PROBLEMS FROM 6.1,6.2
ALGEBRAIC CURVES

problem 6.8
Let U be an open subset of a variety $V, z \in k(V)$ suppose $z \in O_P(V) \forall P \in U$. Show that $U_z = \{P \in U | z(P) \neq 0\}$ is open , and that the function from $U \longrightarrow k = A^1$ defined by $P \longrightarrow z(P)$ is continuous.
solution:
let $f \in \tau(V); S_f = \{p \in V | f(P) = 0\} = V \cap V(f)$ which is a closed set .
now $P \in U_z \implies \exists f, g \in \tau V s.t. z = f/g, g(P) \neq 0$
$P \in V \cap V \setminus S_f \cap V \setminus S_g \subset U_z$[as $\forall P' \in V \cap V \setminus S_f \cap V \setminus S_g$ $z = f/g$ and $z(P) \neq 0$]
so there is an open nbd of P in $U_z \implies U_z$ is open.
let $r \in k$
$U'_z = \{P \in U | z(P) \neq r\}$
similarly $\forall P \in U'_z P \in V \cap V\ S_{(f-rg)} \cap V\ S_g \implies U'_z$ is open $\implies z^{-1}(P)$ is closed.
now in k any closed set S is finite .
so, $z^{-1}(S) = \cup_{p \in S} z^{-1}(p) \implies z^{-1}(S)$ is closed
so, $P \longleftarrow z(P)$ is a continuous function.
problem 6.9
let $X = A^2 - \{(0,0)\}$, an open subvariety of A^2. Show that $\tau(X) = \tau(A^2) = k[X,Y]$
solution:
clearly, $\tau(A^2) \subseteq \tau(X)$
let $z \in \tau(X) = \cap_{P \in X} O_P(A^2), P \neq (0,0)$
so, $z(P)$ defined at $0 \implies \exists f, g \in \tau(A^2) = k[x,y] s.t. z = f/g, g(P) \neq 0$
let $f/g = f_1/g_1$(where g.c.d. of $f_1, g_1 = 1$
so, $g_1(a,b) \neq 0 \forall (a,b) \neq (0,0)$
so, if deg$g_1$in k[X][Y] is non zero then it has infinitely many roots which is not possible.
so, g_1 is a constant polynomial.
so, $z \in k[x,y]$
remark:
$A^2 - \{(0,0)\}$ is not an affine variety.
lets assume $X = A^2 - \{(0,0)\}$ be an affine variety and $I(X) = I$
so, $\tau(X) = k[x,y]/I = k[x,y] \implies I = (0) \implies X = V(I) = A^2$ which is not true.
so, X is not an affine variety.
problem 6.12
let X be a variety, $z \in k(X)$. show that the pole set of z is closed. If

$z \in O_P(X)$, there is a neighborhood U_P of P s.t. $z \in \tau(U_P)$; so, $O_P(X)$ is the union of all $\tau(U_P)$
solution:
$J = \{G | zG \in \tau(X)\}$
clearly pole set of $z = X \cap V(J)$
so, pole set is a closed set.
now $z \in O_P(X) \implies P \in X \smallsetminus V(J) \implies P \in U_P \subset X \smallsetminus V(J)$
so, $z \in O_S(X) \forall S \in U_P \implies z \in \tau(U_P)$
$P \in U_P \implies \tau(U_P) \subseteq O_P(X)$
so, $O_P(X) = \cup_P \tau(U_P)$