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Course: Calculus

Assignment - 1
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Problem 1

Is the function $\log x$ uniformly continuous on $[1, \infty)$?

Solution: For all $x \in [1, \infty)$ we have

$$\log x \leq x - 1$$

Hence for $\epsilon > 0$, $x, y \in [1, \infty)$]let

$$|\log x - \log y| < \epsilon \iff \left|\log \frac{x}{y}\right| < \epsilon$$

Now if $|x - y| < \delta$ where $\delta > 0$ and suppose $x \geq y$ then $x = y + k$ for some $0 \geq k < \delta$ then

$$\log \frac{x}{y} = \log\left(1 + \frac{k}{y}\right) \leq \frac{k}{y} \leq \frac{k}{1} < \delta$$

Hence if we take $\epsilon = \delta$ then for all $x, y \in [1, \infty)$, if $|x - y| < \delta$ then

$$|\log x - \log y| < \epsilon$$

Therefore $\log x$ is uniformly continuous in $[1, \infty)$

$\square$

Problem 2

Let us agree that *a complex-valued function $f = f_r + i f_i$ defined on I will be called Riemann integrable* if its real and imaginary parts f_r, f_i are Riemann Integrable; in which case, we define

$$\int_0^1 f(x)dx \equiv \int_0^1 f_r(x)dx + i\int_0^1 f_i(x)dx\ .$$

Prove that if f is Riemann integrable then $|f|$ is Riemann integrable, and

$$\left|\int_0^1 f(x)dx\right| \leq \int_0^1 |f(x)|dx$$

Solution: As $\sqrt{f_r^2(x) + f_i^2(x)} \geq 0$ for all $x \in [0, 1]$. Now if $\int_0^1 f_r(x)dx = \int_0^1 f_i(x)dx = 0$ then

$$\int_0^1 |f(x)|dx \geq \left|\int_0^1 f(x)dx\right|$$

If at least one of $\int_0^1 f_r(x)dx, \int_0^1 f_i(x)dx$ is nonzero then let

$$a = \frac{\int_0^1 f_r(x)dx}{\sqrt{\left(\int_0^1 f_r(x)dx\right)^2 + \left(\int_0^1 f_i(x)dx\right)^2}} \qquad b = \frac{\int_0^1 f_i(x)dx}{\sqrt{\left(\int_0^1 f_r(x)dx\right)^2 + \left(\int_0^1 f_i(x)dx\right)^2}}$$

Therefore $a^2 + b^2 = 1$. Now by Cauchy Schwarz Inequality

$$\sqrt{a^2 + b^2}\sqrt{f_r^2(x) + f_i^2(x)} = \sqrt{f_r^2(x) + f_i^2(x)} \geq af_r(x) + bf_i(x) \qquad \forall\ x \in I$$

Hence

$$
\begin{aligned}
& a\int_0^1 f_r(x)dx + b\int_0^1 f_i(x)dx = \int_0^1 (af_r(x)+bf_i(x))dx \\
\iff & a\int_0^1 f_r(x)dx + b\int_0^1 f_i(x)dx \le \int_0^1 \sqrt{f_r^2(x)+f_i^2(x)}dx \\
\iff & \sqrt{\left(\int_0^1 f_r(x)dx\right)^2 + \left(\int_0^1 f_i(x)dx\right)^2} \le \int_0^1 \sqrt{f_r^2(x)+f_i^2(x)}dx \\
\iff & \left|\int_0^1 f(x)dx\right| \le \int_0^1 |f(x)|dx
\end{aligned}
$$

□

Problem 3

The following problems are (with minor changes) taken from Rudin. Let p, q be positive real numbers satisfying

$$\frac{1}{p} + \frac{1}{q} = 1$$

(These are said to be *conjugate exponents* to each other. Note that $p = 2, q = 2$ are conjugate. Note also the "limiting cases" $p = 1, q = \infty$, $p = \infty, q = 1$.)

(a) If $u \ge 0,\ v \ge 0$, prove that

$$uv \le \frac{u^p}{p} + \frac{v^q}{q}$$

(Hint: Reduce to proving the case when $u = 1$ and $0 \le v \le 1$. When $v = 0$ or $v = 1$, the inequality is clear; now use convexity.)

(b) If f, g are Riemann integrable non-negative functions on I, then

$$\int_0^1 fg\ dx \le \left\{\int_a^b f^p dx\right\}^{\frac{1}{p}} \left\{\int_a^b g^q dx\right\}^{\frac{1}{q}}$$

(Hint: Reduce to the case when both factors on the right are equal to one. Then use (a).)

(c) If f, g are complex-valued and Riemann integrable on I,

$$\left|\int_0^1 fg\ dx\right| \le \left\{\int_0^1 |f|^p dx\right\}^{\frac{1}{p}} \left\{\int_0^1 |g|^q dx\right\}^{\frac{1}{q}}$$

Solution:

(a) If $u = v = 0$ then we are done. Suppose at least one of is nonzero. Let $v^q \ge u^p$. Hence $v \ne 0$. Then

$$
\begin{aligned}
& uv \le \frac{u^p}{p} + \frac{v^q}{q} \\
\iff & \frac{uv}{v^q} \le \frac{1}{p}\frac{u^p}{v^q} + \frac{1}{q} \\
\iff & \frac{u}{v^{q-1}} \le \frac{1}{p}\frac{u^p}{v^q} + \frac{1}{q}
\end{aligned}
$$

Let $x = \frac{u^p}{v^q}$. Now

$$\frac{1}{p} + \frac{1}{q} = 1 \iff \frac{1}{p} = \frac{q-1}{q} \iff \frac{q}{p} = q - 1$$

Hence $x^{\frac{1}{p}} = \frac{u}{v^q}$. Hence substituting the values we need to prove

$$x^{\frac{1}{p}} \leq \frac{x}{p} + \frac{1}{q} \qquad \forall\, x \in [0,1]$$

Now take the function $f(x) = x^{\frac{1}{p}} - \frac{x}{p}$. Now at $x = 1$ we have

$$f(1) = 1 - \frac{1}{p} = \frac{1}{q}$$

and at $x = 0$ we have $f(0) = 0$. Now f is a differentiable function.

$$f'(x) = \frac{1}{p} x^{\frac{1}{p}-1} - \frac{1}{p} = \frac{1}{p}\left(x^{-\frac{1}{q}} - 1\right)$$

Now since $0 \leq x \leq 1$ we have $0 \leq x^{\frac{1}{q}} \leq 1$ and therefore $x^{-\frac{1}{q}} \geq 1$. Hence $\forall\, x \in [0,1]$ we have $f'(x) \geq 0$. Hence f is increasing. Hence the maximum value f attains on $[0,1]$ is $\frac{1}{q}$. Hence

$$x^{\frac{1}{p}} \leq \frac{x}{p} + \frac{1}{q}$$

Therefore

$$uv \leq \frac{u^p}{p} + \frac{v^q}{q}$$

$\square$

(b) $A = \left(\int_0^1 f^p(x)dx\right)^{\frac{1}{p}}$ and $B = \left(\int_0^1 g^q(x)dx\right)^{\frac{1}{q}}$. Now using part (a) we get

$$\begin{aligned}
&\frac{f(x)}{A}\frac{g(x)}{B} \leq \frac{1}{p}\left(\frac{f(x)}{A}\right)^p + \frac{1}{q}\left(\frac{g(x)}{B}\right)^q \\
\iff & \int_0^1 \frac{f(x)g(x)}{AB}dx \leq \int_0^1 \left[\frac{1}{p}\left(\frac{f(x)}{A}\right)^p + \frac{1}{q}\left(\frac{g(x)}{B}\right)^q\right] dx \\
\iff & \frac{\int_0^1 f(x)g(x)dx}{AB} \leq \frac{1}{p}\frac{\int_0^1 f^p(x)dx}{A^p} + \frac{1}{q}\frac{\int_0^1 g^q(x)dx}{B^q} \\
\iff & \frac{\int_0^1 f(x)g(x)}{AB} \leq \frac{1}{p} + \frac{1}{q} = 1 \\
\iff & \int_0^1 f(x)g(x)dx \leq AB \\
\iff & \int_0^1 f(x)g(x)dx \leq \left(\int_0^1 f^p(x)dx\right)^{\frac{1}{p}} \left(\int_0^1 g^q(x)dx\right)^{\frac{1}{q}}
\end{aligned}$$

$\square$

(c) $A = \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}}$ and $B = \left(\int_0^1 |g(x)|^q dx\right)^{\frac{1}{q}}$. Now using part (a) we get

$$\begin{aligned}
&\frac{|f(x)|}{A}\frac{|g(x)|}{B} \leq \frac{1}{p}\left(\frac{|f(x)|}{A}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{B}\right)^q \\
\iff &\int_0^1 \frac{|f(x)g(x)|}{AB}dx \leq \int_0^1 \left[\frac{1}{p}\left(\frac{|f(x)|}{A}\right)^p + \frac{1}{q}\left(\frac{|g(x)|}{B}\right)^q\right] dx \\
\iff &\frac{\int_0^1 |f(x)g(x)|dx}{AB} \leq \frac{1}{p}\frac{\int_0^1 |f(x)|^p dx}{A^p} + \frac{1}{q}\frac{\int_0^1 |g(x)|^q dx}{B^q} \\
\iff &\frac{\int_0^1 |f(x)g(x)|}{AB} \leq \frac{1}{p} + \frac{1}{q} = 1 \\
\iff &\int_0^1 |f(x)g(x)|dx \leq AB \\
\iff &\int_0^1 |f(x)g(x)|dx \leq \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_0^1 |g(x)|^q dx\right)^{\frac{1}{q}}
\end{aligned}$$

Using (1) we have

$$\left|\int_0^1 f(x)g(x)dx\right| \leq \int_0^1 |f(x)g(x)|dx$$

Hence

$$\left|\int_0^1 f(x)g(x)dx\right| \leq \left(\int_0^1 |f(x)|^p dx\right)^{\frac{1}{p}} \left(\int_0^1 |g(x)|^q dx\right)^{\frac{1}{q}}$$

□

Problem 4

Consider the set $S = \{(x, y) | x, y \in \mathbb{Q}\} \subset I^2$. Is S Jordan-measurable? If yes, compute its area. If not

(a) show directly that χ_S is not integrable *and*

(b) show that ∂S does not have content zero.

Solution: Since $\mathbb{Q} \cap I$ is dense in I, $\mathbb{Q}^2 \cap I^2 = S$ is dense in I^2. FOr any closed rectangle R in I^2 we can write $R = I_1 \times I_2$ where I_1, I_2 are closed intervals in I. Then we will always have a point from $\mathbb{Q} \cap I$ and a point from $(\mathbb{R} \setminus \mathbb{Q}) \cap I$ in eaech of I_1, I_2. Now We take the closed rectangle $R = I^2$ which covers the whole S. Let P be any partition of I^2. Since S is dense in I^2 for any rectangle R_i we have $S \cap R_i \neq \phi$ and $(I^2 \setminus S) \cap R_i \neq \phi$. Hence $M_{R_i}(x) = 1, m_{R_i}(x) = 0$ for all $x \in R_i$. Hence

$$U(P, R) - L(P, R) = \sum_i (M_{R_i}(x) - m_{R_i}(x))Vol(R_i) = \sum_i (1-0)Vol(R_i) = \sum_i Vol(R_i) = Vol(R) = 1$$

for all partitions P of R we have this. Hence χ_S is not integrable. Therefore ∂S does not have measure zero. Hence S is not Jordan-Measurable.

(a) We just show that χ_S is not integrable.

□

(b) Since χ_S is not integrable $\iff$ ∂S is content zero and since χ_S is not integrable we have ∂S is not content zero.

□

Problem 5

What about the set

$$S = \{(x, y)\} \setminus \bigcup_{n=1,\dots} \left\{\left(\frac{1}{n}, y\right)\right\} \subset I^2\ ?$$

Solution: Since

$$S = \{(x, y) \in I^2\} \setminus \left\{\left(\frac{1}{n}, y\right) \mid y \in I,\ n \in \mathbb{N}\right\} = \bigcup_{i=1}^{\infty} \left(\frac{1}{i+1}, \frac{1}{i}\right) \times I$$

Hence the

$$\partial S = S \cup \{\{x\} \times I, I \times \{x\} \mid x = 0, 1\} = \left\{\left(\frac{1}{n}, y\right) \mid y \in I,\ n \in \mathbb{N}\right\} \cup \{\{x\} \times I, I \times \{x\} \mid x = 0, 1\}$$

$$= \{\{x\} \times I, I \times \{x\} \mid x = 0, 1\} \cup \left[\bigcup_{i=1}^{\infty} \left\{\frac{1}{i}\right\} \times I\right]$$

Now for any $\epsilon > 0$ the set $\left\{\frac{1}{i}\right\} \times I$ can be covered by the rectangle $\left[\frac{1}{i} - \frac{\epsilon}{2 \times 2^{i+1}}, \frac{1}{i} + \frac{\epsilon}{2 \times 2^{i+1}}\right]$ Hence

$$Vol\left(\left\{\frac{1}{i}\right\} \times I\right) \le Vol\left(\left[\frac{1}{i} - \frac{\epsilon}{2 \times 2^{i+1}}, \frac{1}{i} + \frac{\epsilon}{2 \times 2^{i+1}}\right] \times I\right) = \frac{\epsilon}{2 \times 2^i} \times 1 = \frac{\epsilon}{2 \times 2^i}$$

Now

$$\bigcup_{i=1}^{\infty} \left\{\frac{1}{i}\right\} \times I \subseteq \bigcup_{i=1}^{\infty} \left[\frac{1}{i} - \frac{\epsilon}{2 \times 2^{i+1}}, \frac{1}{i} + \frac{\epsilon}{2 \times 2^{i+1}}\right] \times I$$

Therefore

$$\begin{aligned} Vol\left(\bigcup_{i=1}^{\infty} \left\{\frac{1}{i}\right\} \times I\right) &\le Vol\left(\bigcup_{i=1}^{\infty} \left[\frac{1}{i} - \frac{\epsilon}{2 \times 2^{i+1}}, \frac{1}{i} + \frac{\epsilon}{2 \times 2^{i+1}}\right] \times I\right) \\ &= \sum_{i=1}^{\infty} Vol\left(\left[\frac{1}{i} - \frac{\epsilon}{2 \times 2^{i+1}}, \frac{1}{i} + \frac{\epsilon}{2 \times 2^{i+1}}\right] \times I\right) \\ &< \sum_{i=1}^{\infty} \frac{\epsilon}{2 \times 2^i} = \frac{\epsilon}{2} \end{aligned}$$

Now each $\{0\} \times I, \{1\} \times I, I \times \{0\}, I \times \{1\}$ can be covered with respectively $\left[-\frac{\epsilon}{8}, \frac{\epsilon}{8}\right] \times I, \left[1 - \frac{\epsilon}{8}, 1 + \frac{\epsilon}{8}\right] \times I, I \times \left[-\frac{\epsilon}{8}, \frac{\epsilon}{8}\right], I \times \left[1 - \frac{\epsilon}{8}, 1 + \frac{\epsilon}{8}\right]$. Therefore there total volume is less than $\frac{\epsilon}{2}$. Hence

$$Vol(\partial S) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence ∂S has measure zero. Therefore S is Jordan-Measurable.

We know that for any closed rectangle R if R^0 is the interior of R then $\int \chi_{R^0} = Vol(R)$ i.e. $Vol(R^0) = Vol(R)$. Now in S for each $\left(\frac{1}{i+1}, \frac{1}{i}\right) \times I$

$$\left(\frac{1}{i+1}, \frac{1}{i}\right) \times (0, 1) \subset \left(\frac{1}{i+1}, \frac{1}{i}\right) \times I \subset \left[\frac{1}{i+1}, \frac{1}{i}\right] \times I$$

Since

$$Vol\left(\left(\frac{1}{i+1}, \frac{1}{i}\right) \times (0, 1)\right) = Vol\left(\left[\frac{1}{i+1}, \frac{1}{i}\right] \times I\right) = \frac{1}{i(i+1)}$$

we have

$$Vol\left(\left(\frac{1}{i+1}, \frac{1}{i}\right) \times I\right) = \frac{1}{i(i+1)}$$

Hence

$$Vol(S) = \sum_{i=1}^{\infty} Vol\left(\left(\frac{1}{i+1}, \frac{1}{i}\right) \times I\right) = \sum_{i=1}^{\infty} \frac{1}{i(i+1)} = \sum_{i=1}^{\infty} \frac{1}{i} - \frac{1}{i+1} = 1$$

$\square$

Problem 6

Let $f : I \to \mathbb{R}$ be a continuous function, and let $\Gamma_f = \{(x, f(x))|x \in I\} \subset \mathbb{R}^2$ be its graph. Show that Γ_f has content zero. What if f is only integrable?

Solution: Since f is continuous on a closed interval f is uniformly continuous. Hence for every $\epsilon > 0$, $\exists\ \delta > 0$ such that for all $x, y \in I$, $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Hence there exists $n \in \mathbb{N}$ such that $\frac{1}{n} \leq \delta < \frac{1}{n-1}$. Now we partition I into $x_0, x_1, \ldots, x_{n-1}, x_n$ where $x_i = \frac{i}{n}$ for $i = 0, 1, 2, \ldots, n$. Now for any closed interval $x, y \in [x_i, x_{i+1}]$ we have $|f(x) - (y)| < \epsilon$ and therefore $M_{f_i}(x) - m_{f_i}(x) < \epsilon$. Now the set Γ_f can be covered with the rectangles $R_i = [m_{f_i}(x), M_{f_i}(x)] \times [x_i, x_i + 1]$ for $i = 0, 1, \ldots, n-1$. Now for each R_i

$$Vol(R_i) < \epsilon \times \frac{1}{n} = \frac{\epsilon}{n}$$

Hence

$$Vol(\Gamma_f) \leq \sum_{i=0}^{n-1} Vol(R_i) < \sum_{i=0}^{n-1} \frac{\epsilon}{n} = \epsilon$$

Hence Γ_f has content zero.

$\square$

Given that f is integrable. Therefore for all $\epsilon > 0$ there exists a partition P of I such that

$$U(P, I) - L(P, I) < \epsilon$$

Let the partition $P = \{x_0, x_1, \ldots, x_n\}$ where $x_0 = 0$ and $x_n = 1$. Now

$$U(P, I) - L(P, I) = \sum_{i=0}^{n-1} \left(M_{f_i}(x) - m_{f_i}(x)\right)(x_{i+1} - x_i)$$

If We choose the rectangles $\{R_i\}$ such that $R_i = [m_{f_i}(x), M_{f_i}(x)] \times [x_i, x_{i+1}]$ then Γ_f is covered by the union of all rectangles R_i. Hence $Vol(\Gamma_f) \leq \sum_{i=0}^{n-1} Vol(R_i) < \epsilon$. Hence Γ_f has content zero.

$\square$

Problem 7

Let $D : I \to \mathbb{R}$ be the function $D(t) = 1 - t$.

1. What is $\int_{\mathbb{R}} \tilde{D}$? (Notation of the notes.)
2. Compute
$$\int_{I \times I} D(x)D(xy)dxdy$$
Justify the steps, even when you have to just refer to a definition.

Solution:

(a) $\tilde{D} = D\chi_I$. Since both D and χ_I are integrable $\tilde{D}$ is also integrable. Then

$$\int_{\mathbb{R}} \tilde{D} = \int_I \tilde{D} = \int_I D\chi_I = \int_0^1 (1-t)dt = \left[t - \frac{t^2}{2}\right]_0^1 = 1 - \frac{1}{2} = \frac{1}{2}$$

$\square$