Political Methodology III: Model Based Inference

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Model Based Inference

1) Likelihood inference
2) Machine Learning
    a) Model Selection
    b) Unsupervised Latent Features
    c) Classification/Prediction/Regression

Supervised Learning Methods

1) Task
    - Classify objects into pre existing categories
    - Predict some future outcome
    - Learn a response surface for causal inference
2) Objective function
    1) Penalized Regressions
        - Ridge regression
        - LASSO regression
3) Optimization
    1) Ridge regression $\rightsquigarrow$ Straightforward modification of OLS
    2) LASSO regression $\rightsquigarrow$ Coordinate Descent
4) Validation
    - Obtain predicted fit for new data $f(\boldsymbol{X}_i, \widehat{\boldsymbol{\theta}})$
    - Examine prediction performance $\rightsquigarrow$ compare prediction/classification to gold standard

Regression models

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Predictions will be variable

Mean Square Error

Suppose θ is some value of the true parameter

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To reduce MSE, we are willing to induce bias to decrease variance $\rightsquigarrow$ methods that shrink coefficeints toward zero

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$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) \quad = \quad \sum_{i=1}^{N} \left(y_i - \ \beta_0 - \sum_{j=1}^{J} \beta_j x_{ij} \right)^2$$

Ridge Regression

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where:

- $\beta_0 \rightsquigarrow$ intercept

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where:

- $\beta_0 \rightsquigarrow$ intercept
- $\lambda \rightsquigarrow$ penalty parameter
- Standardized $\boldsymbol{X}$ (coefficients on same scale)

Ridge Regression$\rightsquigarrow$ Optimization

$$\boldsymbol{\beta}^{\text{Ridge}} \quad = \quad \text{arg min}_{\boldsymbol{\beta}} \left\{ f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) \right\}$$

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Demean the data and set $\beta_0 = \bar{y} = \sum_{i=1}^{N} \frac{y_i}{N}$

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&= \left(\boldsymbol{I}_j + \lambda\boldsymbol{I}_j\right)^{-1}\widehat{\boldsymbol{\beta}} \\
\beta_j^{\textsf{Ridge}} &= \frac{\widehat{\beta}_j}{1+\lambda}
\end{aligned}
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Ridge Regression$\rightsquigarrow$ Intuition (2)

$$\begin{aligned}\boldsymbol{\beta}_j & \sim \textsf{Normal}(0, \tau^2) \\ y_i & \sim \textsf{Normal}(\beta_0 + \boldsymbol{x}_i^{'}\boldsymbol{\beta}, \sigma^2)\end{aligned}$$

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$$p(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) \quad \propto \quad \prod_{j=1}^{J} p(\beta_j) \prod_{i=1}^{N} p(y_i|\boldsymbol{x}_i, \boldsymbol{\beta})$$

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Ridge Regression⇝ Intuition (2)

$$\log p(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) \quad = \quad -\sum_{j=1}^{J} \frac{\beta_j^2}{2\tau^2} - \sum_{i=1}^{N} \frac{(y_i - \beta_0 - \boldsymbol{x}'\boldsymbol{\beta})^2}{2\sigma^2}$$

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where:

- $\lambda = \frac{\sigma^2}{\tau^2}$

Ridge Regression ⇝ Intuition (3)

Definition

Suppose $\boldsymbol{X}$ is an $N \times J$ matrix. Then $\boldsymbol{X}$ can be written as:

$$\boldsymbol{X} = \underbrace{\boldsymbol{U}}_{N\times N}\underbrace{\boldsymbol{S}}_{N\times J}\underbrace{\boldsymbol{V}^{'}}_{J\times J}$$

Where:

$$\begin{aligned}\boldsymbol{U}^{'}\boldsymbol{U} &= \boldsymbol{I}_N \\ \boldsymbol{V}^{'}\boldsymbol{V} &= \boldsymbol{V}\boldsymbol{V}^{'} = \boldsymbol{I}_J\end{aligned}$$

$\boldsymbol{S}$ contains $\min(N, J)$ singular values, $\sqrt{\lambda_j} \geq 0$ down the diagonal and then 0's for the remaining entries

Ridge Regression $\rightsquigarrow$ Intuition (3)

Recall: PCA:

$$\frac{1}{N}\boldsymbol{X}^{'}\boldsymbol{X} = \underbrace{\boldsymbol{W}}_{\text{eigenvectors}} \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \lambda_J \end{pmatrix} \underbrace{\boldsymbol{W}^{'}}_{\text{eigenvectors}}$$

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$$\begin{aligned} \frac{1}{N}\boldsymbol{X}^{'}\boldsymbol{X} &= \boldsymbol{V}\boldsymbol{S}^{'}\underbrace{\left(\boldsymbol{U}^{'}\boldsymbol{U}\right)}_{\boldsymbol{I}_J}\boldsymbol{S}\boldsymbol{V}^{'} \\ &= \frac{1}{N}\boldsymbol{V}\boldsymbol{S}^{'}\boldsymbol{S}\boldsymbol{V}^{'} \\ &= \underbrace{\boldsymbol{V}}_{\text{eigenvectors}} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_J \end{pmatrix} \underbrace{\boldsymbol{V}^{'}}_{\text{eigenvectors}} \end{aligned}$$

Ridge Regression ⇝ Intuition (3)

We can write the predicted values for a regular regression as

$$\begin{aligned}
\hat{Y} &= \boldsymbol{X}\hat{\boldsymbol{\beta}} \\
&= \boldsymbol{X}\left(\boldsymbol{X}^{'}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} \\
&= \boldsymbol{U}\boldsymbol{U}^{'}\boldsymbol{Y} = \sum_{j=1}^{J}\boldsymbol{u}_j\boldsymbol{u}_j^{'}\boldsymbol{Y}
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Where

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Which we can write as:

$$\hat{Y}^{\text{ridge}} = \sum_{j=1}^{J}\boldsymbol{u}_j\frac{\lambda_j}{\lambda_j + \lambda}\boldsymbol{u}_j^{'}\boldsymbol{Y}$$

Degrees of Freedom for Ridge

We will say that the degrees of freedom for Ridge regression with penalty λ is

$$\text{dof}(\lambda) \quad = \quad \sum_{j=1}^{J} \frac{\lambda_j}{\lambda_j + \lambda}$$

Lasso Regression Objective Function

Different Penalty for Model Complexity

$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) = \sum_{i=1}^{N} \left(y_i - \beta_0 - \sum_{j=1}^{J} \beta_j x_{ij} \right)^2 + \lambda \sum_{j=1}^{J} \underbrace{|\beta_j|}_{\text{Penalty}}$$

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Lasso Regression Optimization

Definition

Coordinate Descent Algorithms:
Consider $g : \Re^J \rightarrow \Re$. Our goal is to find $\boldsymbol{x}^* \in \Re^J$ such that $g(\boldsymbol{x}^*) \leq g(\boldsymbol{x})$ for all $\boldsymbol{x} \in \Re$.
To find $\boldsymbol{x}^*$:
Until convergence: for each iteration t and each coordinate j

$$x_j^{t+1} = \text{arg min}_{x_j \in \Re} g(x_1^{t+1}, x_2^{t+1}, \ldots, x_{j-1}^{t+1}, x_j, x_{j+1}^t, \ldots, x_J^t)$$

Lasso Regression Optimization: Coordinate Descent

$$\tilde{f}(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) \quad = \quad \frac{1}{2N}\sum_{i=1}^{N}\left(y_i - \beta_0 - \sum_{j=1}^{J}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{J}|\beta_j|$$

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Lasso Regression$\rightsquigarrow$ Intuition 1, Soft Thresholding

Suppose again $\boldsymbol{X}^{'}\boldsymbol{X} = \boldsymbol{I}_J$

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Lasso Regression$\rightsquigarrow$ Intuition 1, Soft Thresholding

Compare soft assignment

Lasso Regression⇝ Intuition 1, Soft Thresholding

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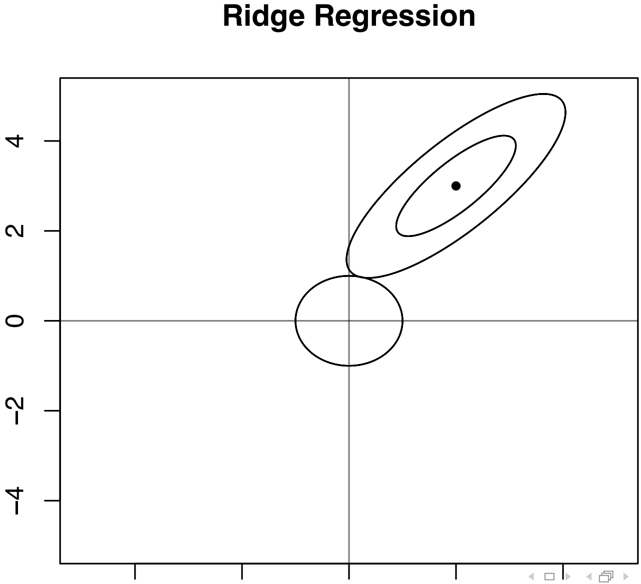
Why does LASSO induce sparsity?

Comparing Ridge and LASSO



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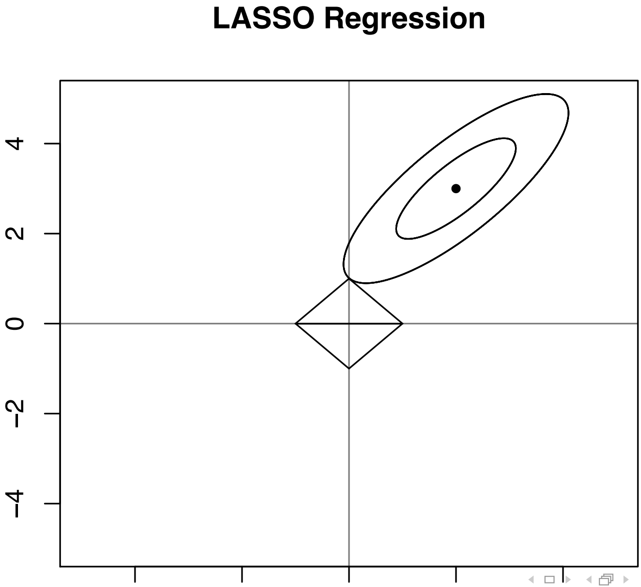

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\sum_{j=1}^{2} \beta_j^2 &= \frac{1}{2} + \frac{1}{2} = 1 \\
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Ridge and LASSO: The Elastic-Net

Combining the two criteria $\rightsquigarrow$ Elastic-Net

$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) \quad = \quad \frac{1}{2N} \sum_{i=1}^{N} \left(y_i - \beta_0 - \sum_{j=1}^{J} \beta_j x_{ij} \right)^2 + \lambda \sum_{j=1}^{J} \left(\frac{1}{2}(1-\alpha)\beta_j^2 + \alpha|\beta_j| \right)$$

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The new update step (for coordinate descent:)

$$\beta_j \quad \leftarrow \quad \frac{\textsf{sign}(r^j)\textsf{max}(|r^j| - \lambda\alpha, 0)}{1 + \lambda(1-\alpha)}$$

Selecting λ

How do we determine λ? $\rightsquigarrow$ Cross validation (Recall code from Monday)

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Applying models gives score (probability) of document belong to class$\rightsquigarrow$ threshold to classify

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Credit Claiming (Grimmer, Westwood, and Messing 2014)

```
library(glmnet)
set.seed(8675309) ##setting seed
folds<- sample(1:10, nrow(dtm), replace=T) ##assigning to fold
out_of_samp<- c() ##collecting the predictions
```

Credit Claiming (Grimmer, Westwood, and Messing 2014)

```
for(z in 1:10){
train<- which(folds!=z) ##the observations we will use to train the model

test<- which(folds==z) ##the observations we will use to test the model
part1<- cv.glmnet(x = dtm[train,], y = credit[train], alpha = 1, family =
b́inomial)́ ##fitting the LASSO model on the data.
## alpha = 1 -> LASSO
## alpha = 0 -> RIDGE
## 0<alpha<1 -> Elastic-Net
out_of_samp[test]<- predict(part1, newx= dtm[test,], s = part1$lambda.min,
type =ćlass)́ ##predicting the labels
print(z) ##printing the labels
}
conf_table<- table(out_of_samp, credit) ##calculating the confusion table
> round(sum(diag(conf_table))/len(credit), 3)
[1] 0.844
```

Generalized Cross Validation and Ridge Regression

In some special cases there are analytic solutions:

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Generalized Cross Validation and Ridge Regression

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Generalized Cross Validation and Ridge Regression

Why do we care?

Generalized Cross Validation and Ridge Regression

Why do we care?
Leave one out cross validation

Generalized Cross Validation and Ridge Regression

Why do we care?
Leave one out cross validation

$$\text{Cross Validation}(1) \quad = \quad \frac{1}{N}\sum_{i=1}^{N}(Y_i - f(\boldsymbol{X}_{-i}, \boldsymbol{Y}_{-i}, \lambda, \hat{\boldsymbol{\beta}}))^2$$

Generalized Cross Validation and Ridge Regression

Why do we care?
Leave one out cross validation

$$\begin{aligned}
\text{Cross Validation}(1) &= \frac{1}{N}\sum_{i=1}^{N}(Y_i - f(\boldsymbol{X}_{-i}, \boldsymbol{Y}_{-i}, \lambda, \hat{\boldsymbol{\beta}}))^2 \\
&= \frac{1}{N}\sum_{i=1}^{N}\left(\frac{Y_i - f(\boldsymbol{X}, \boldsymbol{Y}, \lambda, \hat{\boldsymbol{\beta}})}{1 - H_{ii}}\right)^2
\end{aligned}$$

Generalized Cross Validation and Ridge Regression

Calculating $\boldsymbol{H}$ can be computationally expensive

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