Political Methodology III: Model Based Inference

Justin Grimmer

Associate Professor
Department of Political Science
Stanford University

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Model Based Inference

1) Likelihood inference
2) Logit/Probit
3) Ordered Probit
4) Choice Models:
    - Multinomial Logit
        a) DGP
        b) IIA
        c) Optimization
        d) Quantities of Interest
        e) Interpretation
    - Multinomial Probit
        a) DGP
        b) No IIA, But No Likelihood
        c) Quantities of Interest
        d) Interpretation

Modeling Categorical Outcomes

Categorical outcome (a.k.a. discrete choice) variable:

$$Y_i \in \{1, 2, ..., J\}$$

Example: Multiparty elections

- Voters choose from more than two parties/candidates
- $1 =$ Bush; $2 =$ Clinton; $3 =$ Perot
- $1 =$ Trump; $2 =$ Cruz; $3 =$ Kasich; $4 =$ Rubio; $5 =$ Bush ...

Possible research questions:

- How are candidate characteristics associated with probability of voting? (e.g. past experience, campaign spending)
- What kind of voters tend to choose Perot instead of Bush or Clinton? (e.g. age, race, ideology)
- How would vote shares have changed if Perot had not run?

Goal: Model probability of choosing $Y_i = j$ as function of predictors

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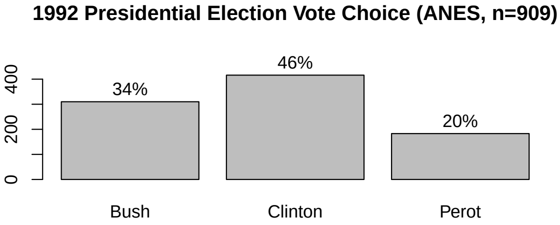
Example: 1992 U.S. Presidential Election

Alvarez and Nagler (1995):

- Y_i: Vote choice in the 1992 U.S. presidential election ($1 =$ Clinton, $2 =$ Bush, $3 =$ Perot)



- Two types of predictors:
  - Voter-specific (V_i): age, gender, education, party, opinions, etc.
  - Candidate-varying (X_{ij}): ideological distance between voter i and candidate j

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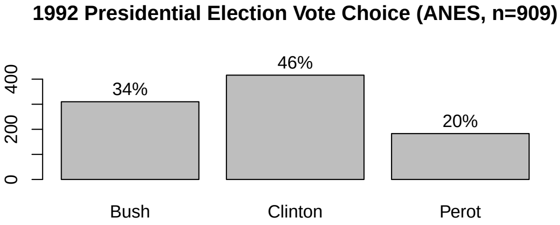

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Definition

Suppose we observe a trial, i, which might result in J outcomes.
And that P(outcome $= i) = \pi_i$
$Y_i = j$ if outcome j occurred and 0 otherwise.
Then Y_i follows a multinomial distribution, with

$$
\begin{aligned}
p(y_i) &= \pi_1^{I(y_i=1)} \pi_2^{I(y_i=2)} \ldots \pi_J^{I(y_i=J)} \\
p(y_i) &= \prod_{j=1}^{J} \pi_j^{I(y_j=j)}
\end{aligned}
$$

Equivalently, we'll write

$$
\begin{aligned}
Y_i &\sim \textit{Multinomial}(1, \boldsymbol{\pi}) \\
Y_i &\sim \textit{Categorical}(\boldsymbol{\pi})
\end{aligned}
$$

Multinomial Distribution

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$$\mathsf{E}[I(Y_i = j)] = \pi_j$$

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\mathsf{E}[I(Y_i = j)] &= \pi_j \\
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\mathsf{Var}[I(Y_i = j)] &= \pi_j(1 - \pi_j) \\
\mathsf{Cov}(I(Y_i = j), I(Y_i = k)] &= -\pi_j \pi_k
\end{aligned}$$

Vote Choice: Parameters of Multinomial Distribution

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&= \prod_{i=1}^{N} f(Y_i|\boldsymbol{\pi}) \\
&= \prod_{i=1}^{N}\prod_{j=1}^{J} \pi_j^{I(Y_i=j)}
\end{aligned}$$

Vote Choice

Obtaining maximum-likelihood estimates:

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\ell(\boldsymbol{\pi}|\boldsymbol{Y}) = \sum_{i=1}^{N}\sum_{j=1}^{J} I(Y_i = j)\log \pi_j + \lambda(\sum_{j=1}^{J} \pi_j - 1)
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\ell(\boldsymbol{\pi}|\boldsymbol{Y}) &= \sum_{i=1}^{N}\sum_{j=1}^{J} I(Y_i = j)\log \pi_j + \lambda(\sum_{j=1}^{J}\pi_j - 1) \\
\frac{\partial \ell(\boldsymbol{\pi}|\boldsymbol{Y})}{\partial \pi_1} &= \frac{\sum_{i=1}^{N} I(Y_i = 1)}{\pi_1} + \lambda \\
\frac{\partial \ell(\boldsymbol{\pi}|\boldsymbol{Y})}{\partial \pi_2} &= \frac{\sum_{i=1}^{N} I(Y_i = 2)}{\pi_2} + \lambda \\
\frac{\partial \ell(\boldsymbol{\pi}|\boldsymbol{Y})}{\partial \pi_3} &= \frac{\sum_{i=1}^{N} I(Y_i = 3)}{\pi_3} + \lambda \\
\vdots \quad &\vdots \quad \vdots \\
\frac{\partial \ell(\boldsymbol{\pi}|\boldsymbol{Y})}{\partial \pi_J} &= \frac{\sum_{i=1}^{N} I(Y_i = J)}{\pi_J} + \lambda \\
\frac{\partial \ell(\boldsymbol{\pi}|\boldsymbol{Y})}{\partial \lambda} &= \sum_{j=1}^{J}\pi_j - 1
\end{aligned}
$$

Set equal to zero
Solve for $\boldsymbol{\pi}^* = (\pi_1^*, \pi_2^*, \ldots, \pi_J^*)$

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Note that: $\sum_{i=1}^{N} \sum_{j=1}^{J} I(y_i = j) = N$

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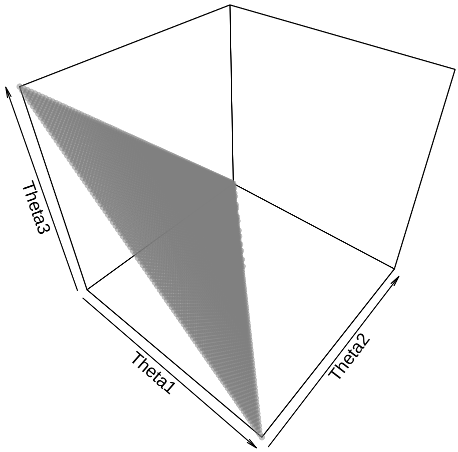
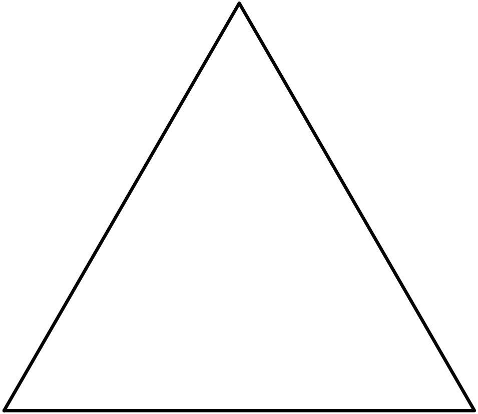
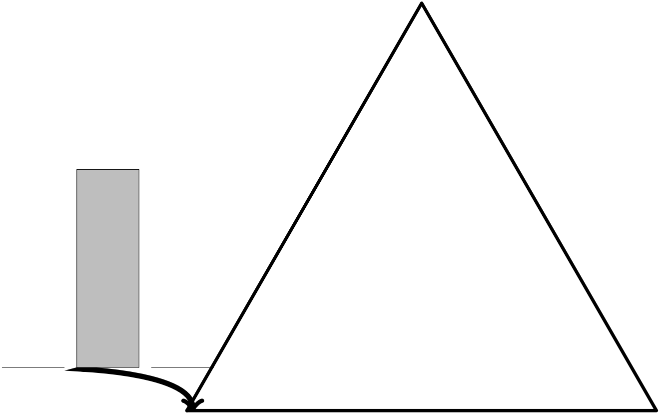
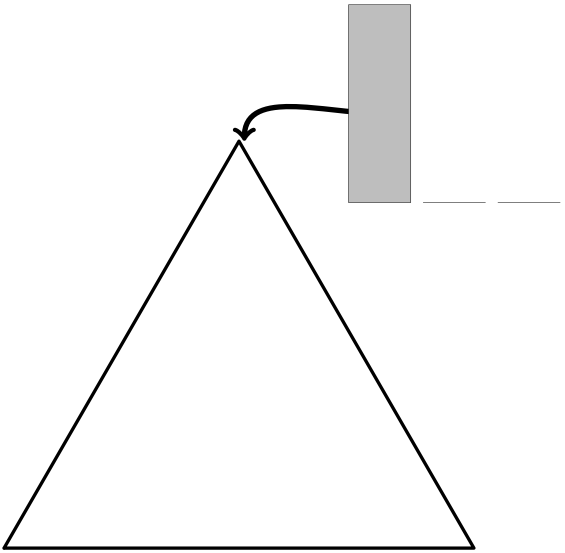
Maximum likelihood estimates$\rightsquigarrow$ Average proportion of time candidate is chosen

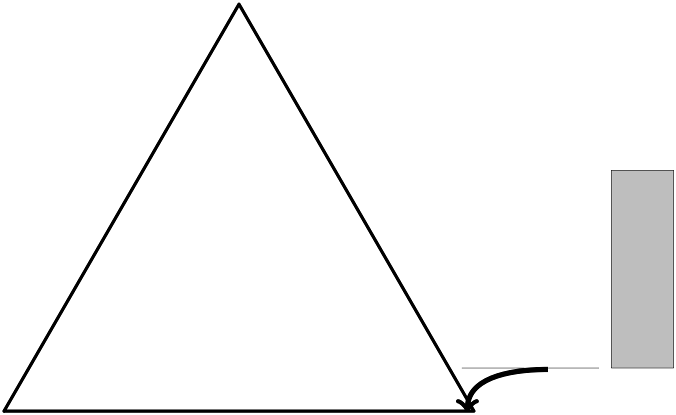
Three Candidate Choice

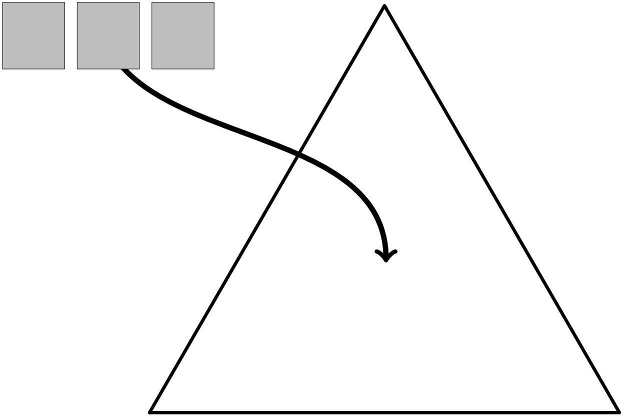

Theta3
Theta1
Theta2











Modeling Choices with Covariates

Multinomial Logit Model

- Generalize the logit model to more than two choices
- The multinomial logit model (MNL):

$$\pi_{ij} = \Pr(Y_i = j \mid V_i) = \frac{\exp(V_i' \delta_j)}{\sum_{k=1}^{J} \exp(V_i' \delta_k)},$$

  where V_i = individual-specific characteristics of unit i (and an intercept)

- Note that $\sum_{j=1}^{J} \pi_{ij} = 1$
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$$\begin{aligned}
\pi_{i1} &= \frac{\exp(V_i^{'}\delta_1)}{\exp(V_i^{'}\delta_1) + 1} \\
&= \frac{1}{1 + \exp(-V_i^{'}\delta_1)}
\end{aligned}$$

Conditional Logit Model

- We can also incorporate alternative-varying predictors X_{ij}
- The conditional logit (CL) model:

$$\pi_{ij} \ = \ \Pr(Y_i = j \mid X_{ij}) \ = \ \frac{\exp(X_{ij}'\beta)}{\sum_{k=1}^{J}\exp(X_{ik}'\beta)}$$

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MNL as a Special Case of CL

- MNL can be written using CL: Create a set of artificial alternative-varying regressors for each V_i:

$$X_{i1} = \begin{pmatrix} V_i \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad X_{i2} = \begin{pmatrix} 0 \\ V_i \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad X_{iJ} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ V_i \end{pmatrix}$$

- Set the element of β for X_{ij} to δ_j and you get the MNL model
- δ_1 must be set to zero for identifiability
- Thus we can write both models (and their mixture) simply as CL:

$$\pi_{ij} = \Pr(Y_i = j \mid X) = \frac{\exp(X'_{ij}\beta)}{\sum_{k=1}^{J} \exp(X'_{ik}\beta)}$$

- We use the names CL and MNL interchangeably from here on

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Predictor Types and Data Formats

Discrete choice data usually come in one of the two formats:

1. Wide format: N rows, $\#V + J \cdot \#X$ predictors

```
 choice women educ idist.Clinton idist.Bush idist.Perot
   Bush     1    3        4.0804     0.1024      0.2601
   Bush     1    4        4.0804     0.1024      0.2601
Clinton     1    2        1.0404     1.7424      0.2401
   Bush     0    6        0.0004     5.3824      2.2201
Clinton     1    3        0.9604    11.0220      6.2001 ...
```

2. Long format: NJ rows, $\#V + \#X$ predictors

```
chid     alt choice women educ  idist
   1    Bush   TRUE     1    3 0.1024
   1 Clinton  FALSE     1    3 4.0804
   1   Perot  FALSE     1    3 0.2601
   2    Bush   TRUE     1    4 0.1024
   2 Clinton  FALSE     1    4 4.0804
   2   Perot  FALSE     1    4 0.2601 ...
```

Predictor Types and Data Formats

- Use `reshape` to change between wide and long
- Some estimation functions (e.g. `mlogit`) can take both formats

Latent Variable Interpretation

- Recall the random utility model:

$$Y_{ij}^* \; = \; X_{ij}^{'}\beta + \epsilon_{ij},$$

where $\begin{cases} Y_{ij}^* & = & \text{latent utility from choosing } j \text{ for } i \\ \epsilon_{ij} & = & \text{stochastic component of the utility} \end{cases}$

- Assume that voter chooses the most preferred candidate, i.e.,

$$Y_i = j \quad \text{if} \quad Y_{ij}^* \geq Y_{ij'}^* \quad \text{for any} \quad j' \in \{1, ..., J\}$$

- Assuming $\epsilon_{ij} \sim_{\text{iid}}$ type I extreme value distribution, this setup implies MNL (McFadden 1974)
- Proof for $J = 2$:

$$\begin{aligned} \Pr(Y_i = 1 \mid X) &= \Pr(Y_{i1}^* \geq Y_{i2}^* \mid X) \\ &= \Pr\left(\epsilon_{i2} - \epsilon_{i1} \leq (X_{i1} - X_{i2})'\beta\right) \\ &= \frac{\exp\left((X_{i1} - X_{i2})'\beta\right)}{1 + \exp\left((X_{i1} - X_{i2})'\beta\right)} = \frac{\exp(X_{i1}'\beta)}{\exp(X_{i1}'\beta) + \exp(X_{i2}'\beta)} \end{aligned}$$

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$$Y_{ij}^* \;=\; X_{ij}^{'}\beta + \epsilon_{ij},$$

where $\begin{cases} Y_{ij}^* & = \text{ latent utility from choosing } j \text{ for } i \\ \epsilon_{ij} & = \text{ stochastic component of the utility} \end{cases}$

- Assume that voter chooses the most preferred candidate, i.e.,

$$Y_i = j \quad \text{if} \quad Y_{ij}^* \geq Y_{ij'}^* \quad \text{for any} \quad j' \in \{1, ..., J\}$$

- Assuming $\epsilon_{ij} \sim_{\text{iid}}$ type I extreme value distribution, this setup implies MNL (McFadden 1974)

- Proof for $J = 2$:

$$\begin{aligned} \Pr(Y_i = 1 \mid X) &= \Pr(Y_{i1}^* \geq Y_{i2}^* \mid X) \\ &= \Pr\left(\epsilon_{i2} - \epsilon_{i1} \leq (X_{i1} - X_{i2})^{'}\beta\right) \\ &= \frac{\exp\left((X_{i1} - X_{i2})^{'}\beta\right)}{1 + \exp\left((X_{i1} - X_{i2})^{'}\beta\right)} = \frac{\exp(X_{i1}^{'}\beta)}{\exp(X_{i1}^{'}\beta) + \exp(X_{i2}^{'}\beta)} \end{aligned}$$

Latent Variable Interpretation

- Recall the random utility model:

$$Y_{ij}^* = X_{ij}^{'}\beta + \epsilon_{ij},$$

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Independence of Irrelevant Alternatives

$$\frac{\pi_{i1}}{\pi_{i2}}$$

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Relative Risk: does not depend on other choices

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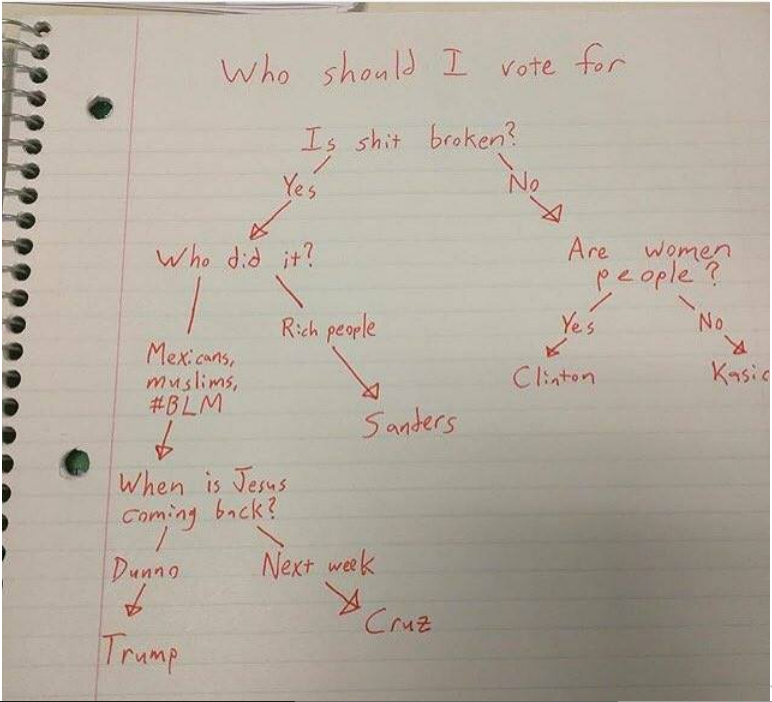
Relative Risk: does not depend on other choices
Multinomial Choice: series of pairwise comparisons


Who should I vote for
Is shit broken?
Yes
No
Who did it?
Are women people?
Yes
No
Clinton
Kasic
Mexicans, muslims, #BLM
Rich people
Sanders
When is Jesus coming back?
Dunno
Next week
Trump
Cruz

Independence of Irrelevant Alternatives (Red Bus, Blue Bus)

Trump, Cruz, and Sanders remaining candidates

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- 25% support Trump
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\text{Trump} &= 0 \\
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\Pr(\text{Sanders}) &= \frac{\exp(\log 2)}{\exp(\log 2) + \exp(0) + \exp(0)} = 2/(2+1+1) = 0.5
\end{aligned}
$$

Independence of Irrelevant Alternatives (Red Bus, Blue Bus)

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Or the model predicts voters choose Sanders $2:1$ over Trump(!) resulting in (66%,33%) split

IIA and Your Work

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What to do?

1) Alternative models: Multinomial Probit, Nested Logit
2) Conditional on included covariates (other information on individual voters would limit the problem)
3) Can overstate size of problem: estimation error (standard errors) often much larger than deviation

Estimation and Inference

- Likelihood for a random sample of size N:

$$
\begin{aligned}
L(\boldsymbol{\beta} \mid Y, X) &= \prod_{i=1}^{N}\prod_{j=1}^{J} \pi_{ij}^{I(Y_i = j)} \\
&= \prod_{i=1}^{N}\prod_{j=1}^{J} \left(\frac{\exp(X_{ij}^{'}\boldsymbol{\beta})}{\sum_{k=1}^{J}\exp(X_{ik}^{'}\boldsymbol{\beta})} \right)^{I(Y_i = j)} \\
\log L(\boldsymbol{\beta} \mid Y, X) &= \sum_{i=1}^{N}\sum_{j=1}^{J} I(Y_i = j) \left(X_{ij}^{'}\boldsymbol{\beta} - \log\left(\sum_{k=1}^{J} \exp(X_{ik}^{'}\boldsymbol{\beta}) \right) \right)
\end{aligned}
$$

Estimation and Inference

- Score:

$$s(\boldsymbol{\beta}|Y_i, \boldsymbol{X}_i) \;=\; \textstyle\sum_{j=1}^{J} I(Y_i = j)\left(X_{ij} - \bar{X}_i\right),$$

  where $\bar{X}_i$ is the weighted average of X_i, i.e. $\bar{X}_i = \sum_{j=1}^{J} \pi_{ij} X_{ij}$
- Solve $\sum_{i=1}^{N} s(\boldsymbol{\beta}|Y_i, \boldsymbol{X}_i) = 0$ (numerically) to get $\boldsymbol{\beta}^{*}_{MLE}$
- It can be shown that the log-likelihood is globally concave $\Rightarrow$ guaranteed convergence to the true (not local) MLE
- Information:

$$I_N(\boldsymbol{\beta}) \;=\; -\mathsf{E}[H(\boldsymbol{\beta})] \;=\; \textstyle\sum_{i=1}^{N}\sum_{j=1}^{J} \pi_{ij}\left(X_{ij} - \bar{X}_i\right)\left(X_{ij} - \bar{X}_i\right)'$$

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Estimation and Inference

$$\boldsymbol{\beta}^{*}_{MLE} \longrightarrow^{D} \text{Normal}\left(\boldsymbol{\beta}, I(\boldsymbol{\beta})^{-1}\right)$$

Interpreting MNL/CL Coefficients

In MNL/CL, β itself is not necessarily informative about the effect of X

1. The coefficients are all with respect to the baseline category
   $\longrightarrow$ Testing $\beta_j = 0$ does not generally make sense
   (unless comparison to the baseline is the goal)
2. Changing X_{ij} has impact on $\Pr(Y_i = k \mid X)$, $k \neq j$:
   - For individual-specific characteristics (V_i), even sign of δ_j may not agree with the direction of the change in response probability for j
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Compute a quantity that has a clear substantive interpretation!

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Calculating Quantities of Interest

1. Choice probability:

$$\pi_j(x) \;=\; \Pr(Y_i = j \mid X = x)$$

e.g. How likely is a female college-educated conservative Republican voter to vote for Perot?

2. Predicted vote share:

$$p_j(x_1) \;\equiv\; \mathsf{E}\left[I\left(\pi_j(X_{i1} = x_1, X_{i2}) \geq \pi_k(X_{i1} = x_1, X_{i2}) \text{ for all } k\right)\right]$$

where X_{i1} is the predictor(s) of interest and X_{i2} is all other predictors

e.g. What would Perot's vote share be if all voters supported abortion?

3. Average partial (treatment) effects:

$$\tau_{jk} \;=\; \mathsf{E}\left[\pi_j(T_{ik} = 1, T_{i*}, W_i) - \pi_j(T_{ik} = 0, T_{i*}, W_i)\right]$$

where T_{ik} is treatment on candidate k, T_{i*} is treatment on others, W_i is pre-treatment covariates

- "Direct effect" if $j = k$; "indirect effect" if $j \neq k$
- If T is individual-specific, $\tau_j = \mathsf{E}[\pi_j(T_i = 1, W_i) - \pi_j(T_i = 0, W_i)]$

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Example: 1992 U.S. Presidential Election

- Model specification (Alvarez and Nagler 1995):

$$\pi_{ij} = \frac{\exp(X_{ij}'\beta + V_i'\delta_j)}{\sum_{k=1}^{J}\exp(X_{ik}'\beta + V_i'\delta_k)}$$

where

$$X_{ij} = \{\text{ideological distance}\}$$
$$V_i = \{1, \text{issue opinions, party, gender, education, age, ...}\}$$

- Estimated coefficients:

$$\hat{\beta} = -0.11\ (0.02)$$

$$\hat{\delta} = \left[\hat{\delta}_{\text{Bush}}\ \hat{\delta}_{\text{Clinton}}\right] = \begin{array}{l} \left[\begin{array}{cc} 0.67\ (0.94) & -0.41\ (0.45) \\ -0.52\ (0.11) & -0.02\ (0.12) \\ 0.54\ (0.23) & 0.30\ (0.22) \\ \vdots & \vdots \end{array}\right] \end{array} \begin{array}{l} \text{(intercept)} \\ \text{(support abortion)} \\ \text{(female)} \\ \vdots \end{array}$$

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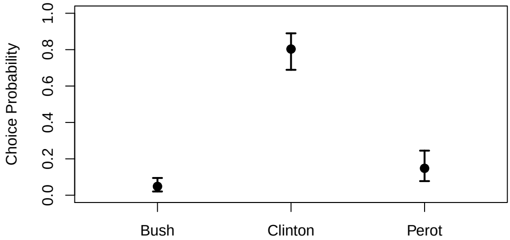

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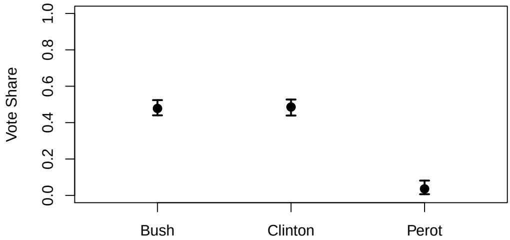

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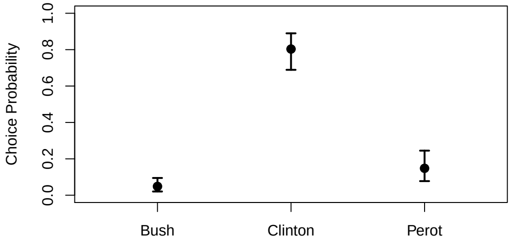

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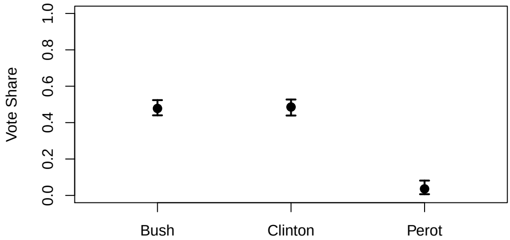

Revisiting The IIA Assumption

- IIA (Trump, Cruz, and Sanders)
- Formally: MNL assumes ϵ_{ij} is i.i.d. $\epsilon_{ij} \perp\!\!\!\perp \epsilon_{ik}$ for $j \neq k$
- This implies that unobserved factors affecting Y_{ij}^* are unrelated to those affecting Y_{ik}^*
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- Example: Multiparty election with parties R, L1 and L2.
- Do voters' unobserved ideological preferences affect Pr($Y_i =$L1) independently of their effect on Pr($Y_i =$L2)? Probably not.

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- Formally: MNL assumes ϵ_{ij} is i.i.d. $\epsilon_{ij} \perp\!\!\!\perp \epsilon_{ik}$ for $j \neq k$
- This implies that unobserved factors affecting Y_{ij}^* are unrelated to those affecting Y_{ik}^*
- When is this assumption plausible?
- Example: Multiparty election with parties R, L1 and L2.
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Multinomial Probit Model

- How can we relax the IIA assumption?
- Instead of assuming ϵ_{ij} to be i.i.d. across alternatives j, we allow ϵ_{ij} to be correlated across j within each voter i
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$$Y_i^* = X_i^{'}\beta + \epsilon_i \quad \text{where} \quad \begin{cases} \boldsymbol{\epsilon}_i \sim_{\text{iid}} \text{MVN}(0, \Sigma_J) \\ Y_i^* = [Y_{i1}^* \ \cdots \ Y_{iJ}^*]' \\ X_i = [X_{i1} \ \cdots \ X_{iJ}]' \end{cases}$$

- Restrictions on the model for identifiability:
  - The (absolute) level of Y_i^* shouldn't matter
    $\longrightarrow$ Subtract the 1st equation from all the other equations and work with a system of $J-1$ equations with $\tilde{\epsilon}_i \sim_{\text{iid}} \text{MVN}(0, \tilde{\Sigma}_{J-1})$
  - The scale of Y_i^* also shouldn't matter
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Limitations of Multinomial Probit

- MNP has no closed-form expression for the likelihood:

$$\pi_{ij} = \int_{-\infty}^{-\ddot{X}_{1j}^\top \beta} \cdots \int_{-\infty}^{-\ddot{X}_{Jj}^\top \beta} \phi(\ddot{\epsilon}_{1j}, ..., \ddot{\epsilon}_{Jj}) d\ddot{\epsilon}_{1j} \cdots d\ddot{\epsilon}_{Jj} \text{ where } \begin{cases} \ddot{X}_{kl} & = & X_{ik} - X_{il} \\ \ddot{\epsilon}_{kl} & = & \epsilon_{ik} - \epsilon_{il} \end{cases}$$

- This makes its estimation computationally costly when J large
- Must use numerical approximation (quadratures) or simulation methods (maximum simulated likelihood or MCMC)
- Moreover, # of parameters in Σ_J increases as J gets large, but data contain little information about Σ_J:

J	3	4	5	6	7
# of elements in Σ_J	6	10	15	21	28
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- Alternative solutions: Nested logit

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Even When You Choose Not To Decide, You Still Have Made a Choice

Lacy and Burden (1999)
Multinomial Probit Model:
Three Choices: Bush, Perot, and Clinton
Four Choices: Bush, Perot, Clinton, and Abstention
What happens if Perot doesn't run?

	Actual	3-choice	4-choice
Bush	32.0	45.7	38.4
Clinton	48.6	54.3	61.6
Abstention	20.9	-	23.7

Perot stole from Clinton!

Model Based Inference

1) Likelihood inference
2) Logit/Probit
3) Ordered Probit
4) Choice Models:
5) Count Models (Poisson, Negative Binomial,...)