Analysis Assignment 1
- Rajeeva L. Karandikar

Soham Chatterjee

Roll: BMC202175

1. (i) We know that if $b_n = \sup\{a_i \mid i \geq n.\ i, n \in \mathbb{N}\}$ then $\alpha = \lim_{n\to\infty} b_n$. Hence $\exists\ N \in \mathbb{N}$ such that

$$|b_n - \alpha| < \epsilon \implies \alpha - \epsilon < b_n < \alpha + \epsilon$$

for all $n > N$. Since $\alpha - \epsilon$ is less than supremum of the set $\{a_i \mid i \geq n,\ i, n \in \mathbb{N}\}$ there exists a a_m where $m \geq n$ such that

$$\alpha - \epsilon < a_m \leq b_n \implies \alpha - \epsilon < a_m < \alpha + \epsilon$$

Here $m \geq n > N \implies m > N$. Now if we take $\max\{N, k\} = N_1$, for all $n > N_1$, $\alpha-\epsilon < b_n < \alpha+\epsilon$ and hence there exists a term of the sequence a_m where $m \geq n$ such that the inequality

$$\alpha - \epsilon < a_m \leq b_n < \alpha + \epsilon$$

satisfies where $m \geq n > N_1 \geq k$. Hence $\forall\ k < \infty$ and $\epsilon > 0\ \exists\ m > k$ such that

$$\alpha - \epsilon < a_m < \alpha + \epsilon$$

(ii) Using the statement of previous problem $\forall \epsilon > 0$ and $k < \infty$ where $k \in \mathbb{N}\ \exists\ m > k$ such that

$$\alpha - \epsilon < a_m < \alpha + \epsilon$$

Now given that $\exists\ \{n_j \mid 1 \leq j \leq t\}$ where $n_1 < n_2 < \cdots < n_t$, $t \in \mathbb{N}$ such that

$$\alpha - \frac{1}{j} < a_{n_j} < \alpha + \frac{1}{j}$$

where $j \in \{1, 2, \cdots, t\}$. Now if we choose $\epsilon = \frac{1}{t+1}$ and $k = n_t + 1$ there exists $m > k$ such that

$$\alpha - \frac{1}{t+1} < a_m < \alpha + \frac{1}{t+1}$$

Now take $m = n_{t+1}$ then we have $n_{t+1} > n_t$ which satisfies the inequality

$$\alpha - \frac{1}{t+1} < a_{n_{t+1}} < \alpha + \frac{1}{t+1}$$

(iii) Using the statement in problem (i) we can say that for $k_1 = 1$ and $\epsilon_1 = 1$ there exists $m > k_1$ such that

$$\alpha - 1 < a_m < \alpha + 1$$

Take $m = n_1$. Now if we choose $k_2 = m$ and $\epsilon_2 = \frac{1}{2}$ using the statement in previous problem there exists a $m' > k_2$ such that

$$\alpha - \frac{1}{2} < a_{m'} < \alpha + \frac{1}{2}$$

Take this $m' = n_2$. Now if there exists $\{n_j \mid 1 \leq j \leq t\}$ where $t \in \mathbb{N}$ and $n_1 < n_2 < \cdots < n_t$ such that

$$\alpha - \frac{1}{j} < a_{n_j} < \alpha + \frac{1}{j}$$

$j \in \{1, 2, \cdots, t\}$ there exists $n_{t+1} > n_t$ such that the following inequality satisfies

$$\alpha - \frac{1}{t+1} < a_{n_{t+1}} < \alpha + \frac{1}{t+1}$$

Hence by Mathematical Induction we can say that $\forall\ k \in \mathbb{N}$ there exists $a_{n_{k+1}} > a_{n_k}$ such that

$$\alpha - \frac{1}{k+1} < a_{n_{k+1}} < \alpha + \frac{1}{k+1}$$

Hence we get a sequence $\{a_{n_k}\}$ where $n_k > n_{k-1}$ such that $\forall\ k \in \mathbb{N}$

$$\alpha - \frac{1}{k} < a_{n_k} < \alpha + \frac{1}{k}$$

Hence the sequence $\{a_{n_k}\}$ converges to α. Therefore $\exists\ \{n_j \mid j \geq 1\}$ such that $n_j < n_{j+1}$ where $j \geq 1$ such that

$$\lim_{j\to\infty} a_{n_j} = \alpha \text{ [Proved]}$$

(iv) First take the sequence $\{c_n\}$. Let $c = \lim_{n\to\infty} \sup c_n$. Therefore using the statement in previous problem we can say that $\exists\{n_j \mid j \geq 1.\ j \in \mathbb{N}\}$ such that $n_j < n_{j+1}$ where $j \geq 1$ such that

$$\lim_{j\to\infty} c_{n_j} = c$$

Given that any subsequence of the sequence $\{c_n\}$ converges to θ. Therefore we can say $\theta = c$.

Now consider the sequence $|-c_n$. Suppose $d = \lim_{n\to\infty} \sup(-c_n)$. Hence $\exists\{m_j \mid j \geq 1,\ j \in \mathbb{N}\}$ such that $m_j < m_{j+1}$ where $j \geq 1$ such that

$$\lim_{j\to\infty} (-c_{m_j}) = d$$

Let $b_n = \sup\{(-c_k) \mid k \geq n,\ k, n \in \mathbb{N}\}$. Hence $b_n \geq (-c_k)\ \forall\ k \geq n$. Hence $-b_n \leq c_k$ $\forall\ k \geq n$. Hence $-b_n = \inf\{c_k \mid\ k \geq n\}$. Hence

$$-d = -\lim_{n\to\infty} \sup(-c_n) = -\lim_{n\to\infty} b_n = \lim_{n\to\infty} (-b_n) = \lim_{n\to\infty} \inf c_n$$

Therefore $\exists\{m_j \mid j \geq 1,\ j \in \mathbb{N}\}$ such that $m_j < m_{j+1}$ where $j \geq 1$ such that

$$\lim_{j\to\infty} c_{m_j} = -d = \lim_{n\to\infty} \inf c_n$$

As any subsequence of the sequence $\{c_n\}$ converges to θ we can say $-d = \theta$. Therefore

$$\lim_{n\to\infty} \sup c_n = \lim_{n\to\infty} \inf c_n = \theta$$

Hence we can say

$$\lim_{n\to\infty} c_n = \theta$$

2. (i) Given that $\{b_n\}$ is a decreasing sequence. Therefore $b_1 \geq b_2 \geq b_3 \geq \cdots$. Now $w_k = u_k + 1 = 2^k + 1$ and $v_k = 2^{k+1}$ hence there are exactly 2^k terms from w_k to v_k. Now $b_{w_k} \geq b_n$ for all $n \geq w_k$. Therefore

$$\sum_{j=w_k}^{v_k} b_j \leq \sum_{j=w_k}^{v_k} b_{w_k} = 2^k b_{w_k}$$

Again $b_{v_k} \leq b_n$ for all $n \in \{1, 2, \cdots, v_k\}$. Hence

$$\sum_{j=w_k}^{v_k} b_j \geq \sum_{j=w_k}^{v_k} b_{v_k} = 2^k b_{v_k}$$

Therefore

$$2^k b_{v_k} \leq \sum_{j=w_k}^{v_k} b_j \leq 2^k b_{w_k}$$

(ii) As $\{b_n\}$ is a decreasing sequence. Therefore $b_1 \geq b_2 \geq b_3 \geq \cdots$. Now $\sum\limits_{n=1}^{\infty} b_n = \lim\limits_{k\to\infty} \sum\limits_{n=1}^{k} b_n$. Let $2^m < k \leq 2^{m+1}$ where $k \in \mathbb{N}$. Suppose Now suppose $\sum\limits_{n=1}^{\infty} 2^n b_{2^n}$ converges. Then

$$\begin{aligned}
&\sum_{n=1}^{k} b_n \leq b_1 + (b_2+b_3) + (b_4+b_5+b_6+b_7) + \cdots + \sum_{n=2^i}^{2^{i+1}-1} b_n + \cdots + \sum_{n=2^m}^{2^{m+1}-1} b_n \\
\implies &\sum_{n=1}^{k} b_n \leq \sum_{i=1}^{k} \left(\sum_{n=2^i}^{2^{i+1}-1} b_n \right) \\
\implies &\sum_{n=1}^{k} b_n \leq \sum_{i=0}^{m} \left(\sum_{n=2^i}^{2^{i+1}-1} b_{2^i} \right) \\
\implies &\sum_{n=1}^{k} b_n \leq \sum_{i=0}^{m} 2^i b_{2^i}
\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} b_n = \lim_{k\to\infty} \sum_{n=1}^{k} b_n \leq \lim_{k\to\infty} \sum_{n=0}^{k} 2^n b_{2^n} = \sum_{n=0}^{\infty} 2^n b_{2^n}$$

Hence $\sum\limits_{n=1}^{\infty} b_n$ converges as $\sum\limits_{n=1}^{\infty} 2^n b_{2^n}$ converges.

Now suppose $\sum\limits_{n=1}^{\infty} b_n$ converges. Then

$$\begin{aligned}
&\sum_{n=1}^{k} b_n \geq b_1 + b_2 + (b_3+b_4) + \cdots + \sum_{n=2^i+1}^{2^{i+1}} b_n + \cdots + \sum_{n=2^{m-1}+1}^{2^m} b_n \\
\implies &\sum_{n=1}^{k} b_n \geq \frac{1}{2} b_1 + b_2 + 2b_4 + \cdots + 2^i b_{2^{i+1}} + \cdots + 2^{m-1} b_{2^m} \\
\implies &\sum_{n=1}^{k} b_n \geq \frac{1}{2} \left[\sum_{i=0}^{m-1} 2^i b_{2^{i+1}} \right]
\end{aligned}$$

Therefore

$$\sum_{n=1}^{\infty} b_n = \lim_{k\to\infty} \sum_{n=1}^{k} b_n \geq \lim_{k\to\infty} \frac{1}{2} \left[\sum_{n=0}^{k-1} 2^n b_{2^{n+1}} \right] = \frac{1}{2} \left[\lim_{k\to\infty} \sum_{n=0}^{k-1} 2^n b_{2^{n+1}} \right] = \frac{1}{2} \left[\sum_{n=1}^{\infty} 2^n b_{2^{n+1}} \right]$$

Hence $\left[\sum\limits_{n=1}^{\infty} 2^n b_{2^{n+1}} \right.$ converges as $\sum\limits_{n=1}^{\infty} b_n$ converges.

Therefore $\sum\limits_{n=1}^{\infty} b_n$ converges if and only if $\sum\limits_{n=1}^{\infty} 2^n b_{2^n}$ converges. [Proved]

(iii) Let $p > 1$. Using the statement in the previous problem we can say that $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $\sum\limits_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p}$ converges. Now

$$\sum_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum_{n=1}^{\infty} 2^{n(1-p)} = \sum_{n=1}^{\infty} (2^{1-p})^n$$

As $p > 1$, $1 - p < 0$ hence $2^{1-p} < 1$. Hence $\sum\limits_{n=1}^{\infty} (2^{1-p})^n$ is a geometric series which converges and therefore $\sum\limits_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p}$ converges. Hence $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges.

Let $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges. Therefore $\sum\limits_{n=1}^{\infty} 2^n \frac{1}{(2^n)^p} = \sum\limits_{n=1}^{\infty} 2^{n(1-p)}$. converges. Now if $p \leq 1$ then $1 - p \geq 0$. Hence

$$\sum_{n=1}^{\infty} 2^{n(1-p)} \geq \sum_{n=1}^{\infty} 2^{n \cdot 0} = \sum_{n=1}^{\infty} 1 = \lim_{k \to \infty} \sum_{n=1}^{k} 1 = \lim_{k \to \infty} k$$

Now $\lim\limits_{k \to \infty} k$ diverges. Then $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ will also diverge but we said that $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges. Contradiction. Hence $p > 1$.

Therefore $\sum\limits_{n=1}^{\infty} \frac{1}{n^p}$ converges if and only if $p > 1$. [Proved]