Fulton Chapter 5: Projective Plane Curves

Applications of Noether's Theorem

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Problem Set - 4
Topic: Algebraic Geometry

Problem 1 Applications of Noether's Theorem: 5.37

C , with the operation $\oplus$, forms an abelian group, with the point O being the identity.

Suppose O is a flex on C in the above proposition.

(a) Show that the flexes form a subgroup of C; as an abelian group, this subgroup is isomorphic to $\mathbb{Z}/(3) \times \mathbb{Z}/(3)$.

(b) Show that the flexes are exactly the elements of order three in the group. (i.e., exactly those elements P such that $P \oplus P \oplus P = O$).

(c) Show that a point P is of order two in the group if and only if the tangent to C at P passes through O.

(d) Let $C = Y^2Z - X(X-Z)(X-\lambda Z)$, $\lambda \neq 0, 1$, $O = [0:1:0]$. Find the points of order two.

(e) Show that the points of order two on a nonsingular cubic form a group isomorphic to $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$

(f) Let C be a nonsingular cubic, $P \in C$. How many lines through P are tangent to C at some point $Q \neq P$? (The answer depends on whether P is a flex.)

Solution:

(d) Let P be a point of order 2. Let $P \in U_2 \implies P = (a, 1, c)$. If $c = 0 \implies a = 0$ so, $c \neq 0$. Let L be the tangent of (a, c) at C_*. As P is order 2 $\implies L$ passes through $(0, 0)$. Let $X = mZ$ be the equation of L.

Now $M \neq \infty$ (then $Z = 0$ be the equation of L which is the tangent at $(0, 1, 0)$. So, $a = mc$. Now,

$$c = a(a-c)(a-\lambda c) \implies c = mc(mc-c)(mc-\lambda c) \implies c^2 m(m-1)(m-\lambda) = 1 \quad (1)$$

So,

$$c^2 = \frac{1}{(m(m-1)(m-\lambda)} \quad [\text{as } (m(m-1)(m-\lambda) \neq 0] \ldots (\text{by } 1)$$

So, c has two distinct values which implies L intersect C_* in three distinct points which is not possible. So, $P \notin U_2$. So, $P = (a, 0, c) = (m, 0, 1) \implies m = 0, 1, \lambda$ tangent at $(m, 0, 1)$

is $X = mZ \ \forall \ m \in \{0, 1, \lambda\}$ and the tangent passes through $(0, 1, 0)$. So there are 3 points of order 2.

$\square$

(e) So, any non-singular irreducible curve is projectively equivalent to the equation given in (c). So, the group has 4 point. So, it is isomorphic to $\mathbb{Z}/(2) \times \mathbb{Z}/(2)$

$\square$