Soham Chatterjee
Email: sohamc@cmi.ac.in
Course: Probability Theory

Assignment - 2
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Problem 1

Suppose that X_1, X_2, X_3 are identical independent random variables taking values in positive integers where $P(X_j = k) = (1-p)p^{k-1}$ (where $0 < p < 1$). Find the probability of that $X_1 \le X_2 \le X_3$.

Solution: We have,

$$P(X_j = k) = (1-p)p^{k-1} \ \forall \ k \in \mathbb{N}, \ j \in \{1,2,3\}$$

Given that X_1, X_2, X_3 are independent. Now

$$\begin{aligned}
P(X_1 \le X_2 \le X_3) =& \sum_{k=1}^{\infty} P(X_1 \le k, X_2 = k, X_3 \ge k) \\
=& \sum_{k=1}^{\infty} P(X_1 \le k)P(X_2 = k)P(X_3 \ge k) \\
=& \sum_{k=1}^{\infty} \left[\left(\sum_{i=1}^{k} P(X_1 = i) \right) P(X_2 = k) \left(\sum_{j=k}^{\infty} P(X_3 = j) \right) \right] \\
=& \sum_{k=1}^{\infty} \left[\left(\sum_{i=1}^{k} (1-p)p^{i-1} \right) (1-p)p^{k-1} \left(\sum_{j=k}^{\infty} (1-p)p^{j-1} \right) \right] \\
=& (1-p)^3 \sum_{k=1}^{\infty} \left[\left(\sum_{i=0}^{k-1} p^i \right) p^{k-1} \left(\sum_{j=k-1}^{\infty} p^j \right) \right] \\
=& (1-p)^3 \sum_{k=1}^{\infty} \left[\left(\sum_{i=0}^{k-1} p^i \right) p^{k-1} \left(p^{k-1} \sum_{j=0}^{\infty} p^j \right) \right] \\
=& (1-p)^3 \sum_{k=1}^{\infty} \left[\frac{1-p^k}{1-p} p^{k-1} \frac{p^{k-1}}{1-p} \right] \\
=& (1-p) \sum_{k=1}^{\infty} p^{2(k-1)}(1-p^k) \\
=& (1-p) \sum_{k=1}^{\infty} \left(p^{2(k-1)} - p \times p^{3(k-1)} \right) \\
=& (1-p) \sum_{k=0}^{\infty} p^{2k} - p \sum_{k=0}^{\infty} p^{3k} \\
=& (1-p) \left[\sum_{k=0}^{\infty} p^{2k} - p \sum_{k=0}^{\infty} p^{3k} \right] \\
=& (1-p) \left[\frac{1}{1-p^2} - \frac{p}{1-p^3} \right] \\
=& \frac{1}{1+p} - \frac{p}{1+p+p^2} \\
=& \frac{1+p+p^2 - p(1+p)}{(1+p)(1+p+p^2)} = \frac{1}{(1+p)(1+p+p^2)}
\end{aligned}$$

$\square$

Problem 2

Suppose that three players A, B, C take turns to throw a biased coin successively in cyclic order $A, B, C, A, B, C, \ldots$. Let $P(H) = p$. Find the probability that A is the first person to throw heads, next B and finally C.

Solution: Let $q = 1-p$. Let X_A, X_B, X_C are three random variables where $X_A = n$ if A gets his first head at $n-$th turn. Similarly $X_B = n$ if B gets his first head at $n-th$ turn and $X_C = n$ if C gets his first head at $n-th$ turn. Hence $P(X_A = n) = (1-p)^{n-1}p = (1-q)q^{n-1}$. Similarly $P(X_B = n) = (1-p)^{n-1} = (1-q)q^{n-1}$ and $P(X_C = n) = (1-p)^{n-1}p = (1-q)q^{n-1}$Now X_A, X_B, X_C are independent. Hence we have to find the probability of $X_A \leq X_B \leq X_C$. Hence by applying the result in Problem 1 we get

$$P(X_A \leq X_B \leq X_C) = \frac{1}{(1+q)(1+q+q^2)} = \frac{1}{1+(1-p)(1+(1-p)+(1-p)^2)} = \frac{1}{(2-p)(3-3p+p^2)}$$

Therefore probability of A getting the first head then B then C is $\frac{1}{(2-p)(3-3p+p^2)}$

□

Problem 3

Each member of a group of n students is assigned a number at random from the set $\{0, 1, 2, \ldots, 9\}$. The sum of the numbers assigned is the total score that the group obtains. Find the mean of the total score. You may assume that any number is equally likely to be assigned to any student.

Solution: Let X be the random variable which takes the values of all possible total scores obtained by adding all the numbers obtained from each student. Hence we need to find $E(X)$. Let $X_1, X_2, \ldots, X_n$ random variables where each X_i takes values from the set $\{0, 1, 2, \ldots, 9\}$ which represents that the number assigned to $i-$th student is X_i. Since all numbers in the set $\{0, 1, 2, \ldots, 9\}$ are equally likely, $P(X_i = k) = \frac{1}{10}$ where $k \in \{0, 1, 2, \ldots, 9\}$. Hence $X = \sum_{i=1}^{n} X_i$ and therefore

$$E(X) = \sum_{i=1}^{n} E(X_i) = \sum_{i=1}^{n} \left[\sum_{k=0}^{9} kP(X_i = k)\right] = \sum_{i=1}^{n} \left[\sum_{k=0}^{9} k\,\frac{1}{10}\right] = \sum_{i=1}^{n} \frac{45}{10} = \frac{9n}{2}$$

□

Problem 4

Suppose that an urn contains n balls numbered from 1 to n. In an experiment k balls are drawn at random and their numbers are added to get S. Find the variance of S.

Solution: Let X be the random variable which takes the value of the sum obtained by adding the numbers on the k balls randomly picked up from the urn. Hence we need find. Let $X_1, X_2, \cdots, X_n$ are the random variables such that for any X_i

$$X_i = \begin{cases} 1 & \text{when i-th ball was picked up} \\ 0 & \text{when i-th ball was not picked up} \end{cases}$$

Therefore for any X_i

$$P(X_i = 1) = \frac{\text{The number of ways to choose other } k-1 \text{ balls from } n-1 \text{ balls}}{\text{The number of ways to choose } k \text{ balls from } n \text{ balls}} = \frac{\binom{n-1}{k-1}}{\binom{n}{k}} = \frac{k}{n}$$

Hence $X = \sum\limits_{i=1}^{n} iX_i$. Hence

$$E(X) = \sum_{i=1}^{n} iE(X_i) = \sum_{i=1}^{n} i\frac{k}{n} = \frac{k}{n}\,\frac{n(n+1)}{2} = \frac{k(n+1)}{2}$$

Now

$$X^2 = \left[\sum_{i=1}^{n} iX_i\right]^2 = \sum_{i=1}^{n} i^2X_i^2 + \sum_{1\le i<j\le n} 2ijX_iX_j \implies E(X^2) = \sum_{i=1}^{n} i^2E(X_i^2) + \sum_{1\le i<j\le n} 2ijE(X_iX_j)$$

$$\sum_{i=1}^{n} i^2E(X_i^2) = \sum_{i=1}^{n} i^2\frac{k}{n} = \frac{k}{n}\,\frac{n(n+1)(2n+1)}{6} = \frac{k(n+1)(2n+1)}{6}$$

Now,

$$\begin{aligned}
E(X_iX_j) &= P(X_iX_j = 1)\\
&= \text{Probability of both } i-\text{th and } j-\text{th ball were picked up}\\
&= \frac{\text{The number of ways to choose other } k-2 \text{ balls from } n-2 \text{ balls}}{\text{The number of ways to choose } k \text{ balls from } n \text{ balls}}\\
&= \frac{\binom{n-2}{k-2}}{\binom{n}{k}} = \frac{k(k-1)}{n(n-1)}
\end{aligned}$$

Therefore

$$\begin{aligned}
\sum_{1\le i<j\le n} 2ijE(X_iX_j) &= \sum_{1\le i<j\le n} 2ij\,\frac{k(k-1)}{n(n-1)}\\
&= \frac{k(k-1)}{n(n-1)}\left[\sum_{1\le i,j\le n} ij - \sum_{i=1}^{n} i^2\right]\\
&= \frac{k(k-1)}{n(n-1)}\left[\left(\sum_{i=1}^{n} i\right)^2 - \sum_{i=1}^{n} i^2\right]\\
&= \frac{k(k-1)}{n(n-1)}\left[\left(\frac{n(n+1)}{2}\right)^2 - \frac{n(n+1)(2n+1)}{6}\right]\\
&= \frac{k(k-1)}{n(n-1)}\left[\frac{n^2(n+1)^2}{4} - \frac{n(n+1)(2n+1)}{6}\right]\\
&= \frac{k(k-1)(n+1)}{2(n-1)}\left[\frac{n(n+1)}{2} - \frac{2n+1}{3}\right]\\
&= \frac{k(k-1)(n+1)}{2(n-1)}\,\frac{3n(n+1)-2(2n+1)}{6}\\
&= \frac{k(k-1)(n+1)}{2(n-1)}\,\frac{3n^2-n-2}{6}\\
&= \frac{k(k-1)(n+1)}{2(n-1)}\,\frac{(n-1)(3n+2)}{6}\\
&= \frac{k(k-1)(n+1)(3n+2)}{12}
\end{aligned}$$

Hence

$$E(X^2) = \sum_{i=1}^{n} i^2E(x_i^2) + \sum_{1\le i<j\le n} ijE(X_iX_j) = \frac{k(n+1)(2n+1)}{6} + \frac{k(k-1)(n+1)(3n+2)}{12}$$

Therefore finally we get the variance

$$
\begin{aligned}
Var(X) &= E(X^2) - E(X)^2 \\
&= \frac{k(n+1)(2n+1)}{6} + \frac{k(k-1)(n+1)(3n+2)}{12} - \frac{k^2(n+1)^2}{4} \\
&= \frac{k(n+1)}{2}\left[\frac{2n+1}{3} + \frac{(k-1)(3n+2)}{6} - \frac{k(n+1)}{2}\right] \\
&= \frac{k(n+1)}{2}\left[\frac{2n+1}{3} + \frac{(k-1)(3n+2)}{6} - \frac{k(n+1)}{2}\right] \\
&= \frac{k(n+1)}{2}\,\frac{2(2n+1)+(k-1)(3n+2)3k(n+1)}{6} \\
&= \frac{k(n+1)}{2}\,\frac{(4n+2)+(3nk+2k-3n-2)-(3nk+3k)}{6} \\
&= \frac{k(n+1)}{2}\,\frac{n-k}{6} = \frac{k(n+1)(n-k)}{12}
\end{aligned}
$$

So $Var(X) = \dfrac{k(n+1)(n-k)}{12}$

$\square$