AERO 320: FUNDAMENTALS OF DYNAMICS AND CONTROL

ICGE 2

Assigned 11/03/2021 – Due 11/10/2021 by 5 pm

One Dimensional Reaction Wheel in a Rocket

Student Outcomes:

- Practice deriving the angular momentum of a system consisting of multiple rigid bodies.
- Apply Euler's equations of motion to determine the rotational dynamics of a system.
- Use $Matlab^{®}$ to study the rotational behavior of a system

Introduction

In class we have derived the rotational equations of motion for a rigid body. When the frame of interest (or body frame) is attached to and moving with the rigid body, and located at the center of mass, these equations are known as *Euler's equations of motion*. In this In-Class Group Exercise, we will extend Euler's Equations to analyze the case of a spinning disc (or reaction wheel) which is located at the center of mass of a rocket. We will use the resulting differential equation(s) to plot and reflect on the angular displacement of the rocket.

1 Problem Statement

The figure below is a schematic of a rocket with a reaction wheel located at its center of mass. The reaction wheel spins along its axis which is along the $\vec{z}_c$ body axis. The body of the rocket is assumed to be a cylinder of length $L = 4$ m and radius $R = 0.25$ m, while its nose is assumed to be a right circular cone of height $h = 1$ m. The mass of the rocket rocket body (i.e., the cylinder) is $M_b = 600\,\text{kg}$ while that of the nose cone is $M_n = 50$ kg. The reaction wheel has a radius of $r = 0.2\,\text{m}$, a thickness of $t = 0.04$ m, and a mass of $m = 10\,\text{kg}$. You may assume that the rocket and the reaction wheel are made of homogeneous material. Additionally, notice that we have assumed that the rocket is solid throughout which is obviously unphysical).

The angle θ is measured positively from the vertical to the current longitudinal axis of the rocket. In this problem, we will assume that the thrust vector is always along the longitudinal axis of the rocket and therefore causes no rotational moment on the rocket.

To analyze this system, you will need to use skills you developed in ME 212 and AERO 300, as well as the material covered in Chapters 1 and 2 of this course. The important concept to keep in mind is that the total angular momentum of the rocket/reaction wheel system about their mutual center of mass, c, is the angular momentum of the rocket due to its absolute angular velocity plus the angular momentum of the reaction wheel (also due to its absolute angular velocity). In equation form, this is

$$\vec{\boldsymbol{h}}_c = \vec{\boldsymbol{h}}_c^r + \vec{\boldsymbol{h}}_c^w \,, \tag{1}$$

where $\vec{\boldsymbol{h}}_c^r$ and $\vec{\boldsymbol{h}}_c^w$ are respectively the angular momentum of the rocket and reaction wheel about their mutual center of mass.

Let $\boldsymbol{h}_c$ denote the components of the total angular momentum of the system about c in the reference frame $\mathcal{F}_c$. Also, let $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_w$ denote the components in $\mathcal{F}_c$ of the absolute angular velocity of the rocket and reaction wheel respectively.

Then,

$$\boldsymbol{h}_c = \boldsymbol{I}^r \boldsymbol{\omega} + \boldsymbol{I}^w \boldsymbol{\omega}_w \,, \tag{2}$$

where $\boldsymbol{I}^r$ and $\boldsymbol{I}^w$ are the moments of inertia of the rocket and reaction wheel respectively about their mutual center of mass in the reference frame $\mathcal{F}_c$.

Note that absolute angular velocity of the reaction wheel, $\vec{\boldsymbol{\omega}}_w$, can be found by considering the angular velocity of the reaction wheel relative to the rocket, $\vec{\boldsymbol{\omega}}_{\text{rel}}$, and the angular velocity of the rocket relative to the inertial frame, $\vec{\boldsymbol{\omega}}$. The angular velocity of the reaction wheel relative to the rocket can be thought of as the speed at which the motor is turning the wheel.

Finally, Euler's equations of motion are still valid for this system and the sum of moments acting on the system is equal to the time rate of change of the total angular momentum,

$$\sum \vec{\boldsymbol{T}} = \vec{\boldsymbol{T}}_c + \vec{\boldsymbol{T}}_d = \dot{\vec{\boldsymbol{h}}}_c = \dot{\vec{\boldsymbol{h}}}_c^r + \dot{\vec{\boldsymbol{h}}}_c^w \tag{3}$$

where $\vec{\boldsymbol{T}}_c$ is an external command torque (which is zero in this case) and $\vec{\boldsymbol{T}}_d$ is a disturbance torque.

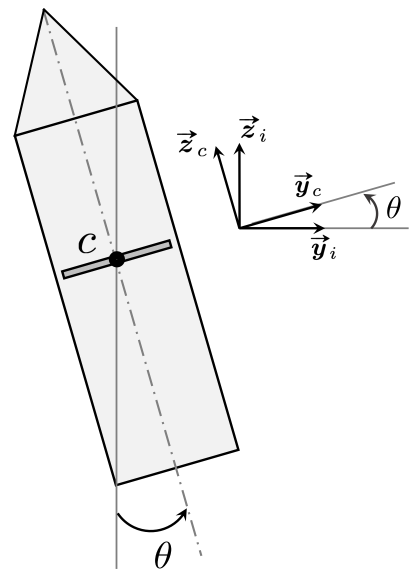

Figure 1: Schematic of the rocket and reaction wheel described in the problem statement.

Analysis (80 points)

1. (30 points) Using the figure above,

   (a) (13 points) Compute the mass moment of inertia matrices of the rocket, $\boldsymbol{I}^r$, and the reaction wheel, $\boldsymbol{I}^w$, about their mutual center of mass in the reference frame $\mathcal{F}_c$. Note that this requires that one first determine the location of the center of mass of the rocket/reaction wheel system.

   (b) (3 points) Draw a Free Body Diagram of the system.

   (c) (2 points) Express the absolute angular velocity of the reaction wheel, $\vec{\boldsymbol{\omega}}_w$ in terms of $\vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\omega}}_{\text{rel}}$.

   (d) (8 points) Starting from Equation (3), write the conservation of angular momentum equation in the reference $\mathcal{F}_c$ in terms of $\boldsymbol{I}^r$, $\boldsymbol{I}^w$, $\boldsymbol{\omega}$, $\boldsymbol{\omega}_{\text{rel}}$, and $\boldsymbol{T}_d$, where $\boldsymbol{\omega}_{\text{rel}}$ denotes the components of the physical vector $\vec{\omega}_{\text{rel}}$ in the reference frame $\mathcal{F}_c$.

   (e) (4 points) What is the rotational kinetic energy of the rocket/reaction wheel system?

2. (7 points) Assume no external moment is applied to the system, i.e., take $\vec{\boldsymbol{T}}_d$ to be zero for now. For a given change in the angular velocity of the reaction wheel relative to the rocket, what is the resulting change in the angular velocity of the rocket? In other words, for $\dot{\boldsymbol{\omega}}_{\text{rel}} \neq 0$, use the result of Question 1(d) to find $\dot{\boldsymbol{\omega}}$.(Assume all angular velocity components in the $\vec{\boldsymbol{x}}_c$- and $\vec{\boldsymbol{y}}_c$-directions are zero.)

3. (23 points) Use $Matlab^{®}$ (or any other numerical tool) to solve for the angular speed and displacement of the rocket if the reaction wheel is accelerating at a constant rate of $0.05\,\frac{\text{rad}}{\text{s}^2}$. Assume the initial angular displacement of the rocket is zero and that the initial angular speed of the rocket and reaction wheel are also zero. Clearly plot your results for $t \in [0\,,\,100\,\text{sec}]$.

4. (20 points) As the rocket ascends, an external disturbance moment, $\vec{\boldsymbol{T}}_d = \vec{\mathfrak{F}}_c^T \begin{pmatrix} 0.1 \\ 0 \\ 0 \end{pmatrix}$ N m, is acting on the rocket due to aerodynamic effects. Solve for the angular velocity and the associated Euler angles of the rocket for the following cases. Clearly plot your results for $t \in [0\,,\, 1000\,\text{sec}]$.

   (a) $\vec{\boldsymbol{\omega}}_{\text{rel}}(t) = \vec{\mathbf{0}}$ and $\vec{\boldsymbol{\omega}}(t=0) = \vec{\mathbf{0}}$,

   (b) $\vec{\boldsymbol{\omega}}_{\text{rel}}(t) = \vec{\mathbf{0}}$ and $\vec{\boldsymbol{\omega}}(t=0) = \vec{\mathfrak{F}}_c^T \begin{pmatrix} 0 \\ 0 \\ 0.1 \end{pmatrix} \frac{\text{rad}}{\text{s}}$,

   (c) $\vec{\boldsymbol{\omega}}_{\text{rel}}(t) = \vec{\mathfrak{F}}_c^T \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix} \frac{\text{rad}}{\text{s}}$ and $\vec{\boldsymbol{\omega}}(t=0) = \vec{\mathbf{0}}$,

   (d) $\vec{\boldsymbol{\omega}}_{\text{rel}}(t) = \vec{\mathfrak{F}}_c^T \begin{pmatrix} 0 \\ 0 \\ 100 \end{pmatrix} \frac{\text{rad}}{\text{s}}$ and $\vec{\boldsymbol{\omega}}(t=0) = \vec{\mathfrak{F}}_c^T \begin{pmatrix} 0 \\ 0 \\ 0.1 \end{pmatrix} \frac{\text{rad}}{\text{s}}$.

Further Analysis (20 points) (This part must be typed or clearly hand-written)

1. (8 points) What is your interpretation of the equation developed in question 2 of the Analysis section?

2. (4 points) Discuss the results of question 3 of the Analysis section.

3. (8 points) Discuss the impact on the angular velocity of the rocket and the reaction wheel in all cases of question 4 of the Analysis section?