Analysis 2 Lecture Notes
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Chapter 1

Maximum and Minimum of Multivariable Functions

For a C^3 function (in a neighborhood of a in $\mathbb{R}$), by Taylor's Theorem

$$f(a+h) = f(a) + f'(a)h + \frac{1}{2}f''(a)h^2 + \underbrace{\frac{1}{6}f'''\begin{pmatrix}\text{some point}\\ \text{between}\\ a \text{ and } a+h\end{pmatrix}h^3}_{\substack{\text{Remainder term } r(h)\\ \frac{r(h)}{h^2}\to 0 \text{ as } h\to 0}}$$

Suppose $f'(a) = 0$ "a is a critical point of f". Then

$$\frac{f(a+h)-f(a)}{h^2} = \frac{1}{2}f''(a) + \frac{r(h)}{h^2}$$

If $f''(a) > 0$ then f has a local minimum at a because choose $\delta > 0$ such that $|h| < \delta$, $\left|\frac{r(h)}{h^2}\right| < \frac{1}{2}f''(a)$. Then $RHS > 0 \ \forall\ h$ such that $|h| < \delta$ and so for $h \in (-\delta, \delta)$, $f(a+h) > f(a)$ i.e. $f(a)$ is minimum value of f in the neighborhood $(a-\delta, a+\delta)$. Similarly $f''(a) < 0$ then f has a local maximum at a.

We want to find an analogy of this for multivariable case

$f : (\text{open } U \text{ in } \mathbb{R}^n) \to \mathbb{R}$ a C^3 function. Then for $h \in$ some open neighborhood W of origin, $a+h \in U$

$$\begin{aligned}
f(a+h) &= f(a) + f'(a)h + \frac{1}{2}f''(a)(h,h) + \underbrace{\frac{1}{6}f'''\begin{pmatrix}\text{some point}\\ \text{between}\\ a \text{ and } a+h\end{pmatrix}(h,h,h)}_{\substack{\text{Remainder term } r(h)\\ \frac{r(h)}{h^2}\to 0 \text{ as } h\to 0}}\\
&= f(a) + \begin{bmatrix}D_1 & \cdots & D_n\end{bmatrix}\begin{bmatrix}h_1\\ \vdots\\ h_n\end{bmatrix} + \frac{1}{2}\sum_{i,j} D_iD_jf(a)h_ih_j + r(h)\\
&= f(a) + \begin{bmatrix}D_1 & \dots & D_n\end{bmatrix}\begin{bmatrix}h_1\\ \vdots\\ h_n\end{bmatrix} + \frac{1}{2}\begin{bmatrix}h_1 & \cdots & h_n\end{bmatrix}\left[D_iD_jf(a)\right]\begin{bmatrix}h_1\\ \vdots\\ h_n\end{bmatrix} + r(h)\\
&= f(a) + \nabla f(a)\cdot h + \frac{1}{2}h^t \underbrace{\left[D_iD_jf(a)\right]}_{\substack{\text{Hessian Matrix}\\ \text{of } f \text{ at } a}} + r(h)
\end{aligned}$$

Definition 1.1: Hessian Matrix of f

Let $f : (\text{open } U \text{ in } \mathbb{R}^n) \to \mathbb{R}$ such that $\begin{cases} f \text{ is } C^1 \iff \dfrac{\partial f}{\partial x_i} \text{ are not continuous on } U \\ f'' \text{ exists at } a \end{cases}$ So components of f'' are $D_iD_jf(a)$. Hessian of f at a = Square matrix $[D_iD_jf(a)]$

When f is C^2, Hessian matrix is Symmetric Matrix

Definition 1.2: Critical Point

Let f be a C^1 function, Open U in $\mathbb{R}^n \xrightarrow{f} \mathbb{R}$. $a \in U$ is called critical point if $f'(a) = 0 \iff \nabla f(a) = 0$

If f has local maximum at a, then along any line through a the same must be hold, so all directional derivative =0 at a.

Definition 1.3: Non-degenerate Point

If f is C^2 then a critical point a is called non-degenerate if the Hessian, $Hf(a)$ is non-singular i.e. $\det(Hf(a)) \neq 0$

Claim 1.0.1

Symmetric Matrix A is positive (semi)definite $\iff \forall$ nonzero vector $x \in \mathbb{R}^n$, $x^tAx > 0$ (resp. ≥ 0)

Proof. **If Part:**

$x = \sum_i c_iv_i$. Where v_i is the eigen-basis. Then

$$x^tAx = \left(\sum_i c_iv_i\right)^t A\left(\sum_j c_jv_j\right) = \left(\sum_i c_iv_i\right)^t \left(\sum_j \lambda_jc_jv_j\right) = \sum_i \lambda_ic_i^2 > 0 \qquad [v_i^tv_j = \delta_{ij}]$$

Only If Part:

Use $x^tAx > 0$ for $x = v_i$ eigenvector $< 0v_i^tAv_i = v_i\lambda_iv_i = \lambda_i$ $\square$

Note:-

Determinant of positive definite matrix > 0 and Determinant of negative definite matrix has sign $(-1)^n$

Theorem 1.1

Let $f : (\text{open } U \text{ in } \mathbb{R}^n) \to \mathbb{R}$. Suppose f has a local maximum or minimum at a then

1. If $f'(a)$ exists then $f'(a) = 0$ i.e. a is a critical point.
2. Suppose in addition to that $f''(a)$ exists then if f has local maximum at a, then $f''(a) \leq 0$ and if f has local minimum at a, then $f''(a) \geq 0$

Proof. 1. For $n = 1$ let we have local minimum at a. Then for small $|h|$

$$\left.\begin{array}{ll} \frac{f(a+h)-f(a)}{h} \geq 0 & \text{for } h > 0 \\ \frac{f(a+h)-f(a)}{h} \leq 0 & \text{for } h < 0 \end{array}\right\} \text{Thus imply respectively that } 1f'(a) \text{ must be } \geq 0 \text{ and } \leq 0$$

For $n > 1$ use $n = 1$ in every direction i.e. for function $f|_{a+tv}$ for $t \in$ open interval to conclude $D_v f(a) = 0$ $\forall$ directions. So $f'(a) = 0$

$\square$

② For $n = 1$

$$f''(a) = \lim_{h \to 0} \frac{f'(a+h) - f'(a)}{h} = \lim_{h \to 0} \frac{f'(a+h)}{h}$$

Observation: If f has local maximum at a then for $0 < |h| < \delta$, $f(a+h) \geq f(a)$. So by MVT there is k between 0 and h such that

$$\frac{f(a+h) - f(a)}{h} = f'(a+k)$$

Using the observation $f''(a) = \lim\limits_{h \to 0} \frac{f'(a+k)}{h} \geq 0$

For $n > 1$ applying this to each $f|_{a+tv}$ $\forall$ direction vectors v we get all $D_v^2 f(a) \geq 0$. In terms of Hessian let $v = \sum c_i e_i \implies D_v f = \sum c_i D_i f \implies D^2 f(a) = \sum_{i,j} c_j c_i D_j D_i f(a)$ in a neighborhood of a.

$$D_v^2 f(a) = \begin{bmatrix} c_1 & \cdots & c_n \end{bmatrix} Hf(a) \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

$\square$

Theorem 1.2

If $f : (\text{open } U \text{ in } \mathbb{R}^n) \to \mathbb{R}$ is a C^3 function and a is a non-generate critical point of f then

$$\begin{aligned} f \text{ has a local minimum at } a &\iff H \text{ is positive definite} \\ &\iff \text{All eigenvalues of } H \text{ are positive} \\ f \text{ has a local maximum at } a &\iff H \text{ is negative definite} \\ &\iff \text{All eigenvalues of } H \text{ are negative} \\ f \text{ has saddle-point otherwise} & \quad\quad\; H \text{ is indefinite} \end{aligned}$$

Proof. **If Part:**

We already proved the if direction in Theorem 1.1

Only If Part:

$\square$