Fulton Chapter 3: Local Properties of Plane Curves

Intersection Numbers

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Problem Set - 6
Topic: Algebraic Geometry

Problem 1 3.18

If P is a simple point on F, then $I(P, F \cap G) = \mathrm{ord}_P^F(G)$. Give a proof of this using properties (1)-(7).

Solution: **Case 1:** F is irreducible. As P is a simple point of F so $O_P(F)$ is a D.V.R.and the uniformizing parameter of $M_P(F)$ is a line passes through P but not the tangent at P. Let the line is L. Let $O_P^F(G) = n$. so,

$$g = G + (F) = L^n u \quad [u \text{ is an unit in } O_P(F)]$$

So, by property 7 $I(P, F \cap G) = I(P, F \cap g)$. By property 6,

$$I(P, F \cap g) = I(P, F \cap L^n u) = nI(P, F \cap L) + I(P, F \cap u)$$

By property 2,

$$I(P, F \cap u) = 0[\text{as } u \text{ is an unit in } O_P(F) \implies u(P) \neq 0]$$

Now tangent of L at P is L. So, F and L do not share their tangents at P. By property 5, $I(P, F \cap L) = 1$[as P is a simple point of $F \implies m_P(F) = 1$]. So,

$$I(P, F \cap G) = I(P, F \cap g) = n = O_P^F(G)$$

Case 2: F is reducible. Let $F = \Pi_{i=1}^n F_i^{a_i}$. P is a simple point of $F \implies m_P(F) = \sum_{i=1}^n a_i m_P(F_i) = 1 \implies$ for some $i, m_P(F_i) = 1; a_i = 1; m_P(F_j) = 0 \forall i \neq j \implies F_i$ is the only irreducible component passes through P. So,

$$O_P^F(G) = O_P^{F_i}(G) = I(P, F_i \cap G) \quad [\text{by case 1}]$$

Now

$$I(P, F \cap G) = \sum_{j=1}^n a_j I(P, F_j \cap G) = I(P, F_i \cap G) \quad [\text{as } \forall\, i \neq j F_j \text{ does not pass through } P \text{ and } a_i = 1]$$

□

Problem 2 3.20

If P is a simple point on F, then $I(P, F \cap (G+H)) \geq \min(I(P, F \cap G), I(P, F \cap H))$. Give an example to show that this may be false if P is not simple on F.

Solution: Let $I(P, F \cap G) = m; I(P, F \cap H) = n$
by the previous problem we know that $m = O_P^F(G); n = O_P^F(H)$
let L be a line which passes through P but not the tangent of F at P
so, $g = L^m u_1; h = L^n u_2$
WLOG $m \geq n$
so, $g + h = L^n(L^{m-n}u_1 + u_2)$
so, $O_P^F(G+H) \geq n$
let $P = (0,0); F = x^2 + y^2; H = x - y; G = x + y$
clearly $I(P, F \cap G) = I(P, F \cap H) = \infty$
but $G + H = 2x$ and $I(P, 2x \cap x^2 - y^2) = I(P, x \cap x^2 - y^2) = 2I(P, x \cap y) = 2$
so the proposition will be failed if P is not a simple point of F.

□

Problem 3 3.21

Let F be an affine plane curve. Let L be a line that is not a component of F. Suppose $L = \{(a+tb, c+td) \mid t \in k\}$. Define $G(T) = F(a+Tb, c+Td)$. Factor $G(T) = \epsilon \prod (T - \lambda_i)^{e_i}, \lambda_i$ distinct. Show that there is a natural one-to-one correspondence between the λ_i and the points $P_i \in L \cap F$. Show that under this correspondence, $I(P_i, L \cap F) = e_i$. In particular, $\sum I(P, L \cap F) \leq \deg(F)$

Solution: Let $P \in L \cap F$. Therefore, $P = (a+kb, c+kd)$ for some $k \in K$.

$$F(a+kb, c+kd) = 0 \implies G(k) = 0 \implies k = \lambda_i$$

for some i. So,

$$P = (a + \lambda_i b, c + \lambda_i d)$$

and for all λ_i, $(a+\lambda_i b, c+\lambda_i d) \in L \cap F$. Ao, there is an one one correspondence between λ_i and P_i and $P_i = (a+\lambda_i b, c+\lambda_i d)$. Now L is not a component of $F \implies I(P_i, L \cap F) = m_{P_i}(L) m_{P_i}(F) = m_P(F)$ [as the tangent at P_i of L_i is L_i]. So,

$$m_{P_i}(F(X,Y)) = m_P(F(X + a + \lambda_i b, Y + c + \lambda_i d)) \quad [\text{where } P = (0,0)]$$

Now either of b, d is non zero [as L is a line]. Let $b \neq 0$. Let $Y = dX/b$.

$$F(X + a + \lambda_i b, dX/b + c + \lambda_i d) = F(a + b(\lambda_i + X/b), c + d(\lambda_i + X/b)) = G(\lambda_i + X/b)$$

Now the lowest degree of X in $G(X/b + \lambda_i) = m_{P_i}(F)$ [as in the least degree homogeneous term if we put $Y = dX/b$ then the degree will be same]. Now

$$G(X/b + \lambda_i) = (X/b)^{e_i} \prod_{i \neq j} (X/b + \lambda_i - \lambda_j)^{e_j}$$

So, the least degree is $e_i \implies m_{P_i}(F) = e_i$ [as $\lambda_i \neq \lambda_j \forall i \neq j$]. So $\sum_i I(P_i, F \cap L_i) \leq \deg(F)$ [as $\deg(G) \leq \deg(F)$]

$\square$

Problem 4 3.23

A point P on a curve F is called a hypercusp if $m_P(F) > 1, F$ has only one tangent line L at P, and $I(P, L \cap F) = m_P(F) + 1$. Generalize the results of the preceding problem to this case.

Solution: Suppose $P = (0,0), L = Y.P$ is a hypercusp if and only if $\frac{\partial F}{\partial^n X}(P) \neq 0$ where $n = m_P(F) + 1$. Let $F = YG + H(X)$ clearly $H(0) = 0$[as $F(0,0) = 0$]. Now $F = Y^{n-1} + F_1$ where $m_P(F_1) \geq n$[as Y is the only tangent at P]. So,$H(x) = X^k(H_1(X)$ where $H_1(0) \neq 0$ and $k \geq n$. $\frac{\partial F}{\partial^n X}(P) \neq 0 \iff$ the coefficient of X^n is non zero. Now $I(P, F \cap Y) = n \iff I(P, Y \cap H(X)) = n \iff I(P, Y \cap X^k) = n \iff k = n \iff$ the coefficient of X^n is non zero. [as $H_1(0) \neq 0$]

<u>2nd Part</u>:

I will show that F has only one irreducible component passing through P. Let assume $P = (0,0)$. Let $F = \Pi_{i=1}^n F_i^{a_i}$ where F_i's are irreducible .

WLOG assume that $F_1, F_2, \ldots, F_k$ passes through P. Let L be the tangent of F at P So, L be the only tangent of F_i at P [as if there is a tangent other than L then it will be a tangent of F as well because the least degree form of F is the product of least degree form of F_i]. So,
$I(P, F \cap L) = \sum_{i=1}^k a_i I(P, F_i \cap L)$
$I(P, F_i \cap L) > m_P(F_i) m_P(L) \implies I(P, F_i \cap L) \geq b_i + 1$[where $b_i = m_P(F_i)$]
$I(P, F \cap L) = \sum_{i=1}^k a_i b_i$
$I(P, F \cap L) = m_P(F) + 1 = \sum_{i=1}^k a_i b_i + 1 \geq \sum_{i=1}^k a_i(b_i + 1)$
so, $\sum_{i=1}^k a_i \leq 1$
but as F passes through $P \implies$ at least one $a_i > 0 \implies \sum_{i=1}^k a_i \geq 1$
so, $\sum_{i=1}^k a_i = 1 \implies a_j = 1; a_i = 0 \forall i \neq j$
so, F has only one irreducible component passing through P

$\square$

Problem 5 3.24

The object of this problem is to find a property of the local ring $O_P(F)$ that determines whether or not P is an ordinary multiple point on F.
Let F be an irreducible plane curve, $P = (0,0), m = m_P(F) > 1$. Let $\mathfrak{m} = \mathfrak{m}_P(F)$. For $G \in k[X,Y]$, denote its residue in $\Gamma(F)$ by g; and for $g \in \mathfrak{m}$, denote its residue in $\mathfrak{m}/\mathfrak{m}^2$ by $\bar{g}$. (a) Show that the map from { forms of degree 1 in $k[X,Y]$} to $\mathfrak{m}/\mathfrak{m}^2$ taking $aX + bY$ to $\overline{ax + by}$ is an isomorphism of vector spaces (see Problem 3.13). (b) Suppose P is an ordinary multiple point, with tangents $L_1, \ldots, L_m$. Show that $I(P, F \cap L_i) > m$ and $\bar{l_i} \neq \lambda \bar{l_j}$ for all $i \neq j$, all $\lambda \in k$. (c) Suppose there are $G_1, \ldots, G_m \in k[X,Y]$ such that $I(P, F \cap G_i) > m$ and

> $\bar{g}_i \neq \lambda \bar{g}_j$ for all $i \neq j$, and all $\lambda \in k$. Show that P is an ordinary multiple point on F. (Hint:: Write $G_i = L_i +$ higher terms. $\bar{l}_i = \bar{g}_i \neq 0$, and L_i is the tangent to G_i, so L_i is tangent to F by Property (5) of intersection numbers. Thus F has m tangents at P.) (d) Show that P is an ordinary multiple point on F if and only if there are $g_1, \ldots, g_m \in \mathfrak{m}$ such that $\bar{g}_i \neq \lambda \bar{g}_j$ for all $i \neq j, \lambda \in k$, and $\dim O_P(F)/(g_i) > m$

Solution: Clearly $M_P(F) = M = (x, y)$ where $x = X + (F), y = Y + (F)$ and both are non zero [as $m_P(F) > 1$]
(a)
let f :forms of degree 1 in $k[X,Y]$=V $\to M/M^2$
s.t. $f(aX + bY) = \overline{ax + by}$
clearly V is a vector space of dimension 2 with bases X, Y and F is a homomorphism of two k-vector space.
let $aX + bY \in ker(f)$
so, $ax + by \in M^2 \implies aX + bY + G \in (F)$ where $m_p(G) > 1$
$\implies m_p(F) = 1$ which is not possible.
so, $a = b = 0 \implies f$ is injective .
by problem 3.13 M/M^2 is a vector space of dimension 2
so, f is an isomorphism of vector space [by rank nullity theorem].
(b)
$I(P, F \cap L_i) > m_P(F)m_P(L_i) = m$[as F and L_i shares tangent at P]
now if $\overline{l_i} = \lambda \overline{l_j} \implies l_i - \lambda l_j \in M^2 \implies L_i - L_j + G \in (F)$
but $M_P(F) > 1 \implies L_i = \lambda L_j$
which is not possible as L_i are distinct tangents at P
(c)
$\overline{g_i} \neq 0 \implies m_P(G_i) \leq 1$
if $m_P(G_i) = 0 \implies I(P, F \cap G) = 0$ which is not possible.
so, $m_P(G_i) = 1 \implies G_i = L_i + H_i, m_P(H_i) > 1$
so, G_i has only i tangent L_i at P and $I(P, F \cap G_i) > m_P(F)m_P(G_i) \implies F and G_i$ share tangent at P
as $\overline{g_i} \neq \lambda \overline{g_j} \implies L_i, L_j$ are distinct [as $\overline{g_i} = \overline{l_i}$]
so, F has m distinct tangents at P and $m_P(F) = m \implies P$ is an ordinary point.
(d)
$dim_k O_P(F)/(g_i) = dim_k O_P(A^2)/(F, G) = I(P, F \cap G)$
so, by (c) if $g_1, g_2..g_m \in M$s.t. $\overline{g_i} \neq \overline{g_j} \forall i \neq j, \lambda \in kk$ and $dim_k O_P(F)/(g_i) = I(P, F \cap G_i) > m \implies P$ is an ordinary point.
if P is an ordinary point take $G_i = L_i$where L_i's are distinct tangent at $P \implies \overline{l_i} \neq \lambda \overline{l_j} \forall i \neq j, \lambda \in k$ and $dim_k O_P(F)/(l_i) > m$

$\square$