Vo Quoc Ba Can   Cosmin Pohoață

Old and New Inequalities

Volume 2

GIL

Preface

"The last thing one knows when writing a book is what to put first."

-Blaise Pascal

Mathematics has been called the science of tautology; that is to say, mathematicians have been accused of spending their time proving that things are equal to themselves. This statement is rather inaccurate on two counts. In the first place, mathematics, although the language of cience, is not a science. More likely it is a creative art, as G. H. Hardy liked to consider it. Secondly, the fundamental results of mathematics are often inequalities rather than equalities.

In the pages that follow, we present a large variety of problems involving such inequalities, questions that became famous in (mathematical) competitions or journals because of their beauty. The most important prerequisite for benefiting from this book is the desire to master the craft of discovery and proof. The formal requirements are quite modest. Anyone who knows basic inequalities such as the ones of **Cauchy-Schwarz**, **Hölder**, **Schur**, **Chebyshev** or **Bernoulli** is well prepared for almost everything to be found here. The student who is not that experienced will also be exposed in the first part to a wide combination of moderate and easy problems, ideas, techniques, and all the ingredients leading to a good preparation for mathematical contests. Some of the problems we chose to discuss are known, but we have included them here with new solutions which show the diversity of ideas pertaining to inequalities. Nevertheless, the book develops many results which are rarely seen, and even experienced readers are likely to find material that is challenging and informative.

To solve a problem is a very human undertaking, and more than a little mystery remains about how we best guide ourselves to the discovery of original solutions. Still, as George Pólya and the others have taught us, there are principles of problem solving. With practice and good coaching we can all improve our skills. Just like singers, actors, or pianists, we have a path toward a deeper mastery of our craft.

About The Authors

Vo Quoc Ba Can is a student at the "Can Tho" University of Medicine and Pharmacy. As a high-school student, he participated in many national contests obtaining several prizes. Though at the moment he is not studying mathematics, his activity in Inequalities has proved to be quite wide lately. Some of his problems were published in specialized journals, but the biggest part of them became popular on the wordwide known MathLinks forum. On the same theme, he (co)authored several manuscripts, which were (unfortunately) published in Vietnamese.

Cosmin Pohoată ¸ is in present a high-school student at the "Tudor Vianu" High School in Bucharest, Romania. During his scholar activity he participated in many (mathematical or not) olympiads and contests. Recently, he was awarded with a Gold Medal at the Sharygin International Mathematical Olympiad, which took place in Dubna, Russia from July 29 to August 1, 2008. In the past few years, he had many important contributions in Euclidean Geometry, distinguishing himself in journals like Forum Geometricorum, Crux Mathematicorum or the American Mathematical Monthly. In Clark Kimberling's Encyclopedia of Triangle Centers, a point appears under his name (X3333 - "The Pohoata Point"). His main mathematical interests besides Euclidean Geometry are Graph Theory, Combinatorial Number Theory and, of course, Inequalities. Beyond mathematics, his activities include computer science, philosophy, music, football (soccer) and tennis.

Problems

1. Prove that for all positive real numbers a, b the following inequality holds

$$\sqrt{2}\left(\sqrt{a(a+b)^3}+b\sqrt{a^2+b^2}\right)\leq 3\left(a^2+b^2\right)$$

Irish MO, 2004

2. Consider real numbers a, b, c contained in the interval $\left[\frac{1}{2},1\right]$. Prove that

$$2\leq\frac{a+b}{1+c}+\frac{b+c}{1+a}+\frac{c+a}{1+b}\leq 3$$

Romanian MO, 2006

3. Let a, b, c be three positive real numbers contained in the interval [0,1]. Prove that

$$\frac{1}{1+ab}+\frac{1}{1+bc}+\frac{1}{1+ca}\leq\frac{5}{a+b+c}$$

Chendi Huang

4. Let x, y, z be positive reals with such that $xyz=1$. Show that the following inequality holds

$$\frac{1}{(1+x)^2+y^2+1}+\frac{1}{(1+y)^2+z^2+1}+\frac{1}{(1+z)^2+x^2+1}\leq\frac{1}{2}$$

Cristinel Mortici, Math. Reflections

5. Let a, b, c be three positive real numbers satisfying $abc=8$. Prove that

$$\frac{a-2}{a+1}+\frac{b-2}{b+1}+\frac{c-2}{c+1}\leq 0$$

Romaninan jBMO Team Preparation Tests, 2008

6. Let a, b, c be the side lengths of an acute-angled triangle. Prove that

$$(a+b+c)(a^2+b^2+c^2)(a^3+b^3+c^3)\geq 4(a^6+b^6+c^6)$$

Vietnamese IMO Team Preparation Tests, 2000

7. Let a, b, c be positive real numbers such that $ab+bc+ca=1$. Prove that

$$a\sqrt{b^2+c^2+bc}+b\sqrt{c^2+a^2+ca}+c\sqrt{a^2+b^2+ab}\geq\sqrt{3}$$

Jose Luis Diaz-Barrero, College Math. Journal

8. Find the maximum value of

$$(x^3+1)(y^3+1)$$

for all real numbers x, y, satisfying the condition that $x+y=1$

Romanian MO, 2006

9. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq\frac{a+b}{b+c}+\frac{b+c}{a+b}+1$$

Bielorussian MO, 1998

10. If x, y, z are positive real numbers, prove that the following inequality holds

$$(x+y+z)^2(xy+yz+zx)^2\leq 3(y^2+yz+z^2)(z^2+zx+x^2)(x^2+xy+y^2)$$

Indian MO, 2007

11. Let a, b, c be positive real numbers such that $a+b+c\geq\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$. Prove that

$$a+b+c\geq\frac{3}{a+b+c}+\frac{2}{abc}$$

Peruvian IMO Team Selection Tests, 2007

12. Let a, b, c be positive real numbers such that $\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\geq 1$. Prove that

$$a+b+c\geq ab+bc+ca$$

Romanian jBMO Team Selection Tests, 2007

13. Let a, b, c be real numbers satisfying $a,\ b,\ c\geq 1$ and $a+b+c=2abc$. Prove that

$$\sqrt[3]{(a+b+c)^2}\geq\sqrt[3]{ab-1}+\sqrt[3]{bc-1}+\sqrt[3]{ca-1}$$

Bruno De Lima Holanda, Math. Reflections

14. Let $a_1,\ a_2,\ \cdots,\ a_n$ be positive real numbers satisfying the condition that $a_1+a_2+\cdots+a_n=1$. Prove that

$$\sum_{j=1}^{n}\frac{a_j}{1+a_1+\cdots+a_j}<\frac{1}{\sqrt{2}}$$

Romaninan IMO Team Preparation Tests, 2008

15. Positive numbers $\alpha,\ \beta,\ x_1,\ x_2,\ \cdots,\ x_n$ $(n\geq 1)$ satisfy the condition $x_1+x_2+\cdots+x_n=1$. Prove that

$$\frac{x_1^3}{\alpha x_1+\beta x_2}+\frac{x_2^3}{\alpha x_2+\beta x_3}+\cdots+\frac{x_n^3}{\alpha x_n+\beta x_1}\geq\frac{1}{n(\alpha+\beta)}$$

Moldavian IMO Team Selection Tests, 2002

16. If three non negative real numbers a, b, c satisfy the condition $\frac{1}{a^2+1}+\frac{1}{b^2+1}+\frac{1}{c^2+1}=2$. Prove that

$$ab+bc+ca\leq\frac{3}{2}$$

Iranian MO, 2005

17. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2}{b}+\frac{b^2}{c}+\frac{c^2}{a}\geq a+b+c+\frac{4(a-b)^2}{a+b+c}$$

Balkan MO, 2005

18. If x, y, z are positive numbers satisfying the condition $xy+yz+zx=1$, show that

$$\frac{27}{4}(x+y)(y+z)(z+x)\geq\left(\sqrt{x+y}+\sqrt{y+z}+\sqrt{z+x}\right)^2\geq 6\sqrt{3}$$

Turkish IMO Team Selection Tests, 2006

19. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 3+\frac{(a-c)^2}{ab+bc+ca}$$

Vo Quoc Ba Can

20. Let a, b, c be non negative real numbers satisfying $ab+bc+ca=3$. Prove that

$$\frac{1}{1+a^2(b+c)}+\frac{1}{1+b^2(c+a)}+\frac{1}{1+c^2(a+b)}\leq\frac{3}{1+2abc}$$

Mathlinks Contest, 2008

21. Let a, b, c be positive real numbers such that $2a+b=1$. Prove that

$$\frac{5a^3}{bc}+\frac{4b^3}{ca}+\frac{3c^3}{ab}\geq 4$$

Tran Van Luan

22. (a) If x, y and z are three real numbers, all different from 1, such that $xyz=1$, prove that

$$\frac{x^2}{(x-1)^2}+\frac{y^2}{(y-1)^2}+\frac{z^2}{(z-1)^2}\geq 1$$

(b) Prove that equality is achieved for infinitely many triples of rational numbers x, y and z.

IMO, 2008

23. Let a, b, c be positive real numbers. Prove that

$$\frac{a}{b(b+c)^2}+\frac{b}{c(c+a)^2}+\frac{c}{a(a+b)^2}\geq\frac{9}{4(ab+bc+ca)}$$

Ho Phu Thai, Math. Reflections

24. Let a, b, c be non negative real numbers such that $a+b+c=1$. Prove that

$$\sqrt{a+\frac{(b-c)^2}{4}}+\sqrt{b}+\sqrt{c}\leq\sqrt{3}$$

Chinese Girls MO, 2008

25. Let a, b, c be the side lengths of a triangle. Prove that

$$\sum_{cyc}\frac{a^3}{a^3+(b+c)^3}+1\geq 2\sum_{cyc}\frac{a^2}{a^2+(b+c)^2}$$

Pham Quang Vu

26. Prove that for any real numbers a, b, c the following inequality holds

$$(a+b-c)^2(b+c-a)^2(c+a-b)^2 \geq (a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2)$$

Japanese MO, 2001

27. Let a, b, c be the side lengths of a triangle. Prove that

$$(a+b)(b+c)(c+a)+(a+b-c)(b+c-a)(c+a-b) \geq 9abc$$

Virgil Nicula and Cosmin Pohoata, Math. Reflections

28. Let a, b, c be positive real numbers. Prove that

$$\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)\left(\frac{1}{a+1}+\frac{1}{b+1}+\frac{1}{c+1}\right) \geq \frac{9}{abc+1}$$

Walther Janous

29. Let a, b, c be positive real numbers contained in the interval [0,1]. Prove that

$$\frac{2a}{1+bc}+\frac{2b}{1+ca}+\frac{2c}{1+ab}+abc \leq 4$$

Adapted after Polish MO, 2005

30. Let a, b, c be non negative real numbers satisfying $Max\{b+c-a,\ c+a-b,\ a+b-c\} \leq 1$. Prove that

$$a^2+b^2+c^2 \leq 1+2abc$$

Chendi Huang

31. If x, y, z are real numbers satisfying $xyz=-1$, prove that

$$x^4+y^4+z^4+3(x+y+z) \geq \frac{y^2+z^2}{x}+\frac{z^2+x^2}{y}+\frac{x^2+y^2}{z}$$

Iranian MO, 2005

32. Let a, b,c, d be positive real numbers satisfying the condition $a+b+c+d=abc+bcd+cda+dab$. Prove that

$$a+b+c+d+\frac{2a}{a+1}+\frac{2b}{b+1}+\frac{2c}{c+1}+\frac{2d}{d+1} \geq 8$$

Vo Quoc Ba Can

33. Let a, b, c be non negative real numbers. Prove that

$$\frac{a^2+2bc}{b^2+c^2}+\frac{b^2+2ca}{c^2+a^2}+\frac{c^2+2ab}{a^2+b^2} \geq 3$$

Russian MO, 1999

34. Let a, b, c be positive real numbers. Prove that

$$\frac{ab}{c(c+a)}+\frac{bc}{a(a+b)}+\frac{ca}{b(b+c)}\geq\frac{a}{c+a}+\frac{b}{a+b}+\frac{c}{b+c}$$

Moldavian MO, 1999

35. Let a, b, c be positive real numbers such that $ab+bc+ca\geq 3$. prove that

$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}}\geq\frac{3}{\sqrt{2}}$$

Pham Huu Duc, Math. Reflections

36. Let x, y, z, t be positive real numbers such that $\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}+\frac{1}{t+1}=1$. Prove that

$$min\left\{\frac{1}{x}+\frac{1}{y}+\frac{1}{z};\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t};\ \frac{1}{z}+\frac{1}{t}+\frac{1}{x};\ \frac{1}{t}+\frac{1}{x}+\frac{1}{y}\right\}\leq 1$$

$$\leq max\left\{\frac{1}{x}+\frac{1}{y}+\frac{1}{z};\ \frac{1}{y}+\frac{1}{z}+\frac{1}{t};\ \frac{1}{z}+\frac{1}{t}+\frac{1}{x};\ \frac{1}{t}+\frac{1}{x}+\frac{1}{y}\right\}$$

Pham Van Thuan

37. Let $a_1,\ a_2,\ \cdots,\ a_n$ be positive real numbers. Prove that

$$\prod_{k=1}^{n}\left(\sum_{j=1}^{n}a_j^{T_k}\right)\geq\left(\sum_{k=1}^{n}a_n^{\frac{T_{n+1}}{3}}\right)^n$$

where $T_k=\dfrac{k(k+1)}{2}$ is the $k-$th triangular number.

Jose Luis Diaz-Barrero, Math. Reflections

38. Let a, b, c, d be positive numbers. Prove that

$$3(a^2-ab+b^2)(c^2-cd+d^2)\geq(a^2c^2-abcd+b^2d^2)$$

Titu Andreescu, Math. Reflections

39. Let a, b, c, d be real numbers such that $a+b+c=1$. Prove that

$$\frac{a}{a^2+1}+\frac{b}{b^2+1}+\frac{c}{c^2+1}\leq\frac{9}{10}$$

Polish MO, 1997

40. Let n be a positive integer, and let x and y be positive real numbers such that $x^n+y^n=1$. Prove that

$$\left(\sum_{k=1}^{n}\frac{1+x^{2k}}{1+x^{4k}}\right)\left(\sum_{k=1}^{n}\frac{1+y^{2k}}{1+y^{4k}}\right)<\frac{1}{(1-x)(1-y)}$$

IMO Shortlist, 2007, proposed by Estonia

41. Let a, b, c be positive real numbers such that $a+b+c+1=4abc$. Prove that

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\geq 3\geq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}}$$

Daniel Campos Salas, Math. Reflections

42. Let a, b, c be nonnegative real numbers such that $a+b+c=3$ Set $x=\sqrt{a^2-a+1}$, $y=\sqrt{b^2-b+1}$ and $z=\sqrt{c^2-c+1}$. Prove that

$$xy+yz+zx\geq 3 \text{ and } x+y+z\leq 2+\sqrt{7}$$

Adapted after the Vietnamese MO, 2007

43. Let $n\geq 2$ be a given integer. Determine

   (a) The largest real c_n such that

$$\frac{1}{1+a_1}+\frac{1}{1+a_2}+\cdots+\frac{1}{1+a_n}\geq c_n$$

   holds for any positive numbers $a_1,\ a_2,\ \cdots,\ a_n$ with $a_1a_2\cdots a_n=1$

   (b) The largest real d_n such that

$$\frac{1}{1+2a_1}+\frac{1}{1+2a_2}+\cdots+\frac{1}{1+2a_n}\geq d_n$$

   holds for any positive numbers $a_1,\ a_2,\ \cdots,\ a_n$ with $a_1a_2\cdots a_n=1$

Italian MO, 2007

44. Let a, b, c be positive real numbers. Prove that

$$\frac{bc}{a^2+bc}+\frac{ca}{ab^2+ca}+\frac{ab}{c^2+ab}\leq \frac{a}{b+c}+\frac{b}{c+a}+\frac{c}{a+b}$$

Pham Huu Duc, Math. Reflections

45. Real numbers $a_1,a_2,\cdots,a_n$ are given. For each i $(1\leq i\leq n)$ define $d_i=max\{a_j\mid 1\leq j\leq i\}-min\{a_j\mid i\leq j\leq n\}$ and let $d=max\{d_i\mid 1\leq i\leq n\}$

   (a) Prove that for any real numbers $x_1\leq x_2\leq\cdots\leq x_n$, we have

$$max\{|x_i-a_i|\mid 1\leq i\leq n\}\geq \frac{d}{2}$$

   (b) Show that there are real numbers $x_1\leq x_2\leq\cdots\leq x_n$ such that we have equality in (a)

IMO, 2007, Proposed by New Zealand

46. Let a, b, c be nonzero positive numbers. Prove that

$$\sqrt{\frac{a^2}{4a^2+ab+4b^2}}+\sqrt{\frac{b^2}{4b^2+bc+4c^2}}+\sqrt{\frac{c^2}{4a^2+ca+4a^2}}\leq 1$$

Zhao Bin, Math. Reflections

47. Let a, b, c be positive numbers such that $4abc = a + b + c + 1$. Prove that

$$\frac{b^2+c^2}{a} + \frac{c^2+a^2}{b} + \frac{a^2+b^2}{c} \geq 2(ab+bc+ca)$$

Andrei Ciupan, Math. Reflections

48. Let a, b, c be positive numbers. Prove that

$$\frac{a^3}{(a+b)^3} + \frac{b^3}{(b+c)^3} + \frac{c^3}{(c+a)^3} \geq \frac{3}{8}$$

Vietnamese IMO Team Selection Tests, 2005

49. Let a, b, c, x, y, z be positive real numbers. Prove that

$$(a^2+x^2)(b^2+y^2)(c^2+z^2) \geq (ayz+bzx+cxy-xyz)^2$$

Titu Andreescu, Math. Reflections

50. Let x, y, z be positive real numbers. Prove that

$$\frac{\sqrt{y+z}}{x} + \frac{\sqrt{z+x}}{y} + \frac{\sqrt{x+y}}{z} \geq \frac{4(x+y+z)}{\sqrt{(x+y)(y+z)(z+x)}}$$

Darij Grinberg

51. Let a, b ,c be nonnegative real numbers such that $abc = 4$ and $a, b, c > 1$. Prove that

$$(a-1)(b-1)(c-1)(\frac{a+b+c}{3} - 1) \leq \left(\sqrt[3]{4} - 1\right)^4$$

Marian Tetiva, Math. Reflections

52. Let a ,b, c be positive real numbers satisfying $abc = 1$. Prove that

$$\frac{1}{b(a+b)} + \frac{1}{c(b+c)} + \frac{1}{a(c+a)} \geq \frac{3}{2}$$

Kazakhstan, Zhautykov Olympiad, 2008

53. Prove that for all positive real numbers a, b, c the following inequality holds:

$$\frac{1}{a+b+c}\left(\frac{1}{b+c} + \frac{1}{c+a} + \frac{1}{a+b}\right) \geq \frac{1}{ab+bc+ca} + \frac{1}{2(a^2+b^2+c^2)}$$

Pham Huu Duc, Math. Reflections

54. Let a, b, c be the sidelengths of a triangle. Prove that

$$\frac{\sqrt{b+c-a}}{\sqrt{b}+\sqrt{c}-\sqrt{a}} + \frac{\sqrt{c+a-b}}{\sqrt{c}+\sqrt{a}-\sqrt{b}} + \frac{\sqrt{a+b-c}}{\sqrt{a}+\sqrt{b}-\sqrt{c}} \leq 3$$

IMO Shortlist, 2006, Proposed by South Korea

55. Let a, b, c be the sidelengths of a triangle with perimeter 1. Prove that

$$1 < \frac{b}{\sqrt{a+b^2}} + \frac{c}{\sqrt{b+c^2}} + \frac{a}{\sqrt{c+a^2}} < 2$$

Vo Quoc Ba Can

56. Prove that for any positive real numbers a, b and c, we have that

$$\sqrt{\frac{b+c}{a}} + \sqrt{\frac{c+a}{b}} + \sqrt{\frac{a+b}{c}} \geq \sqrt{6\frac{a+b+c}{\sqrt[3]{abc}}}$$

Pham Huu Duc, Math. Reflections

57. Let a ,b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{ab+b^2}} + \frac{b}{\sqrt{bc+c^2}} + \frac{c}{\sqrt{ca+a^2}} \geq \frac{3}{\sqrt{2}}$$

Cosmin Pohoata and Michael Rozenberg

58. Let $a_1 \leq a_2 \leq \cdots \leq a_n$ be positive real numbers such that $\dfrac{a_1^2 + a_2^2 + \cdots + a_n^2}{n} = 1$, $\dfrac{a_1 + a_2 + \cdots + a_n}{n} = m$ where $1 \geq m > 0$. Prove that for all i satisfying $a_i \leq m$, we have

$$n - i \geq n(m - a_i)^2$$

Iranian IMO Team Selection Test, 2005

59. Let x ,y, z be positive real numbers. Prove that

$$\frac{3\sqrt{3}}{2} \leq \sqrt{x+y+z}\left(\frac{\sqrt{x}}{y+z} + \frac{\sqrt{y}}{z+x} + \frac{\sqrt{z}}{x+y}\right)$$

Byron Schmuland, Math. Reflections

60. Let a ,b, c be positive real numbers. Prove that

$$\frac{a}{\sqrt{a^2+2bc}} + \frac{b}{\sqrt{b^2+2ca}} + \frac{c}{\sqrt{c^2+2ab}} \leq \frac{a+b+c}{\sqrt{ab+bc+ca}}$$

Ho Phu Thai, Math. Reflections

61. Let a ,b, c be distinct positive real numbers. Prove the following inequality

$$\frac{a^2b + a^2c + b^2a + b^2c + c^2a + c^2b - 6abc}{a^2 + b^2 + c^2 - ab - bc - ca} \geq \frac{16abc}{(a+b+c)^2}$$

Iurie Boreico and Ivan Borsenco, Math. Reflections

62. Let a ,b, c be nonzero positive real numbers. Prove that

$$\frac{a^3+abc}{b+c} + \frac{b^3+abc}{c+a} + \frac{c^3+abc}{a+b} \geq \frac{a(b^3+c^3)}{a^2+bc} + \frac{b(c^3+a^3)}{b^2+ca} + \frac{c(a^3+b^3)}{c^2+ab}$$

Pham Huu Duc and Cosmin Pohoata

63. Let a, b, c, d be real numbers with sum 0. Prove that

$$(ab+ac+ad+bc+bd+cd)^2 + 12 \geq 6(abc+abd+acd+bcd)$$

Kazakhstan, Zhautykov Olympiad, 2006

64. Let a ,b, c be positive real numbers satisfying $a+b+c=1$. Prove that

$$\left(\frac{1}{a}-2\right)^2+\left(\frac{1}{b}-2\right)^2+\left(\frac{1}{c}-2\right)^2 \geq \frac{8(a^2+b^2+c^2)^2}{(1-a)(1-b)(1-c)}$$

Vo Quoc Ba Can, Math. and Youth magazine

65. Let $x_1, x_2, \cdots, x_n$ be real numbers from the interval $[0,1]$ satisfying $x_1x_2\cdots x_n = (1-x_1)^2(1-x_2)^2\cdots(1-x_n)^2$. Find the maximum value of $x_1x_2\cdots x_n$

Chendi Huang

66. Let a, b, c be three positive real numbers with sum 3. Prove that

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} \geq a^2+b^2+c^2$$

Romanian IMO Team Selection Tests, 2006

67. Let a ,b, c be positive real numbers satisfying $a+b+c=1$. Prove that

$$\frac{a}{2b+1}+\frac{b}{2c+1}+\frac{c}{2a+1} \leq \frac{1}{abc}$$

Vo Quoc Ba Can

68. For any three positive numbers a, b, c, prove the inequality

$$(1+abc)\left(\frac{1}{a(1+b)}+\frac{1}{b(1+c)}+\frac{1}{c(1+a)}\right) \geq 3$$

BMO, 2006

69. Let a, b, c, d be real numbers such that $a^2+b^2+c^2+d^2=1$. Prove that

$$\frac{1}{1-ab}+\frac{1}{1-bc}+\frac{1}{1-ca}+\frac{1}{1-da} \leq \frac{16}{3}$$

Vo Quoc Ba Can

70. Let $x_1, x_2, \cdots, x_n$ be positive real numbers such that $x_1+x_2+\cdots+x_n=1$. Prove that

$$\left(\sum_{i=1}^{n}\sqrt{x_i}\right)\left(\sum_{i=1}^{n}\frac{1}{1+\sqrt{x_i}}\right) \leq \frac{n^2}{\sqrt{n+1}}$$

Chinese TST, 2006

71. Let a ,b, c be positive real numbers. Prove that

$$\frac{a^4+b^4+c^4}{ab+bc+ca}+\frac{3abc}{a+b+c} \geq \frac{2}{3}\left(a^2+b^2+c^2\right)$$

Vo Quoc Ba Can, Cosmin Pohoata

72. Let a ,b, c be nonnegative real numbers, from which at least two are nonzero. Prove that

$$\sqrt[3]{\frac{a^2+bc}{b^+c^2}}+\sqrt[3]{\frac{b^2+ca}{c^+a^2}}+\sqrt[3]{\frac{c^2+ab}{a^+b^2}}\geq\frac{\sqrt[3]{9abc}}{a+b+c}$$

Pham Huu Duc, Math. Reflections

73. Let $a_1, a_2, \cdots, a_{100}$ be nonnegative real numbers such that $a_1^2+a_2^2+\cdots+a_{100}^2=1$. Prove that

$$a_1^2a_2+a_2^2a_3+\cdots+a_{100}^2a_1<\frac{12}{25}$$

IMO Shortlist, 2007, Proposed by Poland

74. Let a ,b, c be nonnegative real numbers, no two of which are zero. Prove that

$$\frac{ab+ac+4bc}{b^2+c^2}+\frac{bc+ba+4ca}{c^2+a^2}+\frac{ca+cb+4ab}{a^2+b^2}\geq 4$$

Vasile Cirtoaje

75. Let a ,b, c be positive real numbers. Prove that

$$\frac{a+b^2+c^3}{ab+c^2}+\frac{b+c^2+a^3}{bc+a^2}+\frac{c+a^2+b^3}{ca+b^2}\geq\frac{9}{2}$$

Vo Quoc Ba Can

76. Let a ,b and c be positive real numbers satisfying $a+b+c=2$. Prove that

$$\frac{1}{2}+\sum_{cyc}\frac{a}{b+c}\leq\frac{a^2+bc}{b+c}+\frac{b^2+ca}{c+a}+\frac{c^2+ab}{a+b}\leq\frac{1}{2}+\sum_{cyc}\frac{a^2}{b^2+c^2}$$

Cosmin Pohoata

77. Real number a_i, b_i $(1\leq i\leq n)$ satisfy $\sum\limits_{i=1}^n a_i^2=\sum\limits_{i=1}^n b_i^2=1$ and $\sum\limits_{i=1}^n a_ib_i=0$. Prove that

$$\left(\sum_{i=1}^n a_i\right)^2+\left(\sum_{i=1}^n b_i\right)^2\leq n$$

BMO Shortlist, 2007, Proposed by Romania

78. Let a ,b and c be positive real numbers satisfying $a+b+c=1$. Prove that

$$\frac{ab}{\sqrt{ab+bc}}+\frac{bc}{\sqrt{bc+ca}}+\frac{ca}{\sqrt{ca+ab}}\geq\frac{\sqrt{2}}{2}$$

Chinese MO, 2006

79. Let a ,b and c be nonnegative real numbers satisfying $a^2+b^2+c^2=1$. Prove that

$$1\leq\frac{a}{1+bc}+\frac{b}{1+ca}+\frac{c}{1+ab}\leq\sqrt{2}$$

Faruk Zejnulahi, Crux Math.

80. Let a ,b and c be positive real numbers such that $a \le b \le c$ and $abc = 1$. Prove that

$$a + b^2 + c^3 \ge \frac{1}{a} + \frac{1}{b^2} + \frac{1}{c^3}$$

Vo Quoc Ba Can

81. Given $k+1$ positive real numbers $x_0, x_1, x_2, \cdots, x_k$ and a positive integer n. Show that

$$(x_{\sigma_1} + x_{\sigma_2} + \cdots + x_{\sigma_k})^{-n} \le k^{-n} \sum_{i=0}^{k} x_i^{-n}$$

where the sum on the left is taken of the $k+1$ distinct $k-$element subsets of $\{x_0, x_1, \cdots, x_k\}$

Emre Alkan, Amer. Math. Monthly

82. Let a, b, c be nonnegative real numbers, such that at least two are nonzero and which satisfy the condition $a+b+c = 1$. Prove that

$$\frac{a}{\sqrt{a+2b}} + \frac{b}{\sqrt{b+2c}} + \frac{c}{\sqrt{c+2a}} \le \frac{\sqrt[4]{27}\left(\sqrt{3}-1\right)}{\sqrt{2}}$$

Vo Quoc Ba Can, Pham Kim Hung

83. For Real numbers $x_i > 1$, $1 \le i \le n$, $n \ge 2$ such that

$$\frac{x_i^2}{x_i - 1} \ge S = \sum_{j=1}^{n} x_j, \text{ for all } i = 1, 2, \cdots, n$$

find with proof $sup\ S$

Romanian IMO Team Preparation Tests, 2008

84. Let a ,b, c, d be positive real numbers such that $a+b+c+d = abc+bcd+cda+dab$. Prove that

$$\left(\sqrt{a^2+1} + \sqrt{b^2+1}\right)^2 + \left(\sqrt{c^2+1} + \sqrt{d^2+1}\right)^2 \le (a+b+c+d)^2$$

Vo Quoc Ba Can

85. Let a, b, c be the sidelengths of a triangle. Prove that

$$a^2\left(\frac{b}{c} - 1\right) + b^2\left(\frac{c}{a} - 1\right) + c^2\left(\frac{a}{b} - 1\right) \ge 0$$

Moldavian IMO Team Selection Test, 2006

86. Let x ,y, z be positive real numbers such that $x^2 + y^2 + z^2 \ge 3$. Prove that

$$\frac{x^5 - x^2}{x^5 + y^2 + z^2} + \frac{y^5 - y^2}{y^5 + z^2 + x^2} + \frac{z^5 - z^2}{z^5 + x^2 + y^2} \ge 0$$

Vasile Cirtoaje

87. Let a, b, c be the sidelengths of a triangle. Prove that

$$\frac{a}{b}+\frac{b}{c}+\frac{c}{a}\geq 2\left(\frac{a+b}{b+c}+\frac{b+c}{c+a}+\frac{c+a}{a+b}\right)$$

Vo Quoc Ba Can

88. Let x ,y, z be positive real numbers. Prove that

$$\left(\frac{x+y}{z}+\frac{y+z}{x}+\frac{z+x}{y}\right)-4\left(\frac{x}{y+z}+\frac{y}{z+x}+\frac{z}{x+y}\right)\geq 1-\frac{8xyz}{(x+y)(y+z)(z+x)}$$

Cezar Lupu and Cosmin Pohoata, La Gaceta de la RSME

89. Let x ,y, z be real numbers satisfying $a^2+b^2+c^2=9$. Prove that

$$3\times min\{a,b,c\}\leq 1+abc$$

Virgil Nicula, Crux Mathematicorum

90. Let $x_1,x_2,\cdots,x_{3n}$ be positive real numbers. Prove that

$$2^n\prod_{k=1}^{3n}\frac{1+x_k^2}{1+x_k}\geq\left(1+\prod_{k=1}^{3n}x_k^{\frac{1}{n}}\right)^n$$

Mihaly Bencze, Math. Magazine

91. Given an integer $n\geq 2$, find the largest constant $C(n)$ for which the inequality

$$\sum_{i=1}^{n}x_i\geq C(n)\sum_{1\leq j<i\leq n}2x_ix_j+\sqrt{x_ix_j}$$

holds for all real numbers $x_i\in(0,1)$ satisfying $(1-x_i)(1-x_j)\leq\frac{1}{4}$ for $1\leq j<i\leq n$

Bulgarian IMO Team Selection Tests, 2007

92. Let a, b, c be nonnegative real numbers, from which at least two are nonzero and satisfying the condition $ab+bc+ca=1$. Prove that

$$\sqrt{a^3+a}+\sqrt{b^3+b}+\sqrt{c^3+c}\geq 2\sqrt{a+b+c}$$

Iran TST, 2008

93. Let a, b, c , d be nonnegative real numbers such that $a+b+c+d=3$. Prove that

$$ab(a+2b+3c)+bc(b+2c+3d)+cd(c+2d+3a)+da(d+2a+3b)\leq 6\sqrt{3}$$

Pham Kim Hung

94. For $n\in\mathbb{N}$, $n\geq 2$, determine

$$max\prod_{i=1}^{n}(1-x_i)\,,\ \text{for}\ \ x_i\in\mathbb{R}_+,\ 1\leq i\leq n,\ \sum_{i=1}^{n}x_i^2=1$$

Romanian IMO Team Selection Tests, 2007

95. For integers $n \geq 2$ and real $s > 0$, show that

$$\left(\prod_{i=0}^{n}(s+i)\right)\left(\prod_{j=0}^{n}\frac{1}{s+j}\right) < (n+1)\prod_{k=1}^{n}\left(s+k-\frac{1}{2}\right)$$

Robert E. Shafer, Amer. Math. Monthly

96. Let a, b, c be real numbers contained in the interval $\left[0, \frac{3}{5}\right]$ and also satisfying the condition $a+b+c=1$. Determine the maximum value that the following expression can reach:

$$P(a,b,c) = a^3+b^3+c^3+\frac{3}{4}abc$$

Vo Quoc Ba Can, After Chinese Northern MO, 2007

97. Let a, b, c, d be positive real numbers such that $a \geq b \geq c \geq d$ and $abcd = 1$. Prove that

$$\frac{1}{1+a^3}+\frac{1}{1+b^3}+\frac{1}{1+c^3} \geq \frac{3}{1+abc}$$

MathLinks Contest, 2008

98. Let a, b, c be nonnegative real numbers, from which at least are nonzero. Prove that

$$\frac{a^2(b+c)^2}{b^2+c^2}+\frac{b^2(c+a)^2}{c^2+a^2}+\frac{c^2(a+b)^2}{a^2+b^2} \geq 2(ab+bc+ca)$$

Vo Quoc Ba Can, Cosmin Pohoata

99. Let a, b, c be nonnegative real numbers, from which at least two are nonzero. Prove that

$$\frac{(a+b+c)^2}{ab+bc+ca} \geq \frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab} \geq \frac{(a+b+c)^2}{a^2+b^2+c^2}$$

Tran Quoc Anh

100. Let a, b, c be positive real numbers. Prove that

$$\frac{a^2-bc}{4a^2+4b^2+c^2}+\frac{b^2-ca}{4b^2+4c^2+a^2}+\frac{c^2-ab}{4c^2+4a^2+b^2} \geq 0$$

Vasile Cirtoaje, Math. Reflections

101. Let a, b, c be nonnegative real numbers, from which at least two are nonzero. Prove that

$$\frac{a(b+c)}{b^2+c^2}+\frac{b(c+a)}{c^2+a^2}+\frac{c(a+b)}{a^2+b^2} \geq \frac{a(b+c)}{a^2+bc}+\frac{b(c+a)}{b^2+ca}+\frac{c(a+b)}{c^2+ab}$$

Vo Quoc Ba Can, Cosmin Pohoata

102. Let n be a positive integer. Find the minimum value of

$$\frac{(a-b)^{2n+1}+(b-c)^{2n+1}+(c-a)^{2n+1}}{(a-b)(b-c)(c-a)}$$

for distinct real numbers a, b, c with $bc+ca \geq 1+ab+c^2$.

Michel Bataille, Math. Magazine

103. Let a, b, c be positive real numbers. Prove that

$$\frac{a^{b+c}}{(b+c)^2} + \frac{b^{c+a}}{(c+a)^2} + \frac{c^{a+b}}{(a+b)^2} \geq \frac{3}{4}$$

Vo Quoc Ba Can, Math. and Youth magazine

104. Find the least real number c such that if $n \geq 1$ and $a_1, a_2, \cdots, a_n > 0$ then

$$\sum_{k=1}^{n} \frac{k}{\sum\limits_{j=1}^{k} \frac{1}{a_j}} \leq c \sum_{k=1}^{n} a_k$$

Joel Zinn, American Math. Monthly

105. Let a, b, c be nonnegative real numbers satisfying $a+b+c=1$, and moreover from which at least two are nonzero. Prove that

$$a\sqrt{4b^2+c^2} + b\sqrt{4c^2+a^2} + c\sqrt{4a^2+b^2} \leq \frac{3}{4}$$

Vo Quoc Ba Can

106. Let a, b be real numbers such that $a+b \neq 0$ and let $x, y > 1$ be some given constants. Determine the minimum value of the following expression:

$$f(a,b) = \frac{\left(a^2+1\right)^x \left(b^2+1\right)^y}{(a+b)^2}$$

107. It is given that real numbers $x_1, x_2, \cdots, x_n$ $(n > 2)$ satisfy $\left|\sum\limits_{i=1}^{n} x_i\right| > 1,\ |x_i| \leq 1,\ i = 1, 2, \cdots, n$ Prove that there exists a positive integer k such that

$$\left|\sum_{i=1}^{k} x_i - \sum_{i=k+1}^{n} x_i\right| \leq 1$$

Chinese Western MO, 2005

108. Let x, y, z be real numbers with sum 0, which are contained in the interval $[-1, 1]$. Prove that

$$\sqrt{1+x+y^2} + \sqrt{1+y+z^2} + \sqrt{1+z+x^2} \geq 3$$

Phan Thanh Nam

109. 109 Let a_i be nonzero positive real numbers, for all $i = 1, 2, \cdots, n$ satisfying $a_1 + a_2 + \cdots + a_n = \frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}$. Prove that

$$\frac{1}{n+1-a_1} + \frac{1}{n+1-a_2} + \cdots + \frac{1}{n+1-a_n} \leq 1$$

Vasile Cirtoaje, Amer. Math. Monthly

110. Let a, b, c, d be real numbers contained in the interval $(0, k]$. Prove that

$$\frac{a^4+b^4+c^4+d^4}{abcd} \geq \frac{(2k-a)^4+(2k-b)^4+(2k-c)^4+(2k-d)^4}{(2k-a)(2k-b)(2k-c)(2k-d)}$$

Taiwanese MO, 2002

111. Let a, b, c be positive real numbers. Prove that

$$\sqrt{\frac{a}{a+b}}+\sqrt{\frac{b}{b+c}}+\sqrt{\frac{c}{c+a}} \geq \frac{3}{\sqrt{2}}\sqrt{\frac{ab+bc+ca}{a^2+b^2+c^2}}$$

Nguyen Van Thach

112. Let a, b, c be nonnegative real numbers, from which at least two are nonzero.

$$\frac{a^3}{a^2+b^2}+\frac{b^3}{b^2+c^2}+\frac{c^3}{c^2+a^2} \geq \frac{\sqrt{3\left(a^2+b^2+c^2\right)}}{2}$$

Vo Quoc Ba Can

113. Let a, b, c, d be nonnegative reals. Prove that

$$\sum_{cyc}\left(\frac{a}{a+b+c}\right)^k \geq \left\{1, \frac{1}{2^{k-1}}, \frac{4}{3^k}\right\}$$

for any nonnegative real number k.

Vo Quoc Ba Can

114. Let a, b, c be positive real numbers such that $abc = 1$. Prove that for all (strictly) positive k we have

$$\frac{1}{1+a+b^k}+\frac{1}{1+b+c^k}+\frac{1}{1+c+a^k} \leq 1$$

Vasile Cirtoaje

115. Let a, b, c be nonnegative real numbers, from which at least two are nonzero. Prove that

$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+a}} \leq \frac{5}{4}\sqrt{a+b+c}$$

Jack Garfunkel, Crux Mathematicorum

116. Let a, b, c, d be nonnegative real numbers, from which at least two are nonzero, and which also satisfy the condition that $a+b+c+d = 1$. Show that

$$\frac{a}{\sqrt{a+b}}+\frac{b}{\sqrt{b+c}}+\frac{c}{\sqrt{c+d}}+\frac{d}{\sqrt{d+a}} \leq \frac{3}{2}$$

Mircea Lascu

117. Let a, b, c, d be positive real numbers, from which at least three are nonzero. Prove that

$$\frac{a}{\sqrt{a+b+c}}+\frac{b}{\sqrt{b+c+d}}+\frac{c}{\sqrt{c+d+a}}+\frac{d}{\sqrt{d+a+b}} \leq \frac{5}{4}\sqrt{a+b+c+d}$$

Vo Quoc Ba Can

118. Let $a_1, a_2, \cdots, a_n$ be real numbers satisfying $a_1^2+a_2^2+\cdots+a_n^2 = 1$ Determine the maximum value of the following expression:

$$\min_{i\neq j}|a_i-a_j|$$

Nguyen Kim Cuong, Math. and Youth Magazine

119. Let $a_2, a_3, \cdots, a_n$ be positive real numbers and $s = a_2 + a_3 + \cdots + a_n$. Show that

$$\sum_{k=2}^{n} a_k^{1-\frac{1}{k}} < s + 2\sqrt{s}$$

George Tsintsifas, Amer. Math. Monthly