Analysis Assignment 2
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1. Let the set F is unbounded. Then for any $n \in \mathbb{N}\ \exists\ x_n \in [a,b]$ such that $|f(x_n)| > n$. Now as $\forall\ n \in \mathbb{N}$ $x_n \in [a,b]$, hence the sequence $\{x_n\}$ is bounded.

   Now in Analysis Assignment 1, Question No. 1.(iii) we proved that for a bounded sequence $\{a_n\}$ $\exists\ \{n_j \mid j \geq 1\}$ where $n_j < n_{j+1}$ and $n_j \in \mathbb{N}$ such that

$$\lim_{n\to\infty} a_{n_j} = \lim_{n\to\infty} \sup a_n$$

   Hence there exists a sequence $\{x_{n_k}\}$ where $n_k, k \in \mathbb{N}$ and $n_k < n_{k+1}$ such that $\lim\limits_{k\to\infty} x_{n_k} = \lim\limits_{n\to\infty} \sup x_n$. Let $\alpha = \lim\limits_{n\to\infty} \sup x_n$.

   Since $f(x_n) > n$ hence $f(x_{n_k}) > n_k \geq k$ and the sequence $\{n\}$ diverges. Therefore the sequence $\{f(x_{n_k})\}$ also diverges. But as the sequence $\{x_{n_k}\}$ converges to α and f is continuous in $[a,b]$, $\{f(x_{n_k})\}$ should converge to $f(\alpha)$. Contradiction. Therefore the set F is bounded. [Proved]

2. Since we already proved that f is closed and bounded, suppose M is the least upper bound of the set F where $F = \{f(x) \mid x \in [a,b]\}$. Now we construct a sequence $\{x_n\}$, where $x_n \in [a,b]\ \forall\ n \in \mathbb{N}$ in such that

$$|M - f(x_n)| < \frac{1}{n}$$

   Now such x_n will always exist because if no such x_n exists then $\forall x \in [a,b]\ f(x) < M - \frac{1}{n}$ and then $M - \frac{1}{n}$ would be the upper bound less than least upper bound which is not possible. Hence $\{f(x_n)\}$ converges to M. Therefore x_n is also convergent. Let's say $\{x_n\}$ converges to α. As f is continuous f should converge to $f(\alpha)$. Now $\{x_n\}$ converges to α $\{f(x_n)\}$ converges to M and $f(\alpha)$. As the limit should be unique hence $f(\alpha) = M$.

   Similarly suppose m is the greatest lower bound and we construct a sequence $\{y_n\}$ such that

$$|m - f(y_n)| < \frac{1}{n}$$

   . Hence $f(y_n)$ converges to m. Suppose $\{y_n\}$ converges to β. As f is continuous $f(y_n)$ should converge to $f(\beta)$. Therefore $f(\beta) = m$. Hence f attains its extremes.

   Now f is a continuous function from closed bounded interval $[a,b]$ to a closed bounded interval $[m,M]$. Hence f is uniformly continuous (Source: Lecture Notes of 19.10.2021). Now whenever g is composed upon f its domain becomes the range of f i.e. $[m,M]$. Therefore $g : [m,M] \to \mathbb{R}$ is a continuous function on a closed bounded interval. Therefore g is uniformly continuous.

   As f is uniformly continuous $\forall\ \epsilon_f > 0\ \exists\ \delta_f > 0$ such that

$$|f(x) - f(y)| < \epsilon_f \text{ whenever } |x - y| < \delta_f$$

   where $x, y \in [a,b]$. As g is uniformly continuous $\forall\ \epsilon_g > 0\ \exists\ \delta_g > 0$ such that

$$|g(x) - g(y)| < \epsilon_g \text{ whenever } |x - y| < \delta_g$$

   where $x, y \in [m,M]$. Now if we take $\epsilon_f \leq \delta_g$ then $\forall\ \epsilon_g > 0\ \exists\ \delta_f > 0$ such that

$$|(g \circ f)(x) - (g \circ f)(y)| < \epsilon_g \text{ whenever } |x - y| < \delta_f$$

   where $x, y \in [a,b]$. Hence $(g \circ f)$ is also uniformly continuous. [Proved]

3. (a) As f is continuous in (a,b) and $g(x) = f(x)$ in (a,b), g is also continuous in (a,b). Hence only when G can be discontinuous is at $x = a$, $x = b$.

   Given that

$$g(a) = \alpha = \lim_{x\to\ a} f(x) = \lim_{x\to\ a} g(x)$$

Hence g is continuous at $x = a$. Again

$$g(b) = \beta = \lim_{x \to b} f(x) = \lim_{x \to b} g(x)$$

Therefore g is also continuous at $x = b$. Hence g is continuous on $[a, b]$. [Proved]

(b) Now g is continuous in the closed interval $[a, b]$. Using the result in Problem 1 we can say g is uniformly continuous in $[a, b]$. Therefore g is uniformly continuous in (a, b). Now as $g(x) = f(x)$ in (a, b), f is also uniformly continuous in (a, b). [Proved]