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Course: Analysis 2

Assignment - 4
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Problem 1 Rudin Chapt. 9 Problem 18

Answer analogous questions for the mapping defined by

$$u = x^2 - y^2, \quad v = 2xy$$

Solution:

(a) Let the mapping defined by $f = (f_1, f_2)$ where $f_1(x,y) = x^2 - y^2$ and $f_2(x,y) = 2xy$. Hence $f(x,y) = (u,v)$. Then the range of f is the whole $\mathbb{R}^2$ plane. For any point $(u,v) \in \mathbb{R}^2$ not equal to the origin f maps two distinct maps to (u,v)

$$\left(\sqrt{\frac{\sqrt{u^2+v^2}+u}{2}}, \sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}\right) \quad \left(-\sqrt{\frac{\sqrt{u^2+v^2}+u}{2}}, -\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}\right) \quad \text{when } v \text{ is positive}$$
$$\left(\sqrt{\frac{\sqrt{u^2+v^2}+u}{2}}, -\sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}\right) \quad \left(-\sqrt{\frac{-\sqrt{u^2+v^2}+u}{2}}, \sqrt{\frac{\sqrt{u^2+v^2}-u}{2}}\right) \quad \text{when } v \text{ is negative}$$

(b) The matrix of $f'(x,y)$ is

$$f'(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$$

Therefore the jacobian of f is $4(x^2+y^2)$. Hence at any point $(u,v) \in \mathbb{R}^2$ except the orijin $\mathcal{J}$ is nonzero. Hence if we exclude the origin in both planes, f becomes locally one-one and globally two-one.

(c) Let $r = \sqrt{u^2+v^2}$ which is the distance from origin to the point (u,v). Let $\mathbf{a} = \left(0, \frac{\pi}{3}\right)$, then f maps the pointto $\mathbf{b} = \left(-\frac{\pi^2}{9}, 0\right)$. Hence locally f has the inverse function, $g(u,v) = \left(\sqrt{\frac{r+u}{2}}, \sqrt{\frac{w-u}{2}}\right)$. Hence

$$g'(u,v) = \begin{bmatrix} \frac{u+r}{4r}\sqrt{\frac{2}{u+r}} & \frac{v}{4r}\sqrt{\frac{2}{u+r}} \\ \frac{r-u}{4r}\sqrt{\frac{2}{r-u}} & \frac{v}{4r}\sqrt{\frac{2}{r-u}} \end{bmatrix}$$

Therefore

$$g(u,0) = \begin{bmatrix} \sqrt{\frac{u+|u|}{2}} & \sqrt{\frac{|u|-u}{2}} \end{bmatrix} = \begin{cases} (\sqrt{u}, 0) & \text{when } u \geq 0 \\ (0, \sqrt{-u}) & \text{when } u < 0 \end{cases}$$

Hence

$$\begin{aligned}
\left.\frac{\partial g_1(u,v)}{\partial u}\right|_{\left(-\frac{\pi^2}{9},0\right)} &= \lim_{h\to 0} \frac{g_1\left(-\frac{\pi^2}{9}+h, 0\right) - g_1\left(-\frac{\pi^2}{9}, 0\right)}{h} \\
&= \lim_{h\to 0} \frac{0-0}{h} = 0 \\
\left.\frac{\partial g_1(u,v)}{\partial v}\right|_{\left(-\frac{\pi^2}{9},0\right)} &= \lim_{k\to 0} \frac{g_1\left(-\frac{\pi^2}{9}, k\right) - g_1\left(-\frac{\pi^2}{9}, 0\right)}{k} \\
&= \lim_{k\to 0} \frac{1}{k}\sqrt{\frac{\sqrt{\frac{\pi^4}{81}+k^2} - \frac{\pi^2}{9}}{2}} = \lim_{k\to 0}\sqrt{\frac{\sqrt{a^2+k^2}-a}{2}} \\
&= \frac{1}{\sqrt{2}} \lim_{\theta\to 0} \frac{\sqrt{\sqrt{a^2+a^2\tan^2\theta}-a}}{a\tan\theta} \qquad \left[\text{Assume } k = a\tan\theta, a = \frac{\pi^2}{9}\right] \\
&= \frac{1}{\sqrt{2a}} \lim_{\theta\to 0} \frac{\sqrt{\sec\theta - 1}}{\tan\theta} = \frac{1}{\sqrt{2a}} \lim_{\theta\to 0} \sqrt{\cos\theta}\frac{\sqrt{1-\cos\theta}}{\sin\theta}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2a}} \lim_{\theta\to 0} \sqrt{\cos\theta} \sqrt{\frac{2\sin^2\frac{\theta}{2}}{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}} = \frac{1}{\sqrt{2a}} \lim_{\theta\to 0} \sqrt{\cos\theta} \sqrt{\frac{1}{2\cos^2\frac{\theta}{2}}} \\
&= \frac{1}{2\sqrt{a}} = \frac{3}{2\pi}
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial g_2(u,v)}{\partial u}\right|_{\left(-\frac{\pi^2}{9},0\right)} &= \lim_{h\to 0} \frac{g_2\left(-\frac{\pi^2}{9}+h,0\right) - g_2\left(-\frac{\pi^2}{9},0\right)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{\frac{\pi^2}{9}-h} - \frac{\pi}{3}}{h} = \frac{d}{dx}\left.\left(\sqrt{\frac{\pi^2}{9}-x}\right)\right|_{x=0} \\
&= \frac{1}{2}\left.\frac{-1}{\sqrt{\frac{\pi^2}{9}-x}}\right|_{x=0} = -\frac{3}{2\pi}
\end{aligned}
$$

$$
\begin{aligned}
\left.\frac{\partial g_2(u,v)}{\partial v}\right|_{\left(-\frac{\pi^2}{9},0\right)} &= \lim_{k\to 0} \frac{g_2\left(-\frac{\pi^2}{9},k\right) - g_2\left(-\frac{\pi^2}{9},0\right)}{k} \\
&= \lim_{k\to 0} \frac{\sqrt{\frac{\sqrt{\frac{pi^4}{81}+k^2}+\frac{\pi^2}{9}}{2}} - \frac{\pi}{3}}{k} \\
&= \lim_{\theta\to 0} \frac{\frac{1}{\sqrt{2}}\sqrt{a\sec\theta + a} - \sqrt{a}}{a\tan\theta} \left[\text{ Assume } \frac{\pi^2}{9} = a, k = a\tan\theta\right] \\
&= \frac{1}{\sqrt{a}} \lim_{\theta\to 0} \frac{\frac{1}{\sqrt{2}}\sqrt{\frac{1+\cos\theta}{\cos\theta}} - 1}{\tan\theta} = \frac{1}{\sqrt{a}} \lim_{\theta\to 0} \frac{\sqrt{2}\sqrt{\frac{1+\cos\theta}{\cos\theta}} - 2}{2\tan\theta} \\
&= \frac{1}{\sqrt{a}} \lim_{\theta\to 0} \frac{\cos\theta\left(\sqrt{2}\sqrt{\frac{1+\cos\theta}{\cos\theta}} - 2\right)\left(\sqrt{2}\sqrt{\frac{1+\cos\theta}{\cos\theta}} + 2\right)}{2\sin\theta\left(\sqrt{2}\sqrt{\frac{1+\cos\theta}{\cos\theta}} + 2\right)} \\
&= \frac{1}{\sqrt{a}} \lim_{\theta\to 0} \frac{\cos\theta\left(2\frac{1+\cos\theta}{\cos\theta} - 4\right)}{2\sin\theta\left(\sqrt{2}\sqrt{\frac{1+\cos\theta}{\cos\theta}} + 2\right)} \\
&= \frac{1}{\sqrt{a}} \lim_{\theta\to 0} \frac{2(1-\cos\theta)}{2\sin\theta\left(\sqrt{2}\sqrt{\frac{1+\cos\theta}{\cos\theta}} + 2\right)} \\
&= \frac{1}{\sqrt{a}} \lim_{\theta\to 0} \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\left(\sqrt{2}\sqrt{\frac{1+\cos\theta}{\cos\theta}} + 2\right)} \\
&= \frac{1}{\sqrt{a}} \lim_{\theta\to 0} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}\left(\sqrt{2}\sqrt{\frac{1+\cos\theta}{\cos\theta}} + 2\right)} = 0
\end{aligned}
$$

Therefore

$$
g'\left(-\frac{\pi^2}{9},0\right) = \begin{bmatrix} 0 & \frac{3}{2\pi} \\ \frac{3}{2\pi} & 0 \end{bmatrix}
$$

Now we previously determined $f'(x,y) = \begin{bmatrix} 2x & -2y \\ 2y & 2x \end{bmatrix}$ Therefore

$$
f'(\mathbf{a}) \circ g'(\mathbf{b}) = \begin{bmatrix} 0 & -\frac{2\pi}{3} \\ \frac{2\pi}{3} & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{3}{2\pi} \\ -\frac{3}{2\pi} & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

(d) Thepoints parallel to $x-$axis are of the form (x, c) and the points of the form parallel to $y-$axix are (c, y). Image of f of the lines parallel to $x-$ axis and $y-$axis are

$$f(x,c) = \begin{bmatrix} 2x & -2c \\ 2c & 2x \end{bmatrix} \qquad f(c,y) = \begin{bmatrix} 2c & -2y \\ 2y & 2c \end{bmatrix}$$

□

Problem 2 Rudin Chapt. 9 Problem 19

Show that the system of equations

$$\begin{aligned} 3x + y - z + u^2 &= 0 \\ x - y + 2z + u &= 0 \\ 2x + 2y - 3z + 2u &= 0 \end{aligned}$$

can be solved for x, y, u in terms of z; for x, z, u in terms of y; for y, z, u in terms of x; but not for x, y, z in terms of u.

Solution: Let f be the map from $\mathbb{R}^4$ to $\mathbb{R}^3$ such that

$$f(x,y,z,u) = (3x+y-z+u^2, x-y+2z+u, 2x+2y-3z+2u) = (f_1(x,y,z,u), f_2(x,y,z,u), f_3(x,y,z,u))$$

where each $f_i : \mathbb{R}^4 \to \mathbb{R}$ function. Now $f(0,0,0,0) = (0,0,0)$. Now the matrix of $f'(x,y,z,u)$ is

$$f'(x,y,z,u) = \begin{bmatrix} 3 & 1 & -1 & 2u \\ 1 & -1 & 2 & 1 \\ 2 & 2 & -3 & 2 \end{bmatrix}$$

The determinant of the part x, y, u is $8u - 12 \neq 0$ near $\mathbf{0}$. So by implicit function theorem there exists a solution of $f(x(z), y(z), z, u(z)) = \mathbf{0}$ near $\mathbf{0}$.

The determinant of the part x, z, u is $21 - 14u \neq 0$ near $\mathbf{0}$. So by implicit function theorem there exists a solution of $f(x(y), y, z(y), u(y)) = \mathbf{0}$ near $\mathbf{0}$.

The determinant of the part y, z, u is $3 - 2u \neq 0$ near $\mathbf{0}$. So by implicit function theorem there exists a solution of $f(x, y(x), z(x), u(x)) = \mathbf{0}$ near $\mathbf{0}$.

But in case of x, y, z the determinant is 0. Therefore there exists infinite solutions and they does not depend on u. Hence x, y, z can not be expressed in terms of u

□

Problem 3 Rudin Chapt. 9 Problem 23

Define f in R^3 by

$$f(x, y_1, y_2) = x^2 y_1 + e^x + y_2.$$

Show that $f(0,1,-1) = 0, (D_1 f)(0,1,-1) \neq 0$, and that there exists therefore a differentiable function g in some neighborhood of $(1,-1)$ in R^2, such that $g(1,-1) = 0$ and

$$f(g(y_1, y_2), y_1, y_2) = 0$$

Find $(D_1 g)(1,-1)$ and $(D_2 g)(1,-1)$.

Solution:

$$f(0,1,-1) = 0^2 \times 1 + e^0 - 1 = 0$$

Hence $f(0,1,-1) = 0$. Now $D_1 f(x, y_1, y_2) = 2xy_1 + e^x$. Then

$$D_1 f(0,1,-1) = 2 \times 0 \times 1 + e^0 \neq 0$$

Hence by Implicit Function Theorem there exists a C^1 function $g : \mathbb{R}^2 \to \mathbb{R}$ and a neighbourhood near $(1, -1)$ such that for any point $(y_1, y_2) \in U$ $f(g(y_1, y_2), y_1, y_2) = 0$ and $g(1, -1) = 0$

Let

$$h(y_1, y_2) = f(g(y_1y_2), y_1, y_2) = g^2(y_1, y_2)y_1 + e^{g(y_1,y_2)} + y_2 = 0$$

where $(y_1, y_2) \in U$. Therefore

$$0 = D_1h(y_1, y_2) = \left(2g(y_1, y_2)y_1 + e^{g(y_1,y_2)}\right) D_1g(y_1, y_2) + g(y_1, y_2)^2$$

Similarly

$$0 = D_2h(y_1, y_2) = \left(2g(y_1, y_2)y_1 + e^{g(y_1,y_2)}\right) D_2g(y_1, y_2) + 1$$

Putting $y_1 = 1, y_2 = -1, g(y_1, g_2) = 0$ we get

$$D_1g(1, -1) = 0 \qquad D_2g(1. - 1) = -1$$

$\square$

Problem 4

Find max/min of $x + y + z$ subject to $x^2 - y^2 = 1$ and $2x + z = 1$.

Solution: The constraint $x^2 - y^2 = 1$ is a hyperbolic cylinder. Hence it can be parametrized as $(\theta, z) \to (\pm \cosh \theta, \sinh \theta, z)$. Now given the contraint $2x + z = 1 \iff z = 1 - 2x$ the points can be parametrized as $h_\pm : \theta \to (\pm \cosh \theta, \sinh \theta, 1 \mp 2 \cosh \theta)$. We define function $g_\pm = f \circ h_\pm$. Hence

$$g(\theta) = \pm \cosh \theta + \sin \theta + 1 + \mp \cosh \theta = 1 + \sinh \theta \mp \cosh \theta$$

Hence $g'_\pm(\theta) = \cosh \theta - \mp \sinh \theta = e^{\mp\theta}$ which has no extrema for all $\theta \in \mathbb{R}$. Hence $x + y + z$ has no extrema under the constraints $x^2 - y^2 = 1$ and $2x + z = 1$

$\square$

Problem 5

Show that tangent vectors can be realized as velocity vectors of curves. More precisely, let U be an open set in $\mathbb{R}^n$. Let g be a C^1 map $U \to \mathbb{R}^m$. Let c a point in the image of g, $M = g^{-1}(c)$ and $p \in M$ such that $g'(p)$ is surjective. Recall that $T_pM =$ the kernel of $g'(p)$ is called the tangent space of M at p. Show that this tangent space is spanned by the velocity vectors of all C^1 paths γ in M based at p, i.e., by $\gamma'(0)$, where $\gamma : (-\epsilon, \epsilon) \to M$ is a C^1 function with $\gamma(0) = p$.

Solution: Let $p = (p_1, p_2, \ldots, p_n)$ and $d = n - m$. Now since $g'(p)$ is surjective it spans $\mathbb{R}^m$. Hence the matrix of $g'(p)$ has m linearly independent columns. Suppose the last m columns are linearly independent. Let A be the matrix of $g'(p)$ and

$$A = [A_d \mid A_m]$$

where A_d is a $d \times m$ and A_m is a $m \times m$ matrix. Hence A_m is invertible. Let $p = (P_d, P_m)$ where

$$P_d = (p_1, p_2, \ldots, p_d) \text{ and } P_m = (p_{d+1}, p_{d+2}, \ldots, p_m)$$

By implicit function theorem there $\exists$ a open ball $V \subset \mathbb{R}^d$ containing P_d, an open ball $W \subset \mathbb{R}^m$ containing P_m and a C_1 function $h : \mathbb{R}^d \to \mathbb{R}^m$ such that $h(x) = y$ and $g(x, y) = c$ and $h(P_d) = P_m$

Now let $v = (v_d, v_m) \in T_pM$ where $v_d \in \mathbb{R}^d$ and $v_m \in \mathbb{R}^m$. We define a function $a : \mathbb{R} \to \mathbb{R}^d$ such that

$$a(t) = P_d + tv_d$$

and a function c such that

$$c(t) = (a(t), h(a(t)))$$

Since V is an open ball we can bound $|t|$ by $\epsilon > 0$ such that $c(-\epsilon, \epsilon) \subset V$.

Now

$$c(0) = (a(0), h(a(0))) = (p_d, h(p_d)) = (p_d, p_m)$$

. Now we need to show $c'(0) = v$. Notice that

$$c'(0) = \left(a'(t), \frac{d}{dt} h(a(t)) \Big|_{t=0} \right) = \left(v_d, \frac{d}{dt} h(a(t)) \Big|_{t=0} \right)$$

. Now we will find $\frac{d}{dt} h(a(t))\big|_{t=0}$.

$$\frac{d}{dt} h(a(t)) \Big|_{t=0} = h'(a(0))a'(0) = h'(P_d)v_d$$

Now by implicit theorem we also have $h'(P_d) = -A_m^{-1} A_d$. Hence

$$\begin{aligned} h'(P_d)v_d = -A_m^{-1} A_d v_d = -A_m^{-1}(A_d v_d) = -A_m^{-1}([A_d \mid 0]v) = -A_m^{-1}(g'(p)v - [0 \mid A_m]v) \\ = -A_m^{-1}(-[0 \mid A_m]v) = -A_m^{-1}(-A_m v_m) = v_m \end{aligned}$$

Therefore

$$c'(0) = (v_d, v_m) = v$$

Hence there exists a C^1 function $c : (-\epsilon, \epsilon) \to M$ such that $c(0) = p$ and $c'(0) = v$ where $v \in T_pM$

□