Functions and Chinese Remainder Theorem

Meeting #8

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April 8, 2021

Outline

1. Warm Up
2. Functions
   - Injective Functions
   - Surjective Functions
3. Resources

AMC 2021 12B # 2

Problem

At a math contest, 57 *students are wearing blue shirts, and another* 75 *students are wearing yellow shirts. The* 132 *students are assigned into* 66 *pairs. In exactly* 23 *of these pairs, both students are wearing blue shirts. In how many pairs are both students wearing yellow shirts?*

AMC 2021 12 B # 10

Problem

Two distinct numbers are selected from the set $\{1, 2, 3, 4, \ldots, 36, 37\}$ *so that the sum of the remaining* 35 *numbers is the product of these two numbers. What is the difference of these two numbers?*

Review

Remember that last time, we introduced the idea of modular arithmetic.

Definition (Congruence)

We say

$$a \equiv b \pmod{n}$$

if and only if $a - b$ is divisible by n. That is, $\frac{a-b}{n} \in \mathbb{Z}$

Another way to think of this is that a is congruent to b modulo n if they have the same remainders when divided by n.

Example

$$12 \equiv 9 \equiv 4 \equiv -1 \pmod{5}$$

$$31 \equiv 1 \equiv 4 \pmod{3}$$

Mod Problems Review

Problem

Find the remainder when 4^{13} *is divided by 5.*

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- Don't actually compute 4^{13}
- What is 4 congruent to modulo 5?
- We compute $4 \equiv -1 \pmod 5$
- $4^{13} \equiv (-1)^{13} \equiv -1 \equiv 4 \pmod 5$

Functions

Definition (Function)

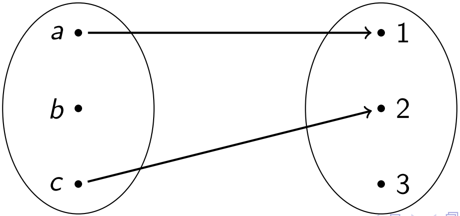
A function an operation which takes a set of input values and *maps* them to outputs value. Each input can have only 1 output.

For example, a line is a function.

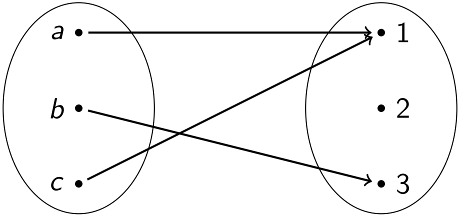
Definition (Injective Function)

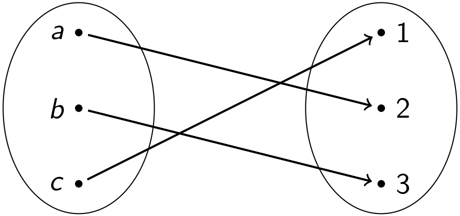
An *injective* function is a function where each output has either 1 or 0 inputs.That is, 2 inputs cannot map to the same output.

Here is an injective function.



Injective of Not?



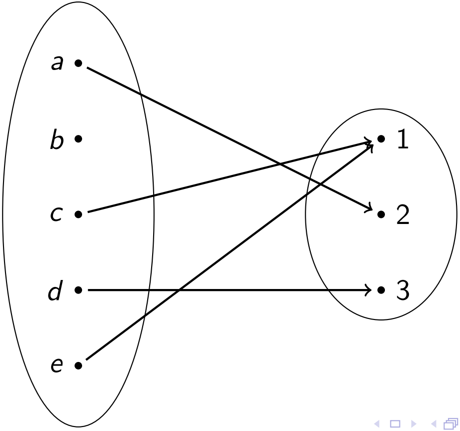


Surjective Functions

Definition (Surjective Function)

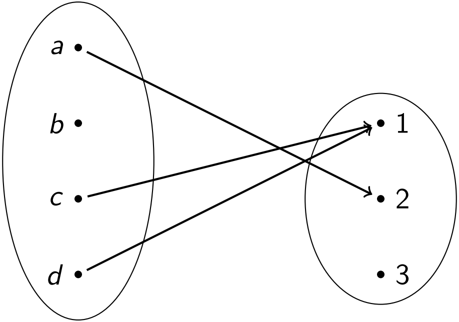
An *surjective* function is a function where each output has at least one input.

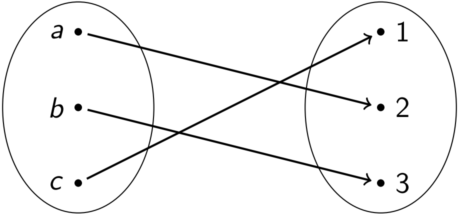
Here is an surjective function.



Surjective of Not?

Determine whether the following functions are surjective.



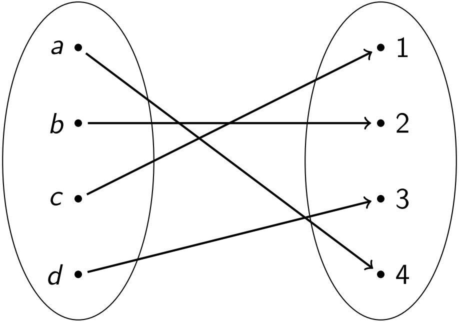


Set Theory

Definition (Bijective Function)

A bijective function is a function where each input has a distinct output and each output has a distinct input. In other words, the function is both injective and surjective.

Here is an example of a bijective function.



Chinese Remainder Theorem

Theorem (CRT)

Let m and n be integers such that $\gcd(m, n) = 1$. Then there is a bijection between residues modulo mn and pairs of residues modulo m and n.

For example, let us take $m, n = 3, 2$, we have the following table.

x modulo 6	x modulo 3	x modulo 2
0	0	0
1	1	1
2	2	0
3	0	1
4	1	0
5	2	1

CRT Destroys AMC Problem

Problem

AMC 2017 12B # 19 Let $N = 123456789101112\ldots4344$ *be the* 79*-digit number obtained that is formed by writing the integers from* 1 *to* 44 *in order, one after the other. What is the remainder when* N *is divided by* 45*?*

- Think about how you can use CRT

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- Think about how you can use CRT
- Take modulo 9 and modulo 5.

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- Conclude that $N \equiv 4 \pmod 5$ and $N \equiv 0 \pmod 9$

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- Conclude that $N \equiv 4 \pmod 5$ and $N \equiv 0 \pmod 9$
- Use CRT to say that there exists a such that $N \equiv a \pmod{45}$
- Find a.
- $a = 9$

Resources

Art of Problem Solving-artofproblemsolving.com

1. Problems
2. Alcumus Game
3. Problem Solving Books
4. Classes