Political Methodology III: Model Based Inference

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Model Based Inference

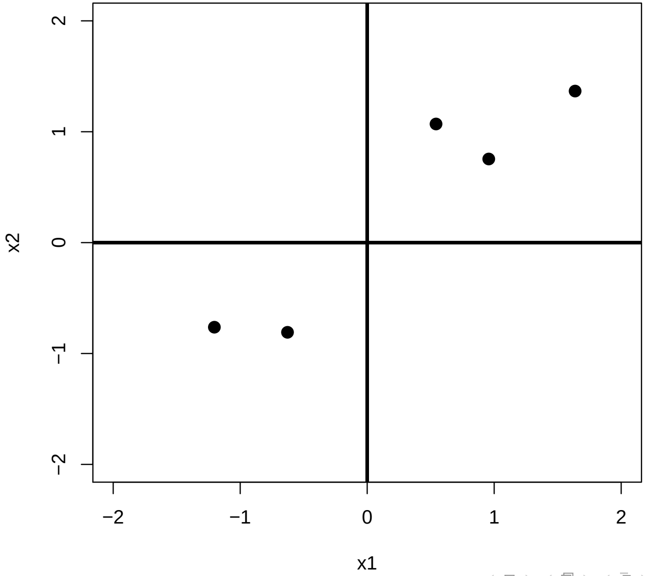
1) Likelihood inference
2) Machine Learning
   a) Model Selection
   b) Unsupervised Latent Features

Principal Component Analysis

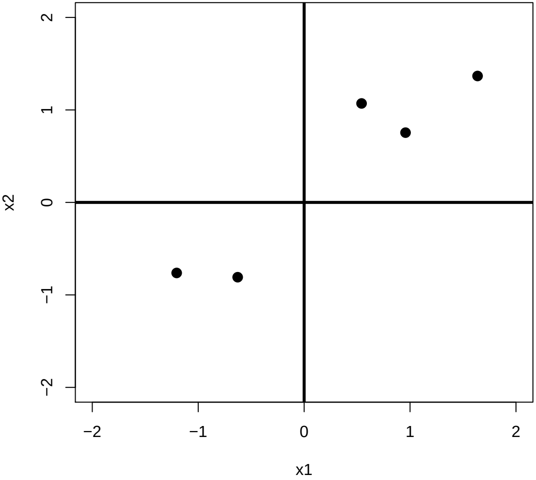
A Simple Two-Dimensional Example

Suppose we have the following observations:

$$\begin{aligned} x_1 &= (0.54, 1.07) \\ x_2 &= (-1.20, -0.76) \\ x_3 &= (-0.63, -0.81) \\ x_4 &= (0.96, 0.75) \\ x_5 &= (1.64, 1.37) \end{aligned}$$

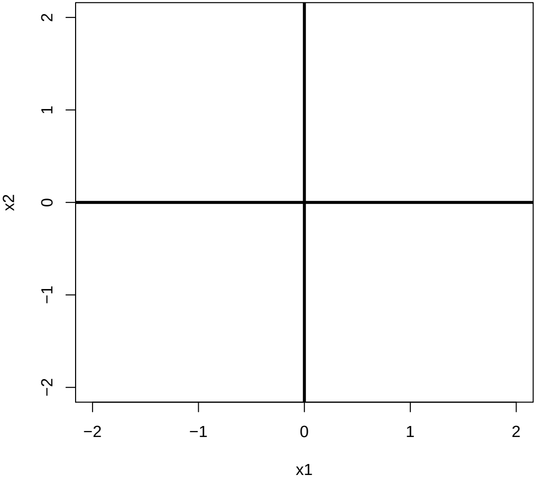

x2
2
1
0
−1
−2
−2
−1
0
1
2
x1

Goal: find line that summarizes bivariate information



Vectors to Draw a Line

Suppose $\boldsymbol{w}_1 = (1, 1)$



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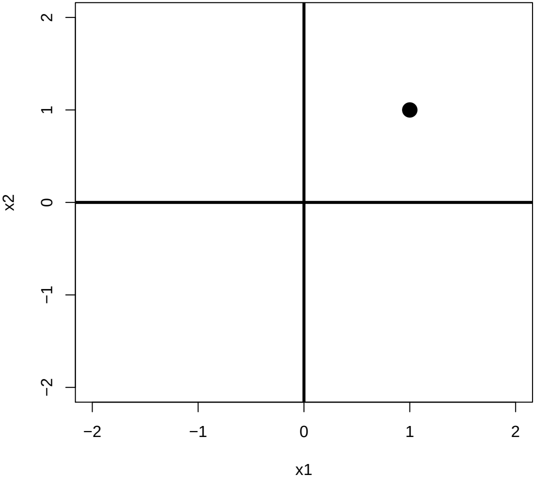

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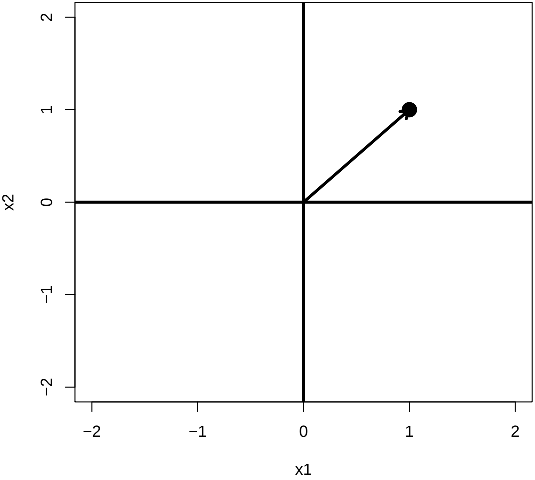
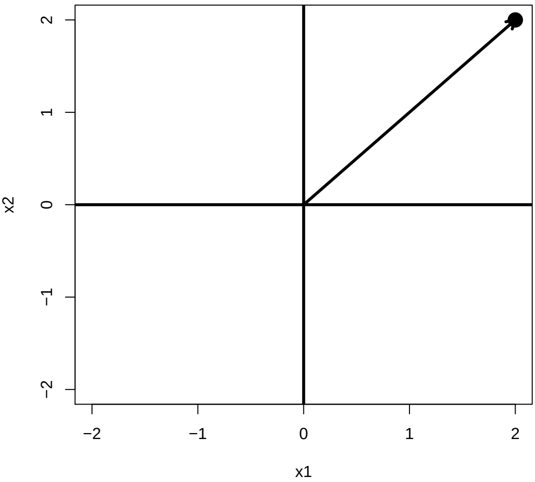


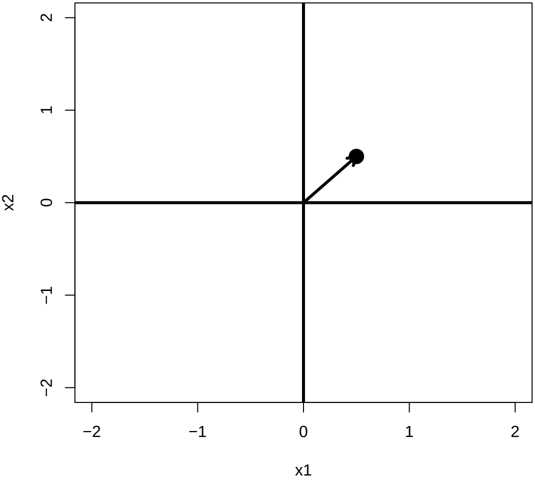
Vectors to Draw a Line

Suppose $\boldsymbol{w}_1 = (1, 1)$ $2\boldsymbol{w}_1 = (2, 2)$



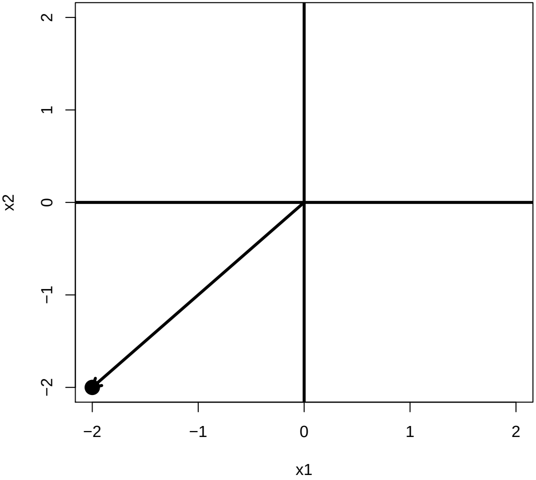
Vectors to Draw a Line

Suppose $\boldsymbol{w}_1 = (1, 1)$ $\frac{1}{2}\boldsymbol{w}_1 = (1/2, 1/2)$



Vectors to Draw a Line

Suppose $\boldsymbol{w}_1 = (1, 1) \quad -2\boldsymbol{w}_1 = (-2, -2)$



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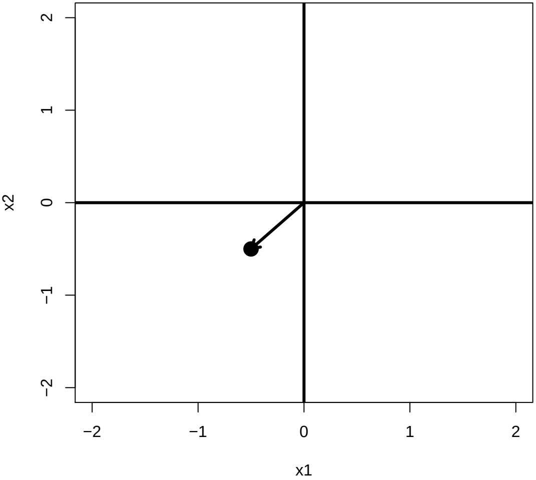
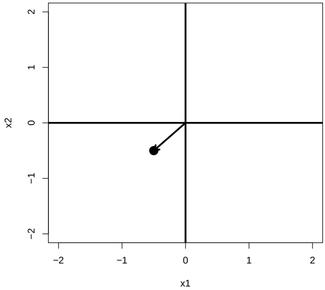


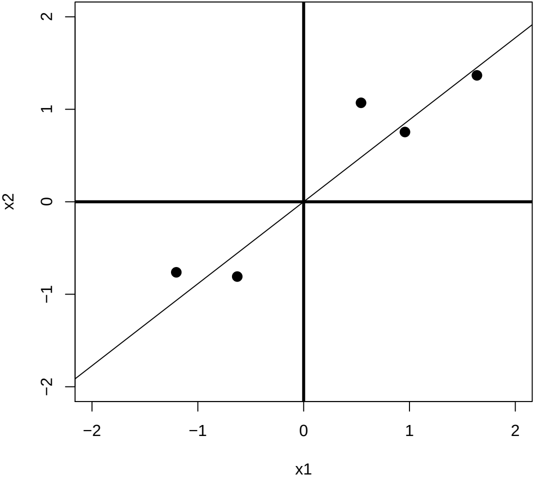
Vectors to Draw a Line

Suppose $\boldsymbol{w}_1 = (1, 1)$

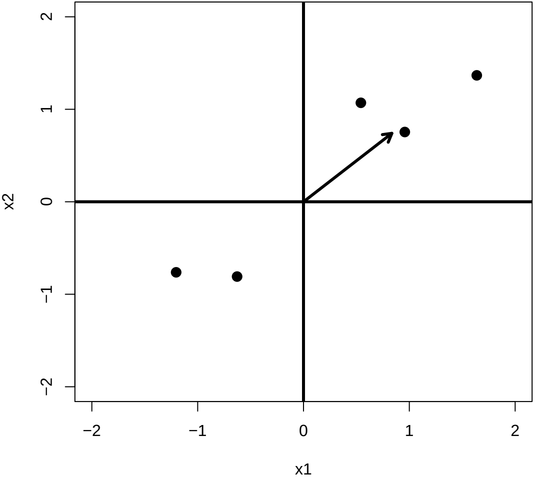
$$z_i = \text{amount we shrink/flip } \boldsymbol{w}_1 \text{ to approximate point } i.$$



$\boldsymbol{w}_1 = (0.75, 0.66)$



$\boldsymbol{w}_1 = (0.75, 0.66)$ $z_1 = 1.12$



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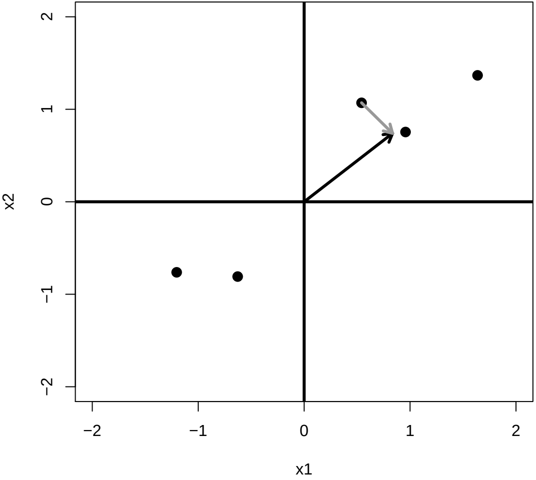

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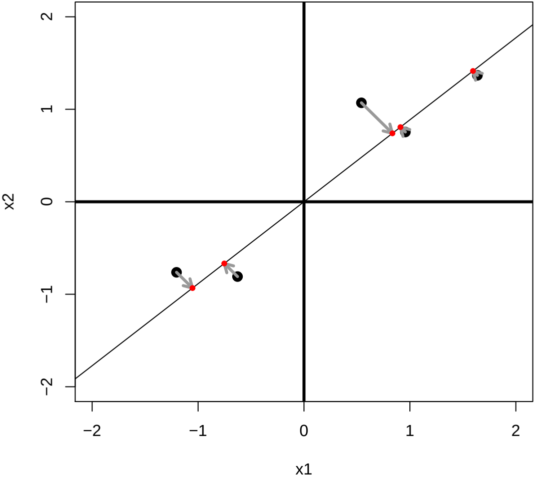

Algebraic Representation

$$\boldsymbol{x}_i = z_i \boldsymbol{w}_1 + \boldsymbol{e}_i$$

Algebraic Representation

$$\begin{aligned} \boldsymbol{x}_i &= z_i \boldsymbol{w}_1 + \boldsymbol{e}_i \\ (x_{i1}, x_{i2}) &= (z_i w_{11} + e_{i1}, z_i w_{12} + e_{i2}) \end{aligned}$$

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$$\text{error} = \frac{1}{N} \sum_{i=1}^{N} ((x_{i1}, x_{i2}) - z_i(w_{11}, w_{12}))^{'} ((x_{i1}, x_{i2}) - z_i(w_{11}, w_{12}))$$

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&= \frac{1}{N} \sum_{i=1}^{N} (x_{i1} - z_i w_{11})^2 + (x_{i2} - z_i w_{12})^2
\end{aligned}
$$

Three Dimensional Approximation

$$
\begin{aligned}
\boldsymbol{x}_1 &= (0.09, -1.02, -0.10) \\
\boldsymbol{x}_2 &= (0.09, 1.41, 0.67) \\
\boldsymbol{x}_3 &= (-0.81, -1.46, -0.54) \\
\boldsymbol{x}_4 &= (1.43, 0.26, 0.61) \\
\boldsymbol{x}_5 &= (1.23, 0.87, 1.33
\end{aligned}
$$

Find $\boldsymbol{w}_1 = (w_{11}, w_{12}, w_{13})$ and z_i to provide best one dimensional approximation.

Three-Dimensional Visualization

Three-Dimensional Visualization

$\boldsymbol{w}_1 = (0.48, 0.75, 0.46)$

$$\boldsymbol{x}_i = z_i \boldsymbol{w}_1 + \boldsymbol{e}_i$$

$$
\begin{aligned}
\boldsymbol{x}_i &= z_i \boldsymbol{w}_1 + \boldsymbol{e}_i \\
(x_{i1}, x_{i2}, x_{i3}) &= (z_i w_{11} + e_{i1}, z_i w_{12} + e_{i2}, z_i w_{13} + e_{i3})
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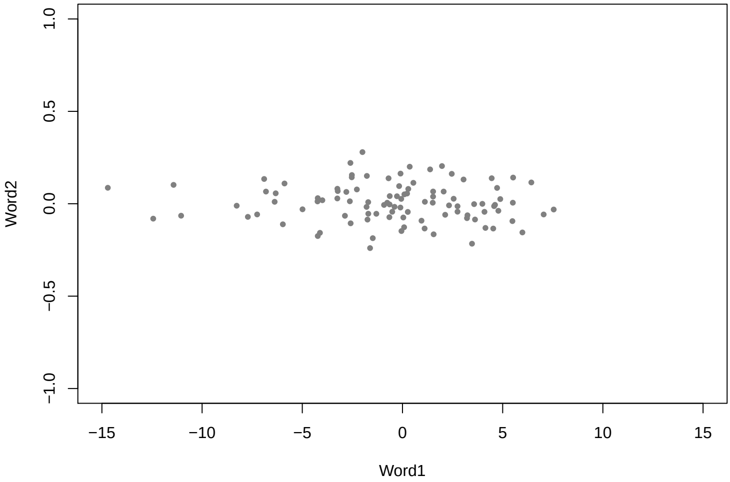
& \quad ((x_{i1}, x_{i2}, x_{i3}) - z_i (w_{11}, w_{12}, w_{13})) \\
&= \frac{1}{N} \sum_{i=1}^{N} (x_{i1} - z_i w_{11})^2 + (x_{i2} - z_i w_{12})^2 + (x_{i3} - z_i w_{13})^2
\end{aligned}
$$

Principal Component Analysis



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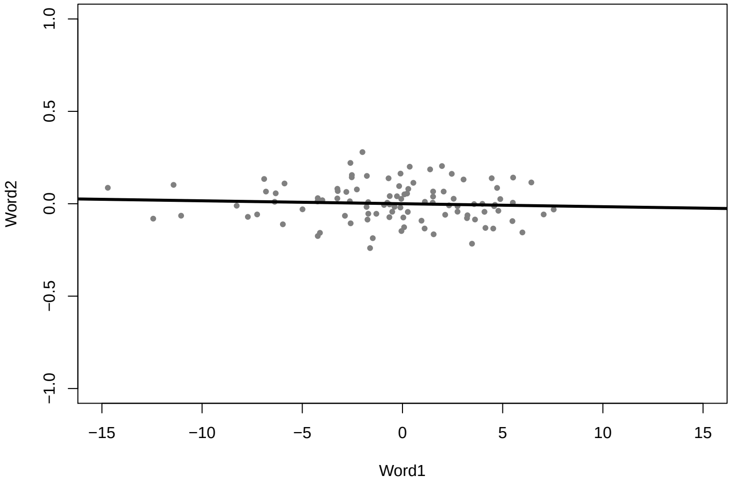

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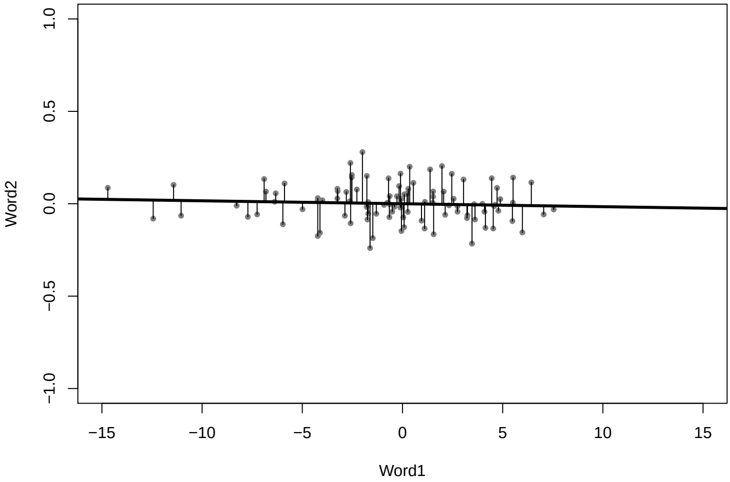

PCA Output

$$\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iJ})$$

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$$\boldsymbol{z}_i = (z_{1i}, z_{2i}, \ldots, z_{Ki})$$

An Introduction to Eigenvectors, Values, and Diagonalization

Definition

Suppose $\boldsymbol{A}$ *is an* $N \times N$ *matrix and* λ *is a scalar.*
If

$$\boldsymbol{A}\boldsymbol{x} = \lambda \boldsymbol{x}$$

Then $\boldsymbol{x}$ *is an eigenvector and* λ *is the associated eigenvalue*

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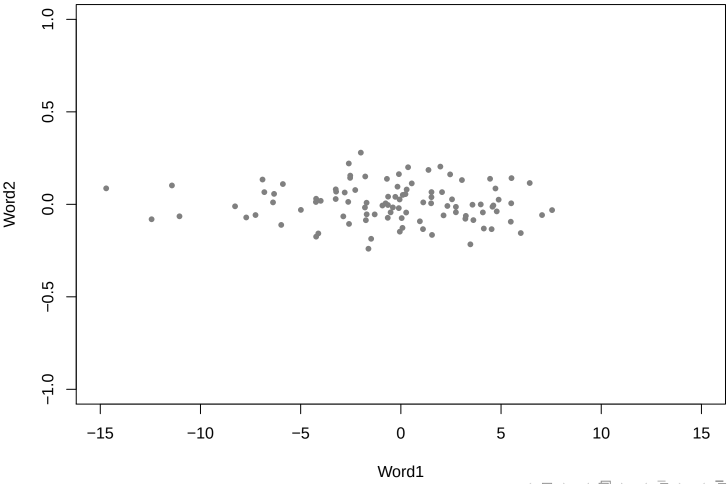
Finding a Lower Dimensional Space (Manifold Learning)



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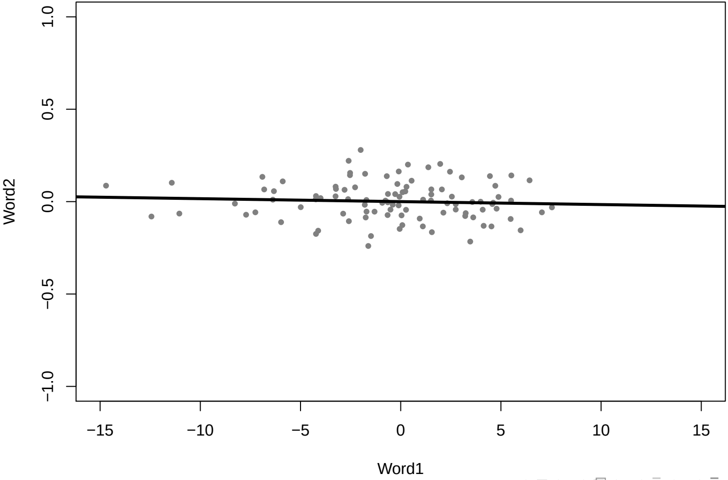

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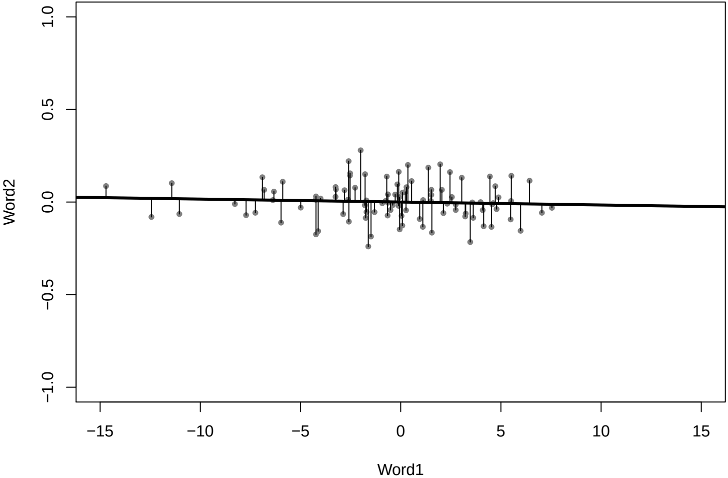

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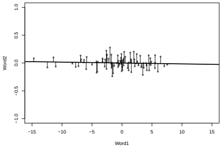
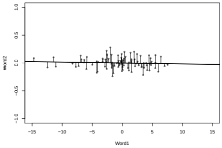


Original data:

Finding a Lower Dimensional Space (Manifold Learning)



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$$\boldsymbol{x}_i \quad = \quad (x_{i1}, x_{i2})$$

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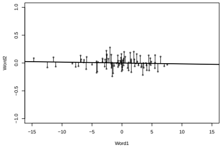

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Which we approximate with

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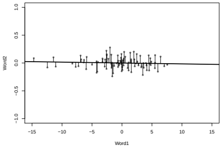

Original data:

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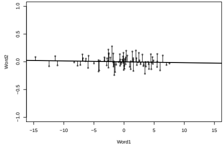
Which we approximate with

$$\begin{aligned}\tilde{\boldsymbol{x}}_i &= z_i \boldsymbol{w}_1 \\ &= z_i(w_{11}, w_{12})\end{aligned}$$

Finding a Lower Dimensional Space (Manifold Learning)



Original data $\boldsymbol{x}_i \in \Re^J$

$$\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iJ})$$

Which we approximate with $L \leq J$ weights z_{il} and vectors $\boldsymbol{w}_l \in \Re^J$

$$\tilde{\boldsymbol{x}}_i = z_{i1}\boldsymbol{w}_1 + z_{i2}\boldsymbol{w}_2 + \ldots + z_{iL}\boldsymbol{w}_L$$

Define $\boldsymbol{\theta} = (\underbrace{\boldsymbol{Z}}_{N \times L}, \underbrace{\boldsymbol{W}_L}_{L \times J})$

Principal Component Analysis $\rightsquigarrow$ Objective function

Consider 1-dimensional case ($L = 1$), centered data, and $||\boldsymbol{w}_1|| = 1$.

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$$f(\boldsymbol{\theta}, \boldsymbol{X}) = \frac{1}{N}\sum_{i=1}^{N} ||\boldsymbol{x}_i - z_{i1}\boldsymbol{w}_1||^2$$

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$$
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&= \frac{1}{N}\sum_{i=1}^{N} (\boldsymbol{x}_i - z_{i1}\boldsymbol{w}_1)'(\boldsymbol{x}_i - z_{i1}\boldsymbol{w}_1)
\end{aligned}
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$$

$\boldsymbol{w}_1^{'}\boldsymbol{w}_1 = 1$

Principal Component Analysis $\rightsquigarrow$ Optimization

Optimization:

Principal Component Analysis $\rightsquigarrow$ Optimization

Optimization:

$$\frac{\partial f(\boldsymbol{\theta}, \boldsymbol{X})}{\partial z_{i1}} \quad = \quad -\frac{2\boldsymbol{w}_1^{'}\boldsymbol{x}_i + 2z_{i1}}{N}$$

Principal Component Analysis⇝ Optimization

Optimization:

$$\frac{\partial f(\boldsymbol{\theta}, \boldsymbol{X})}{\partial z_{i1}} = -\frac{2\boldsymbol{w}_1^{'}\boldsymbol{x}_i + 2z_{i1}}{N}$$

$$0 = -\frac{2\boldsymbol{w}_1^{'}\boldsymbol{x}_i + 2z_{i1}^{*}}{N}$$

Principal Component Analysis$\leadsto$ Optimization

Optimization:

$$
\begin{aligned}
\frac{\partial f(\boldsymbol{\theta}, \boldsymbol{X})}{\partial z_{i1}} &= -\frac{2\boldsymbol{w}_1^{'}\boldsymbol{x}_i + 2z_{i1}}{N} \\
0 &= -\frac{2\boldsymbol{w}_1^{'}\boldsymbol{x}_i + 2z_{i1}^{*}}{N} \\
z_{i1}^{*} &= \boldsymbol{w}_1^{'}\boldsymbol{x}_i
\end{aligned}
$$

Principal Component Analysis$\rightsquigarrow$ Optimization

Substituting in z_{i1}^*

Principal Component Analysis$\rightsquigarrow$ Optimization

Substituting in z_{i1}^{*}

$$= \quad \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i - z_{i1}^{*}\boldsymbol{w}_1)^{'}(\boldsymbol{x}_i - z_{i1}^{*}\boldsymbol{w}_1)$$

Principal Component Analysis$\rightsquigarrow$ Optimization

Substituting in z_{i1}^{*}

$$
\begin{aligned}
&= \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i - z_{i1}^{*}\boldsymbol{w}_1)^{'}(\boldsymbol{x}_i - z_{i1}^{*}\boldsymbol{w}_1) \\
&= \frac{1}{N}\sum_{i=1}^{N}(\underbrace{\boldsymbol{x}_i^{'}\boldsymbol{x}_i}_{\text{Constant}} - 2z_{i1}^{*}\underbrace{\boldsymbol{w}_1^{'}\boldsymbol{x}_i}_{z_{i1}^{*}} + (z_{i1}^{*})^2\underbrace{\boldsymbol{w}_1^{'}\boldsymbol{w}_1}_{1})
\end{aligned}
$$

Principal Component Analysis$\rightsquigarrow$ Optimization

Substituting in z_{i1}^*

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&= -\boldsymbol{w}_1^{'}\boldsymbol{\Sigma}\boldsymbol{w}_1
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$$

Principal Component Analysis⇝ Optimization

$$= \quad -\boldsymbol{w}_1^{'}\boldsymbol{\Sigma}\boldsymbol{w}_1$$

Principal Component Analysis$\rightsquigarrow$ Optimization

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\end{aligned}$$

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Minimize reconstruction error

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Minimize reconstruction error $\rightsquigarrow$ maximize variance of projected data

Principal Component Analysis$\rightsquigarrow$ Optimization

Maximize variance, subject to constraints

Principal Component Analysis$\rightsquigarrow$ Optimization

Maximize variance, subject to constraints

$$g(\boldsymbol{z}^*, \boldsymbol{w}_1, \boldsymbol{X}) \quad = \quad \boldsymbol{w}_1^{'}\boldsymbol{\Sigma}\boldsymbol{w}_1 - \lambda_1(\boldsymbol{w}_1^{'}\boldsymbol{w}_1 - 1)$$

Principal Component Analysis$\rightsquigarrow$ Optimization

Maximize variance, subject to constraints

$$
\begin{aligned}
g(\boldsymbol{z}^{*}, \boldsymbol{w}_1, \boldsymbol{X}) &= \boldsymbol{w}_1^{'}\boldsymbol{\Sigma}\boldsymbol{w}_1 - \lambda_1(\boldsymbol{w}_1^{'}\boldsymbol{w}_1 - 1) \\
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$\boldsymbol{w}_1^*$ = Eigenvector of $\boldsymbol{\Sigma}$

Principal Component Analysis$\leadsto$ Optimization

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$\boldsymbol{w}_1^* =$ Eigenvector of $\boldsymbol{\Sigma}$ (!!!!!!)
We want $\boldsymbol{w}_1$ to maximize variance and

$$\boldsymbol{w}_1^{'}\boldsymbol{\Sigma}\boldsymbol{w}_1 = \lambda_1$$

So $\boldsymbol{w}_1$ is eigenvector associated with the largest eigenvalue λ_1

An Introduction to Eigenvectors, Values, and Diagonalization

Theorem

Suppose $\boldsymbol{A}$ is an invertible $N \times N$ matrix with N linearly independent eigenvectors. Then we can write $\boldsymbol{A}$ as,

$$\boldsymbol{A} = \boldsymbol{W}^{'} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix} \boldsymbol{W}$$

where $\boldsymbol{W} = (\boldsymbol{w}_1, \boldsymbol{w}_2, \ldots, \boldsymbol{w}_N)$ is an $N \times N$ matrix with the N eigenvectors as column vectors.

An Introduction to Eigenvectors, Values, and Diagonalization

Definition

Suppose A is a covariance matrix. Then, we can write A as

$$\boldsymbol{A} = \boldsymbol{W}^{'} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_N \end{pmatrix} \boldsymbol{W}$$

Where $\lambda_1 > \lambda_2 > \ldots > \lambda_N \geq 0$.
We will call $\boldsymbol{w}_1$ the first eigenvector, $\boldsymbol{w}_2$ the second eigenvector, ..., $\boldsymbol{w}_j$ the j^{th} eigenvector.

Back to Principal Components

Theorem

Suppose we want to approximate N observations $\boldsymbol{x}_i \in \Re^J$ with $L < J$ orthogonal-unit length vectors $\boldsymbol{w}_l \in \Re^J$ with associated scores z_{il} to minimize reconstruction error:

Back to Principal Components

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Suppose we want to approximate N *observations* $\boldsymbol{x}_i \in \Re^J$ *with* $L < J$ *orthogonal-unit length vectors* $\boldsymbol{w}_l \in \Re^J$ *with associated scores* z_{il} *to minimize reconstruction error:*

$$f(\boldsymbol{X}, \boldsymbol{\theta}) = \frac{1}{N}\sum_{i=1}^{N} ||\boldsymbol{x}_i - \sum_{l=1}^{L} z_{il}\boldsymbol{w}_l||^2$$

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The optimal solution sets each $\boldsymbol{w}_l$ *to be the* l^{th} *eigenvector of the empirical covariance matrix.*

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$$\boldsymbol{x}_i^L = (\boldsymbol{w}_1^{'} \boldsymbol{x}_i, \boldsymbol{w}_2^{'} \boldsymbol{x}_i, \ldots, \boldsymbol{w}_L^{'} \boldsymbol{x}_i)$$

Application of Principal Components in R

Consider press releases from 2005 US Senators

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Define $\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iJ})$ as the rate senator i uses J words.

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Application of Principal Components in R

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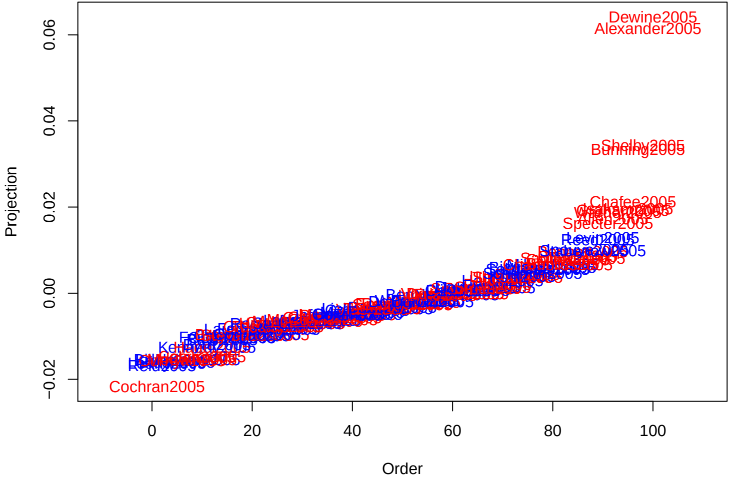
$$x_{ij} = \frac{\text{No. Times } i \text{ uses word } j}{\text{No. words } i \text{ uses}}$$

`dtm`: 100×2796 matrix containing word rates for senators
`prcomp(dtm)` applies principal components

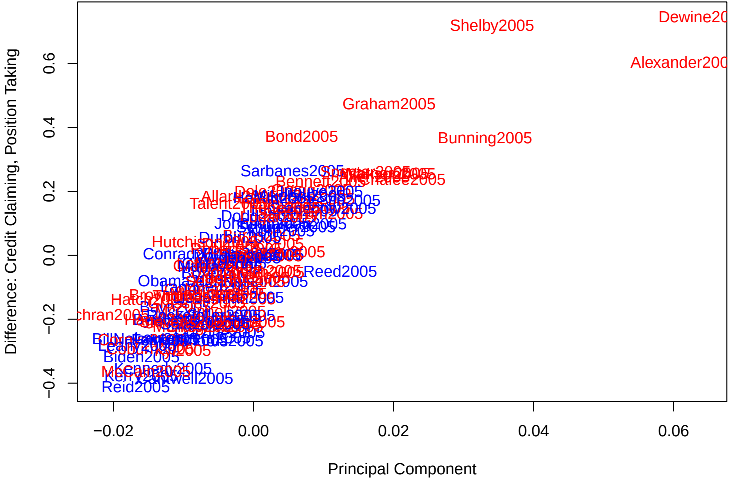
```
load("SenateTDM.RData")
dtm<- t(tdm)
for(z in 1:100){
dtm[z,]<- dtm[z,]/sum(dtm[z,])
}

store<- prcomp(dtm, scale = F)
scores<- store$x[,1]
```

Application of Principal Components in R



Application of Principal Components in R



Probabilistic Principal Components (Tipping and Bishop 1999)

$$
\begin{aligned}
\boldsymbol{x}|\boldsymbol{w} &\sim \text{Multivariate Normal}(\boldsymbol{Z}\boldsymbol{W} + \boldsymbol{\mu}, \sigma^2 \boldsymbol{I}) \\
\boldsymbol{w} &\sim \text{Multivariate Normal}(\boldsymbol{0}, \boldsymbol{I}) \\
\boldsymbol{x} &\sim \text{Multivariate Normal}(\boldsymbol{\mu}, \boldsymbol{\Sigma}) \\
\boldsymbol{\Sigma} &= \boldsymbol{W}\boldsymbol{W}^{'} + \sigma^2 \boldsymbol{I}
\end{aligned}
$$

1) Log-likelihood $\rightsquigarrow$ straightforward
2) Optimization via EM-Algorithm
3) Corresponds to traditional PCA is $\lim_{\sigma^2} \rightarrow 0$
4) Closely related to Factor analysis.

How do we select the number of dimensions L? $\rightsquigarrow$ Model

We want to minimize reconstruction error

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Simplifying:

$$\text{error}(L) \quad = \quad \frac{1}{N}\sum_{i=1}^{N}(\boldsymbol{x}_i - \sum_{l=1}^{L} z_{il}\boldsymbol{w}_l)'(\boldsymbol{x}_i - \sum_{l=1}^{L} z_{il}\boldsymbol{w}_l)$$

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Four types of terms:

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Four types of terms:

1) $\boldsymbol{x}_i^{'}\boldsymbol{x}_i$
2) $z_{ij}z_{ik}\boldsymbol{w}_j^{'}\boldsymbol{w}_k = z_{ij}z_{ik}0 = 0$
3) $z_{ij}z_{ij}\boldsymbol{w}_j^{'}\boldsymbol{w}_j = z_{ij}^2$
4) $\boldsymbol{x}_i^{'}\sum_{l=1}^{L} z_{il}\boldsymbol{w}_l = \sum_{l=1}^{L} z_{il}^2$

How do we select the number of dimensions L? $\rightsquigarrow$ Model

$$\text{error}(L) \quad = \quad \frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{x}_i^{'}\boldsymbol{x}_i - \sum_{l=1}^{L} z_{il}^2\right)$$

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$$0 \quad = \quad \frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{x}_i^{'}\boldsymbol{x}_i\right) - (\sum_{l=1}^{L}\lambda_l + \sum_{j=L+1}^{J}\lambda_l)$$

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\sum_{j=L+1}^{J}\lambda_l &= \frac{1}{N}\sum_{i=1}^{N}\left(\boldsymbol{x}_i^{'}\boldsymbol{x}_i\right) - \sum_{l=1}^{L}\lambda_l \\
\sum_{j=L+1}^{J}\lambda_l &= \text{error}(L)
\end{aligned}$$

How do we select the number of dimensions L? $\rightsquigarrow$ Model

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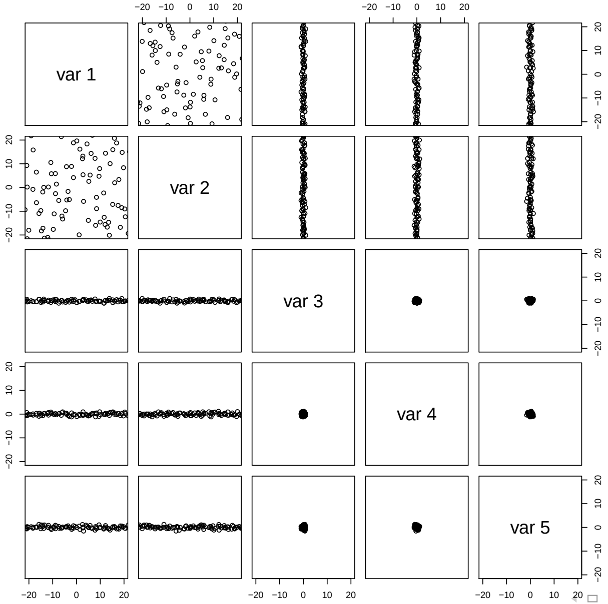
Recommendation $\rightsquigarrow$ look for Elbow

How do we select the number of dimensions L? $\rightsquigarrow$ Model



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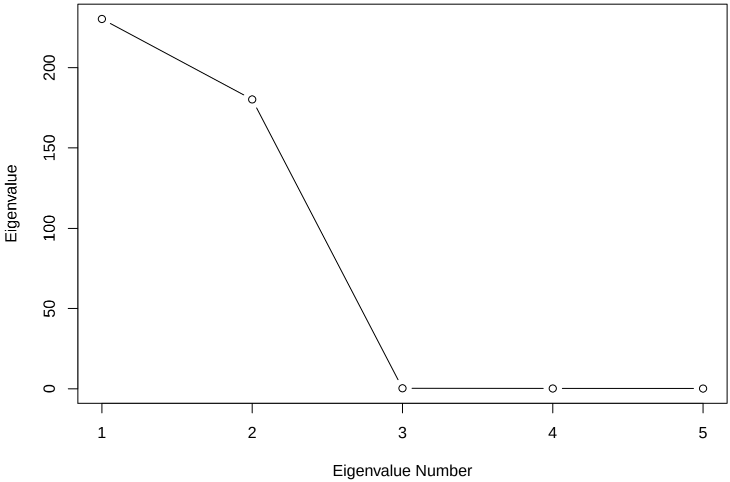

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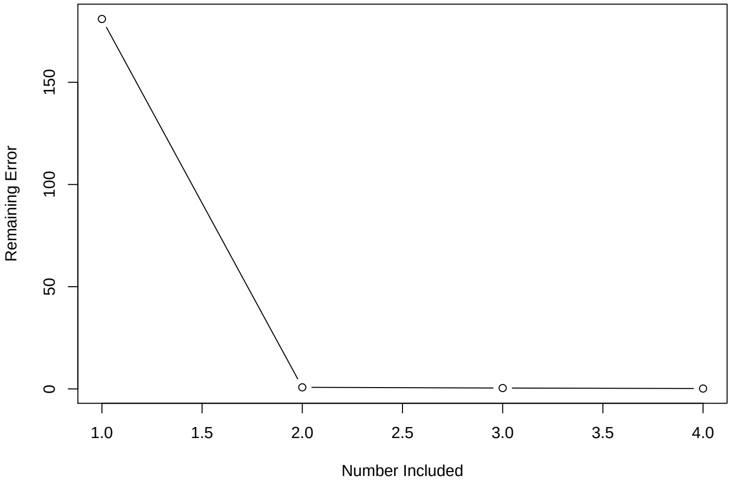

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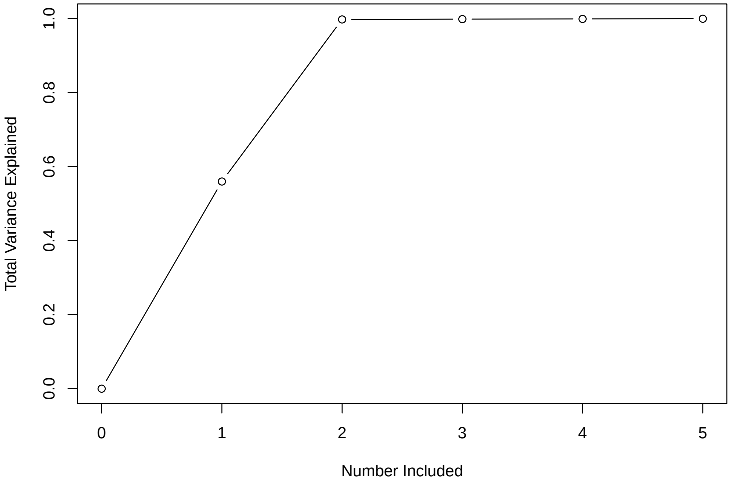

Non-model based evaluations: What's the point?

What is the true underlying dimensionality of $\boldsymbol{X}$?

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Mathematical model $\rightsquigarrow$ insufficient to make modeling decision

Appendix

Kernel Principal Component Analysis

Define a Kernel $(N \times N)$ matrix as:

$$\boldsymbol{K} \quad = \quad \begin{pmatrix} k(\boldsymbol{x}_1, \boldsymbol{x}_1) & k(\boldsymbol{x}_1, \boldsymbol{x}_2) & \dots & k(\boldsymbol{x}_1, x_N) \\ k(\boldsymbol{x}_2, \boldsymbol{x}_1) & k(\boldsymbol{x}_2, \boldsymbol{x}_2) & \dots & k(\boldsymbol{x}_2, \boldsymbol{x}_N) \\ \vdots & \vdots & \ddots & \vdots \\ k(\boldsymbol{x}_N, \boldsymbol{x}_1) & k(\boldsymbol{x}_N, \boldsymbol{x}_2) & \dots & k(\boldsymbol{x}_N, \boldsymbol{x}_N) \end{pmatrix}$$

where $k(\cdot, \cdot)$ is a function that behaves like a similarity function.

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Compute PCA of $\boldsymbol{\Phi}$ from $\boldsymbol{\Phi}\boldsymbol{\Phi}^{'}$

Kernel PCA

PCA of $\boldsymbol{X}$

Kernel PCA

PCA of $\boldsymbol{X}$ Eigenvectors of $\boldsymbol{X}^{'}\boldsymbol{X}$ ($\frac{1}{N}$ doesn't affect eigenvectors)

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$$
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$$
\boldsymbol{w}_1 = \frac{1}{\sqrt{\lambda_1}}\boldsymbol{X}^{'}\boldsymbol{u}_1
$$

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$\boldsymbol{K} = \mathbf{\Phi}\mathbf{\Phi}^{'}$ (assume $\mathbf{\Phi}$ is mean-centered, for now)

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We can obtain $\boldsymbol{u}_1$ and λ_1 from $\boldsymbol{K}$.

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Kernel PCA

Center $\boldsymbol{K}$?
Use centering matrix $\boldsymbol{H}$

$$\begin{aligned}
\boldsymbol{H} &= \boldsymbol{I}_N - \frac{(\mathbf{1}_N \mathbf{1}_N^{'})}{N} \\
\boldsymbol{K}_{\text{center}} &= \boldsymbol{H}\boldsymbol{K}\boldsymbol{H}
\end{aligned}$$

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Spirling and Indian Treaties

How do we preserve word order and semantic language?

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$\boldsymbol{\phi}(\boldsymbol{x}_i) \approx \binom{32}{5}$ element long count vector

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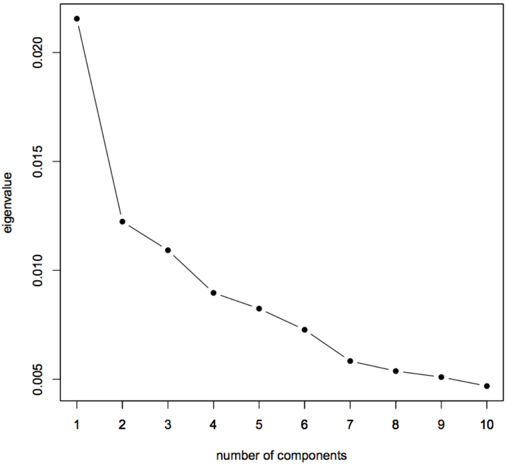

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