Political Methodology III: Model Based Inference

Justin Grimmer

Associate Professor
Department of Political Science
Stanford University

May 23rd, 2017

1) Task:
   - Estimate conditional expectation function (regression) $E[Y|\boldsymbol{X}]$
   - Estimate empirical distribution function $\hat{f}(x)$
   - Estimate conditional empirical distribution function $\hat{f}(x|\boldsymbol{Z})$
2) Objective Function
   - Mean Square Error $\rightsquigarrow$ Average (Predictive) Risk
   - Balance Bias-Variance Tradeoff
3) Optimization
   - Local Linear Regression
4) Validation
   - Bias/Variance Tradeoff in fitting data
   - LOOCV will be a workhorse tool

Nonparametric Regression

Motivating Example: Birth Weight

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$$\boldsymbol{x}_i = (x_{i1}, x_{i2}, \ldots, x_{iJ})$$

Conditional Expectation Functions and Regressions

Definition

For any two random variables Y and X define the regression of Y on X as

$$f(x) \equiv E[Y|X = x] = \int_{-\infty}^{\infty} y f(y|X = x) dy$$

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$$f(x) = \alpha + \beta x$$

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Our estimator $\hat{f}(x)$ will estimate $f(x)$

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And accompanying covariates $(x_1, \ldots, x_N)$, then

$$
\begin{aligned}
\widehat{f}(x) &= \text{average in each bin} \\
&= \frac{\sum_{i=1}^{N} y_i I(x_i = x)}{\sum_{i=1}^{N} I(x_i = x)}
\end{aligned}
$$

Bias-Variance Tradeoff

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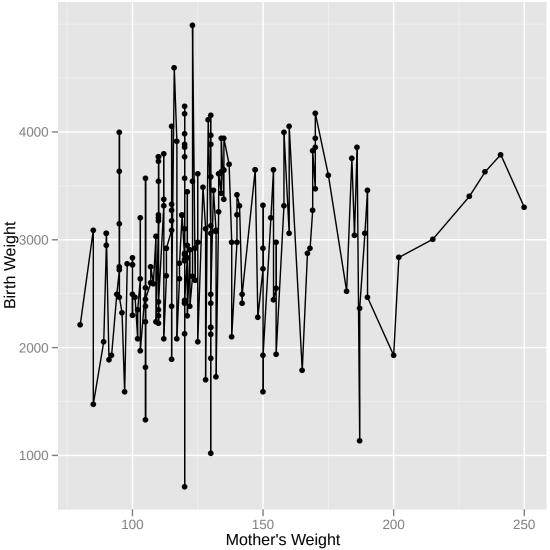
Then all values of $x_1, x_2, \ldots, x_N$ are distinct.
If we calculate conditional mean naively, we get the observed values of $y_1, \ldots, y_N$:

Bias-Variance Tradeoff



< R Code >

```
library(ggplot2)
qplot(lwt, bwt,
geom=c('point'),
xlab='Mothers Weight',
ylab='Birth Weight')
+geom_line()
```

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This yields:

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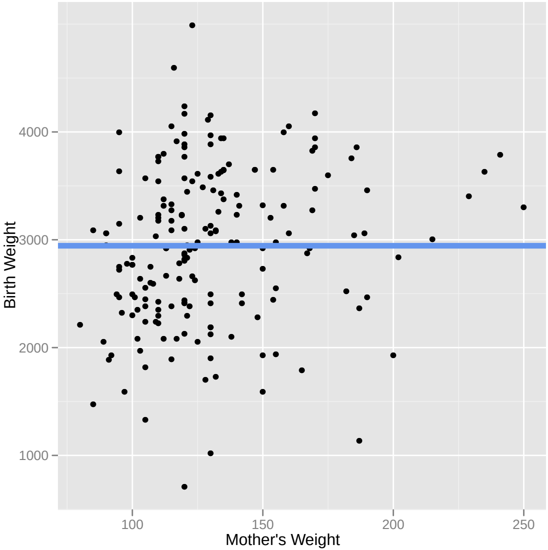


```
library(ggplot2)
qplot(lwt, bwt,
geom=c('point'),
xlab='Motherś Weight',
ylab='Birth Weight')
+geom_abline
(intercept=mean(bwt),
slope=0, lwd=2,
col='cornflowerblue')
```

Compromise

Curse of Dimensionality

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$$N \times \Pr(\boldsymbol{X} \in [-0.1, 0.1] \times [-0.1, 0.1] \times \ldots \times [-0.1, 0.1]) \quad =$$

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$$N \times \text{Pr}(\boldsymbol{X} \in [-0.1, 0.1] \times [-0.1, 0.1] \times \ldots \times [-0.1, 0.1]) \quad = \quad N \times \left(\frac{1}{10}\right)^{10} = \frac{N}{10{,}000{,}000{,}000}$$

Bias-Variance Tradeoff

Modeling the Relationship

$$Y = f(x_i) + \epsilon_i$$

Suppose $f(x_i) \equiv \beta_0 + \beta_{j=1}^{J} x_{ij}$ and $\boldsymbol{x} = (1, x_1, \ldots, x_J)$.

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J + 1 &= \mathrm{tr}(\boldsymbol{H})
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&= \underbrace{\boldsymbol{x}^{'}}_{1\times J}\underbrace{\left(\boldsymbol{X}^{'}\boldsymbol{X}\right)^{-1}}_{J\times J}\underbrace{\boldsymbol{X}^{'}}_{J\times N}\underbrace{\boldsymbol{Y}}_{N\times 1}
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&= \sum_{i=1}^{N} h_i(\boldsymbol{x})Y_i
\end{aligned}$$

Modeling the Relationship

Definition

We will say $\hat{f}(\boldsymbol{x})$, an estimator of $f(\boldsymbol{x})$, is a linear smoother if for each $\boldsymbol{x}$ there exists a vector $\boldsymbol{h}(\boldsymbol{x}) = (h_1(\boldsymbol{x}), h_2(\boldsymbol{x}), \ldots, h_N(\boldsymbol{x}))$ such that

$$\hat{f}(\boldsymbol{x}) = \sum_{i=1}^{N} h_i(\boldsymbol{x}) Y_i$$

This implies that fitted values $\widehat{\boldsymbol{Y}} = (\hat{f}(\boldsymbol{x}_1), \hat{f}(\boldsymbol{x}_2), \ldots, \hat{f}(\boldsymbol{x}_N))$ and

$$\widehat{\boldsymbol{Y}} = \boldsymbol{H}\boldsymbol{Y}$$

where i^{th} row is $\boldsymbol{h}(\boldsymbol{x}_i) = (h_1(\boldsymbol{x}_1), h_2(\boldsymbol{x}_1), \ldots, h_N(\boldsymbol{x}_1))$

Modeling the Relationship

Definition

The matrix $\boldsymbol{H}$ is the smoothing matrix. The row $h(\boldsymbol{x}_i)$ is called the effective kernel for observation i, and the effective number of parameters $\nu =$ tr$(\boldsymbol{H})$.

"Regressogram"

- Suppose we have one variable x_i
- Suppose we divide our data into K bins,$B_k = \{x : a < x < b\}$, $k = 1, 2, \ldots, K$ so that there are 3 observations in each bin $|B_k| = 3$ for all k
- We will say that the fitted value for x_i is then

$$\hat{f}(x_i) = \sum_{x_i \in B_k} \frac{Y_i}{3}$$

"Regressogram"

- In smoothing terms (sorting observations according to x_i values)

$$
\begin{aligned}
\boldsymbol{h}(x_i) &= (0,0,0,\frac{1}{3},\frac{1}{3},\frac{1}{3},0,0,0) \\
\boldsymbol{H} &= \begin{pmatrix}
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{3} & \frac{1}{3} & \frac{1}{3}
\end{pmatrix}
\end{aligned}
$$

Local Averages

- Define $B_x = \{i : |x_i - x| < b\}$ and $|B_x| = n_x$.
- If $n_x > 0$,

$$\hat{f}(x) = \sum_{i \in B_x} \frac{Y_i}{n_x}$$

In smoothing terms

$$\begin{aligned} h_i(x) &= 0 \text{ if } |x_i - x| > b \\ h_i(x) &= \frac{1}{n_x} \text{ if } |x_i - x| < b \end{aligned}$$

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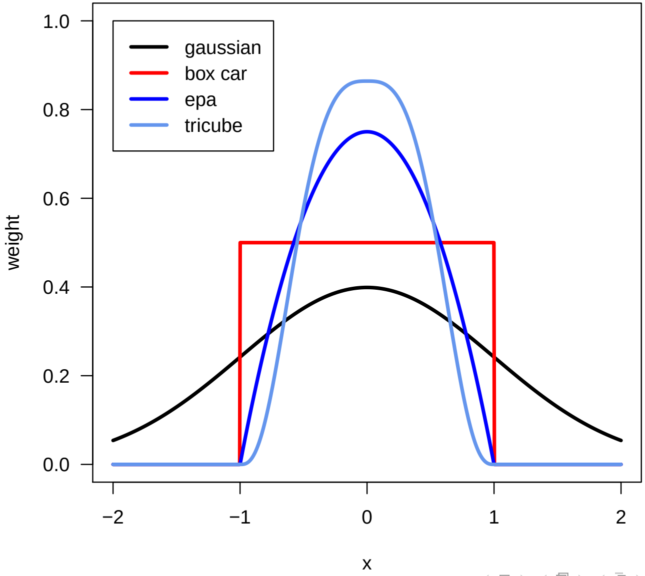
- Previous examples are binary (in/out)
- Continuous weights via Kernel
- Some famous examples (Define $I(x) \equiv 1$ if $|x| < 1,$ otherwise 0)

$$
\begin{aligned}
K(x) &= \frac{1}{2} I(x) \text{ (Box Car)} \\
K(x) &= \frac{1}{\sqrt{2\pi}} \exp\left(\frac{-x^2}{2}\right) \text{ (Gaussian)} \\
K(x) &= \frac{3}{4}(1 - x^2) I(x) \text{ (Epanechnikov)} \\
K(x) &= \frac{70}{81}(1 - |x|^3)^3 I(x) \text{ (Tri-cube)}
\end{aligned}
$$


gaussian
box car
epa
tricube
weight
x
1.0
0.8
0.6
0.4
0.2
0.0
−2
−1
0
1
2

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Define $b > 0$ to be the bandwidth.

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Consider the box-car kernel

$$\frac{1}{2} I\left(\frac{x - x_i}{1}\right) = \frac{1}{2} \text{ if } |x - x_i| < 1$$

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Consider the box-car kernel

$$\begin{aligned}
\frac{1}{2} I\left(\frac{x - x_i}{1}\right) &= \frac{1}{2} \text{ if } |x - x_i| < 1 \\
\frac{1}{2} I\left(\frac{x - x_i}{100}\right) &= \frac{1}{2} \text{ if } |x - x_i| < 100
\end{aligned}$$

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$$\frac{1}{2} I\left(\frac{x - x_i}{0.01}\right) = \frac{1}{2} \text{ if } |x - x_i| < 0.01$$

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\frac{1}{2} I\left(\frac{x - x_i}{100}\right) &= \frac{1}{2} \text{ if } |x - x_i| < 100 \\
\frac{1}{2} I\left(\frac{x - x_i}{0.01}\right) &= \frac{1}{2} \text{ if } |x - x_i| < 0.01
\end{aligned}$$

$b \rightsquigarrow$ controls the sensitivity of Kernel

Local Regression

Definition

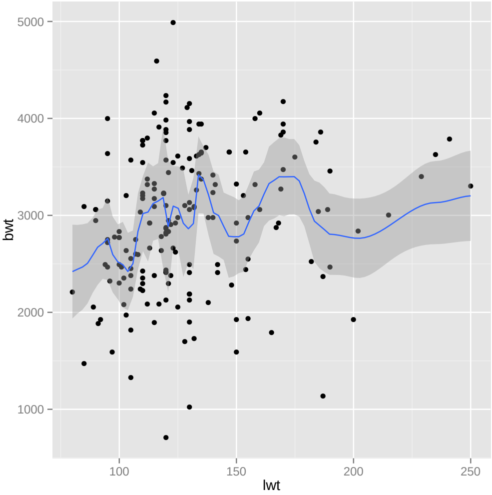
Suppose $b > 0$*. Define the local regression estimator* $\hat{f}(x)$ *of* $f(x)$ *as*

$$\hat{f}(x) = \sum_{i=1}^{N} h_i(x) Y_i$$

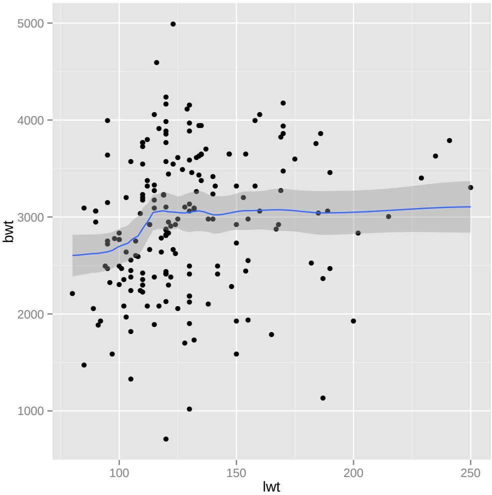
where

$$h_i(x) = \frac{K\left(\frac{x - x_i}{b}\right)}{\sum_{j=1}^{N} K\left(\frac{x - x_j}{b}\right)}$$

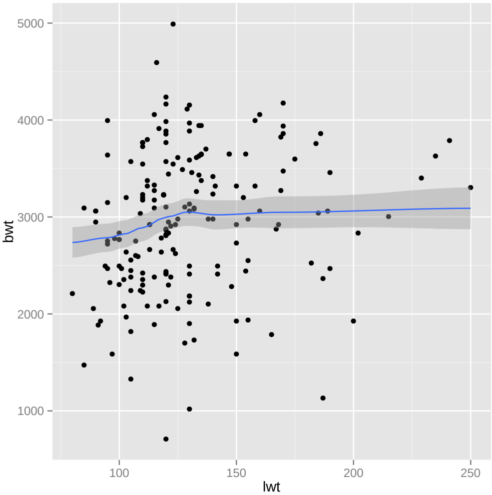
$\boldsymbol{H} = N \times N$ *matrix with* $H_{ij} = h_j(x_i)$



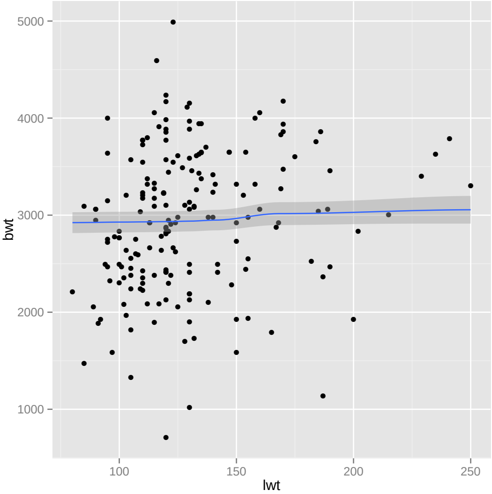

```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 0,
span = 0.1)
```




```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 0,
span = 0.4)
```




```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 0,
span = 0.7)
```




```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 0,
span = 1)
```

Local Regression

$$
\begin{aligned}
w_i(x) &= K\left(\frac{x_i - x}{b}\right) \\
\hat{a} &= \text{arg min}_a \sum_{i=1}^{N} w_i(x)(Y_i - a)^2 \\
\hat{a} &= \frac{\sum_{i=1}^{N} w_i(x) Y_i}{\sum_{i=1}^{N} w_i(x)} \\
\hat{a} &= \hat{f}(x)
\end{aligned}
$$

Local Polynomial regression

$$(\hat{a}, \hat{b}, \hat{c}) = \text{arg min}_{a,b,c} \sum_{i=1}^{N} w_i(x)(Y_i - a - bx_i - cx_i^2)^2$$

Define $\boldsymbol{X}_x$ to be an $N \times 3$ matrix:

$$\boldsymbol{X}_x = \begin{pmatrix} 1 & x_1 - x & (x_1 - x)^2 \\ 1 & x_2 - x & (x_2 - x)^2 \\ \vdots & \vdots & \vdots \\ 1 & x_N - x & (x_N - x)^2 \end{pmatrix}$$

$\boldsymbol{W}_x = N \times N$ matrix where $W_{ii} = w_i(x)$.

$$\hat{f}(x) = \left(\boldsymbol{X}_x^{'} \boldsymbol{W}_x \boldsymbol{X}_x\right)^{-1} \boldsymbol{X}_x^{'} \boldsymbol{W}_x \boldsymbol{Y}$$

Local Polynomial Regression

Definition

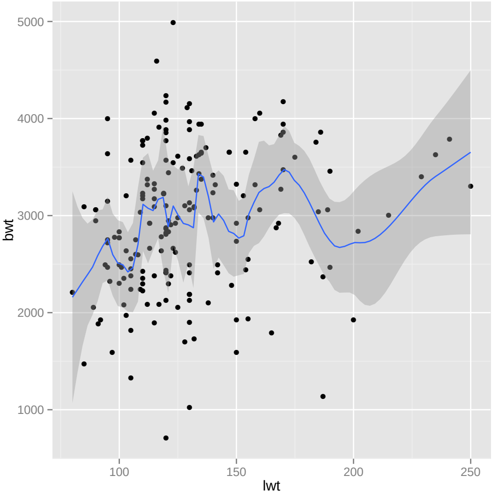
Suppose $b > 0$. Define the local polynomial regression estimator $\hat{f}(x)$ of $f(x)$ as

$$\hat{f}(x) \quad = \quad \left(\boldsymbol{X}_x^{'}\boldsymbol{W}_x\boldsymbol{X}_x\right)^{-1}\boldsymbol{X}_x^{'}\boldsymbol{W}_x\boldsymbol{Y}$$

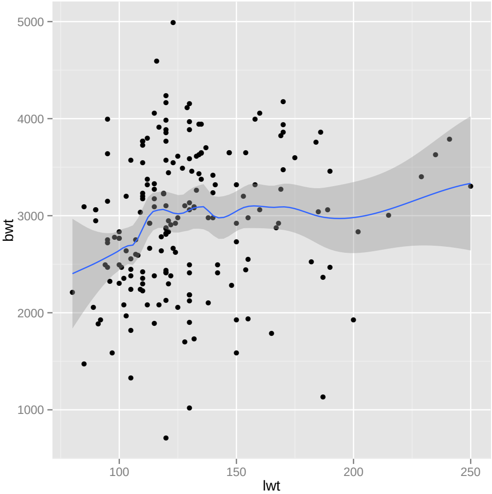
We can write this as $\hat{f}(x) = \sum_{i=1}^{N} h_i(x)Y_i$, where $\boldsymbol{h}(x) = (h_1(x), h_2(x), \ldots, h_N(x))$

$$h(x) \quad = \quad \boldsymbol{e}_1\left(\boldsymbol{X}_x^{'}\boldsymbol{W}_x\boldsymbol{X}_x\right)^{-1}\boldsymbol{X}_x^{'}\boldsymbol{W}_x$$

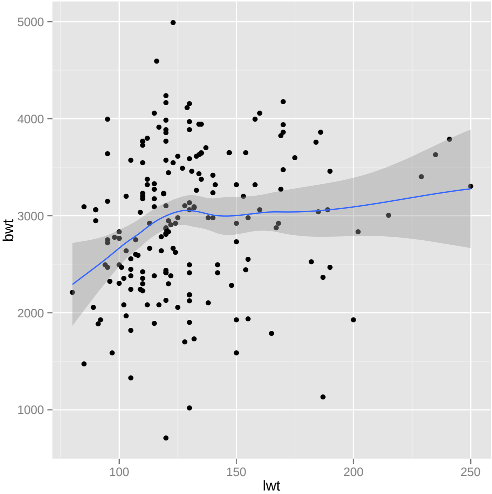
where $\boldsymbol{e}_1 = (1, 0, \ldots, 0)$.
$\boldsymbol{H}$ is an $N \times N$ matrix with typical entry $H_{ij} = h_j(x_i)$.



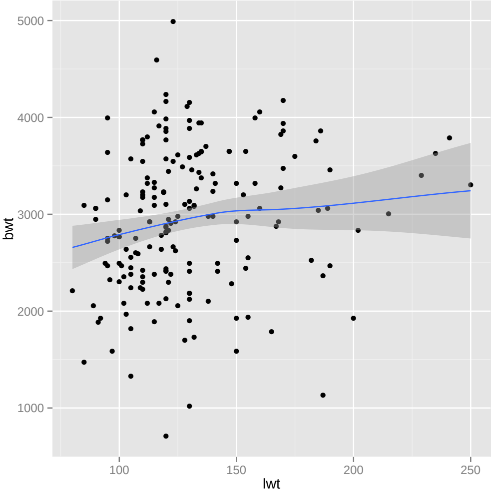

```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 1,
span = 0.1)
```



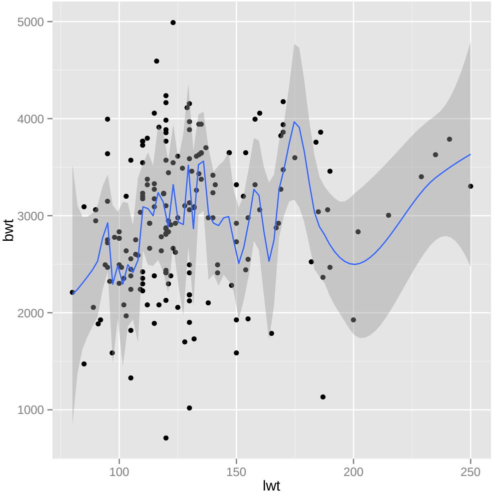

```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 1,
span = 0.4)
```



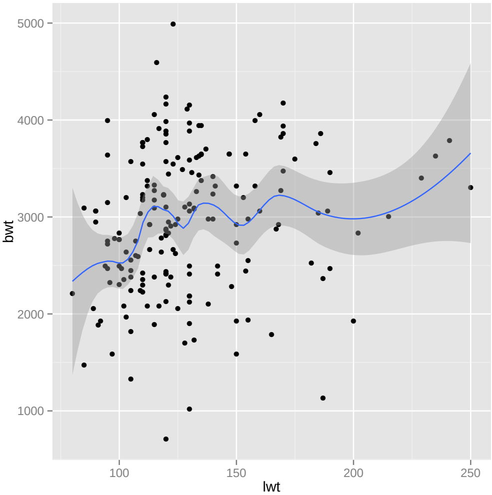

```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 1,
span = 0.7)
```



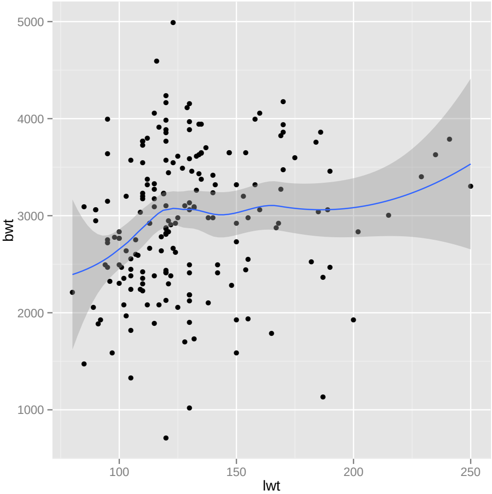

```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 1,
span = 1)
```



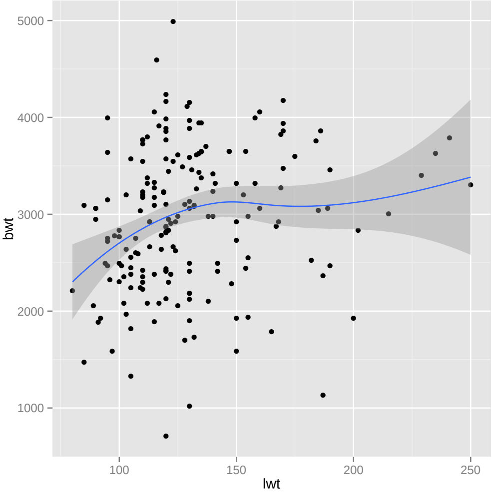

```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 2,
span = 0.1)
```



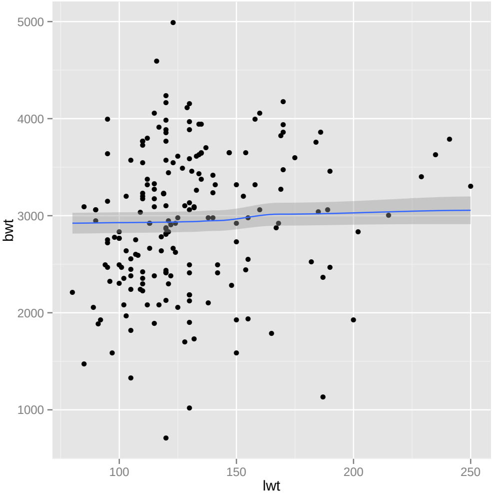

```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 2,
span = 0.4)
```



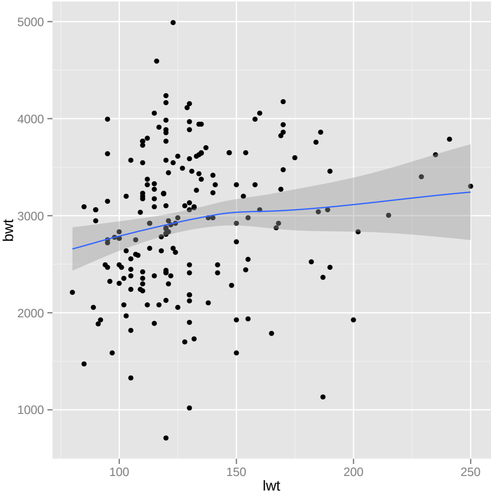

```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 2,
span = 0.7)
```



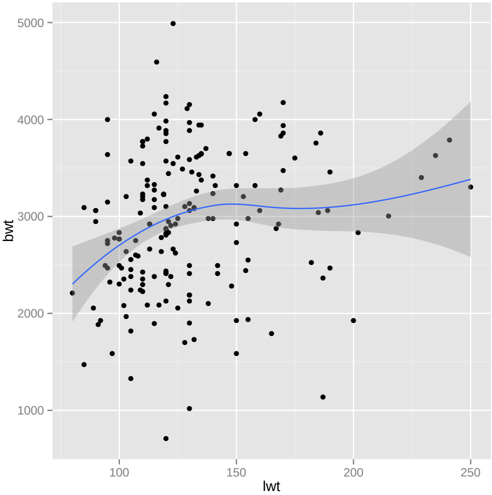

```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 2,
span = 1)
```




```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 0,
span = 1)
```




```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 1,
span = 1)
```




```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 1,
span = 1)
```

Bandwidth Selection

Leave One Out Cross Validation LOOCV

- Define $\hat{f}_{-i,b}(x)$ to be the linear smooth estimtator without observation i and bandwidth b.
- The LOOCV statistic will be

$$\text{CV(b)} = \frac{1}{N}\sum_{i=1}^{N}\left(Y_i - \hat{f}_{-i,b}(x)\right)^2$$

Theorem

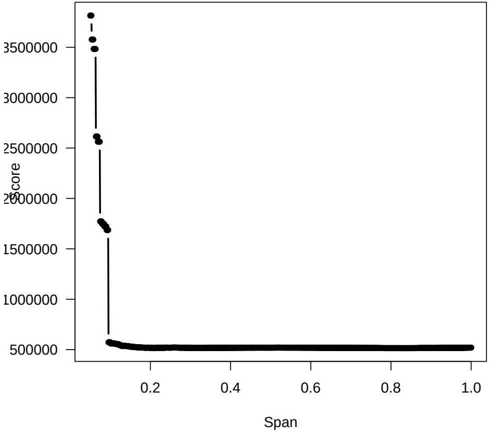
Suppose $\hat{f}(x)$ is a linear smoother. Then CV(b) can be written as

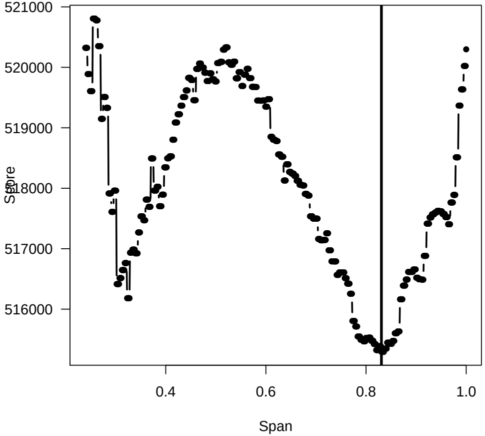
$$\text{CV(b)} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{Y_i - \hat{f}_b(x)}{1 - H_{ii}}\right)^2$$

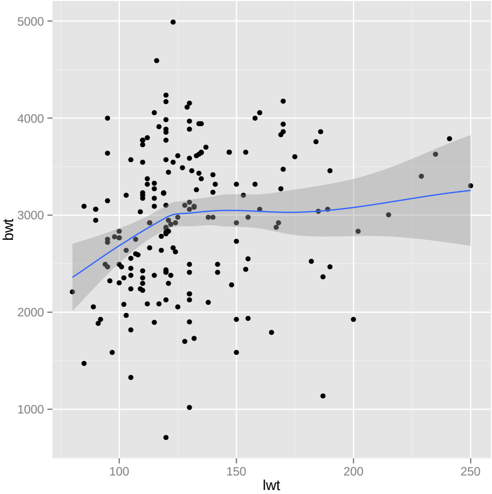
And is often approximated with the Generalized Cross Validation (GCV):

$$\text{GCV(b)} = \frac{1}{N}\sum_{i=1}^{N}\left(\frac{Y_i - \hat{f}_b(x)}{1 - \nu/N}\right)^2$$

where $\nu = tr(\boldsymbol{H})$


3500000
3000000
2500000
2000000
1500000
1000000
500000
Score
0.2
0.4
0.6
0.8
1.0
Span


521000
520000
519000
518000
517000
516000
Score
0.4
0.6
0.8
1.0
Span




```
qplot(lwt, bwt,
geom='point') +
stat_smooth(degree = 1,
span = 0.83)
```

Loess and Conditioning

Many models:

Loess and Conditioning

Many models: condition on many factors in order to learn relationship

Loess and Conditioning

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Possible to visualize relationships in lattice:

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Possible to visualize relationships in lattice:
Suppose we have continuous random variable X and discrete random variables Z_1 and Z_2.

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Suppose we have continuous random variable X and discrete random variables Z_1 and Z_2.
Then we might estimate

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Possible to visualize relationships in lattice:
Suppose we have continuous random variable X and discrete random variables Z_1 and Z_2.
Then we might estimate

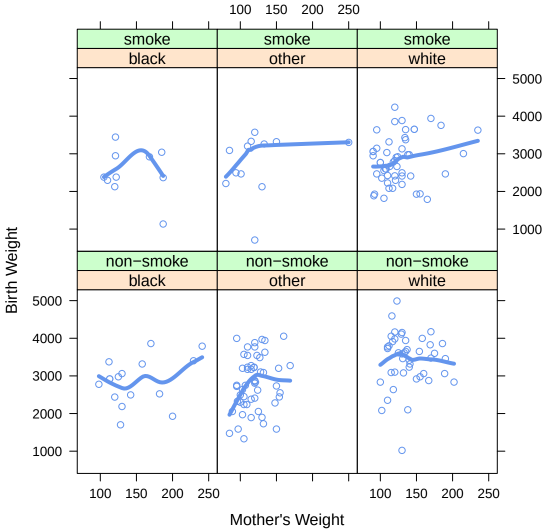
$$f(x, z_1, z_2) = E[Y|X = x, Z_1 = z_1, Z_2 = z_2]$$

Loess and Conditioning

Many models: condition on many factors in order to learn relationship
Possible to visualize relationships in lattice:
Suppose we have continuous random variable X and discrete random variables Z_1 and Z_2.
Then we might estimate

$$
\begin{aligned}
f(x, z_1, z_2) &= E[Y|X = x, Z_1 = z_1, Z_2 = z_2] \\
&= \int_{-\infty}^{\infty} y f(y|X = x, Z_1 = z_1, Z_2 = z_2) dy
\end{aligned}
$$

Loess and Conditioning



< R Code >

```
xyplot(bwt~lwt|race*smoke
lwd=4,
col='cornflowerblue',
xlab='Mother's Weight',
ylab='Birth Weight',
panel= function(x, y,
...){
panel.xyplot(x, y,
...);
panel.loess(x,y,span=.8,
degree=1, ...)  } )
```

Densities

Histogram, Probability Mass Function

Simplest case: probability mass functions

Histogram, Probability Mass Function

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- Suppose we observe $x_1, x_2, \ldots, x_N$

Histogram, Probability Mass Function

Simplest case: probability mass functions

- Suppose we observe $x_1, x_2, \ldots, x_N$
- If X's takes on few unique values, say $K << N$, then an estimator for the pmf is,

Histogram, Probability Mass Function

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- Suppose we observe $x_1, x_2, \ldots, x_N$
- If X's takes on few unique values, say $K << N$, then an estimator for the pmf is,

$$\widehat{p}(X = x) \equiv \widehat{p}(x) \quad = \quad \frac{\sum_{i=1}^{N} I(x_i = x)}{N}$$

Histogram, Probability Mass Function

Simplest case: probability mass functions

- Suppose we observe $x_1, x_2, \ldots, x_N$
- If X's takes on few unique values, say $K << N$, then an estimator for the pmf is,

$$\widehat{p}(X = x) \equiv \widehat{p}(x) \quad = \quad \frac{\sum_{i=1}^{N} I(x_i = x)}{N}$$

Will converge on the probability mass function (by weak law of large numbers)

Histogram, Probability Density Function

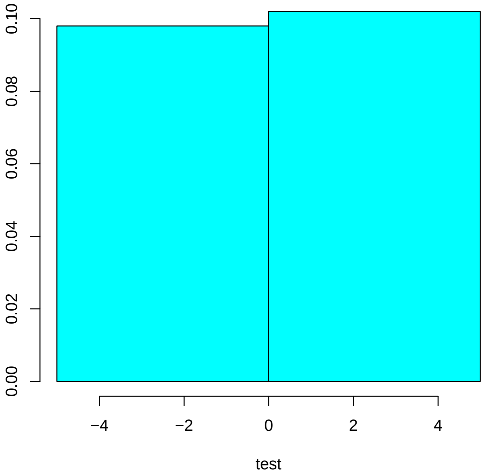
Harder case: probability density functions

- Suppose we observe $x_1, x_2, \ldots, x_N$, with X's continuous
- Goal: estimate the pdf $f(x)$.
- Define a set of bins $b_1, b_2, \ldots, b_M$ such that
  - $b_1 \leq x_1$ and $b_M \geq x_N$
- Then, we can define an estimator for the pdf $f(x)$ as

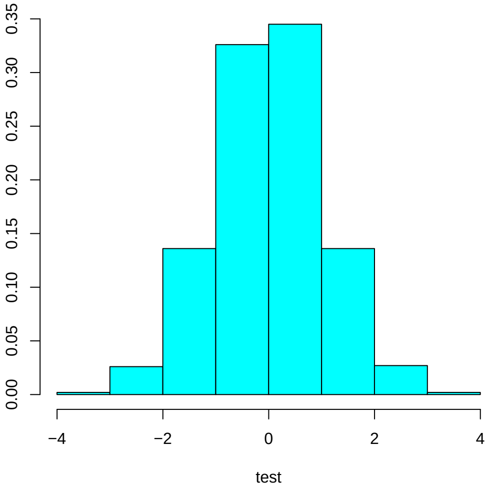
$$\widehat{f(x)} = \frac{\sum_{i=1}^{N} I(x_i > b_z \text{ and } x_i \leq b_{z+1})}{N} \times I(x \in (b_z, b_{z+1}])$$

In words: $\widehat{f(x)}$ is equal to proportion of observations in the bandwidth that x resides in

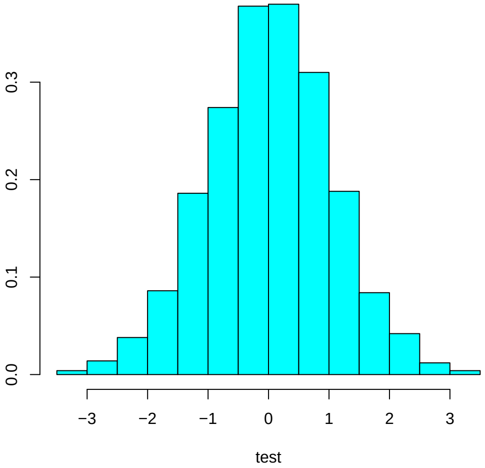
Bias Variance Tradeoff on Bin Size



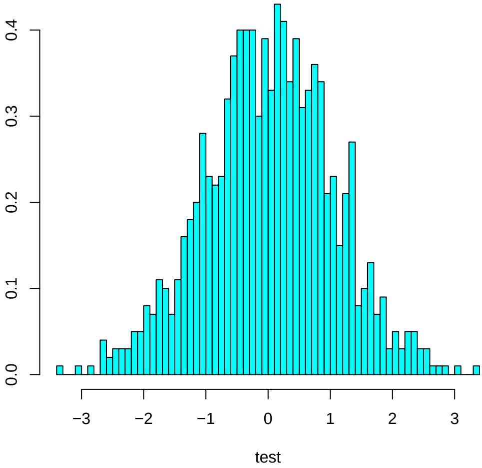
Bias Variance Tradeoff on Bin Size



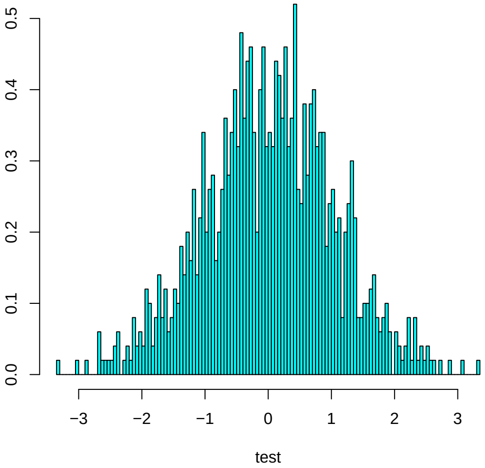
Bias Variance Tradeoff on Bin Size



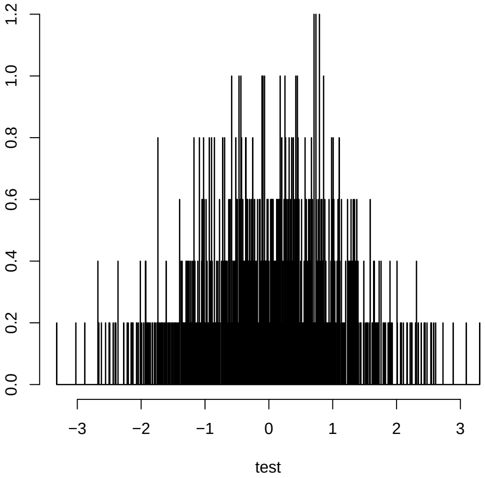
Bias Variance Tradeoff on Bin Size



Bias Variance Tradeoff on Bin Size



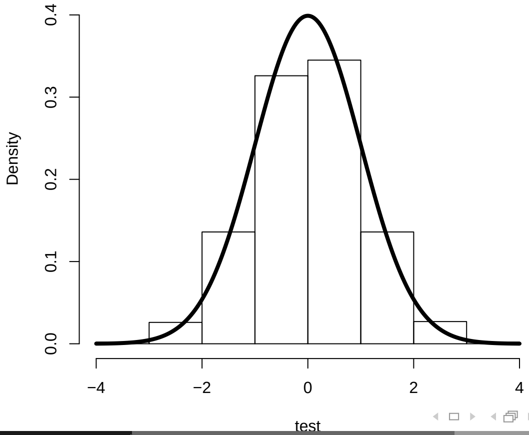
Bias Variance Tradeoff on Bin Size



Bias-Variance Tradeoff/Guidance

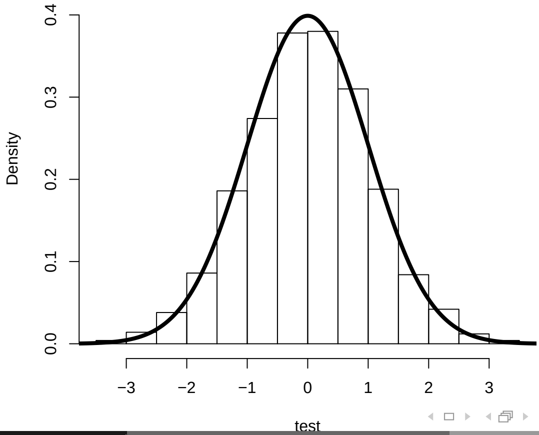
Classic problem:

Bias-Variance Tradeoff



Bias-Variance Tradeoff/Guidance

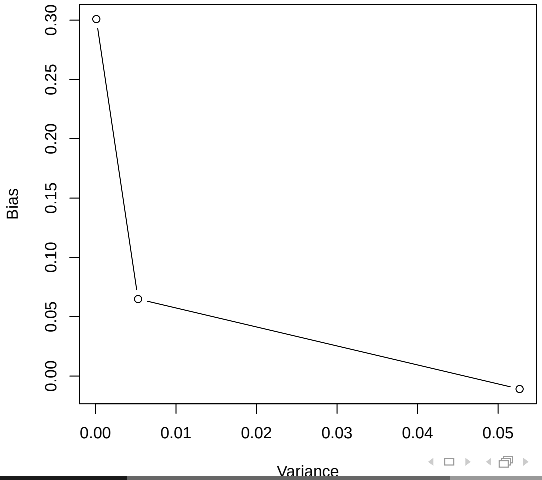
Classic problem:
Bias-Variance Tradeoff



Bias-Variance Tradeoff/Guidance

Classic problem:

Bias-Variance Tradeoff



Navigating the Bias Variance Tradeoff

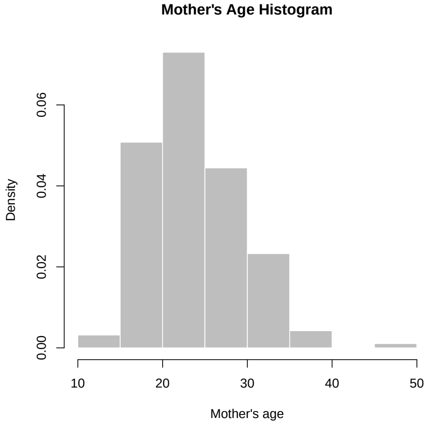
Two formulas for navigating bias-variance tradeoff for data x

1) width= $\frac{3.5 \times \mathsf{sd}(x)}{n^{1/3}}$ (Scott 1979)
2) width= $\frac{2(Q_3 - Q_1)}{n^{1/3}}$ (Freedman and Diaconis (1981))

Histogram Creation

< R Code >

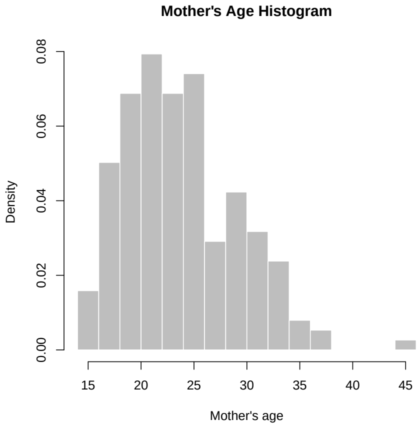
```
    truehist(birthwt$age, xlab='Mothers age',
ylab='Density', main='Mother's Age Histogram', col='gray',
border='white')
```



Histogram Creation

< R Code >

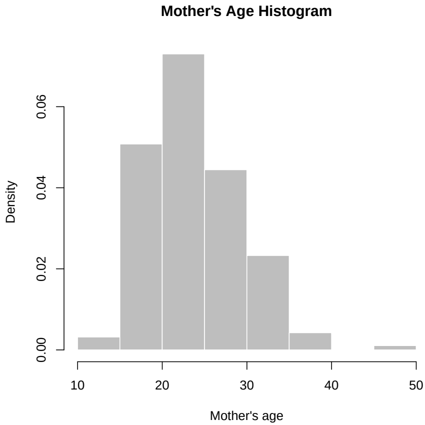
```
    truehist(birthwt$age, xlab='Mothers´ age',
ylab='Density', main='Mother's Age Histogram', col='gray',
border='white', nbins='fd')
```



Histogram Creation

< R Code >

```
truehist(birthwt$age, xlab='Mothers´ age',
ylab='Density', main='Mother's Age Histogram', col='gray',
border='white', nbins='scott')
```



Density Estimation

Histograms make continuous data discrete

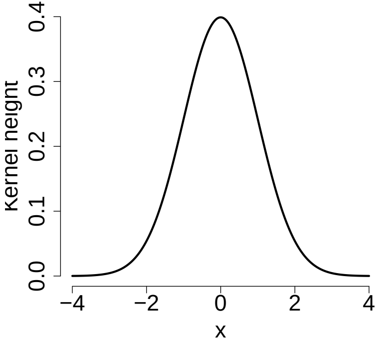
Density estimators "smooth" out histograms.

We will use Kernels again

Gaussian Kernel

Define

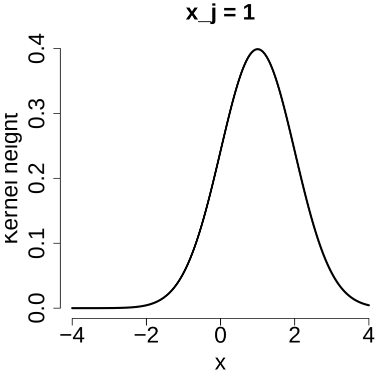
$$K(x) = \frac{\exp[-x^2]}{\sqrt{2\pi}}$$



Gaussian Kernel

Define

$$K\left(\frac{x - x_j}{b}\right) = \frac{\exp[-\left(\frac{x-x_j}{b}\right)^2]}{\sqrt{2\pi}}$$



Gaussian Kernel

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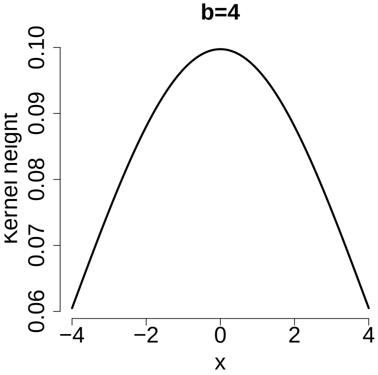
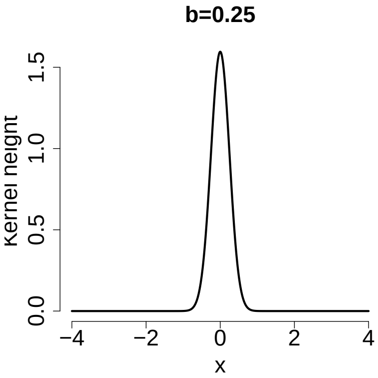
$$K(\frac{x - x_j}{b}) = \frac{\exp[-\left(\frac{x-x_j}{b}\right)^2]}{\sqrt{2\pi}}$$



Gaussian Kernel

Define

$$K\left(\frac{x - x_j}{b}\right) = \frac{\exp[-\left(\frac{x-x_j}{b}\right)^2]}{\sqrt{2\pi}}$$



Density Estimation

For all values $x \in \Re$ define the kernel density estimator as,

$$\widehat{f}(x) \quad = \quad \frac{1}{nb}\sum_{j=1}^{n} K(\frac{x - x_j}{b})$$

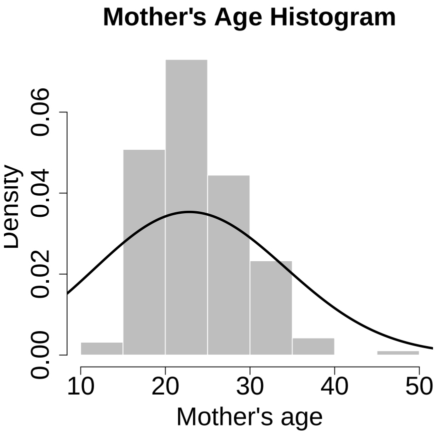
In words: at each point x, calculate weighted average of points, where weight is given by kernel

- As b increases, weight on nearby points is more evenly distributed (redistributed away from x)
- As b decreases, weight on nearby points is more concentrated (redistributed towards x)

Another bias variance tradeoff!

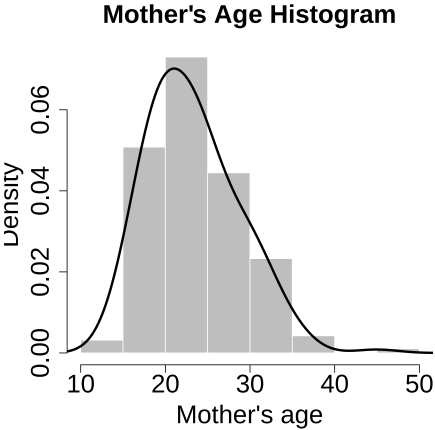
Density Estimation

```
truehist(birthwt$age, xlab='Mothers age',
ylab='Density', main='Mothers Age Histogram', col='gray',
border='white', nbins='scott')
lines(density(birthwt$age, width=40), lwd=3)
```



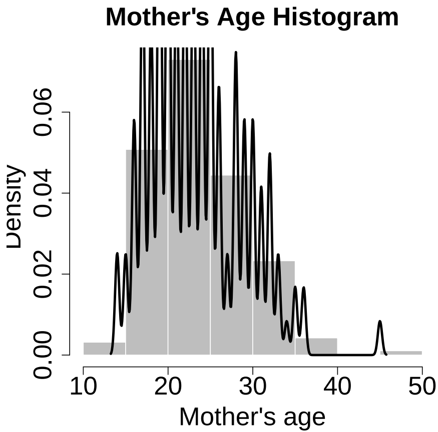
Density Estimation

```
truehist(birthwt$age, xlab='Mothers age',
ylab='Density', main='Mothers Age Histogram', col='gray',
border='white', nbins='scott')
lines(density(birthwt$age, width=10), lwd=3)
```



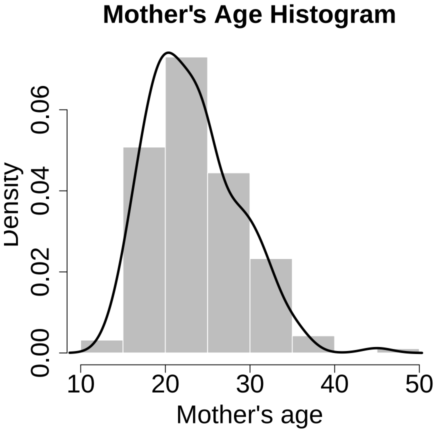
Density Estimation

```
truehist(birthwt$age, xlab='Mothers age',
ylab='Density', main='Mothers Age Histogram', col='gray',
border='white', nbins='scott')
lines(density(birthwt$age, width=1), lwd=3)
```



Density Estimation

```
   truehist(birthwt$age, xlab='Mothers age',
ylab='Density', main='Mothers Age Histogram', col='gray',
border='white', nbins='scott')
    lines(density(birthwt$age, width='SJ-dpi'), lwd=3)
```



Back to the Birth Weight

An initial inference:

Back to the Birth Weight

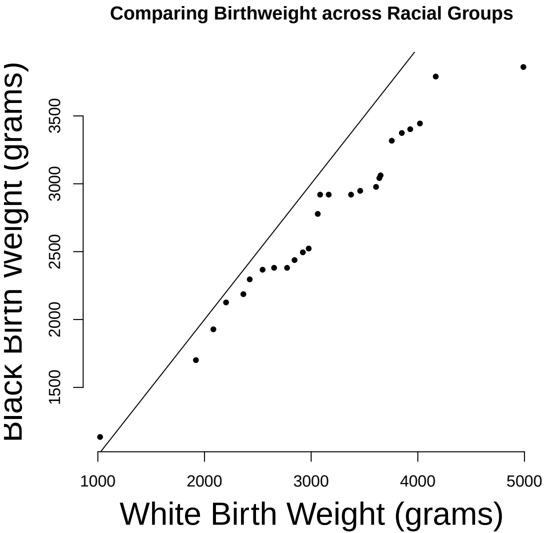
An initial inference:
Compare (contrast):

Back to the Birth Weight

An initial inference:
Compare (contrast):

$$\widehat{p}(y|\text{white}) \quad \text{and} \quad \widehat{p}(y|\text{black})$$

Back to the Birth Weight




```
qqplot(
bwt[which(race2=='white')],
bwt[which(race2=='black')],
xlab='White Birth Weight
(grams)', ylab='Black Birth
Weight (grams)',
frame.plot=F,
main='Comparing Birth
Weight across Racial Groups',
pch=20)
arrows(0,0, 1e7, 1e7)
```

Box Plot

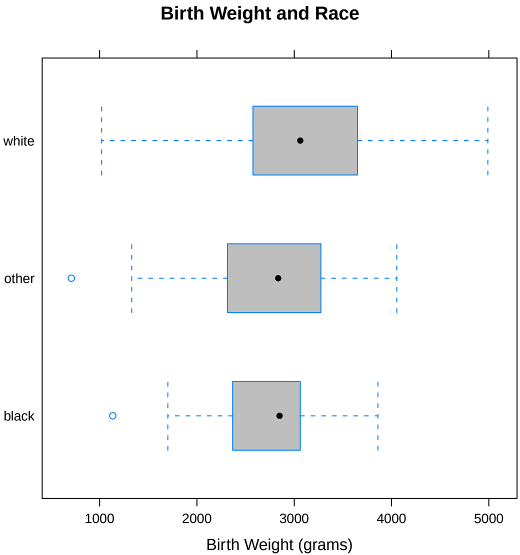
Box Plot

- QQ-plots compare two distributions directly

Box Plot

- QQ-plots compare two distributions directly
- Often want to compare many distributions (and readers are uncomfortable with qq-plots)

Box Plot



< R Code >

```
library(lattice)
bwplot(race2 ~bwt,
xlab='Birth Weight
(grams)',
frame.plot=F,main =
'Birthweight and Race',
panel=function(...){
panel.bwplot(...,
fill='gray', lty=1,
lwd=2, pch=20) } )
```

Violin Plot

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- Box plots can obscure variation in distributions [histogram]

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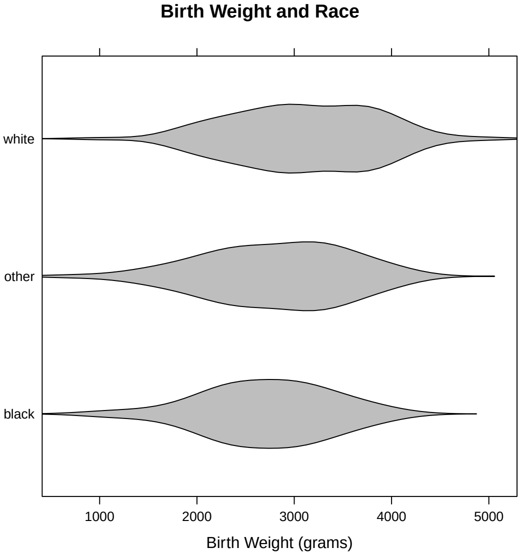
- Violin plots allow smooth estimates of data distribution [density]

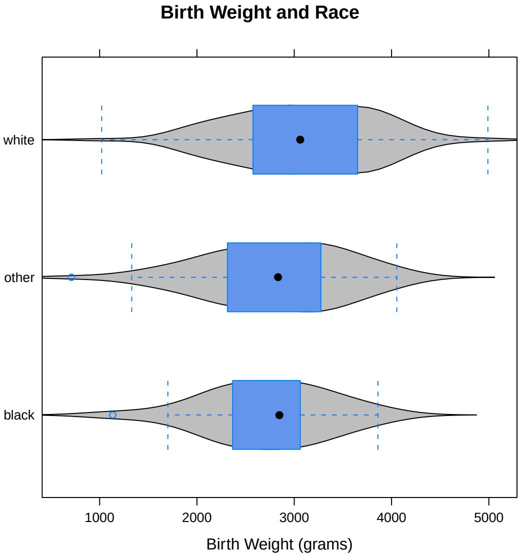
Violin Plot



< R Code >

```
library(lattice)
bwplot(race2~bwt,
xlab='Birth Weight
(grams)', frame.plot=F,
main='Birth Weight and
Race',
panel=function(...){
panel.violin(...,
col='gray',
bw='SJ-dpi') } )
```

Violin Plot



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main='Birth Weight and
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Box (Violin) Plot with Conditioning

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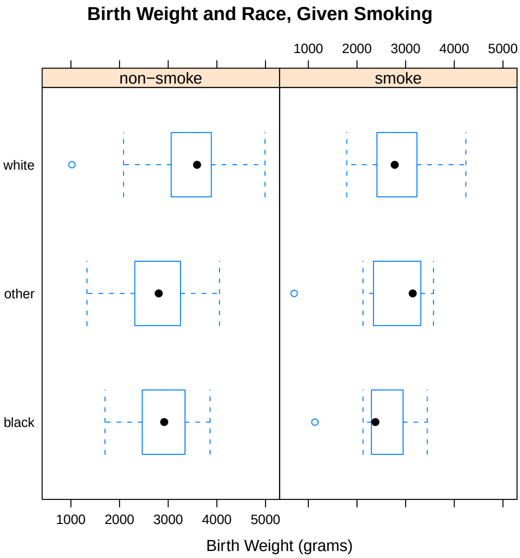
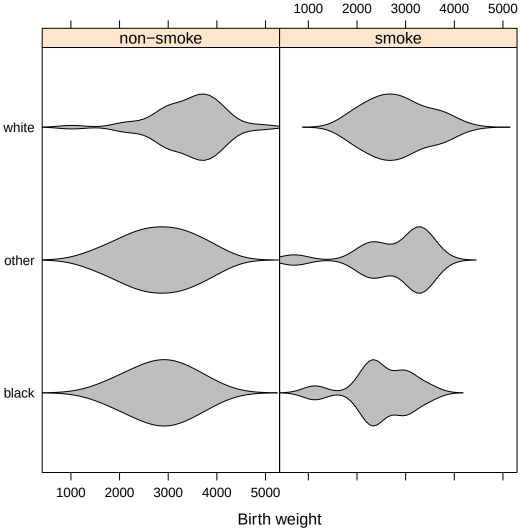
$$\begin{aligned}
\widehat{p}(y|\text{black, smoke}) \quad &; \quad \widehat{p}(y|\text{black, non-smoke}) \\
\widehat{p}(y|\text{white, smoke}) \quad &; \quad \widehat{p}(y|\text{white, non-smoke}) \\
\widehat{p}(y|\text{other, smoke}) \quad &; \quad \widehat{p}(y|\text{other, non-smoke})
\end{aligned}$$

Box (Violin) Plot with Conditioning




```
library(lattice)
bwplot(race2~bwt|smoke,
xlab='Birth Weight
(grams)',
frame.plot=F,main =
'Birth Weight and Race,
Given Smoking')
```

Box (Violin) Plot with Conditioning




```
 library(lattice)
bwplot(race2~bwt|smoke,
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panel=function(...){
panel.violin(...,
col='gray',bw='SJ-dpi')
} )
```

Nonparametrics: Bias/Variance Tradeoff