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Course: Analysis 2

Assignment - 2
Roll: BMC202175
Date: April 4, 2022

Problem 1 Rudin Chapt. 9 Problem 6

If $f(0,0)=0$ and

$$f(x,y)=\frac{xy}{x^2+y^2} \quad \text{ if } (x,y)\neq(0,0)$$

prove that $(D_1f)(x,y)$ and $(D_2f)(x,y)$ exist at every point of R^2, although f is not continuous at $(0,0)$.

Solution: When $(x,y)\neq(0,0)$ then

$$(D_1f)(x,y)=\frac{y(y^2-x^2)}{(x^2+y^2)^2} \text{ and } (D_2f)(X,y)=\frac{x(x^2-y^2)}{(x^2+y^2)^2}$$

Now at $(0,0)$

$$\begin{aligned}(D_1f)(0,0)&=\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{|h|}\\ &=\lim_{h\to 0}\frac{\frac{h\times 0}{h^2+0}-0}{|h|}\\ &=\lim_{h\to 0}\frac{0}{|h|}\\ &=0\end{aligned}$$

$$\begin{aligned}(D_2f)(0,0)&=\lim_{k\to 0}\frac{f(0,k)-f(0,0)}{|k|}\\ &=\lim_{k\to 0}\frac{\frac{0\times k}{0+k^2}-0}{|k|}\\ &=\lim_{k\to 0}\frac{0}{|k|}\\ &=0\end{aligned}$$

Hence $(D_1f)(x,y)$ and $(D_2f)(x,y)$ exists at every point of $\mathbb{R}^2$.

Now if we approach $(0,0)$ along the line then it approaches to 0. But if we approach $(0,0)$ along the line $y=x$ then

$$\lim_{h\to 0}f(h,h)=\lim_{h\to 0}\frac{h^2}{2h^2}=\frac{1}{2}$$

Hence f is not continuous at (0,0)

□

Problem 2 Rudin Chapt. 9 Problem 7

Suppose that f is a real-valued function defined in an open set $E\subset R^n$, and that the partial derivatives $D_1f,\dots,D_nf$ are bounded in E. Prove that f is continuous in E.

Hint: Proceed as in the proof of Theorem 9.21.

Solution: Let a is any arbitrary point in E. Since E is an open set there exists an open ball $B_r(a)$ centered at a of radius $r>0$ inside E. We have to show that $\forall\,\epsilon>0\ \exists\,\delta>0\ |f(a+h)-f(a)|<\epsilon$ wherever $\|h\|<\delta$ where $a+h\in B_r(a)$.

Now any open ball $B_r(a)$ centered at a of radius $r>0$ in $\mathbb{R}^n$ is a convex set. Because if we take any two points $x,y\in B_r(a)$ then for any $\theta\in(0,1)$

$$\|(\theta x+(1-\theta)y)-a\|=\|\theta(x-a)+(1-\theta)(y-a)\|\leq\theta\|x-a\|+(1-\theta)\|y-a\|<\theta r+(1-\theta)r=r$$

and henceforth $\theta x+(1-\theta)y\in B_r(a)$. Therefore any open ball in $\mathbb{R}^n$ is a convex set.

Now. Let $h = \sum_{i=1}^{n} h_i e_i$ where e_i is the $i-$th vector of the standard basis of $\mathbb{R}^m$.

$$
\begin{aligned}
f(a+h) - f(a) &= f\left(a + \sum_{i=1}^{n} h_i e_i\right) - f(a) \\
&= \left[f\left(a + \sum_{i=1}^{n} h_i e_i\right) - f\left(a + \sum_{i=1}^{n-1} h_i e_i\right)\right] + \left[f\left(a + \sum_{i=1}^{n-1} h_i e_i\right) - f\left(a + \sum_{i=1}^{n-2} h_i e_i\right)\right] \\
&\qquad + \cdots + \left[f\left(a + h_1 e_1\right) - f\left(a\right)\right] \\
&= \sum_{k=1}^{n} \left[f\left(a + \sum_{i=1}^{k} h_i e_i\right) - f\left(a + \sum_{i=1}^{k-1} h_i e_i\right)\right]
\end{aligned}
$$

Now each $f\left(a + \sum_{i=1}^{k} h_i e_i\right) - f\left(a + \sum_{i=1}^{k-1} h_i e_i\right)$ is a one-variable function from $\mathbb{R}$ to $\mathbb{R}$. By Mean Value Theorem $\exists\ \theta_k \in (0,1)$ such that

$$
f\left(a + \sum_{i=1}^{k} h_i e_i\right) - f\left(a + \sum_{i=1}^{k-1} h_i e_i\right) = \left.\frac{\partial f}{\partial x_k}\right|_{v_k} h_k
$$

where $v_k = a + \sum_{i=1}^{k-1} h_i e_i + \theta_k h_k e_k$. Hence

$$
f(a+h) - f(a) = \sum_{i=1}^{n} \left.\frac{\partial f}{\partial x_i}\right|_{v_i} h_i = \sum_{i=1}^{n} D_i f(v_i) h_i
$$

Since all $D_i f$ are bounded in, Let $D_i f$ is bounded by M_i where $M_i > 0$. Then take $M = \max\{M_i \mid 1 \leq i\ leqn\}$ Hence $|D_1(v_i)h_i| < M|h_i|$. Hence

$$
|f(a+h) - f(a)| = \left|\sum_{i=1}^{n} D_i f(v_i) h_i\right| = \sum_{i=1}^{k} M|h_i| = M\|h\|_1
$$

Now as $h \to 0$ we can say $\|h\|_1 \to 0$. Hence $M\|h\|_1 \to 0$ Therefore $|f(a+h) - f(a)| \to 0$. Hence $\forall\ \epsilon > 0$ $\exists\ \delta > 0$ such that $\|h\|_1 < \delta$ whenever $|f(a+h) - f(a)| < \epsilon$. Therefore f is continuous in E

□

Problem 3 Rudin Chapt. 9 Problem 8

Suppose that f is a differentiable real function in an open set $E \subset R^n$, and that f has a local maximum at a point $x \in E$. Prove that $f'(x) = 0$.

Solution: Let $u \in \mathbb{R}^n$ and u is nonzero. Then consider the function $g(t) = x + tu$ which is a function from $\mathbb{R}$ to $\mathbb{R}^n$. Now $h = f \circ g$ is a $\mathbb{R} \to \mathbb{R}$ function and $h(t) = (f \circ g)(t) = f(x + ut)$. Hence h has a maximum at $t = 0$. Therefore $h'(0) = 0$. Now by Chain Rule $h'(t) = f'(g(t))g'(t) = f'(g(t))u$. Hence $h'(0) = f'(g(0))u = f'(x)u$. Hence $f'(x)u = 0$ since u is nonzero $f'(x) = 0$

□

Problem 4 Rudin Chapt. 9 Problem 10

If f is a real function defined in a convex open set $E \subset R^n$, such that $(D_1 f)(x) = 0$ for every $x \in E$, prove that $f(x)$ depends only on $x_2, \ldots, x_n$.

Show that the convexity of E can be replaced by a weaker condition, but that some condition is required. For example, if $n = 2$ and E is shaped like a horseshoe, the statement may be false.

Solution: To prove $f(x)$ depends only on $x_2, x_3, \cdots, x_n$ it is enough to show that $f(x, x_2, \cdots, x_n) = f(y, x_2, \cdots, x_n)$ where $(x, x_2, \cdots, x_n), (y, x_2, \cdots, x_n) \in E$. Hence for any $u \in (x, y)$ $(u,_2, \cdots, x_n) \in E$ as E is convex. Now by Mean Value Theorem $f(x, x_2, \cdots, x_n) - f(y, x_2, \cdots, x_n) = (x-y)D_i f(z, x_2, \cdots, x_n)$ for some $z \in (x, y)$. Given that $D_i f(x) = 0$ Hence $f(x, x_2, \cdots, x_n) - f(y, x_2, \cdots, x_n) = 0$. Therefore $f(x)$ depends only on $x_2, \ldots, x_n$.

We need the property where for fixed $x_2, \cdots, x_n$ if $(x, x_2, \cdots, x_n)$ and $(y, x_2, \cdots, x_n)$ are in E then E must contain the line segment joining $(x, x_2, \cdots, x_n)$ and $(y, x_2, \cdots, x_n)$. Hence we can say that if E intersects any line parallel to $X-$axis then it should be an interval on that line.

□

Problem 5 Rudin Chapt. 9 Problem 13

Suppose f is a differentiable mapping of R^1 into R^3 such that $|f(t)| = 1$ for every t. Prove that $f'(t) \cdot f(t) = 0$.

Interpret this result geometrically.

Solution: Let $f = \begin{bmatrix} f_1 \\ f_2 \\ f_3 \end{bmatrix}$ Given that $|f(t)| = 1 \implies f_1^2(t) + f_2^2(t) + f_3^2(t) = 1$ for all $t \in \mathbb{R}$. Now

$$0 = \frac{d}{dt}|f(t)|^2 = \frac{d}{dt}(f_1^(t) + f_2^2(t) + f_3^2(t)) = 2(f_1'(t)f_1(t) + f_2'(t)f_2(t) + f_3'(t)f_3(t)) = 2f'(t) \cdot f(t)$$

Hence $f'(t) \cdot f(t) = 0$ for all $t \in \mathbb{R}$.

$f'(t) \cdot f(t)$ means the vector $f'(t)$ is perpendicular to $f(t)$. If $f(t)$ is the radius vector of a point on an unit sphere then $f'(t)$ is tangent vector on that point to the sphere

□