Political Methodology III: Model Based Inference

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Model Based Inference

1) Likelihood inference
2) Logit/Probit
3) Ordered Probit
4) Choice Models:
5) Count Models
6) Survival Models
7) Hypothesis Tests + Model Checking in Likelihood
    - Likelihood Ratios, Wald, and Score tests
    - Model Checking: analysis of residuals, hat values, etc.

Simple Example: Antiobama Speech

We'll use the speech data from the problem set, as follows:

- $Y_i = 1$ if representative says `obamacare` or `big government` during the year, 0 otherwise
- $\boldsymbol{X}_i = (1, I(\text{Year} = 2010)_i, \text{Democrat}_i, \text{DW-Nom}_i)$

$$\begin{aligned} Y_i &\sim \text{Bernoulli}(\pi_i) \\ \pi_i &= \text{logit}^{-1}(\boldsymbol{X}_i^{\prime}\boldsymbol{\beta}) = \frac{1}{1+\exp(-\boldsymbol{X}_i^{\prime}\boldsymbol{\beta})} \end{aligned}$$

Which covariates do we include? $\rightsquigarrow$ depends on goal.

- Predictive goal $\rightsquigarrow$ replicate task
- Model fitting $\rightsquigarrow$ do covariates increase likelihood? Can we drop them?

Hypothesis Testing — Likelihood Ratio Test

- Null (H_0): $h_1(\boldsymbol{\beta}) = \cdots = h_Q(\boldsymbol{\beta}) = 0$ (Q equality constraints)
- Alternative (H_1): No such constraints
- Let $\widehat{\boldsymbol{\beta}}_R = \widehat{\boldsymbol{\beta}}_{MLE|H_0}$ (restricted MLE) and $\widehat{\boldsymbol{\beta}}_{UR} = \widehat{\boldsymbol{\beta}}_{MLE}$ (original MLE)
- Likelihood ratio (LR) test: If H_0 is true, $L(\widehat{\boldsymbol{\beta}}_R)$ should be equal to $L(\widehat{\boldsymbol{\beta}}_{UR})$ except for sampling variability
- LR statistic:

$$LR(Y) \equiv -2\log\frac{L(\widehat{\boldsymbol{\beta}}_R)}{L(\widehat{\boldsymbol{\beta}}_{UR})} = 2\left[\ell(\widehat{\boldsymbol{\beta}}_{UR}) - \ell(\widehat{\boldsymbol{\beta}}_R)\right]$$

- We can show that $LR(Y) \xrightarrow{d} \chi^2_Q$
- Works for testing any nested models
  — model under H_0 has to be a special case of model under H_1

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```
un_rest_reg<- glm(once~two_10 + dem + dw_nom,
   data = speech_dat, family = binomial(link = logit))

rest_reg<- glm(once~1, family= binomial(link = logit))

##calculating the likelihood ratio
log_lik<- function(pars, X, Y){
   y.tilde<- X%*%pars
   probs<- plogis(y.tilde)
   log_out<- Y%*%log(probs) + (1-Y)%*%log(1 - probs)
   return(log_out)
}
X<- cbind(1, two_10, dem, speech_dat$dw_nom)

un_rest<- log_lik(un_rest_reg$coef, X, once)

rest<- log_lik(rest_reg$coef, as.matrix(rep(1, nrow(X))), once)

> 2 * un_rest - 2*rest
   [,1]
[1,] 433.996
```

```
> 2 * un_rest - 2*rest
   [,1]
[1,] 433.996
##get the same statistic automatically from glm
diff<- un_rest_reg$null.deviance - un_rest_reg$deviance
> diff
[1] 433.996

1 - pchisq(diff, 3) ##very small!
[1] 0
```

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- Wald statistic: Use asymptotic distribution of $\widehat{\beta}$ and representation of restrictions, properties of normal distribution to obtain form

$$W \equiv h(\widehat{\boldsymbol{\beta}}_{UR})' \left[\left(\frac{\partial h(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} \bigg|_{\boldsymbol{\beta}=\widehat{\boldsymbol{\beta}}_{UR}} \right)' \widehat{\text{Var}(\widehat{\boldsymbol{\beta}}_{UR})} \left(\frac{\partial h(\boldsymbol{\beta})}{\partial \boldsymbol{\beta}} \bigg|_{\boldsymbol{\beta}=\widehat{\boldsymbol{\beta}}_{UR}} \right) \right]^{-1} h(\widehat{\boldsymbol{\beta}}_{UR})$$

- The "meat" $\simeq \widehat{\text{Var}(h(\widehat{\boldsymbol{\beta}}_{UR}))}$ (Delta method)
- Choose any $\widehat{\text{Var}(\widehat{\boldsymbol{\beta}}_{UR})}$ as appropriate (e.g. Huber-White)
- We can show that $W \xrightarrow{d} \chi^2_Q$
- An important special case: $Q = 1$ and $H_0 : \beta = 0$
- In this case, we can use the z statistic:

$$z = W^{1/2} = \frac{\widehat{\boldsymbol{\beta}}_{UR}}{\text{s.e.}(\widehat{\boldsymbol{\beta}}_{UR})} \xrightarrow{d} \text{N}(0,1)$$

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```
> un_rest_reg$coef%*%solve(vcov(un_rest_reg))%*%un_rest_reg$coef
[,1]
[1,] 225.2437

> 1 - pchisq(225.2437, 3)
[1] 0
```

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- Score test: If the null is true, $s(\widehat{\boldsymbol{\beta}}_R)$ should also equal zero except for sampling variability
- Score statistic: Use asymptotic distribution and properties of normal distribution to "standardize" $s(\widehat{\boldsymbol{\beta}}_R)$

$$LM = s(\widehat{\boldsymbol{\beta}}_R)' \widehat{\mathsf{Var}(\widehat{\boldsymbol{\beta}}_R)} s(\widehat{\boldsymbol{\beta}}_R) \xrightarrow{d} \chi^2_Q$$

- For $\widehat{\beta}_{QMLE}$, the expression is more complicated
- Also known as the Lagrange multiplier (LM) test due to an alternative derivation

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```
score_func<- function(coef, X, Y){
   y.tilde<- X%*%coef
   probs<- plogis(y.tilde)
   out<- t(Y - probs)%*%X
   return(out) }
```

```
> round(score_func(un_rest_reg$coef, X, once), 2)

[1,] 0 0 0 0

rest_score<- score_func(c(rest_reg$coef, 0, 0, 0), X, once)

> round(rest_score,2)
[1,] 0 -6.30 -128.92 129.51
```

```
hess_func<- function(coef, X, Y){
   y.tilde<- X%*%coef
   probs<- plogis(y.tilde)
   base<- matrix(0, nrow = len(coef), ncol = len(coef))
   for(z in 1:nrow(X)){
    base<- base + probs[z]*(1 - probs[z])* X[z,]%*%t(X[z,])
   }
   return(base)
}

rest_hess<- solve(hess_func(c(rest_reg$coef, 0, 0, 0), X, once))
>rest_score%*%rest_hess%*%t(rest_score)
[1,] 395.0382
> 1- pchisq(395, 3)
[1] 0
```

Comparing The Three Tests

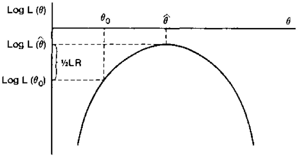

Figure 1. The Likelihood Ratio Test

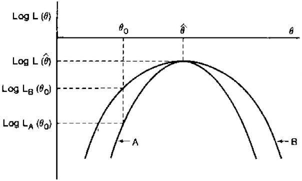

Figure 2. The Wald Test

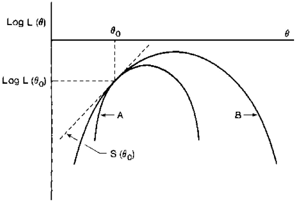

Figure 3. The Lagrange Multiplier Test

- All asymptotically equivalent
- But can be quite different in small samples

	Pros	*Cons*
LR	Most powerful (Neyman-Pearson)	Must compute both $\hat{\theta}_{UR}$ and $\hat{\theta}_R$ Cannot be easily robustified
W	Only need $\hat{\theta}_{UR}$ Easily robustified by sandwich	Not invariant to transformation (e.g. $\theta_1/\theta_2 = 1$ vs. $\theta_1 = \theta_2$)
LM	Only need $\hat{\theta}_R$	$\hat{\theta}_R$ often difficult to estimate

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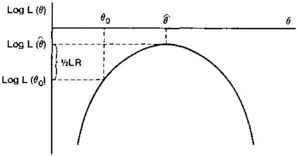

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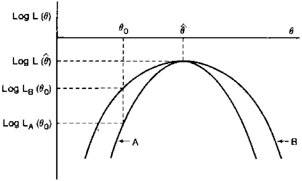

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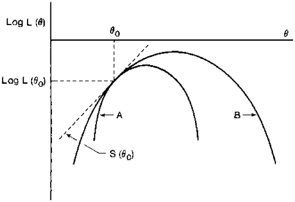

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W	Only need $\hat{\theta}_{UR}$ Easily robustified by sandwich	Not invariant to transformation (e.g. $\theta_1/\theta_2 = 1$ vs. $\theta_1 = \theta_2$)
LM	Only need $\hat{\theta}_R$	$\hat{\theta}_R$ often difficult to estimate

Comparing The Three Tests

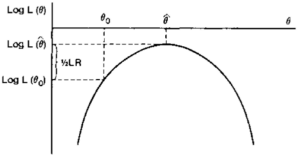

Figure 1. The Likelihood Ratio Test

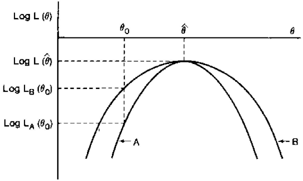

Figure 2. The Wald Test

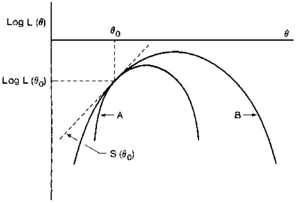

Figure 3. The Lagrange Multiplier Test

- All asymptotically equivalent
- But can be quite different in small samples

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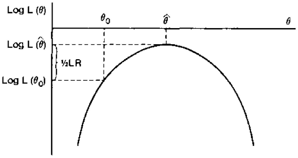

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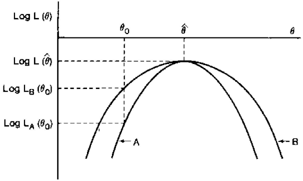

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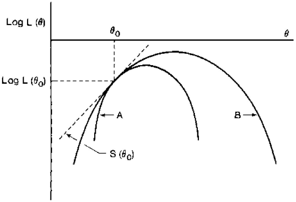

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Generalized Linear Models

- Many of the models we have learned so far all assume that Y_i is a (stochastic) function of the linear predictor, $X_i^\top \beta$
- They also share many characteristics, e.g. the form of the score and Hessian functions
- In fact, many are special cases of the generalized linear model (GLM)
- Here, we provide a general treatment of GLMs to study those models more systematically
- 3 components of a GLM
  1. Systematic component: $X_i^\top \beta$
     - Must be a linear function of X_i
  2. Random component: $f(Y; \theta, \phi)$
     - Must be in the exponential family
     - θ is called the canonical parameter
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Any distribution in the exponential family has the density of the following form:

$$f(y|\theta, \phi) = \exp\left\{\frac{y\theta - b(\theta)}{a(\phi)} + c(y, \phi)\right\}$$

Example: Poisson(λ):

$$\Pr(Y_i = y \mid \lambda) = \frac{\exp(-\lambda)\lambda^y}{y!} = \exp\left\{y \log\lambda - \exp(\log\lambda) - \log y!\right\}$$

$\Longrightarrow \theta = \log\lambda$, $\phi = 1$, $a(\phi) = \phi, b(\theta) = \exp(\theta)$, and $c = -\log y!$

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Properties of the Exponential Family

- Mean is a function of θ and given by

$$\mathsf{E}(Y) \equiv \mu = b'(\theta)$$

- Variance is a function of θ and ϕ and given by

$$\mathsf{Var}(Y) \equiv V = b''(\theta)a(\phi)$$

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- Link function: $g(\mu_i) = X_i^\top \beta$
- Defines the relationship between $X_i^\top \beta$ and the mean μ_i
- Must map the real line onto the possible range of μ_i
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- Therefore, θ_i is always a (often simple) function of $X_i^\top \beta$
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- Must be monotonic and differentiable
- This allows us to express the mean function as: $\mu_i = g^{-1}(X_i^\top \beta)$

Link Functions

- Link function: $g(\mu_i) = X_i^\top \beta$
- Defines the relationship between $X_i^\top \beta$ and the mean μ_i
- Must map the real line onto the possible range of μ_i

- Recall that $\mu_i = b'(\theta_i)$
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- In particular, when $\theta_i = X_i^\top \beta$, the link is called the canonical link
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Likelihood Function, Score, and Information Matrix

- Log-likelihood function:

$$l_n(\theta, \phi; Y) \;=\; \sum_{i=1}^{n} \left\{ \frac{Y_i\theta_i - b(\theta_i)}{a(\phi)} + c(Y_i, \phi) \right\}$$

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Assessing Goodness of Fit for GLMs

- Null model: Predict every observation by sample mean $\bar{Y}$
  $\Rightarrow$ one parameter, maximum data reduction
- Saturated model: Predict every observation by its own value Y_i
  $\Rightarrow$ n parameters, no data reduction
- A useful model sits somewhere in between

$$\boxed{\begin{array}{c}\text{Null Model}\\ (\hat{\mu}_i = \bar{Y})\\ K = 1\end{array}} \longleftarrow \underset{1<K<n}{\text{Useful Models}} \longrightarrow \boxed{\begin{array}{c}\text{Saturated Model}\\ (\hat{\mu}_i = Y_i)\\ K = n\end{array}}$$

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Analysis of Deviance

- Scaled Deviance (Unscaled: $\phi \times$ Deviance $D(Y, \hat{\theta})$):

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\begin{aligned}
D^*(Y;\hat{\theta}) &\equiv 2\{l_n(\tilde{\theta}; Y, \phi) - l_n(\hat{\theta}; Y, \phi)\} \\
&= 2\sum_{i=1}^{n}\left\{Y_i(\tilde{\theta}_i - \hat{\theta}_i) - (b(\tilde{\theta}_i) - b(\hat{\theta}_i))\right\}/a(\phi)
\end{aligned}
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where $\begin{cases} \hat{\theta}_i = \theta(\hat{\mu}_i) & \text{(estimate from the model of interest)} \\ \tilde{\theta}_i = \theta(Y_i) & \text{(``estimate'' from the saturated model)} \end{cases}$

- Note that D^* is a likelihood-ratio statistic
- This implies $D^*(Y;\hat{\theta}) \overset{approx.}{\sim} \chi^2_{n-k}$ if the model fits the data well
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Unscaled Deviance for Normal GLMs

$$
\begin{aligned}
\hat{\theta}_i &= \boldsymbol{X}_i^{'}\boldsymbol{\beta} \\
b(\hat{\theta}_i) &= \hat{\theta}^2/2 \\
\text{Saturated model} &\Rightarrow \tilde{\theta}_i = y_i; b(\tilde{\theta}_i) = y_i^2/2 \\
\text{Deviance} &= 2\sum_{i=1}^{N}\left[y_i(y_i - \hat{\mu}_i) - y_i^2/2 + \hat{\mu}_i^2/2)\right] = \sum_{i=1}^{N}(y_i - \hat{\mu}_i)^2
\end{aligned}
$$

Deviance for Poisson

Saturated model $\tilde{\lambda}_i = y_i$. This implies a log-likelihood of:

$$\begin{aligned}
\tilde{\theta}_i &= \log y_i \\
b(\tilde{\theta})_i &= \exp(\log y_i) = y_i \\
\text{Deviance} &= 2\sum_{i=1}^{N}\left[y_i \log \frac{y_i}{\hat{\lambda}_i} - y_i + \hat{\lambda}_i\right]
\end{aligned}$$

Residuals and Model Checking

- In normal linear regression, $D = \mathcal{X}^2 = \sum_{i=1}^{n} \hat{\epsilon}_i^2 = RSS$
- This suggests the following generalization of residuals for GLM:

  1. Deviance residual:
  $$\hat{\epsilon}_i^D \equiv \text{sign}(Y_i - \hat{\mu}_i)\sqrt{d_i}$$
  where d_i is the deviance for the ith observation
  2. Pearson residual:
  $$\hat{\epsilon}_i^P \equiv \frac{\sqrt{\omega_i}(Y_i - \hat{\mu}_i)}{\sqrt{b''(\hat{\theta}_i)}}$$

- These residuals have approximately the same properties as OLS residuals when N is large
- Thus, most regression diagnostics for linear models also work for GLMs:
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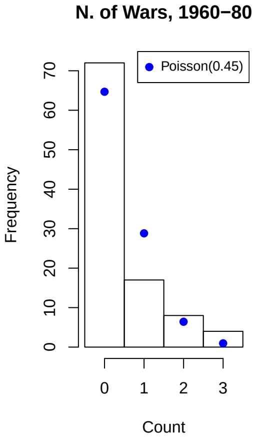
Example: Democracy and War Involvement

Benoit (1996):

- Y_i: $\#$ of involvement in international wars, 1960–80
- X_i: democracy (Freedom House score), population, military capacity, economic interdependence



Coefficients:

	Est.	s.e.		z	p
(Int.)	-3.97	1.62	*	-2.45	0.014
fh73	-0.06	0.04		-1.49	0.136
lpopln70	0.62	0.30	*	2.07	0.039
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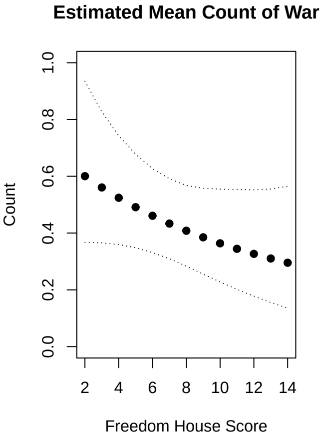

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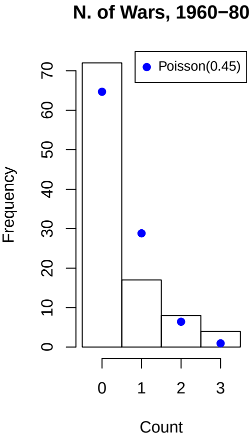

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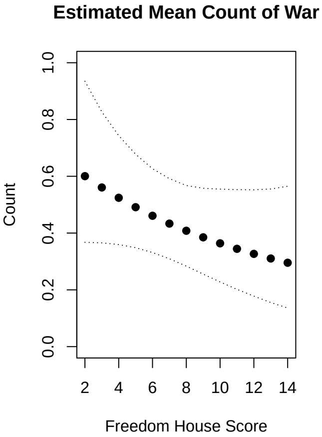

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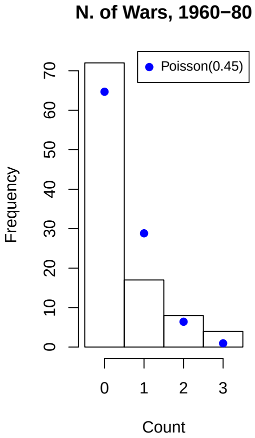

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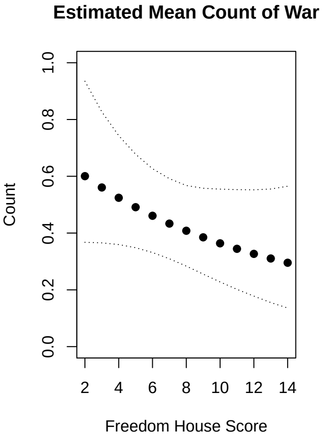

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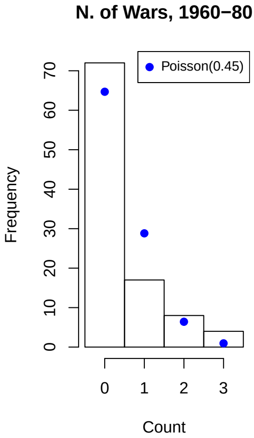
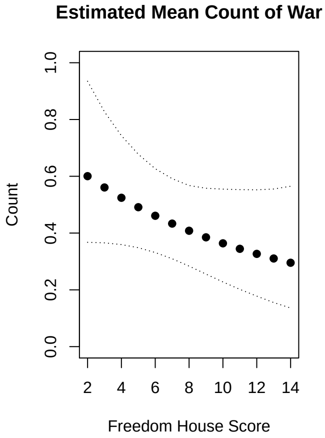


Coefficients:

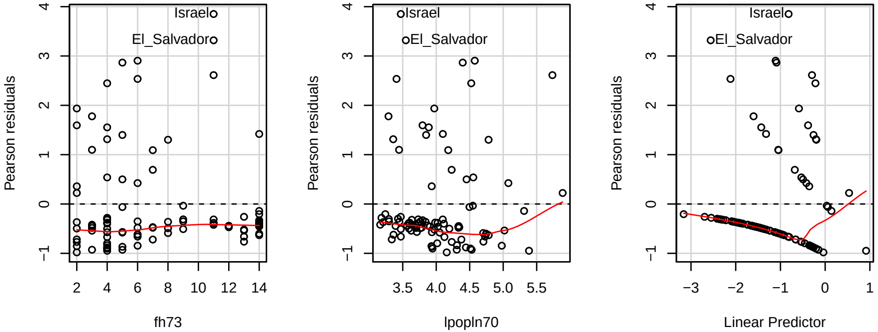
	Est.	s.e.	
(Int.)	-3.97	1.62	*
fh73	-0.06	0.04	
lpopln70	0.62	0.30	*
lmilwp70	1.29	0.45	**
ecintdep	-1.28	1.11	

z	p
-2.45	0.014
-1.49	0.136
2.07	0.039
2.85	0.004
-1.16	0.247



Example: Democracy and War Involvement

Plotting $\hat{\epsilon}_i^P$ against X_{ij} and $X_i^\top \hat{\beta}$:



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- Heavy skew to the right:
  - Nonnegativity of the outcome variable + small sample size
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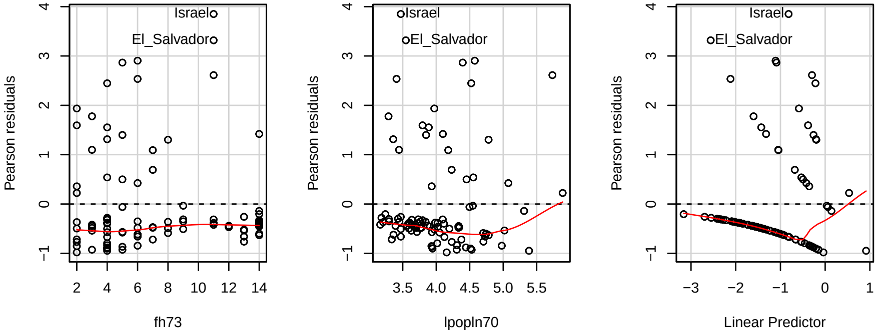

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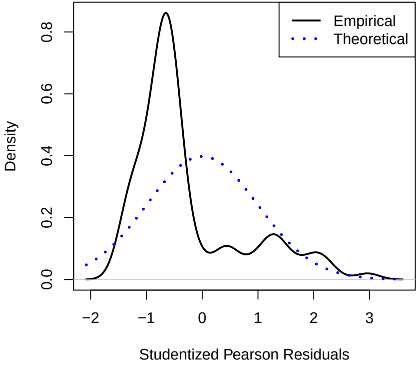

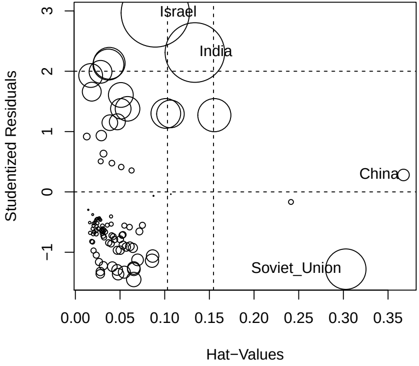

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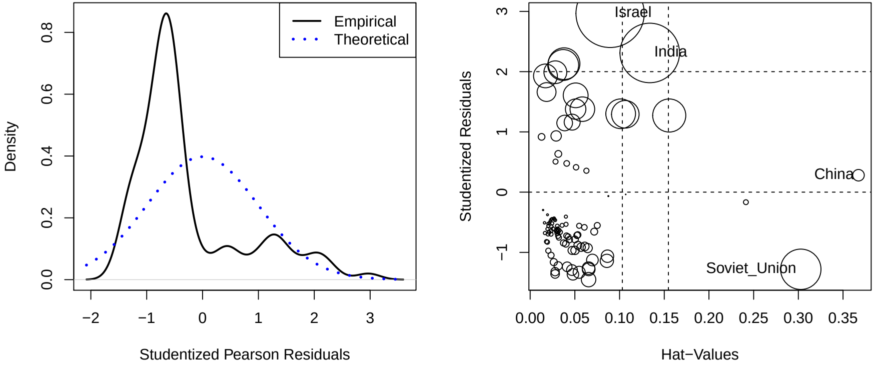

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Model fit