

SOLUTIONS MANUAL  
TO INTRODUCTION TO THE THEORY OF STATISTICS  
(1974)  
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ANONYMOUS

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## CHAPTER I

## Chapter I PROBLEMS

1. (a)  $\Omega = \{(B, W), (B, G), (G, W), (G, G)\}$ . The sample space contains four outcomes; an outcome itself is a 2-tuple where the first component represents the result of drawing from urn one and the second component from urn two.

- (b) The event space is the collection of all subsets of the sample space. There are 16 such subsets.

$$\mathcal{A} = \left\{ \phi, \Omega, \{(B, W)\}, \{(B, G)\}, \{(G, W)\}, \{(G, G)\}, \right. \\ \{(B, W), (B, G)\}, \{(B, W), (G, W)\}, \{(B, W), (G, G)\}, \{(B, G), (G, W)\}, \\ \{(B, G), (G, G)\}, \{(G, W), (G, G)\}, \{(B, W), (B, G), (G, W)\}, \\ \left. \{(B, W), (B, G), (G, G)\}, \{(B, W), (G, W), (G, G)\}, \{(B, G), (G, W), (G, G)\} \right\}$$

- (c)  $1/4$

- (d) 0

2. (a) There are many ways to describe the outcomes of this experiment. For example, one could number the balls in urn one as 1, 2, 3 red; 4, 5 white; and 6 blue and those in urn two as 1 red, 2, 3 white; and 4, 5, 6 blue.

- i. Then  $\Omega = \{(i_1, i_2) : i_1 = 1, \dots, 6 \text{ and } i_2 = 1, \dots, 6, \text{ where}$   
 $i_1$  is the number on the ball drawn from urn 1 and  
 $i_2$  is the number on the ball drawn from urn 2.}
- Note that there are 36 outcomes in this experiment.

- ii. Let A denote the event both balls are red.

B denote the event both balls are white, and

C denote the event both balls are blue.

Then  $P[\text{both balls are same color}] = P[A \cup B \cup C] = P[A] + P[B] + P[C] =$

$$\frac{3}{36} + \frac{4}{36} + \frac{3}{36}.$$

- iii.  $P[A] = \frac{3}{36} < \frac{4}{36} = P[B]$

- (b) (i)  $\frac{12 \cdot 8 \cdot 4}{12^3}$  (ii)  $\frac{12 \cdot 8 \cdot 4}{12 \cdot 11 \cdot 10}$

4. (a)  $\Omega = \{(i_1, i_2) : i_1 = 1, \dots, 5 \text{ and } i_2 = 1, \dots, 5, \text{ where } i_1 \text{ is the number on the first ball drawn and } i_2 \text{ is the number on the second ball drawn}\}.$

$$B_1 = \{(i_1, i_2) : i_1 = 1, 2, 3 \text{ and } i_2 = 1, \dots, 5\}$$

$$B_2 = \{(i_1, i_2) : i_1 = 1, \dots, 5 \text{ and } i_2 = 1, 2, 3\}$$

$$B_1 B_2 = \{(i_1, i_2) : i_1 = 1, 2, 3 \text{ and } i_2 = 1, 2, 3\}$$

$$(b) P[B_1] = \frac{\text{size of } B_1}{\text{size of } \Omega} = \frac{3 \cdot 5}{5 \cdot 5}.$$

$$(c) \Omega = \{(i_1, i_2) : i_1 = 1, \dots, 5 \text{ and } i_2 = 1, \dots, 5 \text{ but } i_1 \neq i_2\}.$$

$$B_1 = \{(i_1, i_2) : i_1 = 1, 2, 3 \text{ and } i_2 = 1, \dots, 5 \text{ but } i_1 \neq i_2\}.$$

$$P[B_1] = \frac{3 \cdot 4}{5 \cdot 4}, \text{ etc.}$$

7. Using H for hit, M for miss, R for right hand and L for left hand, the event the the participant is successful is

$$\{(H, H, H), (H, H, M), (M, H, H)\} = A, \text{ say.}$$

$$\text{Under strategy RLR, } P[A] = p_1 p_2 p_1 + p_1 p_2 (1 - p_1) + (1 - p_1) p_2 p_1 \text{ and}$$

$$\text{under strategy LRL, } P[A] = p_2 p_1 p_2 + p_2 p_1 (1 - p_2) + (1 - p_2) p_1 p_2.$$

$$8. (b) P[A \text{ will beat B in three out of four}] = p^3 + 3p^3(1-p) = \binom{4}{3} p^3 (1-p) + p^4$$

$$P[A \text{ will beat B in five out of seven}] = p^5 + 5p^5(1-p) + 15p^5(1-p)^2 \\ = \binom{7}{5} p^5 (1-p)^2 + \binom{7}{6} p^6 (1-p) + p^7$$

10.  $A = B$  and  $p = 1/2$  is a counterexample.

$$14. P[AB] = P[A] + P[B] - P[A \cup B] \geq P[A] + P[B] - 1 = 1 - \alpha - \beta.$$

$$18. (a) (1/3)^4$$

$$(b) 3(1/3)^4$$

$$(c) 3(1/3)^4 + 4 \cdot 3(1/3)^4 = 5/27$$

$$19. (a) P[\text{total of 9}] = 25/216; P[\text{total of 10}] = 27/216$$

$$(b) P[\text{at least one 6 in 4 tosses}] = 1 - (5/6)^4$$

$$P[\text{at least double 6 in 24 tosses}] = 1 - (35/36)^{24}$$

$$(c) P[\text{at least one 6 with 6 dice}] = 1 - (5/6)^6$$

$$P[\text{at least two 6's with 12 dice}] = 1 - (5/6)^{12} - (12)(1/6)(5/6)^{11}$$

20. This is similar to Problem 27.

$$22. (365)_{25}/(365)^{25}.$$

$$23. \frac{\binom{5}{2} \binom{21}{11}}{\binom{26}{13}} + \frac{\binom{5}{3} \binom{21}{10}}{\binom{26}{13}}.$$

$$24. (a) \frac{\binom{r}{k} (n-1)^{r-k}}{n^r}$$

25. Consider that a single coin is tossed until the first head occurs.

$$P[\text{first head occurs on toss } j] = (1/2)^j.$$

$$P[\text{Ace wins}] = (1/2) + (1/2)^5 + (1/2)^7 + \dots = 4/7.$$

$$P[\text{Bones wins}] = (1/2)^2 + (1/2)^5 + (1/2)^8 + \dots = 2/7.$$

$$P[\text{Clod wins}] = (1/2)^3 + (1/2)^6 + (1/2)^9 + \dots = 1/7.$$

$$26. P[\text{single ring formed}] = (4/5)(2/3).$$

$$P[\text{at least one ring formed}] = 1.$$

27. You might test your intuition on this one and guess the answer before you proceed. Let  $A_1 = \{\text{Mr. Bandit does not get caught under strategy 1}\}$  where strategy 1 is to sell all twenty at once; strategy 2 is to put four stolen cattle in one set of ten; strategy 3 is to put three stolen cattle in one set of ten and one on the other; and strategy 4 is to put two stolen cattle in each set of ten.

$$P[A_1] = \frac{\binom{4}{0} \binom{16}{4}}{\binom{20}{4}}$$

$$P[A_2] = \frac{\binom{4}{0} \binom{6}{2}}{\binom{10}{2}} \cdot \frac{\binom{10}{2}}{\binom{10}{2}}$$

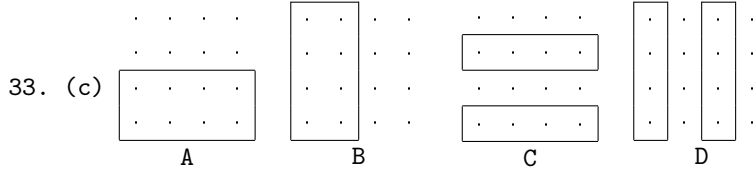
$$P[A_3] = \frac{\binom{3}{0} \binom{7}{1}}{\binom{10}{2}} \cdot \frac{\binom{1}{0} \binom{9}{1}}{\binom{10}{2}}$$

$$P[A_4] = \frac{\binom{2}{0} \binom{8}{2}}{\binom{10}{2}} \cdot \frac{\binom{2}{0} \binom{8}{2}}{\binom{10}{2}}$$

30. (b) Take  $B = C$  and  $P[A] > P[B]$  for a counterexample.

31. Use the corollary of Theorem 29.

32. This is similar to Problem 70. Use Bayes' Formula.



34. Use Theorem 29.

39. Use Bayes' Formula.

40. This problem is known as the "liars problem." It can be varied by changing the number of liars. In fact, the reader might want to try to solve it for only two or three liars before reading the solution. As is the case with most "story" problems some "modelling" is required. Let  $A_T = \{\text{statement that A makes is true}\}$ , and  $D_T = \{D \text{ says that C says that B says that A is telling the truth}\}$ , then  $P[A_T|D_T]$  is what is sought. Also, let

$B_T = \{B \text{ says that A is telling the truth}\}$ , and

$C_T = \{C \text{ says that B says that A is telling the truth}\}$ .

Note that  $\bar{C}_T = \{C \text{ says that B says that A is not telling the truth}\}$  and

similarly for  $\bar{B}_T$  and  $\bar{D}_T$ . Actually some "modelling" has been done in defining these events; for example, it has been assumed that B does say that A's statement is either true or false. Note that

$1/3 = P[A_T] = P[B_T|A_T] = P[C_T|B_T] + P[D_T|C_T] = P[C_T|B_TA_T] = P[D_T|C_TA_T]$ , and

$2/3 = P[\bar{A}_T] = P[B_T|\bar{A}_T] = P[C_T|\bar{B}_T] = P[D_T|\bar{C}_T] = P[C_T|\bar{B_TA}_T] = P[D_T|\bar{C_TA}_T]$ .

Implicitly, it has been assumed that not only does each liar lie with probability  $2/3$  in any given instance, but also the liars lie independently of each other. The solution given here includes the solution for the two and three liars problems.

$P[B_T] = P[B_T|A_T]P[A_T] + P[B_T|\bar{A}_T]P[\bar{A}_T] = (1/3)(1/3) + (2/3)(2/3) = 5/9$ , so

$P[A_T|B_T] = \frac{P[B_T|A_T]P[A_T]}{P[B_T]} = 1/5$ , the solution to the two liar problem.



Now  $P[C_T] = (1/3)(5/9) + (3/2)(4/9) = 13/27$  and

$$\begin{aligned} P[C_T|A_T] &= P[C_TB_TA_T] + P[C_T\bar{B}_TA_T] \\ &= P[C_T|B_TA_T]P[B_T|A_T] + P[C_T|\bar{B}_TA_T]P[\bar{B}_T|A_T] \\ &= 5/9, \text{ hence} \end{aligned}$$

$$P[A_T|C_T] = \frac{(5/9)(1/3)}{(13/27)}, \text{ the solution to the three liar problem.}$$

$$\begin{aligned} \text{Similarly, } P[D_T] &= P[D_TC_T]P[C_T] + P[D_T\bar{C}_T]P[\bar{C}_T] \\ &= (1/3)(13/27) + (2/3)(14/27) = 41/81, \text{ and} \end{aligned}$$

$$\begin{aligned} P[D_T|A_T] &= P[D_TC_TA_T]P[C_T|A_T] + P[D_T\bar{C}_TA_T]P[\bar{C}_T|A_T] \\ &= (1/3)(5/9) + (2/3)(4/9) = 13/27, \text{ and} \end{aligned}$$

finally,

$$P[A_T|D_T] = \frac{(13/27)(1/3)}{41/81} = \frac{13}{41}.$$

42. (a)  $2/3$

(b)  $4/5$

(c)  $1$

46. (b) A and B disjoint and  $P[A] \neq P[B]$  gives a counterexample.

48. Let  $A_j = \{\text{exactly } j \text{ seeds out of the fifty germinate}\}$ .

Model by assuming each seed germinates with probability 0.96.  $P[\text{package will violate guarantee}] =$

$$\sum_{j=0}^{44} P[A_j] = 1 - \sum_{j=45}^{50} P[A_j] = 1 - \sum_{j=45}^{50} (.96)^j (.04)^{50-j}.$$

50. Intuition says the answer ought to be greater than  $1/2$ .

Let  $A = \{\text{tested stone is real}\}$

$B = \{\text{son gets real diamonds}\}$  We want  $P[B|A]$  and  $P[B|\bar{A}]$ . Symmetry suggests that these two conditional probabilities are equal.

Define  $C = \{\text{box with two real diamonds is selected for testing}\}$  and model by assuming  $P[C] = 1/2$ ,  $P[A|C] = 2/3$ , and  $P[A|\bar{C}] = 1/3$ .

Then  $P[A] = P[A|C]P[C] + P[A|\bar{C}]P[\bar{C}] = 2/3 \cdot 1/2 + 1/3 \cdot 1/2 = 1/2$ .

$$\begin{aligned} P[B|A] &= \frac{P[AB]}{P[A]} = \frac{P[AB|C]P[C] + P[AB|\bar{C}]P[\bar{C}]}{P[A]} \\ &= \frac{(2/3)(1/2) + (0)(1/2)}{1/2} = 2/3. \text{ Similarly} \\ P[B|\bar{A}] &= \frac{(0)(1/2) + (2/3)(1/2)}{1/2} = 2/3. \end{aligned}$$

57. Let  $A = \{\text{player wins}\}$ . Let  $B_j = \{\text{total of } j \text{ on first toss}\}$ .

$$P[A] = \sum_{j=2}^{12} P[A|B_j]P[B_j].$$

59. (a)  $p^4 + 4p^3(1-p) + 4p^2(1-p)^2 = a$  (say)

(b)  $p^4 + 4p^3(1-p) + 2p^2(1-p)^2 = b$  (say)

(c)  $pa + (1-p)b$

62. Mark first in a corner. The random player must then mark in the center to keep you from winning. Next mark one of the two spaces adjacent to your first mark, etc. Your opponent's chance of forcing a tie under this strategy is  $(1/8)(1/6)(1/4)(2/2)$ . No other strategy does better. Your chance of winning is  $191/192$ . How does the problem change if you allow your opponent to mark first?

63. Apply Bayes' Formula.

67.  $3/4$ ;  $1/3$

68. (a) Outcomes are yellow-smooth (Y-S), yellow-wrinkled (Y-W), green-smooth (G-S), and green-wrinkled (G-W); they are equally likely.

$$(b) \begin{array}{c|c|c|c} Y-S & Y-W & G-S & G-W \\ \hline 3/8 & 1/8 & 3/8 & 1/8 \end{array}$$

$$(c) \begin{array}{c|c|c|c} Y-S & Y-W & G-S & G-W \\ \hline 9/16 & 3/16 & 3/16 & 1/16 \end{array}$$

70. (a)  $P[B|A] = \frac{P[A|B]P[B]}{P[A|B]P[B] + P[A|\bar{B}]P[\bar{B}]} = \frac{(.95)(.05)}{(.95)(.05) + (.05)(.95)} = \frac{1}{2}$ .

(b)  $.9 = \frac{p(.05)}{p(.05) + (1-p)(.95)}$  implies  $p = \frac{17.1}{17.2} \approx .9942$ .

## CHAPTER II

## Chapter II PROBLEMS

Several of these problems requires showing that a given function is a p.d.f.

This simply involves verifying the conditions of Definition 9.

1. (a)  $f_1(\cdot)$  and  $f_2(\cdot)$  are easily shown to be p.d.f.s. Also, the integral of  $f(x)$  is clearly unity. One can show that  $f(x) \geq 0$ .

- (b) You can disprove this by taking  $\theta_1 = -1, \theta_2 = 2, f_1(x) = I_{(0,1)}(x)$  and  $f_2(x) = I_{(1,2)}(x)$ .

2. The median is  $\alpha$ .

3. Need  $K \int_{-K}^K x^2 dx = 1$ , which gives  $K = \text{fourth root of } 3/2$ .

4. (a) Since  $F_X(x)$  can be written as a function of  $(x-\alpha)/\beta$ , let's do it. That is, write  $F_X(x) = F\left(\frac{x-\alpha}{\beta}\right)$ .

$$\begin{aligned} \text{Now } E[X] &= \int_0^\infty \left[1 - F\left(\frac{x-\alpha}{\beta}\right)\right] dx - \int_{-\infty}^0 F\left(\frac{x-\alpha}{\beta}\right) dx \\ &= \beta \int_{-\alpha/\beta}^\infty (1 - F(y)) dy - \beta \int_{-\infty}^{-\alpha/\beta} F(y) dy \\ &= \beta \left\{ \int_0^\infty (1 - F(y)) dy - \int_{-\infty}^0 F(y) dy + \int_{-\alpha/\beta}^0 (1 - F(y)) dy + \int_{-\alpha/\beta}^0 F(y) dy \right\} \\ &= \beta \left\{ \int_0^\infty (1 - F(y)) dy - \int_{-\infty}^0 F(y) dy \right\} + \alpha. \end{aligned}$$

$E[X]$  equals  $\alpha$  plus a quantity that does not depend on  $\alpha$ ; hence if  $\alpha$  is increased by  $\Delta\alpha$  so is  $E[X]$ .

5. (b)  $X$  is a discrete random variable taking on values 0, 1, 2, and  $P[X=2] = (1/4)^2$ ,  $P[X=1] = 2(1/4)(3/4)$ , and  $P[X=0] = (3/4)^2$ .

- (c)  $E[X] = 1/2$  and  $\text{var}[X] = 3/8$ .

7. (a) The game ends at the first trial if and only if A wins first match; the game ends at the second trial if and only if B wins the first two matches; the game ends at the third trial if and only if B wins the first match and A wins the next two; etc.  
 $P[X=j] = (1/2)^j, j = 1, 2, \dots$

$$(b) E[X] = \sum_{j=1}^{\infty} j(1/2)^j = (1/2) \sum_{j=1}^{\infty} j(1/2)^j = 2.$$

$$\begin{aligned} \text{var}[X] &= E[X^2] - 4 = E[X(X-1)] + 2 - 4 = \\ &= \sum_{j=1}^{\infty} j(j-1)(1/2)^j - 2 = (1/4) \sum_{j=2}^{\infty} j(j-1)(1/2)^{j-2} = 2. \end{aligned}$$

(c) B wins the game if and only if the game ends on an even numbered trial;

$$\text{hence } P[\text{B wins the game}] = (1/2)^2 + (1/2)^4 + \dots = 1/3.$$

Also, let  $p_A$  = probability that A wins the game and  $p_B$  = probability that B wins the game. Note  $p_B = 1 - p_A$ . In order for B to win the game, B must win the first match, having done so B is then in the same position as A at the start of the game, hence  $p_B = (1/2)p_A$  and  $p_B = 1 - p_A$  imply  $p_B = 1/3$ .

Problems 8 and 9 are very similar. The density of 8 is "triangular" whereas that of 9 is "parabolic." Both densities are symmetric about  $\alpha$ .

$$8. (c) E[X] = \alpha \text{ and } \text{var}[X] = \beta^2/6.$$

$$(b) \text{ For } \alpha < q < 1/2, \xi_q = \alpha - \beta + \beta\sqrt{2q}$$

11. Write  $\mu_\theta$  and  $\sigma_\theta^2$  for the mean and variance of  $f(\cdot; \theta)$  including  $\theta = 0$  and  $\theta = 1$ .

$$(b) \mu_\theta = \theta\mu_1 + (1 - \theta)\mu_0$$

$$\sigma_\theta^2 = \theta\sigma_1^2 + (1 - \theta)\sigma_0^2 + \theta(1 - \theta)(\mu_1 - \mu_0)^2$$

$$(c) \theta m_1(t) + (1 - \theta)m_0(t).$$

12. (a) 16/25

(b) Model the problem by assuming that the bombs fall independently of one another. Then if at least one of the three large bombs falls within 40 feet of the track, traffic will be disrupted. Answer is  $1 - (9/25)^3$ .

13. (a)  $E[(X - b)^2] = E[(X - \mu)^2] + (\mu - b)^2$  which is minimized when  $b = \mu$ .

(b) The result follows from the hint by noting that the integral on the right hand side of the equality is non-negative for all  $b$  and zero for  $b = \mu$ .

To prove the hint assume  $m < b$  ( $m > b$  is similar). Write  $E[|X-b|] - E[|X-m|]$

$$= \int_{-\infty}^b (b-x)f(x) dx + \int_b^m (x-b)f(x) dx + \int_m^{\infty} (x-b)f(x) dx -$$

$$\left( \int_{-\infty}^b (m-x)f(x) dx + \int_b^m (m-x)f(x) dx + \int_m^{\infty} (x-m)f(x) dx \right)$$

$$= 2 \int_b^m (x-b)f(x) dx + (b-m)[F(b) + F(m) - F(b) - 1 + F(m)]$$

$$= 2 \int_b^m (x-b)f(x) dx$$

14. (a) 21/25

(b)  $\mu_X = 0$  and  $\sigma_X = 1/2$ , hence

$$P[|X - \mu_X| \geq k\sigma_X] = 1/4 = 1/k^2.$$

(c) See problem 20.

15.  $E[X] = 1$  and  $\text{var}[X] = 1/2$ .

17. No, by Chebyshev inequality.

20.  $P[X \leq \mu t] \geq P[X < \mu t] = 1 - P[(X/\mu) \geq t] \geq 1 - E[(X/\mu)]/t = 1 - (1/t)$  by Chebyshev inequality.

24. (a)  $f_X(x; \theta) \geq 0$  for  $-1/2 \leq \theta \leq 1/2$ .

(b)  $E[X] = (2/3)\theta$ ; median =  $\frac{-1 + (1 + 4\theta^2)^{1/2}}{2\theta}$  for  $\theta \neq 0$ .

(c)  $\theta = 0$ .



## CHAPTER III

## Chapter III PROBLEMS

1. (f) No, the variance of a negative binomial random variable cannot be smaller than its mean.
- (h) Rectangular, normal, logistic, and beta with  $a=b$ . Note that the binomial for  $p=1/2$  and  $n$  even does not work.
- (n) No.
- (o) Yes, if the distribution of  $X$  is symmetric about zero.

2. (b) If  $r \leq 1$ , the mode is zero. If  $r > 1$ , the mode is  $(r-1)/\lambda$ .

4. (b)  $2\Phi(-2)$

(c)  $P[X \leq 0] = \Phi(-\mu/\sqrt{h(\mu)}) = \Phi(-1/\sqrt{a})$  for  $h(\mu) = a\mu^2$ ,  $\mu > 0$ .

6. Let  $X$  be a random variable denoting the low bid of the competition.  $X$  is uniformly distributed over the interval  $((3/4)C, 2C)$ . Let  $P$  denote profit and  $B$  the amount the contractor should bid. Now  $P = (B - C)I_{(B, 2C)}(X)$  and

$$\begin{aligned} E[P] &= \int (B - C)I_{(B, 2C)}(x)f_X(x) dx = (B - C) \int_{(3/4)C}^{2C} I_{(B, 2C)}(x) \left(2C - \frac{3}{4}\right)^{-1} dx \\ &= \frac{(B - C)}{\left(\frac{5}{4}\right)C} (2C - B). \text{ Now maximize with respect to } B \text{ and obtain } B = \frac{3C}{2}. \end{aligned}$$

7. (a) Let  $k$  = number he should stock and  $X$  the number he can sell in 25 days.

Want the minimal  $k$  such that  $P[X \leq k] \geq .95$  where  $X$  has a Poisson distribution with parameter 100; that is, solve for  $k$  in  $\sum_{i=0}^k \frac{e^{-100}(100)^i}{i!} \geq .95$ .

From a table of the Poisson distribution,  $k=117$  is obtained. Using the normal approximation and  $\Phi\left(\frac{k-100}{10}\right) = .95$ ,  $k=117$  is obtained.

- (b) Let  $Z$  = number of days out of 25 that he sells no items.

Under appropriate assumptions (what are they?)  $Z$  has a binomial distribution with  $n=25$  and  $p=c^{-4}$ . Hence,  $E[Z] = 25c^{-4}$ .

8. (a)  $Y$  has a binomial distribution with parameters  $n$  and  $q$ .

- (b)  $X$  has a binomial distribution with parameters  $n$  and  $15/36$ .

- (c)  $(X+n)/2$  has a binomial distribution with parameters  $n$  and  $p$ .

Hence  $E[X] = n(2p - 1)$ .

(d) Show that  $\sum_{j=0}^k \binom{n}{j} (p_1^j q_1^{n-j} - p_2^j q_2^{n-j}) = \sum_{j=0}^k d_j (\text{say}) \geq 0$ .

Note that  $\sum_{j=0}^n d_j = 0$ , hence it suffices to show that the first few  $d_j$ 's are positive, and the remaining are negative. But  $d_j \geq 0$  if and only if

$$j \leq n \log(q_2/q_1) / \log(p_1 q_2 / p_2 q_1).$$

(Use the result of Problem 28 for an alternate proof.)

9.  $\sum_{j=60}^{100} \frac{\binom{2500}{j} \binom{2500}{100-j}}{\binom{5000}{100}}$ . The hypergeometric can be approximated by the binomial and the binomial can in turn be approximated by the normal which gives a numerical answer of approximately  $1 - \Phi(2) = .0228$ .

11. Let  $X$  denote the number of defectives in the sample. Assume that  $X$  has a binomial distribution.

(a)  $P[X \geq 1] = 1 - P[X = 0] = 1 - (.99)^{10}$ .

(b) Want  $P[X \geq 1] \approx .95$ ; or, want  $P[X = 0] \approx .05$ ;

i.e.,  $(.9)^n \approx .05$ , or,  $n \approx 29$ .

15.  $\mu + c \left[ \Phi\left(\frac{a-\mu}{\sigma}\right) - \Phi\left(\frac{b-\mu}{\sigma}\right) \right] / \left[ \Phi\left(\frac{b-\mu}{\sigma}\right) - \Phi\left(\frac{a-\mu}{\sigma}\right) \right]$

17. There is a misprint in this problem. The mean was intended to be 200 rather than 20. Want

$$P[X \geq 150] \geq .90, \text{ or, } \Phi\left(\frac{50}{\sigma}\right) \geq .90, \text{ which implies } \sigma \approx 50/1.282 \approx 39.$$

19. (a)

$$\begin{aligned} E[X] &= \int_0^\infty \beta^{-2} x^2 \exp[-(1/2)(x/\beta)^2] dx \\ &= (1/2) \sqrt{2\pi} \beta^{-1} \int_{-\infty}^\infty x^2 (1/\beta \sqrt{2\pi}) \exp[-(1/2)(x/\beta)^2] dx \\ &= \beta \sqrt{2\pi}/2 \text{ by recognizing that the last integral is the variance of a} \end{aligned}$$

normal distribution with mean 0 and variance  $\beta^2$ , which shows how a little knowledge of probability can be an aid to integration.

$$\text{var}[X] = \beta^2(4 - \pi)/2.$$

(b) No.

25. 

1	2	3	4	5	6	7	8	9
$\frac{9}{81}$	$\frac{12}{81}$	$\frac{16}{81}$	$\frac{12}{81}$	$\frac{12}{81}$	$\frac{10}{81}$	$\frac{6}{81}$	$\frac{3}{81}$	$\frac{1}{81}$



28. Assume true and differentiate both sides with respect to  $p$  to obtain the equality:

$$\sum_{j=k}^n j \binom{n}{j} p^{j-1} q^{n-j} - \sum_{j=k}^n (n-j) \binom{n}{j} p^j q^{n-j-1} = k \binom{n}{k} p^{k-1} q^{n-k}.$$

The inequality is verified by noting the  $(j+1)$ st term of the first sum cancels the  $j$ th term of the second sum. Work backwards.

29. Let  $X$  = # of successes in the first  $n$  Bernoulli trials

and  $Y$  = # of failures prior to the  $r$ th success.

Note that  $(X \leq r-1) \cong (Y > n-r)$  hence  $F_X(r-1) = P[X \leq r-1] = P[Y > n-r] = 1 - F_Y(n-r)$ .

30.  $E[Z_\lambda] = (E[U^\lambda] - E[1 - U^\lambda])/\lambda = 0$  for  $\lambda > -1$ .

$$\begin{aligned} E[Z_\lambda^2] &= (E[U^{2\lambda}] - 2E[U^\lambda(1-U)^\lambda] + E[(1-U)^{2\lambda}])/\lambda^2 \\ &= (2/\lambda^2)([1/(2\lambda+1)] - B(\lambda+1, \lambda+1)) \text{ for } \lambda > -1/2. \end{aligned}$$

$$E[Z_\lambda^3] = 0 \text{ for } \lambda > -1/3.$$

$$E[Z_\lambda^4] = (2/\lambda^4)([1/(4\lambda+1)] - 4B(3\lambda+1, \lambda+1) + 3B(2\lambda+1, 2\lambda+1)) \text{ for } \lambda > -1/4.$$

The last part is misstated. The intent was to get two different  $\lambda$ 's, say  $\lambda_1$  and  $\lambda_2$ , such that  $Z_{\lambda_1}$  and  $Z_{\lambda_2}$  have the same skewness and kurtosis. If  $\lambda_1$  and  $\lambda_2$  are sought so that  $Z_{\lambda_1}$  and  $Z_{\lambda_2}$  have kurtosis equal to zero, then  $\lambda_1 \approx .135$  and  $\lambda_2 \approx 5.20$  will work.



## CHAPTER IV

## Chapter IV PROBLEMS

1. (a) True (b) False (c) True

$$2. (a) E[X] = \int_0^{\infty} [1 - F_X(z)] dz - \int_{-\infty}^0 F_X(z) dz < \int_0^{\infty} [1 - F_Y(z)] dz - \int_{-\infty}^0 F_Y(z) dz = E[Y]$$

Using Eq. 6 of Chapter II (Page 65).

- (b) There are many counterexamples. For example, define

$$F_X(x) = (1/2)I_{[0,1]}(x) + I_{[1,\infty)}(x) \text{ and}$$

$$F_Y(y) = (3/4)I_{[0,4)}(y) + I_{[4,\infty)}(y).$$

- (c) True. (d) False. (e) True.

$$(f) F_X(x) = P[X \leq z] = P[X + 1 \leq z + 1] = P[Y \leq z + 1] = F_Y(z + 1)$$

3. Yes.

4. (b) 1/4

5. (a) 1/36

$$(b) \text{ For } 0 < x < 1, f_{Y|X}(y|x) = [I_{(x,1)}(y)] / (1 - x).$$

6. (b) 1/4

- (c) 1/6

7. (b) No

$$8. E[Y] = E[E[Y|X]] = 1 + p$$

$$\begin{aligned} 10. P[X = Y] &= \sum_{j=0}^{\infty} P[X = Y = j] P[Y = j] \\ &= \sum_{j=0}^{\infty} P[X = j | Y = j] P[Y = j] \\ &= \sum_{j=0}^{\infty} P[X = j] P[Y = j] \text{ (using independence)} \\ &= \sum_{j=0}^{\infty} p^2 (1 - p)^{2j} = p / (2 - p). \end{aligned}$$

11. (a) No. (b) Yes. (c) No. (d) Yes.

$$12. F_X(x) + F_Y(y) - 1 \leq P[X \leq x] + P[Y \leq y] - P[X \leq x \text{ or } Y \leq y] \\ = P[X \leq x; Y \leq y] = F_{X,Y}(x, y).$$

$$F_{X,Y}(x, y) = P[X \leq x; Y \leq y] \leq P[X \leq x] = F_X(x); \text{ also}$$

$$F_{X,Y}(x, y) \leq F_Y(y).$$

$$14. (d) P[Y - \alpha - \beta\mu \leq z] = P[\alpha + \beta X - \alpha - \beta\mu \leq z] = P[X - \mu \leq z/\beta] = P[-(X - \mu) \leq z/\beta] = \\ P[-(Y - \alpha - \beta\mu) \leq z].$$

$$16. (a) \text{ Since } f_X(z) = f_Y(z) = I_{(0,1)}(z), X \text{ and } Y \text{ are independent if and only} \\ \text{if } \alpha = 0.$$

$$\text{cov}[X, Y] = -\alpha \int_0^1 \int_0^1 (x - 1/2)(y - 1/2)(1 - 2x)(1 - 2y) dx dy = 0 \text{ if and} \\ \text{only if } \alpha = 0.$$

$$(b) E[\text{Area}] = E[XY] = \text{cov}[X, Y] + 1/4$$

$$(c) P[2X < 1] = 1/2.$$

$$(d) \text{Length of perimeter} = 2(X + \sqrt{X^2 + Y^2})$$

$$17. (b) 9/16$$

$$(c) E[Y_1] = 15/8; E[Y_2] = 25/8;$$

$$\text{var}[Y_1] = 70/16 - (15/8)^2 \text{ and } \text{var}[Y_2] = 170/16 - (25/8)^2$$

$$(e) 5/11$$

$$18. (c) 3/4 \quad (d) \text{ Solve for } m \text{ in } 1 - e^{-m} - me^{-m} = 1/2.$$

$$(e) 1 - e^{-1} \quad (f) 0$$

$$19. (a) \text{ Do (b) first.}$$

$$(b) f_X(z) = f_Y(z) = ze^{-z}I_{(0,\infty)}(z).$$

$$(c) 1 + (x/2)$$

$$(d) 1 - 4e^{-2} - e^{-4}.$$

$$(e) 1/2.$$

$$(f) f_X(x)f_Y(y).$$

$$\begin{aligned}
20. \quad (a) \quad P[|X+Y| \leq 2|X|] &= \iint_{|x+y| \leq 2|x|} f(x)f(y) \, dx \, dy \\
&= \int_0^\infty \left( \int_{-3x}^x f(y) \, dy \right) f(x) \, dx + \int_{-\infty}^0 \left( \int_x^{-3x} f(y) \, dy \right) f(x) \, dx \\
&= 2 \int_0^\infty \left( \int_{-3x}^{-x} f(y) \, dy \right) f(x) \, dx + 2 \int_0^\infty \left( \int_{-x}^x f(y) \, dy \right) f(x) \, dx \quad (\text{by symmetry}) \\
&= 2 \int_0^\infty \left( \int_{-3x}^{-x} f(y) \, dy \right) f(x) \, dx + 1/2 > 1/2.
\end{aligned}$$

21. Note that  $E[X - Y] = E[E[X|Y]] - E[Y] = 0$ , so

$$\begin{aligned}
\text{var}[X - Y] &= E[(X - Y)^2] \text{ but } E[(X - Y)^2] = E[X^2] - 2E[XY] + E[Y^2] = \\
&E[XE[Y|X]] - 2E[XY] + E[YE[X|Y]] = 0.
\end{aligned}$$

$$22. \quad P\left[\bigcap_{j=1}^m A_j\right] = 1 - P\left[\overline{\bigcap_{j=1}^m A_j}\right] = 1 - P\left[\bigcup_{j=1}^m \overline{A_j}\right] \geq 1 - \sum_{j=1}^m P[\overline{A_j}] \geq 1 - t^{-2}.$$

$$23. \quad (c) \quad f_X(x_0)/[1 - F_X(x_0)]$$

25. Let  $Y$  denote A's score and  $Z$  denote B's score. Then  $X = Y - Z$ .  $Z$  is uniformly distributed over  $(0, 3)$ .

$$P[X \leq x] = P[X \leq x | Y=1]p + P[X \leq x | Y=2](1-p) = P[1-Z \leq x]p + P[2-Z \leq x](1-p).$$

Etc.

$$\begin{aligned}
30. \quad P[X = x] &= \sum_{y=x}^{\infty} P[X = x | Y = y] P[Y = y] = \sum_{y=x}^{\infty} \binom{y}{x} p^x q^{y-x} e^{-\lambda} \lambda^y / y! = (\lambda p)^x e^{-\lambda p} / x!; \\
&\text{i.e., } X \text{ has a Poisson distribution with parameter } \lambda p.
\end{aligned}$$

$$32. \quad (a) \quad Y|X=5 \sim N(10, 25(1 - \rho^2)), \text{ so } .954 = P[4 < Y < 16 | X=5] = \Phi\left(\frac{6}{5\sqrt{1 - \rho^2}}\right) -$$

$$\Phi\left(\frac{-6}{5\sqrt{1 - \rho^2}}\right), \text{ which implies } \frac{6}{5\sqrt{1 - \rho^2}} = 2, \text{ hence } \rho = 4/5.$$

(b) This will be easy after the next chapter when we learn that  $X + Y \sim (15, 26)$ , giving  $P[X+Y \leq 16] = \Phi\left(\frac{16-15}{\sqrt{26}}\right) = \Phi(1/\sqrt{26})$ . For now,  $P[X+Y \leq 16] =$

$$\begin{aligned}
&\iint_{x+y \leq 16} \phi_{5,1}(x) \phi_{10,25}(y) \, dx \, dy = \iint_{u+v \leq 1} \phi(u) \phi(v) \, du \, dv = (\text{using symmetry}) \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{1/\sqrt{26}} \phi(u) \phi(v) \, du \, dv = \Phi(1/\sqrt{26}).
\end{aligned}$$

34. (a) Multinomial with  $k+1=4$ ;  $P[\text{no heads}] = 1/8$ ;  $P[\text{one head}] = 3/8$ ; etc.

$$35. \quad (a) \quad P[X = x, Y = y] = \frac{\binom{4}{x} \binom{4}{y} \binom{44}{6-x-y}}{\binom{52}{6}}$$

$$36. \quad (a) \quad (26 - 9x)/(9 - 3x)$$

$$(e) \quad E[XY|X = x] = xE[Y|X = x].$$

40. No

$$42. \quad m_{Y|X=x}(t) = E[e^{tY}|X = x]. \quad m_Y(t) = E[e^{tY}] = E[E[e^{tY}|X]] = E[m_{Y|X}(t)].$$

$$43. \quad (b) \quad 1 \quad (c) \quad \rho_{X,Y} = 1/2 \quad (d) \quad f_X(x)f_Y(y)$$

$$44. \quad (a) \quad E[Y] = E[E[Y|X]] = E[X + 1/2] = 1$$

$$(b) \quad \text{cov}[X, Y] = 1/12$$

$$(c) \quad 1/4$$

45. Special case of Problem 46.

46. The joint density of  $X$  and  $Y$  might have two, three, or four mass points.

Consider the case of four mass points. Let  $p_{ij} = P[X = x_i; Y = y_j]$  for  $i, j = 1, 2$ , where  $x_1 < x_2$  and  $y_1 < y_2$ .

Write  $p_{1.} = p_{11} + p_{12} = P[X = x_1]$ ,

$$p_{2.} = p_{21} + p_{22} = P[X = x_2],$$

$$p_{.1} = p_{11} + p_{21} = P[Y = y_1], \text{ and}$$

$$p_{.2} = p_{12} + p_{22} = P[Y = y_2].$$

Let  $U = (X - x_1)/(x_2 - x_1)$  and  $V = (Y - y_1)/(y_2 - y_1)$ .

Now  $\text{cov}[X, Y] = 0$  if and only if  $\text{cov}[U, V] = 0$  and  $X$  and  $Y$  are independent if and only if  $U$  and  $V$  are independent.

$$\text{cov}[U, V] = E[UV] - E[U]E[V] = p_{22} - p_{2.}p_{.2}.$$

$\text{cov}[U, V] = 0$  implies  $p_{22} = p_{2.}p_{.2}$  which in turn implies independence.

## CHAPTER V

## Chapter V PROBLEMS

1. (a)  $\text{cov}[X_1 + X_2, X_2 + X_3] = \sigma^2$ ;  $\text{var}[X_1 + X_2] = \text{var}[X_2 + X_3] = 2\sigma^2$ ;  
hence  $\rho[X_1 + X_2, X_2 + X_3] = 1/2$ .  
(b)  $(\sigma_2^2 - \sigma_1^2)/(\sigma_1^2 + \sigma_2^2)$   
(c)  $1/2$ .
3.  $F(x)I_{[0, \infty)}(x)$ .
4. (a)  $P[X = x] = \frac{(M-K)_{x-1}}{(M)_{x-1}} \cdot \frac{K}{M-x+1}$  for  $x = 1, \dots, M-K+1$ .  
(b)  $P[Z = z] = \frac{\binom{K}{r-1} \binom{M-K}{z-r}}{\binom{M}{z-1}} \cdot \frac{\binom{K-r+1}{1}}{\binom{M-z+1}{1}}$ , for  $z = r, \dots, M-K+r$ .  
(c) 
$$\frac{(x, y)}{f_{X,Y}(x, y)} \parallel \begin{array}{c|c|c|c|c} (1, 2) & (1, 3) & (2, 1) & (3, 1) & (4, 1) \\ \hline \frac{2}{5} \cdot \frac{3}{4} & \frac{2}{5} \cdot \frac{1}{4} & \frac{3}{5} \cdot \frac{2}{4} & \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{2}{3} & \frac{3}{5} \cdot \frac{2}{4} \cdot \frac{1}{3} \end{array}$$
5. According to the definition of expectation,  $E[X_1]$  does not exist; however, there is no harm in saying  $E[X_1] = \infty$ ,  $E[Y_1] = n/(n-1)$  for  $n > 1$ .
6. (a) Since  $X \leq \max[X, Y]$ ,  $E[X] \leq E[\max[X, Y]]$ ; similarly,  
 $E[Y] \leq E[\max[X, Y]]$ , hence  $\max[E[X], E[Y]] \leq E[\max[X, Y]]$ .  
(b)  $\max[X, Y] + \min[X, Y] = X + Y$ .
7. (a) Note that  $X$  and  $Y$  are independent and uniformly distributed. Apply the corollary of Theorem 3 on page 180.  
(b) Theorem 8 will do it.
8. The cdf of  $Z = \max[X, Y]$  is given by  
 $(1 - e^{-\lambda_1 z})(1 - e^{-\lambda_2 z})I_{(0, \infty)}(z)$   
so  $E[Z] = E[\max[X, Y]] = \int_0^1 (1 - F_Z(z)) dz = \int_0^1 (e^{-\lambda_1 z} + e^{-\lambda_2 z} - e^{-(\lambda_1 + \lambda_2)z}) dz =$   
 $\frac{1}{\lambda_1} + \frac{1}{\lambda_2} - \frac{1}{\lambda_1 + \lambda_2}$
9.  $X_1 - X_2 \sim N(0, 2)$ . The distribution of  $(X_2 - X_1)^2$  can be found using Example 19. Similarly, for  $Y_2 - Y_1$  and  $(Y_2 - Y_1)^2$ . They are independent so use Equation (26) to find the distribution of  $Z^2 = (X_2 - X_1)^2 + (Y_2 - Y_1)^2$ .

10. (a) Let  $Y_n$  be the life of the fuse that lasts the longest. Find  $n$  such that  $P[Y_n > .8] = .95$ .  $n = 14$  will do.
- (b)  $9/10$ .
11.  $\Phi(\cdot)$ .
12. (a) This problem is starred, not because it is difficult, but because it is messy. The possible values of  $Z = X/(X + Y)$  are zero (if  $X = 0$ ), one (if  $X > 0$  and  $Y = 0$ ), and  $a/b$  where  $a$  and  $b$  are positive integers and  $a < b$ .  $P[Z = (a/b)] = \sum P[X = x; Y = y]$  where the summation is over all pairs  $(x, y)$  for which  $x$  and  $y$  are positive integers and  $y = x(b - a)/a$ .
- (b)  $m_{X, X+Y}(t_1, t_2) = E[e^{t_1 X + t_2 (X+Y)}] = m_{X, Y}(t_1 + t_2, t_2)$ .
13. (a) Write  $E[e^{Y_1 t_1 + Y_2 t_2}]$  in terms of a double integral involving the joint distribution of  $X_1$  and  $X_2$ . Perform the integration by separating the double integral, completing the square, and expressing in terms of integrals of normals.
- (b) Use the joint moment generating function given in (a).
14.  $E[e^{XYt}] = E[E[e^{XYt} | X]] = E[e^{(1/2)Y^2 t^2}] = 1/\sqrt{1 - t^2}$ .
15. (a) Use the moment generation function technique to argue that they are independent standard normals.
16. Let  $S = \sum_{i=1}^{16} X_i$  = weight of beans in box. Assume that the  $X_i$ 's are independent.
- (a) mean =  $16^2$  ounces and variance = 16
- (b)  $P[S > 250] = 1 - \Phi\left(\frac{250 - 16(16)}{4}\right) = \Phi(3/2)$
- (c) Let  $Z$  = number of underweight bags.
- $$Z \sim \text{bin}(16, 1/2), \text{ so } P[Z \leq z] = \sum_0^z \binom{16}{x} (1/2)^{16}.$$
17. (a) Let  $Z$  = number of numbers less than  $1/2$ .  $Z \sim \text{bin}(10, 1/2)$ .  $P[Z = 5] = \binom{10}{5} (1/2)^5$ .
- (b)  $E[Z] = 5$ .
- (c)  $1/2$  using a symmetry argument.



18. (a) Both are  $n\lambda$ .
- (b)  $\Phi(-2)$
19. (a) Buy  $n$  bulbs and again assume independence. Assume that the lifetime are independent (which may not be realistic since the bulbs are burning simultaneously). Want  $n$  such that  $.95 = P[Y_n > 1000] = 1 - [1 - \exp(-10)]^n$ .
- (b) Buy  $n$  bulbs. Want  $n$  such that  $P[S_n > 1000] = .95$ .  $S_n$  has a gamma distribution with parameters  $n$  and  $.01$ . Using Equation (33) of Chapter III and a Poisson table  $n \approx 16$  is obtained.
20. Use the moment generating function technique.
- (a) gamma with parameters  $nr$  and  $\lambda$ .
- (b) gamma with parameters  $\sum r_i$  and  $\lambda$ .
21. (a) negative binomial with parameters  $n$  and  $p$
- (b) negative binomial starting at  $n$  with parameters  $n$  and  $p$
- (c) negative binomial with parameters  $nr$  and  $p$
- (d) negative binomial with parameters  $\sum r_i$  and  $p$
22.  $Z$  can be expressed as  $\sum_{i=1}^Y X_i$  where  $X_i$  is the money received from the  $i$ th location where oil is found.  $Z = 0$  if  $Y = 0$ . Model by assuming the  $X_i$ 's and  $Y$  are independent.  $Y$  has a binomial distribution with  $n = 10$  and  $p = 1/5$ , and the  $X_i$ 's are independent and identically distributed exponential random variables with mean 50000.
- (a)  $E[Z] = E[E[Z|Y]] = E[Y]E[X] = \$100,000$ .
- (b)  $P[Z > 100,000 | Y = 1] = e^{-2}$ .
- $P[Z > 100,000 | Y = 2] = 3e^{-2}$ .
- (c)  $P[Z > 100,000] = \sum_{y=0}^{10} P[Z > 100,000 | Y = y] P[Y = y] = \sum_{y=1}^{10} \left( \sum_{j=0}^{y-1} \frac{e^{-2} 2^j}{j!} \right) \binom{10}{y} \left( \frac{1}{5} \right)^y \left( \frac{4}{5} \right)^{10-y}$   
 using  $Z$  given  $Y = y$  is gamma distributed and Equation (33) of Chapter III.
- $P[Z > 100,000] \approx .4$ .
23. See 24.

$$\begin{aligned}
24. \quad P[X_1 = x_1, \dots, X_k = x_k | X_1 + \dots + X_{k+1} = n] &= \frac{P[X_1 = x_1, \dots, X_k = x_k; X_1 + \dots + X_{k+1} = n]}{P[X_1 + \dots + X_{k+1} = n]} \\
&= \frac{\frac{e^{-\lambda_1} (\lambda_1)^{x_1}}{x_1!} \cdot \frac{e^{-\lambda_2} (\lambda_2)^{x_2}}{x_2!} \cdot \dots \cdot \frac{e^{-\lambda_k} (\lambda_k)^{x_k}}{x_k!} \cdot \frac{e^{-\lambda_{k+1}} (\lambda_{k+1})^{n-x_1-x_2-\dots-x_k}}{(n-x_1-x_2-\dots-x_k)!}}{e^{-\sum \lambda_j} (\sum \lambda_j)^n} \\
&= \frac{n!}{x_1! x_2! \dots x_k! (n-x_1-x_2-\dots-x_k)!} \left(\frac{\lambda_1}{\lambda}\right)^{x_1} \left(\frac{\lambda_2}{\lambda}\right)^{x_2} \dots \left(\frac{\lambda_k}{\lambda}\right)^{x_k} \left(\frac{\lambda_{k+1}}{\lambda}\right)^{n-x_1-\dots-x_k}
\end{aligned}$$

25. Cauchy.

26. Y has a lognormal distribution.  $E[Y] = E[e^X] = m_X(1)$ , the moment generating function of X evaluated at 1. Also  $E[Y^2] = E[e^{2X}] = m_X(2)$ .

27. Exponential with parameter one.

28. Beta with parameters b and a.

29. Write  $Y = 1/X$  then  $f_Y(y) = y^{-2} I_{(1, \infty)}(y)$ .

31. Exponential with parameter one.

32. Beta with parameters reversed.

34. Same as X.

36. Exponential with parameter one.

38.  $P[Y - X = z] = [p/(2-p)]q^z I_{\{0,1,2,\dots\}}(z) + [p/(2-p)]q^{-z} I_{\{-1,-2,\dots\}}(z)$ .

39. Write  $V = Y - X$ , then  $f_V(v) = (\lambda/2)e^{-\lambda|v|}$ .

40. One way of doing it is to transform to, say,  $U = X, V = Y, W = XY/Z$ , find the U, V, W, integrate out u and v and get

$$f_W(w) = \left(\frac{1}{4} - \frac{1}{2} \ln w\right) I_{(0,1)}(w) + \frac{1}{4w^2} I_{[1,\infty)}(w).$$

41. Write  $Z = X+Y$ .  $f_Z(z) = [2z^2 - (2/3)z^3] I_{(0,1)}(z) + [(8/3) - 2z^2 + (2/3)z^3] I_{(1,2)}(z)$ .

$f_Z(z)$  is symmetric about  $z = 1$ .

42. This is starred not because it is difficult, but because the answer, which can be expressed in terms of a Bessel function, is not simple.

$$P[Y - X = z] = \sum_{x=0}^{\infty} P[Y - X = z | X = x] P[X = x] = \sum_{x=\max[0, -z]}^{\infty} P[Y = x+z] P[X = x] \text{ for } z$$

an integer.

44. Let  $X$  have parameters  $a$  and  $b$  and  $Y$  have parameters  $c$  and  $d$ .  $b = d = 1$  and  $a = c + 1$  will suffice.
46. The cdf technique works.  $2z^3 e^{-z^2} I_{(0, \infty)}(z)$ .
47.  $X$  and  $Y$  are independent; hence it suffices to find the marginal distribution of  $X^2$  and  $Y^2$ .
49. The transformation is not one-to-one. See Example 19.
50. The distribution of  $X + Y$  is triangular and given Example 4,  $P[Z \leq z] = P[X+Y \leq z; X+Y \leq 1] + P[X+Y-1 \leq z; X+Y > 1] = P[X+Y \leq z] + P[1 < X+Y \leq 1+z] = z$  for  $0 < z < 1$ . That is  $Z$  is uniformly distributed over  $(0, 1)$ .
53.  $f_{Y_1, Y_2}(y_1, y_2) = \lambda^2 y_2 e^{-\lambda y_2} [1/(1+y_1)^2] I_{(0, \infty)}(y_1) I_{(0, \infty)}(y_2)$ .
54. The transformation is not one-to-one. Use Theorem 14.  $Y_1$  has an exponential distribution with parameter  $1/2$  and  $Y_2$  has a standard Cauchy distribution. They are independent.
57. (a)  $E[X + Y] = E[E[X + Y|Z]] = 1$ .
- (b)  $f_{X, Y}(x, y) = \int f_{X, Y|Z}(x, y|z) f_Z(z) dz = I_{(0, 1)}(x) I_{(0, 1)}(y)$ . Are independent.
- (c)  $f_{X|Z}(x|z) = \int f_{X, Y|Z}(x, y|z) dy = [z + (1 - z)(x + 1/2)] I_{(0, 1)}(x)$  which depends on  $z$  so  $X$  and  $Z$  are not independent.
- (d) Straightforward transformation using distribution of  $X$  and  $Y$  given in (b).
- (e)  $P[\max[X, Y] \leq u|Z = z] = P[X \leq u, Y \leq u|Z = z] = \int_0^u \int_0^u [z + (1 - z)(x + y)] dx dy = zu^2 + (1 - z)u^3$  for  $0 < u < 1$ .
- (f)  $\int f_{(X, Y)|Z}(x, s - x|z) dx = [z + (1 - z)s] [s I_{(0, 1)}(s) + (2 - s) I_{[1, 2]}(s)]$
58. Assume independence of functioning components and capitalize in the forgetfulness of the exponential.
- (a) Let  $Y = Y_3 + Y_2 + Y_1$  be the life of system, where  $Y_j$  is that part of the life when exactly  $j$  components are functioning.  $Y_3$  is the minimum of three independent exponential random variables each with rate parameter  $\lambda/3$ , so  $Y$  has an exponential distribution with rate parameter  $\lambda$ . Similarly for  $Y_2$  and  $Y_1$ .

Furthermore, the  $Y_j$ 's are independent, hence  $Y$  has a gamma distribution with parameters 3 and  $\lambda$ .

(b) Same as answer (a).

59.  $Z$  is the lifetime of the system.  $Z$  has cdf  $(1-2e^{-2z}+e^{-3z})I_{(0,\infty)}(z)$ , mean  $2/3$ , and variance  $1/3$ .

60. Gamma with parameters two and two.

61. Follow the hint and use Equation (33) of Chapter IV for the joint moment generating function of  $X$  and  $Y$ .  $(U, V) = (aX+bY, cX+dY)$  has a bivariate normal distribution with parameters.

$$\mu_U = a\mu_X + b\mu_Y, \quad \mu_V = c\mu_X + d\mu_Y$$

$$\sigma_U^2 = a^2\sigma_X^2 + b^2\sigma_Y^2 + 2ab\sigma_X\sigma_Y\rho_{X,Y}$$

$$\sigma_V^2 = c^2\sigma_X^2 + d^2\sigma_Y^2 + 2cd\sigma_X\sigma_Y\rho_{X,Y}$$

$$\rho_{U,V} = \sigma_U\sigma_V[ac\sigma_X^2 + bd\sigma_Y^2 + (bc+ad)\sigma_X\sigma_Y\rho_{X,Y}].$$

Can you choose  $a, b, c$ , and  $d$  to make  $U$  and  $V$  independent standard normals?

62. (a)  $N(0, u^2 + [1-u]^2)$

(b)  $E[Z] = 0$  and  $\text{var}[Z] = 2/3$  using Theorem 7 of Chapter IV, page 159.

(c) This is starred because the answer is not simple. Use Remark on page 149 and get

$$F_Z(z) = \int P[Z \leq z | U = u] f_U(u) du; \text{ now}$$

both  $P[Z \leq z | U = u]$  and  $f_U(u)$  are known and the problem is reduced to one of integration.

$$f_Z(z) = \int_0^1 \Phi\left(\frac{z}{\sqrt{u^2 + (1-u)^2}}\right) \frac{1}{\sqrt{u^2 + (1-u)^2}} du$$

## CHAPTER VI

## Chapter VI PROBLEMS

3. (a)  $P[|X_2 - X_1| < 1/2] = \int_0^1 P[|X_2 - x_1| < 1/2] dx_1 = 3/4.$   
 (b)  $P[1/4 < (X_1 + X_2)/2 < 3/4] = P[1/2 < X_1 + X_2 < 3/2] = 3/4.$
4. (a)  $f_{X_1, \dots, X_9}(x_1, \dots, x_9) = \sum_{i=1}^9 [(2/3)^{x_i} (1/3)^{1-x_i} I_{\{0,1\}}(x_i)]$   
 $f_{\sum X_j}(s) = \binom{9}{s} (2/3)^s (1/3)^{9-s} I_{\{0,1,\dots,9\}}(s)$   
 (b)  $E[\bar{X}_9] = 2/3, E[S^2] = 2/9.$
5. (a) Yes; it follows from simple algebra.  
 (b) There are various ways to proceed. For example,  

$$\begin{aligned} \text{var}[S^2] &= [1/2n(n-1)]^2 \text{var}[\sum \sum (X_i - X_j)^2] \\ &= [1/2n(n-1)]^2 \sum \sum \sum \sum \text{cov}[(X_i - X_j)^2, (X_\alpha - X_\beta)^2] \\ &\quad (\text{using "variance of a sum is the double sum of the covariances"}) \\ &= [1/2n(n-1)]^2 (2n(n-1) \text{var}[(X_2 - X_1)^2] + 4n(n-1)(n-2) \text{cov}[(X_1 - X_2)^2, (X_1 - X_3)^2]) \\ &= [1/2n(n-1)]^2 (2n(n-1)(2\mu_4 + 2\sigma^4) + 4n(n-1)(n-2)(\mu_4 - \sigma_4)) \\ &= (1/n)(\mu_4 - \frac{n-3}{n-1}\sigma^4). \end{aligned}$$
  
 (c)  $\text{cov}[\bar{X}, S^2] = \text{cov}[\bar{X} - \mu, S^2] = [1/n(n-1)] \text{cov}[\sum (X_k - \mu), \sum (X_i - \mu)^2 - (1/n) \sum \sum (X_i - \mu)(X_j - \mu)]$   

$$\begin{aligned} &= [1/n(n-1)] (\sum \sum \text{cov}[\sum (X_k - \mu), (X_i - \mu)^2] - (1/n) \sum \sum \sum \text{cov}[(X_k - \mu), (X_i - \mu)(X_j - \mu)]) \\ &= [1/n(n-1)] (n\mu_3 - (1/n)(n\mu_3)) = \mu_3/n, \text{ a rather simple answer.} \end{aligned}$$
6. (a)  $M_r = (1/2) \left[ \left( \frac{X_1 - X_2}{2} \right)^r + (-1)^r \left( \frac{X_1 + X_2}{2} \right)^r \right].$   
 For  $r$  odd,  $M_r \equiv 0$  and hence  $E[M_r] = 0$  and  $\text{var}[M_r] = 0$ . For  $r$  even,  
 $M_r = \left( \frac{X_1 - X_2}{2} \right)^r$ , and  $E[M_r] = (1/2^r) \sum_{j=0}^r \binom{r}{j} \mu'_j \mu'_{r-j} (-1)^{r-j}$  and similarly  
 for  $\text{var}[M_r]$ .  
 (b)  $E[(1/n) \sum (X_i - \mu)^r] = (1/n) \sum E[(X_i - \mu)^r] = \mu_r.$
7. (a) Have  $P[-\epsilon < \bar{X}_n - \mu < \epsilon] \geq 1 - \sigma$  for  $n > \sigma^2/\epsilon^2 4$ .  
 Have  $\mu = .5, \sigma^2 = 1/4, \epsilon = .1, \delta = .1$ , hence  $n = 250$ .  
 (b) Use the Central Limit Theorem.  
 $.90 = P[.4 < \bar{X} < .6] \approx \Phi\left(\frac{.6 - .5}{\sqrt{1/4n}}\right) - \Phi\left(\frac{.4 - .5}{\sqrt{1/4n}}\right)$  and so  $n \approx 58$ .

9.  $Y = \bar{X}_1 - \bar{X}_2$  is approximately distributed as a normal distribution with mean = 0 and variance  $2\sigma^2/n$ . Want  $P[|\bar{X}_1 - \bar{X}_2| > \sigma] = .01$ .  $n = 14$ .

10. Want  $.01 = P[\bar{X} < 2200] \approx \Phi\left(\frac{2200 - 2250}{250\sqrt{n}}\right)$ .  $n = 136$ .

11. Want  $.95 = P[|\bar{X} - \mu| \leq .25\sigma]$ .  $n = 62$ .

12. Want  $.01 = P[\bar{X} < 1/2] \approx \Phi\left(\frac{.5 - .52}{\sqrt{.52(.48)/n}}\right)$ .  $n = 3375$ .

15. (a) There are ten equally likely (unordered) samples; compute  $\bar{x}$  for each and the evaluate  $E[\bar{X}]$  and  $\text{var}[\bar{X}]$ . 3 and .75

(b) 1

(c)  $E[\bar{X}] = (N+1)/2$ .

$$\begin{aligned}\text{var}[\bar{X}] &= (1/n^2)\text{var}[\sum X_i] \\ &= (1/n^2)(\sum_i \text{var}[X_i] + \sum_{i \neq j} \text{cov}[X_i, X_j]) \\ &= (1/n^2)(n\sigma^2 + n(n-1)\text{cov}[X_1, X_2]) \\ &= (1/n^2)(n\sigma^2 + n(n-1)\sum_{i \neq j} (i - \mu)(j - \mu)/N(N-1)) \\ &= \frac{\sigma^2}{n} \frac{N-n}{N-1}.\end{aligned}$$

17.  $Z = \sum (X_i - \bar{X})^2 / \sigma^2$  is chi-square distributed with  $n-1$  degrees of freedom.

$$\begin{aligned}S &= \sqrt{\sigma^2 Z / (n-1)}. \quad E[S] = \sqrt{\sigma^2 / (n-1)} E[\sqrt{Z}] \\ &= \sqrt{\sigma^2 / (n-1)} \int_0^\infty \frac{1}{\Gamma((n-1)/2)} (1/2)^{(n-1)/2} z^{(n/2)-1} e^{-(1/2)z} dz \\ &= [(\sigma\sqrt{2})/\sqrt{n-1}] \Gamma(n/2) / \Gamma((n-1)/2).\end{aligned}$$

$$\begin{aligned}\text{var}[S] &= E[S^2] - E^2[S] \\ &= \frac{\sigma^2}{n-1} E[Z] - E^2[S] = \sigma^2 \left\{ 1 - \frac{2}{n-1} \left[ \frac{\Gamma(n/2)}{\Gamma((n-1)/2)} \right] \right\}\end{aligned}$$

18. (b)  $X = (U/m)/(V/n)$  implies  $1/X = (V/n)/(U/m)$ .

(c)  $W = \frac{\frac{mU/m}{nV/n}}{1 + \frac{mU/m}{nV/n}} = \frac{U}{V+U}$  is beta distributed with parameters  $m/2$  and  $n/2$  by Example 25 of Chapter V.

(d)  $E[X] = \frac{n}{m} E\left[\frac{W}{1-W}\right] = \frac{n}{m} \frac{1}{B(m/2, n/2)} \int_0^1 w^{m/2} (1-w)^{(n/2)-2} dw = n/(n-2)$ .  
Similarly for  $E[X^2]$  and  $\text{var}[X]$ .

19. (a) The integral that defines the mean exists for degrees of freedom greater than 1; symmetry shows that the mean is zero. The integral that defines the variance exists for degrees of freedom greater than 2;
- $$\text{var}[T] = E[T^2] = E\left[\frac{(\text{standard normal r.v.})^2}{\text{chi-square r.v./d. of f.}}\right]$$
- $$= E[\text{F-dist'd r.v. with 1 and } k \text{ d. of f.}] = k/(k-2) \text{ for } k > 2.$$

If it seems unfair to use results on the F distribution to obtain results on the t distribution,  $E[T^2]$  can be found directly. For example, the standard normal r.v. of the numerator is independent of the chi-square r.v. in the denominator so the expectation can be factored into the product of the expectation of the square of a standard normal r.v. and the expectation of the reciprocal of a chi-square r.v. divide by degrees of freedom; both factors are known.

- (b) Show  $C(k)[1/(1+t^2/k)^{(k+1)/2}] \xrightarrow[k \rightarrow \infty]{} c e^{-1/2t^2}$ . Assuming that the constant part  $C(k)$  does what it has to do, it is easy to show

$$(1+t^2/k)^{(k+1)/2} \rightarrow e^{-1/2t^2}.$$

- (c)  $\chi = Z/\sqrt{U/k}$  implies  $\chi^2 = Z^2/(U/k)$  which is a ratio of two independent chi-squared distributed r.v.'s divided by their respective degrees of freedom, hence  $\chi^2$  is F-distributed with one and k degrees of freedom.
- (d) According to part (c),  $\chi^2 \sim F(1, k)$ ; according to part (b) of Problem 18,  $1/\chi^2 \sim F(k, 1)$ ; and according to part (c) of Problem 18,  $\frac{1}{1 + (\chi^2/k)} = \frac{k(1/\chi^2)}{1 + k(1/\chi^2)}$  is beta distributed with parameters  $k/2$  and  $1/2$ .

Problems 20 through 24 inclusive are much alike and are intended to give some practice in utilizing the results of Sec. 4.

22. (a) Chi-square with  $n-2$  degree of freedom. (The sum of independent chi-square distributed r.v.'s is chi-square distributed with degrees of freedom equal to the sum of the individual degrees of freedom.)
- (b) Normal with mean  $\mu$  and variance  $n\sigma^2/4k(n-k)$ .
- (c) Chi-square with one degree of freedom.
- (d) F distribution with  $k-1$  and  $n-k-1$  degrees of freedom.
- (e) t-distribution with  $n-1$  degrees of freedom.

23. Don't forget that  $Z_1 + Z_2$  and  $Z_2 - Z_1$  are independent. Similarly for  $X_1 + X_2$  and  $X_2 - X_1$ .
- (b) t-distribution with 2 degrees of freedom.
- (c) Chi-square with 3 degrees of freedom.
- (d) F distribution with 1 and 1 degrees of freedom.
25. Note that  $X_1$  and  $X_2$  are independent identically distributed chi-square random variables with 2 degrees of freedom, so  $X_1/X_2$  has an F distribution with 2 and 2 degrees of freedom.
27.  $U \sim N(\mu, 1/\sum(1/\sigma_j^2))$   
 $V = \sum(X_i - U)^2/\sigma_i^2 = \sum(X_i - \mu)^2/\sigma_i^2 - (U - \mu)\sigma(1/\sigma_j^2)$  which is a difference of two independent chi-square distributed r.v. s., the first with n degrees of freedom, the second with 1 degree of freedom. The result follows using the moment generation function technique. What result does this reduce to if all  $\sigma_j^2$  are equal?
29. The joint distribution of  $(\bar{X}, S_1^2, S_2^2)$  is easily obtained since they are independent. Make a transformation and integrate out the unwanted variable.
30. One could use Theorem 13. On the other hand, note that  $Y_2 - Y_1 = |X_1 - X_2|$  and the distribution of  $X_1 - X_2$  is known and it is easy to find the distribution of the absolute value of a random variable.
31. (a)  $1 - P[\text{both less than median}] = 3/4$ .  
 (b)  $1 - P[\text{all are less than median}] = 1 - (1/2)^n$ .
32.  $E[F(Y_1)]$  is wanted.  $F(Y_1)$  has the same distribution as the smallest observation of a random sample of size n from a uniform distribution over the interval (0, 1).
33.  $E[Y_1] = \mu - [(n-1)/(n+1)]\sqrt{3}\sigma$   
 $E[Y_n] = \mu + [(n-1)/(n+1)]\sqrt{3}\sigma$   
 $\text{var}[Y_1] = \text{var}[Y_n] = 12\sigma^2 n/[(n+1)^2(n+1)]$ .  
 $\text{cov}[Y_1, Y_n] = 12\sigma^2/[(n+1)^2(n+2)]$ .



$$(a) \ E[Y_n - Y_1] = [(n-1)/(n+1)]2\sqrt{3}\sigma.$$

$$\text{var}[Y_n - Y_1] = 24\sigma^2(n-1)/[(n+1)^2(n+2)].$$

$$(b) \ E[(Y_1 + Y_n)/2] = \mu.$$

$$\text{var}[(Y_1 + Y_n)/2] = 6\sigma^2/[(n+1)(n+2)].$$

$$(c) \ E[Y_{k+1}] = \mu.$$

$$\text{var}[Y_{k+1}] = 3\sigma^2/(2k+3).$$

$$(d) \ \frac{3\sigma^2}{n+2} > \frac{\sigma^2}{n} > \frac{6\sigma^2}{(n+1)(n+2)} \text{ for } n > 2.$$

34.  $\bar{X}$  is asymptotically normally distributed with mean  $\alpha$  and variance  $2\beta^2/n$ .

The sample median is asymptotically normally distributed with mean  $\alpha$  and variance  $\beta^2/n$  by Theorem 14. Note that the sample median has the smaller asymptotic variance.

$$35. \ P[(Y_n - a_n)/b_n \leq y] = P[Y_n \leq b_n y + a_n] = (1 - \exp[(b_n y - a_n)/(1 - b_n y - a_n)])^n$$

$$= (1 - \exp[\frac{y + (\log n)^2}{y - \log n}])^n. \text{ Now let } n \rightarrow \infty \text{ and } \exp(-e^{-y}) \text{ results.}$$

36. (a) Similar to Problem 34.

(b) With  $\theta$  replacing  $\lambda$  choose  $a_n$  and  $b_n$  as in Example 9.

(c) We know that  $Y_1^{(n)}$  has exact distribution that is exponential with parameter  $n\lambda$ . So choose  $a_n \geq 0$  and  $b_n = 1/n$  and then  $(Y_1^{(n)} - a_n)/b_n$  has exact (and hence also limiting) distribution that is exponential with parameter  $\lambda$ .



## CHAPTER VII

## Chapter VII PROBLEMS

1. Let  $B$  = number of black balls and  
 $W$  = number of white balls.  
 $R = B/W$ . Set  $p = B/(B+W)$ , so  $R = p/(1+p)$ .  
  - (a) Let  $X_i = 1$  if black ball on  $i$ th draw and  $X_i = 0$  otherwise. MLE of  $p = \sum X_i/n = \bar{X}$  which implies MLE of  $R = \bar{X}/(1 - \bar{X})$ .
  - (b)  $X_i$  has a geometric distribution.  $L(p) = p^n(1-p) \sum x_i$ . MLE of  $p = 1/(1 + \bar{X})$ , so MLE of  $R = 1/\bar{X}$ .
2. MLE of  $p_{ij}$  is  $N_{ij}/n$ .
4. MLE of  $\mu_1 - \mu_2$  is  $\bar{X}_1 - \bar{X}_2$ .  
 $\text{var}[\bar{X}_1 - \bar{X}_2] = \sigma_1^2/n_1 + \sigma_2^2/n_2$ .  $n_1 \approx n[\sigma_1(\sigma_1 + \sigma_2)]$ .
5. MLE of  $a = (\bar{X}_1 + \bar{X}_2 + \bar{X}_3 + \bar{X}_4)/4$ ;  
MLE of  $b = (\bar{X}_1 + \bar{X}_2 - \bar{X}_3 - \bar{X}_4)/4$ ; and  
MLE of  $c = (\bar{X}_1 + \bar{X}_3 - \bar{X}_2 - \bar{X}_4)/4$ .
7. Let  $r$  denote the radius of the circle. Let  $X_i$  denote the  $i$ th measurement.  
 $X_i = r + E_i$  where  $E_i$  is the  $i$ th error of measurement.  $E_i \sim N(0, \sigma^2)$ .  
Now  $\text{var}[X_i] = \text{var}[E_i] = \sigma^2$  so  $S^2 = \sum (X_i - \bar{X})^2/(n-1)$  is an unbiased estimator of  $\sigma^2$ .  $(\pi/n) \sum_{j=1}^n (X_j - S^2)$  is an unbiased estimator of the area  $= \pi r^2$ .
9. Show that  $P_\theta[|(X_1+X_2)/2-\theta| < |X_1-\theta|] > 1/2$  for all  $\theta$ . Make the transformation  $U_1 = X_1 - \theta$  and  $U_2 = X_2 - \theta$  and it suffices to show that  $P[|U_1+U_2| < 2|U_1|] > 1/2$  where  $U_1, U_2$  is a random sample of size two from the Cauchy density  $1/\pi[1+x^2]$ .  
See Problem 20 of Chapter IV.
10. (a)  $\sum (X_i - \hat{\theta}) = 0$  implies  $\hat{\theta} = \bar{X}$ .  
(b)  $\sum (X_i - \hat{\theta})^2$  is minimized for  $\hat{\theta} = \bar{X}$ .
11. (b)  $\text{var}[\sum a_i x_i] = \sigma^2(\sum a_i) = \sigma^2[\sum (a_i - 1/n)^2 + 1/n]$ .
12. (b) MLE of  $\theta$  is  $\min[1/2, \bar{X}]$ .

17. (a)  $X$  is sufficient.  $E[X] = 0$  for all  $\theta$  so  $X$  is not complete.  
 (b) Yes; yes.  
 (c)  $\sum |X_i|/n$ .  
 (d) Yes. (e) Yes.  
 (f)  $|X|$ .
18. (b) Yes.
19. (c) Middle observation for odd sample size and anything between two middle observations for even sample size.  
 (d) No.
21. In computing the means and mean-squared errors use the calculations in Problem 33 of Chapter VI.  
 (a)  $T_1 = 2\bar{X}$ . MSE is  $\theta^2/3n$ .  
 (b)  $T_2 = Y_n$ . MSE is  $2\theta^2/[(n+1)(n+2)]$ .  
 (c)  $T_3 = [(n+2)/(n+1)]Y_n$ . MSE is  $\theta^2/(n+1)^2$ .  
 (d)  $T_4 = [(n+1)/n]Y_n$ . MSE is  $\theta^2/[n(n+2)]$ .  
 (e) MSE is  $2\theta^2/(n+1)(n+2)$ .  
 (g)  $Y_n^2/12$ .
22. (a)  $[(1-2\theta)^2\theta(1-\theta)]/n$   
 (b)  $\sum X$  is a complete sufficient statistic.  $S^2 = \sum (X_i - \bar{X})^2/(n-1)$  is an unbiased estimator of  $\theta(1-\theta)$ , since the sample variance is an unbiased estimator of the population variance; furthermore,  $S^2 = [\sum X_i^2 - n\bar{X}^2]/(n-1) = [\sum X_i - n\bar{X}^2]/(n-1)$  is a function of  $\sum X_i$ ; hence, by the Lehmann-Scheffé Theorem,  $S^2$  is UMVUE of  $\theta(1-\theta)$ .
24.  $-\ln X_i$  has an exponential distribution, so  $-\sum \ln X_i$  has a gamma distribution.  
 (a) MLE of  $\theta$  is  $n/-\sum \ln X_i$ ,  $\therefore$  MLE of  $\mu$  is  $n/(n - \sum \ln X_i)$ .

- (b)  $-\sum \ln X_i$  is complete minimal sufficient by Theorem 9. A minimal sufficient statistic must be a function of every other sufficient statistic.  $-\sum \ln X_i$  is not a function of  $\sum X_i$ , hence  $\sum X_i$  is not sufficient for  $n > 1$ .  $\sum X_i$  is sufficient for  $n = 1$ . Why?
- (c) Yes,  $1/\theta$ .
- (d)  $-\sum \ln X_i/n$  is UMVUE of  $1/\theta$ ;  $(n-1)/-\sum \ln X_i$  is UMVUE of  $\theta$ .  $X_1$  is an unbiased estimator of  $\theta/(1+\theta)$ , hence  $E[X_1 | -\sum \ln X_i]$  is UMVUE of  $\theta/(\theta+1)$ . For  $n > 1$ , following a procedure similar to that in Example 35, the condition distribution of  $X_1$  given  $-\sum \ln X_i$  can be found and then conditional expectation can be obtained. Let  $S = -\sum \ln X_i$ , then  $E[X_1 | S = s] = \int_{e^{-s}}^1 [x_1(n-1) + (s + \ln x_1)^{n-2}/x_1 s^{n-1}] dx_1 = \frac{(n-1)e^{-s}}{s^{n-1}} \int_0^s u^{n-2} e^u du$  which can be integrated and the answer expressed as a finite sum. For  $n = 1$ , what is the UMVUE of  $\theta/(1+\theta)$ ?
26. (a)  $2\bar{X} - 1$ . Mean is  $\theta$  and mean-squared error is  $(\theta^2 - 1)/3n$ .
- (b) MLE is  $Y_n$ . The distribution of  $Y_n$  is given by  $P[Y_n = j] = [(j/\theta)^n - ((j-1)/\theta)^n] I_{\{1, \dots, \theta\}}(j)$  from which the mean and mean-squared error can be found.
- (c)  $Y_n$  is sufficient by the factorization criterion. To show that  $E_\theta[\chi(Y_n)] = 0$  for  $\theta = 1, 2, 3, \dots$  implies that  $\chi(j) = 0$  for  $j = 1, 2, \dots$ . It suffices to substitute in  $\theta = 1, 2, 3$ , etc. successively.
- (d) By the Lehmann-Scheffé Theorem and part (c) it suffices to show that the given statistic is unbiased.
27.  $X$  is sufficient but not complete.
28.  $\tau(\theta) = \text{median} = \ln 2/\theta$ . We already know that the MLE and UMVUE of  $1/\theta$ ; to find the MLE and UMVUE of  $\tau(\theta)$  requires a simple scale adjustment.
29. (b)  $X^2 - 1$ .
- (c)  $I_{(0, \infty)}(X_1)$ .

(d)  $\Phi(\bar{X})$ .

(e)  $\bar{X}^2 - (1/n)$ .

(f)  $X_1 | \bar{X} \sim N(\bar{X}, (n-1)/n)$  and  $E[I_{(0,\infty)}(X_1) | \bar{X}] = \Phi\left(\sqrt{\frac{n}{n-1}}(\bar{X})\right)$  is UMVUE for  $P[X > 0]$ .

30.  $I_{\{0,1\}}(X_1)$  is an unbiased estimator of  $(1+\lambda)e^{-\lambda}$ . See Example 34 for a procedure that will find an UMVUE of  $(1+\lambda)e^{-\lambda}$ .

31. This is a "triangular" density rather than a "rectangular" density as in Problem 21. The results are quite similar.

32. The density given in this problem is a form of the Pareto density. This problem is like Problem 24. In that problem  $-\ln X_i$  has an exponential distribution; in this problem  $\ln(1+X_i)$  has an exponential distribution.

(a)  $(1+\bar{X})/\bar{X}$ .

(b) MLE of  $1/\theta$  is  $\sum \ln(1+X_i)/n$ .

(c)  $\sum \ln(1+X_i)$ .

(d)  $1/n\theta^2$

(e)  $\sum \ln(1+X_i)/n$ .

(f)  $(n-1)/\sum \ln(1+X_i)$ .

33. (a)  $\max[-Y_1, Y_n]$ , or, the absolute value of the observation farthest from zero.

(b)  $X$  is not minimal sufficient since  $|X|$  is sufficient.  $X$  is not complete.

34. (a)  $\sum X_i$  is a complete sufficient statistic and the sample variance is an unbiased estimator so an UMVUE exists.

35. (a)  $\sum X_i$

(b) Find it by using the form given in Equation (16).

37.  $e^{-X_i}$  has an exponential distribution with parameter  $e^\theta$ . See Problem 24 and 32 for similar problems.

(f) The given statistic is a function of the complete sufficient statistics  $\sum e^{-\lambda_i}$  which has a gamma distribution. Verify that the given statistic is unbiased.

39. This is a generalization of Problems 21 and 31.

(a)  $a(\theta) = [\int_0^\theta b(x) dx]^{-1}$  so  $a(\theta)$  is non-increasing. The likelihood function is proportional to  $a^n(\theta)$  for  $\theta > Y_n$ . MLE of  $\theta$  is  $Y_n$ .

(b)  $Y_n$ . See Example 33 for the idea of the completeness proof.

40.  $\theta$  is the mean and variance.

(a)  $\sum X_i^2$ .

(b) It is not a function of a complete sufficient statistic.

(c) No.

41.  $\theta$  should have been assumed positive. Then  $\theta$  is the mean and standard deviation, and is a scale parameter.

42. (a)  $Y_n/2$ .

(b) No,  $E[\frac{n+1}{2n+1}Y_n - \frac{n+1}{n+2}Y_1] = 0$ .

(c)  $\frac{(n+2)[Y_1^{-n-1} - (Y_n/2)^{-n-1}]}{(n+1)[Y_1^{-n-2} - (Y_n/2)^{-n-2}]}$ .

(d)  $(Y_1 + Y_n)/3$ .

43. (a)  $\sum X_i^2$  is complete and sufficient.  $\sum X_i^2/n$  is UMVUE of  $\theta^2$ .

(b)  $c^* = 1/(n+2)$  minimizes MSE in family of estimators of form  $c \sum X_i^2$ .

(c)  $\Gamma(n/2) \sqrt{\sum X_i^2 / [\sqrt{2}\Gamma((n+1)/2)]}$ .

(e) Yes, since both are scale invariant.

44. (a)  $Y_1$

(b)  $Y_1$

(c)  $\bar{X} - 1$

(d)  $Y_1$

(e)  $Y_1 - (1/n)$

(f)  $Y_1 - (1/n)$

(g)  $\frac{e^{(n-1)Y_1} [Y_1 - 1/(n-1)] + 1/(n-1)}{e^{(n-1)Y_1} - 1}$

45. The  $\theta$  in the indicator function should be 0.

(a) Posterior distribution of  $\theta \propto$

$\theta^n \exp[(\theta - 1) \log \sum x_i] \theta^{r-1} e^{-\lambda \theta}$ , hence it is  $\text{gamma}(n+r, \lambda - \sum \log x_i)$ .

(b) Mean posterior is  $(n+r)/(\lambda - \sum \log X_i)$ .

46. Similar to Example 45.

47. (f) Similar, but slightly more tedious, to Example 46.

50. See the last paragraph in Section 7.2.

51. This problem is similar to several others and makes a good review question.

Recall that the sum of geometric distributed r.v.'s have negative binomial distribution. See Problem 21 in Chapter V.

(g)  $\theta = P[X = 0]$ , so  $I_{\{0\}}(X_1)$  is an unbiased estimator, and  $E[I_{\{0\}}(X_1) | \sum X_i]$  is UMVUE.

(h) posterior distribution  $\propto \theta^n (1 - \theta)^{\sum x_i} I_{(0,1)}(\theta)$ , hence the posterior is  $\text{beta}(n+1, \sum x_i + 1)$ .

53. The middle  $\beta$  should be  $1/\beta$ . The factorization criterion shows that  $Y_1$  and  $\sum X_i$  are jointly sufficient.  $Y_1$  and  $\sum (X_i - Y_1)$  are sufficient and complete. Now  $E[Y_1] = \alpha + (\beta/n)$  and  $E[\sum (X_i - Y_1)] = (n-1)\beta$ , so  $\sum (X_i - Y_1)/(n-1)$  is UMVUE of  $\beta$  and  $Y_1 - [\sum (X_i - Y_1)/n(n-1)]$  is UMVUE of  $\alpha$ .

54. (a) Factorization criterion gives  $(\sum X_i, Y_1)$ .

(b)  $L(\theta, \alpha; x_1, \dots, x_n) = (1 - \theta)^n \theta^{\sum x_i} e^{-n\alpha}$  for  $0 \leq \theta \leq 1$  and  $\alpha = y_1$

$y_1 - 1, y_1 - 2, \dots$ . It is monotone increasing in  $\alpha$  for each  $\theta$ , hence MLE of  $\alpha$  is  $Y_1$  and MLE of  $\theta$  is  $(\bar{X} - Y_1)/(\bar{X} - Y_1 + 1)$ .

55. Picture the likelihood function. Between any two consecutive order statistics, the likelihood function is "cusp" shaped. It can be concluded that the maximum of the likelihood function occurs at an order statistic, pick that order statistic that maximizes  $L(y_j)$  for  $j = 1, \dots, n$ .



## CHAPTER VIII

## Chapter VIII PROBLEMS

1. (a)  $Q = -\theta \log X$  has an exponential distribution with parameter one.  
 (b)  $P[Y/2 < \theta < Y] = e^{-1/2} - e^{-1}$ . Using the pivotal quantity given in part (a)  $P[q_1 Y < \theta < q_2 Y]$  is obtained. There are two ways of proceeding to find a better confidence interval; the first is to choose  $q_1$  and  $q_2$  so that the confidence interval has confidence coefficient  $e^{-1/2} - e^{-1}$  and minimum expected length, and the second is to choose  $q_1$  and  $q_2$  so that the confidence interval has expected length  $= (1/2)E[Y]$  and maximum confidence coefficient.
2.  $Q = (n-1)S^2/\theta$ .
3.  $P[T_1 < \tau(\theta) < T_2] = P[T_1 < \tau(\theta)] + P[\tau(\theta) < T_2] - P[T_1 < \tau(\theta) \text{ or } \tau(\theta) < T_2]$   
 $= \gamma + \gamma - 1.$
4. As in Problem 3,  $[Y_1 < \theta < Y_n] = P[Y_1 < \theta] + P[\theta < Y_n] - 1$   
 $= [1 - (1/2)^n] + [1 - (1/2)^n] - 1 = 1 - (1/2)^{n-1}.$
5. (a)  $Q = \theta \sum X_i$  is a pivotal quantity.  
 (b) Use part (a) and the Remark on Page 378.  
 (c)  $\gamma$   
 (d) See part (b).  
 (e)  $n\theta Y_1$ .
6. Similar to Problem 1.
7. (a)  $\gamma = 1/2$ . (See the solution to Problem 4.)  $E[Y_2 - Y_1] = E[|X_2 - X_1|]$   
 $= 2\sqrt{\pi} \approx 1.1284$   
 (b) Have  $P[q_1 < \bar{X} - \theta < q_2] = 1/2$ . Choose  $q_1$  and  $q_2$  symmetric about zero; expected length  $\approx .95$ .
8. (a) Use  $Q = \sqrt{n}(\bar{X} - \mu)/\sigma$  as your pivotal quantity.  
 (b) Use  $Q = \sum (X_i - \mu)^2/\sigma^2$  as your pivotal quantity.

9.  $(-2.09, 2.84)$  for  $\sigma$  known and  $(-1.94, 2.69)$  for  $\sigma$  unknown.
10. (b) Use  $\bar{X} - 1.645S$ .
11. Use  $Q = \sum_{i=1}^5 \sum_{j=1}^{n_i} (X_{ij} - \bar{X}_i.)^2 / \sigma^2$  as your pivotal quantity.  $Q \sim$  chi-square with 23 degrees of freedom.
12. Use 
$$\frac{\left( \sum_1^m (X_i - \bar{X})^2 / \sigma_1^2 \right) / (m-1)}{\left( \sum_1^n (Y_i - \bar{Y})^2 / \sigma^2 \right) / (n-1)} \sim F(m-1, n-1)$$
 as a pivotal quantity.
13. Want  $P[2tS/\sqrt{20} < \sigma]$  where  $t$  is the  $(1-\gamma)/2$  quantile of a  $t$ -distribution with 19 degrees of freedom. Write as  $P[2tS/\sqrt{20} < \sigma] = P[(19)S^2/\sigma^2 < 19(20)/4t^2]$ , where  $(19)S^2/\sigma^2$  is chi-square distributed with 19 degrees of freedom, to complete the calculations for any  $\gamma$ .
14. (a)  $2z\sigma/\text{sqrt}n$  where  $z$  is the  $(1+\gamma)/2$  quantile of a standard normal.  
 (b)  $2tE[S]/\sqrt{n}$  where  $t$  is the  $(1+\gamma)/2$  quantile of a  $t$ -distribution with  $n-1$  degrees of freedom. See Problem 17 of Chapter VI for  $E[S]$ .
15. Want  $P[2tS/\sqrt{n} < \sigma/5] \approx .95$  where  $t$  is .95th quantile of a  $t$ -distribution with  $n-1$  degrees of freedom. Rewrite as  $P[(n-1)S^2/\sigma^2 < (n-1)n/100t^2]$ . Want the minimum  $n$  such that  $(n-1)n \geq 100t_{.95, n-1}^2 \chi_{.95, n-1}^2$ .  $n$  a little over 300 seems to work.
18. Use Equation (1).  $(1.47, 10.03)$
19. The first "the" should be "a". Use  $Q = -\sum \log F(X_i; \theta) = -(1/\theta) \sum \log X_i$  as a pivotal quantity.
20. Use the statistical method and  $\sum X_i$  as a statistic.
21.  $[(Y_1 + Y_2)/2] - \theta$  is good pivotal quantity.
24. The sample size seems large enough to use Equation (10) of Example 8.  
 $.4375 \pm .0408$  for 90%.
25. The UMVUE of  $\tau(\theta)$  is a linear function of  $\bar{X}$  and  $S$ .  $\bar{X}$  and  $S$  are independent and have large sample normal distributions. Hence the large sample distribution of the UMVUE of  $\tau(\theta)$  is normally distributed. Use this to get an appropriate confidence interval.

26. Similar to Example 9.

27. The posterior distribution is given in the solution of Problem 45 of Chapter VII. Use it and Equation 21.

28. The likelihood function is the joint distribution of  $Y_1, \dots, Y_k$  looked at as a function of  $\theta$ .  $L(\theta; y_1, \dots, y_k) = \frac{n!}{(n-k)!} \theta^k e^{-\theta \sum_{i=1}^k y_i} e^{-\theta y_k(n-k)}$  for  $y_1 \leq y_2 \leq \dots \leq y_k$ . MLE of  $1/\theta$  is  $[\sum_{j=1}^k Y_j + (n-k)Y_k]/k$ . Let  $U_i = Y_i - Y_{i-1}$ .  $U_i \sim$  negative exponential with parameters  $\theta(n-i+1)$  using the lack of memory property of exponentially distributed random variables.  $\theta(n-1+1)U_i \sim$  negative exponential with parameter 1.

$$\begin{aligned} \sum Y_i + (n-k)Y_k &= Y_1 + Y_2 + Y_3 + \dots + Y_{k-1} + (n-k+1)Y_k \\ &= U_1 + (U_1 + U_2) + (U_1 + U_2 + U_3) + \dots + (n-k+1)(U_1 + \dots + U_k) \\ &= nU_1 + (n-1)U_2 + \dots + (n-k+1)U_k = \sum_{j=1}^k (n-j+1)U_j. \quad \text{Also,} \end{aligned}$$

$\theta(\sum Y_i + (n-k)Y_k) = \sum_{j=1}^k \theta(n-j+1)U_j$ , which is a sum of  $k$  independent negative exponentially distributed r.v.'s with parameter 1. Use  $Q = \theta(\sum Y_i + (n-k)Y_k) \sim \text{gamma}(k, 1)$  as a pivotal quantity.



## CHAPTER IX

## Chapter IX PROBLEMS

1. (a) (i)  $\Pi_T(\theta) = \sum_{j=1}^{10} \binom{10}{j} \theta^j (1-\theta)^{10-j}$   
(ii)  $\Pi_T(1/2) \approx .377$   
(b) (i)  $C_T = \{(x_1, \dots, x_{10}) : \sum x_j \leq 2\}$   
(ii)  $\Pi_T(1/4) \approx .53$   
(c) (i) Want  $\mathcal{R}(\theta_0) = \mathcal{R}(\theta_1)$ . Reject for  $\sum X_i \leq 4$  does it.  
(ii) maximum risk for minimax  $\approx 305$   
maximum risk for M.P.  $\approx 815$   
(d) Reject for  $\sum X_i \leq 4$ .
2. (a)  $\Pi(\theta) = 1 - (3/4)^\theta + \theta(3/4)^\theta \log(3/4)$ .  
size =  $1/4 + (3/4) \log(3/4)$   
(b) Reject if and only if  $X_1 X_2 \geq 1/2$ .  
(c) Yes  
(d) Reject if and only if  $X_1 X_2 \geq 1/2$   
(e) Reject if and only if  $\prod X_i > 1/2^n$ .  
(f) This is equivalent to finding the minimax test with  $(d_0; \theta_1) = (d_1; \theta_0) =$   
1. Reject if and only if  $X_1 X_2 \geq k$  where  $k$  is solution to  $1 - k + k \log k =$   
 $k^2 - 2k \log k$ .
4. (e) Reject for  $X < k$  where  $k$  is such that  $\alpha + \beta = k^2 + (1 - k)$  is minimized;  
i.e.,  $k = 1/2$ .  
(f) After some manipulation the test reduces to: reject for  $X \log X < k$   
where  $k$  is such that  $P_{\theta=1}[X \ln X < k] = \alpha$ . Note that this test does  
say to reject for "large" and "small"  $x$  which is intuitively appealing.
5. (a) Reject if and only if  $X > 1 - \alpha$ .  
(b)  $\Pi(\theta) = P[X > 1/2] = 1/2 + (1/4)\theta$ . Size is  $1/2$ .  
(c) Yes. Have monotone likelihood ratio in  $X$ . Test is: reject iff  $X >$   
 $1 - \alpha$ .

- (d) Reject if and only if  $|X-1/2| > c$  where  $c$  is such that  $P_{\theta=0}[|X-1/2| > c] = \alpha$ ; i.e.,  $c = (1 - \alpha)/2$ .
- (e)  $\alpha + \beta = P_{\theta=0}[X > k] + P_1[X < k] = 1 - k + k^2$  which is a minimum for  $k = 1/2$ .
6. (a)  $\Pi(\theta) = 1 - P_{\theta}[\theta_0 \alpha^{1/n} \leq Y_n \leq \theta_0] = I_{(0, \theta_0 \alpha^{1/n})}(\theta) + \alpha(\theta_0/\theta)^n I_{(\theta_0 \alpha^{1/n}, \theta_0)}(\theta) + [1 - (1 - \alpha)(\theta_0/\theta)^n] I_{(\theta_0, \infty)}(\theta)$ .
- 7 (a) Reject if and only if  $-\sum \log X_i > (\theta_0/2) \chi_{2n, 1-\alpha}^2$  where  $\chi_{2n, 1-\alpha}^2$  is the  $(1-\alpha)$ -quantile of a chi-square distribution with  $2n$  degrees of freedom.
10. (a)  $\Pi(\theta) = P_{\theta}[X_1 + X_2 \geq 1] = (1/2)[(2\theta-1)/\theta]^2 I_{(1/2, 1)}(\theta) + [1 - (1/2\theta^2)] I_{(1, \infty)}(\theta)$ .  
Size of test =  $\Pi(1) = 1/2$ .
- (b) UMP size  $\alpha = 1/2$  test is given by: reject iff  $Y_2 \geq 1/\sqrt{2}$ . Power of UMP test is  $[1 - (1/2\theta^2)] I_{(1/\sqrt{2}, \infty)}(\theta)$ , which is identical to the power of the given test for  $\theta > 1$ . Note that the test in part (b) is based on a sufficient statistic and the test in part (a) is not.
11. (a)  $\Pi(\theta) = 1 - (1 + \theta)e^{-\theta}$
- (d) Reject if and only if  $X_1 \leq 2 \log 2$ .
12. (a)  $k = 1 - \alpha^{1/n}$
- (b)  $[\alpha + 1 - (1 - \theta)^n] I_{(0, 1 - \alpha^{1/n})}(\theta) + I_{(1 - \alpha^{1/n}, \infty)}(\theta)$
- (c) Maybe this part should have been starred. To prove it, find the most powerful size  $\alpha$  test of  $\theta = 0$  versus  $\theta = \theta_1$  where  $0 < \theta_1 < 1$  (if  $\theta_1 > 1$  you can tell with certainty which hypothesis is true.) It turns out that the power under the alternative  $\theta = \theta_1$  is the same as the power of the given test, so the given test must be uniformly most powerful.

$$\begin{aligned}
13. \quad (a) \quad & \frac{\left( \frac{m+n}{-\sum \log X_i - \sum \log Y_j} \right)^{m+n} [\exp(\sum \log X_i + \sum \log Y_j)]^{[(m+n)/(-\sum \log X_i - \sum \log Y_j)]-1}}{\left( \frac{m}{-\sum \log X_i} \right)^m \left( \frac{n}{-\sum \log Y_j} \right)^n [\exp(\sum \log X_i)]^{[n/(-\sum \log X_i)]-1} [\exp(\sum \log Y_j)]^{[m/(-\sum \log Y_j)]-1}} \\
& = \frac{(m+n)^{m+n}}{m^m n^n} \left( \frac{-\sum \log X_i}{-\sum \log X_i - \sum \log Y_j} \right)^m \left( \frac{-\sum \log Y_j}{-\sum \log X_i - \sum \log Y_j} \right)^n
\end{aligned}$$

(b) Test is of form reject  $\mathcal{H}_0$  if and only if  $T^m(1-T)^n \leq \text{constant}$ .

(c)  $T$  has a beta distribution with parameters  $m$  and  $n$  and does not depend on the common value of  $\theta_1$  and  $\theta_2$  under  $\mathcal{H}_0$ . (See Example 25 of Chapter V)

14. See Example 11. How does the answer change if you test  $\mathcal{H}_0: \theta = 1$  versus  $\mathcal{H}_1: \theta \neq 1$ ?

15. This is a good review or test question. The density is the same as the densities of Problem 2 and 4 with slight reparametrization.

16. See Problem 13.

Problem 17 through 35 cover material allied to that of Section 4 on sampling from the normal distribution.

17. Let  $D_i = X_i - Y_i$ .  $\delta \pm t_{(1+\gamma)/2} \sqrt{\sum (D_i - \bar{D})^2 / (n-1)n}$  is  $\gamma$ -level confidence interval for  $\mu_X - \mu_Y$ . Use test: Reject  $\mathcal{H}_0$  if and only if the confidence interval does not contain zero. Test has size  $\alpha = 1 - \gamma$ .

22.  $C^* = \{(x_1, \dots, x_n) : \sum x_i \leq 6/n + n\sigma^2 z_\alpha\}$ .

24. Use test: reject  $\mathcal{H}_0: \mu = \mu_0$  if and only if  $\bar{X} > k$  where  $\mu_0 < k < \mu_1$ .

$$\begin{aligned}
\alpha &= P_{\mu_0}[\bar{X} > k] = 1 - \Phi\left(\frac{k - \mu_0}{\sigma/\sqrt{n}}\right) \rightarrow 0 \text{ as } n \rightarrow \infty \text{ and } \beta = P_{\mu_1}[\bar{X} < k] = \\
&\Phi\left(\frac{k - \mu_1}{\sigma/\sqrt{n}}\right) \rightarrow 0 \text{ as } n \rightarrow \infty.
\end{aligned}$$

25. Use test based on statistic given in Equation (18).

30. Could use Theorem 7.  $-2\log \lambda_n \approx 4.14 < \chi_{.99}^2(2) = 9.21$

33.  $X_{11}, \dots, X_{1n}$  r.s. from  $N(\mu_1, \sigma^2)$

$\vdots$   $\vdots$

$X_{k1}, \dots, X_{kn}$  r.s. from  $N(\mu_k, \sigma^2)$

$\mathcal{H}_0: \mu_1 = \mu_2 = \dots = \mu_k = 0$ .  $\lambda$  reduces to

$\left( \frac{\sum \sum (x_{ji} - \bar{x}_{j.})^2 + \sum \sum \bar{x}_{j.}^2}{\sum \sum (x_{ji} - \bar{x}_{j.})^2} \right)^{-nk/2}$ ; so the GLR test is equivalent to: reject

$\mathcal{H}_0$  if and only if  $T = \frac{\sum \sum \bar{x}_{j.}^2 / k}{\sum \sum (x_{ji} - \bar{x}_{j.})^2 / k(n-1)}$  is "large". Under  $\mathcal{H}_0$ ,  $T$  is F-distributed with  $k$  and  $k(n-1)$  degrees of freedom.

34.  $L(\mu_X, \mu_Y, \sigma_X^2, \sigma_Y^2, \rho; (x_1, y_1), \dots, (x_n, y_n)) \propto$

$(\sigma_X^2 \sigma_Y^2)^{-n/2} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[ \frac{\sum (x_i - \mu_X)^2}{\sigma_X^2} - 2\rho \frac{\sum (x_i - \mu_X)(y_i - \mu_Y)}{\sigma_X \sigma_Y} + \frac{\sum (y_i - \mu_Y)^2}{\sigma_Y^2} \right] \right\}$ .

The MLE are  $\hat{\mu}_X = \bar{x}$ ,  $\hat{\mu}_Y = \bar{y}$ ,  $\hat{\sigma}_X^2 = \sum (x_i - \bar{x})^2 / n$ ,  $\hat{\sigma}_Y^2 = \sum (y_i - \bar{y})^2 / n$ , and  $\hat{\rho} = \frac{\sum (x_i - \bar{x})(y_i - \bar{y}) / \sqrt{\sum (x_i - \bar{x})^2 \sum (y_i - \bar{y})^2}}{1}$ .  $\lambda$  reduces to  $(1 - \hat{\rho}^2)^{n/2}$ . GLR test is equivalent to: reject  $\mathcal{H}_0$  if and only if  $|\hat{\rho}|$  is "large". Under  $\mathcal{H}_0$ , the distribution of  $\hat{\rho}$  is free of parameters.

35. There are two cases depending on whether or not the common value for the mean under the null hypothesis is known. The generalized likelihood ratio technique gives a test using test statistic  $\sum [(X_i - \mu)^2 / \sigma_i^2]$  for  $\mu$  assumed known and test statistic  $V$  of Problem 27 of Chapter VI for  $\mu$  assumed unknown.

39. (a)  $E[Q] = \sum_{i=1}^{k+1} \frac{1}{np_j} E[(N - np_j)^2] = \sum_{j=1}^{k+1} \frac{1}{np_j} np_j (1 - p_j) = k$ .

$\text{var}[Q] = \sum_i \sum_j \frac{1}{np_i} \frac{1}{np_j} \text{cov}[(N_i - np_i)^2, (N_j - np_j)^2]$ . Certain fourth order central moments of the  $N_i$ 's are needed; these can be found directly or by using the moment generating function. After some manipulation,  $\text{var}[Q]$  reduces to  $2k + (1/n) [\sum (1/p_j) - k^2 - 4k - 1]$ .



(b)  $E[Q_k^\circ] = \sum_1^{k+1} (1/np_j^\circ) [np_j(1 - p_j) + n^2(p_j - p_j^\circ)^2]$   
 $E[Q_k^\circ] \Big|_{p_j = p_j^\circ} = k$ . The answer is no and can be verified by proper choices of  $p_j$  and  $P_j^\circ$ . One might try to minimize  $E[Q_k^\circ]$  with respect to the  $P_j$ 's using Lagrange multipliers and constraint equation  $\sum p_j = 1$ ;  $p_j^* = [(2n + k - 1)p_j^\circ - 1]/2(n - 1)$  results. Furthermore, such  $p_j^*$  will fall between zero and one for  $p_j^\circ$  between  $1/(2n - 1 + k)$  and  $(2n - 1)/(2n - 1 + k)$ .

40. Let  $p$  = proportion of headaches that are psychosomatic. Test  $\mathcal{H}_0: p \geq .4$  versus  $\mathcal{H}_1: p < .4$ . Let  $X$  = # of psychosomatic headaches. Reject  $\mathcal{H}_0$  for small  $X$ . Model assuming  $X$  has a binomial distribution with  $n = 41$ . For  $p = .4$ , and  $X = 12$ ,  $\mathcal{H}_0$  would be accepted at the 5% level.
41. Yes, using results from Theorem 8.
42. Yes; see Example 21.
45. The likelihood function is proportional to  $(p^2)^{n_1} [2p(1-p)]^{n_2} [(1-p)^2]^{n_3}$ . The MLE of  $p$  is  $(2n_1 + n_2)/2n \approx .335$ . Obtain  $\hat{p}_1, \hat{p}_2$ , and  $\hat{p}_3$  and use test statistic  $Q'_2$ . Accept that the data are consistent with the model.
46. Yes.
47. Reject hypothesis.
48. Test  $\mathcal{H}_0: p_{ij} = p_{i.}p_{.j}$ . Reject  $\mathcal{H}_0$ .
49. Use  $Q'_{2k}$  of Equation 30. Note that it reduces to  $\sum_{j=1}^3 \frac{(N_{[ij]} - N_{2j})^2}{N_{1j} + N_{2j}}$ , which has value  $\approx 7.57 > \chi^2_{.95}(2) = 5.99$ .
50. Use approach similar to Problem 45. It is somewhat more difficult to get MLE of  $p, q$ , and  $r = 1 - p - q$ . Compare the computed  $Q'_3$  statistic with  $\chi^2_{1-\alpha}(1)$ .



## CHAPTER X

## Chapter X PROBLEMS

Problems 1 through 6 are solved by using the given data and appropriate formulas in Sections 4 and 5.

1. Equations 7, 8, and 9.
2. See the Corollaries of Theorem 2.
3. Equations 15, 16, and 14.
4. See Page 494.
5. Use the invariance property; see Theorem 2 on page 285.
6. Similar to Problem 8 below.
7.  $P[Y_{x_0} \leq \beta_0 + \beta_1 x_0 + z_p \sigma] = \Phi \left( \frac{\beta_0 + \beta_1 x_0 + z_p \sigma - (\beta_0 + \beta_1 x_0)}{\sigma} \right) = \Phi(z_p) = p.$
8.  $\hat{\beta}_0 + \hat{\beta}_1 x + z_p [\Gamma((n-2)/2)/\sqrt{2}\Gamma((n-1)/2)] [\sum_1^n (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2]^{1/2}.$
10.  $\hat{\beta}_0 \approx .497, \hat{\beta}_1 \approx 2.049, \hat{\sigma}^2 \approx .00117,$  and  $\text{var}[\hat{\beta}_1] \approx .00255.$  A 95% confidence interval estimate for  $\beta_1$  is (1.93, 2.17).  $\beta_1 = 1$  is outside this interval, so according to the confidence interval technique, the hypothesis  $\beta_1 = 1$  may be rejected.
11. Similar to Problem 10.
12. Could set a one-sided confidence interval on  $\mu(.50)$  and use the confidence interval technique.
13. Use the invariance property of confidence intervals. See the Remark on Page 378.
14.  $\hat{\beta}_1(n-4)/\sum(y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2$
15.  $\hat{\beta}_1 = \frac{(\sum a_i)(\sum a_i x_i y_i) - (\sum a_i y_i)(\sum a_i x_i)}{(\sum a_i)(\sum a_i x_i^2) - (\sum a_i x_i)^2},$   
 $\hat{\beta}_0 = (\sum a_i y_i - \hat{\beta}_1 \sum a_i x_i)/\sum a_i,$  and  
 $\hat{\sigma}^2 = \sum a_i (y_i - \hat{\beta}_0 - \hat{\beta}_1 x_i)^2/n.$

16. Recall that  $\hat{B}_0$  and  $\hat{B}_1$  have a bivariate normal distribution. What is required for independence in a bivariate normal?
17.  $\text{cov}[\bar{Y}, \hat{B}_1] = \text{cov}[\hat{B}_0 + \hat{B}_1\bar{x}, \hat{B}_1] = \text{cov}[\hat{B}_0, \hat{B}_1] + \bar{x} \text{var}[\hat{B}_1] = 0$  by Equation (12).  
 $\bar{Y}$  and  $\hat{B}_1$  have a bivariate normal distribution so uncorrelated implies independence.

Problems 19, 20, and 21 can be worked using the theory of Lagrange multipliers as in the proof of Theorem 6.

## CHAPTER XI

## Chapter XI PROBLEMS

2.  $\text{cov}[F_n(B_1), F_n(B_2)] = (1/n)^2 \sum_i \sum_j \text{cov}[I_{B_1}(X_i), I_{B_2}(X_j)] = (1/n) \text{cov}[I_{B_1}(X), I_{B_2}(X)]$   
 $= (1/n) (P[X \in B_1 B_2] - P[X \in B_1] P[X \in B_2]).$
4. (a)  $D_1 = \max[U, 1 - U]$  where  $U$  is uniformly distributed over the interval  $(0, 1)$ .  $F_{D_1}(x) = (2x - 1)I_{[1/2, 1)}(x) + I_{[1, \infty)}(x).$
- (b)  $F_{D_2}(z) = 2(2z - 1/2)^2 I_{(1/2, 1/3)}(z) + [1 - 2(1 - z)^2] I_{[1/3, 1)}(z) + I_{[1, \infty)}(z).$
- (c)  $D_n = \max_{1 \leq i \leq n} [ |F(Y_i) - \frac{i-1}{n}|, |F(Y_i) - \frac{i}{n}| ],$  so  $D_n$  is a function of  $F(Y_1), \dots, F(Y_n)$  which are the order statistics from a uniform over  $(0, 1)$ .
5.  $E[Y_2] = E[(Y_1 + Y_2)/2] + E[|X_1 - X_2|/2] = (1/2)E[|X_1 - X_2|] = 1/\sqrt{\pi}$  using the fact that  $X_1 - X_2 \sim N(0, 2).$
6. Use the same start as in Problem 5.  $X_1 - X_2 \sim N(0, 2(1\rho)).$
7. Yes, see Theorem 14 in Chapter VI.
10.  $n = 15.$
11. The data seemed to be ordered; you might be leary of the two-sample sign test.
13.  $\text{var}[U] = \sum_{j=1}^n \sum_{i=1}^m \sum_{\beta=1}^n \sum_{\alpha=1}^m \text{cov}[I_{[Y_j, \infty)}(X_i), I_{[Y_\beta, \infty)}(X_\alpha)]$   
 $= mn \text{var}[I_{[Y, \infty)}(X)] \quad (j = \beta \text{ and } i = \alpha)$   
 $+ nm(m-1) \text{cov}[I_{[Y, \infty)}(X_1), I_{[Y, \infty)}(X_2)] \quad (j = \beta \text{ and } i \neq \alpha)$   
 $+ n(n-1)m \text{cov}[I_{[Y_1, \infty)}(X), I_{[Y_2, \infty)}(X)] \quad (j \neq \beta \text{ and } i = \alpha)$   
 $+ \text{zero} \quad (j \neq \beta \text{ and } i \neq \alpha)$   
 $= mn [P[X \geq Y] - P^2[X \geq Y]]$   
 $+ nm(m-1) (P[X_1 \geq Y, X_2 \geq Y] - P^2[X \geq Y])$   
 $+ n(n-1)m (P[X \geq Y_1, X \geq Y_2] - P^2[X \geq Y])$   
 $= mn(1/4) + mn(m-1)((1/3) - (1/4)) + mn(n-1)((1/3) - (1/4))$   
 $= mn(m+n+1)/12.$

14.  $m = 1, n = 2$  gives  $P[T_x = 1] = P[T_x = 2] = P[T_x = 3] = 1/3$ .  
 $m = 1, n = 3$  gives  $P[T_x = 1] = P[T_x = 2] = P[T_x = 3] = P[T_x = 4] = 1/4$ .  
 $m = 2, n = 1$  gives  $P[T_x = 3] = P[T_x = 4] = P[T_x = 5] = 1/3$ .  
 $m = 3, n = 1$  gives  $P[T_x = 6] = P[T_x = 7] = P[T_x = 8] = P[T_x = 9] = 1/4$ .  
 $m = n = 2$  gives  $P[T_x = 3] = P[T_x = 4] = P[T_x = 6] = P[T_x = 7] = 1/6$  and  $P[T_x = 5] = 2/6$ .
15.  $U/mn$  is an unbiased estimator of  $p$ . The second question should read: Is  $U/mn$  a consistent estimator of  $p$ ? The answer is yes as can be noted by looking at the intermediate steps in the solution of Problem 13 and letting  $m$  and  $n$  approach infinity.
16. (a) Just algebra noting that  $\bar{r}(X) = \bar{r}(Y) = (n+1)/2$  and  $\sum r^2(X_i) = \sum r^2(Y_i) = \sum i^2 = n(n+1)(2n+1)/6$ .  
 (b)  $S = .9$  and the ordinary correlation coefficient  $\approx .962$ .
17. The ranks of  $X_1, \dots, X_n$  are the same as the ranks of  $F_X(X_1), \dots, F_X(X_n)$ . Likewise for the  $Y_j$ 's. By the probability integral transform the distribution of  $F_X(X_1), \dots, F_X(X_n)$  does not depend on  $F_X(\cdot)$ ; likewise for the  $Y_j$ 's. Hence, the distribution of  $S$  (which is a function only of the ranks of  $F_X(X_1), \dots, F_X(X_n)$  and the ranks of  $F_Y(Y_1), \dots, F_Y(Y_n)$ ) will not depend on  $F_X(\cdot)$  and  $F_Y(\cdot)$ .
18.  $E[S] = 1 - [6n/(n^3 - n)]E[D_1^2]$   
 $= 1 - [6n/(n^3 - n)](E[r^2(X_1)] - 2E[r(X_1)r(Y_1)] + E[r^2(Y_1)])$   
 $= 1 - [6n/(n^3 - n)]((1/n) \sum i^2 - 2(\sum i/n)^2 + (1/n) \sum i^2)$   
 $= 0$  using independence of  $r(X_1)$  and  $r(Y_1)$  and the fact that  $r(X_1)$  and  $r(Y_1)$  have discrete uniform distributions.  
 $\text{var}[S] = [36/(n^3 - n)^2] \sum \sum \text{cov}[D_i^2, D_j^2] = [36/(n^3 - n)](n \text{ var}[D_1^2] + n(n-1) \text{ cov}[D_1^2, D_2^2])$   
 $= [36/(n^3 - n)^2](nE[D_1^4] - n(E[D_1^2])^2 + n(n-1)E[D_1^2 D_2^2] - n(n-1)E[D_1^2]E[D_2^2])$   
 $= [36/(n^3 - n)^2](n \sum_i \sum_j (i-j)^4 (1/n^2) + n(n-1) \sum_i \sum_j \sum_{\alpha \neq i} \sum_{\beta \neq j} (i-j)^2 (\alpha - \beta)^2 (1/n^2 (n-1)^2)$   
 $- n^2 [\sum_i \sum_j (i-j)^2 (1/n^2)]^2)$   
 $= 1/(n-1).$