Classical Mechanics 1, Autumn 2021 CMI
Problem set 4
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1. Given that $\hat{x} \times \hat{y} = \hat{z}$. If θ is the angle between the vectors θ then

$$\begin{aligned} &|\hat{x} \times \hat{y}| = |\hat{x}| \cdot |\hat{y}| \sin\theta \\ \implies &|\hat{z}| = |\hat{x}| \cdot |\hat{y}| \sin\theta \\ \implies &1 = 1 \times 1 \times \sin\theta \\ \implies &\sin\theta = 1 \end{aligned}$$

Therefore $\theta = 90°$. Hence $\hat{x}$ and $\hat{y}$ are perpendicular. Therefore $\hat{x} \cdot \hat{y} = 0$.

Given the identity

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = \boldsymbol{b}(\boldsymbol{a} \cdot \boldsymbol{c}) - \boldsymbol{c}(\boldsymbol{a} \cdot \boldsymbol{b})$$

Now putting $\boldsymbol{a} = \hat{y}$, $\boldsymbol{b} = \hat{x}$ and $\boldsymbol{c} = \hat{y}$ we get

$$\begin{aligned} &\hat{y} \times (\hat{x} \times \hat{y}) = \hat{x}(\hat{x} \cdot \hat{x}) - \hat{x}(\hat{x} \cdot \hat{y}) \\ \implies &\hat{y} \times \hat{z} = \hat{x} - 0 \\ \implies &\hat{y} \times \hat{z} = \hat{x} \text{ [Proved]} \end{aligned}$$

Again putting $\boldsymbol{a} = \hat{x}$, $\boldsymbol{b} = \hat{x}$ and $\boldsymbol{c} = \hat{y}$ we get

$$\begin{aligned} &\hat{x} \times (\hat{x} \times \hat{y}) = \hat{x}(\hat{x} \cdot \hat{y}) - \hat{y}(\hat{x} \cdot \hat{x}) \\ \implies &\hat{x} \times \hat{z} = 0 - \hat{y} \\ \implies &-\hat{x} \times \hat{z} = \hat{y} \\ \implies &\hat{z} \times \hat{x} = \hat{y} \text{ [Proved]} \end{aligned}$$

2. The particle is moving counterclockwise round a circle of length l and constant speed v. Therefore angular speed of the particle ω is also constant and $\omega = \frac{v}{l}$. Hence angle covered in time t from initial position is $\theta = \omega t$. Now at any time t the position vector of the particle

$$\boldsymbol{r} = l(\cos\omega t\hat{x} + \sin\omega t\hat{y})$$

Hence the velocity of the particle is

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = l\omega(-\sin\omega t\hat{x} + \cos\omega t\hat{y})$$

. Hence the acceleration of the particle is

$$\boldsymbol{a} = \dot{\boldsymbol{v}} = l\omega^2(-\cos\omega t\hat{x} - \sin\omega t\hat{y}) = -\omega^2[l(\cos\omega t\hat{x} + \sin\omega t\hat{y})] = -\omega^2\boldsymbol{r} = -\left(\frac{v}{r}\right)^2\boldsymbol{r} = -\frac{v^2}{r}\hat{r}$$

Mass of the particle is m. Hence the force on the particle is $m\boldsymbol{a} = -m\frac{v^2}{r}\hat{r}$

3. (a) The position of the particle can be specified by two parameters θ and ϕ. Hence the particle has two degrees of freedom.
   (b) There are 4 real numbers which have to be specified for initial conditions. Two for initial postion and two for initial velocity.
   (c) For initial position we can specify by θ and ϕ. As the velocity vector is perpendicular to the position vector we can specify the initial velocity by $\hat{\theta}$ and $\hat{\phi}$.

4. Given the force on the particle of mass m moving on real line ($x-$axis) is

$$\boldsymbol{F} = -kx\hat{x}$$

Let the velocity of the particle is $\boldsymbol{v}$. Therefore $\frac{dx}{dt}\hat{x} = \boldsymbol{v}$. Hence

$$\boldsymbol{F} = \frac{d}{dt}\boldsymbol{p} = -k\frac{dx}{dt}\hat{x} = -k\boldsymbol{v}$$

where$\boldsymbol{p}$ is the momentum vector of the particle. Hence $\boldsymbol{v} = \frac{\boldsymbol{p}}{m}$. Now

$$\frac{d}{dt}\begin{pmatrix} x \\ p \end{pmatrix} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dp}{dt} \end{pmatrix} = \begin{pmatrix} \frac{p}{m} \\ -kv \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix}\begin{pmatrix} x \\ p \end{pmatrix}$$

Hence

$$A = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix}$$

Hence

$$A^2 = \begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix}\begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix} = \begin{pmatrix} -\frac{k}{m} & 0 \\ 0 & -\frac{k}{m} \end{pmatrix} = -\frac{k}{m}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = -\frac{k}{m}I$$

Therefore

$$A^3 = (A^2)A = \left(-\frac{k}{m}I\right)A = -\frac{k}{m}A$$

$$A^4 = (A^2)^2 = \left(-\frac{k}{m}I\right)^2 = \frac{k^2}{m^2}I^2 = \frac{k^2}{m^2}I$$

Generalizing the result we get

$$A^{2n} = (A^2)^n = \left(-\frac{k}{m}I\right)^n = (-1)^n\frac{k^n}{m^n}I$$

$$A^{2n+1} = A^{2n}A = \left((-1)^n\frac{k^n}{m^n}I\right)A = (-1)^n\frac{k^n}{m^n}A = (-1)^n\frac{k^n}{m^n}\begin{pmatrix} 0 & \frac{1}{m} \\ -k & 0 \end{pmatrix}$$