Chatterjee_Soham_Solutions

PROMYS 2020 Answersheet
Soham Chatterjee

1. Here :-

$$1^3 + 5^3 + 3^3 = 1 + 125 + 27 = 153$$

$$16^3 + 50^3 + 33^3 = 4096 + 125000 + 33597 = 165033$$

Noticing the pattern it might be possible that

$$\underbrace{(166\cdots 66)^3}_{\{n-1\}\ times\ 6} + \underbrace{(500\cdots 00)^3}_{\{n-1\}\ \ times\ 0} + \underbrace{(333\cdots 33)^3}_{n\ times\ 3} = \underbrace{166\cdots 66}_{\{n-1\}\ times\ 6}\ \underbrace{500\cdots 00}_{\{n-1\}\ times\ 0}\ \underbrace{333\cdots 33}_{n\ times\ 3}$$

Let's try to prove it.

Let $\{A_n\}_{n\geq 1}$ represents a sequence where :-

$$A_n = \underbrace{(166\cdots 66)^3}_{\{n-1\}\ times\ 6} + \underbrace{(500\cdots 00)^3}_{\{n-1\}\ \ times\ 0} + \underbrace{(333\cdots 33)^3}_{n\ times\ 3}$$

Now :-

$$\underbrace{(166\cdots 66)^3}_{\{n-1\}\ times\ 6} = \frac{\overbrace{999\cdots 996}^{\{n-1\}\ times\ 9}}{6} = \frac{10^n - 4}{6} \qquad \cdots(1)$$

$$\underbrace{(500\cdots 00)^3}_{\{n-1\}\ \ times\ 0} = 5 \times 10^{n-1} \qquad \cdots(2)$$

$$\underbrace{(333\cdots 33)^3}_{n\ times\ 3} = \frac{\overbrace{999\cdots 999}^{n\ times\ 9}}{3} = \frac{10^n - 1}{3} \qquad \cdots(3)$$

Hence from (1), (2), (3) :-

$$\begin{aligned}
A_n &= \underbrace{(166\cdots 66)^3}_{\{n-1\}\ times\ 6} + \underbrace{(500\cdots 00)^3}_{\{n-1\}\ \ times\ 0} + \underbrace{(333\cdots 33)^3}_{n\ times\ 3} \\
&= \left(\frac{10^n - 4}{6}\right)^3 + \left(5 \times 10^{n-1}\right)^3 + \left(\frac{10^n - 1}{3}\right)^3 \\
&= \frac{10^{3n} - 3 \times 10^{2n} \times 4 + 3 \times 10^n \times 4^2 - 4^3}{6^3} + 5^3 \times 10^{3n-3} + \frac{10^{3n} - 3 \times 10^{2n} + 3 \times 10^n - 1}{3^3} \\
&= \frac{10^{3n} - 12 \times 10^{2n} + 48 \times 10^n - 64}{6^3} + \frac{10^{3n}}{2^3} + \frac{10^{3n} - 3 \times 10^{2n} + 3 \times 10^n - 1)}{3^3}
\end{aligned}$$

$$
\begin{aligned}
&= 10^{3n}\left(\frac{1}{216}+\frac{1}{8}+\frac{1}{27}\right) - 10^{2n}\left(\frac{12}{216}+\frac{3}{27}\right) + 10^{n}\left(\frac{48}{216}+\frac{3}{27}\right) - \left(\frac{64}{216}+\frac{1}{27}\right)\\
&= \frac{10^{3n}}{6} - \frac{10^{2n}}{6} + \frac{10^{n}}{3} - \frac{1}{3}\\
&= \frac{10^{3n}}{6} - \frac{4\times 10^{2n}}{6} - \frac{10^{2n}}{2} + \frac{10^{n}-1}{3}\\
&= \frac{10^{n}-4}{6}\times 10^{2n} + \frac{10^{n}}{2}\times 10^{n} + \frac{10^{n}-1}{6}\\
&= \underbrace{166\cdots 66}_{\{n-1\}\ times\ 6}\ \underbrace{500\cdots 00}_{\{n-1\}\ times\ 0}\ \underbrace{333\cdots 33}_{n\ times\ 3}
\end{aligned}
$$

Hence, $166^3 + 500^3 + 333^3 = 166500333$ [Ans]

$1666^3 + 5000^3 + 3333^3 = 166650003333$ [Ans]

2. First Part ;-

Given that $\{x_n\}_{n\geq 1}$ is a sequence of positive real numbers where $x_{n-1}x_nx_{n+1} = 1$ for all $n \geq 2$ and $x_1 = 1$, $x_2 = 2$

Therefore :-

$x_1 \times x_2 \times x_3 = 1$

or, $1 \times 2 \times x_3 = 1$

or, $x_3 = \frac{1}{2}$

$x_2 \times x_3 \times x_4 = 1$

or, $2 \times \frac{1}{2} \times x_4 = 1$

or, $x_4 = 1$

$x_3 \times x_4 \times x_5 = 1$

or, $\frac{1}{2} \times 1 \times x_5 = 1$

or, $x_5 = 2$

$x_4 \times x_5 \times x_6 = 1$

or, $1 \times 2 \times x_6 = 1$

or, $x_6 = \frac{1}{2}$

Therefore in the sequence (where $x_1 = 1$ and $x_2 = 2$) 1, 2 and $\frac{1}{2}$ will repeat again and again in this order.

Hence

$$
\{x_n\}_{n\geq 1} = \begin{cases} 1 & when\ n = 3k+1 \\ 2 & when\ n = 3k+2 \\ \frac{1}{2} & when\ n = 3k, \forall\ k \in \mathbb{N} \end{cases} \tag{1}
$$

For other starting values let $x_1 = a$ and $x_2 = b$ where $a, b \in \mathbb{R}$

Now, $x_1 \times x_2 \times x_3 = 1$

or, $a \times b \times x_3 = 1$

or, $x_3 = \frac{1}{ab}$

$x_2 \times x_3 \times x_4 = 1$

or, $b \times \frac{1}{ab} \times x_4 = 1$

or, $x_4 = a$

$x_3 \times x_4 \times x_5 = 1$

or, $\frac{1}{ab} \times a \times x_5 = 1$

or, $x_5 = b$

$x_4 \times x_5 \times x_6 = 1$

or, $a \times b \times x_6 = 1$

or, $x_6 = \frac{1}{ab}$

Therefore in the sequence (where $x_1 = a$ and $x_2 = b$) a, b, $\frac{1}{ab}$ will repeat again and again in this order.

Hence

$$\{x_n\}_{n\geq 1} = \begin{cases} a & when\ n = 3k+1 \\ b & when\ n = 3k+2 \\ \frac{1}{ab} & when\ n = 3k, \forall\ k \in \mathbb{N} \end{cases} \tag{2}$$

Second Part ;-

Given that $\{y_n\}_{n\geq 1}$ is a sequence of positive real numbers where $y_{n-1}y_{n+1} + y_n = 1$ for all $n \geq 2$ and $y_1 = 1$, $y_2 = 2$

Therefore :-

$y_1 \times y_3 + y_2 = 1$

or, $1 \times y_3 + 2 = 1$

or, $y_3 = -1$

$y_2 \times y_4 + y_3 = 1$

or, $2 \times y_4 - 1 = 1$

or, $y_4 = 1$

$y_3 \times y_5 + y_4 = 1$

or, $(-1) \times y_5 + 1 = 1$

or, $y_5 = 0$

$y_4 \times y_6 + y_5 = 1$

or, $1 \times y_6 + 0 = 1$

or, $y_6 = 1$

$y_5 \times y_7 + y_6 = 1$

or, $0 \times y_7 + 1 = 1$

or, $y_7 = a$ where a is any real number

$y_6 \times y_8 + y_7 = 1$

or, $1 \times y_8 + a = 1$

or, $y_8 = 1 - a$

$y_7 \times y_9 + y_8 = 1$

or, $a \times y_9 + 1 = 1$

or, $a \times y_9 = a$

If a is zero then the cycle repeats from y_7 and if a is not zero then $y_9 = 1$

Now, considering $a \neq 0$:-

$y_8 \times y_{10} + y_9 = 1$

or, $(1 - a) \times y_{10} + 1 = 1$

Again if a is 1 then the cycle repeats from y_7 and if a is not equals to 1 then $y_{10} = 0$

So in the sequence (where $y_1 = 1$ and $y_2 = 2$) if $a \neq 0, 1$ then after y_3 1, 0, 1, a, $(1-a)$ will repeat again and again in this order where a is a real number which can vary arbitrarily and whenever $a = 0$ *or* 1 cycle repeats from the 7th term of the sequence

$$\{y_n\}_{n\geq 1} = \begin{cases} -1 & when\ n = 3 \\ 1 & when\ n = 5k - 1\ and\ 5k + 1 \\ 0 & when\ n = 5k, \\ Any\ real\ number\ (Let\ p,\ p \neq 0, 1) & when\ n = 5k + 2 \\ 1 - p & when\ n = 5k + 3 \forall\ k \in \mathbb{N} \end{cases} \tag{3}$$

For other starting values let $y_1 = m$ and $y_2 = n$ where $m,\ n \in \mathbb{R}$:-

$y_1 \times y_3 + y_2 = 1$

or, $m \times y_3 + n = 1$

Here if $m = 0$ then $n = 1$ and y_3 can be any real number

Case 1 [$m = 0$, $n = 1$ and $y_3 = a$ where a is any real number]:-

$y_2 \times y_4 + y_3 = 1$

or, $1 \times y_4 + a = 1$

or, $y_4 = 1 - a$

$y_3 \times y_5 + y_4 = 1$

or, $a \times y_5 + 1 - a = 1$

or, $a \times y_5 = a$

Here if $a = 0$ then the cycle again starts from y_3 where $y_3 = a$ that is a cycle will continue with any real value of y_5 and if a is not equals to 0 then $y_5 = 1$

Considering $a \neq 0$

$y_4 \times y_6 + y_5 = 1$

or, $(1 - a) \times y_6 + 1 = 1$

or, $(1 - a) \times y_6 = 0$

Here if $a = 1$ then the cycle starts from y_3 where $y_3 = a$ that is a cycle will continue with any real value of y_6 and if a is not equals to 1 then $y_6 = 0$

Considering $a \neq 1$

$y_5 \times y_7 + y_6 = 1$

or, $1 \times y_7 + 0 = 1$

or, $y_7 = 1$

$y_6 \times y_8 + y_7 = 1$

or, $0 \times y_8 + 1 = 1$

Here y_8 can be any real number and the cycle repeats from y_3 where $y_3 = a$. That is, the cycle will continue with any real value of y_8

Case 2 [$m \neq 0$ and $y_3 = \frac{1-n}{m}$]:-

$y_2 \times y_4 + y_3 = 1$

or, $n \times y_4 + \frac{1-n}{m} = 1$

or, $n \times y_4 = \frac{m+n-1}{m}$

Here if $n = 0$ then $m = 1$ and the cycle restarts from y_3 of Case 1 where $y_3 = a$. That is, the cycle will continue with any real value of y_4. If n is not equals to 0 and $m = 1$ then $y_4 = 1$. If Here if $n \neq 0$ and $m \neq 1$ then $y_4 = \frac{m+n-1}{m}$

Case 2A [$n \neq 0$, $m = 1$ and $y_4 = 1$]:-

$y_3 \times y_5 + y_4 = 1$

or, $(1 - n) \times y_5 + 1 = 1$

or, $(1 - n) \times y_5 = 0$

Here if $n = 1$ then the cycle restarts from y_3 of Case 1 where $y_3 = a$. That is, the cycle will continue with any real value of y_5 and if $n \neq 1$ then $y_5 = 0$

Considering $n \neq 1$

$y_4 \times y_6 + y_5 = 1$

or, $1 \times y_6 + 0 = 1$

or, $y_6 = 1$

$y_5 \times y_7 + y_6 = 1$

or, $0 \times y_7 + 1 = 1$

or, $0 \times y_7 = 0$

Here y_7 can be any real number and the cycle repeats from y_3 of Case 1 where $y_3 = a$. That is, the cycle will continue with any real value of y_7

Case 2B $[n \neq 0$**,** $m \neq 1$ **and** $y_4 = \frac{m+n-1}{m}$ $]$**:-**

$y_3 \times y_5 + y_4 = 1$

or, $\frac{1-n}{m} \times y_5 + \frac{m+n-1}{m} = 1$

or, $y_5 = \frac{1-m}{n}$

$y_4 \times y_6 + y_5 = 1$

or, $\frac{m+n-1}{m} \times y_6 + \frac{1-m}{n} = 1$

or, $y_6 = m$

$y_5 \times y_7 + y_6 = 1$

or, $\frac{1-m}{n} \times y_7 + m = 1$

or, $y_7 = n$

So if in this sequence (where $y_1 = m \neq 1$and $y_2 = n \neq 0$) m, n, $\frac{1-n}{m}$, $\frac{m+n-1}{m}$, $\frac{1-m}{n}$ will repeat again and again in this order

Considering $m \neq 1$ and $n \neq 0$

$$\{y_n\}_{n\geq 1} = \begin{cases} m & when\ n = 5k+1 \\ n & when\ n = 5k+2 \\ \frac{1-n}{m} & when\ n = 5k+3 \\ \frac{m+n-1}{m} & when\ n = 5k+4 \\ \frac{1-m}{n} & when\ n = 5k\ \forall\ k \in \mathbb{N} \cup \{0\} \end{cases} \tag{4}$$

3. I have calculated manually till t_3 and noticed that t_3 and t_2 have same last two digits 8 & 7 and t_1 and t_2 has same last digit 7.

   So it may be possible that t_n and t_{n-1} has same last $(n-1)$ digits. I assumed it also because in the question we have to prove that t_k has same last 10 digits for $k \geq 10$

   At first we need to show that $3^{10^n} \equiv 1 (mod\ 10^{n+1})$ where $n \geq 2$ and $n \in \mathbb{N} - \{1, 2\}$ $\cdots$ (1)

   To prove this let it be true till $n = k$ where $k \in \mathbb{N} - \{1, 2\}$. Hence $3^{10^k} \equiv 1\ (mod\ 10^{k+1})$

   Now,

$$3^{10^{k+1}} - 1 = \left(3^{10^k}\right)^{10} - 1 = \left(3^{10^k} - 1\right)\left(3^{10^{k^9}} + 3^{10^{k^8}} \cdots + 3^{10^{k^2}} + 3^{10^k} - 1\right)$$

$$= \left(3^{10^k} - 1\right)\left(3^{10^{k^9}} - 1 + 3^{10^{k^8}} - 1 \cdots + 3^{10^{k^2}} - 1 + 3^{10^k} - 1 + 10\right)$$

Now, 10^{k+1} divides $\left(3^{10^k}-1\right)$ and 10 divides $\left(3^{10^{k^9}}-1+3^{10^{k^8}}-1\cdots+3^{10^{k^2}}-1+3^{10^k}-1+10\right)$.
Hence 10^{k+2} divides $3^{10^{k+1}}-1$.
Hence by Mathematical Induction we can say that :-

$$3^{10^n} \equiv 1(mod\ 10^{n+1})\ \forall\ n \geq 2 \text{where} n \in \mathbb{N}-\{1,2\}\ \text{[Proved]}$$

Again, Let $3^a \equiv r\ (mod\ 10^n)$ for some $a, r \in \mathbb{N}$
Hence $3^a = 10^n \times q + r$ for some $r \in \mathbb{N}$
Therefore $3^{3^a} = 3^{10^n+r} \equiv 1 \times 3^r = 3^r (mod\ 10^{n+1})$ as $3^{10^n} \equiv 1(mod\ 10^{n+1})$
Now, $t_2 = 3^{3^3} = 3^{27} = 3^{10\times2+7} \equiv 3^7(\text{mod}\ \ 100)$
$3^7 \equiv 87(mod\ 10)$ Hence $t_2 \equiv 87(mod\ 100)$
Therefore $t_3 = 3^{t_2} \equiv 3^{87}\ (mod\ 1000)$
Now,

$$3^7 \equiv 187\ (mod\ 1000) \implies 3^9 \equiv 683(mod\ 1000) \implies 3^{10} \equiv 49\ (mod\ 1000) \implies 3^{20} \equiv 401\ (mod\ 1000)$$

$$\implies 3^{80} \equiv 601\ (mod\ 1000) \implies 3^{87} \equiv 387\ (mod\ 1000)$$

Therefore $t_3 \equiv 387\ (mod\ 1000)$. Hence the last 2 and 3 digits of $3^{3^{3^3}}$ are respectively 87 and 387
Therefore $t_3 \equiv 87\ (mod\ 100)$
Hence $t_4 \equiv 3^{87}\ (mod\ 1000) \implies t_4 \equiv t_3\ (mod\ 1000)$
Similarly $t_5 \equiv t_4\ (mod\ 10^4)$
In general:-

$$t_{n+1} \equiv t_n\ (mod\ 10^n)\ where\ \forall\ n \in \mathbb{N}$$

Let this result holds till $n = k$ where $k \in \mathbb{N}$
Hence:-

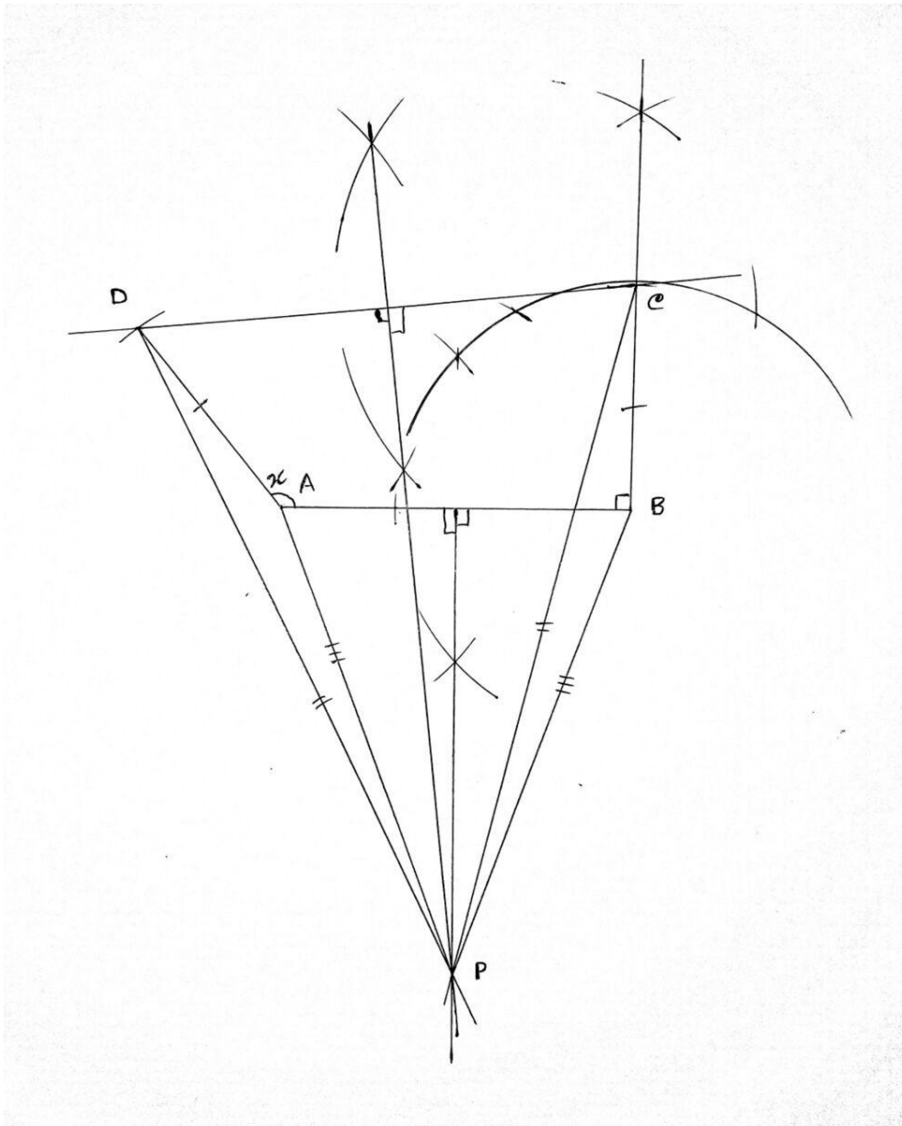
$$t_{k+1} \equiv t_k\ (mod\ 10^k)$$

Now, $t_{k+2} = 3^{k+1} = 3^{10^k \times m + t_k}$ where $m \in \mathbb{N}$
Therefore $t_{k+2} \equiv 3^{t_k} = t_{k+1}\ (mod\ 10^{k+1})$
Hence by Mathematical Induction we can say that

$$t_{n+1} \equiv t_n\ (mod\ 10^n)\ where\ \forall\ n \in \mathbb{N}$$

Hence $t_{11} \equiv t_{10}\ (mod\ 10^{10})$
$t_{12} \equiv t_{11}\ (mod\ 10^{11}) \implies t_{12} \equiv t_{11}\ (mod\ 10^{10}) \implies t_{12} \equiv t_{10}\ (mod\ 10^{10})$
Continuing this we can say $t_n \equiv_{10}\ \ (mod\ 10^{10})$ where $n \geq 10\ \forall\ n \in \mathbb{N}$
That means last 10 digits of t_n are the same for all $k \geq 10$ [Proved]

4. In the given "proof" the step where "$x = \angle PAD - \angle PAB$" written is wrong.



Because unlike the figure in the question, PD will be outside the quadrilateral $ABCD$. So $\angle PAD - \angle PAB$ does not equals to x.

5. Here it will be easy to solve the problem by Euler's Totient Function

   I learned about Euler's Totient Function from [1] and from[2]

   Another approach is to apply Inclusion–Exclusion Principle but that approach is easy for less number of primes.

   I learned about this principle from [3] and from [4]

   For larger number of primes Euler's Totient Function is most suitable. Here we can not use directly the fuction so first we need to prove it's properties

   Let $\phi(n)$ denotes a function such that $\phi(n) =$ number of positive numbers less than n that are coprime to n where $n \in \mathbb{N}$

   Hence for a prime number $p,\ \ \phi(p) = p - 1$

   For p^n there are p^{n-1} numbers which divide p^n. These numbers are $p,\ p^2,\ p^3,\ \cdots,\ p^{n-1}$

   Hence:-

   $$\phi(p^n) = p^n - p^{n-1} = p^n\left(1 - \frac{1}{p}\right)$$

   Now, for two primes p & q , $\phi(p) = p - 1$ and $\phi(q) = q - 1$

   Hence the number of numbers coprime to pq and less than pq is $(p - 1)(q - 1)$

   Therefore:- $\phi(pq) = (p - 1)(q - 1) = \phi(p)\phi(q)$

   Let n be a positive number and $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ be the prime distribution of n.

   Hence:- $\phi(n) = \phi(p_1^{a_1})\phi(p_2^{a_2}) \cdots \phi(p_k^{a_k}) = p_1^{a_1}\left(1 - \frac{1}{p_1}\right) p_2^{a_2}\left(1 - \frac{1}{p_2}\right) \cdots p_k^{a_k}\left(1 - \frac{1}{p_k}\right)$

   $$\phi(n) = n\left(1 - \frac{1}{p_1}\right)\left(1 - \frac{1}{p_2}\right) \cdots \left(1 - \frac{1}{p_k}\right)$$

   The L.C.M. of 2, 3, 5 is = 30

   Hence the number of quite prime positive integers less than 30 is $= \phi(30) = (2 - 1)(3 - 1)(5 - 1) = 8$

   First Part :-

   Hence $= \phi(90) = 3 \times \phi(30) = 24$

   Now between 90 and 100 there exist 3 quite prime positive numbers

   Hence the number of quite prime positive integers less than 100 is $24 + 4 = 27$ [Ans]

   Now, the number of quite prime positive integers less than 960 is

   $$= \phi(960) = 32 \times \phi(30) = 256$$

   Now between 960 and 1000 eliminating all the even numbers :-

961	963	965	967	969
971	973	975	977	979
981	983	985	987	989
991	993	995	997	999

   Now simply striking out the multiples of 3 then 5 :-

[1] https://en.wikipedia.org/wiki/Euler%27s_totient_function

[2] An Excursion In Mathematics by M. R. Modak, S. A. Katre, V. V. Acharya and V. M. Sholapurkar

[3] https://en.wikipedia.org/wiki/Inclusion%E2%80%93exclusion_principle

[4] Problem-Solving Strategies by Aurther Engel

961	~~963~~	~~965~~	967	~~969~~
971	973	~~975~~	977	979
~~981~~	983	~~985~~	~~987~~	989
991	~~993~~	~~995~~	997	~~999~~

Hence the number of very quite prime positive integers less than $1000 = 256 + 10 = 266$ [Ans]

Second Part :-

The primes less than 15 are 2, 3, 5, 7, 11, 13

Their L.C.M. $= 2 \times 3 \times 5 \times 7 \times 11 \times 13 = 30030$

Hence the number of very quite prime positive integers less than 30030 is

$$= \phi(30030) = (2-1)(3-1)(5-1)(7-1)(11-1)(13-1) = 5760$$

Therefore $\phi(90090) = 3 \times \phi(30030) = 3 \times 5760 = 17280$

We need to consider less than 90000

Now from 90001 to 90090 eliminating all the even number we get:-

90001	90011	90021	90031	90041	90051	90061	90071	90081
90003	90013	90023	90033	90043	90053	90063	90073	90083
90005	90015	90025	90035	90045	90055	90065	90075	90085
90007	90017	90027	90037	90047	90057	90067	90077	90087
90009	90019	90029	90039	90049	90059	90069	90079	90089

Now simply striking out the multiples of 3 then 5 then 7 then 11 and then 13 :-

90001	90011	~~90021~~	90031	~~90041~~	~~90051~~	90061	90071	~~90081~~
~~90003~~	~~90013~~	90023	~~90033~~	90043	90053	~~90063~~	90073	~~90083~~
~~90005~~	~~90015~~	~~90025~~	~~90035~~	~~90045~~	~~90055~~	~~90065~~	~~90075~~	~~90085~~
90007	90017	~~90027~~	90037	90047	~~90057~~	90067	~~90077~~	~~90087~~
~~90009~~	90019	90029	~~90039~~	90049	90059	~~90069~~	~~90079~~	90089

Hence between 90000 and 90090 there exists 19 very quite prime positive integers present

Hence there are $17280 - 19 = 17261$ very quite prime positive integers present less than 90000

In general we can say the number of very quite prime positive integers less than a number n is approximately $= \dfrac{\phi(30030)}{30030} \times n$

Hence the number of very quite prime positive integers less than 10^{10} is $\approx \dfrac{\phi(30030)}{30030} \times 10^{10}$ [Ans]

And the number of very quite prime positive integers less than 10^{100} is $\approx \dfrac{\phi(30030)}{30030} \times 10^{100}$ [Ans]

6. Here the monkey have to fill in the $n \times n$ grid in such a way that the cat and the dog have same set of numbers that is the set of 3 numbers obtained by multiplying the numbers in each row is equals to the set of 3 numbers obtained by multiplying the numbers in each column.

   A solution for the 3×3 grid ;-

9	2	4
1	5	6
8	3	7

   Hence for 3×3 grid it is possible to arrange the numbers

   For 5×5 grid a solution is :-

17	2	7	15	20
14	25	9	11	16
6	21	13	3	12
10	22	18	23	1
5	24	4	8	19

   Hence 5×5 has a solution

   From the arrangements we can conclude the primes p which have only one multiple which is its own value that is $\frac{n^2}{2} < p < n^2$ would be placed at the diagonal [Here only one of the two diagonals can be used]

   If the number of such primes is greater than n then it has no solution.

   For 11×11 grid the number of primes between 60 to 121 is greater than 11. Hence 11×11 grid has no solution

   For $n \times n$ grid the condition on n is

   $$\left| \{p \ : \ p \text{ is a prime such that } \frac{n^2}{2} < p < n^2\} \right| < n$$

   Now we can progress further using the Prime Number Theorem[5]

   Let $\pi(x)$ denotes the prime computing fuction.

   That means :-

   $$\pi(x) = \text{The number of primes less than or equal to } x \text{ where } x \in \mathbb{R}$$

   Now, with the help of the third inequality under the section of Non-asymptotic bounds on the prime-counting function in [6] which states that :-

   $$\frac{x}{\log x + 2} < \pi(x) < \frac{x}{\log x - 4} \ \forall \ x \geq 55 \text{ where } x \in \mathbb{R} \cdots (1)$$

   we can find any bound for n.

   Now,

   $$\frac{n^2}{\log(n^2) + 2} - \frac{\frac{n^2}{2}}{\log\left(\frac{n^2}{2}\right) - 4} > n$$

[5] https://en.wikipedia.org/wiki/Prime_number_theorem

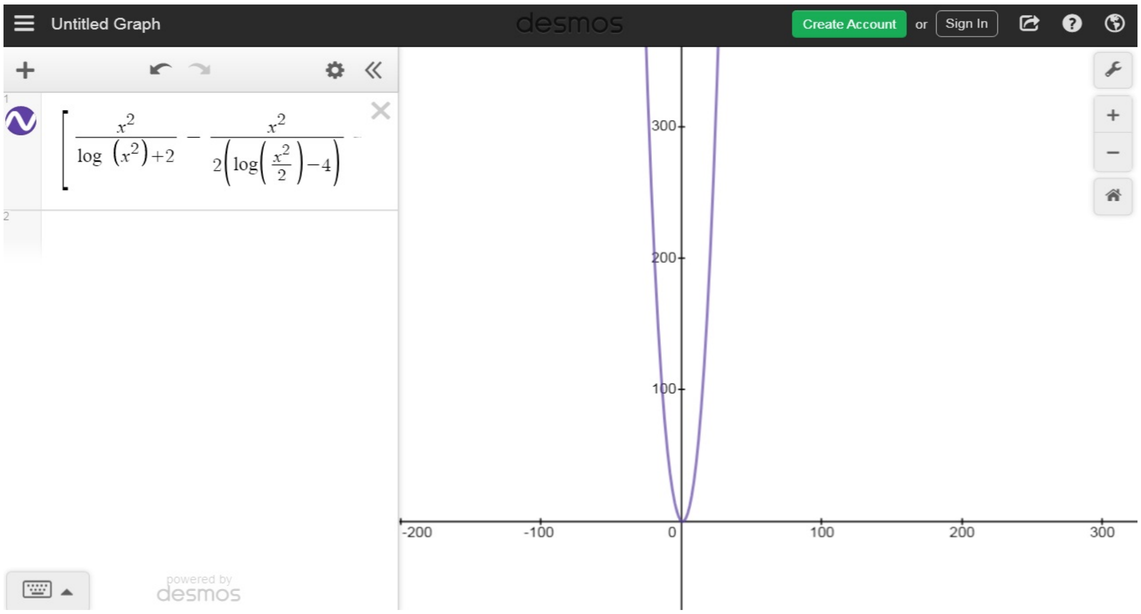
[6] https://en.wikipedia.org/wiki/Prime_number_theorem#Non-asymptotic_bounds_on_the_prime-counting_function

This inequality is satisfied by $n = 11,\ 12,\ 13,\ \cdots 18$.

Now consider the function :-

$$f(x) = \frac{x^2}{\log(x^2)+2} - \frac{\frac{x^2}{2}}{\log\left(\frac{x^2}{2}\right)-4} - x$$

The graph of the function :-



Now, I am unable to prove that this function is always greater than 0 when $x \geq 11$

If it is proven then we can conclude that there is no solution for an $n \times n$ grid so that the cat and the dog obtain the same set of numbers.

7. The set S contains some real numbers following the 3 rules

   (a) $1 \in S$

   (b) If $\frac{a}{b} \in S$ [where $\frac{a}{b}$ is in lowest forms and a, b are coprime] then $\frac{a}{2b} \in S$

   (c) $\frac{a}{b},\ \frac{c}{d} \in S$ [where $\frac{a}{b}$ and $\frac{c}{d}$ is in lowest forms and a, b and c, d are coprimes] then $\frac{a+c}{b+d} \in S$

As $1 \in S$ hence $\frac{1}{2},\ \frac{1}{4},\ \frac{1}{8} \cdots \in S$

Hence till now S contains 1 and some fractions $\frac{a}{b}$ where $a < b$

Now we need to prove that by those rules a rational number $\frac{p}{q}$ where $p > q$ does not belongs to S

Let a fraction $\frac{m}{n}$ which is less than 1 is in S

Hence $\frac{m+1}{n+1}$ also is in S

$$\frac{m}{n} < 1 \implies m < n \implies m+1 < n+1 \implies \frac{m+1}{n+1} < 1$$

Hence with the 3rd rule there can not be a fraction greater than 1 belongs to S.

Now in second rule a fraction is halfed and the new fraction is in S.

Let a fraction $\frac{x}{y}$ which is less than 1 is in S

$$\frac{x}{y} < 1 \implies x < y \implies x < 2y \implies \frac{x}{2y} < 1$$

Hence it is proved that S does not contains any rational number greater than 1.

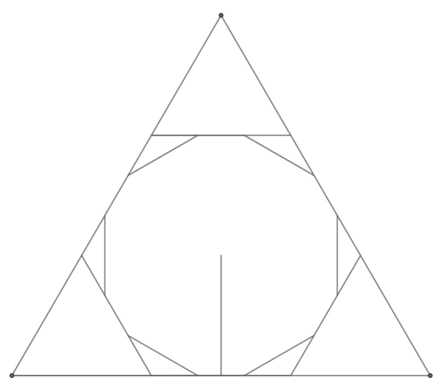
Hence $S = \{x\ : x \in (0,\ 1) \text{ where } x \in \mathbb{Q}\}$

8. Let the side of the triangle is a.

   Now, the area of the triangle is 10

   Hence $\dfrac{\sqrt{3}}{4}a^2 = 10$

   or, $a^2 = \dfrac{40}{\sqrt{3}}$



Now after every trisection of sides of previous figure and snipping out the corners a regular polygon is created and the distance between the centre of the polygon and the sides remain same.

Hence P_1 is a regular hexagon with side $\dfrac{a}{3}$

The distance between the centre and the sides $= \dfrac{\sqrt{3}}{2}a \times \dfrac{1}{3} = \dfrac{a}{2\sqrt{3}}$

Hence area of P_1:-

$$P_1 = 6\times\frac{1}{2} \times \frac{a}{2\sqrt{3}} \times 2\frac{a}{2\sqrt{3}} \tan\left(\frac{2\pi}{2\times 6}\right) = \frac{a^2}{2}\tan\frac{\pi}{6} = \frac{40}{\sqrt{3}} \times \frac{1}{2} \times \frac{1}{\sqrt{3}} = \frac{20}{3}$$

Now, P_2 is a regular polygon with 12 sides formed as mentioned above.

Area of P_2:-

$$P_2 = 12 \times \frac{a^2}{12}\tan\left(\frac{2\pi}{2\times 12}\right) = \frac{40}{\sqrt{3}}\tan\frac{\pi}{12}$$

After every trisection of the sides of the previous polygon and snipping out the corner the number of sides of the new polygon is twice the previous one.

Hence P_n has $3 \times 2^{n-1}$ sides.

In this way the figure tends to reduce into the incircle of the given equilateral triangle.

Hence area of P_∞ or P_n where $n \to \infty$:-

$$P_\infty = \lim_{n\to\infty} P_n = \lim_{n\to\infty} 3 \times 2^{n-1} \times \frac{a^2}{12} \tan\left(\frac{2\pi}{2 \times 3 \times 2^{n-1}}\right) = \lim_{n\to\infty} 3 \times 2^{n-1} \times \frac{a^2}{12} \tan\left(\frac{\pi}{3 \times 2^{n-1}}\right)$$

$$= \lim_{n\to\infty} \frac{a^2}{12} \times \frac{\tan\left(\frac{\pi}{3\times 2^{n-1}}\right)}{\frac{\pi}{3\times 2^{n-1}}} \times \pi = \pi \times \frac{a^2}{12} = \pi \times \frac{40}{\sqrt{3}} \times \frac{1}{12} = \frac{10\pi}{3\sqrt{3}}$$

P_∞ or P_n where $n \to \infty$ is the incircle of the equilateral triangle.

Hence the area of P_∞ = Area of the incircle = $\dfrac{10\pi}{3\sqrt{3}}$.

10. For 3 trains :-

Final remaining linked up trains	Number of times from all possible arrangements
1	2
2	3
3	1

Hence average of all outcomes = $\dfrac{1\times 2 + 2\times 3 + 3 \times 1}{3!} = \dfrac{11}{6} = 1 + \dfrac{1}{2} + \dfrac{1}{3}$

For 4 trains :-

Final remaining linked up trains	Number of times from all possible arrangements
1	6
2	11
3	6
4	1

Hence average of all outcomes= $\dfrac{1\times 6 + 2 \times 11 + 3 \times 6 + 4 \times 1}{4!} = \dfrac{50}{24} = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4}$

Noticing the pattern in case of 5 trains average of all outcomes may be $= 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5}$

Let $C(n)$ be avarage of all outcomes when there are n trains in the line.

We need to prove that

$$C(n) = \sum_{i=1}^{n} \frac{1}{i}$$

Let this is true for $n = k$

Hence

$$C(k) = \sum_{i=1}^{k} \frac{1}{i}$$

For $n = k + 1$ we can say that a new train is introduced behind all the k trains in case of $n = k$

Let $v_1,\ v_2, \cdots,\ v_k$ denotes the speeds of the k trains where $v_1 > v_2 > \cdots > V_k$

Hence there are $(n + 1)$ intervals of speeds; $(\infty,\ v_1]$, $(v_1,\ v_2]$, $(v_2,\ v_3]$, $\cdots$, $(v_k,\ 0)$

Therefore the speed of $(k + 1)$th train can be from any of these intervals

The slowest train and the trains behind it forms a cluster which is always at the tail end. Then the trains between the slowest and the second slowest trains forms a cluster with the second slowest train and so on

Now two cases arise :-

(a) The speed of $(k + 1)$th train is less than v_k. Then the $(k + 1)$th train will form a single cluster of its own.

   The probability of this event $= \dfrac{1}{n + 1}$ as the speed of $(k + 1)$th train only belongs to $(v_k,\ 0)$

(b) The speed of $(k + 1)$th train is more than slowest train. Then it will join the last cluster.

   The probability for this event is $= \dfrac{n}{n + 1}$ as the speed of $(k + 1)$th train belongs to any of the intervals $(v_1,\ v_2]$, $(v_2,\ v_3]$, $\cdots$, $(v_{k-1},\ v_k]$

Hence

$$C(k+1) = \frac{C(k) + 1}{n + 1} + \frac{C(k) \times n}{n + 1} = C(k)\left(\frac{1}{n + 1} + \frac{n}{n + 1}\right) + \frac{1}{n + 1} = C(k) + \frac{1}{n + 1}$$

Hence by Mathematical Induction we can say that $C(n) = \displaystyle\sum_{i=1}^{n} \frac{1}{i}$ is true for all $n \geq 3$ [Proved]

Therefore $C(5) = 1 + \dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \dfrac{1}{5}$

Hence for n trains average of all possible outcomes $= \displaystyle\sum_{i=1}^{n} \frac{1}{i}$

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