Fulton chapter 3: Local Properties of Plane Curves

Intersection Numbers

Aritra Kundu
Email: aritra@cmi.ac.in

Problem Set - 2
Topic: Algebraic Geometry

k is an algebraically closed field. F, G are two affine plane curves in $k[X,Y]$. $P \in A^2$

Property 1: $I(P, F \cap G) \geq 0$,the equality holds if and only if $P \notin V(F) \cap V(G)$

Property 2: Assume F, G both passes through P. $I(P, F \cap G) < \infty \iff F, G$ has no common component passes through P. Otherwise $I(P, F \cap G) = \infty$

Property 3: For any affine transformation T $I(P, F \cap G) = I(Q, F^T \cap G^T)$ where $T(P) = Q$

Property 4: $I(P, F \cap G) = I(P, G \cap F)$. Let P be a simple point of both F, G. F and G are said to be intersected transversely if they do not share the tangent at P.

Property 5: $I(P, F \cap G) \geq m_P(F)m_P(G)$ the equality holds if and only if F and G have no common tangent at P. This property requires to ensure the condition that F, G intersect transversely if and only if $I(P, F \cap G) = 1$

Property 6: $I(P, F \cap GH) = I(P, F \cap G) + I(P, F \cap H)$

Property 7: $I(P, F \cap G) = I(P, F \cap G + AF) \forall$ polynomial A in $k[x, y]$. If F is irreducible then $I(P, F \cap G)$ only depends on the image of G in $\tau(F)$

Definition: If F, G has no common component passing through P then F, G has said to be intersected properly. Lets assume $I(P, F \cap G)$ exists for any two affine curves.

Problem 1 Claim

Intersection number of F, G at P,$I(P, F \cap G)$ which has the 7 properties is unique.

Solution: We can assume $P = (0, 0)$[as by property 3 we can apply an affine transformation to make P to be origin by keeping unchanged $I(P, F \cap G)$]

1. If F, G has a common component passing through P then $I(P, F \cap G) = \infty$[by property 2]
2. $I(P, F \cap G) = 0 \iff$ either $F(P) \neq 0$ or $G(P) \neq 0$
3. $I(P, F \cap G) = m_P(F)m_p(G) \iff F, G$ do not share any tangent at P.

So we can assume F, G has intersected properly and $I(P, F \cap G)$ can not be computed directly from the 3 properties mentioned. Let $F(X,0), G(X,0)$ are polynomials of degree of degree r, s respectively. Let's assume $r = 0$. $P(n)$ be the statement that if $I(P, F \cap G)$ has a value less than n then it is unique.

$P(1)$ is true [by property 1][as then $I(P, F \cap G) = 0 \iff P \notin V(F) \cap V(G)$]. Let $P(n)$ be true. Let F, G be affine curves such that $I(P, F \cap G)$ has a value equal to n. So, $F = YH_1$ and $G = Yg_1 + g_2(X)$ where

$$g_2(X) = X^{m_1}(a_0 + a_1X ... a_{s-m_1}X^{s-m_1}.a_0 \neq 0 \; [m_1 > 0 \text{ as } P \in V(G)]$$

Now by property 6 $I(P, F \cap G) = I(P, Y \cap G) + I(P, H_1 \cap G)$

$$\begin{aligned} I(P, Y \cap G) = I(P, Y \cap X^{m_1}) + I(P, Y \cap (a_0 + a_1X + ..a_{s-m_1}X^{s-m_1})) &= m_1(I(P, Y \cap X) \\ &= m_1 \; [\text{as } a_0 \neq 0 \implies I(P, Y \cap (a_0 + a_1X + ..a_{s-m_1}X^{s-m_1})) = 0] \end{aligned}$$

so,

$$I(P, H_1 \cap G) + m_1 = I(P, F \cap G)$$

Now $I(P, H_1 \cap G) < I(P, F \cap G)$ so $I(P, H_1 \cap G) < n$ so $I(P, H_1 \cap G)$ is unique and so $I(P, F \cap G)$ is unique. Therefore $P(n+1)$ is true. So, $P(n)$ is true $\forall n \in \mathbb{N}$[by principle of mathematical induction].

Let $r > 0$. WLOG assume that $r \leq s$[by property 4]. a, b be the leading coefficients of $F(X,0)$ and $G(X,0)$. Let $H_1 = G - (b/a)X^{s-r}F$. So,

$$I(P, F \cap G) = I(P, F \cap H_1)$$

clearly $deg(H_1(X,0)) < deg(G(X,0))$. If $deg(H_1(X,0)) > deg(F(X,0))$ then repeating the process for finite number of times we get H s.t.$deg(H(X,0)) < deg(F(X,0))$ and $I(P, F \cap G) = I(P, F \cap H)$. Then interchanging the role of F, H and after repeating the process for finitely many times we can make the minimum of $degA(X,0), degB(X,0)$ is 0 and then go to the previous case.

$\square$