Classical Mechanics 1, Autumn 2021 CMI
Problem set 3
- Govind S. Krishnaswami

Soham Chatterjee

Roll: BMC202175

1. Given that

$$\phi = \frac{1}{\sqrt{x^2+y^2+z^2}}$$

Hence

$$\begin{aligned}
\nabla\phi &= \nabla\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) \\
&= \left(\frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}\right)\left(\frac{1}{\sqrt{x^2+y^2+z^2}}\right) \\
&= -\frac{1}{2}\frac{2x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{i} - \frac{1}{2}\frac{2y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{j} - \frac{1}{2}\frac{2z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{k} \\
&= -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{i} - \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{j} - \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{k}
\end{aligned}$$

Therefore in Cartesian Coordinates the gradient of ϕ is

$$\nabla\phi = -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{i} - \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{j} - \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{k}$$

In Spherical Polar Coordinates

$$r = \sqrt{x^2+y^2+z^2} \text{ and } \boldsymbol{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

Substituting these in the expression of $\nabla\phi$we get

$$\begin{aligned}
\nabla\phi &= -\frac{x}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{i} - \frac{y}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{j} - \frac{z}{(x^2+y^2+z^2)^{\frac{3}{2}}}\hat{k} \\
&= -\frac{\boldsymbol{r}}{(r^2)^{\frac{3}{2}}} = -\frac{\boldsymbol{r}}{r^3} = -\frac{1}{r^2}\hat{r}
\end{aligned}$$

Therefore in Spherical Polar Coordinates gradient of ϕ is

$$\nabla\phi = -\frac{1}{r^2}\hat{r}$$

2. In Spherical Polar Coordinates we have

$$\hat{r} = \sin\theta\cos\phi\hat{i} + \sin\theta\sin\phi\hat{j} + \cos\theta\hat{k}$$

$$\hat{\theta} = \cos\theta\cos\phi\hat{i} + \cos\theta\sin\phi\hat{j} - \sin\theta\hat{k}$$

$$\hat{\phi} = -\sin\phi\hat{i} + \cos\phi\hat{j}$$

Hence

$$\dot{\hat{r}} = (\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)\hat{i} + (\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)\hat{j} - \dot{\theta}\sin\theta\hat{k}$$

Now

$$\boldsymbol{v} = \dot{\boldsymbol{r}} = \dot{r}\hat{r} + r\dot{\hat{r}}$$

Hence

$$\begin{aligned}
\boldsymbol{r} \times \boldsymbol{v} &= r\hat{r} \times (\dot{r}\hat{r} + r\dot{\hat{r}}) \\
&= r\hat{r} \times (r\dot{\hat{r}}) \\
&= r^2(\hat{r} \times \dot{\hat{r}}) \\
&= r^2(\sin\theta\cos\phi\hat{\imath} + \sin\theta\sin\phi\hat{\jmath} + \cos\theta\hat{k}) \times ((\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)\hat{\imath} \\
&\qquad\qquad + (\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)\hat{\jmath} - \dot{\theta}\sin\theta\hat{k}) \\
&= r^2\Big[[\sin\theta\sin\phi\hat{(} - \dot{\theta}\sin\theta) - \cos\theta(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi)]\hat{\imath} + \\
&\quad [\cos\theta((\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi) - \sin\theta\cos\phi(-\dot{\theta}\sin\theta)]\hat{\jmath} + \\
&\quad [\sin\theta\cos\phi(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) - \sin\theta\sin\phi((\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)]\hat{k}\Big]
\end{aligned}$$

Now Angular Momentum (L) is

$$\boldsymbol{L} = \boldsymbol{r} \times \boldsymbol{p}$$

where $\boldsymbol{p}$ is the linear momentum vector. Assuming the mass of the particle (m) is constant, $\boldsymbol{p} = m\boldsymbol{v}$
Hence

$$\boldsymbol{L} = \boldsymbol{r} \times (m\boldsymbol{v}) = m(\boldsymbol{r} \times \boldsymbol{v})$$

Since we need the z component of angular momentum

$$\begin{aligned}
\boldsymbol{L}_z &= m(\boldsymbol{r} \times \boldsymbol{v})_z \\
&= mr^2[\sin\theta\cos\phi(\dot{\theta}\cos\theta\sin\phi + \dot{\phi}\sin\theta\cos\phi) - \sin\theta\sin\phi((\dot{\theta}\cos\theta\cos\phi - \dot{\phi}\sin\theta\sin\phi)]\hat{k} \\
&= [\dot{\theta}(\sin\theta\sin\phi\cos\theta\cos\phi - \sin\theta\sin\phi\cos\theta\cos\phi) + \dot{\phi}(\sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi)]\hat{k} \\
&= mr^2\dot{\phi}\sin^2\theta\hat{k}
\end{aligned}$$

Therefore the z component of the angular momentum of the particle is

$$L_z = mr^2\dot{\phi}\sin^2\theta \text{ [Proved]}$$

3. To specify a line segment of fixed length l, first we need to specify one end point of the line segment. For that we need 3 parameters to specify the end point (r, θ, ϕ). Now the other endpoint lies in the circumference of the sphere of radius l whose centre is at the first end point. Now in this spherical coordinate system we only need θ and ϕ to specify the other end point.

   Therefore in total we need 5 parameters to specify a line segment. Hence a line segment has 5 Degrees of Freedom