# Analysis 2 Lecture Notes – Upendra Kulkarni

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# Contents

# Chapter 1

# Normed Linear Space

> **Definition 1.0.1: Limit of Sequence in $\mathbb{R}$**
>
> Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We say
> $$\lim_{n \to \infty} s_n = s$$
> where $s \in \mathbb{R}$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number $N$ such that for $n > N$
> $$s - \epsilon < s_n < s + \epsilon \text{ i.e. } |s - s_n| < \epsilon$$

Want to generalize this to a sequence in $\mathbb{R}^n$ i.e. $s_n \in \mathbb{R}^n$ $\forall$ $n \in \mathbb{N}$. Now the $s - s_n$ makes no sense. So it is useful to have a notion of whether vectors are big or small. We have magnitude of a vector. So lets revisit this

> **Definition 1.0.2: Limit of Sequence in $\mathbb{R}^n$**
>
> Let $\{s_n\}$ be a sequence in $\mathbb{R}^n$. We say
> $$\lim_{n \to \infty} s_n = s$$
> where $s \in \mathbb{R}^n$ if $\forall$ real numbers $\epsilon > 0$ $\exists$ natural number $N$ such that for $n > N$
> $$\|s - s_n\| < \epsilon$$

The same definition works if we interpret $\|v\| = $ length of the vector $v$.

From school, for v=$v_1, v_2, \cdots, v_n$ we had
$$\|v\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}$$
But it will be useful to have a more general notion of length (of which the above will be an example)

## 1.1 Defination

> **Definition 1.1.1: Normed Linear Space and Norm $\|\cdot\|$**
>
> Let $V$ be a vector space over $\mathbb{R}$ (or $\mathbb{C}$). A norm on $V$ is function $\|\cdot\|$ $V \to \mathbb{R}_{\geq 0}$ satisfying
>
> **(1)** $\|x\| = 0 \iff x = 0 \ \forall \ x \in V$
>
> **(2)** $\|\lambda x\| = |\lambda| \|x\| \ \forall \ \lambda \in \mathbb{R}(\text{or } \mathbb{C}), \ x \in V$
>
> **(3)** $\|x + y\| \leq \|x\| + \|y\| \ \forall \ x, y \in V$ (Triangle Inequality/Subadditivity)
>
> And $V$ is called a normed linear space.
> • Same definition works with $V$ a vector space over $\underline{\mathbb{C}}$ (again $\|\cdot\| \to \mathbb{R}_{\geq 0}$) where **(2)** becomes $\|\lambda x\| = |\lambda| \|x\|$
> $\forall \ \lambda \in \mathbb{C}, \ x \in V$, where for $\lambda = a + ib$, $|\lambda| = \sqrt{a^2 + b^2}$

    **Special Case $p = 1$**: $\|x\|_1 = |x_1| + |x_2| + \cdots + |x_m|$ is clearly a norm by usual triangle inequality.
    **Special Case $p \to \infty$ ($\mathbb{R}^m$ with $\|\cdot\|_\infty$)**: $\|x\|_\infty = \max\{|x_1|, |x_2|, \cdots, |x_m|\}$
For $m = 1$ these $p-$norms are nothing but $|x|$. Now exercise

### Question 1

Prove that triangle inequality is true if $p \geq 1$ for $p-$norms. (What goes wrong for $p < 1$ ?)

*Solution:* **For Property ③ for norm-2**

**When field is $\mathbb{R}$ :**

We have to show

$$\sum_i (x_i + y_i)^2 \leq \left( \sqrt{\sum_i x_i^2} + \sqrt{\sum_i y_i^2} \right)^2$$

$$\implies \sum_i (x_i^2 + 2x_i y_i + y_i^2) \leq \sum_i x_i^2 + 2\sqrt{\left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]} + \sum_i y_i^2$$

$$\implies \left[\sum_i x_i y_i\right]^2 \leq \left[\sum_i x_i^2\right]\left[\sum_i y_i^2\right]$$

So in other words prove $\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$ where

$$\langle x, y \rangle = \sum_i x_i y_i$$

> **Note:-**
>
> - $\|x\|^2 = \langle x, x \rangle$
>
> - $\langle x, y \rangle = \langle y, x \rangle$
>
> - $\langle \cdot, \cdot \rangle$ is $\mathbb{R}-$linear in each slot i.e.
>
>   $$\langle rx + x', y \rangle = r\langle x, y \rangle + \langle x', y \rangle \text{ and similarly for second slot}$$
>
>   Here in $\langle x, y \rangle$ $x$ is in first slot and $y$ is in second slot.

    Now the statement is just the Cauchy-Schwartz Inequality. For proof

$$\langle x, y \rangle^2 \leq \langle x, x \rangle \langle y, y \rangle$$

expand everything of $\langle x - \lambda y, x - \lambda y \rangle$ which is going to give a quadratic equation in variable $\lambda$

$$\langle x - \lambda y, x - \lambda y \rangle = \langle x, x - \lambda y \rangle - \lambda \langle y, x - \lambda y \rangle$$
$$= \langle x, x \rangle - \lambda \langle x, y \rangle - \lambda \langle y, x \rangle + \lambda^2 \langle y, y \rangle$$
$$= \langle x, x \rangle - 2\lambda \langle x, y \rangle + \lambda^2 \langle y, y \rangle$$

Now unless $x = \lambda y$ we have $\langle x - \lambda y, x - \lambda y \rangle > 0$ Hence the quadratic equation has no root therefore the discriminant is greater than zero.

**When field is $\mathbb{C}$ :**

Modify the definition by

$$\langle x, y \rangle = \sum_i \overline{x_i} y_i$$

Then we still have $\langle x, x \rangle \geq 0$

## 1.2  Open and Closed Ball

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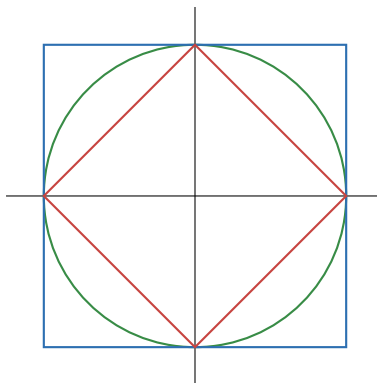**Definition 1.2.1: Open and Closed Ball in Normed Linear Space**

An open Ball of radius $r$ with center $x$ in Normed Linear Space $V$ is the set

$$\{y \in V \mid \|x - y\| < r\} = B_r(x)$$

and Closed ball is the set

$$\{y \in V \mid \|x - y\| \leq r\} = \overline{B_r(x)}$$

---

Now take $B_r(0)$ w.r.t $\| \cdot \|_1$, $\| \cdot \|_2$, $\| \cdot \|_\infty$. Now imagine a sequence converging to origin. So if I



draw a ordinary circle around the origin then no matter how small the circle the points of the sequence are eventually land inside the circle. If instead of that circle can same be said for diamond w.r.t norm 2. Then i can take circle that is inside that diamond. Same is true for $\infty-$norm. Hence convergence with respect to all norm 1 and norm 2 and even $\infty$ results for convergence.

Now there is no reason why we can not consider a norm on an infinite dimensional vector space. It will work. Perhaps i can define only for some sequences where the morm converges.

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**Example 1.2.1**

Suppose for set of all bounded infinite sequences a vector space because every number in a vector is less than some number so if you add two vectors then add the bound and if you scale then scale the bound. Now the $\infty$ norm works on that.

Now suppose you take all continuous real valued functions on closed interval $[0, 1]$, such a function is bounded and this is a vector space and we can define $\infty-$norm even for that because for all $f$ in this space attains its maximum value so just take that maximum value. Its an extremely infinite dimensional space.

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**Note:-**

$\mathbb{R}^\infty$ is the space of all sequences.

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**Question 2**

Modify the above proof for field $\mathbb{C}$

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Show that the following are normed linear spaces.

(a) $l^\infty$ = Set of all bounded infinite sequences $(x_1, x_2, \cdots)$ $x_i \in \mathbb{R}$ with norm $\|x\| = \sup |x_i|$

(b) $C[0,1]$ = Set of all continuous functions $[0,1] \to \mathbb{R}$ with norm $\|f\| = \sup\limits_{x \in [0,1]} |f(x)|$

## 1.3 Limit of a Sequence

**Definition 1.3.1: Limit of Sequence in Normed Linear Space**

A sequence $\{s_n\}$ in a normed linear space $V$ converge to $s$ means $\forall$ real number $\epsilon > 0$ $\exists$ natural number $N$ such that for $\forall$ $n > N$ $\|s - s_n\| < \epsilon$

## 1.4 Continuity

**Definition 1.4.1: Continuity in Normed Linear Space**

Let $S$ be a subset of $V$ and $f : S \to W$ where $V, W$ are normed linear space. $f$ is continuous at $v \in V$ means $\forall$ $\epsilon > 0$, $\exists$ $\delta > 0$, st whenever $\|x - v\| < \delta$ for $x \in S$ one has $\|f(x) - f(v)\| < \epsilon$

Distance in a normed linear space for $x, y \in V$ is

$$d(x, y) = \|x, y\|$$

Hence properties of this $d$ are

① $d(x, y) = 0 \iff x = y$

② $d(\lambda x, \lambda y) = |\lambda| d(x, y)$ for any scalar $\lambda$

③ $d(u, v) + d(u, v) \geq d(u, w)$

# Chapter 2

# Metric Space

## 2.1 Definition

**Definition 2.1.1: Metric Space $X$**

A set $X$ with a function $d \; X \times X \to \mathbb{R}_{\geq 0}$ such that

① $d(x,y) = 0 \iff x = y$

② $d(x,y) = d(y,x)$

③ $d(x,z) \leq d(x,y) + d(y,z)$

Notice that there is no homogeneity condition, and it does ot make sense as we don't have a field. In fact there is no notion of addition. But the condition ① of norm has to be satisfied by this distance. Also we don't have a translational condition i.e. distance between $x, y$ and distance between $x + v, y + v$ has to be same. Hence

**Note:-**
A metric space need not be a vector space. So it doesn't need a zero, or a notion of addition or scalar multiplication.

If I take a metric space and take any subset of it. And those three conditions of distance functions are still satisfied.

**Note:-**
Any subset of metric space is a metric space under the same distance function.

## 2.2 Open and Closed Ball and Set

**Definition 2.2.1: Open Ball and Closed Ball in a Metric Space**

An open ball of radius $r$ with center $c \in X$ in a metric space $X$ is

$$B_r(c) = \{x \in X \mid d(c,x) < r\}$$

and a closed ball is

$$\overline{B_r(c)} = \{x \in X \mid d(c,x) \leq r\}$$

**Definition 2.2.2: Open Set and Closed Ball in a Metric Space**

An open set in a metric space $X$ is one of the form of union of some open balls and a closed set in a metric space $X$ is one of the form of $X\backslash$some open sets

**Note:-**

We will do topology in Normed Linear Space (Mainly $\mathbb{R}^n$ and occasionally $\mathbb{C}^n$)using the language of Metric Space

**Example 2.2.1** (Open Set and Close Set)

Open Set:
- $\phi$
- $\bigcup_{x \in X} B_r(x)$ (Any $r > 0$ will do)
- $B_r(x)$ is open

Closed Set:
- $X$, $\phi$
- $\overline{B_r(x)}$

$x-$axis $\cup$ $y-$axis

**Question 4**

Is the set $x-$axis$\backslash\{$Origin$\}$ a closed set

*Solution:* We have to take its complement and check whether that set is a open set i.e. if it is a union of open balls

Now this works well for points which are above or below the $x-$axis. But for origin no matter how small the ball we take it willl have points from $x-$axis. Hence the set is not a closed set.

**Question 5**

Any continuous path in $\mathbb{R}^2$ is closed where path $= f : [0, 1] \to \mathbb{R}^2$

*Solution:* This is true. To be proved later.
Analogous to: For continuous function $f : [0, 1] \to \mathbb{R}$, the image is a closed interval

**Question 6**

If i take $X = x-$axis $\cup$ $y-$axis then is it open

*Solution:* Yes because here the space is only the union of those two axis. So any ball would be like a cross or line but it just as the metric space given to us. [It is open for this metric space but not open in $\mathbb{R}^2$]
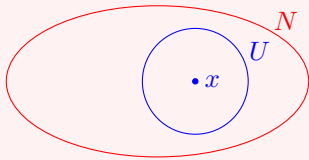
**Note:-**

If $S \subset X$, then $S$ itself has a collection of open sets of $S$ by containing $S$ as a metric space.

**Definition 2.2.3: Neighborhood**

For a point $x$ in metric space $X$, a neighborhood of $x$ is a set $N$ such that $x \in$an open set $U \subset N$
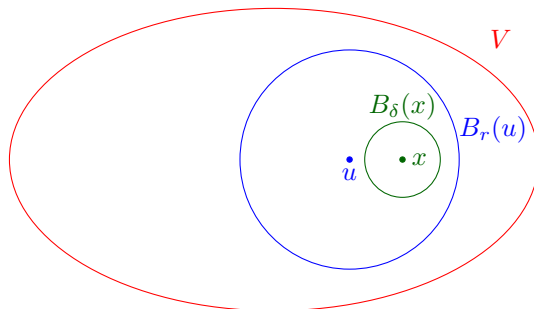


If $N$ itself is open. then we say that $N$ is an open neighborhood of $x$

## Theorem 2.2.1

If $x \in$ open set $V$ then $\exists\, \delta > 0$ such that $B_\delta(x) \subset V$

**Proof:**  By openness of $V$, $x \in B_r(u) \subset V$



Given $x \in B_r(u) \subset V$, we want $\delta > 0$ such that $x \in B_\delta(x) \subset B_r(u) \subset V$. Let $d = d(u, x)$. Choose $\delta$ such that $d + \delta < r$ (e.g. $\delta < \frac{r-d}{2}$)

If $y \in B_\delta(x)$ we will be done by showing that $d(u, y) < r$ but

$$d(u, y) \leq d(u, x) + d(x, y) < d + \delta < r$$

$\square$

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**Note:-**

$V$ is open $\iff \bigcup\limits_{x \in V} B_r(x)$ (where $r$ depends on $x$)

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## Theorem 2.2.2

Let $X$ be a metric space.

1. Union of open sets is open

2. Intersection of two open sets is open

Analogues to these as we are just taking complement of the open sets

$1'$. Arbitrary intersection of closed sets is closed

$2'$. Finite union of closed sets is closed.

**Proof:**  1. Let $\{V_\alpha\}_{\alpha \in I}$ be a collection of open sets where $I$ is an index set. We want ti show $\bigcup\limits_{\alpha \in I} V_\alpha$ is open in $X$. Since each $V_\alpha$ is open $V_\alpha = \bigcup\limits_{\beta \in J_\alpha} B_{r_\beta}(c_\beta)$ Then

$$\bigcup_{\alpha \in I} V_{alpha} = \bigcup_{\alpha \in I} \bigcup_{\beta \in J_\alpha} B_{r_\beta}(c_\beta)$$
$$= \bigcup_{\beta \in \sqcup J_\alpha} B_{r_\beta}(c_\beta)$$

which is still a union of balls

$\square$

2. The statement implies intersection of finite number of open sets is open. We can prove this by induction.

We will do by showing that for each $x \in V_1 \cap V_2$ $\exists\, r > 0$ s.t. $B_r(x) \subset V_1 \cap V_2$
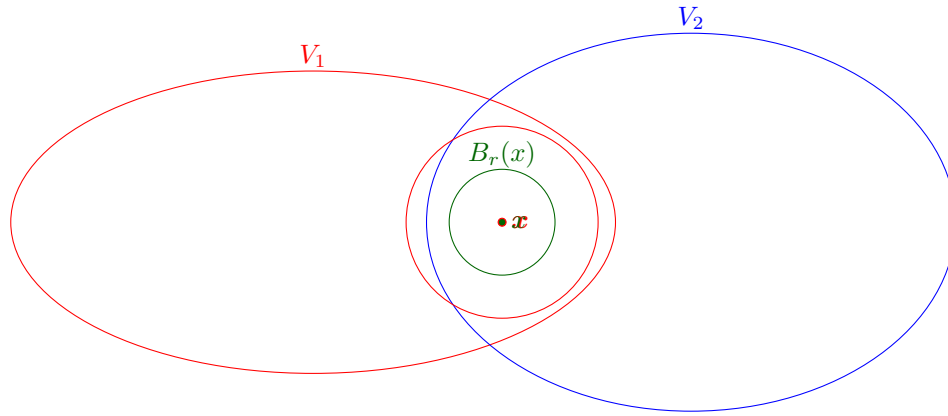
As $x \in V_1 \ \exists \ r_1$ such that $x \in B_{r_1}(x) \subset V_1$. Similarly $x \in V_2 \ \exists \ r_2$ such that $x \in B_{r_2}(x) \subset V_2$. Take $r = \min\{r_1, r_2\}$. Thus we have $x \in B_r(x) \subset V_1 \cap V_2$

The second part for closed sets are left as exercise $\qquad\square$

## 2.3 Topological Space

> **Definition 2.3.1: Topological Space**
>
> A topological space is a set $X$ together with a collection of subsets of $X$ (i.e. a subset of the power set of $X$) that is closed under taking arbitrary unions and finite intersections. This collection is called a topology on $X$

> **Note:-**
>
> Union means $\bigcup_{\alpha \in I} S_\alpha = \{x \in X \mid \exists \ \alpha \ \text{s.t.} \ x \in S_\alpha\}$
>
> Intersection means $\bigcap_{\alpha \in I} S_\alpha = \{x \in X \mid \forall \alpha, \ x \in S_\alpha\}$

> **Question 7**
>
> Suppose i have a topological space $X$ under given some topology. Is the entire set open ? And that the empty set is open ?

***Solution:*** If $I = \phi$, $\bigcup_{\alpha \in I} S_\alpha = \{x \in X \mid \exists \ \alpha \in I \ \text{s.t.} \ x \in S_\alpha\}$ gives $\phi$ and

$\bigcap_{\alpha \in I} S_\alpha = \{x \in X \mid \forall \alpha \in I, \ x \in S_\alpha\}$ gives $X$ because $\forall \ \alpha \in I$ condition is vacuously true for each $x \in X$.

> **Note:-**
>
> Intersection of empty families are not defined in set theory. This brings a very important point. In a set theory you have to have a universe. (Set theory have to avoid paradoxes, Russel Paradox) At the beginning you construct a large enough universe and you taking subsets only from that universe. Notice all subsets we are considering here are subsets of $X$ and here we defined how we union and intersection mean. Though it still this asks what our axioms of set theory. So you can change the part of the definition of topological space like this "... with a collection of subsets of $X$ including the empty set and the whole space..."

(If you don't like this as it is)

> **Note:-**
>
> If $S$ is a subset of metric space $X$, then $S$ is itself a metric space and as such open/closed sets as subsets of metric space

## Question 8

Is there any connection between being open in $X$ and being open in $S$ (Similar question for closed)

**Solution:** Let $x \in S$. Now, Ball of radius $r$ in $S = S \cap$ Ball of radius $r$ in $X$. Therefore

$$\text{Open Set in } S = \bigcup \text{ Balls in } S$$
$$= \bigcup (\text{Balls in } X \cap S)$$
$$= \left( \bigcup \text{ Balls in } X \right) \cap S$$
$$= \text{Open set } X \cap S$$

Part 2 is left as exercise

**Corollary 2.3.1**

If $S \subset X$ is open in $X$ then a subset $T$ of $S$ is open in $S \iff T$ is open in $X$

**Corollary 2.3.2**

If $S \subset X$ is closed in $X$ then a subset $T$ of $S$ is closed in $S \iff T$ is closed in $X$

**Definition 2.3.2: Subspace of a Topological Space $X$**

For any subset $S$ of a topological space $X$, the collection $S \cap U$, $U$ open in $X$ is called a subspace.

## Question 9

Prove that subspace of a metric space $X$ defines a topology on $X$

## Wrong Concept 2.1

If $x \in$ open $V$ then there exists $r > 0$ such that $x \in B_r(x) \subset V$
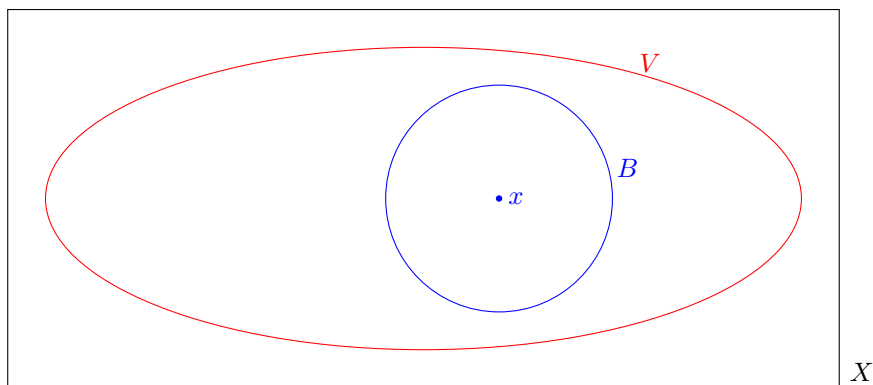


**Idea:** Why not we take $r = \inf\{$distance from $x$ to boundary of ball $B\}$.

Now we first have to ensure $r > 0$. Suppose that's true.

Then we have to define boundary. What is boundary, We can give a reasonable definition (Boundary has already a definition but we don't know that for now). Let boundary of $B = \{x \in X \mid d(c, x) = \delta\}$ Now this definition is not proper for our purpose.Because if we take union of all balls in $V$ then we will have lots of points as boundary but part of them should not be considered as boundary. Even if we take this definition.

Then the big question comes/ We are taking a infimum of a certain set of real numbers. The very first question arises is whether this set is nonempty. For example if we take $B$ to be the metric space it

self we have no boundary.

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Questions which come thorough this.

- Is there a meaningful way to define boundary

- Can we modify the idea

# Chapter 3

# Continuity in Metric Space

## 3.1 Limit Point and Closure

> **Definition 3.1.1: Limit Point**
>
> $S \subset X$ is a metric space. We say that $x \in X$ is a limit point of $S$ if $\exists$ a sequence $\{s_n\}$ with all $s_n \in S \setminus \{x\}$ such that $s_n \to x$ (each $s_n$ is different from $x$)

> **Theorem 3.1.1**
>
> $x$ is a limit point of $S \iff$ every neighborhood of $x$ in $X$ contains a point of $S$ other than $S$.

**Proof:** **If Part:**

Let $x$ be a limit point of $S$. Therefore take a sequence $\{s_n\}$ in $S \setminus \{x\}$ with $s_n \to x$ .

To prove what we want it is enough to show that $B_r(x) \cap S$ contains a point other than $x$. As $s_n \to x$, $\exists$ $N$ s.t. $\forall$ $n > N$ $d(x, s_n) < r$ i.e. $s_n \in B_r(x)$. In particular $s_n]inB_r(x) \cap (S \setminus \{x\})$

**Only If Part:**

We need to produce a sequence $\{s_n\} \in S \setminus \{x\}$ with $\lim s_n = x$. Take $s_n \in B_{\frac{1}{n}}(x) \cap (S \setminus \{x\})$ See that $\lim_{n \to \infty} s_n = x$.

This is essentially because $\frac{1}{n} \to 0$.

Complete the rest of the proof.

$\square$

> **Definition 3.1.2: Closure**
>
> Given a topological space $X$ and $S \subset X$, the closure of the set $S$ is $\overline{S}$ the smallest closed set containing $S$.

> **Theorem 3.1.2**
>
> $\overline{S}$ = Smallest closed set of $X$ containing $S$ = A
> = $S \cup$ (limit points of $S$) = B
> = $\{x \in X \mid x = \lim_{n \to \infty}$ for some sequence $\{s_n\}$ in $S\}$ = C
> = $\{x \in X \mid$ Every neighborhood of $x$ intersects $S\}$ = D

**Proof:** **A $\subset$ D**

$A^c = \bigcup$ (All open set $V$ s.t. $V \cap S = \phi$)
$D^c = \{x \in X \mid \exists$ open neighborhood of $x, B$ s.t. $B \cap S = \phi\}$
Clearly for all $x \in D^c$, $x \in A^c$. Hence $D^c \subset A^c \implies A \subset D$

**_D ⊂ B_**

Take $x \in D$. Suppose $x \notin S$. Now any neighborhood of $x$ intersects $S$ in a point hence it has to be a different point from $x$ since $x \notin S$. Therefore $x$ is a limit point of $S$. $D \subset B$

**_B ⊂ C_**

If $x \in S$ then take a sequence

□

### Question 10

What does it mean to be smallest closed set containing the set $S$ here ?

**_Solution:_** $\bigcap$ All closed sets containing $S$ is automatically closed and hence the smallest closed set containing $S$.

**_Proof:_** For proof of Theorem 3.1.2 notice A,B,C,D all contains $S$ (obvious).

> **Note:-**
> We don't need to show B,C,D are closed. We can also take the sets element wise and show each set is a subset of the other. This may simplify our way of proof. (exercise)

Now see $A$ and $D$ completely deal with topology. $A$ is about closed sets and $D$ is about open sets. So $A$ and $D$ close to each other. Now by the 3.1.1 we have equivalence of $C$ and $D$. So we can prove like this

$$A \iff D \iff B \ \& \ C$$

Left as exercise □

> **Note:-**
> For these kind of proofs instead of looking for the most efficient way try to find a path that allows you to go from anywhere to anywhere

## 3.2 Continuity

> **Definition 3.2.1: Continuity**
>
> $f : X \to Y$ function between metric spaces is continuous at $a \in X$ if $\forall \ \epsilon > 0 \ \exists \ \delta > 0$ s.t.
>
> $$d(x, a) < \delta \implies d(f(a), f(x)) < \epsilon$$
> $$\Updownarrow \qquad\qquad\qquad \Updownarrow$$
> $$x \in B_\delta(x) \implies f(x) \in B_\epsilon(f(a))$$

That means $f^{-1}(\text{Any ball around } f(a)) \supset \text{Ball around } a$.
So $f : X \to Y$ is continuous at all points $\iff f^{-1}(\text{Any ball intersecting the range}) \supset \text{A ball}$

> **Note:-**
> We can not say $f^{-1}(\text{Any ball})$ because because we need a ball that contains a point in the range

> **Theorem 3.2.1**
>
> $f$ is continuous $\iff f^{-1}(\text{Any open set in } Y)$ is open in $X$

***Proof:*** **If Part:-**

It is enough to show $f^{-1}$(Any ball) is open on $X$ because $f^{-1}$ preserves unions $f^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha \left(f^{-1}(V_\alpha)\right)$

Let $B$ is any open set (as its conceptually simpler to take open set here instead of a ball) in $Y$. Let $a \in f^{-1}(B)$. Hence we can say $f(a) \in B$. Since $B$ is an open set we can say there is a ball $B_\epsilon(f(a)) \subset B$. Since $f$ is continuous $\exists\ \delta$ such that $f(x) \in B_\epsilon(f(a))$ whenever $x \in B_\delta(a)$. Now $f^{-1}(B) \supset f^{-1}\left(B_\epsilon(f(a))\right) \supset B_\delta(a)$ Hence $f^{-1}(B)$ is open.

**Only If Part:-**

Lets prove continuity ar $a \in X$. We are given that $f^{-1}\left(B_\epsilon(f(a))\right)$ is open and obviously contains $a$. Therefore $f^{-1}\left(B_\epsilon(f(a))\right)$ contains a ball around $a$. Take $\delta =$ Radius of the ball. $\qquad\square$

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**Question 11**

For a metric space $X$, show that $\overline{S} = \{x \in X \mid \lim_{n\to\infty} s_n = x\}$ for some sequence $\{s_n\}$ in $S$.

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**Question 12**

For a function $f : X \to Y$ between metric spaces, show that the followings are equivalent.

1. $f$ is continuous

2. $f^{-1}$(Open Set) is open

3. $f^{-1}$(Closed Set) is closed

4. $f(\overline{S}) = \overline{f(S)}$

5. $x_n \to x \implies f(x_n) \to f(x)$

One or more of the above are wrong so check if they are true and if not then find the true statement.

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***Solution:*** 4 is wrong. How to correct and rest is left as exercise

**Question 13**

For $f : X \to Y$ any set map

(i) $f^{-1}$ preserves unions, intersections, complements

(ii) Is there any condition on $f$ under which $f$ possesses the property above ?

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**Example 3.2.1** (Continuous Function)

1. Any constant function.

2. $X \xrightarrow{f} Y \xrightarrow{g} Z$ $f, g$ continuous $\implies g \cdot f$ is continuous

3. Is $S \subset X$ then $S \xrightarrow{\text{Inclusion}} X$ is continuous

4. Projection $\underset{(x_1, x_2, \cdots, x_n) \mapsto x_i}{\mathbb{R}^n \to \mathbb{R}}$

   More generally for example $\underset{(x,y,z) \mapsto (x,x,y,y)}{\mathbb{R}^3 \to \mathbb{R}^4}$

---

5. Map from metric space to euclidean space.

$$\left.\begin{array}{l} X \to \mathbb{R}^n \\ x \mapsto (f_1(x), f_2(x), \cdots, f_n(x)) \end{array}\right\} \begin{array}{l} f \text{ is continuous} \\ \overset{\Longleftrightarrow}{\text{each } f_i \text{is continuous}} \end{array}$$

6. $\mathbb{R} \times \mathbb{R} \to \mathbb{R} : (x, y) \mapsto x \pm y, xy$ are continuous.

We need to prove $x_n \to x$ and $y_n \to y$ in $\mathbb{R} \implies \begin{cases} x_n \pm y_n \to x \pm y \\ x_n y_n \to xy \end{cases}$

$\mathbb{R} \setminus \{0\} \to \mathbb{R} : x \mapsto \frac{1}{x}$ is continuous

7. sum and product of two continuous real valued function on $X$ are continuous

$$f, g : X \xrightarrow{f,g} \mathbb{R} \text{ continuous} \implies X \underset{x}{\overset{f,g}{\longrightarrow}} \underset{(f(x),g(x))}{\mathbb{R} \times \mathbb{R}} \xrightarrow{+} \mathbb{R}$$

$$f : X \to \mathbb{R} \implies \frac{1}{f} : \underbrace{X \setminus f^{-1}(0)}_{\text{open set in } X} \to \mathbb{R} \text{ is continuous}$$

$\{0\}$ is closed in $\mathbb{R}$, so $f^{-1}(0)$ is closed in $X$ by continuity of $f$

Therefore any polynomial in continuous real valued functions on $X$ is continuous.

8. **Special Case:**

- $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ linear map is continuous where $(x_1, x_2, \cdots, x_n) \longmapsto (a_{11}x_1 + \cdots + a_{1n}x_n, a_{21}x_1 + \cdots + a_{2n}x_n, \cdots, a_{m1}x_1 + \cdots + a_{mn}x_n)$

$$\text{Matrix of } T = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- $M_{n \times n}(\mathbb{R}) \underset{A}{\to} \underset{\det(A)}{\mathbb{R}}$ is continuous

$\frac{1}{\det} : GL_n(\mathbb{R}) \to \mathbb{R}$

Here $M_{n \times n}$ is a vector space of dimension $n^2$ in which $GL_n(\mathbb{R}) = \{A \mid \det(A) \neq 0\}$ is an open set.

- $GL_n(\mathbb{R}) \to GL_n(\mathbb{R})$ is continuous.
  $A \longmapsto A^{-1}$

9. Any norm $(f)$ on $\mathbb{R}^n$ is uniformly continuous w.r.t usual topology on $\mathbb{R}^n$ i.e. $f : \mathbb{R}^n \to \mathbb{R}$ is continuous w.r.t usual norms $(\| \cdot \| = p0\text{norm for } p = 1, 2, \infty)$ on $\mathbb{R}^n (\| \cdot \|)$ and $\mathbb{R}(| \cdot |)$

**Theorem 3.2.2**

Any norm $(f)$ on $\mathbb{R}^n$ is uniformly continuous w.r.t usual topology on $\mathbb{R}^n$ i.e. $\forall \, \epsilon > 0 \, \forall \, x, y \in \mathbb{R}^n \, \exists \, \delta > 0$ s.t. $\|x - y\| < \delta \implies |f(x) - f(y) < \epsilon$

*Proof:*

$$\left.\begin{array}{l} f(x) \leq f(y) + f(x - y) \\ \qquad \| \\ f(y) \leq f(x) + f(y - x) \end{array}\right\} |f(x) - f(y)| \leq f(x - y)$$

Let $x = \sum x_i e_i$ and $y = \sum y_i e_i$ where $\{e_i\}$ is the standard basis of $\mathbb{R}^n$.

$$f(x - y) = f\left(\sum (x_i - y_i)e_i\right) \leq \sum f((x_i - y_i)e_i) = |x_i - y_i| f(e_i)$$

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Notice $\sum |x_i - y_i| = \|x - y\|_1$. Let $M = \max\{f(e_i)\}$ Then

$$|f(x) - f(y)| \leq f(x - y) \leq M\|x - y\|_1$$

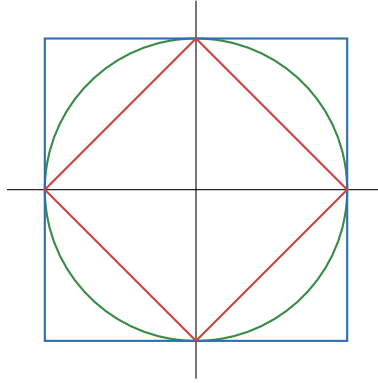Thus $\|x - y\| < \frac{\epsilon}{M} \implies |f(x) - f(y)| < \epsilon$      $\square$

# Chapter 4

# Equivalence of Norms

We back to Normed Linear Space for a little while.

In $\mathbb{R}^n$, $u = (u_1, u_2, \cdots, u_n)$ where each $u_i \in \mathbb{R}$. we have $p-$norm: $\|u\|_p = \left( \sum_i |u_i|^p \right)^{\frac{1}{p}}$ where $1 \le p \le \infty$. Balls in $\mathbb{R}^2$ w.r.t. $\|\cdot\|_1$, $\|\cdot\|_2$, $\|\cdot\|_\infty$.



<u>Observe</u>: A set $V$ in $\mathbb{R}^2$ is

$$\begin{aligned}
\text{open w.r.t. } \|\cdot\|_1 &\iff V = \bigcup_{u \in V} \text{ Box in } V \text{ centered box} \\
\text{open w.r.t. } \|\cdot\|_2 &\iff V = \bigcup_{u \in V} \text{ Diamond in } V \text{ centered box} \\
\text{open w.r.t. } \|\cdot\|_\infty &\iff V = \bigcup_{u \in V} \text{ Circle in } V \text{ centered box}
\end{aligned}$$

---

**Definition 4.1: Equivalence of Norms**

Suppose $\|\cdot\|$, $\|\cdot\|'$ are two norms in vector space $V$, We say that the two norms are equivalent if there are constants $\alpha, \beta > 0$ s.t.
$$\alpha\|x\|' \le \|x\| \le \beta\|x\|'$$

---

**Example 4.0.1** (Norm Equivalence)

1. **$p = \infty$ and $p = 1$**

   $\|x\|_\infty = \max\{|x_i| \mid 1 \le i \le n\} \le \|x\|_1 = \sum_i |x_i|$

   $\|x\|_\infty \ge \text{ each } |x_i| \implies n\|x\|_\infty \ge \|x\|_1$

   Hence
   $$\|x\|_\infty| \le \|x\|_1 \le n\|x\|_\infty \text{ and } \frac{1}{n}\|x\|_1 \le \|x\|_\infty \le \|x\|_1$$

2. $p = \infty$ and $p = 2$

$$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty$$

**Theorem 4.1**

All norms on a finite dimensional vector space are equivalent

**Proof:** Proved in Theorem 5.2.7 ☐

**Theorem 4.2**

Suppose $\|\cdot\|$ and $\|\cdot\|'$ are equivalent on a vector space $V$. Then

(i) $\{x_n\} \to x$ w.r.t. $\|\cdot\|$ $\iff$ $\{x_n\} \to x$ w.r.t $\|\cdot\|'$

(ii) $S \subset V$ is open w.r.t $\|\cdot\|$ $\iff$ $S$ is open w.r.t $\|\cdot\|'$

**Proof:** For both proofs if we just prove one direction the we are done actually since we can just replace the words to prove for opposite direction,

(i) **If Part:-**

Since $\|\cdot\|, \|\cdot\|'$ are equivalent we have $\exists\, \alpha, \beta$ such that $\alpha\|x\|' \le \|x\| \le \beta\|x\|'$. So if we show $\alpha\,\|x_n - x\| < \|x_n - x\| < \alpha\epsilon$ we are done.

Let $\{x_n\} \to x$ w.r.t $\|\cdot\|$ i.e. $\forall\, \epsilon > 0\ \exists N$ s.t. $\forall\, n > N$ $\|x_n - x\| < \alpha\epsilon$. Hence we have $\alpha\,\|x_n - x\|' < \alpha\epsilon$. Hence $\forall\, \epsilon > 0\ \exists\, N$ such that $\forall\, n > N$ $\|x_n - x\|' < \epsilon$

☐

(ii) **Only If Part:-**

$V$ is open w.r.t $\|\cdot\|$ $\iff$ $\bigcup_{x \in V} B_r(x)$ and $V'$ is open w.r.t $\|\cdot\|'$ $\iff$ $\bigcup_{x \in V} B_r'(x)$

Now we have

$$B_r(x) = \{y \mid \|y - x\| < r\} \text{ and } B_r'(x) = \{y \mid \|y - x\|' < s\}$$

Hence by equivalence of the norms for any $v$

$$\alpha\|v\|' \le \|v\| \le \beta\|v\|'$$

Since $\|v\| < r$ we have

$$\|v\|' \le \frac{r}{\beta} \implies B'_{\frac{r}{\beta}}(x) \subset B_r(x)$$

☐

**Corollary 4.1**

$p = 1$ and $p = \infty$ on $\mathbb{R}^n$ (and $\mathbb{C}^n$) give the same topology as $p = 2$ norm

**Corollary 4.2**

Let $x_m$ be a square in $\mathbb{R}^n$. $\overline{x_m} = (x_{m_1}, x_{m_2}, \cdots, x_{m_n})$. Then $\{\overline{x_m}\} \to x = (x_1, x_2, \cdots, x_n)$ w.r.t $\|\cdot\|_2$ $\iff$ $\{x_{m_i}\} \to x_i$ in $\mathbb{R}$ for each $i$.

We can check this w.r.t $\| \cdot \|_\infty$

$$
\begin{aligned}
\overline{x_m} \to \overline{x} \text{ w.r.t } \| \cdot \|_\infty \quad &\iff \quad \forall \, \epsilon > 0 \, \exists \, N \text{ s.t. } \forall \, m > N \, \max\{|x_{m_i} - x_i| \mid 1 \le i \le n\} \\
&\iff \quad \text{each } |x_{m_i} - x_i| < \epsilon \; \forall \, i \\
&\iff \quad \lim_{n \to \infty} x_{m_i} = x_i \; \forall \, i
\end{aligned}
$$

# Chapter 5

# Compactness

## 5.1 Sequentially Compact

**Definition 5.1.1: Sequentially Compact**

Let $(X, d)$ be a metric space. $X$ is called sequentially compact if every sequence in $X$ has a convergent subsequence. (Often applied to a subset $S$ of $X$)

**Note:-**

For $S$ to be sequentially compact the limit of subsequence must be in $S$

**Definition 5.1.2: Boundedness**

A subset $S$ of $(X, d)$ is bounded if $S \subset B_r(x)$ for some $x \in X$ and $r > 0$

**Note:-**

Boundedness depends on the metric but if two metrics are "equivalent" analogous to norms)

**Theorem 5.1.1**

A subset $K$ of $\mathbb{R}^n$ is sequentially compact $\iff$ $K$ is closed and bounded

**Proof:** Proof in steps

1. **A closed interval $[a, b]$ in $\mathbb{R}$ is sequentially compact**

   **Proof:** Given a sequence $x_1, x_2, \cdots$ in $\mathbb{R}$ in $[a, b]$ we can extract a monotonic subsequence as follows:

   We call $x_i$ to be a peak if $x_i > x_j \ \forall \ j > i$. Now there are two cases. If number of peaks is infinite then the next peak comes after the previous one so smaller than the previous one. So its a strictly decreasing sequence. If number of peaks are finite then at some point we cant find a peak with this property that means no matter which term i peak there is at least one term after that which is greater than or equal to that term. $y_1 =$a term after the last peak. and $y_{i+1} =$a term after $y_i$ such that $y_{i+1} \geq y_i$. Hence $y_1, y_2, \cdots$ is a weakly increasing sequence.

   When $\{x_n\}$ contained in $[a, b]$ by boundedness of the monotonic subsequence, it converges to its sup/inf and the limit is in $[a, b]$ $\qquad \square$

2. $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n] \subset \mathbb{R}^n$ **is sequentially compact (w.r.t $p-$norm for $p = 1, 2, \infty$. Later for any norm)**

    ***Proof:***   Recall a sequence $\{x_m\} \to x$ in $\mathbb{R}^n$ $\iff$ The sequence converges in each coordinate i.e. $x_{m_i} \to x_i$

    Take a sequence in the given box. Extract a subsequence whose entries in 1st slot converge (necessarily to $x_i$ in $[a_1, b_1]$ by step 1 From this sequence, extract a further subsequence whose entries in second slot converge to $x_2 \in [a_2, b_2]$. Continue      □

3. **Every closed subset of a sequentially compact set is sequentially compact**

    ***Proof:***   Exercise                                                       □

    This will show each closed and bounded subset of the Euclidean Space $\mathbb{R}^n$ is sequentially compact. (because such a set will be contained in a box)

4. **If K is sequentially compact then K is closed and bounded**

    ***Proof:***   If $K$ is not closed then some limit point $x$ of $K$ will not be in $K$. Then there is a sequence $\{y_m\}$ in $K$ converges to $x \notin K$ violating sequential compactness of $K$.

    If $K$ is not bounded take $\{x_m\} \in K$ with $\|x_m\| \geq n$ then $\{x_m\}$ can not be convergent      □

                                                                          □

**Note:-**

Step 4 works for any metric space. Then we need to have a ball instead of norm

> **Theorem 5.1.2**
>
> If $K$ is a sequentially compact of a metric space $X$, then $K$ is closed and bounded

***Proof:***   Same argument as step 4 use $x_m$ such that $d(x_m, x) \geq m$      □

> **Question 14**
>
> If $K$ is closed and bounded in $(X, d)$ $\implies$ $K$ is sequentially compact

***Solution:*** No. Any counter-example. Define a metric on real number which induces same topology as the normal topology in such a way that there is a closed and bounded set that is not compact.

> **Question 15**
>
> 1. If $V$, $W$ are normed linear spaces can we define a norm on $V \times W$ ?
>
> 2. If $V$, $W$ are metric spaces can we define a metric on $V \times W$ ?
>
> 3. If $V$, $W$ are topological spaces can we define a topology on $V \times W$ ?

## 5.2 Open Cover and Compactness

> **Definition 5.2.1: Open Cover**
>
> Let $\{V_\alpha\}_{\alpha \in I}$ be a family of subsets of metric space $X$ we say that $\{V_\alpha\}_{\alpha \in I}$ is a cover of $X$ if $\bigcup_\alpha V_\alpha = X$ and we say that $\{V_\alpha\}_{\alpha \in I}$ is an open cover if each $V_\alpha$ is open (in $X$)

> **Definition 5.2.2: Compact**
>
> $X$ is called compact if each open cover of $X$ has a finite subcover i.e. $\{V_{\alpha_1}, V_{\alpha_2}, \cdots, V_{\alpha_n}\} \subset \{V_\alpha\}_{\alpha \in I}$ with $V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots \cup V_{\alpha_n} = X$

> **Note:-**
>
> 1. This definition makes sense for any topological space $X$.
>
>    If $X$ is a metric space then it is a fact that $X$ is compact $\iff$ $X$ is sequentially compact. This is not true for general topological spaces. Both implications fail.
>
> 2. Reformulation of compactness for subset $K$ of $X$ in terms of open sets of $X$
>
>    $K$ is compact $\iff$ Every cover of $K$ bt open sets of $K$ has a finite subcover.
>    As open sets of $K$ are precisely (open sets of $X$) $\cap K$. We have the following
>    $K$ is compact $\iff$ For any family $\{V_\alpha \cap K\}_{\alpha \in I}$ where $V_\alpha$ are open in $X$ whose union is $K$, there is a finite subcover.
>    $\iff$ For any family $\{V_\alpha \cap K\}_{\alpha \in I}$ of open sets in $X$ such that $\bigcup_{\alpha \in I} V_\alpha \supset K$, there must be a finite subfamily $V_{\alpha_1}, V_{\alpha_2}, \cdots, V_{\alpha_n}$ with $\bigcup_{i=1}^n V_{\alpha_i} \supset K$
>
>    If i take this definition of compactness of a subset $K$ of metric space $X$ then $K$ is compact as subset of $X$ $\iff$ $K$ is compact as a subset of it itself

**Theorem 5.2.1** Haine Borel Theorem

$$K \subset \mathbb{R}^n \text{ is compact} \quad \iff \quad K \text{ is closed and boundeded}$$

(w.r.t $p = 1, 2$ or $\infty$ norm as they are equivalent.)

***Proof:*** **Only If Part:-**

Proof in steps

① Closed interval $[a, b]$ is compact in $\mathbb{R}$. ***Proof:*** Theorem 5.2.4

② Closed box $[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is compact in $\mathbb{R}^n$. ***Proof:*** Theorem 5.2.6

③ A closed subset of a compact set is compact. ***Proof:*** Theorem 5.2.3

These steps would give the backward direction of Haine Borel Theorem i.e. suppose $K$ is closed and bounded in $\mathbb{R}^n \implies K \in [-M.M]^n \implies$ compact by ②

**If Part:-**

**Bounded:** First we have to show that $K$ is compact $\implies K$ is bounded an i.e. $K \subset B_r(x)$ in $(X, d)$ for some $x \in X$, $r > 0$
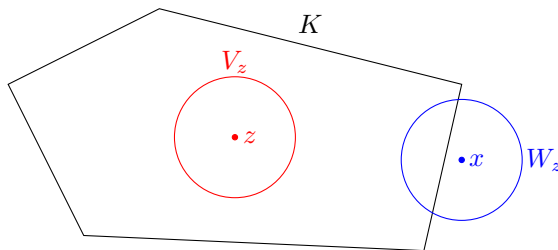
Consider open cover $\{B_n(x)\}_{n\in\mathbb{Z}^+}$ of $X$ and hence of $K$. This must have a finite subcover $B_{n_1}(x), B_{n_2}(x),$ $\cdots, B_{n_k}(x)$. Take $r = \max\{n_1, n_2, \cdots, n_k\}$ Hence

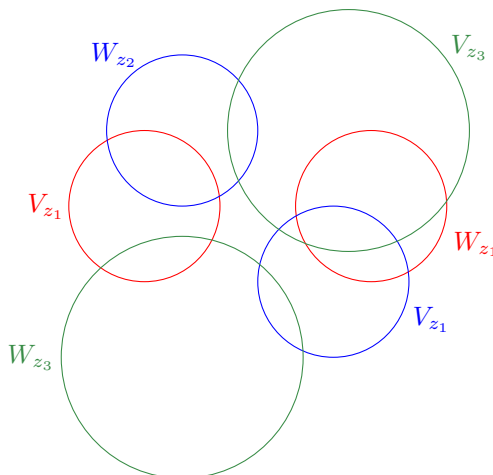$$K \text{ is compact} \implies K \text{ is closed}$$

**Closed:** We will show that $X \setminus K$ is open. Pick $x \notin K$. Enough to construct an open neighborhood $U_x \ni x$ such that $U_x \cap x = \phi$

Take $z \in K$. Let $c = d(x, z)$ then

$$\underset{\substack{\| \\ W_z}}{B_{\frac{c}{3}}(x)} \quad \cap \quad \underset{\substack{\| \\ V_z}}{B_{\frac{c}{3}}(z)} = \phi \quad \text{by triangle inequality}$$



Now $\bigcup\limits_{z\in K} V_z \supset K$. So $\{V_z\}$ is an open cover of $K$. By compactness we have $V_{z_1} \cup V_{z_2} \cup \cdots \cup V_{z_n} \supset K$.



As $W_z \cap V_z = \phi \; \forall z \in K$. We have $\underbrace{(W_{z_1} \cup W_{z_2} \cup \cdots \cup W_{z_n})}_{\substack{\text{Finite intersection of} \\ \text{open neighborhoods of } x, \\ \text{so call this } U_x}} \cap K = \phi$ □

Key fact that made this work: For $x \neq z$ in $X$, we could find open neighborhoods of $V$ and $W$ (of $x$ and $z$ respectively) such that $V \cap W = \phi$. Topological spaces that satisfy this property are called Housdorff.
What we proved is the following

> **Theorem 5.2.2**
>
> For a Housdorff Topological space $X$ any compact subset $K$ is closed and bounded

> **Theorem 5.2.3**
>
> $C$ is a closed subset of compact set $X \implies C$ is compact.

***Proof:*** Take any open cover $\{V_\alpha\}_{\alpha \in I}$ of $C$ by open sets in $X$ i.e $\bigcup\limits_\alpha V_\alpha \supset C$. Now $\{V_\alpha\}_{\alpha \in I} \cup \{X \setminus C\}$ is an open cover of $x$. We have a finite subcover by compactness of $X$. The same subcover (after dropping $X \setminus C$ if necessary) works for $C$. $\qquad\square$

---

## Wrong Concept 5.1: Closed interval $[a, b]$ is compact in $\mathbb{R}$

Suppose $\{V_\alpha\}_{\alpha \in I}$ is an open cover of $[a, b]$ by open sets in $\mathbb{R}$.

Hence every one of the points in the interval is covered by one of the $V_\alpha$. Hence there is some interval contained in the $V_\alpha$
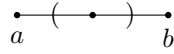


So i could just ignore the $V_\alpha$ and say for each point in the interval we can get an open interval that is part of a $V_\alpha$. So how can i find a subcover. I could simply travel from one end to the other.

So i start with $a$ so $a$ must be contained in some open interval
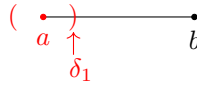


Not only that i have covered up a small segment of the closed interval, upto a point, $a + \delta_1$. Say $[a, a + \delta_1) \subset V_1$.

Let $a + \delta_1$ is contained in some open interval which is contained in $V_2$ upto the point $a + \delta_2$
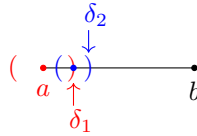


Now continue.

- - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - - -

What is wrong with this ?

We could have smaller and smaller intervals. For example length of first interval can be $\frac{1}{3}$, length of second interval can be $\frac{1}{9}$, length of third interval can be $\frac{1}{27}$ and so on. So its a geometric progression and it will sum less than 1. So i just may not get there in finite number of steps.

---

### Question 16

Suppose $X$ is a topological space that is compact and 5.2 (Take $x$ ro be a compact metric space if you like). Prove that given disjoint compact subsets $K$ and $L$, there are disjoint open sets $U$ and $V$ with $K \subset U$ and $L \subset V$ (First do it for $K =$ single point)

In the above exercise we could have replaced the word compact with another word which is closed because $X$ is given to be compact so any closed set will be compact and in a Housdorff space compact subset is also closed.

**Note:-**

Cauchy Sequence in Metric space need not converge. For example $(0, 1)$ and take the sequence $\frac{1}{n}$. It wants to converge to 0 but 0 is not there.

**Theorem 5.2.4** Haine Borel Theorem - If Part: Step ①

$[0, 1]$ is compact in $\mathbb{R}$

***Proof:*** Let $\{V_\alpha\}_{\alpha \in I}$ be a family of open sets in $\mathbb{R}$ covering $[0, 1]$.

Let $S = \{a \in [0,1] \mid [0,a]$ can be covered by a finite number of $V_\alpha$'s$\}$. Our goal is to prove $1 \in S$.

Let $0 \le x < y \le 1$. So $[0,x] \subset [0,y]$. This $y \in S \implies x \in S$ i.e $x \notin S \implies y \notin S$. Now $S$ is nonempty because $0 \in S$ and $S$ is bounded. Let $u = $ lub of $S$. Clearly $0 \le u \le 1$. Hence it is enough to show $u = 1$ and $u \in S$.

$0 \in $ some open set $V_\alpha$. Hence $\exists \epsilon > 0$ $B_\epsilon(0) \subset V_\alpha$. Hence $\forall$ point $x \in [0,\epsilon)$ $x \in S$

For $a \in [0,u)$, $a \in S$ (otherwise $a$ itself would be an upper bound for $S$). As $\{V_\alpha\}_{\alpha \in I}$ cover $[0,1]$, $u \in V_\beta$. So $\exists \epsilon > 0$ such that $(u - \epsilon, u + \epsilon) \subset V_\beta$ As $u - \epsilon \in S$ we have $V_{\alpha_1} \sup V_{\alpha_2} \sup \cdots V_{\alpha_k} \supset [0, u - \epsilon]$ Then $V_{\alpha_\beta} \cup V_{\alpha_1} \cup V_{\alpha_2} \cup \cdots V_{\alpha_k} \supset \left[0, u + \frac{\epsilon}{2}\right]$. So $u = 1$ because otherwise some $u + \delta \in S$ contradicting that $u$ is an upper bound. $\square$

---

## Question 17

Can the strategy from the last time be made to work ti actually extract a finite subcover of a given cover.

---

**Theorem 5.2.5**

Suppose $X \xrightarrow{f} Y$ continuous and $K \subset X$ is compact. Then $f(K)$ is compact

**Proof:** Let $\{V_\alpha\}_{\alpha \in I}$ be an open cover of $f(k)$ by open sets $V_\alpha$ of $Y$. So

$$\bigcup_\alpha V_\alpha \supset f(K) \implies f^{-1}\left(\bigcup_\alpha V_\alpha\right) = \bigcup_\alpha f^{-1}(V_\alpha) \supset f^{-1}(f(K)) \supset K$$

Thus $\left\{f^{-1}(V_\alpha)\right\}_{\alpha \in I}$ is an open (because of continuity Theorem 3.2.1) cover of $K$.
Extract a finite subcover

$$f^{-1}(V_{\alpha_1}) \cup f^{-1}(V_{\alpha_2}) \cup \cdots f^{-1}(V_{\alpha_m}) \supset K$$
$$\implies f\left(f^{-1}(V_{\alpha_2}) \cup \cdots f^{-1}(V_{\alpha_m})\right) \supset f(K)$$
$$\implies \bigcup_{i=1}^m f\left(f^{-1}(V_{\alpha_i})\right) \supset f(K)$$

As $V_{\alpha_i} \supset f\left(f^{-1}(V_{\alpha_i})\right)$ we have $\bigcup_{i=1}^m V_{\alpha_i} \supset f(K)$ $\square$

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## Question 18

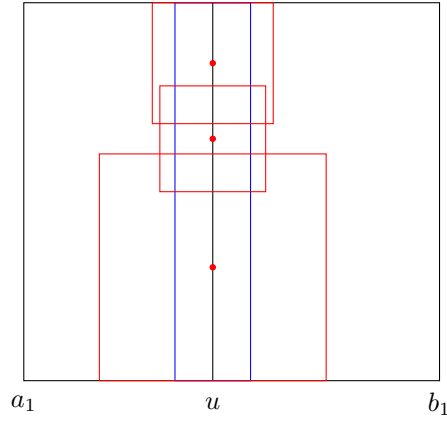$f(\text{Sequentially compact } K)$ is sequentially compact

---

**Theorem 5.2.6** Haine Borel Theorem - If Part: Step ②

$K = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is compact in $\mathbb{R}^n$

**Proof:** Induction on $n$. $n = 1$ we already proved in Theorem 5.2.4. Let $\mathcal{F} = \{V_\alpha\}_{\alpha \in I}$ be a cover of $K$ by open sets in $\mathbb{R}^n$. Fix $u \in [a_1, b_1]$ and consider $\{u\} \times \underbrace{[a_2, b_2] \times \cdots \times [a_n, b_n]}_{\substack{=C \text{ is compact} \\ \text{by induction on } n}}$ Hence $\{u\} \times C$ is compact because

$\mathbb{R}^{n-1} \to \mathbb{R}^n$ which maps $(y_2, y \cdots, y_n) \mapsto (u, y_2, \cdots, y_n)$ or $f(C) = \{u\} \times C$ is continuous.

For each $p = (u, y_2, \cdots, y_n)$ in $\{u\} \times C$ pick an open neighborhood $V_p \in \mathcal{F}$. Hence $V_p \supset (x - \epsilon, x + \epsilon) \times \underbrace{(y_2 - \epsilon, y_2 + \epsilon) \times \cdots \times (y_n - \epsilon, y_n + \epsilon)}_{W_p}$ for some $\epsilon = \epsilon_p$ depending on $p$

$$a_1 \qquad\qquad u \qquad\qquad b_1$$

By compactness of $\{u\} \times C$, extract a finite subcover of the cover $\{W_p\}$. Hence $W_{p_1} \cup W_{p_2} \cup \times \cup W_{p_k} \supset \{u\} \times C$. Since its a union of open sets we have in fact $W_{p_1} \cup W_{p_2} \cup \times \cup W_{p_k} \supset (u - \epsilon, u + \epsilon) \times C$ where $\epsilon = \min\{\epsilon_{p_1}, \epsilon_{p_2}, \cdots, \epsilon_{p_k}\}$. Let $\mathcal{F}_u = \{V_{p_1}, V_{p_2} < \cdots, V_{p_k}\}$. So

$$V_{p_1} \cup V_{p_2} \cup \cdots \cup V_{p_k} \supset W_{p_1} \cup W_{p_2} \cup \cdots \cup W_{p_k} \supset (u - \epsilon, u + \epsilon) \times C$$

i.e. this finite subcover $\mathcal{F}_u$ cover not just the slice but a tube around it.

Now as $u$ varies in $[a_1, b_1]$, $(u - \epsilon_u, u + \epsilon_u)$ gives an open cover. Extract a finite subcover $(u_1 - \epsilon_{u_1}, u_1 + \epsilon_{u_1}), (u_2 - \epsilon_{u_2}, u_2 + \epsilon_{u_2}), \cdots, (u_l + \epsilon_{u_l}, u_l + \epsilon_{u_l})$. Then $\mathcal{F}_{u_1} \cup \mathcal{F}_{u+2} \cup \cdots \cup \mathcal{F}_{u_l}$ is a finite subcover of $[a_1, b_1] \times C = K$ $\quad\square$

---

**Question 19**

Why the map $\mathbb{R}^{n-1} \to \mathbb{R}^n$ which maps $(y_2, y \cdots, y_n) \mapsto (u, y_2, \cdots, y_n)$ or $f(C) = \{u\} \times C$ is continuous ?

---

**Question 20**

$X, Y$ are topological spaces. $K \subset X$ and $Y \subset Y$ are compact subsets. Then $K \times L$ is compact subset of $X \times Y$ where Open sets of $X \times Y$ are $\bigcup$(Open set of $X$)$\times$(Open set in $Y$)

---

**Theorem 5.2.7**

All norms on $\mathbb{R}^n$ are equivalent

**Proof:** Enough to show any norm $f \sim \|\cdot\|$

$$\text{i.e } \alpha\|u\| \le f(u) \le \beta\|u\| \forall\ u$$

$$\text{i.e } \alpha \le \frac{f(u)}{\|u\|} \le \beta \ \forall\ u \forall\ u \ne 0$$

Note that $\frac{f(x)}{\|x\|} = f\left(\frac{x}{\|x\|}\right) = f(u)$ where $u = \frac{x}{\|x\|}$, so $\|u\| = 1$. Hence it is enough to show that

$$\alpha \le f(u) \le \beta$$

for any $u$ with $\|u\| = 1$

Let $S = \{u \mid \|u\| = 1\}$ is the unit sphere in $\mathbb{R}^n$, which is closed and bounded

$$\begin{aligned}
S \text{ is closed and bounded} \quad &\Longrightarrow \quad S \text{ is compact} \\
&\Longrightarrow \quad f(S) \text{ is compact in } \mathbb{R} \\
&\Longrightarrow \quad f(S) \text{ is closed and bounded in } \mathbb{R} \\
&\Longrightarrow \quad f(S) \text{ has largest element in } \beta \text{ and smallest} \\
&\qquad\qquad \text{element } \alpha \text{ such that } \alpha \le f(S) \le \beta
\end{aligned}$$

$\square$

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# Chapter 6

# Differentiation

Derivative of $f$ at $a \in \mathbb{R}$ is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

To take this limit $f$ should be defined in some $(a - \epsilon, a + \epsilon)$ i.e. $f :$ neighborhood of $a \to \mathbb{R}$

**Goal:** Definition of $f'(a)$ for $a \in$(Some open $U$ in $\mathbb{R}^m) \xrightarrow{f} \mathbb{R}^n$, $a, h \in \mathbb{R}^m$

$f(a+h) - f(a)$ makes sense in $\mathbb{R}^n$ but can't divide by $h$, which is a vector in $\mathbb{R}^m$. If $m = 1$ can use the same definition. $f :$ Open $U$ in a $\mathbb{R} \to \mathbb{R}^n$ which maps $a \mapsto (f_1(a), f_2(a), \cdots, f_n(a))$. If $n = 1$ i.e. $\mathbb{R}^m \supset U \xrightarrow{f} \mathbb{R}$ we have partial derivatives.

> **Example 6.0.1** (Derivative of $f : \mathbb{R}^m \to \mathbb{R}$)
>
> $f(x, y, z) = x^4 \sin(yz)$. Here
>
> $$\frac{\partial f}{\partial x} = 4x^3 \sin(yz)$$
> $$= \lim_{h \to 0} \frac{f(x+h, y, z) - f(x, y, z)}{h}$$
>
> Hence
>
> $$\frac{\partial f}{\partial x}\bigg|_{p=(r,s,t)} = \lim_{h \to 0} \frac{f(r+h, s, t) - f(r, s, t)}{h}$$
> $$= \lim_{h \to 0} \frac{f(p + he_1) - f(p)}{h} \qquad [p = re_1 + se_2 + te_3 \text{ using standard basis } e_1, e_2, e_3]$$
>
> Its a real number if the limit exists

## 6.1 Partial Derivatives

> **Definition 6.1.1: Partial Derivative of $f : \mathbb{R}^n \supset U \to \mathbb{R}$**
>
> For $f :$(Open $U$ in $\mathbb{R}^m) \to \mathbb{R}^n$, define "$i-$th partial derivative of $f$ at $a \in U$" to be $\left(\text{Notation } \frac{\partial f}{\partial x_i}\big|_a,\right.$
> $\left. \frac{\partial f}{\partial x_i}(a), D_i f(a)\right)$ $\lim_{h \to a} \frac{f(a + he_i) - f(a)}{h}$ $(i = 1, 2, \cdots, m)$

Note that this limit (if exists) is in $\mathbb{R}^n$.

$$\left.\begin{array}{l} \text{If } f = (f_1, f_2, \cdots, f_n) \text{ ($f_i$ real} \\ \text{values in function } U \to \mathbb{R}) \end{array}\right\} \frac{\partial f}{\partial x_i}(a) = \left(\frac{\partial f_1}{\partial x_i}(a), \frac{\partial f_2}{\partial x_i}(a), \cdots, \frac{\partial f_n}{\partial x_i}(a)\right)$$
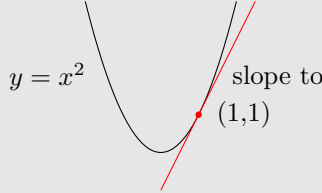
So for $f : U \to \mathbb{R}^m$, $a \in U$ we get $\frac{\partial f_j}{\partial x_i}$ where $1 \leq j \leq n$ and $1 \leq i \leq m$. We can arrange these in a matrix of dimensions $n \times m$

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(a) & \cdots & \dfrac{\partial f_1}{\partial x_m}(a) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial f_n}{\partial x_1}(a) & \cdots & \dfrac{\partial f_n}{\partial x_m}(a) \end{bmatrix}$$

**Note:-**

$f'(a)$ can be defined as a linear map $\mathbb{R}^n \to \mathbb{R}^m$

In the old situation $f : \mathbb{R} \to \mathbb{R}$, $f'(a) \in \mathbb{R}$ is a $1 \times 1$ matrix, as such it encodes a linear map $\underset{x \mapsto f'(a)x}{\mathbb{R} \to \mathbb{R}}$



$y = x^2$

slope to

$(1,1)$

The tangent line can be though of as the graph of linear map $\underset{t \mapsto 10t}{\mathbb{R} \to \mathbb{R}}$

## 6.2 Differentiation

$f'(a)$ is a number such that $\lim\limits_{h \to 0} \frac{f(a+h) - f(a) - f'(a)h}{h} = 0$. Inspired by this for $a \in U(\text{Open in } \mathbb{R}^m) \xrightarrow{f} \mathbb{R}^n$, we define $f'(a)$ is a linear map $\mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - f'(a)h\|}{\|h\|} = 0 \text{ in } \mathbb{R}$$

($h$ is a small vector $\in \mathbb{R}$)

---

**Definition 6.2.1: Differentiation of $f : \mathbb{R}^m \supset U \to \mathbb{R}^n$**

$U$ open set in $\mathbb{R}^m$, $f : U \to \mathbb{R}^n$, $a \in U$ given. We say that $f$ is differentiable at $a$ if there is a linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ such that

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} = 0$$

i.e $\forall \, \epsilon > 0$, $\exists \, \delta > 0$ s.t.

$$\|h\| < \delta \implies \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} < \epsilon$$

We call such a linear map $T$ the derivative of $f$ at $a$, denoted by $f'(a), D(f(a))$

---

Note that $f'(a)h = $ Value of linear map $f'(a)$ applied to a vector $h$

**Note:-**

If the above limit is 0 w.r.t any norm on $\mathbb{R}^M$ (respectively $\mathbb{R}^n$) then the same limit is 0 w.r.t any other norm on $\mathbb{R}^m$ (respectively $\mathbb{R}^n$) because all norms are equivalent

**Theorem 6.2.1**

Derivative is unique i.e. if $a \in U(\text{Open in } \mathbb{R}^m) \xrightarrow{f} \mathbb{R}^n$, and

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} = 0 = \lim_{h \to 0} \frac{\|f(a+h) - f(a) - S(h)\|}{\|h\|}$$

then $T = S$ i.e $T(v) = S(v) \; \forall \, v \in \mathbb{R}^m$

***Proof:*** Let $R = S - T$. Want to show $R(v) = 0 \ \forall \ v \in \mathbb{R}^m$

$$\frac{\|R(h)\|}{\|h\|} = \frac{\|S(h) - T(h)\|}{\|h\|} = \frac{\|(f(a+h) - f(a) - T(h)) - (f(a+h) - f(a) - S(h))\|}{\|h\|}$$

$$\leq \frac{\|(f(a+h) - f(a) - T(h))\|}{\|h\|} + \frac{\|(f(a+h) - f(a) - S(h))\|}{\|h\|}$$

Taking $\lim\limits_{h \to 0}$, we get $\lim\limits_{h \to 0} \frac{\|R(h)\|}{\|h\|} = 0$.

Fix any nonzero $v$, $\lim\limits_{\lambda \to 0} \lambda v = 0$. Take $h = \lambda v \ (\lambda \neq 0)$

$$\frac{\|R(h)\|}{\|h\|} = \frac{|\lambda| \|R(v)\|}{|\lambda| \|v\|} = \frac{\|R(v)\|}{\|v\|}$$

Hence

$$0 = \lim_{h \to 0} \frac{\|R(h)\|}{\|h\|} = \lim_{h \to 0} \frac{\|R(v)\|}{\|v\|} \implies \|R(v)\| = 0 \implies Rv = 0$$

$\square$

### Question 21

If $f : \mathbb{R}^m \to \mathbb{R}^n$ is a linear map then what is $f' : \mathbb{R}^m \to \mathbb{R}^n$
$\quad v \mapsto Av$

***Solution:*** See that $f'(a) = f$. (Immediate from definition)

$\square$

### Question 22

For an affine map $\mathbb{R}^m \xrightarrow{g} \mathbb{R}^n$ for some $c \in \mathbb{R}^n$. Calculate $g'(a)$.
$\quad v \mapsto Av + c$

***Solution:*** $g'(a) =$ the map $h \mapsto Ah$

$\square$

**Theorem 6.2.2** Matrix of $f'(a)$

Prove that the matrix of $f'(a)$ w.r.t standard basis of $\mathbb{R}^m$ and $\mathbb{R}^n$ is the Jacobian Matrix

***Proof:*** $j$th column of matrix of $T = T(e_j) \in \mathbb{R}^n$

$$\lim_{\lambda \to 0} \frac{\|f(a + \lambda e_j) - f(a) - T(\lambda e_j)\|}{\|\lambda e_j\|} \quad \text{by definition of } T = f'(a)$$

$$= \lim_{\lambda \to 0} \frac{\|f(a + \lambda e_j) - f(a) - \lambda T(e_j)\|}{|\lambda|} \quad \begin{cases} T(\lambda e_j) = \lambda T(e_j) \text{ by linearity} \\ \|\lambda e_j\| = |\lambda| \|e_j\| = |\lambda| \end{cases}$$

$$= \lim_{\lambda \to 0} \left\| \frac{f(a + \lambda e_j) - f(a)}{|\lambda|} - \frac{\lambda T(e_j)}{|\lambda|} \right\|$$

Hence for $\lambda > 0$

$$\lim_{\lambda \to 0} \frac{f(a + \lambda e_j) - f(a)}{|\lambda|} - T(e_j) = 0$$

Let $f = \begin{bmatrix} f_1 \\ f_2 \\ \vdots \\ f_n \end{bmatrix}$. Hence

$$T(e_j) = \lim_{\lambda \to 0} \frac{f(a + \lambda e_j) - f(a)}{\lambda}$$

$$= \lim_{\lambda \to 0} \frac{\begin{bmatrix} f_1(a + \lambda e_j) \\ f_2(a + \lambda e_j) \\ \vdots \\ f_n(a + \lambda e_j) \end{bmatrix} - \begin{bmatrix} f_1(a) \\ f_2(a) \\ \vdots \\ f_n(a) \end{bmatrix}}{\lambda} = \begin{bmatrix} \dfrac{\partial f_1}{\partial x_j}(a) \\[2mm] \dfrac{\partial f_2}{\partial x_j}(a) \\[2mm] \vdots \\[2mm] \dfrac{\partial f_n}{\partial x_j}(a) \end{bmatrix}$$

Matrix of $f'(a)$ = Jacobian Matrix

$$\begin{bmatrix} \dfrac{\partial f_1}{\partial x_1}(a) & \cdots & \dfrac{\partial f_1}{\partial x_m}(a) \\[2mm] \vdots & \ddots & \vdots \\[2mm] \dfrac{\partial f_n}{\partial x_1}(a) & \cdots & \dfrac{\partial f_n}{\partial x_m}(a) \end{bmatrix}$$

We have proved if $f'(a)$ exists, then all partial derivatives $\frac{\partial f_i}{\partial x_j}$ exists at $x = a$ and make up the matrix of $f'(a)$ $\quad\square$

If all $\frac{\partial f_i}{\partial x_j}$ exists at $x = a$, does not imply $f$ is differentiable at $x = a$?

### Question 23

Under Which conditions if all $\frac{\partial f_i}{\partial x_j}$ exists at $x = a$, it implies that $f$ is differentiable at $x = a$?

### Theorem 6.2.3

If $f'(a)$ exists then $f$ is continuous at $x = a$

**Proof:** If $f'(a)$ exists then $f$ is continuous at $x = a$ $\iff \lim_{h \to 0} f(a+h) = f(a) \iff \lim_{h \to 0} \|f(a+h) - f(a)\| = 0$

$$\|f(a+h) - f(a) + T(h) - T(h)\| \le \|f(a+h) - f(a) - T(h)\| + \|T(h)\|$$

Now $\lim_{h \to 0} \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} \|h\| = 0 \cdot 0 = 0$ and

$$\lim_{h \to 0} \|T(h)\| = 0 \text{ because } \begin{cases} T \text{ is continuous (being linear) so} \\ T(h) \to T(0) = 0 \end{cases}$$

$\square$

# Chapter 7

# Chain Rule of Differentiation and Operator Norm

## 7.1 Operator Name

$V, W$ are vector spaces. $\mathcal{L}(V, W) =$ Set of linear maps $V \to W$ is a vector space via $(A + B)(v) = A(v) + B(v)$ and $A(\lambda v) = \lambda A(v)$

If $V = \mathbb{R}^m$ and $\mathbb{R}^n$, $\dim(\mathbb{R}^m, \mathbb{R}^n) = mn$. We can identity $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ with $n \times m$ matrices.

$$\|A\|_{\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)} = \|A\| = \sup_{\|u\|=1} \|A(u)\|$$

This gives a norm because $\|A\| \geq 0$ and $\|A\| = 0 \implies A = 0$ and $\|\lambda A\| = |\lambda| \|A\|$. As $(A + B)(u) = A(u) + B(u)$ we have $\|(A + B)(u)\| \leq$
$A(u)\| + \|B(u)\|$ in $W$ and hemce $\|A + B\| \leq \|A\| + \|B\|$.

> **Question 24**
>
> Why this is well defined ?

**Solution:** The set $S = \{u \mid \|u\| = 1\}$ is closed and bounded in $V$, therefore compact. $A$ being linear is continuous. $\therefore A(S)$ is a compact subset of $W \implies A(S)$ is bounded.

$\square$

**Basic Properties:-**

1. $\|Av\| \leq \|A\| \|v\|$ i.e. $\|Av\|_W \leq \|A\|_{\mathcal{L}} \|v\|_V$

   **Proof:** If $v = 0$ then we are done. If $v \neq 0$, $u = \frac{v}{\|v\|}$ so $\|u\| = 1$. Hence

   $$\|A\| \geq \|Au\| = \left\| A\left(\frac{v}{\|v\|}\right) \right\| = \frac{\|Av\|}{\|v\|}$$

   $\square$

2. $\|A(v)\| \leq M\|v\|$ for all $v \implies \|A\| \leq M$ in fact $\inf\{M \mid \|A(v)\| \leq M\|v\| \, \forall \, v\}$

   **Proof:** Suppose $\|A(v)\| \leq M\|v\| \, \forall \, v$. In particular $\forall \, v$ with $\|v\| = 1$. So $\|A(v)\| \leq M$.

   Rest exercise: If $L < \inf$ of the set show $\exists \, u$ of norm=1 with $\|A(v)\| > L$. $\square$

3. $U \xrightarrow{A} V \xrightarrow{B} W$ linear maps between finite dimensional vector spaces then $\|BA\| \leq \|B\| \|A\|$

   **Proof:** Take $u$ with $\|u\| = 1$. Then

   $$\|BA(u)\| \leq \|B\| \|A(u)\| \leq \|B\| \|A\| \|u\| = \|B\| \|A\|$$

   Now take sup over $u$. $\square$

> **Note:-**
>
> $A, B \underset{\mathcal{L}(U,V)\oplus\mathcal{L}(V,W)}{} \mapsto \underset{\mathcal{L}(U,W)}{BA}$ is continuous because each slot of matrix of $BA$ is obtained by adding/multi-plying entries of $A$ and $B$.

---

**Question 25**

Show $A_n \to A$ in $\mathcal{L}(U,V)$, $B_n \to B$ in $\mathcal{L}(V,W)$ then $B_n A_n \to BA$ in $\mathcal{L}(U,W)$

## 7.2 Chain Rule

> **Theorem 7.2.1**
>
> Let
>
> $$\mathbb{R}^n$$
> $$\cup$$
> $$\mathbb{R}^m \supset U \xrightarrow{\ f\ } V \xrightarrow{\ g\ } \mathbb{R}^k$$
> $$\cup \qquad \cup$$
> $$a \longmapsto b$$
>
> $f$ is differentiable at $a$. and $g$ is differentiable at $b = f(a)$. Then $g \circ f$ is differentiable at $a$ and
>
> $$(g \circ f)'(a) = \underbrace{g'(f(a))f'(a)}_{\substack{\text{Multiplication} \\ \text{of matrices}}}$$

***Proof:*** **1-Variable Case**

$\frac{dz}{dx} = \frac{dz}{dy}\frac{dy}{dx}$

$$\frac{dz}{dx} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y}\frac{\Delta y}{\Delta x}$$
$$= \lim_{\Delta x \to 0} \frac{\Delta z}{\Delta y} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$
$$= \lim_{\Delta y \to 0} \frac{\Delta z}{\Delta y} \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \qquad [\text{As } \Delta x \to 0,\ \Delta y \to 0]$$

**Multi Variable Case**

Note that $\mathbb{R}^m \xrightarrow{f'(a)} \mathbb{R}^n \xrightarrow{g'(f(a))} \mathbb{R}^k$

If $T = f'(a)$ then

$$\lim_{h \to 0} \frac{\|f(a+h) - f(a) - T(h)\|}{\|h\|} = 0$$

and $S = g'(f(a)) = g'(b)$ then

$$\lim_{h \to 0} 7\frac{\|fg(b+k) - g(b) - S(k)\|}{\|k\|} = 0$$

Let

$$\alpha(h) = f(a+h) - f(a) - T(h) \qquad\qquad \epsilon(h) = \frac{\|\alpha(h)\|}{\|h\|} \to 0 \text{ as } h \to 0$$

$$\beta(k) = g(b+k) - g(b) - S(k) \qquad\qquad \eta(k) = \begin{cases} \dfrac{\|\beta(k)\|}{\|k\|} \to 0 \text{ as } k \to 0 \\ 0 \text{ when } k = 0 \end{cases}$$

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$\eta$ is continuous at $k = 0$. Now note that $\eta : V - b \to \mathbb{R}$ because we ae always taking $b + k$ for $\eta$. We want to show that
$$\lim_{h \to 0} \frac{\|g(f(a+h)) - g(f(a)) - ST(h)\|}{\|h\|} = 0 \iff \lim_{k \to 0} \frac{\|g(b+k) - g(b) - ST(h)\|}{\|h\|} = 0$$
where $f(a+h) = b+k \iff k = f(a+h) - f(a)$. We have taken a specific value of $k$ depending on $h$. So now $k$ is a function of $h$. Hence $T(h) = f(a+h) - f(a) - \alpha(h) = k - \alpha(h)$

$$g(b+k) - g(b) - ST(h)$$
$$= g(b+k) - g(b) - S(k - \alpha(h))$$
$$= g(b+k) - g(b) - S(k) + S(\alpha(h))$$

Therefore
$$\frac{\|g(b+k) - g(b) - ST(h)\|}{\|h\|} \leq \frac{\|g(b+k) - g(b) - S(k)\|}{\|h\|} + \frac{\|S(\alpha(h))\|}{\|h\|}$$
want to bound each of these separately

$$\frac{\|S(\alpha(h))\|}{\|h\|} \leq \|S\| \frac{\|\alpha(h)\|}{\|h\|} \to 0$$

Now how to bound the first term. In the first term $\frac{\|\beta(k)\|}{\|h\|} = \eta(k) \frac{\|k\|}{\|h\|}$. Now

$$k = T(h) + \alpha(h)$$
$$\implies \|k\| \leq \|T(h)\| + \|\alpha(h)\|$$
$$\implies \frac{\|k\|}{\|h\|} \leq \frac{\|T(h)\|}{\|h\|} + \frac{\|\alpha(h)\|}{\|h\|} \leq \frac{\|T\|\|h\|}{\|h\|} + \frac{\|\alpha(h)\|}{\|h\|} = \|T\| + \frac{\|\alpha(h)\|}{\|h\|}$$

Hence
$$\frac{\|\beta(k)\|}{\|k\|} = \eta(k) \frac{\|k\|}{\|h\|} \leq \eta(k) \left[ \|T\| + \frac{\|\alpha(h)\|}{\|h\|} \right]$$

As $h \to 0$ $\|T\| + \frac{\|\alpha(h)\|}{\|h\|} \to \|T\| + 0$ which is finite. And as $h \to 0$, $k \to 0 \implies \eta(k) \to 0$ because $\eta$ is continuous at 0. $\qquad\square$

## 7.3 Special Case of Chain Rule: When $m = k = 1$

Open interval in $\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^n \xrightarrow{g} \mathbb{R}$. $\gamma$ = parameterized curve in $\mathbb{R}^n$
$$(g \circ \gamma)'(t) = g'(\gamma(t)) \cdot \gamma'(t)$$

$\mathbb{R} \xrightarrow{\gamma} \mathbb{R}^n$ maps $t \to \begin{bmatrix} \gamma_1(t) \\ \vdots \\ \gamma_n(t) \end{bmatrix}$ hence $\gamma'(t) = \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix}$

Now $g'(y) = \begin{bmatrix} \left.\frac{\partial g}{\partial x_1}\right|_y & \cdots & \left.\frac{\partial g}{\partial x_n}\right|_y \end{bmatrix}$. Hence

$$(g \circ \gamma)'(t) = \begin{bmatrix} \left.\frac{\partial g}{\partial x_1}\right|_y & \cdots & \left.\frac{\partial g}{\partial x_n}\right|_y \end{bmatrix} \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix}$$

$$= \begin{bmatrix} \left.\frac{\partial g}{\partial x_1}\right|_y \\ \vdots \\ \left.\frac{\partial g}{\partial x_n}\right|_y \end{bmatrix} \cdot \begin{bmatrix} \gamma_1'(t) \\ \vdots \\ \gamma_n'(t) \end{bmatrix} \qquad \text{[Usual dot product of vectors in } \mathbb{R}^n\text{]}$$

Call $\begin{bmatrix} \frac{\partial g}{\partial x_1}\big|_y \\ \vdots \\ \frac{\partial g}{\partial x_n}\big|_y \end{bmatrix}$ $\nabla g(\gamma(t)) = $ Gradient of $g$ at the point $\gamma(t)$. Hence $(g \circ \gamma)'(t) = \nabla g(\gamma(t)) \cdot \gamma'(t)$

### Question 26

Fix $u, v \in \mathbb{R}^n$ and take parametrized curve $\gamma(t) = u + tv$. What does the above equation give at $t = 0$ (for a given function $g$)

# Chapter 8

# Mean Value Theorem

We will use Euclidean norm on $\mathbb{R}^n$ and have Cauchy-Schwarz Inequality $|v \cdot w| \leq \|v\|\|w\|$.

> **Theorem 8.1** 1-Variable MVT
>
> If $f : [a, b] \to \mathbb{R}$ continuous and $f'$ exists on $(a, b)$, then $\exists ? c \in (a, b)$ s.t
>
> $$f(a) - f(b) = f'(c)(b - a)$$

**Proof:**  1-variable MVT
$\Uparrow$ Via Rolle's Theorem, using $f'(\text{extremum}) = 0$
Extreme Value Theorem
$\Uparrow$ $[a, b]$ is compact, $f$ is continuous $\implies$ $f([a, b])$ is compact in $\mathbb{R}$ $\implies$ closed and bounded
Heine Borel Theorem $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

---

**Question 27**

First consider $f : [a, b] \to \mathbb{R}^n$ continuous and $f'$ exists on $(a, b)$. Is there a $c \in (a, b)$ s.t.

$$\|f(b) - f(a)\| \overset{?}{=} \|f'(c)\|(b - a)$$

---

**Solution:**  No. For example $f : [0, 2\pi] \to \mathbb{R}^2$ which maps $t \mapsto (\sin t, \cos t)$. Then $f'(t) = (\cos t, -\sin t)$ and $\|f'(t)\| = 1$. $f(2\pi) - f(0) = (0, 0)$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

> **Theorem 8.2** MVT of Real-Valued Functions
>
> Let $f$ be a continuous function $[a, b] \to \mathbb{R}^n$ and $f'(c)$ exists $\forall\, c \in (a, b)$. Then $\exists\, c \in (a, b)$ s.t
>
> $$\|f(b) - f(a)\| \leq (b - a)\|f'(c)\|$$
>
> (Here the norm is Euclidean norm. For the inequality which norm we take does matter.)

**Proof:**  Clever use if $1-$Variable MVT. We want to bound norm of $f(b) - f(a) = z$.
**Idea:** Dot with $z$ and then use Cauchy Schwarz

$$[a, b] \xrightarrow{\ f\ } \mathbb{R}^n \xrightarrow{\ g = \langle -, z \rangle\ } \mathbb{R}$$
$$t \longmapsto f(t) \longmapsto \langle f(t), z \rangle$$
$$\phi(t)$$

Notice $g : \mathbb{R}^n \to \mathbb{R}$ where $g$ maps $x \longmapsto \langle x, z \rangle = x_1 z_1 + \cdots + x_n z_n$. Hence $g$ is differentiable and $g'(x) = \begin{bmatrix} z_1 & z_2 & \cdots & z_n \end{bmatrix}$. Now we can apply $MVT$ to $\phi$.

$$\phi(b) - \phi(a) = (b - a)\phi'(c)$$

$$
\begin{aligned}
LHS = \phi(b) &= \phi(a) \\
&= \langle f(b), z \rangle - \langle f(a), z \rangle \\
&= \langle f(b) - f(a), z \rangle = \langle z, z \rangle = \|z\|^2
\end{aligned}
$$

And

$$\phi'(c) = (g \circ f)'(c) = g'(f(c)) \circ f'(c) = \langle z, f'(c) \rangle$$

Therefore

$$\|z\|^2 = \langle z, f'(c) \rangle \leq (b - a)\|z\|\|f'(c)\|$$

Nothing to prove if $\|z\| = 0$ and else cancel $\|z\|$ from both sides. $\qquad\square$

---

**Theorem 8.3** General Multivariable MVT

Let $\mathbb{R}^m \supset$ Convex Open $U \xrightarrow{f} \mathbb{R}^n$, $f$ differentiable on $U$ and $\|f'(x)\| < M \ \forall \ x \in U$. Then $\forall \ a, b \in U$

$$\|f(b) - f(a)\| \leq M\|b - a\|$$

---

**Proof:** Given $a, b \in U$ holds $\gamma : [0,1] \to U$ which maps $t \to a + t(b - a)$ [This is valid by convexity]. Apply MVT to

$$[0,1] \xrightarrow{\ \gamma\ } U \xrightarrow{\ f\ } \mathbb{R}^n$$
$$\underbrace{\qquad\qquad}_{g}$$

We get $\in (0, 1)$ such that $\|g(1) - g(0)\| \leq \|g'(t)\|$ i.e

$$\|f(b) - f(a)\| \leq \| \underbrace{f'(\gamma(t)) \circ \gamma'(t)}_{\substack{\text{Matrix Vector} \\ \text{Multiplication}}} \| \underbrace{\leq}_{\substack{\downarrow \\ \text{Justify}}} M\|b - a\|$$

$\qquad\square$

Suppose $U$ is convex in $\mathbb{R}^m$, $f : U \to \mathbb{R}^n$ and $f'(a) = 0 \ \forall \ a \in U$. Then $f = $ Constant because

$$\|f(b) - f(a)\| \leq 0\|b - a\|$$

$\forall \ a, b \in U$ by MVT. What happens if $U$ is open but not convex. If $U$ is connected the conclusion is again true.

---

**Definition 8.1: Connected Set in $\mathbb{R}^n$**

A set $S$ in $\mathbb{R}^n$ is (path)connected if $\forall \ a, b \in S \ \exists$ continuous function

$$
\begin{aligned}
\gamma \ [0,1] &\longrightarrow S \\
0 &\longmapsto a \\
1 &\longmapsto b
\end{aligned}
$$

---

**Theorem 8.4**

If $S$ is connected open set in $\mathbb{R}^n$ and $f : S \to \mathbb{R}^n$ is differentiable on $U$ with $f'(a) = 0 \ \forall \ a \in U$ then $f(a) = $ Constant.

---

**Proof:** $\forall \ x \in \gamma([0,1])$ find $B_r(x) \subset S$. $\gamma([0,1])$ is compact. So $\exists$ finite subcover of $\gamma([0,1])$ by balls around $\gamma(t_1), \gamma(t_2), \ldots, \gamma(t_N)$.

ORder these alls so that $a \in$ first ball and $b \in$ last ball, any two consecutive balls overlap. This gives piecewise linear path from $a \to b$. Use MVT for each segment. $\qquad\square$

# Chapter 9

# Higher Derivatives

## 9.1 Class $C^1$ Functions

Open $U$ in $\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n$. $D(f(a))$ is a linear map $\mathbb{R}^m \to \mathbb{R}^n$ i.e $D(f(a)) \in \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. If $f$ is differentiable at each $a \in U$, then we get a function $Df : U \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ which maps $a \longmapsto D(f(a)) = f'(a)$. We can ask about continuity and differentiability of this map $Df$.

We want to consider $C^1(U)$ functions which are all functions that are differentiable at each $a \in U$ and $Df$ is continuous i.e $C^1(U) = $ Set of continuously differentiable functions

> **Definition 9.1.1: $C^1$ Functions**
>
> $U$ open in $\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n$. Suppose $f'(a)$ exists $\forall \, a \in U$ then we get a function
>
> $$f : U \longrightarrow \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \backsimeq \mathbb{R}^{mn}$$
> $$a \longmapsto f'(a)$$
>
> which maps $a \longmapsto f'(a)$. We say $f \in C^1(U)$, "$f$ is continuously differentiable" if $f'(a)$ exists for each $a$ and $f'$ is a continuous function

> **Theorem 9.1.1**
>
> A function $f : U$ Open in $\mathbb{R}^m \to \mathbb{R}^n$ is $C^1(U) \iff \left. \dfrac{\partial f_i}{\partial x_j} \right|_a$ exists at each $a \in U$ and are continuous functions

***Proof:*** **If Part:-**

① $a \in U$ Open in $\mathbb{R}^m \xrightarrow{f} \mathbb{R}^n$ s.t $f(a) = \begin{bmatrix} f_1(a) \\ \vdots \\ f_n(a) \end{bmatrix}$, $\displaystyle\lim_{h \to 0} \dfrac{f(a+h) - f(a) - Th}{\|h\|} = 0$ Matrix of $T$ w.r.t standard

basis of $\mathbb{R}^m$ and $\mathbb{R}^n$ is

$$T = \left[ \left.\dfrac{\partial f}{\partial x_1}\right|_a \quad \cdots \quad \left.\dfrac{\partial f}{\partial x_m}\right|_a \right] = \begin{bmatrix} \left.\dfrac{\partial f_1}{\partial x_1}\right|_a & \cdots & \left.\dfrac{\partial f_1}{\partial x_m}\right|_a \\ \vdots & \ddots & \vdots \\ \left.\dfrac{\partial f_n}{\partial x_1}\right|_a & \cdots & \left.\dfrac{\partial f_n}{\partial x_m}\right|_a \end{bmatrix}$$

② $\displaystyle\lim_{x \to v} f(x) = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix} \iff \displaystyle\lim_{x \to v} f_I(x) = b_i$ for each $i = 1, 2, \ldots, n$

By ① and ② the proof of forward direction is obvious

**Only If Part:-**

If we prove that $f'(a)$ exists for each $a \in U$ then $f'$ is automatically continuous because by ① the matrix of $f'(a)$ must be the Jacobian Matrix and we are given that all entries of this matrix namely the functions $\dfrac{\partial f_i}{\partial x_j}$ are continuous so apply ②

    **Another reduction:** We may assume that $n = 1$ because this case in general, it follows immediately that for $f == \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}$,

$$f'(a) = \begin{bmatrix} f_1(a) \\ \vdots \\ f_n(a) \end{bmatrix} = \begin{bmatrix} \dfrac{\partial f}{\partial x_1}\Big|_a & \cdots & \dfrac{\partial f}{\partial x_m}\Big|_a \end{bmatrix}$$

Hence

$$\frac{f(a+h) - f(a) - Th}{\|h\|} = \frac{1}{\|h\|}\left( \begin{bmatrix} f_1(a+h) \\ \vdots \\ f_n(a+h) \end{bmatrix} - \begin{bmatrix} f_1(a) \\ \vdots \\ f_n(a) \end{bmatrix} - \begin{bmatrix} f_1'(a)h \\ \vdots \\ f_n'(a)h \end{bmatrix} \right)$$

$\lim\limits_{h \to 0}$ of this $= 0$ because in each slot the limits is $0$ by $n = 1$ case which we have assumed, and will prove now.

> **Note:-**
>
> Proof of the fact that in case of $n = 1$ if $\dfrac{\partial f_i}{\partial x_j}$ $(j = 1, 2, \ldots, m)$ are continuous functions $U \to \mathbb{R}$ then $f'(a)$ exists for each $a \in U$

We want to show $f'(a) = \begin{bmatrix} \dfrac{\partial f}{\partial x_1}\Big|_a & \cdots & \dfrac{\partial f}{\partial x_m}\Big|_a \end{bmatrix}$ i.e $\lim\limits_{h \to 0} \dfrac{f(a+h) - f(a) - Th}{\|h\|} = 0$. We want to bound the numerator. Fix $a = \begin{bmatrix} a_1 & \cdots & a_m \end{bmatrix}^T$. Let $h = \begin{bmatrix} h_1 & \cdots & h_m \end{bmatrix}^T$. Now choose $r > 0$ such that $B_r(a) \subset U$ and restrict $h$ such that $\|h\| < r$.
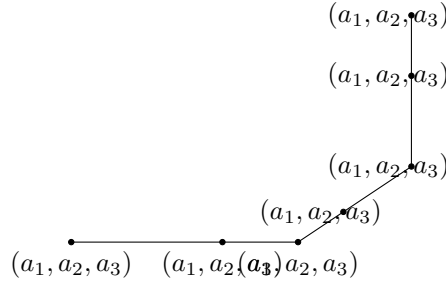
$$Th = \sum_j \frac{\partial f}{\partial x_j}\Big|_a h_j = \left\langle \underbrace{\begin{bmatrix} \dfrac{\partial f}{\partial x_1}\Big|_a \\ \vdots \\ \dfrac{\partial f}{\partial x_m}\Big|_a \end{bmatrix}}_{\overset{\|}{q}}, \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} \right\rangle$$

And $f(a+h) - f(a) = f(a_1, h_1, \ldots, a_m + h_m) - f(a_1, \ldots, a_n)$

    **Idea:** Bound this in terms of partial derivatives using the mean value theorem (ordinary 1-variable version, which is applicable because each $\frac{\partial f}{\partial x_j}$ is continuous)

$$
\begin{aligned}
f(a+h) - f(a) =\ & f(a_1 + h_1, a_2, \ldots, a_m) - f(a) \\
& + f(a_1 + h_1, a_2 + h_2, \ldots, a_m) - f(a_1 + h_1, a_2, \ldots, a_m) \\
& \vdots \qquad\qquad\qquad \vdots \\
& + f(a_1 + h_1, a_2 + h_2, \ldots, a_m + h_m) - f(a_1 + h_1, a_2 + h_2, \ldots, a_{m-1} + h_{m-1}, a_m)
\end{aligned}
$$

$$\overset{\text{MVT}}{=} \frac{\partial f}{\partial x_1}\Big|_{v_1} h_1 + \frac{\partial f}{\partial x_2}\Big|_{v_2} h_2 + \cdots + \frac{\partial f}{\partial x_m}\Big|_{v_m} h_m = \left\langle \underbrace{\begin{bmatrix} \dfrac{\partial f}{\partial x_1}\Big|_{v_1} \\ \vdots \\ \dfrac{\partial f}{\partial x_m}\Big|_{v_m} \end{bmatrix}}_{\overset{\|}{p}}, \begin{bmatrix} h_1 \\ \vdots \\ h_m \end{bmatrix} \right\rangle$$

$(a_1, a_2, a_3)$

$(a_1, a_2, a_3)$

$(a_1, a_2, a_3)$

$(a_1, a_2, a_3)$

$(a_1, a_2, a_3)$   $(a_1, a_2, (a_3)a_2, a_3)$

Notice that $v_1, v_2, \ldots, v_m$ are functions of $h$. Putting together we get

$$\frac{f(a+h) - f(a) - Th}{\|h\|} = \frac{1}{\|h\|} \left\langle h, \begin{bmatrix} \left.\frac{\partial f}{\partial x_1}\right|_{v_1} - \left.\frac{\partial f}{\partial x_1}\right|_a \\ \vdots \\ \left.\frac{\partial f}{\partial x_m}\right|_{v_m} - \left.\frac{\partial f}{\partial x_1}\right|_a \end{bmatrix} \right\rangle$$

$$= \frac{1}{\|h\|} \|h\| \|p - q\| = \|p - q\|$$

Showing $\lim\limits_{h \to 0} \|p - q\| = 0$ is enough to complete the proof

$$\lim_{h \to 0} \|p - q\| = 0 \impliedby p - q \to 0 \text{ as } h \to 0 \impliedby \left.\frac{\partial f}{\partial x_i}\right|_{v_i} - \left.\frac{\partial f}{\partial x_i}\right|_a \to 0 \text{ as } h \to 0$$

which is true because $\frac{\partial f}{\partial x_i}$ is a continuous function. More formally choose $\|h\| < \delta$ s.t

$$\left\| \frac{\partial f}{\partial x_i}(b) - \frac{\partial f}{\partial x_1}(a) \right\| < \frac{\epsilon}{m} \ \forall \ b \in B_\delta(a)$$

That ensures $\|p - q\| < \epsilon$ by triangle inequality.

---

**Question 28**

$$f(x, y) = \begin{cases} \frac{xy^2}{xx^2 + y^4} & (x, y) \neq (0, 0) \\ 0 & \text{else} \end{cases}$$

1. Calculate all directional derivatives in particular $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

2. Does $f'(a, b)$ exists at all $a, b \in \mathbb{R}$

3. Is $f$ continuous everywhere

$\square$

## 9.2 Higher Derivatives and Class $C^k$ functions

Now we want to define class $C^k$ of functions for $k \geq 0$. [Like in 1-Variable case. Usefull for Taylor's theorem].

Now second derivative of $f : U \to \mathbb{R}^n$ at $a \in U \subset \mathbb{R}^m$ = derivative at $a$ of $f' : U \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n) \simeq \mathbb{R}^{mn}$.

$\therefore f''(a)$ or $D^2 f(a) : \mathbb{R}^m \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$. Matrix of $f''(a)$ has $mn$ rows and $m$ columns and equals to $\dfrac{\partial f'_{pq}}{\partial x_j}$ where

$p = 1, \ldots, n$, $q = 1, \ldots, m$ and $j = 1, \ldots, m$

**Question 29**

Do $\dfrac{\partial}{\partial x_l}$ and $\dfrac{\partial}{\partial x_k}$ commute ?

**Solution:** No but actually yes under some good conditions. We discussed here

$\square$

If $f''(a)$ exists for all $a \in U$, then we get a function $f''$ or $D^2 f : U \to \mathcal{L}(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$ which maps $a \mapsto f''(a)$. Dimension of the RHS is $m^2 n$. Now we can ask about continuity and differentiability of $f''$

---

**Definition 9.2.1: $C^k$ Functions**

Open $U \subset \mathbb{R}^m$, $f : U \to \mathbb{R}^n$. Then

$$
\begin{array}{lll}
f \text{ is } C^0 \text{ if} & & f \text{ is continuous} \\
f \text{ is } C^1 \text{ if} & f'(a) \text{ exists } \forall\, a \in U \text{ and} & f' \text{ is continuous} \\
f \text{ is } C^2 \text{ if} & f''(a) \text{ exists } \forall\, a \in U \text{ and} & f'' \text{ is continuous} \\
\quad\vdots & \quad\vdots & \quad\vdots \\
f \text{ is } C^k \text{ if} & f^{(k)}(a) \text{ exists } \forall\, a \in U \text{ and} & f^{(k)} \text{ is continuous}
\end{array}
$$

---

**Note:-**

How to understand $\mathcal{L}(V, \mathcal{L}(U, W))$ where $U, V, W$ are vector spaces. Just set theoretically

$$
\text{Maps(}\ A\ ,\ \text{Maps(}\ B\ ,\ C\ \text{))} \ \simeq\ \text{Maps(}\ A \times B\ ,\ C\ )
$$

$$
\downarrow \qquad\qquad\qquad\qquad\qquad\qquad \downarrow
$$

$$
f(a)\ :\quad B \longrightarrow C \ \Longleftrightarrow\ \phi \longmapsto \phi(a, b)
$$

$$
b \longmapsto \phi(a, b)
$$

Under this dictionary, maps in $\mathcal{L}(V, \mathcal{L}(U, W))$ must correspond to some special kind of maps $V \times U \to W$.

---

Hence we can say $\mathcal{L}(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n))$ is equivalent to the space of maps $\mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ which is space of bilinear maps from $\mathbb{R}^m \times \mathbb{R}^m$ to $\mathbb{R}^n$

Component functions of $f' : U \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ are precisely $\frac{\partial f_i}{\partial x_j}$ where $i = 1, \ldots, n$ and $j = 1, \ldots, m$. So matrix of $f''(a)$ w.r.t standard basis of $\mathbb{R}^m$ and $\mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ will consist of numbers $\left( \frac{\partial}{\partial x_k} \left( \frac{\partial}{\partial x_j} f_i \right) \right)(a)$. If $f''(a)$ exists then these are generated to exist. If $f''(a)$ exists at each $a \in U$ then we have the function

$$
f'' : U \longrightarrow \mathcal{L}(\mathbb{R}^m, \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)) \simeq \text{Space of bilinear maps } \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n
$$
$$
a \longmapsto f''(a)
$$

$f''$ is $C^2(U) \overset{\text{definition}}{\Longleftrightarrow} f''$ is continuous on $U \iff \dfrac{\partial}{\partial x_k} \dfrac{\partial}{\partial x_j} f_i$ are continuous functions $U \to \mathbb{R}$