Political Methodology III: Model Based Inference

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IN GOD WE TRUST

Going Public (Franco, Grimmer, and Whang 2017)

- Presidents interrupt prime time coverage $\rightsquigarrow$ why? what effect?
- There are at least 6 explanations (Canes Wrone 2001, 2006) (others: credit claim, veto bargain, beauty contest, highlight obstruction, stupid)
- Gathered serendipitous surveys $\rightsquigarrow$ happen to be in field when presidents go public (before $\rightsquigarrow$ control; after $\rightsquigarrow$ treatment) (we also have social media data and newspapers)
- Does going public increase probability respondents identify topic of president's speech as salient problem?
- Do respondents identify it as most important problem?

Modeling Bivariate Responses

$$\begin{aligned} Y_i &\in \{0, 1\} \\ X_{i1} &= \text{Treatment status (0/1)} \\ X_{i2} &= \text{Republican (0/1)} \end{aligned}$$

Infer effect of going public, condition on Republican as well

Modeling Bivariate Responses

$$\begin{aligned} Y_i &\in \{0,1\} \\ X_{i1} &= \text{ Treatment status (0/1)} \\ X_{i2} &= \text{Republican (0/1)} \end{aligned}$$

Infer effect of going public, condition on Republican as well

	Control	Treat
$\bar{Y}$	0.373	0.367

Linear Probabilty Model

$$
\begin{aligned}
Y_i & \sim \textsf{Normal}(\mu_i, \sigma^2) \\
E[Y_i|X_i, \boldsymbol{\beta}] & = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2}
\end{aligned}
$$

Linear Probabilty Model

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Y_i &\sim \textsf{Normal}(\mu_i, \sigma^2) \\
E[Y_i|X_i, \boldsymbol{\beta}] &= \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2}
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$\beta_1 =$ Increase in the probability respondent i identifies problem as most important problem.

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\end{aligned}
$$

$\beta_1 =$ Increase in the probability respondent i identifies problem as most important problem.
Issues:

1) Predictions outside of 0 and 1
2) Constant effect
3) Efficiency loss

Probit Model

Suppose
$Y_i \sim \text{Bernoulli}(\pi_i)$

Probit Model

Suppose

$Y_i \sim$ Bernoulli(π_i)

Assume:

$$\tilde{Y}_i \sim \text{Normal}(\mu_i, 1)$$

Probit Model

Suppose
$Y_i \sim$ Bernoulli(π_i)
Assume:

$$\begin{aligned}
\tilde{Y}_i &\sim \text{Normal}(\mu_i, 1) \\
\tilde{Y}_i &= \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i
\end{aligned}$$

Probit Model

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$Y_i \sim$ Bernoulli(π_i)
Assume:

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\tilde{Y}_i &\sim \text{Normal}(\mu_i, 1) \\
\tilde{Y}_i &= \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i \\
\epsilon_i &\sim \text{Normal}(0, 1)
\end{aligned}$$

Probit Model

Suppose
$Y_i \sim \text{Bernoulli}(\pi_i)$
Assume:

$$
\begin{aligned}
\tilde{Y}_i &\sim \text{Normal}(\mu_i, 1) \\
\tilde{Y}_i &= \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i \\
\epsilon_i &\sim \text{Normal}(0, 1) \\
Y_i &= I(\tilde{Y}_i > 0)
\end{aligned}
$$

Probit Model

Suppose
$Y_i \sim \text{Bernoulli}(\pi_i)$
Assume:

$$
\begin{aligned}
\tilde{Y}_i &\sim \text{Normal}(\mu_i, 1) \\
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\epsilon_i &\sim \text{Normal}(0, 1) \\
Y_i &= I(\tilde{Y}_i > 0)
\end{aligned}
$$

We will write:

Probit Model

Suppose
$Y_i \sim \text{Bernoulli}(\pi_i)$
Assume:

$$
\begin{aligned}
\tilde{Y}_i &\sim \text{Normal}(\mu_i, 1) \\
\tilde{Y}_i &= \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i \\
\epsilon_i &\sim \text{Normal}(0, 1) \\
Y_i &= I(\tilde{Y}_i > 0)
\end{aligned}
$$

We will write:
$\boldsymbol{X}_i = (1, X_{i1}, X_{i2})$
$\boldsymbol{\beta} = (\beta_0, \beta_1, \beta_2)$
$\boldsymbol{X}_i \boldsymbol{\beta} = \beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2}$

Probit Model

$$\pi_i = P(Y_i = 1) \quad = \quad P(\tilde{Y}_i > 0)$$

Probit Model

$$\begin{aligned}\pi_i = P(Y_i = 1) &= P(\tilde{Y}_i > 0) \\ &= P(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i > 0)\end{aligned}$$

Probit Model

$$\begin{aligned}
\pi_i = P(Y_i = 1) &= P(\tilde{Y}_i > 0) \\
&= P(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i > 0) \\
&= P(\epsilon_i > -\boldsymbol{X}_i^{'}\boldsymbol{\beta})
\end{aligned}$$

Probit Model

$$
\begin{aligned}
\pi_i = P(Y_i = 1) &= P(\tilde{Y}_i > 0) \\
&= P(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i > 0) \\
&= P(\epsilon_i > -\boldsymbol{X}_i^{'}\boldsymbol{\beta}) \\
&= P(\epsilon_i < \boldsymbol{X}_i^{'}\boldsymbol{\beta})
\end{aligned}
$$

Probit Model

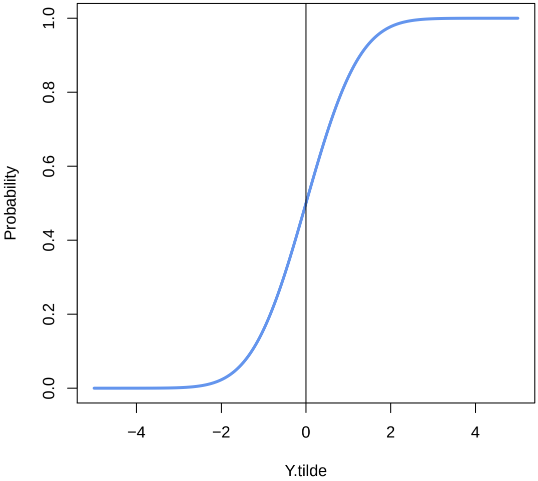
$$
\begin{aligned}
\pi_i = P(Y_i = 1) &= P(\tilde{Y}_i > 0) \\
&= P(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i > 0) \\
&= P(\epsilon_i > -\boldsymbol{X}_i^{'}\boldsymbol{\beta}) \\
&= P(\epsilon_i < \boldsymbol{X}_i^{'}\boldsymbol{\beta}) \\
&= \int_{-\infty}^{\boldsymbol{X}_i^{'}\boldsymbol{\beta}} \phi(\epsilon_i) d\epsilon_i
\end{aligned}
$$

Probit Model

$$
\begin{aligned}
\pi_i = P(Y_i = 1) &= P(\tilde{Y}_i > 0) \\
&= P(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i > 0) \\
&= P(\epsilon_i > -\boldsymbol{X}^{'}_i \boldsymbol{\beta}) \\
&= P(\epsilon_i < \boldsymbol{X}^{'}_i \boldsymbol{\beta}) \\
&= \int_{-\infty}^{\boldsymbol{X}^{'}_i \boldsymbol{\beta}} \phi(\epsilon_i) d\epsilon_i \\
&= \Phi(\boldsymbol{X}^{'}_i \boldsymbol{\beta})
\end{aligned}
$$

Probit Model

$$
\begin{aligned}
\pi_i = P(Y_i = 1) &= P(\tilde{Y}_i > 0) \\
&= P(\beta_0 + \beta_1 X_{i1} + \beta_2 X_{i2} + \epsilon_i > 0) \\
&= P(\epsilon_i > -\boldsymbol{X}_i^{'}\boldsymbol{\beta}) \\
&= P(\epsilon_i < \boldsymbol{X}_i^{'}\boldsymbol{\beta}) \\
&= \int_{-\infty}^{\boldsymbol{X}_i^{'}\boldsymbol{\beta}} \phi(\epsilon_i) d\epsilon_i \\
&= \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})
\end{aligned}
$$



Probability
0.0
0.2
0.4
0.6
0.8
1.0
−4
−2
0
2
4
Y.tilde

Probit Model

$$\begin{aligned} Y_i &\sim \text{Bernoulli}(\pi_i) \\ \pi_i &= \Phi(\boldsymbol{X}^{'}\boldsymbol{\beta}) \end{aligned}$$

Probit Model

$$
\begin{aligned}
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\pi_i &= \Phi(\boldsymbol{X}^{'}\boldsymbol{\beta})
\end{aligned}
$$

Then our likelihood:

Probit Model

$$\begin{aligned} Y_i &\sim \text{Bernoulli}(\pi_i) \\ \pi_i &= \Phi(\boldsymbol{X}^{'}\boldsymbol{\beta}) \end{aligned}$$

Then our likelihood:

$$L(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) = f(\boldsymbol{Y}|\boldsymbol{X}, \boldsymbol{\beta})$$

Probit Model

$$
\begin{aligned}
Y_i &\sim \textsf{Bernoulli}(\pi_i) \\
\pi_i &= \Phi(\boldsymbol{X}^{'}\boldsymbol{\beta})
\end{aligned}
$$

Then our likelihood:

$$
\begin{aligned}
L(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) &= f(\boldsymbol{Y}|\boldsymbol{X}, \boldsymbol{\beta}) \\
&= \prod_{i=1}^{N} f(Y_i|\boldsymbol{X}_i\boldsymbol{\beta})
\end{aligned}
$$

Probit Model

$$\begin{aligned} Y_i &\sim \text{Bernoulli}(\pi_i) \\ \pi_i &= \Phi(\boldsymbol{X}'\boldsymbol{\beta}) \end{aligned}$$

Then our likelihood:

$$\begin{aligned} L(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) &= f(\boldsymbol{Y}|\boldsymbol{X}, \boldsymbol{\beta}) \\ &= \prod_{i=1}^{N} f(Y_i|\boldsymbol{X}_i \boldsymbol{\beta}) \\ &= \prod_{i=1}^{N} \pi_i^{Y_i}(1-\pi_i)^{1-Y_i} \end{aligned}$$

Probit Model

$$\begin{aligned} Y_i &\sim \textsf{Bernoulli}(\pi_i) \\ \pi_i &= \Phi(\boldsymbol{X}^{'}\boldsymbol{\beta}) \end{aligned}$$

Then our likelihood:

$$\begin{aligned} L(\boldsymbol{\beta}|\boldsymbol{X},\boldsymbol{Y}) &= f(\boldsymbol{Y}|\boldsymbol{X},\boldsymbol{\beta}) \\ &= \prod_{i=1}^{N} f(Y_i|\boldsymbol{X}_i\boldsymbol{\beta}) \\ &= \prod_{i=1}^{N} \pi_i^{Y_i}(1-\pi_i)^{1-Y_i} \\ &= \prod_{i=1}^{N} \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})^{Y_i}(1-\Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}))^{1-Y_i} \end{aligned}$$

Probit Model

$$L(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) \quad = \quad \prod_{i=1}^{N} \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})^{Y_i}(1 - \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}))^{1-Y_i}$$

Probit Model

$$\begin{aligned}
L(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) &= \prod_{i=1}^{N} \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})^{Y_i}(1 - \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}))^{1-Y_i} \\
\log L(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) &= \sum_{i=1}^{N} \left(Y_i \log \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}) + (1 - Y_i)\log(1 - \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}))\right)
\end{aligned}$$

Probit Model

$$
\begin{aligned}
L(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) &= \prod_{i=1}^{N} \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})^{Y_i}(1 - \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}))^{1-Y_i} \\
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\end{aligned}
$$

Maximize likelihood with respect to $\boldsymbol{\beta}$

Probit Model

$$l(\boldsymbol{\beta}|\boldsymbol{X},\boldsymbol{Y}) \quad = \quad \sum_{i=1}^{N}\left(Y_i \log \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}) + (1-Y_i)\log(1-\Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}))\right)$$

Probit Model

$$l(\boldsymbol{\beta}|\boldsymbol{X},\boldsymbol{Y}) \quad = \quad \sum_{i=1}^{N}\Big(Y_i \log \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}) + (1-Y_i)\log(1-\Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}))\Big)$$

Score Function for β_k:

$$\frac{\partial l(\boldsymbol{\beta}|\boldsymbol{X}_i, Y_i)}{\partial \beta_k} \quad = \quad \left(Y_i \frac{\phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})}{\Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})} - (1-Y_i)\frac{\phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})}{1-\Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})}\right) X_k$$

Probit Model

$$l(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) = \sum_{i=1}^{N} \left(Y_i \log \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}) + (1 - Y_i) \log(1 - \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})) \right)$$

Score Function for β_k:

$$\frac{\partial l(\boldsymbol{\beta}|\boldsymbol{X}_i, Y_i)}{\partial \beta_k} = \left(Y_i \frac{\phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})}{\Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})} - (1 - Y_i) \frac{\phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})}{1 - \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})} \right) X_k$$

Hessian (typical entry h_{kj}):

$$\begin{aligned}\frac{\partial^2 l(\boldsymbol{\beta}|\boldsymbol{X}_i, Y_i)}{\partial \beta_k \partial \beta_j} &= \phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})[Y_i \frac{\phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}) + \boldsymbol{X}_i\boldsymbol{\beta}\Phi(\boldsymbol{X}_i\boldsymbol{\beta})}{\Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta})^2} \\ &\quad + (1 - Y_i) \frac{\phi(\boldsymbol{X}_i\boldsymbol{\beta}) - \boldsymbol{X}_i^{'}\boldsymbol{\beta}(1 - \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}))}{(1 - \Phi(\boldsymbol{X}_i\boldsymbol{\beta}))^2}] X_{ik} X_{ij}\end{aligned}$$

Probit Model

Maximum Likelihood Estimates

$$0 = \sum_{i=1}^{N} \left(Y_i \frac{\phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}^*)}{\Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}^*)} - (1 - Y_i) \frac{\phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}^*)}{1 - \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}^*)} \right) X_k$$

Solve for $\boldsymbol{\beta}$?

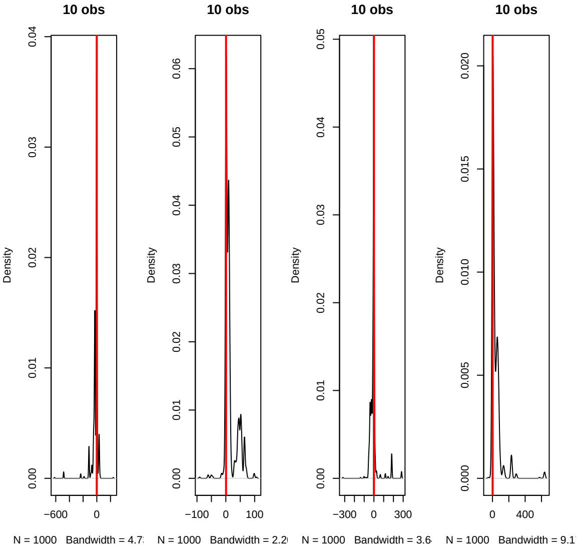
Probit Model

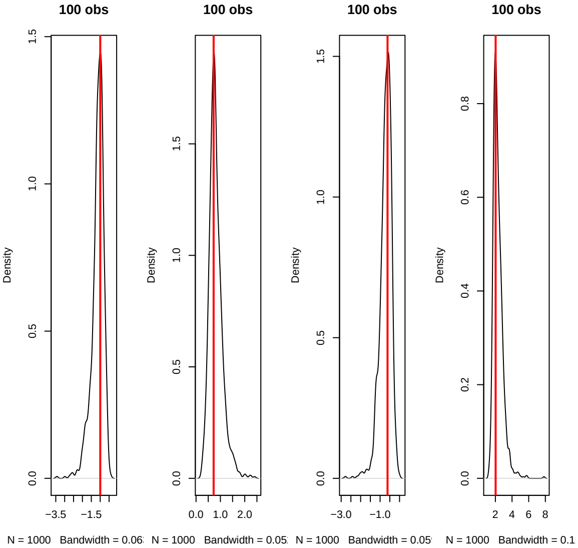
Maximum Likelihood Estimates

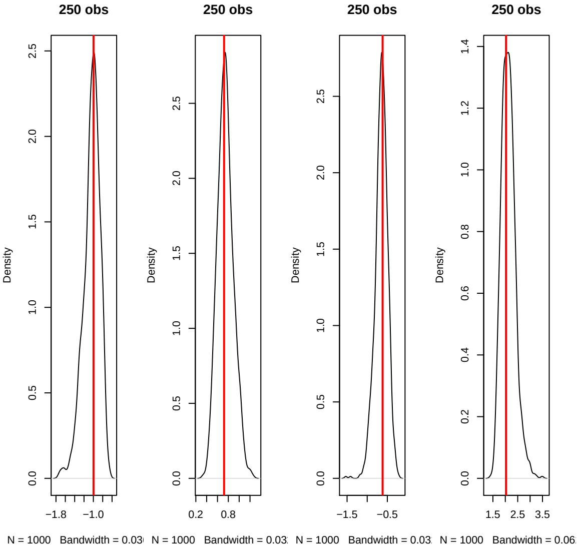
$$0 = \sum_{i=1}^{N} \left(Y_i \frac{\phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}^*)}{\Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}^*)} - (1 - Y_i)\frac{\phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}^*)}{1 - \Phi(\boldsymbol{X}_i^{'}\boldsymbol{\beta}^*)} \right) X_k$$

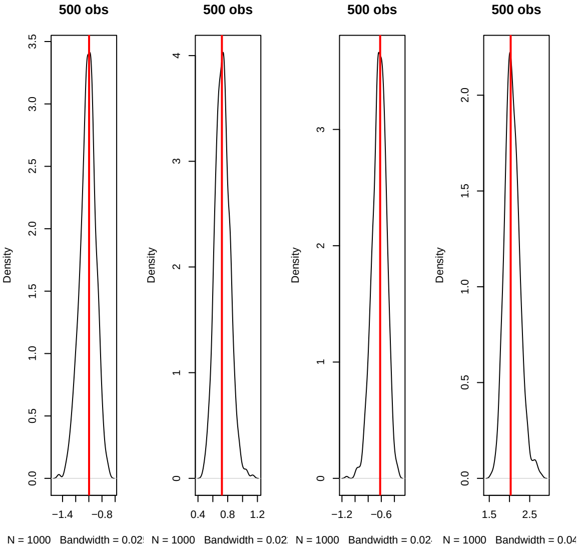
Solve for $\boldsymbol{\beta}$? Computational Approaches

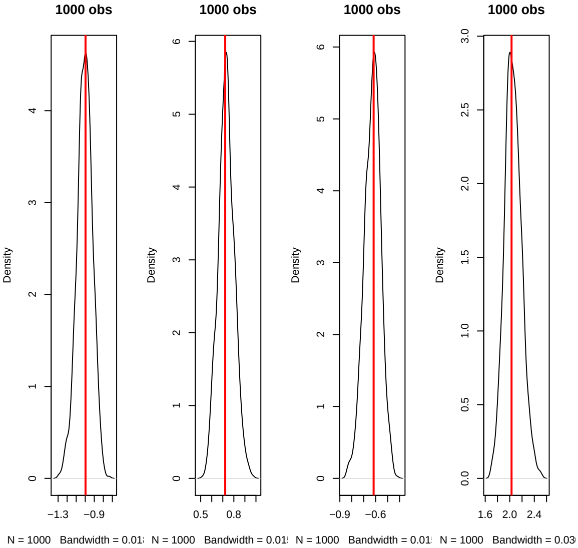
- Newton Raphson
- BFGS (Quasi-Newton. Requires only an approximate Hessian.)


10 obs
10 obs
10 obs
10 obs
Density
Density
Density
Density
N = 1000 Bandwidth = 4.7
N = 1000 Bandwidth = 2.2
N = 1000 Bandwidth = 3.6
N = 1000 Bandwidth = 9.1


100 obs
100 obs
100 obs
100 obs
Density
Density
Density
Density
N = 1000 Bandwidth = 0.06
N = 1000 Bandwidth = 0.05
N = 1000 Bandwidth = 0.05
N = 1000 Bandwidth = 0.1


250 obs
250 obs
250 obs
250 obs
Density
Density
Density
Density
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.5
1.0
1.5
2.0
2.5
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
−1.8
−1.0
0.2
0.8
−1.5
−0.5
1.5
2.5
3.5
N = 1000 Bandwidth = 0.03
N = 1000 Bandwidth = 0.03
N = 1000 Bandwidth = 0.03
N = 1000 Bandwidth = 0.06


500 obs
500 obs
500 obs
500 obs
Density
Density
Density
Density
N = 1000 Bandwidth = 0.02
N = 1000 Bandwidth = 0.02
N = 1000 Bandwidth = 0.02
N = 1000 Bandwidth = 0.04


1000 obs
1000 obs
1000 obs
1000 obs
Density
Density
Density
Density
0
1
2
3
4
0
1
2
3
4
5
6
0
1
2
3
4
5
6
0.0
0.5
1.0
1.5
2.0
2.5
3.0
−1.3
−0.9
0.5
0.8
−0.9
−0.6
1.6
2.0
2.4
N = 1000 Bandwidth = 0.01
N = 1000 Bandwidth = 0.01
N = 1000 Bandwidth = 0.01
N = 1000 Bandwidth = 0.03

Probit Model: Maximum Likelihood Estimates

R Code

Probit Model: Most Important Problem

	Most Important Problem
Intercept	-0.26
	(0.03)
Post-Speech	-0.01
	(0.05)
Republican	-0.20
	(0.05)

How do we interpret the coefficients?

Probit Model: Most Important Problem

	Most Important Problem
Intercept	-0.26
	(0.03)
Post-Speech	-0.01
	(0.05)
Republican	-0.20
	(0.05)

How do we interpret the coefficients?

They are on the latent scale $\rightsquigarrow$ Need to define all values when determining predicted probabilities

Probit Model: Interpreting the Coefficients

Expected Value:

Probit Model: Interpreting the Coefficients

Expected Value:

$$E[Y_i|\boldsymbol{X}_i] = \Phi(\beta_0^* + \beta_1^* X_{i1} + \beta_2^* X_{i2})$$

Probit Model: Interpreting the Coefficients

Expected Value:

$$\begin{aligned} E[Y_i|\boldsymbol{X}_i] &= \Phi(\beta_0^* + \beta_1^* X_{i1} + \beta_2^* X_{i2}) \\ E[Y_i|X_{i1} = 0, X_{i2} = 0] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 0) = 0.40 \end{aligned}$$

Probit Model: Interpreting the Coefficients

Expected Value:

$$
\begin{aligned}
E[Y_i|\boldsymbol{X}_i] &= \Phi(\beta_0^* + \beta_1^* X_{i1} + \beta_2^* X_{i2}) \\
E[Y_i|X_{i1} = 0, X_{i2} = 0] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 0) = 0.40 \\
E[Y_i|X_{i1} = 0, X_{i2} = 1] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 1) = 0.32
\end{aligned}
$$

Probit Model: Interpreting the Coefficients

Expected Value:

$$
\begin{aligned}
E[Y_i|\boldsymbol{X}_i] &= \Phi(\beta_0^* + \beta_1^* X_{i1} + \beta_2^* X_{i2}) \\
E[Y_i|X_{i1} = 0, X_{i2} = 0] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 0) = 0.40 \\
E[Y_i|X_{i1} = 0, X_{i2} = 1] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 1) = 0.32
\end{aligned}
$$

First difference:

Probit Model: Interpreting the Coefficients

Expected Value:

$$\begin{aligned} E[Y_i|\boldsymbol{X}_i] &= \Phi(\beta_0^* + \beta_1^* X_{i1} + \beta_2^* X_{i2}) \\ E[Y_i|X_{i1}=0, X_{i2}=0] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 0) = 0.40 \\ E[Y_i|X_{i1}=0, X_{i2}=1] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 1) = 0.32 \end{aligned}$$

First difference:

$$\begin{aligned} E[Y_i|X_{i1}=1, X_{i2}=0] - E[Y_i|X_{i1}=0, X_{i2}=0] &= \\ \Phi(-0.26 - 0.01) - \Phi(-0.26) &= -0.0039 \end{aligned}$$

Probit Model: Interpreting the Coefficients

Expected Value:

$$\begin{aligned} E[Y_i|\boldsymbol{X}_i] &= \Phi(\beta_0^* + \beta_1^* X_{i1} + \beta_2^* X_{i2}) \\ E[Y_i|X_{i1} = 0, X_{i2} = 0] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 0) = 0.40 \\ E[Y_i|X_{i1} = 0, X_{i2} = 1] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 1) = 0.32 \end{aligned}$$

First difference:

$$\begin{aligned} E[Y_i|X_{i1} = 1, X_{i2} = 0] - E[Y_i|X_{i1} = 0, X_{i2} = 0] &= \\ \Phi(-0.26 - 0.01) - \Phi(-0.26) &= -0.0039 \\ E[Y_i|X_{i1} = 1, X_{i2} = 1] - E[Y_i|X_{i1} = 0, X_{i2} = 1] &= \\ \Phi(-0.26 - 0.01 - 0.2) - \Phi(-0.26 - 0.2) &= -0.0036 \end{aligned}$$

Probit Model: Interpreting the Coefficients

Expected Value:

$$\begin{aligned}
E[Y_i|\boldsymbol{X}_i] &= \Phi(\beta_0^* + \beta_1^* X_{i1} + \beta_2^* X_{i2}) \\
E[Y_i|X_{i1}=0, X_{i2}=0] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 0) = 0.40 \\
E[Y_i|X_{i1}=0, X_{i2}=1] &= \Phi(-0.26 - 0.01 \times 0 - 0.20 \times 1) = 0.32
\end{aligned}$$

First difference:

$$\begin{aligned}
E[Y_i|X_{i1}=1, X_{i2}=0] - E[Y_i|X_{i1}=0, X_{i2}=0] &= \\
\Phi(-0.26 - 0.01) - \Phi(-0.26) &= -0.0039 \\
E[Y_i|X_{i1}=1, X_{i2}=1] - E[Y_i|X_{i1}=0, X_{i2}=1] &= \\
\Phi(-0.26 - 0.01 - 0.2) - \Phi(-0.26 - 0.2) &= -0.0036
\end{aligned}$$

We will perform inference on these next lecture

Probit Model: Interpreting the Coefficients

$$\begin{aligned}
\frac{\partial \Phi(\boldsymbol{X}_i \boldsymbol{\beta})}{\partial X_{ij}} &= \phi(\boldsymbol{X}_i \beta)\beta_j \\
\textsf{Max Effect} &= 0.4 \times \beta_j
\end{aligned}$$

Logit Model

Define:

$$
\begin{aligned}
\mathsf{Odds}(\pi) &= \frac{\pi}{1-\pi} \\
\log \mathsf{Odds}(\pi) &= \log\left(\frac{\pi}{1-\pi}\right) \\
\mathsf{logit}(\pi) &\equiv \log\left(\frac{\pi}{1-\pi}\right)
\end{aligned}
$$

Logit Model

$$\log\left(\frac{\pi}{1-\pi}\right) = \alpha$$

Logit Model

$$
\begin{aligned}
\log\left(\frac{\pi}{1-\pi}\right) &= \alpha \\
\left(\frac{\pi}{1-\pi}\right) &= \exp(\alpha)
\end{aligned}
$$

Logit Model

$$
\begin{aligned}
\log\left(\frac{\pi}{1-\pi}\right) &= \alpha \\
\left(\frac{\pi}{1-\pi}\right) &= \exp(\alpha) \\
\pi &= \exp(\alpha)(1-\pi)
\end{aligned}
$$

Logit Model

$$
\begin{aligned}
\log\left(\frac{\pi}{1-\pi}\right) &= \alpha \\
\left(\frac{\pi}{1-\pi}\right) &= \exp(\alpha) \\
\pi &= \exp(\alpha)(1-\pi) \\
\pi(1+\exp(\alpha)) &= \exp(\alpha)
\end{aligned}
$$

Logit Model

$$
\begin{aligned}
\log\left(\frac{\pi}{1-\pi}\right) &= \alpha \\
\left(\frac{\pi}{1-\pi}\right) &= \exp(\alpha) \\
\pi &= \exp(\alpha)(1-\pi) \\
\pi(1+\exp(\alpha)) &= \exp(\alpha) \\
\pi &= \frac{\exp(\alpha)}{1+\exp(\alpha)} = \mathsf{Logit}^{-1}(\alpha)
\end{aligned}
$$

Logit Model

$$\begin{aligned}
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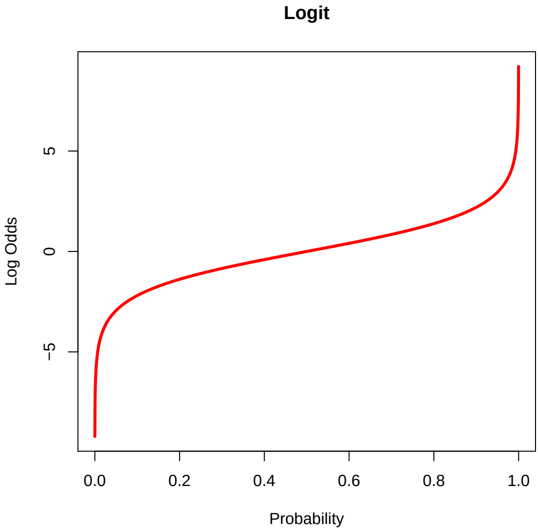
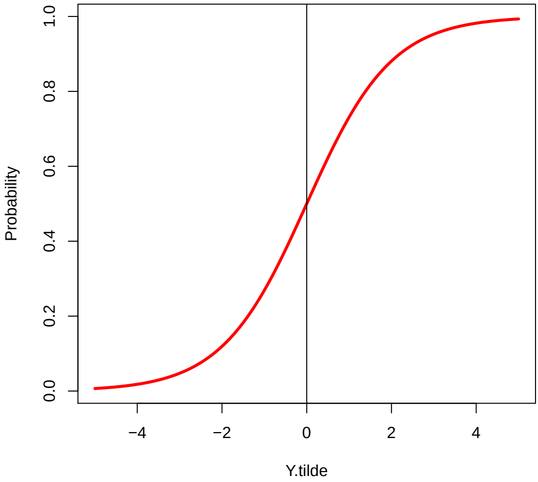
\pi(1+\exp(\alpha)) &= \exp(\alpha) \\
\pi &= \frac{\exp(\alpha)}{1+\exp(\alpha)} = \textsf{Logit}^{-1}(\alpha) \\
\frac{\exp(\alpha)}{1+\exp(\alpha)} &= \frac{1}{1+\exp(-\alpha)} = \textsf{Logistic}(\alpha)
\end{aligned}$$

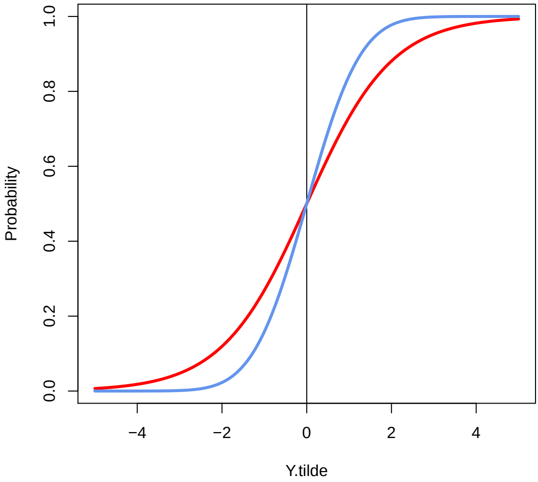
Logit Model



Logit Model



Logit Model



$$Z \sim \text{Logistic}(\mu, s)$$

$$
\begin{aligned}
Z &\sim \textsf{Logistic}(\mu, s) \\
f(z) &= \frac{\exp(\frac{z-\mu}{s})}{s(1+\exp(\frac{z-\mu}{s}))^2}
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Z &\sim \underbrace{\mathsf{Logistic}(0, 1)}_{\mathsf{Standard\ Logistic}} \\
f(z) &= \frac{\exp(z)}{(1 + \exp(z))^2}
\end{aligned}
$$

Logit Model

$$\begin{aligned}\tilde{Y}_i &= \boldsymbol{X}_i^{'}\boldsymbol{\beta} + \epsilon_i \\ \epsilon_i &\sim \textsf{Logistic}(0, 1)\end{aligned}$$

Logit Model

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\tilde{Y}_i &= \boldsymbol{X}_i^{'}\boldsymbol{\beta} + \epsilon_i \\
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P(Y_i = 1|\boldsymbol{X}_i, \boldsymbol{\beta}) &= I(\tilde{Y}_i > 0)
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\end{aligned}
$$

Logit Model

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\epsilon_i &\sim \textsf{Logistic}(0, 1) \\
P(Y_i = 1|\boldsymbol{X}_i, \boldsymbol{\beta}) &= I(\tilde{Y}_i > 0) \\
&= I(\boldsymbol{X}_i^{'}\boldsymbol{\beta} + \epsilon_i > 0) \\
&= I(\epsilon_i > -\boldsymbol{X}_i^{'}\boldsymbol{\beta}) \\
&= I(\epsilon_i < \boldsymbol{X}_i^{'}\boldsymbol{\beta}) \\
&= \textsf{Logistic}(\boldsymbol{X}_i^{'}\boldsymbol{\beta}) = F(\boldsymbol{X}_i^{'}\boldsymbol{\beta})
\end{aligned}
$$

Logit Model

$$
\begin{aligned}
Y_i &\sim \text{Bernoulli}(\pi_i) \\
\pi_i &= F(\boldsymbol{X}_i^{'}\boldsymbol{\beta})
\end{aligned}
$$

Likelihood function

$$
\begin{aligned}
L(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) &= f(\boldsymbol{Y}|\boldsymbol{X}, \boldsymbol{\beta}) \\
&= \prod_{i=1}^{N} f(Y_i|\boldsymbol{X}_i, \boldsymbol{\beta}) \\
&= \prod_{i=1}^{N} F(\boldsymbol{X}_i^{'}\boldsymbol{\beta})^{Y_i}(1 - F(\boldsymbol{X}_i^{'}\boldsymbol{\beta}))^{1-Y_i}
\end{aligned}
$$

Logit Model

Log-likelihood:

$$l(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) \quad = \quad \sum_{i=1}^{N} Y_i \log F(\boldsymbol{X}_i^{'}\boldsymbol{\beta}) + (1 - Y_i) \log(1 - F(\boldsymbol{X}_i^{'}\boldsymbol{\beta}))$$

Homework: Derive Score + Hessian for Logistic Normal + Implement with Optim

R Code

Logit Model: Most Important Problem

Most Important Problem

	Probit	Logit
Intercept	-0.26	-0.41
	(0.03)	(0.05)
Post-Speech	-0.01	-0.02
	(0.05)	(0.09)
Republican	-0.20	-0.32
	(0.05)	(0.08)

Logit Model: Interpreting the Coefficients

Expected Value:

Logit Model: Interpreting the Coefficients

Expected Value:

$$E[Y_i|\boldsymbol{X}_i] = \frac{1}{1 + \exp(-\beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2})}$$

Logit Model: Interpreting the Coefficients

Expected Value:

$$
\begin{aligned}
E[Y_i|\boldsymbol{X}_i] &= \frac{1}{1+\exp(-\beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2})} \\
E[Y_i|X_{i1}=0, X_{i2}=0] &= \frac{1}{1+\exp(0.41)} = 0.40
\end{aligned}
$$

Logit Model: Interpreting the Coefficients

Expected Value:

$$
\begin{aligned}
E[Y_i|\boldsymbol{X}_i] &= \frac{1}{1+\exp(-\beta_0-\beta_1 X_{i1}-\beta_2 X_{i2})} \\
E[Y_i|X_{i1}=0, X_{i2}=0] &= \frac{1}{1+\exp(0.41)} = 0.40 \\
E[Y_i|X_{i1}=0, X_{i2}=1] &= \frac{1}{1+\exp(0.41+0.32)} = 0.33
\end{aligned}
$$

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E[Y_i|\boldsymbol{X}_i] &= \frac{1}{1+\exp(-\beta_0 - \beta_1 X_{i1} - \beta_2 X_{i2})} \\
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\end{aligned}
$$

First Differences

Logit Model: Interpreting the Coefficients

Expected Value:

$$
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E[Y_i|\boldsymbol{X}_i] &= \frac{1}{1+\exp(-\beta_0-\beta_1 X_{i1}-\beta_2 X_{i2})} \\
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\end{aligned}
$$

First Differences

$$
\begin{aligned}
E[Y_i|X_{i1}=1, X_{i2}=0] - E[Y_i|X_{i1}=0, X_{i2}=0] &= \\
0.3941 - 0.3989 &= -0.005
\end{aligned}
$$

Logit Model: Interpreting the Coefficients

Expected Value:

$$
\begin{aligned}
E[Y_i|\boldsymbol{X}_i] &= \frac{1}{1+\exp(-\beta_0-\beta_1 X_{i1}-\beta_2 X_{i2})} \\
E[Y_i|X_{i1}=0, X_{i2}=0] &= \frac{1}{1+\exp(0.41)} = 0.40 \\
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First Differences

$$
\begin{aligned}
E[Y_i|X_{i1}=1, X_{i2}=0] - E[Y_i|X_{i1}=0, X_{i2}=0] &= \\
0.3941 - 0.3989 &= -0.005 \\
E[Y_i|X_{i1}=1, X_{i2}=1] - E[Y_i|X_{i1}=0, X_{i2}=1] &= \\
0.3208 - 0.32519 &= -0.0043
\end{aligned}
$$

Logit Model: Interpreting the Coefficients

$$
\begin{aligned}
\frac{\partial F(\boldsymbol{X}_i^{'}\boldsymbol{\beta})}{\partial X_{ij}} &= f(\boldsymbol{X}_i^{'}\boldsymbol{\beta})\beta_j \\
\text{Max Effect} &= 0.25 \times \beta_j
\end{aligned}
$$

Most Important Problem: Linear Probability Model

	Probit	Logit	LPM
Intercept	-0.26	-0.41	0.40
	(0.03)	(0.05)	(0.13)
Post-Speech	-0.01	-0.02	-0.004
	(0.05)	(0.09)	(0.02)
Republican	-0.20	-0.32	-0.07
	(0.05)	(0.08)	(0.02)

LPM might (usually) be ok, but

1) Probit/Logit = easy to run
2) Base for more complicated models