Basic Counting

Meeting #2

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Outline

End Goal

Today's goal is to introduce the $\binom{n}{k}$ principle. This is simply just an extension of the factorial principle.

Where will we see factorials in the future?

Example (Taylor Series)

$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots$
$= \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$

Warm Up-Dealing With Restrictions

Problem

The McRoberts Debate Club has 6 English Delegates, 5 French Delegates and 3 Chinese Delegates. In how many ways can these 14 delegates sit in a row of 14 chairs, if each country's delegates must all sit next to each other?

Correcting for Overcounting

Problem

How many possible distinct arrangements are there of the letters in the word BIG*?*

Problem

How many possible distinct arrangements are there of the letters in the word BALL*?*

A Harder Problem

Problem

How many distinct arrangements are there of MAMA*?*

Introduction

Definition

A combination is counting the number of ways to form a committee from a group of people.

Example

In how many ways can a 2-person committee be chosen from a group of 4 people? The order in which we chose the 2 people does not matter.

Challenging Problem for Committee forming

Problem

In the McRoberts Student Council, how many ways can a a group of 3 representatives be chosen from a group of 8 students?

Definition (The principle)

$\binom{n}{r}$ is the number of ways to choose an r-person committee from a total of n people, where order does not matter.

Example

In the previous 2 examples, we have shown that $\binom{4}{2} = 6$ and $\binom{8}{3} = 56$

Finding the formula

Problem

Consider a math circle that has n students. Find the number of ways to choose r different students from the n students, where order matters.

Problem

Find the number of ways that any given r people can be assigned to be volunteers.

The formula

Theorem

$$\binom{n}{r} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4) \cdot \ldots \cdot (n-r+1)}{r!}$$

Finding a Better Formula

That was pretty messy. Can we come up with a better formula?
Notice that $n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4) \cdot \ldots \cdot (n-r+1) = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4) \cdot \ldots \cdot (n-r+1) \cdot (n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1}{(n-r) \cdot (n-r-1) \cdot \ldots \cdot 2 \cdot 1} = \frac{n!}{(n-r)!}$ So,

Theorem

$$\binom{n}{r} = \frac{n \cdot (n-1) \cdot (n-2) \cdot (n-3) \cdot (n-4) \cdot \ldots \cdot (n-r+1)}{r!} = \frac{\frac{n!}{(n-r)!}}{r!} = \frac{n!}{r!(n-r)!}$$

Problems

Problem (Computational)

Compute $\binom{10}{3}$

Problem (Another computation)

Compute $\binom{10}{7}$

A first combinatorial identity

Theorem

$$\binom{n}{r} = \binom{n}{n-r}$$

Resources

Art of Problem Solving-artofproblemsolving.com

1. Problems
2. Alcumus Game
3. Problem Solving Books
4. Classes