Analysis-I Exercise
- Rajeeva L. Karandikar

Soham Chatterjee

Roll: BMC202175

1 Exercise A

A1. Given that

$$a + t = b + t$$

where $a, b, t \in \mathbb{R}$. Now adding $-t$ both sides we get

$$\begin{aligned} (a+t) - t &= (b+t) - t \\ \text{or, } a + (t - t) &= b + (t - t) \\ \text{or, } a + 0 &= b + 0 \\ \text{or, } a &= b \text{ [Proved]} \end{aligned}$$

A2. Given that

$$a \cdot s = b \cdot s$$

where $a, b, s \in \mathbb{R}$. Now multiplying $\frac{1}{s}$ both sides we get

$$\begin{aligned} (a \cdot s) \cdot \frac{1}{s} &= (b \cdot s) \cdot \frac{1}{s} \\ \text{or, } a \cdot (s \cdot \frac{1}{s}) &= b \cdot (s \cdot \frac{1}{s}) \\ \text{or, } a \cdot 1 &= b \cdot 1 \\ \text{or, } a &= b \text{ [Proved]} \end{aligned}$$

A3. Given that

$$a + t < b + t$$

where $a, b, t \in \mathbb{R}$. Now adding $-t$ both sides we get

$$\begin{aligned} (a+t) - t &< (b+t) - t \\ \text{or, } a + (t - t) &< b + (t - t) \\ \text{or, } a + 0 &< b + 0 \\ \text{or, } a &< b \text{ [Proved]} \end{aligned}$$

A4. We know that $x - x = 0$. If $x > 0$ then $x - x > 0 - x \implies 0 > -x$. Similarly if $x < 0$ then $x - x < 0 - x \implies 0 < -x$.

Let $x \cdot y > 0$ and $x > 0$. Let $y < 0$. Hence $z > 0$. Hence $x \cdot -y > 0$. Hence

$$x \cdot y + x \cdot (-y) > 0 \implies x \cdot (y - y) > 0 \implies x \cdot 0 > 0$$

which is not true. Hence $y > 0$. If $x < 0$ assume $y > 0$. Hence $-x > 0$. Hence $(-x) \cdot y > 0$. Hence

$$x \cdot y + (-x) \cdot y > 0 \implies (x - x) \cdot y > 0 \implies 0 \cdot y > 0$$

which is not true. Hence $y < 0$.

Now for the given problem

$$\begin{aligned}
&a \cdot s < b \cdot s \\
&\text{or, } a \cdot s + (-a) \cdot s < b \cdot s + (-a) \cdot s \\
&\text{or, } (a - a) \cdot s < (b - a) \cdot s \\
&\text{or, } 0 \cdot s < (b - a) \cdot s \\
&\text{or, } 0 < (b - a) \cdot s
\end{aligned}$$

As $s < 0$ hence

$$b - a < 0 \implies b - a + a < 0 + a \implies b < a \text{ [Proved]}$$

A5. Gicen that $x \cdot y = x$. Multiplying both sides by $\frac{1}{x}$

$$\begin{aligned}
&x \cdot y = x \\
&\text{or, } (x \cdot y) \cdot \frac{1}{x} = x \cdot \frac{1}{x} \\
&\text{or, } y \cdot \left(x \cdot \frac{1}{x} \right) = 1 \\
&\text{or, } y \cdot 1 = 1 \\
&\text{or, } y = 1 \text{ [Proved]}
\end{aligned}$$

2 Exercise B

B1. Given $y > 1$. As $y > 0$, $\frac{1}{y}$ multiplying both sides by $\frac{1}{y}$ we get

$$y \cdot \frac{1}{y} > 1 \cdot \frac{1}{y} \implies 1 > \frac{1}{y} \text{ [Proved]}$$

B2. Consider the real number $\frac{1}{z}$. By Archimedean Property $\exists\, n \in \mathbb{N}$ such that $n \cdot \frac{1}{z} > 1$ Now multiplying z in both sides we get

$$\left(n \cdot \frac{1}{z} > 1 \right) \cdot z > 1 \cdot z \implies n \cdot \left(\frac{1}{z} \cdot z \right) > z \implies n > z$$

Hence $\exists n \in \mathbb{N}$ such that $n > z$.[Proved]

B3. Let A be a non-empty set of integers which is bounded above. As $A \subset \mathbb{Z} \subset \mathbb{R}$, A has a least upper bound. Let's say it is s. Hence $\forall \epsilon > 0\ \exists\, a \in A$ such that

$$s - \epsilon < a \leq s$$

Because if it is not true then there exists no such $a \in A$ which is greater than $s - \epsilon$. Then $s - \epsilon$ would be an upper which is less than s which is not possible since s is the least upper bound. Now a is an integer. Take $\epsilon = 1$. If $s > a$ then a is not the upper bound $\exists\ b \in A$ such that

$$s \geq b > a > s - 1$$

Hence there exists two distinct integers such that $0 < b - a < 1$ which is not possible. Hence $s = a$. Therefore the least upper bound of a bounded non-empty set of integers is also an integer and belongs to that set.

Now, let S be the set of all integers n such that $n \leq z$. Then $\exists$ a least upper bound of B. Let b be the least upper bound of B. As we previously proved b is an integer and $b \in B$. Hence $b \leq z$.

Now, if $b + 1 \leq z$ then $b + 1 \in B$. Then there exists an upper bound of the set B which is greater than the least upper bound and also an element of B which is not possible. Hence $b + 1 > z$. Hence $\exists\ t \in \mathbb{Z}$ such that $t - 1 \leq z < t$.[Proved]

B4. $\forall\ z > 1,\ z \in \mathbb{R}\ \exists\ t \in \mathbb{Z}$ such that $t - 1 \leq z < t$. As $t - 1 \leq z$ we can say

$$t - 1 + 1 \leq z + 1 \implies t \leq z + 1$$

Hence

$$z < t \leq z + 1$$

Therefore $\exists\ s \in \mathbb{Z}$ such that $z < s \leq z + 1$.[Proved]

B5. Given that $y > x$. Hence $y - x > 0$. Now by Archimedean Property $\exists\ k \in \mathbb{N}$ such that

$$k \cdot (y - x) > 1 \implies k \cdot y > 1 + k \cdot x \text{ [Proved]}$$

B6. Now, $\exists\ k \in \mathbb{N}$ such that

$$1 + k \cdot x < k \cdot y$$

Now $\exists\ m \in \mathbb{N}$ such that $k \cdot x < m \leq k \cdot x + 1$. Hence

$$k \cdot x < m < k \cdot y$$

There $\exists\ k, m \in \mathbb{N}$ such that $k \cdot x < m < k \cdot y$. [Proved]

B7. We know that there $\exists\ k, m \in \mathbb{N}$ such that $k \cdot x < m < k \cdot y$. Multiplying $\frac{1}{k}$ we get

$$\frac{1}{k} \cdot k \cdot x < \frac{1}{k} \cdot m < \frac{1}{k} \cdot k \cdot y \implies x < \frac{m}{k} < y$$

As m, k are integers $\frac{m}{k} \in \mathbb{Q}$. Therefore there $\exists\ r \in \mathbb{Q}$ such that $x < r < y$.[Proved]

B8. AS $\beta > \alpha$, $\beta - \alpha > 0$. By Archimedean Property $\exists\ n \in \mathbb{N}$ such that $n \cdot (\beta - \alpha) > 1 \implies \beta - \alpha > \frac{1}{n}$. Now let S be the set of all integers k such that $k > n \cdot \alpha$. Hence the set S has a greatest lower bound. Let m be the greatest lower bound. Hence $m > n \cdot \alpha \implies \frac{m}{n} > \alpha$.

Now $n \cdot \alpha + 1 > m$ because if not then $m - 1$ be an element of S which is less than the greatest lower bound which is not possible. Hence

$$n \cdot x + 1 > m \implies \alpha + \frac{1}{n} > \frac{m}{n}$$

Therefore

$$\alpha + (\beta - \alpha) > \alpha + \frac{1}{n} > \frac{m}{n} > \alpha \implies y > \frac{m}{n} > \alpha$$

Hence $\forall\ \alpha, \beta \in \mathbb{R}\ \exists\ r \in \mathbb{Q}$ such that $\alpha < r < \beta$.[Proved]