Let k be fixed. Consider tossing an unbiased coin until you get to see either $k+1$ Heads or $k+1$ Tails. It is evident, by the pigeonhole, that this experiment will last for at most $2k+1$ rounds (and also at least $k+1$) rounds.

For any $k+1+\leq i \leq 2k+1$, let P_i denotes the probability that the experiment ends at round i. Then, it is the case that either (a) round i is the first time we see exactly $k+1$ Heads or (b) it is the first time we see exactly $k+1$ Tails. Both events are equiprobable with probability. Suppose we finally get heads. Then the remaining k heads arrange in $i-1$ tosses in $\binom{i-1}{k}$ ways. Therefore

$$P_i = \binom{i-1}{k} 2^{-i} \times 2 = \binom{i-1}{k} 2^{-i+1}$$

Now we know $\sum\limits_{i=k+1}^{2k+1} P_i = 1$ and notice

$$\sum_{i=k+1}^{2k+1} P_i = \sum_{i=k+1}^{2k+1} \binom{i-1}{k} 2^{-(i-1)} = 2^{-2k} \sum_{i=0}^{k} \binom{k+i}{i} 2^{k-i} = 1$$

Hence the given sum is equal to $2^{2k} = 4^k$. [Proved