Classical Mechanics 1, Autumn 2021 CMI
Problem set 1
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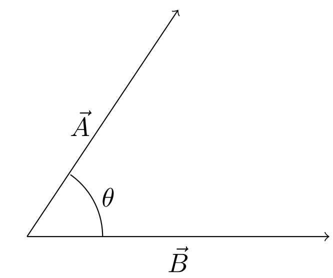
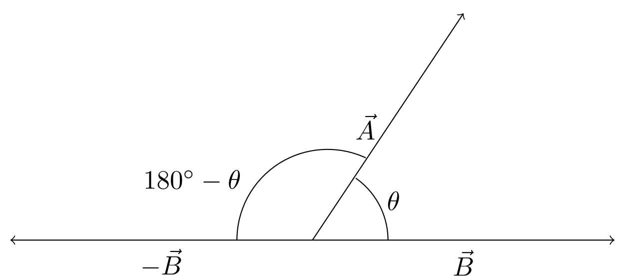
1. **Geometric Interpretation of Non-collinear Vector Subtraction:-**

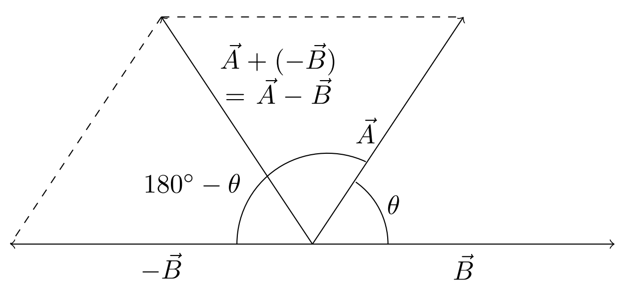
   Let $\vec{A}$ and $\vec{B}$ are two non-collinear vectors emanating from origin in $2d$ plane which makes an angle θ with each other.



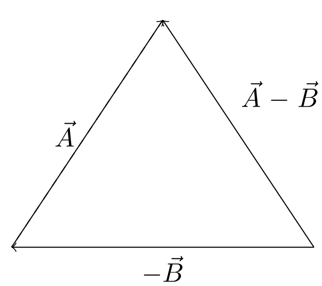
   Now for $\vec{A} - \vec{B}$ we can write this as $\vec{A} + (-\vec{B})$ where $-\vec{B}$ represents the vector which has same magnitude as $\vec{B}$ but the direction is oppesite. Hence this $-\vec{B}$ vector makes an agle $180° - \theta$ with $\vec{A}$



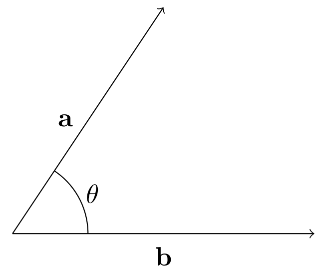
   Now addition of this two vectors will be the diagonal of the parallelogram whose two adjacent sides are $\vec{A}$ and $-\vec{B}$.



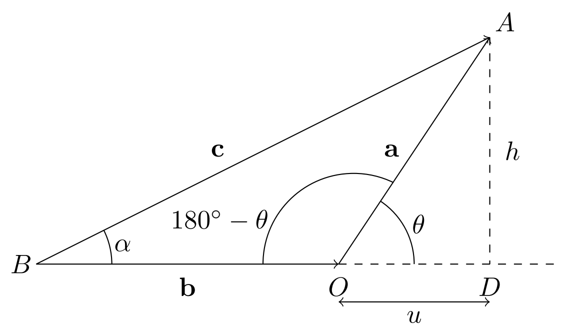
   Hence we can say geometrically that $\vec{A} - \vec{B}$ is the third side of the triangle made by $\vec{A}$ and $-\vec{B}$.



2. Let $\mathbf{a}$ and $\mathbf{b}$ are two vectors in $\mathbb{R}^3$ which makes an angle θ with each other.



Hence the the vector $\mathbf{c} = \mathbf{a} + \mathbf{b}$ is the resultant vector



Now by Pythagorean's theorem

$$OD^2 + AD^2 = OA^2 \implies u^2 + h^2 = a^2$$

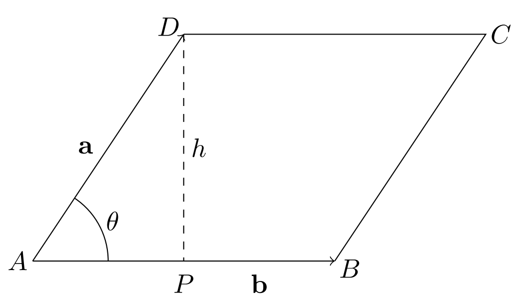
and

$$\begin{aligned} BD^2 + AD^2 = AB^2 &\implies (b+u)^2 + h^2 = c^2 \\ &\implies b^2 + 2ub + u^2 + h^2 = c^2 \\ &\implies b^2 + 2ub + a^2 = c^2 \\ &\implies b^2 + 2(a\cos\theta)b + a^2 = c^2 \\ &\implies b^2 + 2ab\cos\theta + a^2 = c^2 \text{ [Proved]} \end{aligned}$$

and

$$\alpha = \tan^{-1}\left(\frac{h}{b+u}\right) = \tan^{-1}\left(\frac{a\sin\theta}{b + a\cos\theta}\right)$$

3. Let $ABCD$ be a parallelogram where $\angle DAB = \theta$



Here the height of the parllelogram is the length of the perpendicular drawn from D on the line AB, $DP = h$. Now take the sides AD and AB as two vectors **a** and **b** respectively. Then the base of the parallelogram is b. Hence the area of the parallelogram is

$$\begin{aligned} h \times b &= (a \sin\theta) \times b \\ &= |\mathbf{a} \times \mathbf{b}| \text{ [Proved]} \end{aligned}$$

Now if we take any other two adjacent sides instead of AD and AB we can shift the vectors **a** and **b** to those sides and the area will be same.

(a) If we choose AD and DC then $AD = \mathbf{a}$ and $DC = \mathbf{b}$. Hence the area of the parallelogram will be still same.

(b) If we choose DC and BC then $BC = \mathbf{a}$ and $DC = \mathbf{b}$. Hence the area of the parallelogram will be still same.

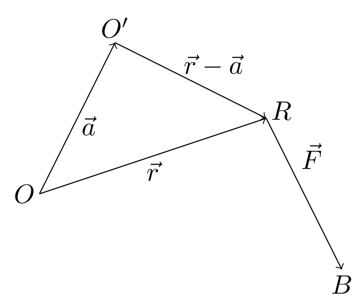
(c) If we choose BC and AB then $BC = \mathbf{a}$ and $AB = \mathbf{b}$. Hence the area of the parallelogram will be still same.

Therefore no matter which two adjacent sides we choose as the vectors the area of the parallelogram remains same, $|\mathbf{a} \times \mathbf{b}|$

4. Let the initial position vector of a particle on which the force $\vec{F}$ is acted upon is $\vec{r}$. Hence the initial torque (τ_i) on the particle is

$$\tau_i = \vec{r} \times \vec{F}$$

Now the origin is shifted by a vector $\vec{a}$ the new position vector of the particle is $\vec{r} - \vec{a}$



Hence after shifting the origin the new torque (τ_f) will be

$$\tau_f = (\vec{r} - \vec{a}) \times \vec{F}$$

Hence the difference between final and initial torque is

$$\tau_f - \tau_i = (\vec{r} - \vec{a}) \times \vec{F} - \vec{r} \times \vec{F} = -\vec{a} \times \vec{F} = \vec{F} \times \vec{a}$$

Therefore if the origin is shifted by $\vec{a}$ then the change in torques is $\vec{F} \times \vec{a}$.

5. Let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

and

$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$

Therefore

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}
\end{aligned}$$

Hence the cartesian components of $\vec{a} \times \vec{b}$ is $(a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}$

6. Let

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$
$$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$$
$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

Therefore

$$\begin{aligned}
\vec{a} \times \vec{b} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \\
&= (a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}
\end{aligned}$$

and

$$\begin{aligned}
\vec{b} \times \vec{c} &= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \\
&= (b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}
\end{aligned}$$

Now

$$\begin{aligned}
LHS = \vec{a} \cdot (\vec{b} \times \vec{c}) &= (a_1\hat{i} + a_2\hat{j} + a_3\hat{k}) \cdot \Big((b_2c_3 - b_3c_2)\hat{i} + (b_3c_1 - b_1c_3)\hat{j} + (b_1c_2 - b_2c_1)\hat{k}\Big) \\
&= a_1(b_2c_3 - b_3c_2) + a_2(b_3c_1 - b_1c_3) + a_3(b_1c_2 - b_2c_1) \\
&= c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)
\end{aligned}$$

and

$$\begin{aligned}
RHS = \vec{c} \cdot (\vec{a} \times \vec{b}) &= (c_1\hat{i} + c_2\hat{j} + c_3\hat{k}) \cdot \Big((a_2b_3 - a_3b_2)\hat{i} + (a_3b_1 - a_1b_3)\hat{j} + (a_1b_2 - a_2b_1)\hat{k}\Big) \\
&= c_1(a_2b_3 - a_3b_2) + c_2(a_3b_1 - a_1b_3) + c_3(a_1b_2 - a_2b_1)
\end{aligned}$$

Therefore $LHS = RHS$. Hence

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{c} \cdot (\vec{a} \times \vec{b}) \text{ [Proved]}$$

7. Let

$$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$$
$$\vec{b} = 2\hat{i} + 3\hat{j} + 4\hat{k}$$
$$c = 3\hat{i} + 4\hat{j} + 5\hat{k}$$

Then

$$\begin{aligned}
\vec{a} \times \vec{b} &= (\hat{i} + 2\hat{j} + 3\hat{k}) \times (2\hat{i} + 3\hat{j} + 4\hat{k}) \\
&= (2 \times 4 - 3 \times 3)\hat{i} + (3 \times 2 - 1 \times 4)\hat{j} + (1 \times 3 - 2 \times 2)\hat{k} \\
&= -\hat{i} + 2\hat{j} - \hat{k}
\end{aligned}$$

hence

$$\begin{aligned}(\vec{a}\times\vec{b})\times\vec{c} &= (-\hat{i}+2\hat{j}-\hat{k})\times(3\hat{i}+4\hat{j}+5\hat{k})\\ &= (2\times 5-(-1)\times 4)\hat{i}+((-1)\times 3-(-1)\times 5)\hat{j}+((-1)\times 4-2\times 3)\hat{k}\\ &= 14\hat{i}+8\hat{j}-10\hat{k}\end{aligned}$$

and

$$\begin{aligned}\vec{b}\times\vec{c} &= (2\hat{i}+3\hat{j}+4\hat{k})\times(3\hat{i}+4\hat{j}+5\hat{k})\\ &= (3\times 5-4\times 4)\hat{i}+(4\times 3-2\times 5)\hat{j}+(2\times 4-3\times 3)\hat{k}\\ &= -\hat{i}+2\hat{j}-\hat{k}\end{aligned}$$

hence

$$\begin{aligned}\vec{a}\times(\vec{b}\times\vec{c}) &= (\hat{i}+2\hat{j}+3\hat{k})\times(-\hat{i}+2\hat{j}-\hat{k})\\ &= (2\times(-1)-3\times 2)\hat{i}+(3\times(-1)-1\times(-1))\hat{j}+(1\times 2-2\times(-1))\hat{k}\\ &= -8\hat{i}-2\hat{j}+4\hat{k}\end{aligned}$$

Therfore

$$(\vec{a}\times\vec{b})\times\vec{c}\neq\vec{a}\times(\vec{b}\times\vec{c})$$

Hence the cross product is not associative. [Proved]