Fulton Chapter 4: Projective Varieties

Projective Algebraic Sets

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Problem Set - 7
Topic: Algebraic Geometry

Problem 1 4,7

Show that each irreducible component of a cone is a cone.

Solution: Let V is an algebraic set over P^n.

$$C(V) = \{(x_1, x_2, ...x_{n+1}) | (x_1, x_2, ...x_{n+1}) \in A^{n+1} \text{ or } (x_1, x_2, ...x_{n+1}) = (0, 0...0)\}$$

is defined to be the cone over V. Let $V = \cup_{i=1}^{n} V_i$ where V_i is an irreducible component of V.

Claim: $C(V) = \bigcup_{i=1}^{n} C(V_i)$

Proof: Let $a \in C(V) \implies a = (0, 0, ..0)$ or $a \in V$. In both of the cases $a \in \cup_{i=1}^{n} C(V_i)$. If $b \in \bigcup_{i=1}^{n} C(V_i) \implies b \in C(V_i) \implies b = (0, 0, ...0)$ or $b \in V_i \implies b \in C(V)$. So, $C(V) = \bigcup_{i=1}^{n} C(V_i)$.

□

Now $I_a(C(V_i)) = I_p(V_i)$ as V_i is an irreducible projective space. $I_p(V_i)$ is prime $\implies C(V_i)$ is irreducible. So, irreducible component of $C(V)$ is also a cone [as the decomposition is unique]

□

Problem 2 4,12

Let $H_1, H_2..H_m$ be hypersurfaces in $P^n, m \leq n$.Show that $H_1 \cap H_2 \cap ...H_m \neq \phi$

Solution: Hyperplane is a hypersurface defined by a form of degree 1.i.e ,V is a hypersurface if $V = V(F)$ where $deg(F) = 1$ and F is a form. $V = \cap_{i=1}^{n} H_i$. Let $H_i = V(F_i)$, $V = V(F_1, F_2...F_m)$. Let

$$F_i(X_1, X_2..X_{n+1}) = \sum_{j=1}^{n+1} a_{ji} X_j$$

Let $A = (a_{ij})$ which is a $(n+1) \times m$ order matrix. So rank of $A = r, 1 < r < n+1$ [as $m < n+1$]. So, by the problem 4.11 $\exists$ a projective change of co-ordinates T s.t.$V^T = V(X_{r+1}, ...X_{n+1})$. So,$V^T \neq \phi$. So, $V \neq \phi$

$\square$

Problem 3 4,13

Let $P = (a_1, a_2..a_{n+1}), Q = (b_1, b_2...b_{n+1})$ be two distinct points of P^n.The line L through P and Q is defined by

$$L = \{(\lambda a_1 + \mu b_1...\lambda a_{n+1} + \mu b_{n+1}) | \lambda, \mu \neq 0\}$$

Prove the projective analogue of Problem 2.15.

Solution:

(a) If T is a projective change of co-ordinates then $T(L)$ is the line passing through $T(P), T(Q)$.

$$T(L) = \{(T(\lambda P + \mu Q))|\lambda, \mu \neq 0\} = \{(\lambda T(P) + \mu T(Q))|\lambda, \mu \neq 0\}$$

[as T maps linearly to the co-ordinates]So, $T(L)$ is the line passing through $T(P), T(Q)$

(b) A line is a linear subvariety of dimension 1 and a linear subvariety of dimension 1 is a line passing through any two of its point.

let L be a line passing through $P = (a_1, a_2..a_{n+1}), Q = (b_1, b_2...b_{n+1})$
as P, Q are distinct point in $P^n \implies (a_1, a_2..a_{n+1}), (b_1, b_2...b_{n+1})$ are linearly independent vectors in k^{n+1}
so, there is an invertible matrix A of $n+1 \times n+1$ s.t. $A(1, 0, ...0) = (a_1, a_2..a_{n+1}), A(0, 1, ...0) = (b_1, b_2...b_{n+1})$
so, there corresponding projective change of co ordinate T will transform e_1 to P, e_2 to Q.
now $L^T = T^{-1}(L)$ is the line passing through $T^{-1}(P) = e_1, T^{-1}(Q) = e_2$
so, $T^{-1}(L) = (\lambda, \mu, 0, 0, ...0) = V(X_3, ...X_{n+1})$[as $\lambda, \mu \neq 0$] which is a linear subvariety of dimension 1
similarly if V is a linear subvariety of dimension 1 then $\exists$ a projective change of co-ordinates T s.t. $T^{-1}(L) = V(X_3, ..., X_{n+1})$ which is the line passing through $e_1, e_2 \implies L$ is a line passing through $T(e_1), T(e_2)$
c)In P^2 a line is the same thing as a hyperplane .
If L is a line in P^2
so, $T^{-1}(L) = V(X_3) = \{(\lambda, \mu, 0)|\lambda, \mu \neq 0\} \implies L = V(X_3(T_1, T_2, T_3)) \implies L$is a hyperplane.
d)let$P, P' \in P^1, L1, L2$ are two distinct lines passing through P and $L'1, L'2$ are two distinct passing through P' show that there is an projective change of co-ordinates T s.t.$T(P) = P', T(Li) = L'i.i = 1, 2$

$\square$

4.14)
let P_1, P_2, P_3 (resp. Q_1, Q_2, Q_3) be three points in P^2 not lying on a line .Show that $\exists$ a projective

change of co-ordinates $T : P^2 \to P^2$ s.t. $T(P_i) = Q_i$
Solution:
let $P_i = (a_i1, a_i2, a_i3)$
since P_1, P_2, P_3(resp Q_1, Q_2, Q_3) are not lying in a line so, they are linearly independent in $K^3 \implies$ forms a basis in K^3 .
so, $\exists$ an invertible matrix A s.t. $A(P_i) = Q_i$
let T be the corresponding projective change of co-ordinates w.r.t A
so, $T(P_i) = Q_i$

4.15)
Show that any two distinct lines in P^2 intersect in one point.
Solution:
let $L_1 = (\lambda, \mu, 0)$(ie, the line passing through $(1, 0, 0) = e_1; e_2 = (0, 1, 0)$, $L_2 = (\lambda P + \mu Q)$
let $P = (a_1, a_2, a_3), Q = (b_1, b_2, b_3)$
so, $L_2 = \{(\lambda a_1 + \mu b_1), (\lambda a_2 + \mu b_2), (\lambda a_3 + \mu b_3)\}$
if both a_3, b_3 are zero then L_1, L_2 becomes the same line.
let $a_3 \neq 0$
if $b_3 = 0$ then $Q = b_1 e_1 + b_2 e_2 \in L_1$
so, L_1, L_2 intersect in Q
let $b_3 \neq 0$
$b_3(P) - a_3(Q) = (b_3 a_1 - a_3 b_1, b_3 a_2 - a_3 b_2, 0) \in L_1$
so, L_1, L_2 intersect in a point.
let A, B be two lines
so, $\exists$ a projective change of co-ordinates T s.t. $T(A) = L_1$
let $T(B) = L_2$
so,let R be the intersection point of L_1, L_2
so, $T^{-1}(R)$ is the intersection point of A, B

4.16)
Let L_1, L_2, L_3(resp.M_1, M_2, M_3) are three line in P^2 s.t. not all 3 passes through a same point .show that there is a projective change of co-ordinates T s.t. $T(L_i) = M_i$)
Solotion:
let P_{ij} is the point of intersection of L_i and L_j and Q_{ij} is the point of intersection of M_i and M_j where $i < j$
so, as P_{12}, P_{13}, P_{23}(resp,(Q_{12}, Q_{13}, Q_{23}) does not lie in a line so, by problem 4.14 $\exists$ a projective change of co-ordinates T s.t.$T(P_{ij}) = Q_{ij}$
and so by the problem 4.13 part a $T(L_i) = M_i$

4.18
let $H = V(\sum a_i X_i)$ be a hyperpalne in P^n.$(a_1, a_2 ... a_{n+1})$ is determined by H upto constant.
a)show that assigning $(a_1, a_2, ... a_{n+1}) = P \in P^n$, to H sets a natural one to one correspondence between {hyperplanes in P^n} and P^n.
Solution:
$\phi : P^n \to \{\text{hyperplanes in } P^n\}$ s.t. $\phi(a_1, a_2 .. a_{n+1}) = V(a_1 X_1 + .. a_{n+1} X^{n+1})$
clearly ϕ_1 is well defined.

$\psi : \{\text{hyperplanes in } P^n\} \to P^n$ s.t. $V(F) = V(a_1X_1 + ..a_{n+1}X^{n+1}) = (a_1, a_2..a_{n+1})$
let $V(a_1X_1 + ..a_{n+1}X^{n+1}) = V(b_1X_1 + ..b_{n+1}X^{n+1}) \implies I(V(a_1X_1 + ..a_{n+1}X^{n+1})) = I(V(b_1X_1 + ..b_{n+1}X^{n+1}))$
$\implies a_1X_1 + ..a_{n+1}X^{n+1} = \lambda(b_1X_1 + ..b_{n+1}X^{n+1}), \lambda \neq 0$[as forms of deg 1 are irreducible]
$\implies (a_1, a_2..a_{n+1}) = (b_1, b_2..b_{n+1})$ in P^n
and $\phi o \psi$ and $\psi o \phi$ both are identity. so, assigning $(a_1, a_2, ...a_{n+1}) = P \in P^n$, to H sets a natural one to one correspondence between $\{\text{hyperplanes in } P^n\}$ and P^n.
$P \in P^n, P^* = \phi(P)$, H is a hyperplane then $H^* = \psi(H)$
b)Show that $P^{**} = P; H^{**} = H$.Show that $P \in H \iff H^* \in P^*$
Solution:
clearly by part a $P^{**} = P; H^{**} = H$
let $P = (p_1, p_2..p_{n+1} \in H = V(a_1X_1 + ..a_{n+1}X^{n+1}) \iff a_1p_1 + ...a_{n+1}p_{n+1} = 0 \iff (a_1, a_2, ...a_{n+1}) \in V(p_1X_1 + ..p_{n+1}X^{n+1}) \iff H^* \in P^*$