Chapter 1

Tutorial 1

Problem 1 Let $\mathbb{R}$ denote the set of real numbers. Put $\mathbb{H} := \mathbb{R} \cdot 1 \oplus \mathbb{R} \cdot i \oplus \mathbb{R} \cdot j \oplus \mathbb{R} \cdot k$ where $i^2 = j^2 = -1$, $ij = -ji = k$. Show that $\mathbb{H}$ is a division ring

Solution: Any element $p \in \mathbb{H}$, $p \neq 0$ can be written as $p = a + bi + cj + dk$. Now If we take $q = \frac{1}{a^2+b^2+c^2+d^2}(a - bi - cj - dk)$ then

$$pq = \frac{(a^2 + b^2 + c^2 + d^2) + (ab - cd + cd - ab)i + (ac - ac + bd - bd)j + (ad - ad - ac + ac)k}{a^2 + b^2 + c^2 + d^2} = 1$$

Hence for any nonzero element in bbH there exists an multiplicative inverse of that element. Hence nonzero elements of $\mathbb{H}$ are units. Hence $\mathbb{H}$ is a division ring.

$\square$

Problem 2

Problem 3

Solution: Let $f(x), g(x)$ elements of $R[X]$. Suppose

$$f(x) = \sum_{i=0}^{n} a_i x^i, \quad g(x) = \sum_{i=0}^{m} b_i x^i$$

where $a_n \neq 0$ and $b_m \neq 0$. Hence

$$f(x)g(x) = cx^{n+m} + \cdots + a_0 b_0$$

where $a_n b_n = c$. Since $R[X]$ is an integral domain $c \neq 0$. Hence for any two elements $f, g \in R[X]$

$$\deg(fg) = \deg(f) + \deg(g)$$

Any unit in R is also an unit of $R[X]$. Hence $U(R) \subseteq U(R[X])$. Now Let f be an unit in $R[X]$. Suppose f' is the multiplicative inverse of f. If $\deg(f(x)) \geq 1$ then

$$0 = \deg(1) = \deg(ff') = \deg(f) + \deg(f') \geq 1$$

Hence All the units in $R[X]$ are constant polynomials which are also units in R. Hence $U(R[X]) \subseteq U(R)$. Therefore $U(R) = U(R[X])$

$\square$

Problem 4

Solution:

$$(2x+1)^2 = 4x^2 + 4x + 1 = 1$$

Hence $(2x+1)$ is an unit of $\mathbb{Z}/4\mathbb{Z}[X]$ but it is not in $\mathbb{Z}/4\mathbb{Z}$

□

Problem 5

Solution: Consider the surjective group homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}/n\mathbb{Z}$. Then the kernel of the homomorphism is $n\mathbb{Z}$. Hence by the correspondence theorem the set of ideals of $\mathbb{Z}$ containing $n\mathbb{Z}$ is isomorphic to the set of ideals of $\mathbb{Z}/n\mathbb{Z}$

Now the ideals containg $n\mathbb{Z}$ in $\mathbb{Z}$ are the ideals of the form $d\mathbb{Z}$ where $d \mid n$. Hence

□

Problem 6

Problem 7

Solution: Let R is the commutative ring. Suppose R is integral domain. Now

$$ab = ac \implies a(b-c) = 0 \implies b - c = 0 \implies b = c$$

Now suppose $ab = ac \implies b = c$. Suppose R is not an integral domain. Then there exists at least one element $x \in R$ such that x is an zero divisor. Let $xy = 0$. Now $zb = zc$ does not imply $b = c$ because b can be equal to y and c can be equal to 0. Hence contradiction. THere does not exists any zero divisors in R. Hence R is an integral domain.

□

Problem 8

Solution: For left zero divisor $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. No nonzero element exists such that it becomes a right zero divisor

□

Problem 9

Solution:

(a) Given that $(a,b) \sim (c,d)$ iff $ad = bc$. Let $(a,b) \in S(R)$. Then $(a,b) \sim (a,b)$ iff $ab = ba$. Given that R is a commutative ring. Hence $ab = ba$. Therefore $(a,b) \sim (a,b)$.

Now let $(a,b),(c,d) \in S(R)$ and $(a,b) \sim (c,d) \iff ad = bc$. Now $(c,d) \sim (a,b)$ if $cb = da$. Now we know R is a commutative ring. Hence $ad = da$ and $bc = cb$. Therefore

$$ad = bc \iff da = cb \iff (c,d) \sim (a,b)$$

Again suppose $(a,b),(c,d),(e,f) \in S(R)$ and $(a,b) \sim (c,d)$ and $(c,d) \sim (e,f)$. Therefore $ad = bc$ and $cf = de$. Now $(a,b) \sim (e,f)$ if $af = be$. Now multiplying both sides of $ad = bc$ by f from right and both sides of $cf = de$ by b from left we get

$$adf = bcf \qquad bcf = bde \iff adf = bde \iff afd - bed = 0 \iff (af - be)d = 0$$

Since R is an integral domain and $f \neq 0$we have $af - be = 0 \iff af = be$

(b) Let $\overline{(a,b)} = \overline{(a',b')}$ and $\overline{(c,d)} = \overline{(c',d')}$. Hence $(a,b) \sim (a',b')$ and $(c,d) \sim (c',d')$ therefore $ab' = a'b$ and $cd' = c'd$. Now

$$\overline{(a,b)} + \overline{(c,d)} = \overline{(ad+bc, bd)} \qquad \overline{(a',b')} + \overline{(c',d')} = \overline{(a'd'+b'c', b'd')}$$

Now $\overline{(ad+bc,bd)} = \overline{(a'd'+b'c',b'd')}$ iff $(ad+bc,bd) = (a'd'+b'c',b'd')$ if $(ad+bc)b'd' = (a'd'+b'c')bd$. Now since R is integral doamin R is commutative hence

$$(ad+bc)b'd' - (a'd'+b'c')bd = adb'd' + bcb'd' - a'd'bd - b'c'bd = ab'dd' - a'bdd' + bb'cd' - bb'c'd = 0$$

Therefore $(ad+bc)b'd' = (a'd'+b'c')bd$. Hence $\overline{(ad+bc,bd)} = \overline{(a'd'+b'c',b'd')}$.

Now

$$\overline{(a,b)} \circ \overline{(c,d)} = \overline{(ac,bd)} \qquad \overline{(a',b')} \circ \overline{(c',d')} = \overline{(a'c',b'd')}$$

Now $\overline{(ac,bd)} = \overline{(a'c',b'd')} \iff (ac,bd) \sim (a'c',b'd')$ if $acb'd' = bda'c'$. Now

$$acb'd' - bda'c' = acb'd' - a'bcd' + a'bcd' - bda'c' = cd'(ab' - a'b) + a'b(cd' - c'd) = 0$$

Hence $acb'd' = bda'c'$. Therefore $\overline{(ac,bd)} = \overline{(a'c',b'd')}$

Therefore the sum and the product operations are well-defined

□

Problem 10

Solution: Let $\frac{a}{b}, \frac{c}{d} \in R$. Hence $\frac{-c}{d} \in R$. Now

$$\frac{a}{b} + \frac{-c}{d} = \frac{ad - cb}{bd} = \frac{a}{b} + \frac{-c}{d}$$

Now since $p \nmid b$ and $p \nmid d$ then $p \nmid bd$. Hence $\frac{ad-cb}{bd} \in R$. Hence $(R,+)$ is an additie abelian group.

Now let $\frac{a}{b}, \frac{c}{d} \in R$. Then

$$\frac{a}{b}\frac{c}{d} = \frac{ac}{bd}$$

Now as $p \nmid b$ and $p \nmid d$ we have $p \nmid bd$. Therefore $\frac{ac}{bd} \in R$. Hence R is closed under multiplication.

Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in R$. Now

$$\frac{a}{b}\left(\frac{c}{d}\frac{e}{f}\right) = \frac{a}{b}\frac{ce}{df} = \frac{ace}{bdf} = \frac{ac}{bd}\frac{e}{f} = \left(\frac{a}{b}\frac{c}{d}\right)\frac{e}{f}$$

Hence multiplication is associative.

Let $\frac{a}{b}, \frac{c}{d}, \frac{e}{f} \in R$. Now

$$\begin{aligned}
\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right) &= \frac{a}{b}\frac{cf+ed}{df} \\
&= \frac{a(cf+de)}{bdf} \\
&= \frac{acf+ade}{bfd} \\
\frac{a}{b}\frac{c}{d}+\frac{a}{b}\frac{e}{f} &= \frac{ac}{bd}+\frac{ae}{bf} \\
&= \frac{acf+ade}{bdf}
\end{aligned}$$

Hence

$$\frac{a}{b}\left(\frac{c}{d}+\frac{e}{f}\right) = \frac{a}{b}\frac{c}{d}+\frac{a}{b}\frac{e}{f}$$

Hence elements in R follows distributive property. Therefore R is a ring

$\square$

Problem 11

Solution: Given that for any element $a \in R$

$$a^2 = a$$

Now

$$2a = (a+a) = (a+a)^2 = a^2 + 2a + a^2 = a + 2a + a \iff a + a = 0$$

Then

$$a + b = (a+b)^2 = a^1 + ab + ba + b^2 = a + ab + ba + b \iff ab + ba = 0 \iff ab = -ba \iff ab = ba$$

Therefore the ring is commutative

$\square$

Problem 12

Solution: We have $0 \in Z(R)$ since $0 \cdot r = 0 = r \cdot 0$ for all $r \in R$; in particular, $Z(R)$ is nonempty. Next, if $x, y \in Z(R)$ and $r \in R$, then

$$(x-y)r = xr - yr = rx - ry = r(x-y).$$

Hence $Z(R) \leq R$. Now $xyr = xry = rxy$, so that $xy \in Z(R)$ then by definition, $Z(R)$ is a subring.

$\square$

Problem 13

Solution: Same is P7

$\square$

Problem 14

Solution: The ideal of a ring is a subgroup. Hence the ideal of $\mathbb{Z}$ is a subgroup of Z. Now we know the only subgroup of $\mathbb{Z}$ is $n\mathbb{Z}$ where $n \in \mathbb{Z}$. Hence the ideals of R of the form $n\mathbb{Z}$ where $n \in \mathbb{Z}$

$\square$

Chapter 2

Tutorial 2

Problem 15

Solution:

(a) If $x \in IJ$, then $x = \sum a_i b_i$ where $a_i \in I$ and $b_i \in J$. Thus for any fixed i, we have that since $a_i \in I$, we have that $a_i b_i \in I$, and the same argument shows that $a_i b_i \in J$, thus $\sum a_i b_i \in I$ and $\sum a_i b_i \in J$, this means that $x = \sum a_i b_i \in I \cap J$, and thus $IJ \subseteq I \cap J$.

Now

$$\begin{aligned} I \cap J &= (I \cap J)R \\ &= (I \cap J)(I + J) \\ &= (I \cap J)I + (I \cap J)J \\ &\subseteq IJ + IJ = IJ \end{aligned}$$

Hence $I \cap J = IJ$

(b) Since $1 \in R$ and $I + J = R$, $\exists\ x \in I$, $y \in J$ such that $x + y = 1$. Now consider the element $bx + ay$ in R.

$$bx + ay - a = bx - ax = (b - a)x \in I \quad bx + ay - b = -by + ay = (a - b)y \in J$$

(c) Consider the homomorphism $\varphi : R \to (R/I) \times (R/J)$ which maps any element $r \in R$ to $(r + I, r + J)$. Now this homomorphism is injective becasue if not then suppose for $r, s \in R$, $r \neq s$ $(r, I, r + J) = (s + I, s + J)$. Hence that means $((r - s) + I, (r - s) + J) = (I, J)$. Hence $r - s \in I, J$. Therefore $r - s \in I \cap J$. But $I \cap J = IJ$ and given that $IJ = (0)$. Hence $r - s = 0$ But we assumed $r \neq s$. Hence contradiction. Therefore φ is injective.

Now φ is surjective. Because let $(a + I, b + J)$ be any element of $(R/I) \times (R/J)$. Then by part (b) we have an element x in R such that $x - a \in I$ and $x - b \in J$. Hence $(x - a) + a = x \in a + I$ and $(x - b) + b = x \in b + J$. Therefore $a + I = x + I$ and $b + J = x + J$. Hence $(a + I, b + J) = (x + I, x + J)$. Now since $x \in R$. $\varphi(x) = (x + I, x + J) = (a + I, b + J)$. Hence φ is surjective.

Hence φ is isomorphism. Thereofer $R \cong (R/I) \times (R/J)$

$\square$

Problem 16

Solution:

$\square$

Problem 17

Solution:

□

Problem 18

Solution:

□

Problem 19

Solution:

□

Problem 20

Solution:

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Problem 21

Solution:

□

Problem 22

Solution:

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