Algebra 2 Week 6 – Artin 5.1, 5.2, 5.6 Chapter 8

Name: Soham Chatterjee **Roll:** BMC202175

5.1. (a) Since its an Euclidean Space the form is positive definite and symmetric. Hence $\forall\ \lambda \in \mathbb{R}$, $\langle v - \lambda w, v - \lambda w\rangle \geq 0$. Hence

$$\langle v - \lambda w, v - \lambda w\rangle = \langle v, v\rangle - \lambda\langle v, w\rangle + \lambda^2\langle w\rangle$$

Now $\langle v, v\rangle - \lambda\langle v, w\rangle + 2\lambda^2\langle w, w\rangle \geq 0$ is a quadratic equation of λ where it gives positive value when $v \neq \lambda w$ and 0 when $v = \lambda w$. Therefore the discriminant must be negative or zero. Hence

$$\begin{aligned} &4\langle v, w\rangle^2 - 4\langle v, v\rangle\langle w, w\rangle \leq 0 \\ \iff &\langle v, w\rangle^2 \leq \langle v, v\rangle\langle w, w\rangle \\ \iff &|\langle v, w\rangle| \leq |v||w| \qquad \text{[Proved]} \end{aligned}$$

(b) We have for any $v \in V$ $|v|^2 = \langle v, v\rangle$. Now

$$\begin{aligned} |v + w|^2 + |v - w|^2 &= \langle v + w, v + w\rangle + \langle v - w, v - w\rangle \\ &= [\langle v, v\rangle + 2\langle v, w\rangle + \langle w, w\rangle] + [\langle v, v\rangle - 2\langle v, w\rangle + \langle w, w\rangle] \\ &= 2\langle v, v\rangle + 2\langle w, w\rangle \\ &= 2|v|^2 + 2|w|^2 \qquad \text{[Proved]} \end{aligned}$$

(c) Given that $|v| = |w| \implies |v|^2 = |w|^2 \implies \langle v, v\rangle = \langle w, w\rangle$. To prove $(v + w) \perp (v - w)$ if we show that $\langle v + w, v - w\rangle = 0$ we are done. Now

$$\begin{aligned} \langle v + w, v - w\rangle &= \langle v, v\rangle + \langle v, w\rangle - \langle w, v\rangle - \langle w, w\rangle \\ &= \langle v, v\rangle - \langle w, w\rangle \\ &= 0 \qquad \text{[Proved]} \end{aligned}$$

5.2. $W^\perp = \{v \mid v \in V,\ \langle v, w\rangle = 0\ \forall\ w \in W\}$. Hence $W^{\perp\perp} = \{w' \mid w' \in V,\ \langle w', v\rangle = 0\ \forall\ v \in W^\perp\}\}$. Let $w \in W$ then $\langle w, v\rangle = 0\ \forall v \in W^\perp$. Hence $w \in W^{\perp\perp}$. Therefore $W \subseteq W^{\perp\perp}$. Now we know that $V = W \oplus W = W^\perp \oplus W^{\perp\perp}$ Hence $\dim W^\perp = \dim W^{\perp\perp}$. We can say then $W = W^{\perp\perp}$ [Proved]

5.6. Let $\lambda \in \mathbb{C}$ be an eigen value of the unitary matrix A. Hence $AX = \lambda X$ for some column vector X. Now

$$(PX)^*(PX) = X^*P^*PX = X^*X \qquad (PX)^*(PX) = (\lambda X)^*(\lambda X) = \overline{\lambda}X^*\lambda X = \overline{\lambda}\lambda X^*X$$

Hence $X^*X = \overline{\lambda}\lambda X^*X \implies \overline{\lambda}\lambda = 1 \implies |\lambda|^2 = 1$. Hence the complex numbers with unit modulous 1 will appear as eigenvalues of a unitary matrix.