Analysis 2 Lecture Notes – Upendra Kulkarni

Soham Chatterjee

Contents

Chapter 1

Normed Linear Space:-

Definition 1.0.1: Limit of Sequence in $\mathbb{R}$

Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say
$$\lim_{n\to\infty} s_n = s$$
where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$
$$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Want to generalize this to a sequence in $\mathbb{R}^n$ i.e. $s_n \in \mathbb{R}^n$ $\forall$ $n \in \mathbb{N}$. Now the $s - s_n$ makes no sense. So it is useful to have a notion of whether vectors are big or small. We have magnitude of a vector. So lets revisit this

Definition 1.0.2: Limit of Sequence in $\mathbb{R}^n$

Let $\{s_n\}$ be a sequence in $\mathbb{R}^n$. We say
$$\lim_{n\to\infty} s_n = s$$
where $s \in \mathbb{R}^n$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number N such that for $n > N$
$$\|s - s_n\| < \epsilon$$

The same definition works if we interpret $\|v\|$ = length of the vector v.

From school, for v=$v_1, v_2, \cdots, v_n$ we had
$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$
But it will be useful to have a more general notion of length (of which the above will be an example)

1.1 Defination:-

Definition 1.1.1: Normed Linear Space and Norm $\|\cdot\|$

Let V be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on V is function $\|\cdot\|\ V \to \mathbb{R}_{\geq 0}$ satisfying

1. $\|x\| = 0 \iff x = 0\ \forall\ x \in V$
2. $\|\lambda x\| = |\lambda|\|x\|\ \forall\ \lambda \in \mathbb{R}(\text{or } \mathbb{C}),\ x \in V$
3. $\|x+y\| \leq \|x\| + \|y\|\ \forall\ x, y \in V$ (Triangle Inequality/Subadditivity)

And V is called a normed linear space.
• Same definition works with V a vector space over $\mathbb{C}$ (again $\|\cdot\| \to \mathbb{R}_{\geq 0}$) where 2 becomes $\|\lambda x\| = |\lambda|\|x\|\ \forall\ \lambda \in \mathbb{C},\ x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2+b^2}$

Example 1.1.1 (***p***−Norm)
$V = \mathbb{R}^m$, $p \in \mathbb{R}_{\geq 0}$. Define for $x = (x_1, x_2, \cdots, x_m) \in \mathbb{R}^m$

$$\|x\|_p = \Big(|x_1|^p + |x_2|^p + \cdots + |x_m|^p\Big)^{\frac{1}{p}}$$

(In school $p = 2$)

Special Case $p = 1$: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.
Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$): $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these p−norms are nothing but $|x|$. Now exercise

Question 1

Prove that triangle inequality is true if $p \geq 1$ for p−norms. (What goes wrong for $p < 1$?)

1.2 Open and Closed Ball:-

Definition 1.2.1: Open and Closed Ball in Normed Linear Space

An open Ball of radius r with center x in Normed Linear Space V is the set

$$\{y \in V \mid \|x-y\| < r\} = B_r(x)$$

and Closed ball is the set

$$\{y \in V \mid \|x-y\| \leq r\} = \overline{B_r(x)}$$

Now take $B_r(0)$ w.r.t $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$. Now imagine a sequence converging to origin. So if I draw a ordinary circle around the origin then no matter how small the circle the points of the sequence are eventually land inside the circle. If instead of that circle can same be said for diamond w.r.t norm 2. Then i can take circle that is inside that diamond. Same is true for ∞−norm. Hence convergence with respect to all norm 1 and norm 2 and even ∞ results for convergence.

Theorem 1.2.1
For a finite dimensional normed linear space, notion of convergence is independent of norm chosen

Proof: **For Property 3 for norm-2**

When field is $\mathbb{R}$:

We have to show

$$\begin{aligned}
&\sum_i (x_i+y_i)^2 \leq \left(\sqrt{\sum_i x_i^2}+\sqrt{\sum_i y_i^2}\right)^2 \\
\implies &\sum_i (x_i^2+2x_iy_i+y_i^2) \leq \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]}+\sum_i y_i^2 \\
\implies &\left[\sum_i x_iy_i\right]^2 \leq \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]
\end{aligned}$$

So in other words prove $\langle x,y\rangle^2 \leq \langle x,x\rangle\langle y,y\rangle$ where

$$\langle x,y\rangle = \sum_i x_iy_i$$

Note:-

- $\|x\|^2 = \langle x,x\rangle$
- $\langle x,y\rangle = \langle y,x\rangle$
- $\langle\cdot,\cdot\rangle$ is $\mathbb{R}-$linear in each slot i.e.

$$\langle rx+x',y\rangle = r\langle x,y\rangle + \langle x',y\rangle \text{ and similarly for second slot}$$

Here in $\langle x,y\rangle$ x is in first slot and y is in second slot.

Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x,y\rangle^2 \leq \langle x,x\rangle\langle y,y\rangle$$

expand everything of $\langle x-\lambda y, x-\lambda y\rangle$ which is going to give a quadratic equation in variable λ

$$\begin{aligned}
\langle x-\lambda y, x-\lambda y\rangle &= \langle x, x-\lambda y\rangle - \lambda\langle y, x-\lambda y\rangle \\
&= \langle x,x\rangle - \lambda\langle x,y\rangle - \lambda\langle y,x\rangle + \lambda^2\langle y,y\rangle \\
&= \langle x,x\rangle - 2\lambda\langle x,y\rangle + \lambda^2\langle y,y\rangle
\end{aligned}$$

Now unless $x=\lambda y$ we have $\langle x-\lambda y, x-\lambda y\rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

When field is $\mathbb{C}$:

Modify the definition by

$$\langle x,y\rangle = \sum_i \overline{x_i}y_i$$

Then we still have $\langle x,x\rangle \geq 0$ $\square$

Now there is no reason why we can not consider a norm on an infinite dimensional vector space. It will work. Perhaps i can define only for some sequences where the some converges.

Example 1.2.1 (Example of Theorem 1.1.2)

Suppose for set of all bounded infinite sequences a vector space because every number in a vector is less than some number so if you add two vectors then add then bound and if you scale then scale the bound. Now the ∞ norm works on that.

Now suppose you take all continuous real valued functions on closed interval $[0,1]$, such a function is bounded and this is a vector space and we can define $\infty-$norm even for that because for all f in this space attains its maximum value so just take that maximum value. Its an extremely infinite dimensional space.

Note:-

$\mathbb{R}^\infty$ is the space of all sequences.

Question 2

Modify the above proof for field $\mathbb{C}$

Question 3

Show that the following are normed linear spaces.

(a) l^∞ = Set of all bounded infinite sequences $(x_1, x_2, \cdots)$ $x_i \in \mathbb{R}$ with norm $\|x\| = \sup |x_i|$

(b) $C[0,1]$ = Set of all continuous functions $[0,1] \to \mathbb{R}$ with norm $\|f\| = \sup\limits_{x\in[0,1]} |f(x)|$

1.3 Limit of a Sequence:-

Definition 1.3.1: Limit of Sequence in Normed Linear Space

A sequence $\{s_n\}$ in a normed linear space V converge to s means $\forall$ real number $\epsilon > 0$ $\exists$ natural number N such that for $\forall$ $n > N$ $\|s - s_n\| < \epsilon$

1.4 Continuity:-

Definition 1.4.1: Continuity in Normed Linear Space

Let S be a subset of V and $f : S \to W$ where V, W are normed linear space. f is continuous at $v \in V$ means $\forall$ $\epsilon > 0$, $\exists$ $\delta > 0$, st whenever $\|x - v\| < \delta$ for $x \in S$ one has $\|f(x) - f(v)\| < \epsilon$

Distance in a normed linear space for $x, y \in V$ is

$$d(x,y) = \|x, y\|$$

Hence properties of this d are

1. $d(x,y) = 0 \iff x = y$
2. $d(\lambda x, \lambda y) = |\lambda| d(x,y)$ for any scalar λ
3. $d(u,v) + d(u,v) \geq d(u,w)$

Chapter 2

Metric Space:-

2.1 Definition:-

Definition 2.1.1: Metric Space X

A set X with a function $d\ X \times X \to \mathbb{R}_{\geq 0}$ such that

1. $d(x,y) = 0 \iff x = y$
2. $d(x,y) = d(y,x)$
3. $d(x,z) \leq d(x,y) + d(y,z)$

Notice that there is no homogeneity condition, and it does ot make sense as we don't have a field. In fact there is no notion of addition. But the condition 1 of norm has to be satisfied by this distance. Also we don't have a translational condition i.e. distance between x, y and distance between $x+v, y+v$ has to be same. Hence

Note:-

A metric space need not be a vector space. So it doesn't need a zero, or a notion of addition or scalar multiplication.

If I take a metric space and take any subset of it. And those three conditions of distance functions are still satisfied.

Note:-

Any subset of metric space is a metric space under the same distance function.

2.2 Open and Closed Ball and Set:-

Definition 2.2.1: Open Ball and Closed Ball in a Metric Space

An open ball of radius r with center $c \in X$ in a metric space X is

$$B_r(c) = \{x \in X \mid d(c,x) < r\}$$

and a closed ball is

$$\overline{B_r(c)} = \{x \in X \mid d(c,x) \leq r\}$$

Definition 2.2.2: Open Set and Closed Ball in a Metric Space

An open set in a metric space X is one of the form of union of some open balls and a closed set in a metric space X is one of the form of $X\backslash$some open sets

Note:-

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

Example 2.2.1 (Open Set and Close Set)

Open Set:
- ϕ
- $\bigcup\limits_{x\in X} B_r(x)$ (Any $r>0$ will do)
- $B_r(x)$ is open

Closed Set:
- $X,\ \phi$
- $\overline{B_r(x)}$

$x-$axis $\cup$ $y-$axis

Question 4

Is the set $x-$axis$\backslash\{$Origin$\}$ a closed set

Solution: We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Now this works well for points which are above or below the $x-$axis. But for origin no matter how small the ball we take it willl have points from $x-$axis. Hence the set is not a closed set.

Question 5

Any continuous path in $\mathbb{R}^2$ is closed where path $= f:[0,1]\to\mathbb{R}^2$

Solution: This is true. To be proved later.
Analogous to: For continuous function $f:[0,1]\to\mathbb{R}$, the image is a closed interval

Question 6

If i take $X=x-$axis $\cup$ $y-$axis then is it open

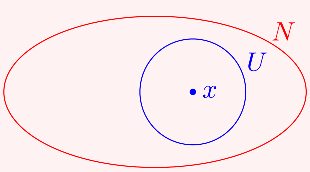
Solution: Yes because here the space is only the union of those two axis. So any ball would be like a cross or line but it just as the metric space given to us. [It is open for this metric space but not open in $\mathbb{R}^2$]

Note:-

If $S\subset X$, then S itself has a collection of open sets of S by containing S as a metric space.

Definition 2.2.3: Neighborhood

For a point x in metric space X, a neighborhood of x is a set N such that $x\in$an open set $U\subset N$

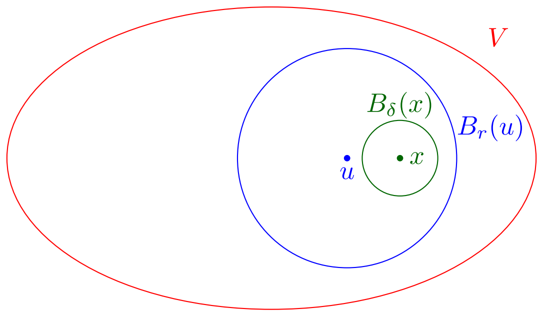


If N itself is open. then we say that N is an open neighborhood of x

Theorem 2.2.1

If $x \in$ open set V then $\exists\, \delta > 0$ such that $B_\delta(x) \subset V$

Proof: By openness of V, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose δ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leq d(u, x) + d(x, y) < d + \delta < r$$

$\square$

Note:-

V is open $\iff \bigcup_{x \in V} B_r(x)$ (where r depends on x)

Theorem 2.2.2

Let X be a metric space.

1. Union of open sets is open
2. Intersection of two open sets is open

Analogues to these as we are just taking complement of the open sets

$1'$. Arbitrary intersection of closed sets is closed

$2'$. Finite union of closed sets is closed.

Proof:

1. Let $\{V_\alpha\}_{\alpha \in I}$ be a collection of open sets where I is an index set. We want ti show $\bigcup_{\alpha \in I} V_\alpha$ is open in X. Since each V_α is open $V_\alpha = \bigcup_{\beta \in J_\alpha} B_{r_\beta}(c_\beta)$ Then

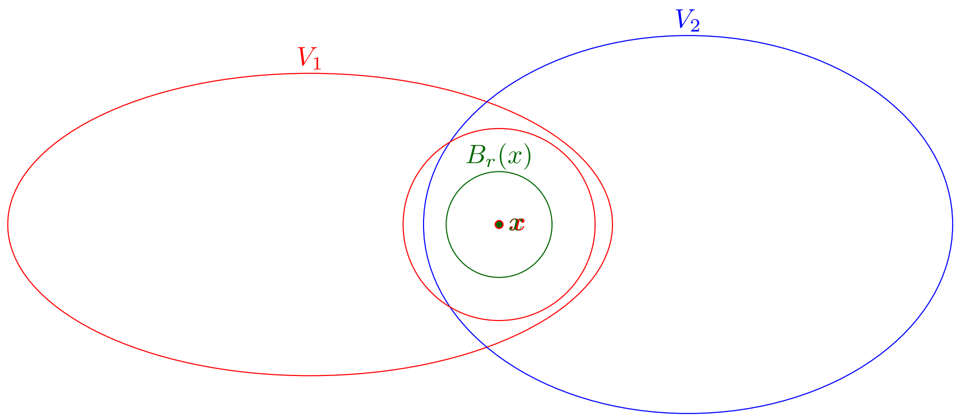
$$\begin{aligned}\bigcup_{\alpha \in I} V_{alpha} &= \bigcup_{\alpha \in I} \bigcup_{\beta \in J_\alpha} B_{r_\beta}(c_\beta) \\ &= \bigcup_{\beta \in \sqcup J_\alpha} B_{r_\beta}(c_\beta)\end{aligned}$$

which is still a union of balls

$\square$

2. The statement implies intersection of finite number of open sets is open. We can prove this by indution.

We will do by showing that for each $x \in V_1 \cap V_2\ \exists\, r > 0$ s.t. $B_r(x) \subset V_1 \cap V_2$



As $x \in V_1$ $\exists$ r_1 such that $x \in B_{r_1}(x) \subset V_1$. Similarly $x \in V_2$ $\exists$ r_2 such that $x \in B_{r_2}(x) \subset V_2$. Take $r = \min\{r_1, r_2\}$. Thus we have $x \in B_r(x) \subset V_1 \cap V_2$

The second part for closed sets are left as exercise $\square$

2.3 Topological Space:-

Definition 2.3.1: Topological Space

A topological space is a set X together with a collection of subsets of X (i.e. a subset of the power set of X) that is closed under taking arbitrary unions and finite intersections. This collection is called a topology on X

Note:-

Union means $\bigcup\limits_{\alpha \in I} S_\alpha = \{x \in X \mid \exists\ \alpha \text{ s.t. } x \in S_\alpha\}$

Intersection means $\bigcap\limits_{\alpha \in I} S_\alpha = \{x \in X \mid \forall \alpha,\ x \in S_\alpha\}$

Question 7

Suppose i have a topological space X under given some topology. Is the entire set open ? And that the empty set is open ?

Solution: If $I = \phi$, $\bigcup\limits_{\alpha \in I} S_\alpha = \{x \in X \mid \exists\ \alpha \in I \text{ s.t. } x \in S_\alpha\}$ gives ϕ and $\bigcap\limits_{\alpha \in I} S_\alpha = \{x \in X \mid \forall \alpha \in I,\ x \in S_\alpha\}$ gives X because $\forall\ \alpha \in I$ condition is vacuously true for each $x \in X$.

Note:-

Intersection of empty families are not defined in set theory. This brings a very important point. In a set theory you have to have a universe. (Set theory have to avoid paradoxes, Russel Paradox) At the beginning you construct a large enough universe and you taking subsets only from that universe. Notice all subsets we are considering here are subsets of X and here we defined how we union and intersection mean. Though it still this asks what our axioms of set theory. So you can change the part of the definition of topological space like this "...with a collection of subsets of X including the empty set and the whole space..."

(If you don't like this as it is)

Note:-

If S is a subset of metric space X, then S is itself a metric space and as such open/closed sets as subsets of metric space

Question 8

Is there any connection between being open in X and being open in S (Similar question for closed)

Solution: Let $x \in S$. Now, Ball of radius r in $S = S \cap$ Ball of radius r in X. Therefore

$$\begin{aligned}\text{Open Set in } S &= \bigcup \text{ Balls in } S\\ &= \bigcup (\text{Balls in } X \cap S)\\ &= \left(\bigcup \text{ Balls in } X\right) \cap S\\ &= \text{Open set } X \cap S\end{aligned}$$

Part 2 is left as exercise

Corollary 2.3.1

If $S \subset X$ is open in X then a subset T of S is open in $S \iff T$ is open in X

Corollary 2.3.2

If $S \subset X$ is closed in X then a subset T of S is closed in $S \iff T$ is closed in X

Definition 2.3.2: Subspace of a Topological Space X

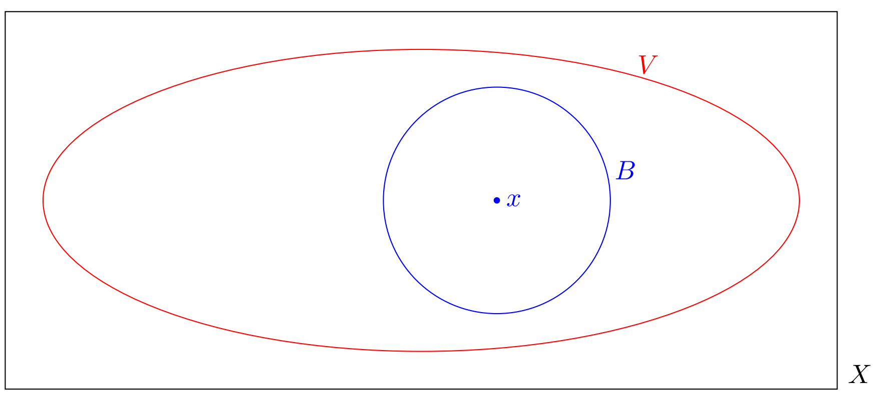
For any subset S of a topological space X, the collection $S \cap U$, U open in X is called a subspace.

Question 9

Prove that subspace of a metric space X defines a topology on X

Wrong Concept 2.1

If $x \in$ open V then there exists $r > 0$ such that $x \in B_r(x) \subset V$



Idea: Why not we take $r = \inf\{\text{distance from } x \text{ to boundary of ball } B\}$.

Now we first have to ensure $r > 0$. Suppose that's true.

Then we have to define boundary. What is boundary, We can give a reasonable definition (Boundary has already a definition but we don't know that for now). Let boundary of $B = \{x \in X \mid d(c, x) = \delta\}$ Now this definition is not proper for our purpose.Because if we take union of all balls in V then we will have lots of points as boundary but part of them should not be considered as boundary. Even if we take this definition.

Then the big question comes/ We are taking a infimum of a certain set of real numbers. The very first question arises is whether this set is nonempty. For example if we take B to be the metric space it self we have no boundary.

Questions which come thorough this.

- Is there a meaningful way to define boundary
- Can we modify the idea

Chapter 3

Continuity in Metric Space

3.1 Limit Point and Closure:-

Definition 3.1.1: Limit Point

$S \subset X$ is a metric space. We say that $x \in X$ is a limit point of S if $\exists$ a sequence $\{s_n\}$ with all $s_n \in S \setminus \{x\}$ such that $s_n \to x$ (each s_n is different from x)

Theorem 3.1.1

x is a limit point of $S \iff$ every neighborhood of x in X contains a point of S other than S.

Proof: **If Part:**

Let x be a limit point of S. Therefore take a sequence $\{s_n\}$ in $S \setminus \{x\}$ with $s_n \to x$.

To prove what we want it is enough to show that $B_r(x) \cap S$ contains a point other than x. As $s_n \to x$, $\exists\ N$ s.t. $\forall\ n > N$ $d(x, s_n) < r$ i.e. $s_n \in B_r(x)$. In particular $s_n]inB_r(x) \cap (S \setminus \{x\})$

Only If Part:

We need to produce a sequence $\{s_n\} \in S \setminus \{x\}$ with $\lim s_n = x$. Take $s_n \in B_{\frac{1}{n}}(x) \cap (S \setminus \{x\})$ See that $\lim\limits_{n\to\infty} s_n = x$. This is essentially because $\frac{1}{n} \to 0$.

Complete the rest of the proof.

□

Definition 3.1.2: Closure

Given a topological space X and $S \subset X$, the closure of the set S is $\overline{S}$ the smallest closed set containing S.

Theorem 3.1.2

$$\begin{aligned}\overline{S} &= \text{Smallest closed set of } X \text{ containing } S = \text{A}\\ &= S \cup (\text{limit points of } S) = \text{B}\\ &= \{x \in X \mid x = \lim_{n\to\infty} \text{ for some sequence } \{s_n\} \text{ in } S\} = \text{C}\\ &= \{x \in X \mid \text{Every neighborhood of } x \text{ intersects } S\} = \text{D}\end{aligned}$$

Proof: $\boldsymbol{A \subset D}$

$A^c = \bigcup$ (All open set V s.t. $V \cap S = \phi$)
$D^c = \{x \in X \mid \exists$ open neighborhood of x, B s.t. $B \cap S = \phi\}$
Clearly for all $x \in D^c$, $x \in A^c$. Hence $D^c \subset A^c \implies A \subset D$

$\boldsymbol{D \subset B}$

Take $x \in D$. Suppose $x \notin S$. Now any neighborhood of x intersects S in a point hence it has to be a different point from x since $x \notin S$. Therefore x is a limit point of S. $D \subset B$

$\boldsymbol{B \subset C}$

If $x \in S$ then take a sequence

$\square$

Question 10

What does it mean to be smallest closed set containing the set S here ?

Solution: $\bigcap$ All closed sets containing S is automatically closed and hence the smallest closed set containing S.

Proof: For proof of 3.1.2 notice A,B,C,D all contains S (obvious).

Note:-

We don't need to show B,C,D are closed. We can also take the sets element wise and show each set is a subset of the other. This may simplify our way of proof. (exercise)

Now see A and D completely deal with topology. A is about closed sets and D is about open sets. So A and D close to each other. Now by the 3.1.1 we have equivalence of C and D. So we can prove like this

$$A \iff D \iff B \;\&\; C$$

Left as exercise $\square$

Note:-

For these kind of proofs instead of looking for the most efficient way try to find a path that allows you to go from anywhere to anywhere

3.2 Continuity:-

Definition 3.2.1: Continuity

$f : X \to Y$ function between metric spaces is continuous at $a \in X$ if $\forall\, \epsilon > 0\ \exists\, \delta > 0$ s.t.

$$\begin{array}{ccc} d(x,a) < \delta & \implies & d(f(a), f(x)) < \epsilon \\ \Updownarrow & & \Updownarrow \\ x \in B_\delta(x) & \implies & f(x) \in B_\epsilon(f(a)) \end{array}$$

That means f^{-1}(Any ball around $f(a)$) $\supset$ Ball around a.
So $f : X \to Y$ is continuous at all points $\iff f^{-1}$(Any ball intersecting the range) $\supset$ A ball

Note:-

We can not say f^{-1}(Any ball) because because we need a ball that contains a point in the range

Theorem 3.2.1

f is continuous $\iff$ f^{-1}(Any open set in Y) is open in X

Proof: **If Part:-**

It is enough to show f^{-1}(Any ball) is open on X because f^{-1} preserves unions $f^{-1}\left(\bigcup_{\alpha} V_\alpha\right) = \bigcup_{\alpha}\left(f^{-1}(V_\alpha)\right)$

Let B is any open set (as its conceptually simpler to take open set here instead of a ball) in Y. Let $a \in f^{-1}(B)$. Hence we can say $f(a) \in B$. Since B is an open set we can say there is a ball $B_\epsilon(f(a)) \subset B$. Since f is continuous $\exists\ \delta$ such that $f(x) \in B_\epsilon(f(a))$ whenever $x \in B_\delta(a)$. Now $f^{-1}(B) \supset f^{-1}\Big(B_\epsilon(f(a))\Big) \supset B_\delta(a)$ Hence $f^{-1}(B)$ is open.

Only If Part:-

Lets prove continuity ar $a \in X$. We are given that $f^{-1}\Big(B_\epsilon(f(a))\Big)$ is open and obviously contains a. Therefore $f^{-1}\Big(B_\epsilon\big(f(a)\big)\Big)$ contains a ball around a. Take $\delta =$ Radius of the ball. □

Question 11

For a metric space X, show that $\overline{S} = \{x \in X \mid \lim\limits_{n\to\infty} s_n = x\}$ for some sequence $\{s_n\}$ in S.

Question 12

For a function $f : X \to Y$ between metric spaces, show that the followings are equivalent.

1. f is continuous
2. f^{-1}(Open Set) is open
3. f^{-1}(Closed Set) is closed
4. $f(\overline{S}) = \overline{f(S)}$
5. $x_n \to x \implies f(x_n) \to f(x)$

One or more of the above are wrong so check if they are true and if not then find the true statement.

Solution: 4 is wrong. How to correct and rest is left as exercise

Question 13

For $f : X \to Y$ any set map

(i) f^{-1} preserves unions, intersections, complements

(ii) Is there any condition on f under which f possesses the property above ?

Example 3.2.1 (Continuous Function)

- $X \xrightarrow{f} Y \xrightarrow{g} Z$ f, g continuous $\implies$ $g \cdot f$ is continuous
- Is $S \subset X$ then $S \xrightarrow{\text{Inclusion}} X$ is continuous

Question 14

$\mathbb{R}^n \to \mathbb{R}$ projects any coordinate is continuous

Question 15

$X \xrightarrow{f} \mathbb{R}^n$ f is continuous $\iff$ $f(x) \to (f_1(x), f_2(x), \cdots, f_n(x))$ where each f_i is continuous