\mu^2$, $\mu > 0$.

6. Let X be a random variable denoting the low bid of the competition. X is uniformly distributed over the interval $((3/4)C, 2C)$. Let P denote profit and B the amount the contractor should bid. Now $P = (B - C)I_{(B,2C)}(X)$ and

$$E[P] = \int (B - C)I_{(B,2C)}(x)f_X(x)\,dx = (B - C)\int_{(3/4)C}^{2C} I_{(B,2C)}(x)\left(2C - \frac{3}{4}\right)^{-1} dx$$

$$= \frac{(B - C)}{\left(\frac{5}{4}\right)C}(2C - B).$$ Now maximize with respect to B and obtain $B = \frac{3C}{2}$.

7. (a) Let k = number he should stock and X the number he can sell in 25 days.
   Want the minimal k such that $P[X \leq k] \geq .95$ where X has a Poisson distribution with parameter 100; that is, solve for k in $\sum_{i=0}^{k} \frac{e^{-100}(100)^i}{i!} \geq .95$.
   From a table of the Poisson distribution, $k = 117$ is obtained. Using the normal approximation and $\Phi\left(\frac{k - 100}{10}\right) = .95$, $k = 117$ is obtained.

   (b) Let Z = number of days out of 25 that he sells no items.
   Under appropriate assumptions (what are they?) Z has a binomial distribution with $n = 25$ and $p = c^{-4}$. Hence, $E[Z] = 25c^{-4}$.

8. (a) Y has a binomial distribution with parameters n and q.

   (b) X has a binomial distribution with parameters n and 15/36.

   (c) $(X + n)/2$ has a binomial distribution with parameters n and p.
   Hence $E[X] = n(2p - 1)$.

(d) Show that $\sum_{j=0}^{k} \binom{n}{j} \left(p_1^j q_1^{n-j} - p_2^j q_2^{n-j} \right) = \sum_{j=0}^{k} d_j \text{(say)} \geq 0$.

Note that $\sum_{j=0}^{n} d_j = 0$, hence it suffices to show that the first few d_j's are positive, and the remaining are negative. But $d_j \geq 0$ if and only if $j \leq n \log(q_2/q_1)/\log(p_1 q_2/p_2 q_1)$.

(Use the result of Problem 28 for an alternate proof.)

9. $\sum_{j=60}^{100} \frac{\binom{2500}{j}\binom{2500}{100-j}}{\binom{5000}{100}}$. The hypergeometric can be approximated by the binomial and the binomial can in turn be approximated by the normal which gives a numerical answer of approximately $1 - \Phi(2) = .0228$.

11. Let X denote the number of defectives in the sample. Assume that X has a binomial distribution.

(a) $P[X \geq 1] = 1 - P[X = 0] = 1 - (.99)^{10}$.

(b) Want $P[X \geq 1] \approx .95$; or, want $P[X = 0] \approx .05$;

i.e., $(.9)^n \approx .05$, or, $n \approx 29$.

15. $\mu + c\left[\Phi\left(\frac{a-\mu}{\sigma}\right) - \Phi\left(\frac{b-\mu}{\sigma}\right)\right] / \left[\Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right)\right]$

17. There is a misprint in this problem. The mean was intended to be 200 rather than 20. Want

$P[X \geq 150] \geq .90$, or, $\Phi\left(\frac{50}{\sigma}\right) \geq .90$, which implies $\sigma \approx 50/1.282 \approx 39$.

19. (a)

$$\begin{aligned} E[X] &= \int_0^\infty \beta^{-2} x^2 \exp[-(1/2)(x/\beta)^2]\, dx \\ &= (1/2)\sqrt{2\pi}\beta^{-1} \int_{-\infty}^{\infty} x^2 (1/\beta\sqrt{2\pi}) \exp[-(1/2)(x/\beta)^2]\, dx \\ &= \beta\sqrt{2\pi}/2 \text{ by recognizing that the last integral is the variance of a} \end{aligned}$$

normal distribution with mean 0 and variance β^2, which shows how a little knowledge of probability can be an aid to integration.

$\text{var}[X] = \beta^2(4 - \pi)/2$.

(b) No.

25.

1	2	3	4	5	6	7	8	9
$\frac{9}{81}$	$\frac{12}{81}$	$\frac{16}{81}$	$\frac{12}{81}$	$\frac{12}{81}$	$\frac{10}{81}$	$\frac{6}{81}$	$\frac{3}{81}$	$\frac{1}{81}$

28. Assume true and differentiate both sides with respect to p to obtain the equality:

$$\sum_{j=k}^{n} j\binom{n}{j}p^{j-1}q^{n-j} - \sum_{j=k}^{n}(n-j)\binom{n}{j}p^{j}q^{n-j-1} = k\binom{n}{k}p^{k-1}q^{n-k}.$$

The inequality is verified by noting the (j+1)st term of the first sum cancels the jth term of the second sum. Work backwards.

29. Let X = # of successes in the first n Bernoulli trials
and Y = # of failures prior to the rth success.
Note that $(X \le r-1) \cong (Y > n-r)$ hence $F_X(r-1) = P[X \le r-1] = P[Y > n-r] = 1 - F_Y(n-r)$.

30. $E[Z_\lambda] = (E[U^\lambda] - E[1-U^\lambda])/\lambda = 0$ for $\lambda > -1$.

$E[Z_\lambda^2] = (E[U^{2\lambda}] - 2E[U^\lambda(1-U)^\lambda] + E[(1-U)^{2\lambda}])/\lambda^2$

$= (2/\lambda^2)([1/(2\lambda+1)] - B(\lambda+1, \lambda+1))$ for $\lambda > -1/2$.

$E[Z_\lambda^3] = 0$ for $\lambda > -1/3$.

$E[Z_\lambda^4] = (2/\lambda^4)([1/(4\lambda+1) - 4B(3\lambda+1, \lambda+1) + 3B(2\lambda+1, 2\lambda+1)])$ for $\lambda > -1/4$.

The last part is misstated. The intent was to get two different λ's, say λ_1 and λ_2, such that Z_{λ_1} and Z_{λ_2} have the same skewness and kurtosis. If λ_1 and λ_2 are sought so that Z_{λ_1} and Z_{λ_2} have kurtosis equal to zero, then $\lambda_1 \approx .135$ and $\lambda_2 \approx 5.20$ will work.

CHAPTER IV

Chapter IV PROBLEMS

1. (a) True (b) False (c) True

2. (a) $E[X] = \int_0^{\infty} [1-F_X(z)]\ dz - \int_{-\infty}^{0} F_X(z)\ dz < \int_0^{\infty} [1-F_Y(z)]\ dz - \int_{-\infty}^{0} F_Y(z)\ dz = E[Y]$
   Using Eq. 6 of Chapter II (Page 65).

   (b) There are many counterexamples. For example, define

   $F_X(x) = (1/2)I_{[0,1]}(x) + I_{[1,\infty)}(x)$ and

   $F_Y(y) = (3/4)I_{[0,4)})(y) + I_{[4,\infty)}(y)$.

   (c) True. (d) False. (e) True.

   (f) $F_X(x) = P[X \leq z] = P[X+1 \leq z+1] = P[Y \leq z+1] = F_Y(z+1)$

3. Yes.

4. (b) 1/4

5. (a) 1/36

   (b) For $0 < x < 1$, $f_{Y|X}(y|x) = [I_{(x,1)}(y)]/(1-x)$.

6. (b) 1/4

   (c) 1/6

7. (b) No

8. $E[Y] = E[E[Y|X]] = 1 + p$

10. $$\begin{aligned} P[X=Y] &= \sum_{j=0}^{\infty} P[X=Y|Y=j]P[Y=j] \\ &= \sum_{j=0}^{\infty} P[X=j|Y=j]P[Y=j] \\ &= \sum_{j=0}^{\infty} P[X=j]P[Y=j] \text{ (using independence)} \\ &= \sum_{j=0}^{\infty} p^2(1-p)^{2j} = p/(2-p). \end{aligned}$$

11. (a) No. (b) Yes. (c) No. (d) Yes.

12. $F_X(x) + F_Y(y) - 1 \le P[X \le x] + P[Y \le y] - P[X \le x \text{ or } Y \le y]$

$$= P[X \le x; Y \le y] = F_{X,Y}(x, y).$$

$F_{X,Y}(x, y) = P[X \le x; Y \le y] \le P[X \le x] = F_X(x)$; also
$F_{X,Y}(x, y) \le F_Y(y)$.

14. (d) $P[Y - \alpha - \beta\mu \le z] = P[\alpha + \beta X - \alpha - \beta\mu \le z] = P[X - \mu \le z/\beta] = P[-(X - \mu) \le z/\beta] =$
$P[-(Y - \alpha - \beta\mu) \le z]$.

16. (a) Since $f_X(z) = f_Y(z) = I_{(0,1)}(z)$, X and Y are independent if and only if $\alpha =$ 0.
$\text{cov}[X, Y] = -\alpha \int_0^1 \int_0^1 (x - 1/2)(y - 1/2)(1 - 2x)(1 - 2y)\, dx\, dy = 0$ if and only if $\alpha = 0$.

(b) E[Area] = E[XY] = cov[X,Y] + 1/4

(c) $P[2X < 1] = 1/2$.

(d) Length of perimeter = $2(X + \sqrt{X^2 + Y^2})$

17. (b) 9/16

(c) $E[Y_1] = 15/8; E[Y_2] = 25/8$;
$\text{var}[Y_1] = 70/16 - (15/8)^2$ and $\text{var}[Y_2] = 170/16 - (25/8)^2$

(e) 5/11

18. (c) 3/4 (d) Solve for m in $1 - e^{-m} - me^{-m} = 1/2$.

(e) $1 - e^{-1}$ (f) 0

19. (a) Do (b) first.

(b) $f_X(z) = f_Y(z) = ze^{-z}I_{(0,\infty)}(z)$.

(c) $1 + (x/2)$

(d) $1 - 4e^{-2} - e^{-4}$.

(e) 1/2.

(f) $f_X(x)f_Y(y)$.

20. (a) $P[|X+Y| \leq 2|X|] = \iint_{|x+y|\leq 2|x|} f(x)f(y)\,dx\,dy$

$$= \int_0^\infty \left(\int_{-3x}^{x} f(y)\,dy\right) f(x)\,dx + \int_{-\infty}^{0} \left(\int_{x}^{-3x} f(y)\,dy\right) f(x)\,dx$$

$$= 2\int_0^\infty \left(\int_{-3x}^{-x} f(y)\,dy\right) f(x)\,dx + 2\int_0^\infty \left(\int_{-x}^{x} f(y)\,dy\right) f(x)\,dx \text{ (by symmetry)}$$

$$= 2\int_0^\infty \left(\int_{-3x}^{-x} f(y)\,dy\right) f(x)\,dx + 1/2 > 1/2.$$

21. Note that $E[X - Y] = E[E[X|Y]] - E[Y] = 0$, so

$\text{var}[X - Y] = E[(X - Y)^2,]$ but $E[(X - Y)^2] = E[X^2] - 2E[XY] + E[Y^2] =$

$E[XE[Y|X]] - 2E[XY] + E[YE[X|Y]] = 0$.

22. $P[\bigcap_{j=1}^{m} A_j] = 1 - P[\overline{\bigcap_{j=1}^{m} A_j}] = 1 - P[\bigcup_{j=1}^{m} \overline{A_j}] \geq 1 - \sum_{j=1}^{m} P[\overline{A_j}] \geq 1 - t^{-2}$.

23. (c) $f_X(x_0)/[1 - F_X(x_0)]$

25. Let Y denote A's score and Z denote B's score. Then $X = Y\text{-}Z$. Z is uniformly distributed over $(0,3)$.

$P[X \leq x] = P[X \leq x|Y=1]p + P[X \leq x|Y=2](1 - p) = P[1 - Z \leq x]p + P[2 - Z \leq x](1 - p)$.

Etc.

30. $P[X = x] = \sum_{y=x}^{\infty} P[X = x|Y = y]P[Y = y] = \sum_{y=x}^{\infty} \binom{y}{x} p^x q^{y-x} e^{-\lambda}\lambda^y/y! = (\lambda p)^x e^{-\lambda p}/x!$; i.e., X has a Poisson distribution with parameter λp.

32. (a) $Y|X = 5 \sim N(10, 25(1 - \rho^2))$, so $.954 = P[4 < Y < 16|X = 5] = \Phi\left(\frac{6}{5\sqrt{1 - \rho^2}}\right) - \Phi\left(\frac{-6}{5\sqrt{1 - \rho^2}}\right)$, which implies $\frac{6}{5\sqrt{1 - \rho^2}} = 2$, hence $\rho = 4/5$.

(b) This will be easy after the next chapter when we learn that $X+Y \sim (15, 26)$, giving $P[X + Y \leq 16] = \Phi\left(\frac{16 - 15}{\sqrt{26}}\right) = \Phi\left(1/\sqrt{26}\right)$. For now, $P[X + Y \leq 16] = \iint_{x+y\leq 16} \phi_{5,1}(x)\phi_{10,25}(y)\,dx\,dy = \iint_{u+v\leq 1} \phi(u)\phi(v)\,du\,dv =$ (using symmetry) $= \int_{-\infty}^{\infty}\int_{-\infty}^{1/\sqrt{26}} \phi(u)\phi(v)\,du\,dv = \Phi\left(1/\sqrt{26}\right)$.

34. (a) Multinomial with $k + 1 = 4$; P[no heads] = 1/8; P[one head] = 3/8; etc.

35. (a) $P[X = x, Y = y] = \dfrac{\binom{4}{x}\binom{4}{y}\binom{44}{6 - x - y}}{\binom{52}{6}}$

36. (a) $(26 - 9x)/(9 - 3x)$

(e) $E[XY|X = x] = xE[Y|X = x]$.

40. No

42. $m_{Y|X=x}(t) = E[e^{tY}|X = x]$. $m_Y(t) = E[e^{tY}] = E[E[e^{tY}|X]] = E[m_{Y|X}(t)]$.

43. (b) 1 (c) $\rho_{X,Y} = 1/2$ (d) $f_X(x)f_Y(y)$

44. (a) $E[Y] = E[E[Y|X]] = E[X + 1/2] = 1$

(b) $\text{cov}[X, Y] = 1/12$

(c) 1/4

45. Special case of Problem 46.

46. The joint density of X and Y might have two, three, or four mass points. Consider the case of four mass points. Let $p_{ij} = P[X = x_i; Y = y_j]$ for $i, j = 1, 2$, where $x_1 < x_2$ and $y_1 < y_2$.

Write $p_{1.} = p_{11} + p_{12} = P[X = x_1]$,

$p_{2.} = p_{21} + p_{22} = P[X = x_2]$,

$p_{.1} = p_{11} + p_{21} = P[Y = y_1]$, and

$p_{.2} = p_{12} + p_{22} = P[Y = y_2]$.

Let $U = (X - x_1)/(x_2 - x_1)$ and $V = (Y - y_1)/(y_2 - y_1)$.

Now $\text{cov}[X, Y] = 0$ if and only if $\text{cov}[U, V] = 0$ and X and Y are independent if and only if U and V are independent.

$\text{cov}[U, V] = E[UV] - E[U]E[V] = p_{22} - p_{2.}p_{.2}$.

$\text{cov}[U, V] = 0$ implies $p_{22} = p_{2.}p_{.2}$ which in turn implies independence.

CHAPTER V

Chapter V PROBLEMS

1. (a) $\operatorname{cov}[X_1 + X_2, X_2 + X_3] = \sigma^2$; $\operatorname{var}[X_1 + X_2] = \operatorname{var}[X_2 + X_3] = 2\sigma^2$;

   hence $\rho[X_1 + X_2, X_2 + X_3] = 1/2$.

   (b) $(\sigma_2^2 - \sigma_1^2)/(\sigma_1^2 + \sigma_2^2)$

   (c) 1/2.

3. $F(x)I_{[0,\infty)}(x)$.

4. (a) $P[X = x] = \dfrac{(M - K)_{x-1}}{(M)_{x-1}} \cdot \dfrac{K}{M - x + 1}$ for $x = 1, \ldots, M - K + 1$.

   (b) $P[Z = z] = \dfrac{\binom{K}{r-1}\binom{M-K}{z-r}}{\binom{M}{z-1}} \cdot \dfrac{\binom{K-r+1}{1}}{\binom{M-z+1}{1}}$, for $z = r, \ldots, M - K + r$.

   (c)

(x, y)	$(1, 2)$	$(1, 3)$	$(2, 1)$	$(3, 1)$	$(4, 1)$
$f_{X,Y}(x, y)$	$\frac{2}{5}\cdot\frac{3}{4}$	$\frac{2}{5}\cdot\frac{1}{4}$	$\frac{3}{5}\cdot\frac{2}{4}$	$\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{2}{3}$	$\frac{3}{5}\cdot\frac{2}{4}\cdot\frac{1}{3}$

5. According to the definition of expectation, $E[X_1]$ does not exist; however, there is no harm in saying $E[X_1] = \infty$, $E[Y_1] = n/(n - 1)$ for $n > 1$.

6. (a) Since $X \le \max[X, Y]$, $E[X] \le E[\max[X, Y]]$; similarly,

   $E[Y] \le E[\max[X, Y]]$, hence $\max[E[X], E[Y]] \le E[\max[X, Y]]$.

   (b) $\max[X, Y] + \min[X, Y] = X + Y$.

7. (a) Note that X and Y are independent and uniformly distributed. Apply the corollary of Theorem 3 on page 180.

   (b) Theorem 8 will do it.

8. The cdf of $Z = \max[X, Y]$ is given by

   $(1 - e^{-\lambda_1 z})(1 - e^{-\lambda_2 z})I_{(0,\infty)}(z)$

   so $E[Z] = E[\max[X, Y]] = \int_0^1 (1 - F_Z(z))\, dz = \int_0^1 \left(e^{-\lambda_1 z} + e^{-\lambda_2 z} - e^{-(\lambda_1+\lambda_2)z}\right) dz =$

   $\dfrac{1}{\lambda_1} + \dfrac{1}{\lambda_2} - \dfrac{1}{\lambda_1 + \lambda_2}$

9. $X_1 - X_2 \sim N(0, 2)$. The distribution of $(X_2 - X_1)^2$ can be found using Example 19. Similarly, for Y_2-Y_1 and $(Y_2$-$Y_1)^2$. They are independent so use Equation (26) to find the distribution of $Z^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2$.

10. (a) Let Y_n be the life of the fuse that lasts the longest. Find n such that $P[Y_n > .8] = .95$. $n = 14$ will do.

(b) 9/10.

11. $\Phi(\cdot)$.

12. (a) This problem is starred, not because it is difficult, but because it is messy. The possible values of $Z = X/(X+Y)$ are zero (if $X = 0$), one (if $X > 0$ and $Y = 0$), and a/b where a and b are positive integers and $a < b$. $P[Z = (a/b)] = \sum P[X = x; Y = y]$ where the summation is over all pairs (x, y) for which x and y are positive integers and $y = x(b-a)/a$.

(b) $m_{X,X+Y}(t_1, t_2) = E[e^{t_1X+t_2(X+Y)}] = m_{X,Y}(t_1+t_2, t_2)$.

13. (a) Write $E[e^{Y_1t_1+Y_2t_2}]$ in terms of a double integral involving the joint distribution of X_1 and X_2. Perform the integration by separating the double integral, completing the square, and expressing in terms of integrals of normals.

(b) Use the joint moment generating function given in (a).

14. $E[e^{XYt}] = E[E[e^{XYt}|X]] = E[e^{(1/2)Y^2t^2}] = 1/\sqrt{1-t^2}$.

15. (a) Use the moment generation function technique to argue that they are independent standard normals.

16. Let $S = \sum_{1}^{16} X_i$ = weight of beans in box. Assume that the X_i's are independent.

(a) mean = 16^2 ounces an variance = 16

(b) $P[S > 250] = 1 - \Phi\left(\frac{250 - 16(16)}{4}\right) = \Phi(3/2)$

(c) Let Z = number of underweight bags.

$Z \sim \text{bin}(16, 1/2)$, so $P[Z \le z] = \sum_{0}^{2} \binom{16}{x} (1/2)^{16}$.

17. (a) Let Z = number of numbers less than 1/2. $Z \sim \text{bin}(10, 1/2)$. $P[Z = 5] = \binom{10}{5}(1/2)^5$.

(b) $E[Z] = 5$.

(c) 1/2 using a symmetry argument.

18. (a) Both are $n\lambda$.

(b) $\Phi(-2)$

19. (a) Buy n bulbs and again assume independence. Assume that the lifetime are independent(which may not be realistic since the bulbs are burning simultaneously). Want n such that $.95 = P[Y_n > 1000] = 1 - [1 - \exp(-10)]^n$.

(b) Buy n bulbs. Want n such that $P[S_n > 1000] = .95$. S_n has a gamma distribution with parameters n and .01. Using Equation (33) of Chapter III and a Poisson table $n \approx 16$ is obtained.

20. Use the moment generating function technique.

(a) gamma with parameters nr and λ.

(b) gamma with parameters $\sum r_i$ and λ.

21. (a) negative binomial with parameters n and p

(b) negative binomial starting at n with parameters n and p

(c) negative binomial with parameters nr and p

(d) negative binomial with parameters $\sum r_i$ and p

22. Z can be expressed as $\sum_1^Y X_i$ where X_i is the money received from the ith location where oil is found. $Z = 0$ if $Y = 0$. Model by assuming the X_i's and Y are independent. Y has a binomial distribution with $n = 10$ and $p = 1/5$, and the X_i's are independent and identically distributed exponential random variables with mean 50000.

(a) $E[Z] = E[E[Z|Y]] = E[Y]E[X] = \$100,000$.

(b) $P[Z > 100{,}000 | Y = 1] = e^{-2}$.

$P[Z > 100{,}000 | Y = 2] = 3e^{-2}$.

(c) $P[Z > 100{,}000] = \sum_{y=0}^{10} P[Z > 100{,}000 | Y = y]P[Y = y] = \sum_{y=1}^{10} \left(\sum_0^{y-1} \frac{e^{-2}2^j}{j!} \right) \binom{10}{y} \left(\frac{1}{5}\right)^y \left(\frac{4}{5}\right)^{10-y}$

using Z given $Y = y$ is gamma distributed and Equation (33) of Chapter III.

$P[Z > 100{,}000] \approx .4$.

23. See 24.

24. $P[X_1 = x_1, \ldots, X_k = x_k \mid X_1 + \cdots + X_{k+1} = n] = \dfrac{P[X_1 = x_1, \ldots, X_k = x_k;\ X_1 + \cdots + X_{k+1} = n]}{P[X_1 + \cdots + X_{k+1} = n]}$

$$= \frac{\dfrac{e^{-\lambda_1}(\lambda_1)^{x_1}}{x_1!} \cdot \dfrac{e^{-\lambda_2}(\lambda_2)^{x_2}}{x_2!} \cdot \ \cdots\ \cdot \dfrac{e^{-\lambda_k}(\lambda_k)^{x_k}}{x_k!} \cdot \dfrac{e^{-\lambda_{k+1}}(\lambda_{k+1})^{n-x_1-x_2-\ldots-x_k}}{(n - x_1 - x_2 - \ldots - x_k)!}}{\dfrac{e^{-\sum\lambda_j}(\sum\lambda_j)^n}{n!}}$$

$$= \frac{n!}{x_1!x_2!\cdot\ldots\cdot x_k!(n - x_1 - x_2 - \ldots - x_k)!}\left(\frac{\lambda_1}{\lambda}\right)^{x_1}\left(\frac{\lambda_2}{\lambda}\right)^{x_2}\cdot\ldots\cdot\left(\frac{\lambda_k}{\lambda}\right)^{x_k}\left(\frac{\lambda_{k+1}}{\lambda}\right)^{n-x_1-\ldots-x_k}$$

25. Cauchy.

26. Y has a lognormal distribution. $E[Y] = E[e^X] = m_X(1)$, the moment generating function of X evaluated at 1. Also $E[Y^2] = E[e^{2X}] = m_X(2)$.

27. Exponential with parameter one.

28. Beta with parameters b and a.

29. Write $Y = 1/X$ then $f_Y(y) = y^{-2}I_{(1,\infty)}(y)$.

31. Exponential with parameter one.

32. Beta with parameters reversed.

34. Same as X.

36. Exponential with parameter one.

38. $P[Y - X = z] = [p/(2 - p)]q^z I_{\{0,1,2,\ldots\}}(z) + [p/(2 - p)]q^{-z} I_{\{-1,-2,\ldots\}}(z)$.

39. Write $V = Y - X$, then $f_V(v) = (\lambda/2)e^{-\lambda|v|}$.

40. One way of doing it is to transform to, say, $U = X, V = Y, W = XY/Z$, find the U, V, W, integrate out u and v and get

$$f_W(w) = \left(\frac{1}{4} - \frac{1}{2}\ln w\right) I_{(0,1)}(w) + \frac{1}{4w^2}I_{[1,\infty]}(w).$$

41. Write $Z = X + Y$. $f_Z(z) = [2z^2 - (2/3)z^3]I_{(0,1)}(z) + [(8/3) - 2z^2 + (2/3)z^3]I_{(1,2)}(z)$. $f_Z(z)$ is symmetric about $z = 1$.

42. This is starred not because it is difficult, but because the answer, which can be expressed in terms of a Bessel function, is not simple.

$P[Y - X = z] = \sum_{x=0}^{\infty} P[Y - X = z \mid X = x]P[X = x] = \sum_{x=\max[0,-z]}^{\infty} P[Y = x + z]P[X = x]$ for z an integer.

44. Let X have parameters a and b and Y have parameters c and d. $b = d = 1$ and $a = c + 1$ will suffice.

46. The cdf technique works. $2z^3e^{-z^2}I_{(0,\infty)}(z)$.

47. X and Y are independent; hence it suffices to find the marginal distribution of X^2 and Y^2.

49. The transformation is not one-to-one. See Example 19.

50. The distribution of $X + Y$ is triangular and given Example 4, $P[Z \le z] = P[X + Y \le z; X + Y \le 1] + P[X + Y - 1 \le z; X + Y > 1] = P[X + Y \le z] + P[1 < X + Y \le 1 + z] = z$ for $0 < z < 1$. That is Z is uniformly distributed over $(0, 1)$.

53. $f_{Y_1,Y_2}(y_1, y_2) = \lambda^2 y_2 e^{-\lambda y_2}[1/(1 + y_1)^2]I_{(0,\infty)}(y_1)I_{(0,\infty)}(y_2)$.

54. The transformation is not one-to-one. Use Theorem 14. Y_1 has an exponential distribution with parameter 1/2 and Y_2 has a standard Cauchy distribution. They are independent.

57. (a) $E[X + Y] = E[E[X + Y|Z]] = 1$.

(b) $f_{X,Y}(x, y) = \int f_{X,Y|Z}(x, y|z)f_Z(z)\, dz = I_{(0,1)}(x)I_{(0,1)}(y)$. Are independent.

(c) $f_{X|Z}(x|z) = \int f_{X,Y|Z}(x, y|z)\, dy = [z + (1 - z)(x + 1/2)]I_{(0,1)}(x)$ which depends on z so X and Z are not independent.

(d) Straightforward transformation using distribution of X and Y given in (b).

(e) $P[\max[X, Y] \le u|Z = z] = P[X \le u, Y \le u|Z = z] = \int_0^u \int_0^u [z+(1-z)(x+y)]\, dx\, dy = zu^2 + (1 - z)u^3$ for $0 < u < 1$.

(f) $\int f_{(X,Y)|Z}(x, s - x|z)\, dx = [z + (1 - z)s][sI_{(0,1)}(s) + (2 - s)I_{[1,2]}(s)]$

58. Assume independence of functioning components and capitalize in the forgetfulness of the exponential.

(a) Let $Y = Y_3+Y_2+Y_1$ be the life of system, where Y_j is that part of the life when exactly j components are functioning. Y_3 is the minimum of three independent exponential random variables each with rate parameter $\lambda/3$, so Y has an exponential distribution with rate parameter λ. Similarly for Y_2 and Y_1.

Furthermore, the Y_j's are independent, hence Y has a gamma distribution with parameters 3 and λ.

(b) Same as answer (a).

59. Z is the lifetime of the system. Z has cdf $(1-2e^{-2z}+e^{-3z})I_{(0,\infty)}(z)$, mean 2/3, and variance 1/3.

60. Gamma with parameters two and two.

61. Follow the hint and use Equation (33) of Chapter IV for the joint moment generating function of X and Y. $(U,V) = (aX+bY, cX+dY)$ has a bivariate normal distribution with parameters.

$\mu_U = a\mu_X + b\mu_Y$, $\mu_V = c\mu_X + d\mu_Y$

$\sigma_U^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho_{X,Y}$

$\sigma_V^2 = c^2\sigma_X^2 + d^2\sigma_Y^2 + 2cd\sigma_X\sigma_Y\rho_{X,Y}$

$\rho_{U,V} = \sigma_U\sigma_V[ac\sigma_X^2 + bd\sigma_Y^2 + (bc+ad)\sigma_X\sigma_Y\rho_{X,Y}]$.

Can you choose a, b, c, and d to make U and V independent standard normals?

62. (a) $N(0, u^2 + [1-u]^2)$

(b) $E[Z] = 0$ and $var[Z] = 2/3$ using Theorem 7 of Chapter IV, page 159.

(c) This is starred because the answer is not simple. Use Remark on page 149 and get

$F_Z(z) = \int P[Z \le z | U = u] f_U(u)\ du$; now

both $P[Z \le z | U = u]$ and $f_U(u)$ are known and the problem is reduced to one of integration.

$$f_Z(z) = \int_0^1 \Phi\left(\frac{z}{\sqrt{u^2 + (1-u)^2}}\right) \frac{1}{\sqrt{u^2 + (1-u)^2}}\ du$$

CHAPTER VI

Chapter VI PROBLEMS

3. (a) $P[|X_2 - X_1| < 1/2] = \int_0^1 P[|X_2 - x_1| < 1/2]\ dx_1 = 3/4.$

   (b) $P[1/4 < (X_1 + X_2)/2 < 3/4] = P[1/2 < X_1 + X_2 < 3/2] = 3/4.$

4. (a) $f_{X_1,\ldots,X_9}(x_1, \ldots, x_9) = \sum_{i=1}^{9} [(2/3)^{x_i}(1/3)^{1-x_i} I_{\{0,1\}}(x_i)]$

   $f_{\sum X_j}(s) = \binom{9}{5}(2/3)^s(1/3)^{9-s} I_{\{0,1,\ldots,9\}}(s)$

   (b) $E[\overline{X}_9] = 2/3,\ \ E[S^2] = 2/9.$

5. (a) Yes; it follows from simple algebra.

   (b) There are various ways to proceed. For example,

   $var[S^2] = [1/2n(n - 1)]^2\ var[\sum\sum(X_i - X_j)^2]$

   $= \ [1/2n(n - 1)]^2\ \sum\sum\sum\sum\ cov[(X_i - X_j)^2, (X_\alpha - X_\beta)^2]$

   (using "variance of a sum is the double sum of the covariances")

   $= \ [1/2n(n-1)]^2(2n(n-1)var[(X_2-X_1)^2]+4n(n-1)(n-2)cov[(X_1-X_2)^2, (X_1-X_3)^2])$

   $= \ [1/2n(n - 1)]^2(2n(n - 1)(2\mu_4 + 2\sigma^4) + 4n(n - 1)(n - 2)(\mu_4 - \sigma_4))$

   $= \ (1/n)(\mu_4 - \frac{n - 3}{n - 1}\sigma^4).$

   (c) $cov[\overline{X}, S^2] = cov[\overline{X}-\mu, S^2] = [1/n(n-1)]cov[\sum(X_k-\mu), \sum(X_i-\mu)^2-(1/n)\sum\sum(X_i-$
   $\mu)(X_j - \mu)]$

   $= \ [1/n(n - 1)](\sum\sum cov[\sum(X_k - \mu), (X_i - \mu)^2] - (1/n)\sum\sum\sum cov[(X_k - \mu), (X_i -$
   $\mu)(X_j - \mu)])$

   $= \ [1/n(n - 1)](n\mu_3 - (1/n)(n\mu_3)) = \mu_3/n$, a rather simple answer.

6. (a) $M_r = (1/2)\left[\left(\frac{X_1 - X_2}{2}\right)^r + (-1)^r\left(\frac{X_1 - X_2}{2}\right)\right]^r.$

   For r odd, $M_r \equiv 0$ and hence $E[M_r] = 0$ and $var[M_r] = 0$. For r even, $M_r =$
   $\left(\frac{X_1 - X_2}{2}\right)^r$, and $E[M_r] = (1/2^r)\sum_{j=0}^{r}\binom{r}{j}\mu'_j\mu'_{r-j}(-1)^{r-j}$ and similarly for $var[M_r]$.

   (b) $E[(1/n)\sum(X_i - \mu)^r] = (1/n)\sum E[(X_i - \mu)^r] = \mu_r.$

7. (a) Have $P[-\epsilon < \overline{X}_n - \mu < \epsilon] \geq 1 - \sigma$ for $n > \sigma^2/\epsilon^2 4$.

   Have $\mu = .5, \sigma^2 = 1/4, \epsilon = .1, \delta = .1$, hence $n = 250$.

   (b) Use the Central Limit Theorem.

   $.90 = P[.4 < \overline{X} < .6] \approx \Phi\left(\frac{.6 - .5}{\sqrt{1/4n}}\right) - \Phi\left(\frac{.4 - .5}{\sqrt{1/4n}}\right)$ and so $n \approx 58$.

9. $Y = \overline{X}_1 - \overline{X}_2$ is approximately distributed as a normal distribution with mean = 0 and variance $2\sigma^2/n$. Want $P[|\overline{X}_1 - \overline{X}_2| > \sigma] = .01$. $n = 14$.

10. Want $.01 = P[\overline{X} < 2200] \approx \Phi\left(\dfrac{2200 - 2250}{250\sqrt{n}}\right)$. $n = 136$.

11. Want $.95 = P[|\overline{X} - \mu| \leq .25\sigma]$. $n = 62$.

12. Want $.01 = P[\overline{X} < 1/2] \approx \Phi\left(\dfrac{.5 - .52}{\sqrt{.52(.48)/n}}\right)$. $n = 3375$.

15. (a) There are ten equally likely (unordered) samples; compute $\bar{x}$ for each and the evaluate $E[\overline{X}]$ and $\text{var}[\overline{X}]$. 3 and .75

(b) 1

(c) $E[\overline{X}] = (N+1)/2$.

$$\begin{aligned}
\text{var}[\overline{X}] &= (1/n^2)\text{var}[\textstyle\sum X_i] \\
&= (1/n^2)(\sum_i \text{var}[X_i] + \sum\sum_{i \neq j} \text{cov}[X_i, X_j]) \\
&= (1/n^2)(n\sigma^2 + n(n-1)\text{cov}[X_1, X_2]) \\
&= (1/n^2)(n\sigma^2 + n(n-1)\sum\sum_{i \neq j}(i-\mu)(j-\mu)/N(N-1)) \\
&= \frac{\sigma^2}{n}\,\frac{N-n}{N-1}.
\end{aligned}$$

17. $Z = \sum(X_i - \overline{X})^2/\sigma^2$ is chi-square distributed with $n-1$ degrees of freedom.

$S = \sqrt{\sigma^2 Z/(n-1)}$. $E[S] = \sqrt{\sigma^2/(n-1)}E[\sqrt{Z}]$

$$\begin{aligned}
&= \sqrt{\sigma^2/(n-1)}\int_0^\infty \frac{1}{\Gamma((n-1)/2)}(1/2)^{(n-1)/2}z^{(n/2)-1}e^{-(1/2)z}\,dz \\
&= [(\sigma\sqrt{2})/\sqrt{n-1}]\Gamma(n/2)/\Gamma((n-1)/2).
\end{aligned}$$

$$\begin{aligned}
&\text{var}[S] = E[S^2] - E^2[S] \\
&= \frac{\sigma_2}{n-1}E[Z] - E^2[S] = \sigma^2\left\{1 - \frac{2}{n-1}\left[\frac{\Gamma(n/2)}{\Gamma((n-1)/2)}\right]\right\}
\end{aligned}$$

18. (b) $X = (U/m)/(V/n)$ implies $1/X = (V/n)/(U/m)$.

(c) $W = \dfrac{\frac{m}{n}\frac{U/m}{V/n}}{1 + \frac{m}{n}\frac{U/m}{V/n}} = \dfrac{U}{V+U}$ is beta distributed with parameters $m/2$ and $n/2$ by Example 25 of Chapter V.

(d) $E[X] = \dfrac{n}{m}E[\dfrac{W}{1-W}] = \dfrac{n}{m}\dfrac{1}{B(m/2, n/2)}\displaystyle\int_0^1 w^{m/2}(1-w)^{(n/2)-2}\,dw = n/(n-2)$.
Similarly for $E[X^2]$ and $\text{var}[X]$.

19. (a) The integral that defines the mean exists for degrees of freedom greater than 1; symmetry shows that the mean is zero. The integral that defines the variance exists for degrees of freedom greater than 2;

$\text{var}[T] = E[T^2] = E[\frac{(\text{standard normal r.v.})^2}{\text{chi-square r.v./d. of f.}}]$

$= E[\text{F-dist'd r.v. with 1 and k d. of f.}] = k/(k-2)$ for $k > 2$.

If it seems unfair to use results on the F distribution to obtain results on the t distribution, $E[T^2]$ can be found directly. For example, the standard normal r.v. of the numerator is independent of the chi-square r.v. in the denominator so the expectation can be factored into the product of the expectation of the square of a standard normal r.v. and the expectation of the reciprocal of a chi-square r.v. divide by degrees of freedom; both factors are known.

(b) Show $C(k)[1/(1+t^2/k)^{(k+1)/2})] \underset{k\to\infty}{\longrightarrow} c\ e^{-1/2t^2}$. Assuming that the constant part $C(k)$ does what is has to do, it is easy to show

$(1+t^2/k)^{(k+1)/2} \longrightarrow e^{-1/2t^2}$.

(c) $X = Z/\sqrt{U/k}$ implies $X^2 = Z^2/(U/k)$ which is a ratio of two independent chi-squared distributed r.v.'s divided by their respective degrees of freedom, hence X^2 is F-distributed with one and k degrees of freedom.

(d) According to part (c), $X^2 \sim F(1,k)$; according to part (b) of Problem 18, $1/X^2 \sim F(k,1)$; and according to part (c) of Problem 18, $\frac{1}{1+(X^2/k)} = \frac{k(1/X^2)}{1+k(1/X^2)}$ is beta distributed with parameters $k/2$ and $1/2$.

Problems 20 through 24 inclusive are much alike and are intended to give some practice in utilizing the results of Sec. 4.

22. (a) Chi-square with n-2 degree of freedom. (The sum of independent chi-square distributed r.v.'s is chi-square distributed with degrees of freedom equal to the sum of the individual degrees of freedom.)

(b) Normal with mean μ and variance $n\sigma^2/4k(n-k)$.

(c) Chi-square with one degree of freedom.

(d) F distribution with $k-1$ and $n-k-1$ degrees of freedom.

(e) t-distribution with $n-1$ degrees of freedom.

23. Don't forget that Z_1+Z_2 and Z_2-Z_1 are independent. Similarly for X_1+X_2 and $X_2 - X_1$.

(b) t-distribution with 2 degrees of freedom.

(c) Chi-square with 3 degrees of freedom.

(d) F distribution with 1 and 1 degrees of freedom.

25. Note that X_1 and X_2 are independent identically distributed chi-square random variables with 2 degrees of freedom, so X_1/X_2 has an F distribution with 2 and 2 degrees of freedom.

27. $U \sim N(\mu, 1/\sum(1/\sigma_j^2))$
$V = \sum(X_i - U)^2/\sigma_i^2 = \sum(X_i - \mu)^2/\sigma_i^2 - (U - \mu)\sigma(1/\sigma_j^2)$ which is a difference of two independent chi-square distributed r.v. s., the first with n degrees of freedom, the second with 1 degree of freedom. The result follows using the moment generation function technique. What result does this reduce to if all σ_j^2 are equal?

29. The joint distribution of $(\overline{X}, S_1^2, S_2^2)$ is easily obtained since they are independent. Make a transformation and integrate out the unwanted variable.

30. One could use Theorem 13. On the other hand, note that $Y_2 - Y_1 = |X_1 - X_2|$ and the distribution of $X_1 - X_2$ is known and it is easy to find the distribution of the absolute value of a random variable.

31. (a) $1 - P[\text{both less than median}] = 3/4$.

(b) $1 - P[\text{all are less than median}] = 1 - (1/2)^n$.

32. $E[F(Y_1)]$ is wanted. $F(Y_1)$ has the same distribution as the smallest observation of a random sample of size n from a uniform distribution over the interval $(0, 1)$.

33. $E[Y_1] = \mu - [(n - 1)/(n + 1)]\sqrt{3}\,\sigma$
$E[Y_n] = \mu + [(n - 1)/(n + 1)]\sqrt{3}\,\sigma$
$\text{var}[Y_1] = \text{var}[Y_n] = 12\sigma^2 n/[(n + 1)^2(n + 1)]$.
$\text{cov}[Y_1, Y_n] = 12\sigma^2/[(n + 1)^2(n + 2)]$.

(a) $E[Y_n - Y_1] = [(n-1)/(n+1)]2\sqrt{3}\,\sigma$.

$\operatorname{var}[Y_n - Y_1] = 24\sigma^2(n-1)/[(n+1)^2(n+2)]$.

(b) $E[(Y_1 + Y_n)/2] = \mu$.

$\operatorname{var}[(Y_1 + Y_n)/2] = 6\sigma^2/[(n+1)(n+2)]$.

(c) $E[Y_{k+1}] = \mu$.

$\operatorname{var}[Y_{k+1}] = 3\sigma^2/(2k+3)$.

(d) $\dfrac{3\sigma^2}{n+2} > \dfrac{\sigma^2}{n} > \dfrac{6\sigma^2}{(n+1)(n+2)}$ for $n > 2$.

34. $\overline{X}$ is asymptotically normally distributed with mean α and variance $2\beta^2/n$. The sample median is asymptotically normally distributed with mean α and variance β^2/n by Theorem 14. Note that the sample median has the smaller asymptotic variance.

35. $P[(Y_n - a_n)/b_n \le y] = P[Y_n \le b_n y + a_n] = (1 - \exp[(b_n y - a_n)/(1 - b_n y - a_n)])^n$
$= (1 - \exp[\dfrac{y + (\log n)^2}{y - \log n}])^n$. Now let $n \to \infty$ and $\exp(-e^{-y})$ results.

36. (a) Similar to Problem 34.

(b) With θ replacing λ choose a_n and b_n as in Example 9.

(c) We know that $Y_1^{(n)}$ has exact distribution that is exponential with parameter $n\lambda$. So choose $a_n \ge 0$ and $b_n = 1/n$ and then $(Y_1^{(n)} - a_n)/b_n$ has exact (and hence also limiting) distribution that is exponential with parameter λ.

CHAPTER VII

Chapter VII PROBLEMS

1. Let B = number of black balls and

   W = number of white balls.

   $R = B/W$. Set $p = B/(B+W)$, so $R = p/(1+p)$.

   (a) Let $X_i = 1$ if black ball on ith draw and $X_i = 0$ otherwise. MLE of $p = \sum X_i/n = \overline{X}$ which implies MLE of $R = \overline{X}/(1-\overline{X})$.

   (b) X_i has a geometric distribution. $L(p) = p^n(1-p)^{\sum x_i}$. MLE of $p = 1/(1+\overline{X})$, so MLE of $R = 1/\overline{X}$.

2. MLE of p_{ij} is N_{ij}/n.

4. MLE of $\mu_1 - \mu_2$ is $\overline{X}_1 - \overline{X}_2$.

   $\mathrm{var}[\overline{X}_1 - \overline{X}_2] = \sigma_1^2/n_1 + \sigma_2^2/n_2$. $n_1 \approx n[\sigma_1(\sigma_1+\sigma_2)]$.

5. MLE of $a = (\overline{X}_1 + \overline{X}_2 + \overline{X}_3 + \overline{X}_4)/4$;

   MLE of $b = (\overline{X}_1 + \overline{X}_2 - \overline{X}_3 - \overline{X}_4)/4$; and

   MLE of $c = (\overline{X}_1 + \overline{X}_3 - \overline{X}_2 - \overline{X}_4)/4$.

7. Let r denote the radius of the circle. Let X_i denote the ith measurement.

   $X_i = r + E_i$ where E_i is the ith error of measurement. $E_i \sim N(0, \sigma^2)$.

   Now $\mathrm{var}[X_i] = \mathrm{var}[E_i] = \sigma^2$ so $S^2 = \sum(X_i - \overline{X})^2/(n-1)$ is an unbiased estimator of σ^2. $(\pi/n)\sum_{j=1}^{n}(X_j - S^2)$ is an unbiased estimator of the area $= \pi r^2$.

9. Show that $P_\theta[|(X_1+X_2)/2-\theta| < |X_1-\theta|] > 1/2$ for all θ. Make the transformation $U_1 = X_1 - \theta$ and $U_2 = X_2 - \theta$ and it suffices to show that $P[|U_1+U_2| < 2|U_1|] > 1/2$ where U_1, U_2 is a random sample of size two from the Cauchy density $1/\pi[1+x^2]$. See Problem 20 of Chapter IV.

10. (a) $\sum(X_i - \hat{\theta}) = 0$ implies $\hat{\theta} = \overline{X}$.

    (b) $\sum(X_i - \hat{\theta})^2$ is minimized for $\hat{\theta} = \overline{X}$.

11. (b) $\mathrm{var}[\sum a_i x_i] = \sigma^2(\sum a_i) = \sigma^2[\sum(a_i - 1/n)^2 + 1/n]$.

12. (b) MLE of θ is $\min[1/2, \overline{X}]$.

17. (a) X is sufficient. $E[X] = 0$ for all θ so X is not complete.

(b) Yes; yes.

(c) $\sum |X_i|/n$.

(d) Yes. (e) Yes.

(f) $|X|$.

18. (b) Yes.

19. (c) Middle observation for add sample size and anything between two middle observations for even sample size.

(d) No.

21. In computing the means and mean-squared errors use the calculations in Problem 33 of Chapter VI.

(a) $T_1 = 2\overline{X}$. MSE is $\theta^2/3n$.

(b) $T_2 = Y_n$. MSE is $2\theta^2/[(n+1)(n+2)]$.

(c) $T_3 = [(n+2/(n+1))]Y_n$. MSE is $\theta^2/(n+1)^2$.

(d) $T_4 = [(n+1)/n]Y_n$. MSE is $\theta^2/[n(n+2)]$.

(e) MSE is $2\theta^2/(n+1)(n+2)$.

(g) $Y_n^2/12$.

22. (a) $[(1-2\theta)^2\theta(1-\theta)]/n$

(b) $\sum X$ is a complete sufficient statistic. $S^2 = \sum(X_i-\overline{X})^2/(n-1)$ is an unbiased estimator of $\theta(1-\theta)$, since the sample variance is an unbaised estimator of the population variance; furthermore, $S^2 = [\sum X_i^2 - n\overline{X}^2]/(n-1) = [\sum X_i - n\overline{X}^2]/(n-1)$ is a function of $\sum X_i$; hence, by the Lehmann-Scheffé Theorem, S^2 is UMVUE of $\theta(1-\theta)$.

24. $-\ln X_i$ has an exponential distribution, so $-\sum \ln X_i$ has a gamma distribution.

(a) MLE of θ is $n/-\sum \ln X_i$, $\therefore$ MLE of μ is $n/(n - \sum \ln X_i)$.

(b) $-\sum \ln X_i$ is complete minimal sufficient by Theorem 9. A minimal sufficient statistic must be a function of every other sufficient statistic. $-\sum \ln X_i$ is not a function of $\sum X_i$, hence $\sum X_i$ is not sufficient for $n > 1$. $\sum X_i$ is sufficient for $n = 1$. Why?

(c) Yes, $1/\theta$.

(d) $-\sum \ln X_i/n$ is UMVUE of $1/\theta$; $(n-1)/-\sum \ln X_i$ is UMVUE of θ. X_1 is an unbiased estimator of $\theta/(1+\theta)$, hence $E[X_1 | -\sum \ln X_i]$ is UMVUE of $\theta/(\theta+1)$. For $n > 1$, following a procedure similar to that in Example 35, the condition distribution of X_1 given $-\sum \ln X_i$ can be found and then conditional expectation can be obtained. Let $S = -\sum \ln X_i$, then $E[X_1 | S = s] = \int_{e^{-s}}^{1} [x_1(n-1)+(s+\ln x_1)^{n-2}/x_1 s^{n-1}]\, dx_1 = \frac{(n-1)e^{-s}}{s^{n-1}} \int_0^s u^{n-2}e^u\, du$ which can be integrated and the answer expressed as a finite sum. For $n = 1$, what is the UMVUE of $\theta/(1+\theta)$?

26. (a) $2\overline{X} - 1$. Mean is θ and mean-squared error is $(\theta^2 - 1)/3n$.

(b) MLE is Y_n. The distribution of Y_n is given by $P[Y_n = j] = [(j/\theta)^n - ((j-1)/\theta)^n]\, I_{\{1,\ldots,\theta\}}(j)$ from which the mean and mean-squaed error can be found.

(c) Y_n is sufficient by the factorization criterion. To show that $E_\theta[\chi(Y_n)] = 0$ for $\theta = 1, 2, 3, \ldots$ implies that $\chi(j) = 0$ for $j = 1, 2, \ldots$. It suffices to substitute in $\theta = 1, 2, 3$, etc. successively.

(d) By the Lehmann-Scheffé Theorem and part (c) it suffices to show that the given statistic is unbiased.

27. X is sufficient but not complete.

28. $\tau(\theta) = \text{median} = \ln 2/\theta$. We already know that the MLE and UMVUE of $1/\theta$; to find the MLE amd UMVUE of $\tau(\theta)$ requires a simple scale adjustment.

29. (b) $X^2 - 1$.

(c) $I_{(0,\infty)}(X_1)$.

(d) $\Phi(\overline{X})$.

(e) $\overline{X}^2 - (1/n)$.

(f) $X_1|\overline{X} \sim N(\overline{X}, (n - 1)/n)$ and $E[I_{(0,\infty)}(X_1)|\overline{X}] = \Phi\left(\sqrt{\frac{n}{n-1}}(\overline{X})\right)$ is UMVUE for $P[X > 0]$.

30. $I_{\{0,1\}}(X_1)$ is an unbiased estimator of $(1+\lambda)e^{-\lambda}$. See Example34 for a procedure that will find an UMVUE of $(1+\lambda)e^{-\lambda}$.

31. This is a "triangular" density rather than a "rectangular" density as in Problem 21. The results are quite similar.

32. The density given in this problem is a form of the Pareto density. This problem is like Problem 24. In that problem $-\ln X_i$ has an exponential distribution; in this problem $\ln(1+X_i)$ has an exponential distribution.

(a) $(1+\overline{X})/\overline{X}$.

(b) MLE of $1/\theta$ is $\sum \ln(1+X_i)/n$.

(c) $\sum \ln(1+X_i)$.

(d) $1/n\theta^2$

(e) $\sum \ln(1+X_i)/n$.

(f) $(n-1)/\sum \ln(1+X_i)$.

33. (a) $\max[-Y_1, Y_n]$, or, the absolute value of the observation farthest from zero.

(b) X is not minimal sufficient since $|X|$ is sufficient. X is not complete.

34. (a) $\sum X_i$ is a complete sufficient statistic and the sample variance is an unbiased estimator so an UMVUE exists.

35. (a) $\sum X_i$

(b) Find it by using the form given in Equation (16).

37. e^{-X_i} has an exponential distribution with parameter e^{θ}. See Problem 24 and 32 for similar problems.

(f) The given statistic is a function of the complete suffifient statistics $\sum e^{-\lambda_i}$ which has a gamma distribution. Verify that the given statistic is unbiased.

39. This is a generalization of Problems 21 and 31.

(a) $a(\theta) = [\int_0^\theta b(x)\ dx]^{-1}$ so $a(\theta)$ is non-increasing. The likelihood function is proportional to $a^n(\theta)$ for $\theta > Y_n$. MLE of θ is Y_n.

(b) Y_n. See Example 33 for the idea of the completeness proof.

40. θ is the mean and variance.

(a) $\sum X_i^2$.

(b) It is not a function of a complete sufficient statistic.

(c) No.

41. θ should have been assumed positive. Then θ is the mean and standard deviation, and is a scale parameter.

42. (a) $Y_n/2$.

(b) No, $E[\frac{n+1}{2n+1}Y_n - \frac{n+1}{n+2}Y_1] = 0$.

(c) $\frac{(n+2)[Y_1^{-n-1} - (Y_n/2)^{-n-1}]}{(n+1)[Y_1^{-n-2} - (Y_n/2)^{-n-2}]}$.

(d) $(Y_1 + Y_n)/3$.

43. (a) $\sum X_i^2$ is complete and sufficient. $\sum X_i^2/n$ is UMVUE of θ^2.

(b) $c^* = 1/(n+2)$ minimizes MSE in family of estimators of form $c\sum X_i^2$.

(c) $\Gamma(n/2)\sqrt{\sum X_i^2}/[\sqrt{2}\Gamma((n+1)/2)]$.

(e) Yes, since both are scale invariant.

44. (a) Y_1

(b) Y_1

(c) $\overline{X} - 1$

(d) Y_1

(e) $Y_1 - (1/n)$

(f) $Y_1 - (1/n)$

(g) $\frac{e^{(n-1)Y_1}[Y_1 - 1/(n-1)] + 1/(n-1)}{e^{(n-1)Y_1} - 1}$

45. The θ in the indicator function should be 0.

(a) Posterior distribution of $\theta \propto$

$\theta^n \exp[(\theta - 1) \log \sum x_i]\theta^{r-1}e^{-\lambda\theta}$, hence it is gamma$(n + r, \lambda - \sum \log x_i)$.

(b) Mean posterior is $(n + r)/(\lambda - \sum \log X_i)$.

46. Similar to Example 45.

47. (f) Similar, but slightly more tedious, to Example 46.

50. See the last paragraph in Section 7.2.

51. This problem is similar to several others and makes a good review question. Recall that the sum of geometric distributed r.v.'s have negative binomial distribution. See Problem 21 in Chapter V.

(g) $\theta = P[X = 0]$, so $I_{\{0\}}(X_1)$ is an unbiased estimator, and $E[I_{\{0\}}(X_1) \mid \sum X_i]$ is UMVUE.

(h) posterior distribution $\propto \theta^n(1-\theta)^{\sum x_i} I_{(0,1)}(\theta)$, hence the posterior is beta$(n+1, \sum x_i + 1)$.

53. The middle β should be $1/\beta$. The factorization criterion shows that Y_1 and $\sum X_i$ are jointly sufficient. Y_1 and $\sum(X_i - Y_1)$ are sufficient and complete. Now $E[Y_1] = \alpha + (\beta/n)$ and $E[\sum(X_i - Y_1)] = (n - 1)\beta$, so $\sum(X_i - Y_1)/(n - 1)$ is UMVUE of β and $Y_1 - [\sum(X_i - Y_1)/n(n - 1)]$ is UMVUE of α.

54. (a) Factorization criterion gives $(\sum X_i, Y_1)$.

(b) $L(\theta, \alpha; x_1, \ldots, x_n) = (1 - \theta)^n \theta^{\sum x_i} \theta^{-n\alpha}$ for $0 \leq \theta \leq 1$ and $\alpha = y_1$ $y_1-1, y_1-2, \ldots$. It is monotone increasing in α for each θ, hence MLE of α is Y_1 and MLE of θ is $(\overline{X} - Y_1)/(\overline{X} - Y_1 + 1)$.

55. Picture the likelihood function. Between any two consecutive order statistics, the likelihood function is "cusp" shaped. It can be concluded that the maximum of the likelihood function occurs at an order statistic, pick that order statistic that maximizes $L(y_j)$ for $j = 1, \ldots, n$.

CHAPTER VIII

Chapter VIII PROBLEMS

1. (a) $Q = -\theta \log X$ has an exponential distribution with parameter one.

   (b) $P[Y/2 < \theta < Y] = e^{-1/2} - e^{-1}$. Using the pivotal quantity given in part (a) $P[q_1 Y < \theta < q_2 Y]$ is obtained. There are two ways of proceeding to find a better confidence interval; the first is to choose q_1 and q_2 so that the confidence interval has confidence coefficient $e^{-1/2}-e^{-1}$ and minimum expected length, and the second is to choose q_1 and q_2 so that the confidence interval has expected length = (1/2)E[Y] and maximum confidence coefficient.

2. $Q = (n-1)S^2/\theta$.

3. $P[T_1 < \tau(\theta) < T_2] = P[T_1 < \tau(\theta)] + P[\tau(\theta) < T_2] - P[T_1 < \tau(\theta) \text{ or } \tau(\theta) < T_2]$

   $= \gamma + \gamma - 1$.

4. As in Problem 3, $[Y_1 < \theta < Y_n] = P[Y_1 < \theta] + P[\theta < Y_n] - 1$

   $= [1 - (1/2)^n] + [1 - (1/2)^n] - 1 = 1 - (1/2)^{n-1}$.

5. (a) $Q = \theta \sum X_i$ is a pivotal quantity.

   (b) Use part (a) and the Remark on Page 378.

   (c) γ

   (d) See part (b).

   (e) $n\theta Y_1$.

6. Similar to Problem 1.

7. (a) $\gamma = 1/2$. (See the solution to Problem 4.) $E[Y_2 - Y_1] = E[|X_2 - X_1|]$

   $= 2\sqrt{\pi} \approx 1.1284$

   (b) Have $P[q_1 < \bar{X}-\theta < q_2] = 1/2$. Choose q_1 and q_2 symmetric about zero; expected length $\approx .95$.

8. (a) Use $Q = \sqrt{n}(\bar{X} - \mu)/\sigma$ as your pivotal quantity.

   (b) Use $Q = \sum(X_i - \mu)^2/\sigma^2$ as your pivotal quantity.

9. (-2.09, 2.84) for σ known and (-1.94, 2.69) for σ unknown.

10. (b) Use $\overline{X} - 1.645S$.

11. Use $Q = \sum_{i=1}^{5} \sum_{j=1}^{n_i} (X_{ij} - \overline{X}_{i.})^2/\sigma^2$ as your pivotal quantity. $Q \sim$ chi-square with 23 degrees of freedom.

12. Use $\dfrac{\left(\sum_1^m (X_i - \overline{X})^2/\sigma_1^2\right)/(m-1)}{\left(\sum_1^n (Y_i - \overline{Y})^2/\sigma^2\right)/(n-1)} \sim F(m-1, n-1)$ as a pivotal quantity.

13. Want $P[2tS/\sqrt{20} < \sigma]$ where t is the $(1-\gamma)/2$ quantile of a t-distribution with 19 degrees of freedom. Write as $P[2tS/\sqrt{20} < \sigma] = P[(19)S^2/\sigma^2 < 19(20)/4t^2]$, where $(19)S^2/\sigma^2$ is chi-square distributed with 19 degrees of freedom, to complete the calculations for any γ.

14. (a) $2z\sigma$/sqrtn where z is the $(1+\gamma)/2$ quantile of a standard normal.

    (b) $2tE[S]/\sqrt{n}$ where t is the $(1+\gamma)/2$ quantile of a t-distribution with n-1 degrees of freedom. See Problem 17 of Chapter VI for E[S].

15. Want $P[2tS/\sqrt{n} < \sigma/5] \approx .95$ where t is .95th quantile of a t-distribution with n-1 degrees of freedom. Rewrite as $P[(n-1)S^2/\sigma^2 < (n-1)n/100t^2]$. Want the minimum n such that $(n-1)n \geq 100t^2_{.95,n-1}\chi^2_{.95,n-1}$. n a little over 300 seems to work.

18. Use Equation (1). (1.47,10.03)

19. The first "the" should be "a". Use $Q = -\sum \log F(X_i;\theta) = -(1/\theta)\sum \log X_i$ as a pivotal quantity.

20. Use the statistical method and $\sum X_i$ as a statistic.

21. $[(Y_1 + Y_2)/2] - \theta$ ias good pivotal quantity.

24. The sample size seems large enough to use Equation (10) of Example 8. $.4375 \pm .0408$ for 90%.

25. The UMVUE of $\tau(\theta)$ is a linear function of $\overline{X}$ and S. $\overline{X}$ and S are independent and have large sample normal distributions. Hence the large sample distribution of the UMVUE of $\tau(\theta)$ is normally distributed. Use this to get an appropriate confidence interval.

26. Similar to Example 9.

27. The posterior distribution is given in the solution of Problem 45 of Chapter VII. Use it and Equation 21.

28. The likelihood function is the joint distribution of $Y_1, \ldots, Y_k$ looked at as a function of θ. $L(\theta; y_1, \ldots, y_k) = \frac{n!}{(n-k)!}\theta^k e^{-\theta \sum_i y_i} e^{-\theta y_k (n-k)}$ for $y_1 \leq y_2 \leq \cdots \leq y_k$. MLE of $1/\theta$ is $[\sum_1^k Y_j + (n-k)Y_k]/k$. Let $U_i = Y_i - Y_{i-1}$. $U_i \sim$ negative exponential with parameters $\theta(n-i+1)$ using the lack of memory property of exponentially distributed random variables. $\theta(n-1+1)U_i \sim$ negative exponential with parameter 1.

$\sum Y_i + (n-k)Y_k = Y_1 + Y_2 + Y_3 + \cdots + Y_{k-1} + (n-k+1)Y_k$

$= U_1 + (U_1 + U_2) + (U_1 + U_2 + U_3) + \cdots + (n-k+1)(U_1 + \cdots + U_k)$

$= nU_1 + (n-1)U_2 + \cdots + (n-k+1)U_k = \sum_{j=1}^{k}(n-j+1)U_j$. Also,

$\theta(\sum Y_i + (n-k)Y_k) = \sum_{j=1}^{k}\theta(n-j+1)U_j$, which is a sum of k independent negative exponentially distributed r.v.'s with parameter 1. Use $Q = \theta(\sum Y_i + (n-k)Y_k) \sim$ gamma$(k, 1)$ as a pivotal quantity.

CHAPTER IX

Chapter IX PROBLEMS

1. (a) (i) $\Pi_\Upsilon(\theta) = \sum_{i}^{10} \binom{10}{j} \theta^j (1-\theta)^{10-j}$

   (ii) $\Pi_\Upsilon(1/2) \approx .377$

   (b) (i) $C_\Upsilon = \{(x_1, \ldots, x_{10}) : \sum x_j \le 2\}$

   (ii) $\Pi_\Upsilon(1/4) \approx .53$

   (c) (i) Want $\mathcal{R}(\theta_0) = \mathcal{R}(\theta_1)$. Reject for $\sum X_i \le 4$ does it.

   (ii) maximum risk for minimax ≈ 385

   maximum risk for M.P. ≈ 815

   (d) Reject for $\sum X_i \le 4$.

2. (a) $\Pi(\theta) = 1 - (3/4)^\theta + \theta(3/4)^\theta \log(3/4)$.

   size $= 1/4 + (3/4)\log(3/4)$

   (b) Reject if and only if $X_1X_2 \ge 1/2$.

   (c) Yes

   (d) Reject if and only if $X_1X_2 \ge 1/2$

   (e) Reject if and only if $\Pi X_i > 1/2^n$.

   (f) This is equivalent to finding the minimax test with $\ell(d_0;\theta_1) = \ell(d_1;\theta_0) = 1$. Reject if and only if $X_1X_2 \ge k$ where k is solution to $1 - k + k\log k = k^2 - 2k^2 \log k$.

4. (e) Reject for $X < k$ where k is such that $\alpha+\beta = k^2+(1-k)$ is minimized; i.e., $k = 1/2$.

   (f) After some manipulation the test reduces to: reject for $X \log X < k$ where k is such that $P_{\theta=1}[X \ln X < k] = \alpha$. Note that this test does say to reject for "large" and "small" x which is intuitively appealing.

5. (a) Reject if and only if $X > 1 - \alpha$.

   (b) $\Pi(\theta) = P[X > 1/2] = 1/2 + (1/4)\theta$. Size is 1/2.

   (c) Yes. Have monotone likelihood ratio in X. Test is: reject iff $X > 1 - \alpha$.

(d) Reject if and only if $|X - 1/2| > c$ where c is such that $P_{\theta=0}[|X - 1/2| > c] = \alpha$; i.e., $c = (1 - \alpha)/2$.

(e) $\alpha + \beta = P_{\theta=0}[X > k] + P_1[X < k] = 1 - k + k^2$ which is a minimum for $k = 1/2$.

6. (a) $\Pi(\theta) = 1 - P_\theta[\theta_0\alpha^{1/n} \le Y_n \le \theta_0] = I_{(0,\theta_0\alpha^{1/n})}(\theta) + \alpha(\theta_0/\theta)^n I_{(\theta_0\alpha^{1/n},\theta_0)}(\theta) + [1 - (1 - \alpha)(\theta_0/\theta)^n] I_{(\theta_0,\infty)}(\theta)$.

7. (a) Reject if and only if $-\sum \log X_i > (\theta_0/2)\chi^2_{2n,1-\alpha}$ where $\chi^2_{2n,1-\alpha}$ is the $(1-\alpha)$-quantile of a chi-square distribution with 2n degrees of freedom.

10. (a) $\Pi(\theta) = P_\theta[X_1+X_2 \ge 1] = (1/2)[(2\theta-1)/\theta]^2 I_{(1/2,1)}(\theta) + [1-(1/2\theta^2)] I_{(1,\infty)}(\theta)$. Size of test $= \Pi(1) = 1/2$.

(b) UMP size $\alpha = 1/2$ test is given by: reject iff $Y_2 \ge 1/\sqrt{2}$. Power of UMP test is $[1-(1/2\theta^2)] I_{(1/\sqrt{2},\infty)}(\theta)$, which is identical to the power of the given test for $\theta > 1$. Note that the test in part (b) is based on a sufficient statistic and the test in part (a) is not.

11. (a) $\Pi(\theta) = 1 - (1 + \theta)e^{-\theta}$

(d) Reject if and only if $X_1 \le 2\log 2$.

12. (a) $k = 1 - \alpha^{1/n}$

(b) $[\alpha + 1 - (1 - \theta)^n] I_{(0,1-\alpha^{1/n})}(\theta) + I_{(1-\alpha^{1/n},\infty)}(\theta)$

(c) Maybe this part should have been starred. To prove it, find the most powerful size α test of $\theta = 0$ versus $\theta = \theta_1$ where $0 < \theta_1 < 1$ (if $\theta_1 > 1$ you can tell with certainty which hypothesis is true.) It turns out that the power under the alternative $\theta = \theta_1$ is the same as the power of the given test, so the given test must be uniformly most powerful.

13. (a) $$\frac{\left(\frac{m+n}{-\sum \log X_i - \sum \log Y_j}\right)^{m+n} [\exp(\sum \log X_i + \sum \log Y_j)]^{[(m+n)/(-\sum \log X_i - \sum \log Y_j)]-1}}{\left(\frac{m}{-\sum \log X_i}\right)^{m} \left(\frac{n}{-\sum \log Y_j}\right)^{n} [\exp(\sum \log X_i)]^{[n/-\sum \log X_i]-1} [\exp(\sum \log Y_j)]^{[n/-\sum \log Y_j]-1}}$$
$$= \frac{(m+n)^{m+n}}{m^m n^n} \left(\frac{-\sum \log X_i}{-\sum \log X_i - \sum \log Y_j}\right)^{m} \left(\frac{-\sum \log Y_j}{-\sum \log X_i - \sum \log Y_j}\right)^{n}$$

(b) Test is of form reject $\mathscr{H}_0$ if and only if $T^m(1 - T)^n \leq$ constant.

(c) T has a beta distribution with parameters m and n and does not depend on the common value of θ_1 and θ_2 under $\mathscr{H}_0$. (See Example 25 of Chapter V)

14. See Example 11. How does the answer change if you test $\mathscr{H}_0 : \theta = 1$ versus $\mathscr{H}_1 : \theta \neq 1$?

15. This is a good review or test question. The density is the same as the densities of Problem 2 and 4 with slight reparametrization.

16. See Problem 13.

Problem 17 through 35 cover material allied to that of Section 4 on sampling from the normal distribution.

17. Let $D_i = X_i - Y_i$. $\delta \pm t_{(1+\gamma)/2}\sqrt{\sum(D_i - \overline{D})^2/(n-1)n}$ is γ-level confidence interval for $\mu_X - \mu_Y$. Use test: Reject $\mathscr{H}_0$ if and only if the confidence interval does not contain zero. Test has size $\alpha = 1 - \gamma$.

22. $C^* = \{(x_1, \ldots, x_n) : \sum x_i \leq 6/n + n\sigma^2 z_\alpha\}$.

24. Use test: reject $\mathscr{H}_0 : \mu = \mu_0$ if and only if $\overline{X} > k$ where $\mu_0 < k < \mu_1$.
$\alpha = P_{\mu_0}[\overline{X} > k] = 1 - \Phi\left(\frac{k - \mu_0}{\sigma/\sqrt{n}}\right) \to 0$ as $n \to \infty$ and $\beta = P_{\mu_1}[\overline{X} < k] = \Phi\left(\frac{k - \mu_1}{\sigma/\sqrt{n}}\right) \to 0$ as $n \to \infty$.

25. Use test based on statistic given in Equation (18).

30. Could use Theorem 7. $-2 \log \lambda_n \approx 4.14 < \chi^2_{.99}(2) = 9.21$

33. $X_{11}, \ldots, X_{1n}$ r.s. from $N(\mu_1, \sigma^2)$

$\vdots \qquad\qquad \vdots$

$X_{k1}, \ldots, X_{kn}$ r.s. from $N(\mu_k, \sigma^2)$

$\mathscr{H}_0: \mu_1 = \mu_2 = \cdots = \mu_k = 0$. λ reduces to $\left(\frac{\sum\sum(x_{ji}-\bar{x}_{j.})^2+\sum\sum\bar{x}_{j.}^2}{\sum\sum(x_{ji}-\bar{x}_{j.})^2}\right)^{-nk/2}$; so the GLR test is equivalent to: reject $\mathscr{H}_0$ if and only if $T = \frac{\sum\sum\bar{X}_{j.}^2/k}{\sum\sum(X_{ji}-\bar{X}_{j.})^2/k(n-1)}$ is "large". Under $\mathscr{H}_0$, T is F-distributed with k and $k(n-1)$ degrees of freedom.

34. $L(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho; (x_1, y_1), \ldots, (x_n, y_n)) \propto$
$$(\sigma_X^2\sigma_Y^2)^{-n/2} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{\sum(x_i-\mu_X)^2}{\sigma_X^2} - 2\rho\frac{\sum(x_i-\mu_X)(y_i-\mu_Y)}{\sigma_X\sigma_Y} + \frac{\sum(y_i-\mu_Y)^2}{\sigma_Y^2}\right]\right\}.$$
The MLE are $\hat{\mu}_X = \bar{x}$, $\hat{\mu}_Y = \bar{y}$, $\hat{\sigma}_X^2 = \sum(x_i - \bar{x})^2/n$, $\hat{\sigma}_Y^2 = \sum(y_i - \bar{y})^2/n$, and $\hat{\rho} = \sum(x_i - \bar{x})(y_i-\bar{y})/\sqrt{\sum(x_i-\bar{x})^2(y_i-\bar{y})^2}$. λ reduces to $(1-\hat{\rho}^2)^{n/2}$. GLR test is equivalent to: reject $\mathscr{H}_0$ if and only if $|\hat{\rho}|$ is "large". Under $\mathscr{H}_0$, the distribution of $\hat{\rho}$ is free of parameters.

35. There are two cases depending on whether or not the common value for the mean under the null hypothesis is knwon. The generalized likelihood ratio technique gives a test using test statistic $\sum[(X_i-\mu)^2/\sigma_i^2]$ for μ assumed known and test statistic V of Problem 27 of Chapter VI for μ assumed unknown.

39. (a) $E[Q] = \sum_{i}^{k+1}\frac{1}{np_j}E[(N-np_j)^2] = \sum_{1}^{k+1}\frac{1}{np_j}np_j(1-p_j) = k$.

$\mathrm{var}[Q] = \sum_i\sum_j\frac{1}{np_i}\frac{1}{np_j}\mathrm{cov}[(N_i-np_i)^2, (N_j-np_j)^2]$. Certain fourth order central moments of the N_i's are needed; these can be found directly or by using the moment generating function. After some manipulation, var[Q] reduces to $2k + (1/n)[\sum(1/p_j) - k^2 - 4k - 1]$.

(b) $E[Q_k^\circ] = \sum_1^{k+1} (1/np_j^\circ)[np_j(1-p_j) + n^2(p_j - p_j^\circ)^2]$

$E[Q_k^\circ]\Big|_{p_j=p_j^\circ} = k$. The answer is no and can be verified by proper choices of p_j and p_j°. One might try to minimize $E[Q_k^\circ]$ with respect to the P_j's using Lagrange multipliers and constraint equation $\sum p_j = 1$; $p_j^* = [(2n+k-1)p_j^\circ - 1]/2(n-1)$ results. Furthermore, such p_j^* will fall between zero and one for p_j° between $1/(2n-1+k)$ and $(2n-1)/(2n-1+k)$.

40. Let p = proportion of headaches that are psychosomatic. Test $\mathscr{H}_0 : p \geq .4$ versus $\mathscr{H}_1 : p < .4$. Let X = # of psychosomatic headaches. Reject $\mathscr{H}_0$ for small X. Model assuming X has a binomial distribution with $n = 41$. For $p = .4$, and X = 12, $\mathscr{H}_0$ would be accepted at the 5% level.

41. Yes, using results from Theorem 8.

42. Yes; see Example 21.

45. The likelihood function is proportional to $(p^2)^{n_1}[2p(1-p)]^{n_2}[(1-p)^2]^{n_3}$. The MLE of p is $(2n_1+n_2)/2n \approx .335$. Obtain $\hat{p}_1, \hat{p}_2$, and $\hat{p}_3$ and use test statistic Q_2'. Accept that the data are consistent with the model.

46. Yes.

47. Reject hypothesis.

48. Test $\mathscr{H}_0 : p_{ij} = p_{i.}p_{.j}$. Reject $\mathscr{H}_0$.

49. Use Q_{2k}' of Equation 30. Note that it reduces to $\sum_{j=1}^{3} \frac{(N_{[ij]} - N_{2j})^2}{N_{1j} + N_{2j}}$, which has value $\approx 7.57 > \chi^2_{.95}(2) = 5.99$.

50. Use approach similar to Problem 45. It is somewhat more difficult to get MLE of p, q, and $r = 1 - p - q$. Compare the computed Q_3' statistic with $\chi^2_{1-\alpha}(1)$.

CHAPTER X

Chapter X PROBLEMS

Problems 1 through 6 are solved by using the given data and appropriate formulas in Sections 4 and 5.

1. Equations 7, 8, and 9.

2. See the Corollaries of Theorem 2.

3. Equations 15, 16, and 14.

4. See Page 494.

5. Use the invariance property; see Theorem 2 on page 285.

6. Similar to Problem 8 below.

7. $P[Y_{x_0} \leq \beta_0 + \beta_1 x_0 + z_p\sigma] = \Phi\left(\dfrac{\beta_0 + \beta_1 x_0 + z_p\sigma - (\beta_0 + \beta_1 x_0)}{\sigma}\right) = \Phi(z_p) = p.$

8. $\hat{\beta}_0 + \hat{\beta}_1 x + z_p[\Gamma((n-2)/2)/\sqrt{2}\Gamma((n-1)/2)]\,[\sum_1^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2]^{1/2}.$

10. $\hat{\beta}_0 \approx .497, \hat{\beta}_1 \approx 2.049, \hat{\sigma}^2 \approx .00117$, and $\hat{\text{var}}[\hat{\beta}_1] \approx .00255$. A 95% confidence interval estimate for β_1 is (1.93, 2.17). $\beta_1 = 1$ is outside this interval, so according to the confidence interval technique, the hypothesis $\beta_1 = 1$ may be rejected.

11. Similar to Problem 10.

12. Could set a one-sided confidence interval on $\mu(.50)$ and use the confidence interval technique.

13. Use the invariance property of confidence intervals. See the Remark on Page 378.

14. $\hat{\beta}_1(n-4)/\sum(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$

15. $\hat{\beta}_1 = \dfrac{(\sum a_i)(\sum a_i x_i y_i) - (\sum a_i y_i)(\sum a_i x_i)}{(\sum a_i)(\sum a_i x_i^2) - (\sum a_i x_i)^2},$
    $\hat{\beta}_0 = (\sum a_i y_i - \hat{\beta}_1 \sum a_i x_i)/\sum a_i$, and
    $\hat{\sigma}^2 = \sum a_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2/n.$

16. Recall that $\hat{B}_0$ and $\hat{B}_1$ have a bivariate normal distribution. What is required for independence in a bivariate normal?

17. $\mathrm{cov}[\overline{Y}, \hat{B}_1] = \mathrm{cov}[\hat{B}_0 + \hat{B}_1\overline{x}, \hat{B}_1] = \mathrm{cov}[\hat{B}_0, \hat{B}_1] + \overline{x}\ \mathrm{var}[\hat{B}_1] = 0$ by Equation (12).
$\overline{Y}$ and $\hat{B}_1$ have a bivariate normal distribution so uncorrelated implies independence.

Problems 19, 20, and 21 can be worked using the theory of Lagrange multipliers as in the proof of Theorem 6.

CHAPTER XI

Chapter XI PROBLEMS

2. $\mathrm{cov}[F_n(B_1), F_n(B_2)] = (1/n)^2 \sum_i \sum_j \mathrm{cov}[I_{B_1}(X_i), I_{B_2}(X_j)] = (1/n)\mathrm{cov}[I_{B_1}(X), I_{B_2}(X)]$
$= (1/n)(P[X \epsilon B_1 B_2] - P[X \epsilon B_1][X \epsilon B_2])$.

4. (a) $D_1 = \max[U, 1-U]$ where U is uniformly distributed over the interval $(0,1)$.
$F_{D_1}(x) = (2x-1)I_{[1/2,1)}(x) + I_{[1,\infty]}(x)$.

   (b) $F_{D_2}(z) = 2(2z-1/2)^2 I_{(1/2,1/3)(z)} + [1-2(1-z)^2]I_{[1/3,1)}(z) + I_{[1,\infty]}(z)$.

   (c) $D_n = \max_{1 \le i \le n}[\left|F(Y_i) - \frac{i-1}{n}\right|, \left|F(Y_i) - \frac{i}{n}\right|]$, so D_n is a function of $F(Y_1), \ldots, F(Y_n)$ which are the order statistics from a uniform over $(0,1)$.

5. $E[Y_2] = E[(Y_1+Y_2)/2] + E[|X_1 - X_2|/2] = (1/2)E[|X_1 - X_2|] = 1/\sqrt{\pi}$ using the fact that $X_1 - X_2 \sim N(0,2)$.

6. Use the same start as in Problem 5. $X_1 - X_2 \sim N(0, 2(1\rho))$.

7. Yes, see Theorem 14 in Chapter VI.

10. n = 15.

11. The data seemed to be ordered; you might be leary of the two-sample sign test.

13. $\mathrm{var}[U] = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{\beta=1}^{n}\sum_{\alpha=1}^{m} \mathrm{cov}[I_{[Y_j,\infty)}(X_i), I_{[Y_\beta,\infty)}(X_\alpha)]$

$= mn\ \mathrm{var}[I_{[Y,\infty)}(X)]$ $\quad (j = \beta$ and $i = \alpha)$

$+ nm(m-1)\ \mathrm{cov}[I_{[Y,\infty)}(X_1), I_{[Y,\infty)}(X_2)]$ $\quad (j = \beta$ and $i \neq \alpha)$

$+ n(n-1)m\ \mathrm{cov}[I_{[Y_1,\infty]}(X), I_{[Y_2,\infty]}(X)]$ $\quad (j \neq \beta$ and $i = \alpha)$

+ zero $\quad (j \neq \beta$ and $i \neq \alpha)$

$= mn\ [P[X \ge Y] - P^2[X \ge Y]]$

$+ nm(m-1)\ (P[X_1 \ge Y, X_2 \ge Y] - P^2[X \ge Y])$

$+ n(n-1)m\ (P[X \ge Y_1, X \ge Y_2] - P^2[X \ge Y])$

$= mn(1/4) + mn(m-1)((1/3) - (1/4)) + mn(n-1)((1/3) - (1/4))$

$= mn(m+n+1)/12$.

14. $m = 1, n = 2$ gives $P[T_x = 1] = P[T_x = 2] = P[T_x = 3] = 1/3$.

$m = 1, n = 3$ gives $P[T_x = 1] = P[T_x = 2] = P[T_x = 3] = P[T_x = 4] = 1/4$.

$m = 2, n = 1$ gives $P[T_x = 3] = P[T_x = 4] = P[T_x = 5] = 1/3$.

$m = 3, n = 1$ gives $P[T_x = 6] = P[T_x = 7] = P[T_x = 8] = P[T_x = 9] = 1/4$.

$m = n = 2$ gives $P[T_x = 3] = P[T_x = 4] = P[T_x = 6] = P[T_x = 7] = 1/6$ and $P[T_x = 5] = 2/6$.

15. U/mn is an unbiased estimator of p. The second question should read: Is U/mn a consistent estimator of p? The answer is yes as can be noted by looking at the intermediate steps in the solution of Problem 13 and letting m and n approach infinity.

16. (a) Just algebra noting that $\bar{r}(X) = \bar{r}(Y) = (n+1)/2$ and $\sum r^2(X_i) = \sum r^2(Y_i) = \sum i^2 = n(n+1)(2n+1)/6$.

(b) $S = .9$ and the ordinary correlation coefficient $\approx .962$.

17. The ranks of $X_1, \ldots, X_n$ are the same as the ranks of $F_X(X_1), \ldots, F_X(X_n)$. Likewise for the Y_j's. By the probability integral transform the distribution of $F_X(X_1), \ldots, F_X(X_n)$ does not depend on $F_X(\cdot)$; likewise for the Y_j's. Hence, the distribution of S (which is a function only of the ranks of $F_X(X_1), \ldots, F_X(X_n)$ and the ranks of $F_Y(Y_1), \ldots, F_Y(Y_n)$) will not depend on $F_X(\cdot)$ and $F_Y(\cdot)$.

18. $E[S] = 1 - [6n/(n^3 - n)]E[D_1^2]$

$= 1 - [6n/(n^3 - n)](E[r^2(X_1)] - 2E[r(X_1)r(Y_1)] + E[r^2(Y_1)])$

$= 1 - [6n/(n^3 - n)]((1/n)\sum i^2 - 2(\sum i/n)^2 + (1/n)\sum i^2)$

$= 0$ using independence of $r(X_1)$ and $r(Y_1)$ and the fact that $r(X_1)$ and $r(Y_1)$ have discrete uniform distributions.

$\text{var}[S] = [36/(n^3-n)^2]\sum\sum \text{cov}[D_i^2, D_j^2] = [36/(n^3-n)](n\ \text{var}[D_1^2]) + n(n-1)\ \text{cov}[D_1^2, D_2^2])$

$= [36/(n^3 - n)^2](nE[D_1^4] - n(E[D_1^2])^2 + n(n-1)E[D_1^2 D_2^2] - n(n-1)E[D_1^2]E[D_2^2])$

$= [36/(n^3-n)^2](n\sum_i\sum_j (i-j)^4(1/n^2) + n(n-1)\sum_i\sum_j\sum_{\alpha\neq i}\sum_{\beta\neq j}(i-j)^2(\alpha-\beta)^2(1/n^2(n-1)^2)$

$- n^2[\sum_i\sum_j (i-j)^2(1/n^2)]^2)$

$= 1/(n-1)$.