# Classical Mechanics 1, Autumn 2021 CMI
# Problem set 8
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1. Given that $\ddot{\theta} = -\omega^2 \sin\theta$. Let $f(\theta) = -\omega^2 \sin\theta$ Now

$$f(\theta) \approx f(\theta_0) + \frac{d}{d\theta} f(\theta)\bigg|_{\theta=\theta_0} (\theta - \theta_0) + \frac{1}{2}\ \frac{d^2}{d\theta^2} f(\theta)\bigg|_{\theta=\theta_0} (\theta - \theta_0)^2$$

Given that $\theta_0 = \pi$. Hence

$$\begin{aligned} f(\theta) &\approx f(\pi) + \frac{d}{d\theta} f(\theta)\bigg|_{\theta=\pi} (\theta - \pi) + \frac{1}{2}\ \frac{d^2}{d\theta^2} f(\theta)\bigg|_{\theta=\pi} (\theta - \pi)^2 \\ &= -\omega^2 \sin\pi + (-\omega^2 \cos\pi)(\theta - \pi) + \frac{1}{2}(\omega^2 \sin\pi)(\theta - \pi)^2 \\ &= -\omega^2 \cos(\pi)(\theta - \pi) \\ &= \omega^2(\theta - \pi) \end{aligned}$$

(a) Putting $\theta = \pi + \phi$ we get

$$\dot{\theta} = \dot{\phi} \implies \ddot{\theta} = \ddot{\phi}$$

Hence

$$f(\theta) = f(\pi + \phi) = \omega^2(\pi + \phi - \pi) = \omega^2\phi = \frac{g}{l}\phi$$

Therefore we get

$$\ddot{\phi} = \frac{g}{l}\phi$$

(b) Suppose the solution of the equation $\ddot{\phi} = \omega^2\phi$ will be proportional to $e^{\lambda t}$ for some real number $\lambda \in \mathbb{R}$. Hence $\phi = e^{\lambda t}$. Therefore

$$\frac{d^2}{dt^2}(e^{\lambda t} - \omega^2 e^{\lambda t}) \implies \lambda^2 e^{\lambda t} - \omega^2 e^{\lambda t} = 0 \implies (\omega + \lambda)(\omega - \lambda)e^{\lambda t} = 0$$

As $e^{\lambda t} \neq 0$ the solutions of $\lambda$ are $\omega$ and $-\omega$.

Hence the linearly solutions for $\phi$ are $c_1 e^{\omega t}$ and $c_2 e^{-\omega t}$ where $c_1, c_2$ are two real numbers. Hence the general solution of $\phi$ is

$$\phi(t) = c_1 e^{\sqrt{\frac{g}{l}}t} + c_2 e^{-\sqrt{\frac{g}{l}}t}$$

Now $\phi(0) = c_1 + c_2$ and $\dot{\phi}(0) = \omega(c_1 - c_2)$ hence $c_1 = \frac{1}{2}(\phi(0) + \frac{\dot{\phi}(0)}{\omega})$ and $c_2 = \frac{1}{2}(\phi(0) - \frac{\dot{\phi}(0)}{\omega})$.

Therefore the general solution of the equation is

$$\phi(t) = c_1 e^{\sqrt{\frac{g}{l}}t} + c_2 e^{-\sqrt{\frac{g}{l}}t}$$

where $c_1 = \frac{1}{2}(\phi(0) + \frac{\dot{\phi}(0)}{\omega})$ and $c_2 = \frac{1}{2}(\phi(0) - \frac{\dot{\phi}(0)}{\omega})$.

(c) $\phi$ is positive when it rotates counter-clock wise and negative when it rotates clockwise. Therefore

- When $c_1 = c_2 = 0$ we have $\phi(0) = \dot{\phi}(0) = 0$. Hence $\theta(t) = \pi$ for all $t$. Hence it is a static solution.
- When $c_1 = 0, c_2 > 0$ we have $\phi(0) > 0$ and $\dot{\phi}(0) < 0$. Hence initial angle is positive and it is decreasing i.e. the bob is rotating clockwise and approaches $\phi = 0$ but never reaches it in finite time.
- When $c_1 = 0, c_2 < 0$ we have $\phi(0) < 0$ and $\dot{\phi}(0) > 0$. Hence initial angle is negative and it is increasing i.e. the bob is rotating counter-clockwise and approaches $\phi = 0$ but never reaches it in finite time.

- When $c_1 > 0, c_2 = 0$ we have $\phi(0) > 0$ and $\dot{\phi}(0) > 0$. hence initial angle is positive and it is increasing i.e. the bob is rotating counter-clockwise.
- When $c_1 > 0, c_2 > 0$ and $c_1 = c_2$ we have $\phi(0) > 0$ and $\dot{\phi}(0) = 0$. Hence initial angle is positive and the bob will start rotating counter-clockwise
- When $c_1 > 0, c_2 > 0$ and $c_1 > c_2$ we have $\phi(0) > 0$ and $\dot{\phi}(0) > 0$. Hence initial angle is positive and it is increasing i.e. the bob is rotating counter-clockwise.
- When $c_1 > 0, c_2 > 0$ and $c_1 < c_2$ we have $\phi(0) > 0$ and $\dot{\phi}(0) < 0$. Hence initial angle is positive and it is decreasing i.e. the bob is rotating clockwise and eventually angle becomes negative.
- When $c_1 > 0, c_2 < 0$ and $|c_1| = |c_2|$ we have $\phi(0) = 0$ and $\dot{\phi}(0) > 0$. Hence initial angle is zero and it keeps increasing i.e. the bob is rotating counter-clockwise.
- When $c_1 > 0, c_2 < 0$ and $|c_1| > |c_2|$ we have $\phi(0) > 0$ and $\dot{\phi}(0) > 0$. Hence initial angle is positive and it keeps increasing i.e. the bob is rotating counter-clockwise.
- When $c_1 > 0, c_2 < 0$ and $|c_1| < |c_2|$ we have $\phi(0) < 0$ and $\dot{\phi}(0) > 0$. Hence initial angle is negative and it keeps increasing i.e. the bob is rotating counter-clockwise and eventually angle becomes positive.
- When $c_1 < 0, c_2 = 0$ we have $\phi(0) < 0$ and $\dot{\phi}(0) < 0$. hence initial angle is negative and it is decreasing i.e. the bob is rotating clockwise.
- When $c_1 < 0, c_2 > 0$ and $|c_1| = |c_2|$ we have $\phi(0) = 0$ and $\dot{\phi}(0) < 0$. Hence initial angle is zero and it is decreasing i.e. the bob is rotating clockwise.
- When $c_1 < 0, c_2 > 0$ and $|c_1| > |c_2|$ we have $\phi(0) < 0$ and $\dot{\phi}(0) < 0$. Hence initial angle is negative and it is decreasing i.e. the bob is rotating clockwise.
- When $c_1 < 0, c_2 > 0$ and $|c_1| < |c_2|$ we have $\phi(0) > 0$ and $\dot{\phi}(0) < 0$. Hence initial angle is positive and it is decreasing i.e. the bob is rotating clockwise and eventually angle becomes negative.
- When $c_1 < 0, c_2 < 0$ and $c_1 = c_2$ we have $\phi(0) < 0$ and $\dot{\phi}(0) = 0$. Hence initial angle is negative and the bob will start rotating clockwise
- When $c_1 < 0, c_2 < 0$ and $c_1 > c_2$ we have $\phi(0) < 0$ and $\dot{\phi}(0) > 0$. Hence initial angle is negative and it keeps increasing i.e. the bob is rotating counter-clockwise and eventually angle becomes positive.
- When $c_1 < 0, c_2 < 0$ and $c_1 < c_2$ we have $\phi(0) < 0$ and $\dot{\phi}(0) < 0$. Hence initial angle is negative and it keeps decreasing i.e. the bob is rotating clockwise.

2. (a) Gravitational Forces acting between

   - point masses $m_e, m_m$ is $\dfrac{G\,m_e\,m_m}{|\boldsymbol{r}_e - \boldsymbol{r}_m|^2}$
   - point masses $m_e, m_s$ is $\dfrac{G\,m_e\,m_s}{|\boldsymbol{r}_e - \boldsymbol{r}_s|^2}$
   - point masses $m_m, m_s$ is $\dfrac{G\,m_m\,m_s}{|\boldsymbol{r}_m - \boldsymbol{r}_s|^2}$

   Hence for the particle with mass $m_e$ Newton's 2nd law equations of motion is

$$m_e\ddot{\boldsymbol{r}}_e = -\frac{G\,m_e\,m_m}{|\boldsymbol{r}_e - \boldsymbol{r}_m|^3}(\boldsymbol{r}_e - \boldsymbol{r}_m) - \frac{G\,m_e\,m_s}{|\boldsymbol{r}_e - \boldsymbol{r}_s|^3}(\boldsymbol{r}_e - \boldsymbol{r}_s)$$
$$\implies \ddot{\boldsymbol{r}}_e = \frac{G\,m_m}{|\boldsymbol{r}_e - \boldsymbol{r}_m|^3}(\boldsymbol{r}_m - \boldsymbol{r}_e) + \frac{G\,m_s}{|\boldsymbol{r}_e - \boldsymbol{r}_s|^3}(\boldsymbol{r}_s - \boldsymbol{r}_e)$$

For the particle with mass $m_m$ Newton's 2nd law equations of motion is

$$\begin{aligned} m_m \ddot{\boldsymbol{r}}_m &= -\frac{G\, m_m\, m_e}{|\boldsymbol{r}_m - \boldsymbol{r}_e|^2}(\boldsymbol{r}_m - \boldsymbol{r}_e) - \frac{G\, m_m\, m_s}{|\boldsymbol{r}_m - \boldsymbol{r}_s|^2}(\boldsymbol{r}_m - \boldsymbol{r}_s) \\ \Longrightarrow \ddot{\boldsymbol{r}}_m &= \frac{G\, m_e}{|\boldsymbol{r}_m - \boldsymbol{r}_e|^2}(\boldsymbol{r}_e - \boldsymbol{r}_m) + \frac{G\, m_s}{|\boldsymbol{r}_m - \boldsymbol{r}_s|^2}(\boldsymbol{r}_s - \boldsymbol{r}_m) \end{aligned}$$

For the particle with mass $m_s$ Newton's 2nd law equations of motion is

$$\begin{aligned} m_s \ddot{\boldsymbol{r}}_s &= -\frac{G\, m_s\, m_e}{|\boldsymbol{r}_s - \boldsymbol{r}_e|^2}(\boldsymbol{r}_s - \boldsymbol{r}_e) - \frac{G\, m_s\, m_m}{|\boldsymbol{r}_s - \boldsymbol{r}_m|^2}(\boldsymbol{r}_s - \boldsymbol{r}_m) \\ \Longrightarrow \ddot{\boldsymbol{r}}_s &= \frac{G\, m_e}{|\boldsymbol{r}_s - \boldsymbol{r}_e|^2}(\boldsymbol{r}_e - \boldsymbol{r}_s) + \frac{G\, m_m}{|\boldsymbol{r}_s - \boldsymbol{r}_m|^2}(\boldsymbol{r}_m - \boldsymbol{r}_s) \end{aligned}$$

(b) Now

$$\begin{aligned} m_e \ddot{\boldsymbol{r}}_e + m_m \ddot{\boldsymbol{r}}_m + m_s \ddot{\boldsymbol{r}}_s = & \left[ -\frac{G\, m_e\, m_m}{|\boldsymbol{r}_e - \boldsymbol{r}_m|^3}(\boldsymbol{r}_e - \boldsymbol{r}_m) - \frac{G\, m_e\, m_s}{|\boldsymbol{r}_e - \boldsymbol{r}_s|^3}(\boldsymbol{r}_e - \boldsymbol{r}_s) \right] \\ & + \left[ -\frac{G\, m_m\, m_e}{|\boldsymbol{r}_m - \boldsymbol{r}_e|^2}(\boldsymbol{r}_m - \boldsymbol{r}_e) - \frac{G\, m_m\, m_s}{|\boldsymbol{r}_m - \boldsymbol{r}_s|^2}(\boldsymbol{r}_m - \boldsymbol{r}_s) \right] \\ & + \left[ -\frac{G\, m_s\, m_e}{|\boldsymbol{r}_s - \boldsymbol{r}_e|^2}(\boldsymbol{r}_s - \boldsymbol{r}_e) - \frac{G\, m_s\, m_m}{|\boldsymbol{r}_s - \boldsymbol{r}_m|^2}(\boldsymbol{r}_s - \boldsymbol{r}_m) \right] \\ \Longrightarrow & \frac{d}{dt}(m_e \dot{\boldsymbol{r}}_e + m_m \dot{\boldsymbol{r}}_m + m_s \dot{\boldsymbol{r}}_s) = 0 \\ \Longrightarrow & \; m_e \dot{\boldsymbol{r}}_e + m_m \dot{\boldsymbol{r}}_m + m_s \dot{\boldsymbol{r}}_s = \text{Constant} \end{aligned}$$

Therefore total linear momentum of the system is conserved.

3. (a) Magnitude of Relativistic effect on the tennis ball served at the speed $v = 100$ km/h $= \dfrac{100 \times 10^3}{3600}$m/s $= \dfrac{1000}{36}$m/s is

$$\left( \frac{\frac{1000}{36}}{3 \times 10^8} \right)^2 = \left( \frac{1}{108 \times 10^5} \right)^2 \approx 8.57 \times 10^{-15}$$

(b) Magnitude of Relativistic effect of Earth's motion at a speed of 30 km/s=$3 \times 10^4$ m/s around the sun is

$$\left( \frac{3 \times 10^4}{3 \times 10^8} \right)^2 = (10^{-4})^2 = 10^{-8}$$