Political Methodology III: Model Based Inference

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Model Based Inference

1) Likelihood inference
2) Machine Learning
    a) Unsupervised Latent Features
    b) Classification/Prediction/Regression

Regression models

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Problem:

- J will likely be large (perhaps $J > N$)

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Predictions will be variable

Mean Square Error

Suppose θ is some value of the true parameter

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$$\mathsf{E}[(\hat{\theta} - \theta)^2] = E[\hat{\theta}^2] - 2\theta E[\hat{\theta}] + \theta^2$$

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To reduce MSE, we are willing to induce bias to decrease variance$\leadsto$ methods that shrink coefficeints toward zero

Ridge Regression

Penalty for model complexity

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$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) \;=\; \sum_{i=1}^{N} \left(y_i - \ \beta_0 - \sum_{j=1}^{J} \beta_j x_{ij} \right)^2$$

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where:

- $\beta_0 \rightsquigarrow$ intercept

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where:

- $\beta_0 \rightsquigarrow$ intercept
- $\lambda \rightsquigarrow$ penalty parameter
- Standardized $\boldsymbol{X}$ (coefficients on same scale)

Ridge Regression$\rightsquigarrow$ Optimization

$$\boldsymbol{\beta}^{\textsf{Ridge}} = \textsf{arg min}_{\boldsymbol{\beta}} \{ f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) \}$$

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&= \textsf{arg min}_{\boldsymbol{\beta}} \left\{ (\boldsymbol{Y} - \boldsymbol{X}^{'}\boldsymbol{\beta})^{'}(\boldsymbol{Y} - \boldsymbol{X}^{'}\boldsymbol{\beta}) + \lambda \boldsymbol{\beta}^{'}\boldsymbol{\beta} \right\}
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Demean the data and set $\beta_0 = \bar{y} = \sum_{i=1}^{N} \frac{y_i}{N}$

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Ridge Regression $\rightsquigarrow$ Intuition (1)

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&= \left(\boldsymbol{I}_j + \lambda\boldsymbol{I}_j\right)^{-1}\widehat{\boldsymbol{\beta}} \\
\beta_j^{\textsf{Ridge}} &= \frac{\widehat{\beta}_j}{1+\lambda}
\end{aligned}
$$

Ridge Regression $\rightsquigarrow$ Intuition (2)

$$\begin{aligned}
\boldsymbol{\beta}_j &\sim \textsf{Normal}(0, \tau^2) \\
y_i &\sim \textsf{Normal}(\beta_0 + \boldsymbol{x}_i^{'}\boldsymbol{\beta}, \sigma^2)
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$$p(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) \propto \prod_{j=1}^{J} p(\beta_j) \prod_{i=1}^{N} p(y_i|\boldsymbol{x}_i, \boldsymbol{\beta})$$

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Ridge Regression⇝ Intuition (2)

$$\log p(\boldsymbol{\beta}|\boldsymbol{X}, \boldsymbol{Y}) = -\sum_{j=1}^{J} \frac{\beta_j^2}{2\tau^2} - \sum_{i=1}^{N} \frac{(y_i - \beta_0 - \boldsymbol{x}'\boldsymbol{\beta})^2}{2\sigma^2}$$

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$$-2\sigma^2\log p(\boldsymbol{\beta}|\boldsymbol{X},\boldsymbol{Y}) = \sum_{i=1}^{N}(y_i-\beta_0-\boldsymbol{x}^{'}\boldsymbol{\beta})^2 + \sum_{j=1}^{J}\frac{\sigma^2}{\tau^2}\beta_j^2$$

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where:

- $\lambda = \frac{\sigma^2}{\tau^2}$

Ridge Regression $\rightsquigarrow$ Intuition (3)

Definition

Suppose $\boldsymbol{X}$ *is an* $N \times J$ *matrix. Then* $\boldsymbol{X}$ *can be written as:*

$$\boldsymbol{X} = \underbrace{\boldsymbol{U}}_{N \times N} \underbrace{\boldsymbol{S}}_{N \times J} \underbrace{\boldsymbol{V}^{'}}_{J \times J}$$

Where:

$$\begin{aligned} \boldsymbol{U}^{'}\boldsymbol{U} &= \boldsymbol{I}_N \\ \boldsymbol{V}^{'}\boldsymbol{V} &= \boldsymbol{V}\boldsymbol{V}^{'} = \boldsymbol{I}_J \end{aligned}$$

$\boldsymbol{S}$ *contains* $\min(N, J)$ *singular values,* $\sqrt{\lambda_j} \geq 0$ *down the diagonal and then 0's for the remaining entries*

Ridge Regression $\rightsquigarrow$ Intuition (3)

Recall: PCA:

$$\frac{1}{N}\boldsymbol{X}^{'}\boldsymbol{X} = \underbrace{\boldsymbol{W}}_{\text{eigenvectors}} \begin{pmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_J \end{pmatrix} \underbrace{\boldsymbol{W}^{'}}_{\text{eigenvectors}}$$

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$$\frac{1}{N}\boldsymbol{X}^{'}\boldsymbol{X} \quad = \quad \boldsymbol{V}\boldsymbol{S}^{'}\underbrace{\left(\boldsymbol{U}^{'}\boldsymbol{U}\right)}_{\boldsymbol{I}_J}\boldsymbol{S}\boldsymbol{V}^{'}$$

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Ridge Regression $\rightsquigarrow$ Intuition (3)

We can write the predicted values for a regular regression as

$$\begin{aligned}
\hat{Y} &= \boldsymbol{X}\hat{\boldsymbol{\beta}} \\
&= \boldsymbol{X}\left(\boldsymbol{X}^{'}\boldsymbol{X}\right)^{-1}\boldsymbol{X}^{'}\boldsymbol{Y} \\
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Ridge Regression $\rightsquigarrow$ Intuition (3)

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$$\hat{Y}^{\text{ridge}} = \sum_{j=1}^{J}\boldsymbol{u}_j\frac{\lambda_j}{\lambda_j + \lambda}\boldsymbol{u}_j^{'}\boldsymbol{Y}$$

Degrees of Freedom for Ridge

We will say that the degrees of freedom for Ridge regression with penalty λ is

$$\text{dof}(\lambda) = \sum_{j=1}^{J} \frac{\lambda_j}{\lambda_j + \lambda}$$

Lasso Regression Objective Function

Different Penalty for Model Complexity

$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) = \sum_{i=1}^{N} \left(y_i - \beta_0 - \sum_{j=1}^{J} \beta_j x_{ij} \right)^2 + \lambda \sum_{j=1}^{J} \underbrace{|\beta_j|}_{\text{Penalty}}$$

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Lasso Regression Optimization

Definition

Coordinate Descent Algorithms:
Consider $g : \Re^J \rightarrow \Re$. *Our goal is to find* $\boldsymbol{x}^* \in \Re^J$ *such that* $g(\boldsymbol{x}^*) \leq g(\boldsymbol{x})$ *for all* $\boldsymbol{x} \in \Re$.
To find $\boldsymbol{x}^*$*:*
Until convergence: for each iteration t *and each coordinate* j

$$x_j^{t+1} = \textit{arg min}_{x_j \in \Re} g(x_1^{t+1}, x_2^{t+1}, \ldots, x_{j-1}^{t+1}, x_j, x_{j+1}^{t}, \ldots, x_J^{t})$$

Lasso Regression Optimization: Coordinate Descent

$$\tilde{f}(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{2N}\sum_{i=1}^{N}\left(y_i - \beta_0 - \sum_{j=1}^{J}\beta_j x_{ij}\right)^2 + \lambda\sum_{j=1}^{J}|\beta_j|$$

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Compare soft assignment

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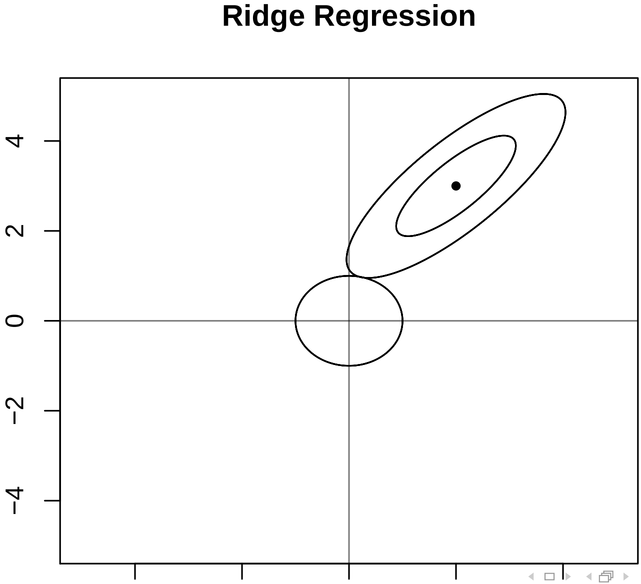
Why does LASSO induce sparsity?

Comparing Ridge and LASSO



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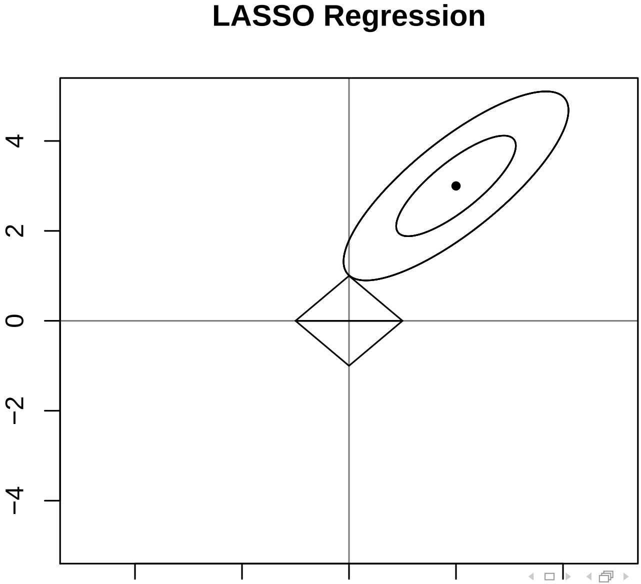

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Ridge and LASSO: The Elastic-Net

Combining the two criteria $\rightsquigarrow$ Elastic-Net

$$f(\boldsymbol{\beta}, \boldsymbol{X}, \boldsymbol{Y}) = \frac{1}{N}\sum_{i=1}^{N}\left(y_i - \beta_0 - \sum_{j=1}^{J}\beta_j x_{ij}\right)^2 + \lambda \sum_{j=1}^{J}\left(\frac{1}{2}(1-\alpha)\beta_j^2 + \alpha|\beta_j|\right)$$

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$$\beta_j \quad \leftarrow \quad \frac{\text{sign}(r^j)\text{max}(|r^j| - \lambda\alpha, 0)}{1 + \lambda(1-\alpha)}$$

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How do we determine λ? $\rightsquigarrow$ Cross validation (Recall code from Monday)

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Credit Claiming (Grimmer, Westwood, and Messing 2014)

```
library(glmnet)
set.seed(8675309) ##setting seed
folds<- sample(1:10, nrow(dtm), replace=T) ##assigning to fold
out_of_samp<- c() ##collecting the predictions
```

Credit Claiming (Grimmer, Westwood, and Messing 2014)

```
for(z in 1:10){
train<- which(folds!=z) ##the observations we will use to train the model

test<- which(folds==z) ##the observations we will use to test the model
part1<- cv.glmnet(x = dtm[train,], y = credit[train], alpha = 1, family =
'binomial') ##fitting the LASSO model on the data.
## alpha = 1 -> LASSO
## alpha = 0 -> RIDGE
## 0<alpha<1 -> Elastic-Net
out_of_samp[test]<- predict(part1, newx= dtm[test,], s = part1$lambda.min,
type ='class') ##predicting the labels
print(z) ##printing the labels
}
conf_table<- table(out_of_samp, credit) ##calculating the confusion table
> round(sum(diag(conf_table))/len(credit), 3)
[1] 0.844
```

Generalized Cross Validation and Ridge Regression

In some special cases there are analytic solutions:

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Why do we care?

Generalized Cross Validation and Ridge Regression

Why do we care?
Leave one out cross validation

Generalized Cross Validation and Ridge Regression

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$$\text{Cross Validation}(1) = \frac{1}{N}\sum_{i=1}^{N}(Y_i - f(\boldsymbol{X}_{-i}, \boldsymbol{Y}_{-i}, \lambda, \hat{\boldsymbol{\beta}}))^2$$

Generalized Cross Validation and Ridge Regression

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Calculating $\boldsymbol{H}$ can be computationally expensive

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Generalized Cross Validation and Ridge Regression

Calling $\boldsymbol{H}$ can be computationally expensive

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