HW#3

1) Prove that $3^{2^n} = o(2^{3^n})$

$$\lim_{n\to\infty} \frac{3^{2^n}}{2^{3^n}} \Rightarrow \lim_{n\to\infty} 2^{-3^n} 3^{2^n} = \frac{1}{\infty} \infty = O(\infty) = 0$$

Thus $3^{2^n} = o(2^{3^n})$ ✓

2) $\binom{2n}{n} = \Theta\left(\frac{4^n}{\sqrt{n}}\right)$

$$\frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n^2)!}$$

$$= \frac{\sqrt{2\pi 2n} \cdot \left(\frac{2n}{e}\right)^{2n} \left(1 + \Theta\left(\frac{1}{2n}\right)\right)}{\left(\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n \cdot \left(1 + \Theta\left(\frac{1}{n}\right)\right)\right)^2}$$

$$= \frac{4^n}{\pi n} \left(\frac{\left(1 + \Theta\left(\frac{1}{2n}\right)\right)}{\left(1 + \Theta\left(\frac{1}{n}\right)\right)^2}\right)$$

$$\lim_{n\to\infty} \frac{4^n}{\pi n} \left(\frac{\left(1+\Theta\left(\frac{1}{2n}\right)\right)}{\left(1+\Theta\left(\frac{1}{n}\right)\right)^2}\right) \cdot \frac{\sqrt{n}}{4^n} = \frac{1}{\sqrt{\pi}}$$

Because $\frac{1}{\sqrt{\pi}}$ is greater than 0 & less than ∞ $\boxed{\text{True ✓}}$

3)a) Ia. Base step $n=1$

$$\sum_{i=1}^{n+1} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$= \left(\frac{1(1+1)}{2}\right)^2$$

$$1 = 1$$

IIa. Induction step $n \geq 1$

$$\sum_{i=1}^{n+1} i^3 = \sum_{i=1}^{n} i^3 + (n+1)^3$$

$$= \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3$$

$$= \frac{n^2(n+1)^2 + 4(n+1)^3}{4}$$

$$= \frac{(n+1)^2(n+2)^2}{4} = \left(\frac{(n+1)(n+2)}{2}\right)^2$$

$$= \left(\frac{1(1+1)}{2}\right)^2 = 1 = 1 \checkmark$$

Thus $P(n)$ is true for all $n \geq 1$

3 (cont) b) Ib. Basic Step $n=0$

$$\sum_{i=0}^{0} i^3 = \left(\frac{n(n+1)}{2}\right)^2$$

$$0 = \left(\frac{0(1)}{2}\right)^2$$

$$0 = 0 \checkmark$$

IIb. Induction Step $n>0$

$$\sum_{i=0}^{n} i^3 = \sum_{i=0}^{n-1} i^3 + n^3$$

$$= \left(\frac{(n-1)n}{2}\right)^2 + n^3$$

$$= \left(\frac{n^2-n}{2}\right)^2 + n^3$$

$$= \frac{n^4-n^2}{4} + n^3$$

$0=0$ Thus $P(n)$ is true for all $n \geq 0$

4) I. Base Step $n=1$

$$S(1)=0 \qquad \log(1)=0$$

$0 \geq 0$ so $p(1)$ is true

II. $n \geq 2$ Induction Step

$$S(n) = S(\lceil n/2 \rceil)+1$$

$$\geq \lg \lceil n/2 \rceil + 1$$

$$\geq \lg(n/2)+1$$

$$= \lg(n) - \lg(2) + 1$$

$$S(n) = \lg(n)$$

$S(n) \geq \lg(n)$ for all $n \geq 1$ ✓

Thus $S(n) = \Omega(\lg n)$

5) I. Base Step $n=1$

$$T(1)=1 \qquad \frac{4}{3}(1)^2 = \frac{4}{3}$$

$1 \leq \frac{4}{3}$ so $P(1)$ is true

II. Induction Step $n \geq 2$

$$T(n) = T(\lfloor n/2 \rfloor)+n^2$$

5 cont) II cont.

$$\leq T(\lfloor n/2 \rfloor) + n^2$$

$$\leq \frac{4}{3} \lfloor n/2 \rfloor^2 + n^2$$

$$\leq \frac{4}{3} (n/2)^2 + n^2$$

$$\leq \frac{4}{12} n^2 + n^2$$

$$\leq \frac{4}{3} n^2$$

$T(n) \leq \frac{4}{3} n^2$ for all $n \geq 1$

Thus $T(n) = O(n^2)$

6) I Base Case 1 $n=1$

$T(1) = 2$ $\qquad$ $3(1)^2 - 1 = 2$

$2 \leq 2$ ✓

Base Case 2 $n=2$

$T(1) = 2$ $\qquad$ $3(2)^2 - 1 = 11$

$2 \leq 11$ ✓

Cont Next Page

6 (cont) II Induction Step

$$T(n) = 9T(\lfloor n/3 \rfloor) + 1$$

$$T(n) \leq 9(3\lfloor n/3 \rfloor^2) + 1$$

$$\leq 27(n/3)^2 + 1$$

$$\leq \frac{27}{9} n^2 + 1$$

$$T(n) \leq 3n^2 + 1$$

$T(n) \leq 3n^2+1$ for any value k in $1 \leq k < n$

7) I Base Case $n=1$

$T(n) = 6$ $\quad$ $6(1) = 6$

$6 \leq 6$ ✓

Base Case $n=3$

$T(n) = 6$ $\quad$ $6(3) = 18$

$6 \leq 18$ ✓

II Induction Step $n \geq 3$

$$T(n) = 2T(\lfloor n/3 \rfloor) + n$$

$$\leq 2(6(\lfloor n/3)) + n$$

7 (cont) II cont. Induction Step $n \geq 3$

$\leq 12(n/3) + n$

$\leq 4n + n$

$T(n) = 5n$

$5n \leq 6n$

Thus $T(n) = O(n)$