SOME PROBLEMS FROM CHAPTER 3 SECTION 3
INTERSECTION NUMBERS

problem 3.18
case 1.
F is irreducible.
As P is a simple point of F so $O_P(F)$ is a D.V.R.and the uniformizing parameter of $M_P(F)$ is a line passes through P but not the tangent at P.
let the line is L let $O_P^F(G) = n$
so,$g = G + (F) = L^n u$[u is an unit in $O_P(F)$]
so, by property 7 $I(P, F \cap G) = I(P, F \cap g)$
by property 6,$I(P, F \cap g) = I(P, F \cap L^n u) = nI(P, F \cap L) + I(P, F \cap u)$
by property 2, $I(P, F \cap u) = 0$[as u is an unit in $O_P(F) \implies u(P) \neq 0$]
now tangent of L at P is L.
so, F and L do not share their tangents at P
by property 5, $I(P, F \cap L) = 1$[as P is a simple point of $F \implies m_P(F) = 1$]
so, $I(P, F \cap G) = I(P, F \cap g) = n = O_P^F(G)$
case 2.
F is reducible.
let $F = \Pi_{i=1}^n F_i^{a_i}$
P is a simple point of $F \implies m_P(F) = \sum_{i=1}^n a_i m_P(F_i) = 1 \implies$ for some $i, m_P(F_i) = 1; a_i = 1; m_P(F_j) = 0 \forall i \neq j \implies F_i$ is the only irreducible component passes through P.
so,$O_P^F(G) = O_P^{F_i}(G) = I(P, F_i \cap G)$[by case 1]
now $I(P, F \cap G) = \sum_{j=1}^n a_j I(P, F_j \cap G) = I(P, F_i \cap G)$[as $\forall i \neq j F_j$ does not pass through P and $a_i = 1$]

problem 3.20
let $I(P, F \cap G) = m; I(P, F \cap H) = n$
by the previous problem we know that $m = O_P^F(G); n = O_P^F(H)$
let L be a line which passes through P but not the tangent of F at P
so, $g = L^m u_1; h = L^n u_2$
WLOG $m \geq n$
so, $g + h = L^n(L^{m-n}u_1 + u_2)$
so, $O_P^F(G + H) \geq n$
let $P = (0, 0); F = x^2 + y^2; H = x - y; G = x + y$
clearly $I(P, F \cap G) = I(P, F \cap H) = \infty$
but $G + H = 2x$ and $I(P, 2x \cap x^2 - y^2) = I(P, x \cap x^2 - y^2) = 2I(P, x \cap y) = 2$
so the proposition will be failed if P is not a simple point of F.

problem 3.21
let $P \in L \cap F$
therefore, $P = (a+kb, c+kd)$ for some $k \in K$
$F(a+kb, c+kd) = 0 \implies G(k) = 0 \implies k = \lambda_i$ for some i
so, $P = (a+\lambda_i b, c+\lambda_i d)$
and for all $\lambda_i, (a+\lambda_i b, c+\lambda_i d) \in L \cap F$
so, there is an one one correspondence between λ_i and P_i
and $P_i = (a+\lambda_i b, c+\lambda_i d)$
now L is not a component of $F \implies I(P_i, L \cap F) = m_{P_i}(L)m_{P_i}(F) = m_P(F)$[as the tangent at P_i of L_i is L_i]
so, $m_{P_i}(F(X,Y)) = m_P(F(X+a+\lambda_i b, Y+c+\lambda_i d))$[where $P = (0,0)$]
now either of b, d is non zero [as L is a line]
let $b \neq 0$
let $Y = dX/b$
$F(X+a+\lambda_i b, dX/b+c+\lambda_i d) = F(a+b(\lambda_i+X/b), c+d(\lambda_i+X/b)) = G(\lambda_i+X/b)$
now the lowest degree of X in $G(X/b+\lambda_i) = m_{P_i}(F)$[as in the least degree homogeneous term if we put $Y = dX/b$ then the degree will be same]
now $G(X/b+\lambda_i) = (X/b)^{e_i}\Pi_{i\neq j}(X/b+\lambda_i-\lambda_j)^{e_j}$
so, the least degree is $e_i \implies m_{P_i}(F) = e_i$[as $\lambda_i \neq \lambda_j \forall i \neq j$]
so $\sum_i I(P_i, F \cap L_i) \leq deg(F)$ [as $degG \leq degF$]

problem 3.23
suppose $P = (0,0), L = Y.P$ is a hypercusp if and only if $\frac{\partial F}{\partial^n X}(P) \neq 0$ where $n = m_P(F)+1$
let $F = YG + H(X)$ clearly $H(0) = 0$[as $F(0,0) = 0$]
now $F = Y^{n-1} + F_1$ where $m_P(F_1) \geq n$[as Y is the only tangent at P]
so,$H(x) = X^k(H_1(X)$ where $H_1(0) \neq 0$ and $k \geq n$
$\frac{\partial F}{\partial^n X}(P) \neq 0 \iff$ the coefficient of X^n is non zero.
now $I(P, F \cap Y) = n \iff I(P, Y \cap H(X)) = n \iff I(P, Y \cap X^k) = n \iff k = n \iff$ the coefficient of X^n is non zero.[as $H_1(0) \neq 0$]
2nd part: I will show that F has only one irreducible component passing through P
let assume $P = (0,0)$
let $F = \Pi_{i=1}^n F_i^{a_i}$ where F_i's are irreducible
WLOG assume that $F_1, F_2..F_k$ passes through P
let L be the tangent of F at P
so, L be the only tangent of F_i at P[as if there is a tangent other than L then it will be a tangent of F as well because the least degree form of F is the product of least degree form of F_i]
so, $I(P, F \cap L) = \sum_{i=1}^k a_i I(P, F_i \cap L)$
$I(P, F_i \cap L) > m_P(F_i)m_P(L) \implies I(P, F_i \cap L) \geq b_i + 1$[where $b_i = m_P(F_i)$]
$I(P, F \cap L) = \sum_{i=1}^k a_i b_i$

$I(P, F \cap L) = m_P(F) + 1 = \sum_{i=1}^{k} a_i b_i + 1 \geq \sum_{i=1}^{k} a_i(b_i + 1)$
so, $\sum_{i=1}^{k} a_i \leq 1$
but as F passes through $P \implies$ at least one $a_i > 0 \implies \sum_{i=1}^{k} a_i \geq 1$
so, $\sum_{i=1}^{k} a_i = 1 \implies a_j = 1; a_i = 0 \forall i \neq j$
so, F has only one irreducible component passing through P

problem 3.24
clearly $M_P(F) = M = (x, y)$ where $x = X + (F), y = Y + (F)$ and both are non zero [as $m_P(F) > 1$]
(a)
let f :forms of degree 1 in $k[X, Y]$=V $\to M/M^2$
s.t. $f(aX + bY) = \overline{ax + by}$
clearly V is a vector space of dimension 2 with bases X, Y and F is a homomorphism of two k- vector space.
let $aX + bY \in ker(f)$
so, $ax + by \in M^2 \implies aX + bY + G \in (F)$ where $m_p(G) > 1$
$\implies m_p(F) = 1$ which is not possible.
so, $a = b = 0 \implies f$ is injective .
by problem 3.13 M/M^2 is a vector space of dimension 2
so, f is an isomorphism of vector space [by rank nullity theorem].
(b)
$I(P, F \cap L_i) > m_P(F)m_P(L_i) = m$[as F and L_i shares tangent at P]
now if $\overline{l_i} = \lambda\overline{l_j} \implies l_i - \lambda l_j \in M^2 \implies L_i - L_j + G \in (F)$
but $M_P(F) > 1 \implies L_i = \lambda L_j$
which is not possible as L_i are distinct tangents at P
(c)
$\overline{g_i} \neq 0 \implies m_P(G_i) \leq 1$
if $m_P(G_i) = 0 \implies I(P, F \cap G) = 0$ which is not possible.
so, $m_P(G_i) = 1 \implies G_i = L_i + H_i, m_P(H_i) > 1$
so, G_i has only i tangent L_i at P and $I(P, F \cap G_i) > m_P(F)m_P(G_i) \implies FandG_i$ share tangent at P
as $\overline{g_i} \neq \lambda\overline{g_j} \implies L_i, L_j$ are distinct [as $\overline{g_i} = \overline{l_i}$]
so, F has m distinct tangents at P and $m_P(F) = m \implies P$ is an ordinary point.
(d)
$dim_k O_P(F)/(g_i) = dim_k O_P(A^2)/(F, G) = I(P, F \cap G)$
so, by (c) if $g_1, g_2..g_m \in M$s.t. $\overline{g_i} \neq \overline{g_j} \forall i \neq j, \lambda \in kk$ and $dim_k O_P(F)/(g_i) = I(P, F \cap G_i) > m \implies P$ is an ordinary point.
if P is an ordinary point take $G_i = L_i$where L_i's are distinct tangent at $P \implies \overline{l_i} \neq \lambda\overline{l_j} \forall i \neq j, \lambda \in k$ and $dim_k O_P(F)/(l_i) > m$