Analysis 2 Lecture Notes – Upendra Kulkarni

Soham Chatterjee

Chapter 1

Examples on Multivariable Differentiation

Example 1.1 (Example where all partial derivatives exist and function is continuous but f' does not exists.)

$$f(x,y) = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

(i) Is f continuous at origin ?

(ii) Do $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}$ exist at origin ? elsewhere ?

Solution:

(i) Want $|f(x,y) - f(0,0)| \to 0$ as $(x,y) \to (0,0)$

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq \sqrt{\frac{x^2+y^2}{2}} \to 0$$

as $(x,y) \to (0,0)$

(ii)
$$\frac{\partial f}{\partial x} = \frac{y^3}{(x^2+y^2)^{\frac{3}{2}}} \qquad \frac{\partial f}{\partial y} = \frac{x^3}{(x^2+y^2)^{\frac{3}{2}}}$$

Now
$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = \lim_{h \to 0} \frac{0-0}{h} = 0$$

Similarly $\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$. SO if $f'(0)$ exists then it will be the matrix $\begin{bmatrix} 0 & 0 \end{bmatrix}$. So it will be the zero operator $\implies D_v f(\text{origin}) = 0$ for any direction for any vector v. Let's test for $v = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$D_v f(\text{origin}) = \lim_{t\to 0} \frac{f(0+tv) - f(0,0)}{t} = \lim_{t \to 0} \frac{f(t,t)}{t} = \lim_{t\to 0} \frac{t^2}{t\sqrt{2t^2}} \neq 0$$

Thus f is not differentiable at origin. Therefore at least one of the partial derivatives must be discontinuous at origin (here by symmetry both are discontinuous). $\frac{\partial f}{\partial x} = 0$ at origin but $= 1$ at $y-$axis.

Example 1.2 (Example where f' exists but not continuous)

Recall one-variable example $g(x) = \begin{cases} x^2 \sin\frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$. Define $f(x,y) = g(\sqrt{x^2+y^2})$

(i) Is f continuous ?

(ii) Is f differentiable ?

(iii) Is f' continuous at origin ?

Solution:

(i) Because f is composition of two continuous functions. f is continuous.

(ii) Need to check at origin only

Example 1.3

$f(x,y) = \begin{cases} \frac{x^2 y}{x^6+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$

(i) Is f continuous at origin ?

(ii) Calculate the directional derivatives for unit vectors $u = (\cos\theta, \sin\theta)$

(iii) Is f differentiable at origin ?

Solution:

(i)

$$f(x,x^3) = \frac{x^5}{2x^6} = \frac{1}{2x}$$

It has no limit as $x \to 0$. Hence f is not continuous at origin.

(ii)

$$\begin{aligned} D_u f(0) &= \lim_{h\to 0} \frac{f(0+hu) - f(0)}{h} \\ &= \lim_{h\to 0} \frac{f(h\cos\theta, h\sin\theta)}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \frac{h^3\cos^2\theta\sin\theta}{h^6\cos^6\theta + h^2\sin^2\theta} \\ &= \lim_{h\to 0} \frac{\cos^4\theta\sin\theta}{h^4\cos^6\theta + \sin^2\sin\theta} = \frac{\cos^2\theta}{\sin\theta} \qquad \text{when } \sin\theta \neq 0 \end{aligned}$$

When $\sin\theta = 0$, $f = 0$ on $x-$axis. So $D_u f(0) = 0$ for $\theta = 0, \pi, \ldots$. SO $D_u f()$ exists for all u

(iii) If $f'(0)$ exists then it's matrix would be $\begin{bmatrix} & 0 \end{bmatrix}$. But then all directional derivatives would have to be zero because $D_u f(a) = f'(a)v$ which is not possible