Fulton Chapter 5: Projective Plane Curves

Linear System of Curves and Bézout's Theorem

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Problem Set - 3
Topic: Algebraic Geometry

Problem 1 Linear System of Curves: 5.19

$A = \{(a, b, 1) \mid \forall\ a, b \in \{0, 1, 2\}\}$. Show that there is infinitely many cubics passing through the 9 points in A

Solution: Let F be a cubic passing through the 9 points in A. So, F_* will pass through

$$B = \{(a, b) | \forall a, b \in \{0, 1, 2\}\} \quad [* \text{ taking w.r.t } Z]$$

Now $F_* = g(X) + Y(H(X, Y))$ as

$$F_*(a, 0) = 0\ \forall\ a \in \{0, 1, 2\} \implies g(x) = \lambda(X - 2)(X - 1)X$$
$$[\text{as } F \text{ is cubic } \implies g \text{ is a deg 3 polynomial}]$$

.

Similarly

$$F_* = g_1(Y) + XH(X, Y) \implies g_1(Y) = \mu(Y - 2)(Y - 1)Y$$

So,

$$F_* = \lambda(X - 2)(X - 1)X + \mu(Y - 2)(Y - 1)Y + XY(aX + bY + c)$$

But

$$F_*(1, 2) = 0 = F_*(2, 1) = F_*(1, 1) \implies a + 2b + c = 0 = 2a + b + c = a + b + c \implies a = b = c = 0$$

So, any polynomial passing through B must be of the form

$$\lambda(X - 2)(X - 1)X + \mu(Y - 2)(Y - 1)Y$$

where $\lambda, \mu \in k$. So, any F passing through the points in A will be of the form

$$\lambda(X - 2Z)(X - Z)X + \mu(Y - 2Z)(Y - Z)Y$$

so there are infinitely curves passing through 9 points in A

□

Problem 2 Bézout's Theorem: 5.23

F is a projective plane curve of degree n ,it contains no lines and char(k)=0 then

(a) if $P \in H \cap F$ then either it is a multiple point or a flex.

(b) $I(P, H \cap F) = 1 \iff P$ is an ordinary flex.

Solution:

Claim 1 : T be a projective change of co-ordinates then hessian of

$$F^T = det(A)^2 H^T$$

Proof: Let $T = (T_1, T_2, T_3); T_i = a_i X + b_i Y + c_i Z$. So the matrix of T is $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

$$\begin{aligned}(F(T_1,T_2,T_3))_x &= a_1 F_{T_1}(T_1,T_2,T_3) + a_2 F_{T_2}(T_1,T_2,T_3) + a_3 F_{T_3}(T_1,T_2,T_3) \\ &= \begin{bmatrix} F_{T_1} & F_{T_2} & F_{T_3} \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \quad \text{[by chain rule]}\end{aligned}$$

$$(F(T_1,T_2,T_3))_{xx} = \sum_{i=1}^{3} \sum_{j=1}^{3} a_i (a_j F_{T_i T_j})$$

So, $F_{xx}(T_1, T_2, T_3)$ is the (1,1) position of $\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} F_{T_1T_1} & F_{T_1T_2} & F_{T_1T_3} \\ F_{T_2T_1} & F_{T_2T_2} & F_{T_2T_3} \\ F_{T_3T_1} & F_{T_3T_2} & F_{T_3T_3} \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$. So,

$$\begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \begin{bmatrix} F_{T_1T_1} & F_{T_1T_2} & F_{T_1T_3} \\ F_{T_2T_1} & F_{T_2T_2} & F_{T_2T_3} \\ F_{T_3T_1} & F_{T_3T_2} & F_{T_3T_3} \end{bmatrix} \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} (F^T)_{xx} & (F^T)_{xy} & (F^T)_{xz} \\ (F^T)_{yx} & (F^T)_{yy} & (F^T)_{yz} \\ (F^T)_{zx} & (F^T)_{zy} & (F^T)_{zz} \end{bmatrix}$$

So, $R =$ hessian of $F^T = det(A)^2 H^T$. So,

$$P \in H \cap F \iff T(P) \in R \cap F^T \iff T(P) \in H^T \cap F^T$$

and

$$I(P, F \cap H) = 1 \iff I(T(P), F^T \cap H^T) = 1 \iff I(T(P), F^T \cap R) = 1$$

so we can assume $P = (0, 0, 1)$

$\square$

Claim 2: $(n-1)F_j = \sum_i X_i F_{ij}$

Proof: Let

$$F = \sum_{k=0}^{r} X_j^k F_k \quad [\text{where } \deg F_k = n-k] \implies F_j = \sum_{k=1}^{r} k X_j^{k-1} F_k$$
$$\implies deg(F_j) = n-1 \quad [\text{as char } K = 0]$$

Applying Euler's theorem on F_j we get the relation

$$(n-1)F_j = \sum_i X_i F_{ji} = \sum_i X_i F_{ij} \quad [\text{as } F_{ij} = F_{ji}]$$

□

Claim 3: $I(P, f \cap h) = I(P, f \cap g)$ where $g = f_y^2 f_{xx} + f_x^2 f_{yy} - 2f_x f_y f_{xy}$

Proof: Let

$$F(X,Y,Z) = \sum_{k=0}^{r} X^k F_k(Y,Z) \implies F_X(X,Y,1) = \sum_{k=1}^{r} k X^{k-1} F_k(Y,1) = f_x(X,y,1)$$

so, $F_{XY}(X,Y,1) = f_{XY}(X,Y); F_{XX}(X,Y,1) = f_{XX}(X,Y,1)$

$$H(X,Y,Z) = \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ F_{zx} & F_{zy} & F_{zz} \end{vmatrix} = \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ xF_{xx} + yF_{yx} + zF_{zx} & xF_{xy} + yF_{yy} + zF_{zy} & xF_{xz} + yF_{zy} + zF_{zz} \end{vmatrix}$$

$$= \begin{vmatrix} F_{xx} & F_{xy} & F_{xz} \\ F_{yx} & F_{yy} & F_{yz} \\ (n-1)F_x & (n-1)F_y & (n-1)F_z \end{vmatrix} = \begin{vmatrix} F_{xx} & F_{xy} & zF_{xz} + yF_{xy} + xF_{xx} \\ F_{yx} & F_{yy} & zF_{yz} + yF_{yy} + xF_{yx} \\ (n-1)F_x & (n-1)F_y & (n-1)(zF_z + yF_y + xF_x) \end{vmatrix}$$

$$= \begin{vmatrix} F_{xx} & F_{xy} & (n-1)F_x \\ F_{yx} & F_{yy} & (n-1)F_y \\ (n-1)F_x & (n-1)F_y & (n-1)nF \end{vmatrix}$$

So,

$$h(x,y) = H(x,y,1) = \begin{vmatrix} f_{xx} & f_{xy} & (n-1)f_x \\ f_{yx} & f_{yy} & (n-1)f_y \\ (n-1)f_x & (n-1)f_y & (n-1)nf \end{vmatrix}$$

$$= (n-1)nf[f_{yy}f_{xx} - f_{yx}f_{xy}] - (n-1)^2 f_y[f_{xx}f_y - f_x f_{xy}] + (n-1)^2 f_x[f_y f_{yx} - f_{yy}f_x]$$
$$= (n-1)nf[f_{yy}f_{xx} - f_{yx}f_{xy}] - (n-1)^2[f_y^2 f_{xx} + f_x^2 f_{yy} - 2f_x f_y f_{xy}] \quad [\text{as } f_{xy} = f_{yx}]$$

So, $I(P, f \cap g) = I(P, f \cap h)$

□

Claim 4: If P is a multiple point then $I(P, f \cap g) \geq 2$

Proof: $P \in F \cap H$ P is a multiple point of $F \implies I(P, F \cap H) = I(P, f \cap h) \geq 2$ [as $m_P(f) \geq 2$]. So, $I(P, f \cap g) \geq 2$

$\square$

Claim 5: $P = (0, 0)$ be a simple point of $f = y + ax^2 + bxy + cy^2 + dx^3+$ higher terms $y = 0$ be the tangent at P. P is a flex iff $a = 0$; P is an ordinary flex iff $a = 0; d \neq 0$

Proof: As P is a simple point $O_P(f)$ is a d.v.r and the maximal ideal is generated by x. P is a flex iff $ord(y) \geq 3$. So,

$$y(1 + bx + cy + higherterms) = x^2(a + dx + higherterms)$$

so $ord(y) \geq 3 \iff a = 0$

P is an ordinary flex iff $ord(y) = 3$. So,

$$y(1 + bx + cy + higherterms) = x^2(a + dx + higherterms)$$

so $ord(y) = 3 \iff a = 0; d \neq 0$. Now

$$\begin{aligned} f_x &= 2ax + by + 3dx^2 + \text{ other terms} \\ f_y &= 1 + bx + 2cy + \text{ other terms} \\ f_{xx} &= 2a + 6dx + \text{ other terms} \\ f_{yy} &= 2c + \text{ other terms} \\ f_{xy} &= b + \text{ other terms} \end{aligned}$$

Now

$$\begin{aligned} f_x f_y f_{xy} &= (2ax + by)b + \text{ higher terms} \\ f_y^2 f_{xx} &= 2a + 6dx + \text{ higher terms} \\ f_x^2 f_{yy} & \quad \text{has no 1 degree terms} \\ g &= 2a + (6d - 4ab)x - 2b^2 y + \text{ higher terms} \end{aligned}$$

Now

$$P = (0, 0) \in f \cap g \iff a = 0 \iff P \text{ is a flex}$$

$I(P, g \cap f) = 1 \iff$ they do not share tangent at $P \iff d \neq 0 \iff P$ is an ordinary flex.

Corollary: A non singular cubic has 9 flexes all are ordinary.

Proof: Let F be a cubic non singular curve H be its hessian. So H is also cubic. $\sum_P (I(P, F \cap H) = 9$

Now let for some $P, I(p.F \cap H) \neq 0$. As P is non singular so P is simple so by the previous theorem P is a flex. It must be ordinary as F is cubic. So, $I(P, H \cap F) = 1$. So, there are 9 flexes of F

$\square$

Problem 3 Bézout's Theorem: 5.24

(a) Let (0,1.0) be a flex and $Z = 0$ be the tangent at that point. char k=0. Show that $F = ZY^2 + bYZ^2 + cYXZ+$ terms in X, Z find a projective change of co-ordinates s.t F reduced to a form $Y^2Z =$ cubic in X, Z

(b) Show that any irreducible cubic is projectively equivalent to one of the following $G_1 = Y^2Z - X^3$, $G_2 = Y^2Z - X^2(X + 1)$, $G_3 = Y^2Z = X(X - Z)(X - \lambda Z)$ where $\lambda \neq 0, 1$

Solution:

(a) $P = (0, 0)$. So, $O_P(F_*)$ is a d.v.r. and generated by X. So,

$$Z(1 + bZ + cX + dXZ + eX^2 + fZ^2) = X^3k \quad \text{[as } F \text{ is a cubic]}$$

so,

$$F_*(X, Z) = Z + bZ^2 + cXZ + dXZ^2 + eX^2Z + fZ^3 - kX^3$$
$$\implies F = ZY^2 + bZ^2Y + cXYZ + dXZ^2 + X^2Z + fZ^3 - kX^3$$

Now, using $Y \to (Y - b/2Z - c/2X)$ and keeping others same. $ZY^2 + bZ^2Y + cXYZ$ becomes

$$ZY^2 - \frac{b^2Z^3}{4} - \frac{c^2X^2Z}{4} - \frac{bcXZ^2}{4}$$

so, we get F to the form ZY^2 =cubic in X, Z

□

(b) Assume char F=0. Let F be an irreducible curve. So, it has at most finitely many singular point.

Let $Q_1 = (a, b, c) \in H \cap F$ be a simple point of F and L be its tangent [as $H \cap F$ is infinite]. Let $Q_2 = (d, e, f)$ be another point on L. So, $\exists$ a projective change of co-ordinates T s.t. $T(Q_1) = (0, 1, 0), T(Q_2) = (1, 0, 0)$. So, (0,1,0) a simple point of F^T and its tangent is $Z = 0$. By problem 5.23 it is a flex. So, by problem (a) F^T is projectively equivalent to Y^2Z-cubic in X, Z. Let

$$G = Y^2Z - (X - \lambda_1 Z)(X - \lambda_2 Z)(X - \lambda_3 Z)$$

Case 1: If all $\lambda_i's$ are equal then using $X \to (X + \lambda Z)$ and keeping others unchanged we get $X^3 = Y^2Z$

Case 2. If $\lambda = \lambda_1 = \lambda_2 \neq \lambda_3$. Then using $X \to (X + \lambda Z)$ and keeping others unchanged we get

$$Y^2Z = X^2(X + (\lambda - \lambda_3)Z)$$

using $Z \to \frac{Z}{\lambda - \lambda_3}$, $Y \to \sqrt{\lambda - \lambda_3}Y$ keeping X unchanged we get

$$Y^2Z = X^2(X + 1) \quad [\text{as } \lambda - \lambda_3 \neq 0]$$

Case 3. All λ_i's are distinct. Then using $X \to (X + \lambda_1 Z)$ and keeping others unchanged we get

$$Y^2 Z = X(X - (\lambda_2 - \lambda_1)Z)(X - (\lambda_3 - \lambda_1)Z)$$

again using

$$Z \to \frac{Z}{\lambda_2 - \lambda_1},\ Y \to \sqrt{\lambda_2 - \lambda_1}Y$$

keeping X unchanged we get

$$Y^2 Z = X(X - Z)(X - \frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1}Z)$$

and also

$$\frac{\lambda_3 - \lambda_1}{\lambda_2 - \lambda_1} \neq 0, 1$$

so, it is projectively equivalent to $Y^2 Z = X(X - Z)(X - \lambda Z)$ where $\lambda \neq 0, 1$.

So, F is projectively equivalent to $G_1 = Y^2 Z - X^3$ or $G_2 = Y^2 Z - X^2(X + 1)$ or $G_3 = Y^2 Z = X(X - Z)(X - \lambda Z)$ where $\lambda \neq 0, 1$

Remark:

Claim 1: G_1 has a cusp at $(0, 0, 1)$. $G_{1*}(X, Z) = Y^2 - X^3$ which has a cusp at $(0, 0)$ and the tangent is $Y = 0$

Claim 2: G_2 has a node at $(0, 0, 1)$. $G_{2*}(X, Z) = Y^2 - X^2 - X^3$ which has a node at $(0, 0)$ and the tangents are $X + Y = 0, X - Y = 0$

Claim 3: If F is an irreducible conic then $\forall P \in F$ $m_P(F) \leq 2$. If $\exists P \in V(F)$ s.t. $m_P(F) > 2$ $\exists$ a projective change of co-ordinates T s.t.$T(P) = (0, 1, 0)$ $G = F^T$ $G_* = L_1 L_2 L_3 \implies G$ is reducible which is not possible.

So, if F is an irreducible conic then either is non singular or it has cusp or it has a node. Now both claim 1 and problem 5.10 implies that if F has a cusp then it is projectively equivalent to G_1. Both claim 2 and problem 5.11 implies that if F has a node then it is projectively equivalent to G_2. So, if F is non singular then it is projectively equivalent to G_3.

$\square$