Fulton Chapter 3: Local Properties of Plane Curves

Intersection Numbers

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Problem Set - 1
Topic: Algebraic Geometry

Problem 1 Property 6

If $F = \prod F_i^{r_i}$, and $G = \prod G_j^{s_j}$, then $I(P, F \cap G) = \sum_{i,j} r_i s_j I(P, F_i \cap G_j)$

Solution: It is enough to show that

$$I(P, F \cap GH) = I(P, F \cap G) + I(P, F \cap H)$$

if one of $I(P, F \cap G)$ or $I(P, F \cap H)$ is infinite then F has a common component with G or H. So, F has a common component with GH so, $I(P, F \cap GH) = \infty$

Assume that both of $I(P, F \cap G)$ or $I(P, F \cap H)$ is finite.

$$F, GH \in (F, G) \implies (F, GH) \subseteq (F, G) \qquad O = O_P(A^2)$$

so, the natural map ϕ from $O/(F, GH)$ to $O/(F, G)$ is onto $\psi : O/(F, H) \to O/(F, GH)$ s.t. $\psi(\overline{z}) = \overline{Gz}$. So clearly ψ is a $k-$ linear map.

Now

$$\forall\ \overline{z} \in O/(F, H) \phi(\psi(\overline{z})) = \phi(\overline{Gz}) = 0$$

in $O/(F, G)$ [As $Gz \in (G) \subseteq (F, G)$]. So, $\text{im}(\psi) \subseteq ker(\phi)$

If

$$\overline{z} \in O/(F, GH) \in ker(\phi) \implies z \in (F, G) \implies z = aF + bG$$

so, $\psi(\overline{b}) = \overline{bG} = \overline{z}$ so, $\text{im}(\psi) = ker(\phi)$. Let $\psi(\overline{z}) = 0$. So,

$$Gz \in (F, GH) \implies Gz = a_1 F + a_2 GH$$

Now $\exists\ D \in k[x, y]$ s.t.

$$Dz = c; Da_1 = b_1; Da_2 = b_2$$

where $c, b_1, b_2 \in k[x, y]$ and $D(P) \neq 0$. So,

$$Gc = b_1F + b_2GH \implies G(b_2H - c) = b_1F \implies F|(b_2H - c)$$
$$\implies b_2H - c = Fr \quad \text{[As } F, G \text{ has no common factor]}$$

So, $z = a_2H - (r/D)F$ as $D(P) \neq 0 \implies D$ is an unit in O. So, $z \in (F, H)$. So, ψ is one-one.

$$0 \longrightarrow O/(F, H) \xrightarrow{\psi} O/(F, GH) \xrightarrow{\phi} O/(F, G) \longrightarrow 0$$

this sequence is exact. So,

$$dim_k(O/(F, GH) = dim_k(O/(F, G)) + dim_k(O/(F, H))$$

So,

$$(P, F \cap GH) = I(P, F \cap G) + I(P, F \cap H)$$

$\square$