

Fulton chapter 5

Linear System of Curves and Bézout's Theorem

Aritra Kundu

Email: aritra@cmi.ac.in

Problem Set - 3

Topic: Algebraic Geometry

problem 5.19

$$A = \{(a, b, 1) | \forall a, b \in \{0, 1, 2\}\}$$

show that there is infinitely many cubics passing through the 9 points in A

solution:

let F be a cubic passing through the 9 points in A

so, F_* will pass through $B = \{(a, b) | \forall a, b \in \{0, 1, 2\}\}$ [* taking w.r.t Z]

now $F_* = g(X) + Y(H(X, Y))$

as $F_*(a, 0) = 0 \forall a \in \{0, 1, 2\} \implies g(x) = \lambda(X - 2)(X - 1)X$ [as F is cubic $\implies g$ is a deg 3 polynomial]

similarly $F_* = g_1(Y) + XH(X, Y) \implies g_1(Y) = \mu(Y - 2)(Y - 1)Y$

so, $F_* = \lambda(X - 2)(X - 1)X + \mu(Y - 2)(Y - 1)Y + XY(aX + bY + c)$

but $F_*(1, 2) = 0 = F_*(2, 1) = F_*(1, 1) \implies a + 2b + c = 0 = 2a + b + c = a + b + c \implies a = b = c = 0$

so, any polynomial passing through B must be of the form $\lambda(X - 2)(X - 1)X + \mu(Y - 2)(Y - 1)Y$ where $\lambda, \mu \in k$

so, any F passing through the points in A will be of the form $\lambda(X - 2Z)(X - Z)X + \mu(Y - 2Z)(Y - Z)Y$

so there are infinitely curves passing through 9 points in A

problem 5.23

F is a projective plane curve of degree n , it contains no lines and $\text{char}(k) = 0$ then

(a) if $P \in H \cap F$ then either it is a multiple point or a flex.

(b) $I(P, H \cap F) = 1 \iff P$ is an ordinary flex.

solution:

claim 1 : T be a projective change of co-ordinates then hessian of $F^T = \det(A)^2 H^T$

let $T = (T_1, T_2, T_3); T_i = a_i X + b_i Y + c_i Z$

so the matrix of T is $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$(F(T_1, T_2, T_3))_x = a_1 F_{T_1}(T_1, T_2, T_3) + a_2 F_{T_2}(T_1, T_2, T_3) + a_3 F_{T_3}(T_1, T_2, T_3) = \begin{bmatrix} F_{T_1} & F_{T_2} & F_{T_3} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ [by}$$

chain rule]

$$(F(T_1, T_2, T_3))_{xx} = \sum_{i=1}^3 \sum_{j=1}^3 a_i(a_j F_{T_i T_j})$$

so, $F_{xx}(T_1, T_2, T_3)$ is the (1,1) position of

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} F_{T_1 T_1} & F_{T_1 T_2} & F_{T_1 T_3} \\ F_{T_2 T_1} & F_{T_2 T_2} & F_{T_2 T_3} \\ F_{T_3 T_1} & F_{T_3 T_2} & F_{T_3 T_3} \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$\text{so, } \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} F_{T_1 T_1} & F_{T_1 T_2} & F_{T_1 T_3} \\ F_{T_2 T_1} & F_{T_2 T_2} & F_{T_2 T_3} \\ F_{T_3 T_1} & F_{T_3 T_2} & F_{T_3 T_3} \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$$

$$= \begin{bmatrix} (F^T)_{xx} & (F^T)_{xy} & (F^T)_{xz} \\ (F^T)_{yx} & (F^T)_{yy} & (F^T)_{yz} \\ (F^T)_{zx} & (F^T)_{zy} & (F^T)_{zz} \end{bmatrix}$$

so, $R = \text{hessian of } F^T = \det(A)^2 H^T$

$$\text{so, } P \in H \cap F \iff T(P) \in R \cap F^T \iff T(P) \in H^T \cap F^T$$

$$\text{and } I(P, F \cap H) = 1 \iff I(T(P), F^T \cap H^T) = 1 \iff I(T(P), F^T \cap R) = 1$$

so we can assume $P = (0, 0, 1)$

$$\text{claim 2: } (n-1)F_j = \sum_i X_i F_{ij}$$

$$\text{let } F = \sum_{k=0}^r X_j^k F_k \text{ [where } \deg F_k = n - k]$$

$$\implies F_j = \sum_{k=1}^r k X_j^{k-1} F_k \implies \deg(F_j) = n - 1 \text{ [as } \text{char } K = 0]$$

applying Euler's theorem on F_j we get the relation.

$$(n-1)F_j = \sum_i X_i F_{ji} = \sum_i X_i F_{ij} \text{ [as } F_{ij} = F_{ji}]$$

$$\text{claim 3: } I(P, f \cap h) = I(P, f \cap g) \text{ where } g = f_y^2 f_{xx} + f_x^2 f_{yy} - 2f_x f_y f_{xy}$$

$$\text{let } F(X, Y, Z) = \sum_{k=0}^r X^k F_k(Y, Z) \implies F_X(X, Y, 1) = \sum_{k=1}^r k X^{k-1} F_k(Y, 1) = f_x(X, y, 1)$$

$$F(X, Y, Z) = \sum_{k=0}^r X^k F_k(Y, Z) \implies F_Y(X, Y, 1) = \sum_{k=1}^r k X^{k-1} F_k(Y, 1) = f_y(X, y, 1)$$

$$\text{so, } F_{XY}(X, Y, 1) = f_{XY}(X, Y); F_{XX}(X, Y, 1) = f_{XX}(X, Y, 1)$$

$$H(X, Y, Z) = \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{vmatrix} = \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ xF_{xx} + yF_{yx} + zF_{zx} & xF_{xy} + yF_{yy} + zF_{zy} & xF_{xz} + yF_{zy} + zF_{zz} \end{vmatrix}$$

$$= \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ (n-1)F_x & (n-1)F_y & (n-1)F_z \end{vmatrix} = \begin{vmatrix} F_{xx} & F_{xy} & zF_{xz} + yF_{xy} + xF_{xx} \\ F_{yx} & F_{yy} & zF_{yz} + yF_{yy} + xF_{yx} \\ (n-1)F_x & (n-1)F_y & (n-1)(zF_z + yF_y + xF_x) \end{vmatrix} = \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \\ (n-1)F_x & (n-1)F_y \end{vmatrix}$$

$$\text{so, } h(x, y) = H(x, y, 1) = \begin{vmatrix} f_{xx} & f_{xy} & (n-1)f_x \\ f_{yx} & f_{yy} & (n-1)f_y \\ (n-1)f_x & (n-1)f_y & (n-1)nf \end{vmatrix}$$

$$= (n-1)nf[f_{yy}f_{xx} - f_{yx}f_{xy}] - (n-1)^2 f_y[f_{xx}f_y - f_xf_{xy}] + (n-1)^2 f_x[f_yf_{yx} - f_{yy}f_x]$$

$$= (n-1)nf[f_{yy}f_{xx} - f_{yx}f_{xy}] - (n-1)^2 [f_y^2 f_{xx} + f_x^2 f_{yy} - 2f_x f_y f_{xy}] \text{ [as } f_{xy} = f_{yx}]$$

$$\text{so, } I(P, f \cap g) = I(P, f \cap h)$$

claim 4: If P is a multiple point then $I(P, f \cap g) \geq 2$

$$P \in F \cap H \text{ } P \text{ is a multiple point of } F \implies I(P, F \cap H) = I(P, f \cap h) \geq 2 \text{ [as } m_P(f) \geq 2]$$

$$\text{so, } I(P, f \cap g) \geq 2$$

claim 5: $P = (0, 0)$ be a simple point of $f = y + ax^2 + bxy + cy^2 + dx^3 + \text{higher terms}$ $y = 0$ be the tangent at P

P is a flex iff $a = 0$; P is an ordinary flex iff $a = 0$; $d \neq 0$

As P is a simple point $O_P(f)$ is a d.v.r and the maximal ideal is generated by x

P is a flex iff $\text{ord}(y) \geq 3$

so, $y(1 + bx + cy + \text{higher terms}) = x^2(a + dx + \text{higher terms})$

so $\text{ord}(y) \geq 3 \iff a = 0$

P is an ordinary flex iff $\text{ord}(y) = 3$

so, $y(1 + bx + cy + \text{higher terms}) = x^2(a + dx + \text{higher terms})$

so $\text{ord}(y) = 3 \iff a = 0$; $d \neq 0$

now $f_x = 2ax + by + 3dx^2 + \text{other terms}$

$f_y = 1 + bx + 2cy + \text{other terms}$

$f_{xx} = 2a + 6dx + \text{other terms}$

$f_{yy} = 2c + \text{other terms}$

$f_{xy} = b + \text{other terms}$

now $f_x f_y f_{xy} = (2ax + by)b + \text{higher terms}$

$f_y^2 f_{xx} = 2a + 6dx + \text{higher terms}$

$f_x^2 f_{yy}$ has no 1 degree terms

$g = 2a + (6d - 4ab)x - 2b^2y + \text{higher terms}$

now $P = (0, 0) \in f \cap g \iff a = 0 \iff P$ is a flex

$I(P, g \cap f) = 1 \iff$ they do not share tangent at $P \iff d \neq 0 \iff P$ is an ordinary flex.

corollary :

A non singular cubic has 9 flexes all are ordinary

let F be a cubic non singular curve H be its hessian .

so H is also cubic

$\sum_P (I(P, F \cap H)) = 9$

now let for some $P, I(P, F \cap H) \neq 0$

as P is non singular so P is simple so by the previous theorem P is a flex.

it must be ordinary as F is cubic.

so, $I(P, H \cap F) = 1$

so, there are 9 flexes of F

problem 5.24

a) let $(0, 1, 0)$ be a flex and $Z = 0$ be the tangent at that point . char $k = 0$. show that $F = ZY^2 + bYZ^2 + cYXZ + \text{terms in } X, Z$

find a projective change of co-ordinates s.t F reduced to a form $Y^2Z = \text{cubic in } X, Z$

solution:

$P = (0, 0)$

so, $O_P(F_*)$ is a d.v.r. and generated by X

so, $Z(1 + bZ + cX + dXZ + eX^2 + fZ^2) = X^3k$ [as F is a cubic]

so, $F_*(X, Z) = Z + bZ^2 + cXZ + dXZ^2 + eX^2Z + fZ^3 - kX^3 \implies F = ZY^2 + bZ^2Y + cXYZ + dXZ^2 + X^2Z + fZ^3 - kX^3$

now, using $Y \rightarrow (Y - b/2Z - c/2X)$ and keeping others same

$ZY^2 + bZ^2Y + cXYZ$ becomes $ZY^2 - \frac{b^2Z^3}{4} - \frac{c^2X^2Z}{4} - \frac{bcXZ^2}{4}$

so, we get F to the form $ZY^2 = \text{cubic in } X, Z$

b) show that any irreducible cubic is projectively equivalent to one of the following $G_1 = Y^2Z - X^3$, $G_2 = Y^2Z - X^2(X + 1)$, $G_3 = Y^2Z = X(X - Z)(X - \lambda Z)$ where $\lambda \neq 0, 1$

solution:

assume $\text{char } F = 0$

let F be an irreducible curve .

so, it has at most finitely many singular point

let $Q_1 = (a, b, c) \in H \cap F$ be a simple point of F and L be its tangent [as $H \cap F$ is infinite]

let $Q_2 = (d, e, f)$ be another point on L

so, \exists a projective change of co-ordinates T s.t. $T(Q_1) = (0, 1, 0), T(Q_2) = (1, 0, 0)$

so, $(0, 1, 0)$ a simple point of F^T and its tangent is $Z = 0$

by problem 5.23 it is a flex.

so, by problem (a)

F^T is projectively equivalent to Y^2Z -cubic in X, Z

let $G = Y^2Z - (X - \lambda_1 Z)(X - \lambda_2 Z)(X - \lambda_3 Z)$

case 1.

if all λ_i 's are equal then using $X \rightarrow (X + \lambda Z)$ and keeping others unchanged we get $X^3 = Y^2Z$

case 2.

if $\lambda = \lambda_1 = \lambda_2 \neq \lambda_3$

then using $X \rightarrow (X + \lambda Z)$ and keeping others unchanged we get

$Y^2Z = X^2(X + (\lambda - \lambda_3)Z)$

using $Z \rightarrow \frac{Z}{\lambda - \lambda_3}$, $Y \rightarrow \sqrt{\lambda - \lambda_3}Y$ keeping X unchanged we get

$Y^2Z = X^2(X + 1)$ [as $\lambda - \lambda_3 \neq 0$]

case 3.

all λ_i 's are distinct

then using $X \rightarrow (X + \lambda_1 Z)$ and keeping others unchanged we get

$Y^2Z = X(X - (\lambda_2 - \lambda_1)Z)(X - (\lambda_3 - \lambda_1)Z)$

again using $Z \rightarrow \frac{Z}{\lambda_2 - \lambda_1}$, $Y \rightarrow \sqrt{\lambda_2 - \lambda_1}Y$ keeping X unchanged we get

$Y^2Z = X(X - Z)(X - \frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1}Z)$

and also $\frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1} \neq 0, 1$

so, it is projectively equivalent to $Y^2Z = X(X - Z)(X - \lambda Z)$ where $\lambda \neq 0, 1$

so, F is projectively equivalent to $G_1 = Y^2Z - X^3$ or $G_2 = Y^2Z - X^2(X + 1)$ or $G_3 = Y^2Z = X(X - Z)(X - \lambda Z)$ where $\lambda \neq 0, 1$

remark:

claim 1: G_1 has a cusp at $(0, 0, 1)$

$G_{1*}(X, Z) = Y^2 - X^3$ which has a cusp at $(0, 0)$ and the tangent is $Y = 0$

claim 2: G_2 has a node at $(0, 0, 1)$

$G_{2*}(X, Z) = Y^2 - X^2 - X^3$ which has a node at $(0, 0)$ and the tangents are $X + Y = 0, X - Y = 0$

claim 3: if F is an irreducible conic then $\forall P \in F \ m_P(F) \leq 2$

if $\exists P \in V(F)$ s.t. $m_P(F) > 2$

\exists a projective change of co-ordinates T s.t. $T(P) = (0, 1, 0)$

$G = F^T G_* = L_1 L_2 L_3 \implies G$ is reducible

which is not possible

so, if F is an irreducible conic then either is non singular or it has cusp or it has a node.

now both claim 1 and problem 5.10 implies that if F has a cusp then it is projectively equivalent to G_1

both claim 2 and problem 5.11 implies that if F has a node then it is projectively equivalent to G_2

so, if F is non singular then it is projectively equivalent to G_3