HW #4

1) Base case $n=1$

Each edge must have a vertex on each end. Because there is only one vertex an edge may not exist.

Thus satisfying that $m=0$ when $n=1$ in the formula $m=n-1$. ✓

Induction Step

If you take any tree with n_T verticies and m_T edges and remove one edge it must split int two seperate trees. This is due to the fact that each edge must have a vertex on each end.

T_1 has n_1 verticies and m_1 edges. T_2 has n_2 verticies and m_2 edges.

The total verticies in the two trees is n_1+n_2

By the original equation

$m_T = (n_1+n_2)-1$

Which means that this is $\boxed{\text{true}}$

2)

In order to disconnect a graph an edge ~~vertex~~ must be removed. Thus, the number of edges is the number of vertices minus the connected components. This can be seen through the result of problem one.

3) $T(n) = \begin{cases} 1 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + 5 & n \ge 3 \end{cases}$

$T(n) = 5 + 2T(\lfloor n/3 \rfloor)$

$= 5 + 2(2T(\lfloor \frac{\lfloor n/3 \rfloor}{3} \rfloor) + 5$

$= 15 + 2^2 T(\lfloor \frac{n}{3^2} \rfloor)$

$= 15 + 2^2 (2T(\lfloor \frac{\lfloor n/3 \rfloor}{3^2} \rfloor) + 5)$

$= 15 + 2^3 T(\lfloor \frac{n}{3^3} \rfloor) + 5(2^2)$

$= 35 + 2^3 T(\lfloor \frac{n}{3^3} \rfloor)$

$T(n) = \sum_{i=0}^{k-1} 5(2^i) + 2^k T(\lfloor \frac{n}{3^k} \rfloor)$ { dep

Recursion halts when

$1 \le \lfloor \frac{n}{3^k} \rfloor < 3$

$\lfloor \frac{n}{3^k} \rfloor < 3$

$\frac{n}{3^k} < 3 \Rightarrow \frac{n}{3} < 3^k \Rightarrow \lg(\frac{n}{3}) <$

$k - 1 \le \lg(\frac{n}{3}) < k \Rightarrow k = \lfloor \lg($

3 cont) Thus

$$T(n) = \sum_{i=0}^{\lg(\frac{n}{3})} 5(2^i) + 2^{\lg(\frac{n}{3})+1} T\left(\left\lfloor \frac{n}{3^{\lg(n/3)+1}} \right\rfloor\right)$$

$$\boxed{T(n) = \sum_{i=0}^{\lg(n)} 5(2^i) + 2^{\lg(\frac{n}{3})+1}}$$

4)

$$T(n) = \begin{cases} 3 & 1 \le n < 5 \\ 4T(\lfloor n/5 \rfloor) + n & n \ge 5 \end{cases}$$

$$T(n) = 4T(\lfloor n/5 \rfloor) + n$$

$$= n + 4\left(4T\left(\left\lfloor \frac{\lfloor n/5 \rfloor}{5} \right\rfloor\right) + n\right)$$

$$= n + 4^2 T\left(\left\lfloor \frac{n}{5^2} \right\rfloor\right) + 4n$$

$$= 5n + 4^2 T\left(\left\lfloor \frac{n}{5^2} \right\rfloor\right)$$

$$= 5n + 4^2\left(4T\left(\left\lfloor \frac{\lfloor n/5 \rfloor}{5^2} \right\rfloor\right) + n\right)$$

$$= 21n + 4^3 T\left(\left\lfloor \frac{n}{5^3} \right\rfloor\right)$$

$$T(n) = \sum_{i=0}^{k-1} 4^i \left\lfloor \frac{n}{5^i} \right\rfloor + 4^k T\left(\left\lfloor \frac{n}{5^k} \right\rfloor\right)$$

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4 (cont) Recursion halts when

$$1 \le \left\lfloor \frac{n}{5^k} \right\rfloor < 5$$

$$\left\lfloor \frac{n}{5^k} \right\rfloor < 5$$

$$\frac{n}{5^k} < 5 \Rightarrow \frac{n}{5} < 5^k \Rightarrow \lg\left(\frac{n}{5}\right) < k$$

$$k - 1 \le \lg\left(\frac{n}{5}\right) < k$$

$$k = \lg\left(\frac{n}{5}\right) + 1$$

$$k = \lg(n)$$

$$T(n) = \sum_{i=0}^{\lg(n)-1} 4^i \left\lfloor \frac{n}{5^i} \right\rfloor + 4^{\lg(n)} T\left(\left\lfloor \frac{n}{5^{\log(n)}} \right\rfloor\right)$$

$$= \sum_{i=0}^{\lg(n)-1} 4^i \left\lfloor \frac{n}{5^i} \right\rfloor + 4^{\lg(n)} \cdot 3$$

Hence $T(n) = \theta(n)$

5) a)
$$T(n) = \begin{cases} 6 & 1 \le n < 3 \\ 2T(\lfloor n/3 \rfloor) + n & n \ge 3 \end{cases}$$

$$T(n) = 2T(\lfloor n/3 \rfloor) + n$$
$$= n + 2\left(2T\left(\left\lfloor \frac{\lfloor n/3 \rfloor}{3} \right\rfloor\right) + n\right)$$
$$= n + 2^2 T\left(\left\lfloor \frac{n}{3^2} \right\rfloor\right) + 2n$$
$$= 3n + 2^2 T\left(\left\lfloor \frac{n}{3^2} \right\rfloor\right)$$
$$= 3n + 2^2\left(2T\left(\left\lfloor \frac{\lfloor n/3 \rfloor}{3^2} \right\rfloor\right) + n\right)$$
$$= 7n + 2^3 T\left(\left\lfloor \frac{n}{3^3} \right\rfloor\right)$$

$$T(n) = \sum_{i=0}^{k-1} 2^i + 2^k T\left(\left\lfloor \frac{n}{3^k} \right\rfloor\right) \quad \left\{ \text{depth at } k \right.$$

Recursion halts when

$$1 \le \left\lfloor \frac{n}{3^k} \right\rfloor < 3$$

As we can see in problem 3 this means

$$k = \lfloor \lg\left(\tfrac{n}{3}\right) \rfloor + 1$$

Thus
$$T(n) = \sum_{i=0}^{\lg\left(\frac{n}{3}\right)} 2^i + 2^{\lg\left(\frac{n}{3}\right)} T\left(\left\lfloor \frac{n}{3^{\lg \frac{n}{3}}} \right\rfloor\right)$$

5 cont) a cont)

$$\boxed{T(n) = \sum_{i=0}^{\lg(n)} 2^i + 2^{\lg(n)}}$$

b) Because there is no n in the summation it grows linearly. This can be represented as $O(n)$.

c) Compare n to $n^{\log_2 2} = n$

By case 3 $T(n) = \Theta(n)$

6) a) $T(n) = 7T(n/4) + n$

$f(n) = n$ compare to $n^{\log_4 7}$

Case 1 thus $\boxed{T(n) = \Theta(n^{\log_4(7)})}$

b) $T(n) = 9T(n/3) + n^2$

$f(n) = n^2$ compare to $n^{\log_3 9}$

Case 2 thus $\boxed{T(n) = \cancel{(n^{\log_3 9} \log(n}\,(n^3 \log(n))}$

c) $T(n) = 6T(n/5) + n^2$

$f(n) = n^2$ compare to $n^{\log_5 6}$

case 3 thus $T(n) = \Theta(n^2)$

6 cont) d) $T(n) = 6T(n/5) + n\log(n)$

$f(n) = n\log(n)$ compare to $n^{\log_5 6}$

case 1 thus $\boxed{T(n) = \Theta(n^{\log_5 6})}$

e) $T(n) = 7T(n/2) + n^2$

$f(n) = n^2$ compare to $n^{\log_2 7}$

case 1 thus $\boxed{T(n) = \Theta(n^{\log_2 7})}$

f) $S(n) = aS(n/4) + n^2$

$f(n) = n^2$ compare to $n^{\log_4 a}$

if $a < 4$ case 3

thus $\boxed{T(n) = \Theta(n^2)}$

if $a = 4$ case 2

thus $\boxed{T(n) = \Theta(n^2 \log(n))}$

if $a > 4$ case 1

thus $\boxed{T(n) = \Theta(n^{\log_4 a})}$

7)

Compare $f(n) = n^{deg(f)}$ to $n^{\log_b a}$

We know $deg(f) > \log_b(a)$ because the problem says so

Thus the qualifications for case 3 are met.

The regularity condition holds because this is true for any value of c.