Fulton Chapter 6: Varieties, Morphisms, and Rational Maps

The Zariski Topology and Varieties

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Problem Set - 5
Topic: Algebraic Geometry

Problem 1 The Zariski Topology: 6.8

Let U be an open subset of a variety $V, z \in k(V)$. Suppose $z \in O_P(V)\ \forall\ P \in U$. Show that $U_z = \{P \in U \mid z(P) \neq 0\}$ is open , and that the function from $U \to k = A^1$ defined by $P \mapsto z(P)$ is continuous.

Solution: Let $f \in \tau(V)$; $S_f = \{p \in V | f(P) = 0\} = V \cap V(f)$ which is a closed set. Now $P \in U_z \implies \exists\ f, g \in \tau(V)$ s.t. $z = f/g, g(P) \neq 0$.

$$P \in V \cap V \setminus S_f \cap V \setminus S_g \subset U_z \quad [\text{as } \forall\ P' \in V \cap V \setminus S_f \cap V \setminus S_g,\ z = f/g \text{ and } z(P) \neq 0]$$

So there is an open neighborhood of P in $U_z \implies U_z$ is open.

Let $r \in k$. $U_z' = \{P \in U | z(P) \neq r\}$. Similarly

$$\forall\ P \in U_z' P \in V \cap V\ S_{(f-rg)} \cap V\ S_g \implies U_z' \text{ is open } \implies z^{-1}(P) \text{ is closed}$$

Now in k any closed set S is finite. So,

$$z^{-1}(S) = \bigcup_{p \in S} z^{-1}(p) \implies z^{-1}(S) \text{ is closed}$$

So, $P \mapsto z(P)$ is a continuous function.

□

Problem 2 Varieties: 6.9

Let $X = A^2 - \{(0,0)\}$, an open subvariety of A^2. Show that $\tau(X) = \tau(A^2) = k[X, Y]$

Solution: Clearly, $\tau(A^2) \subseteq \tau(X)$. Let $z \in \tau(X) = \bigcap_{P \in X} O_P(A^2), P \neq (0,0)$. So, $z(P)$ defined at $0 \implies \exists\ f, g \in \tau(A^2) = k[x, y]$ s.t. $z = f/g, g(P) \neq 0$. Let $f/g = f_1/g_1$ (where g.c.d. of $f_1, g_1 = 1$). So,

$$g_1(a, b) \neq 0\ \forall\ (a, b) \neq (0, 0)$$

So, if $\deg(g_1)$ in $k[X][Y]$ is non zero then it has infinitely many roots which is not possible. So, g_1 is a constant polynomial. So, $z \in k[x,y]$

$\boxed{\textbf{\textit{Remark:}}}$

$A^2 - \{(0,0)\}$ is not an affine variety. Lets assume $X = A^2 - \{(0,0)\}$ be an affine variety and $I(X) = I$. So,

$$\tau(X) = k[x,y]/I = k[x,y] \implies I = (0) \implies X = V(I) = A^2$$

which is not true. So, X is not an affine variety.

$\square$

Problem 3 Varieties: 6.12

Let X be a variety, $z \in k(X)$. show that the pole set of z is closed. If $z \in O_P(X)$, there is a neighborhood U_P of P s.t. $z \in \tau(U_P)$; so, $O_P(X)$ is the union of all $\tau(U_P)$

Solution: $J = \{G|zG \in \tau(X)\}$. Clearly pole set of $z = X \cap V(J)$. So, pole set is a closed set. Now

$$z \in O_P(X) \implies P \in X \setminus V(J) \implies P \in U_P \subset X \setminus V(J)$$

So,

$$z \in O_S(X) \forall S \in U_P \implies z \in \tau(U_P)$$

$$P \in U_P \implies \tau(U_P) \subseteq O_P(X)$$

So, $O_P(X) = \cup_P \tau(U_P)$

$\square$