Improving Scheduling Outcomes with a Dutch Auction

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Abstract

Employers often fail to take into account the preferences of shift workers when assigning them shifts. In this paper I discuss the flaws of modern scheduling and propose two mechanisms—trading cycles and a Dutch auction mechanism, where less demanded shifts offer increased wages—which improve outcomes for both employers and employees. For each mechanism I include optimal strategy for employees, welfare results, and use stability and acceptability measures to discuss which mechanisms are employer-optimal. The permission of trades strictly improves welfare for both employees and the employer unless the costs of trading are high. The Dutch auction mechanism guarantees stability and that each employee is weakly satisfied with her shift.

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Contents

1 Introduction

Roughly 25 million Americans are shift workers with schedules that vary week to week.[1] These jobs are most prominent in the service and retail sectors, but also appear elsewhere: emergency medicine, steelworking, even airline crews. On the surface, employers provide schedules that vary by the week because they want to evenly disperse the shifts that nobody seems to want. But for many businesses, these variant schedules have unintended effects, or hardly provide flexibility in the first place. There is an 87 percent fluctuation in shift-working schedules by the week—meaning workers can hardly plan weeks in advance—and over half of shift workers report that their schedules are generated without any input on their end.[2] I propose that both employees and employers would benefit from scheduling mechanisms that take into account employee preferences.

Employees benefit from the chance to plan their work schedule around their wants and needs. At the same time, businesses benefit from providing flexibility to their employees, specifically because it decreases absenteeism and turnover. Employees are more likely to miss shifts when they are assigned shifts that they do not want. And employee dissatisfaction with inflexible work schedules is a leading cause of turnover across all industries.[3] The cost of turnover is substantial: replacing employees costs nearly equal 20 percent of an employee's annual salary.[4] So, given that inflexible scheduling causes turnover, and that turnover is so costly, employers should find ways to make schedules more flexible. My

[1] These data are based on 2014 survey data from the General Social Survey, a project of the National Opinion Research Center at the University of Chicago. The survey showed that about 19 percent of employed survey respondents work irregular, shift, or rotating schedules.

[2] Both percentages taken from "Restoring a Fair Workweek," prepared by the Center for Popular Democracy.

[3] Neil Kokemuller, "How to Manage Retail Payroll Budgets."

[4] Boushey and Glynn (2012). This datum holds across jobs paying \$30K or less, between \$30K and \$50K, and between \$50K and \$70K.

research objective is to analyze scheduling mechanisms that discourage turnover while satisfying both employees and employers.

The difficulty in scheduling lies in what I call the "Friday Night Problem." Barring the inclusion of tips, shift workers get paid the same amount on a Friday night as they would on, say, a Tuesday afternoon. This is patently unfair: Friday nights generally have more opportunity outside of the shift than does a Tuesday afternoon, and its proposed wage should thereby reflect that truth. For most businesses, though, the wage at each shift is the same. Barring irregularities in preferences, most employees will strictly prefer working a Tuesday afternoon shift to a Friday night shift.[5]

The Friday Night Problem is indicative of a broader problem in scheduling: that the downside, or "cost," of working each shift is unequal. Indeed, most workers would agree that shifts on Friday nights are much costlier than those on Tuesday afternoons, but the problem spans much wider. Many shift workers have other obligations—about half of the 25 million shift workers have a child; about the same number are married—and so their costs for shifts are going to vary weekly due to their other needs, and will especially vary from their coworkers.[6] With these data in mind, it is sensible to maximize employees' chances to submit preferences for working certain shifts over others.

It must be considered: what do businesses currently use to make schedules? It has already been noted that about half of shift workers are assigned shifts without any input on their end. Many retail and restaurant businesses take into account when employees are *not* able to come in, specifically through web and mobile scheduling platforms.[7] For other industries, like in medicine, scheduling

[5] The inclusion of tips may flip this inequality; Friday nights may be preferable then.

[6] This estimate is derived from the Bureau of Labor Statistics's 2015 National Longitudinal Survey of Youth.

[7] Popular apps that accomplish this are MakeShift, 7Shifts, HomeBase, and even holistic

is more dynamic. At St. Peter's Hospital in Albany, New York, extra shifts are auctioned off.[8] Each nurse enters the minimum price that she will work the shift that she desires, and an algorithm assigns shifts to the lowest bidder. Robinson (2003) finds that this system is successful: the average bid is \$37/hour, more than the per-hour rate for a standard nurse, but less than the per-hour rate for a contract nurse, who were paid roughly \$49/hour around the time of the research.[9] Further, Robinson finds that the auction program saved over \$1.7 million in costs, while reducing the hospital's nurse vacancy rate from 11 percent in 2000 to below 5 percent by 2003.

Another critical question arises: what is the best way to allocate shifts while taking into account the preferences of workers? The random assignment of shifts may suffice if employees are able to trade shifts aftermarket, especially if those trades are not very costly.[10] The mechanism at St. Peter's Hospital was effective and suggests that the use of auctions in problems of this kind would be valuable. In this paper I prepare an auction mechanism that allocates shifts in a similar fashion to that at St. Peter's, and prove its usefulness for both employees and employers.

In their other applications, auctions are useful in organically generating prices for goods, especially when the preferences of potential buyers are largely unknown. In a standard auction, each potential buyer submits a bid, which stipulates a price that she is willing to pay for the given object. There are many variations of auctions; in this paper I use the descending-price ("Dutch") auc-

CRM platforms like Workday and Monday.com.

[8] Nurses can bid on shifts in any department, provided that they have the proper skills and certification. Nurses are bidding on extra shifts, that is, they are picking up extra hours in addition to their regularly scheduled shifts. See section **2.1** for more on the mechanisms through which shifts are preliminarily assigned to nurses.

[9] See De Grano (2007) for further exposition on the mechanism used at St. Peter's.

[10] Trades can be costly nonmonetarily, too. For example, it may take a long time to figure out with which coworker the employee should trade shifts.

tion. In this auction, a seller posts a high price of sale, and the price decreases over time until a potential buyer declares that she is willing to purchase the good. The auction for a single good thus closes after one bid.

In scheduling matters, the auctioneer has the goal to sell labor. Thus scheduling proposes a flipped problem: an employer is auctioning off shifts to employees, who are "potential sellers." Though these auctions appear as wholly different mechanisms, the use case for each is roughly the same.[11]

In this paper I explain the difficulties of modern scheduling, specifically that incorporating employee preferences improves both employee and employer welfare. Trading cycles (or merely allowing employees to trade) and a Dutch auction are two mechanisms that incorporate preferences and would accomplish this. I compare employee and employer welfare in each mechanism, and include the optimal strategy that employees should use in the auction mechanism and when making trades.

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[11] It is worth adding that Vickrey (1961), and later Myerson (1981) proved that, *ceteris paribus*, auctioneers generate equal revenue across all standard auction mechanisms. Each, at least theoretically, offers equal utility to the seller.

2 Literature Review

This paper discusses mechanisms that would potentially ameliorate problems with shift worker scheduling via the application of economic theory. I first review the presence of shift worker scheduling in literature. Then, I give a brief overview of auctions in literature, in both theory and practice. I lastly present the contemporary use of auctions to schedule employees, and leverage the nursing industry as the lead example.

2.1 Scheduling

Scheduling entails the allocation of shifts, each usually defined by a few-hour time block, to shift workers.

The first general survey of shift worker scheduling is in Aggarwal (1982). He is keenly aware of the constraints which scheduling attempts to resolve: nurses face union regulations, a finite number of operating rooms and limited operating budgets; fire protection and police face city budgets, the need for swift response, and public pressure for improved services; transport systems face safety regulation, unions, and rest period requirements. His findings that are most critical to this paper are the restrictions for employee-to-customer individualized services: fulfillment of hourly demand for services, limited number of service channels at each location, and limited operating budgets. Special management objectives for scheduling are to minimize employee costs and idle time, while minimizing customer wait time. Aggarwal categorizes the scheduling heuristics in common use that businesses can use to optimize schedules.

Following Aggarwal's research into scheduling objectives, there was an influx of literature that procured optimal methodology for meeting these objectives. On any occasion where agents have ordinal preferences over single items, optimal assignment is soluble via simple mathematical programming.[12] But in reality, preferences are interdependent and often highly complex. Linear integer programs that assign goods to large numbers with more complex preferences require high computation times in order to arrive at an exact solution.[13] There is little demand for high-runtime algorithms that find the lone solution in scheduling, though, especially when contending with budget constraints.[14] Research including Randhawa and Sitompul (1993) and Lourenco et al. (2001) explore heuristic scheduling methods—performed entirely by computers—that are able to produce near-optimal results with much less computation time.

Other research has sought to find if the complexity or runtime of computation is even worth the outcome. Williams et al. (2018) explore "stable scheduling": allocating shifts many weeks in advance and over-supplying the number of employees to shifts such that someone can miss a shift and not significantly penalize herself or the business. A randomized controlled experiment at Gap explored the benefits of stable scheduling: branches with stable scheduling saw a 7 percent rise in sales and a 5 percent rise in labor productivity, each significantly above growth of the control group.

There is a large number of commercial software packages that may assist with scheduling. But these software packages, while they provide significant optimization capabilities, generally target specific application areas.[15] To understand the different patterns of scheduling, then, it is not so useful to analyze one-by-one the different techniques, and how they are applied to different industries. Instead, it is much more intuitive to analyze an industry which might

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[12] Algorithms that assign agents to goods are especially prominent in modern matching research.

[13] Ernst et al. (2004).

[14] Indeed, Aggarwal (1982) cites budgeting as the most prominent constraint for scheduling in businesses whose employees and customers interface frequently.

[15] This was the case in Ernst et al. (2004), and still holds today.

see the various types of scheduling.

The nursing industry hosts varied kinds of scheduling, each of which is common to the working sector. The first kind of scheduling, cyclic scheduling, entails little computational effort: many schedules that satisfy hospital requirements are generated, each nurse is assigned a functional schedule for the week, and schedules eventually rotate. In an empirical report, Bard and Purnomo (2005) espouse that many nurses think that cyclic scheduling does not provide enough flexibility.

The next kind of scheduling is non-cyclic scheduling: new schedules that satisfy hospital requirements are generated for each scheduling block and assigned to nurses more than a week in advance.[16]

Another mechanism by which scheduling can occur is through self-scheduling, where nurses book shifts on a first-come, first-serve basis. Silvestro and Silvestro (2000) discuss that self-scheduling is conducive to staff empowerment, motivation and roster effectiveness, though it may be unfair. Koning (2014) found that self-scheduling increases job satisfaction specifically for nurses.

Self-scheduling is strongly related to preference scheduling—the final standard mechanism through which shifts are assigned to nurses—because each factors in the desires of nurses for certain allocations over others. In preference scheduling, nurses submit their desires for shifts, either manually or via technology. The employer then tries to satisfy as many preferences and requests as possible while abiding by business constraints. Pryce et al. (2006) studied a Danish psychiatric hospital where preferences scheduling was in place. There, nurses reported that they were more satisfied with their work hours, less likely to swap shifts, and were more pleased with their work-life balance when compared

[16] De Grano (2007).

with nurses in the control groups.

Scheduling as a discipline thus has many routes of assignment: automated versus manual; cyclic versus non-cyclic; and preference-incorporating versus first-come first-serve. I now explore auction literature to determine if auctions could be useful as a means of incorporating preferences in schedule allocation.

2.2 Auctions

There are two standard ways to model agent behavior in auction theory. First, there is the basic private value model: each agent is risk-neutral and has a value for an object, prescribed in theory as independently and identically distributed from the values of other agents. This value is "private," in the sense that agents do not know the values of other agents, and their values are unaffected by the values of other agents. Thus, bids are functions exclusively of one's own value. Alternatively, the common value model imposes that the good has a singular, objective value, and that agents have varied estimates for its true value. The cognition of other agents' estimates thus affects one's own estimate.

The private value model befits the scheduling problem. Not every shift is equal; likewise, not every employee values each shift equally. The core difference between scheduling and the previously defined basic private value is the number of goods which are auctioned off. In the literature there are single object, multi-unit, and multi-object auctions.

Trivially, single object auctions facilitate transactions of one good at a time. Multi-unit auctions facilitate transactions of many homogeneous goods at once. Multi-object auctions are multi-unit auctions where the goods are heterogeneous. In the scheduling case—again, because not every shift is equal, and not every employee values each shift equally—the relevant literature is in the multi-object domain. Multi-object auctions in the private value model (hereon just called "multi-object auctions") are particularly popular in the privatization of assets. Ausubel (2004) explores multi-object auctions for the allocation of Treasury bills and telecommunications spectra; Benoit and Krishna (2000) for the sale of public transportation systems in Britain and Scandinavia, for example.

Ausubel (2004) notes that auction literature has provided us with two guiding principles. First, an auction should encourage a bidder to truthfully report her value for the good. It follows that the price paid by the winning bidder should be as independent as possible of her own bids and should depend on competition, specifically the opposing participants' bids.[17] Second, an auction should maximize the information available to each bidder at the time of her bid.[18] In the multi-object setting, these guiding principles remain largely applicable: nontruthful reporting leads to destructive auction outcomes like those described in Klemperer (2002); violation of the second principle leads to expected revenue that is more unpredictable than in an open, complete information auction.[19]

Bidding strategy is one focal point of multi-object auction research; outcome efficiency (revenue maximization) is the other. Benoit and Krishna (2000) observe nontrivial bidding behavior in multi-object auctions. It is sometimes beneficial for a bidder to submit a bid above her value for one object, attempting to raise the price paid by another agent for the object, thus depleting the other agent's budget, and decreasing the competition for other objects. Unlike in single-object auction theory, a particular agent's payoff is affected by the

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[17] Vickrey (1961).

[18] Milgrom and Weber (1982).

[19] Maskin and Riley (2000). This threatens the revenue-equivalence theorem, which imposes that seller revenue is independent of the auction type deployed. Maskin and Riley find that the hypothesis of the theorem may be too strong, in that it requires symmetry of buyers' information.

price paid by a rival bidder.

Similarly, bidders in multi-object auctions can communicate with others via their bids, sending signals about their values or about future behavior. Klemperer (2002) describes a handful, though his description of the occurrence in Cramton and Schwartz (2000) is most indicative of the potential for bids to send signals:

> In a multi-license U.S. spectrum auction in 1996–1997, U.S. West was competing vigorously with McLeod for lot number 378: a license in Rochester, Minnesota. Although most bids in the auction had been in exact thousands of dollars, U.S. West bid $313,378 and $62,378 for two licenses in Iowa in which it had earlier shown no interest, overbidding McLeod, who had seemed to be the uncontested high bidder for these licenses. McLeod got the point that it was being punished for competing in Rochester and dropped out of that market. Since McLeod made subsequent higher bids on the Iowa licenses, the "punishment" bids cost U.S. West nothing.

It is immediately observable that optimal bidding strategies in multi-object auctions are nontrivial. Ortega-Reichert (1968) produced the seminal piece on multi-object auctions, where he found that, in the auction of two goods sequentially, that the presence of a second auction decreases the equilibrium bid for the good in the first auction. Fatima (2008) finds that this holds for the general case. Ausubel (2004) focuses exclusively on multi-unit auctions but proposes an English auction for multiple objects which induces strategyproofness, and more broadly, follows the two auction-guiding principles.

Outcome efficiency is the other domain in multi-auction literature. Noting that outcome inefficiency (the generation of revenue that is sub-maximum) is generally common in the auction of multiple objects, Ausubel (2004) poses a new ascending-bid auction in the multi-unit setting—common to Treasury bill or spectra auctions—that may retain efficiency when bidders have interdependent values for goods. Since then, researchers like Kazumura et al. (2020) have specifically contended with the difficulty of generating a revenue maximizing mechanism in the multi-object allocation problems. Though a precise description is elusive, Kazumura et al. provide a partial solution by imposing additional restraints, maximizing *ex post* revenue. Maskin and Riley (1989) provide optimal selling procedures with an eye to revenue maximization in certain settings. Nonetheless, the conclusion which Jehiel and Moldovanu (2001) draw still holds: it is very difficult to characterize revenue maximization for the general multi-object setting.

2.3 Auctions for Scheduling

Auctions have been studied extensively in economics as a means of allocating goods to buyers. Scheduling imposes a flipped problem: instead of agents buying goods, they sell labor at the price of the offered wage. In the literature, these kinds of auctions are known as procurement auctions.

It is standard that only industries where participant behavior is predictable should deploy auction mechanisms, in order to ensure that goods and services are not sold at too low (or high) of a price. Specifically, auctions are at least weakly better than other forms of markets when the value of a good or service to potential buyers is largely unknown by the seller, or when the buyer thinks competition would drive up the demand for the good or service.[20] The auctioning of multiple objects at once is effective when there are multiple participants, demand is variant across multiple objects, and the exact demand for each is unknown.

Scheduling in the procurement context meets all these criteria. Failure of agents to bid at all would obstruct theoretical outcomes the most, but this would be rare in the practice of scheduling, especially because jobs are shift workers' primary sources of income.[21]

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[20] Klemperer (2002).

[21] From the Bureau of Labor Statistics' 2015 National Longitudinal Survey of Youth, the

A prominent bidding software, StayStaffed, describes the use of auctions to allocate schedules:

> Employees may bid on available shifts in advance of [shift] openings. The candidate submits the lowest hourly rate they will accept for that shift. As bids are received, the manager selects a candidate based upon bid price, seniority, experience or other factors they wish to consider. Once the manager has made a candidate selection, all employees who had bid on that shift are notified via email that the shift has been filled. More savings for you and more freedom for your employees.[22]

Auctions have been especially successful in the allocation of schedules to nurses. Robinson (2003) describes the auction of shifts at St. Peter's Hospital. There, nurses can bid on the posted vacancies for shifts, provided that they have the proper certifications. The shift bidding program saved over $1.7 million. At Rio Grande Regional Hospital, the faculty used a bidding software package to fill additional shifts starting July of 2003 and saved $150,000 in its first six months of use.[23] The hospital also encouraged agency nurses to transition to full-time workers.

The scheduling problem is especially complex for industries like nursing because personnel skill levels must be matched with operational requirements, further complicated by conflicting constraints and preference considerations. For my model I assume that all employees are equally qualified and certified.

I avoid the prescription of a decentralized mechanism which consumes much time to effect schedules. Each mechanism must be simply implementable, either via a software or manually. For example, the mechanism used for course scheduling at schools like Princeton University, Yale School of Management, is

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salaries of shift workers constitute nearly half of their families' income.

[22] "Stay Staffed Staffing Management," Staffing Management, Web based Management Software.

[23] De Grano (2007).

highly useful but not very practical.[24] The process there is strategically nontrivial: each student has a nonzero "point endowment" and may allocate these points to each course she desires; a higher allocation to a course increases her probability of enrollment in each. Though this mechanism of assignment would be useful in the allocation of shifts, it requires much more time to optimally allocate an endowment than it does to orchestrate a few bids.[25] In general, I am looking for a functional heuristic mechanism, rather than an optimal one.

[24] See Sonmez and Ünver (2007) and Budish and Cantillon (2012) for more on these.

[25] Recall that half of shift workers are married, and another inclusive half have at least one child. The goal is to create mechanisms that produce satisfactory results in minimal time.

3 Theoretical Model

There is an employer who posts S shifts to a lot of N employees, each of whom can obtain at most one shift from the posted lot. Both shifts and employees are indivisible, meaning that employees cannot work portions of shifts. Each shift pays out a positive wage to a single employee. Employees have costs for shifts, which are independently and identically distributed from the costs which other employees have for shifts. This befits the assumption that not every shift is equal, and that not every employee values each shift equally. An employee always prefers working a low-cost shift to a high-cost shift. Her expected utility is generally defined as the wage for the shift less her cost for the shift.

Employees do not automatically truthfully disclose their costs for a given shift. Indeed, I assume that each is interested in engaging in strategic behavior to maximize her expected utility. I likewise assume that each employee's preferences concern solely her own assignment. Shifts have no preference for the employees assigned to them.

I begin by drafting functional notation for evaluating the scheduling problem. I then investigate three different mechanisms for shift allocation: random assignment, random assignment with trades, and a Dutch auction mechanism.

I pose two general heuristics for assessing the effectiveness of these three mechanisms: *acceptability* and *stability*. An individual assignment of a shift to an employee is acceptable if her cost for the shift does not exceed the wage she is paid for the shift. An aggregate allocation of shifts to employees is acceptable if there is no employee-shift assignment which is not acceptable. An aggregate allocation of shifts to employees is stable if there is not the possibility of a trade which makes each participating employee better off.

The first and most common mechanism for shift allocation is random assignment. In this, employees have no say over the shifts they are assigned. I investigate general utility for the employee and determine the probabilities that the aggregate allocation is stable and acceptable.

I then present the benefits of allowing employees to trade shifts. In this scenario, shifts are still assigned via random assignment, but employees can trade with their fellow employees when each employee wants the other employee's shift more than she wants her own. I calculate the number of trades that should be expected, and discuss the improved outcomes that trades bring.

Lastly, I pose a Dutch auction mechanism which decentralizes the scheduling process. In this, each shift constitutes its own auction. The employer sets a minimum wage at which each auction begins, and which increases over time until an employee claims that she would like to work the shift for the proposed wage. At this point, the auction concludes and the shift is assigned to the employee who claimed the shift at the given wage. I offer optimal strategy for the employee as she seeks to work shifts which maximize her utility, and discuss the improved outcomes that a Dutch auction brings.

I define an "absenteeism rate" equal to the probability that an employee will miss a shift to which she is assigned. When an employee earns a wage equal to the cost of her shift, I estimate that she will miss her shift with probability 0.06. I acquire this absenteeism rate from the following: the Bureau of Labor Statistics reported that of the relevant shift worker industries, there were 3.0 absences from work of employed salary workers per 100 shifts. It is also known from the Bureau of Labor Statistics that employees are not asked for their preferences on the shifts that they are assigned about half of the time. Random assignment thus accounts for at most 6.0 absences from work per 100 shifts. Conservatively, I assume that, on average, the random assignment of a shift means that the employee will attend with 0.94 probability. I reference this rate in the results to draw further conclusions on the usefulness of each assignment mechanism.

3.1 Notation

An employer posts S shifts, each denoted s_j with $j = (1, ..., S)$. There are N employees, each denoted e_i with $i = (1, ..., N)$, who are looking to obtain no more than one shift. It must be true that $N \geq S$, and each shift needs to be assigned.

Each shift pays wage $w = 1$. Each employee e_i has private cost $c_i^j \sim Uni\,(0, 2)$ for working s_j, which is independently and identically distributed from the cost c_k^j which employee e_k has for s_j; likewise, her cost for s_j is independently and identically distributed from her cost for shift s_ℓ.[26]

Define the assignment of s_j to e_i as the ordered pair (e_i, s_j). Define the utility u_i^j that employee e_i receives from the allocation (e_i, s_j) as

$$u_i^j\left(c_i^j\right) = 1 - c_i^j.$$

The utility that e_i receives from obtaining no shift, defined as the matching $(e_i, \emptyset)$, is similarly defined as

$$u_i^\emptyset = 0.$$

Note that if $w > c_i^j$, then $u_i^\emptyset > u_i^j$.

Even if (e_i, s_j) exists, e_i will not show up to s_j if the cost c_i^j of working it significantly exceeds w. Define the probability of e_i missing s_j given (e_i, s_j) as

[26] The expectation is that the average allocation should be baseline acceptable, hence uniformity of costs about the wage makes sense. If the wage is much greater than the average cost for a shift, the employer is overspending on labor and can decrease the wage. If the wage is much less than the average cost of a shift, employees will find jobs where their assignments are acceptable.

A_i^j, where

$$A_i^j\left(c_i^j, w\right) = \frac{0.06c_i^j}{w}.$$

See the introduction to section **3** for an explanation of this parametrization. Trivially, with $w = 1$, $A_i^j = 0.06c_i^j$.

An allocation is *acceptable* for e_i if $w \geq c_i^j$ holds for such (e_i, s_j). An allocation is acceptable if every individual's allocation is acceptable. An allocation is *unacceptable* if and only if it is not acceptable.

Define an allocation of shifts as *stable* if there is no set of employees who are better off if they traded shifts. Trades are desirable when (e_i, s_j) and (e_k, s_ℓ) exist where $1 \leq k \neq i \leq N$ and where $1 \leq \ell \neq j \leq S$ such that $c_i^j > c_i^\ell$ and $c_k^l > c_k^j$, or when (e_i, s_j) exists such that $c_i^j > w$ and $(e_k, \emptyset)$, where $1 \leq k \neq i \leq N$ and where $c_k^j < w < c_i^j$. Trades are also desirable when cycles exist. Say that (e_i, s_j) and (e_k, s_ℓ) exist. e_i "points" to the shift s_ℓ of e_k if $c_i^j > c_i^\ell$. A cycle is a sequence of at least three employees $e_i, e_{i+1}, ..., e_z$ such that e_i points to the shift of e_{i+1} for all $i \in \{1, ..., z-1\}$ and $e_i = e_z$, or the first and last person in the sequence are the same. An allocation is *unstable* if and only if it is not stable.

Appendices A, B, and C hold the proofs of the formal results.

3.2 Random Assignment

Each employee e_i is assigned a shift s_j. There is no optimal strategy for employees in this mechanism; they are merely given a shift at random. I investigate the probability that the allocation is acceptable and stable.

Proposition 3.2.1. *Each allocation is individually acceptable with probability $\frac{1}{2}$. The aggregate allocation is acceptable with probability $\left(\frac{1}{2}\right)^S$.*

Proof: The average cost c_i^j that each employee e_i has for shift s_j is the wage

w.[27] So $Pr\left(c_i^j \le w\right) = \frac{1}{2}$. By definition, an individual allocation is acceptable if $c_i^j \le w$ when (e_i, s_j), so an individual allocation is acceptable with probability $\frac{1}{2}$.

For the aggregate allocation to be acceptable, each individual allocation must be acceptable. Because the number of assignments is equal to the number of shifts, efficient aggregate assignment occurs at probability $\left(\frac{1}{2}\right)^S$.

Note that the probability of acceptability decreases exponentially in the number of shifts. Half of the employees are expected to find any given shift acceptable, but the probability that each shift is assigned in an acceptable manner is especially low with a high number of shifts. Note also that the probability of acceptability is independent of the number of employees in the market.

In random assignment, acceptability is a rather trivial calculation. The probability that the aggregate allocation is stable, however, is more complex. I establish a bound for stability at the two-sized group level, starting with the simple case of $N = S$.

Proposition 3.2.2. *When $N = S$, the probability that random assignment leads to a stable outcome is bounded above by $\frac{3}{4}^{\binom{S}{2}}$.*

Proof: See Appendix A.

Stability is generally understood as the optimal matching such that no employees are better off trading shifts. Trades are possible among r-sized employee groups, where $r \le N$. At the two-sized group level, the trade is merely a swap of shifts. But at the r-sized group level, trades are possible in cycles.

Briefly take that (e_1, s_1), (e_2, s_2), and (e_3, s_3) exist. e_1 may be unwilling to trade with e_2 but willing to trade with e_3; likewise e_2 may be unwilling to trade

[27] Uniformity of costs about the wage was established in section **3.1**.

with e_3 but willing to trade with e_1; likewise e_3 may be unwilling to trade with e_1 but willing to trade with e_2. In such a case, a singular swap is not possible, but a series of trades such that e_1 acquires s_3, e_2 acquires s_1, and e_3 acquires s_2 is possible.

The difficulty, however, is that the probability that three employees want to trade among themselves, a la the previous example, is conditioned on the nonexistence of a two-cycle between any of the employees in the three-cycle.

It is much simpler to establish a bound at the two-sized group level. There is a $\frac{3}{4}$ probability that any two employees will not want to swap shifts, and $\binom{S}{2}$ possible pairs of employees who can trade shifts among themselves. This is the origin of the upper bound.

Stability becomes more complex when there are employees who are not assigned shifts. Recall that if $w > c_i^j$ for e_i, then $u_i^{\emptyset} > u_i^j$. Thus an employee e_i who is assigned shift s_j, but who has cost c_i^j for s_j such that $c_i^j < w$, would trade allocations with an employee e_k who does not have a shift, but who has cost c_k^j for s_j such that $c_k^j > w$.

It is thus clear that the calculation slightly changes when employees are added. I now investigate stability when $N > S$.

Proposition 3.2.3. *When $N > S$, the probability that random assignment leads to a stable outcome is bounded above by*

$$\frac{3}{4}^{S\left(N-\frac{S+1}{2}\right)}. \tag{1}$$

Proof: See Appendix A.

Recall from **Proposition 3.2.2** that stability decreased in S when $N = S$. When N increases while S is constant, the probability that the aggregate allocation is stable decreases.

Consider the case where $N = 3$ and $S = 2$, and that (e_1, s_1), (e_2, s_2), and

$(e_3, \emptyset)$ exist. The probability of instability between e_1 and e_2 is the same as that of the two-sized group level from equation (1). But now there is the chance that e_3 has a lower cost for s_1 than e_1, or a lower cost for s_2 than e_2, and that the costs which e_1 and e_2 have for their respective assignments exceeds the wage. If this is the case, the allocation is unstable. There is instability between e_1 and e_3 with probability $\frac{3}{4}$. By symmetry, this holds for e_2 and e_3 also. Thus the probability of stability for $N = 3$ and $S = 2$ is bounded above by $\frac{3}{4} * \frac{3}{4}^2$.[28]

In the general $N > S$ case, there are $N - S$ possible employees who are unassigned shifts, and S employees for whom employee e_q (where $N - S < q < N$) can cause instability. With a few susbtituions available in Appendix A, the probability that the allocation is stable is bounded above by $\frac{3}{4}^{S\left(N - \frac{S+1}{2}\right)}$.

I have thus prepared a discrete calculation for acceptability and an upper bound for stability in random assignment, viewable in Figure 1. The probability of acceptability strictly decreases in the number of shifts but is independent of the number of employees; meanwhile, stability strictly decreases in both. When $N = S$, acceptability and stability are equally likely outcomes only when each nears 0 probability, when six shifts must be allocated. Stability is a much likelier outcome than acceptability up until then.

The outcomes differ when more employees are included. It was noted in **Proposition 3.2.3** that the addition of employees discounted the probability dramatically. Figure 2 demonstrates this change. Clearly the addition of extra employees makes stability a much more unlikely outcome. The presence of many employees who are unassigned shifts brings about instability with high likelihood, especially if S is large.

[28] Recall that instability calculations at the three-sized group level and above require conditioning probabilities of instability at these levels on the probability of stability at smaller group levels. The two-sized group provides a sufficient upper bound.

Figure 1:

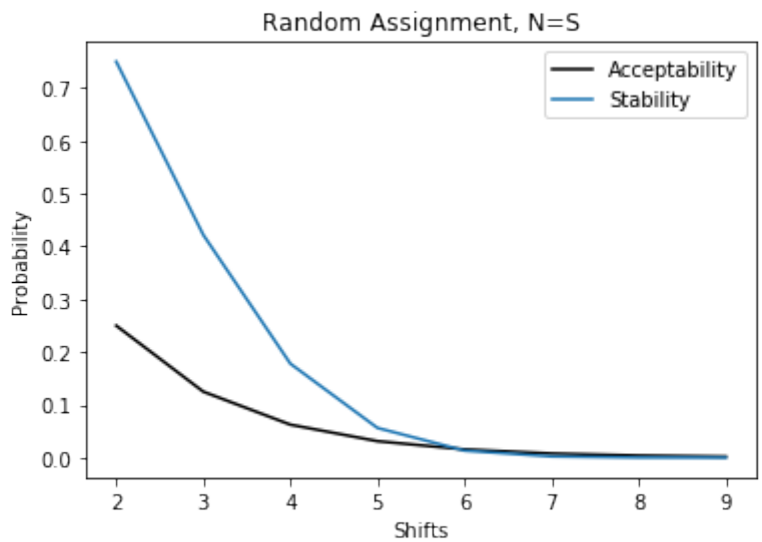
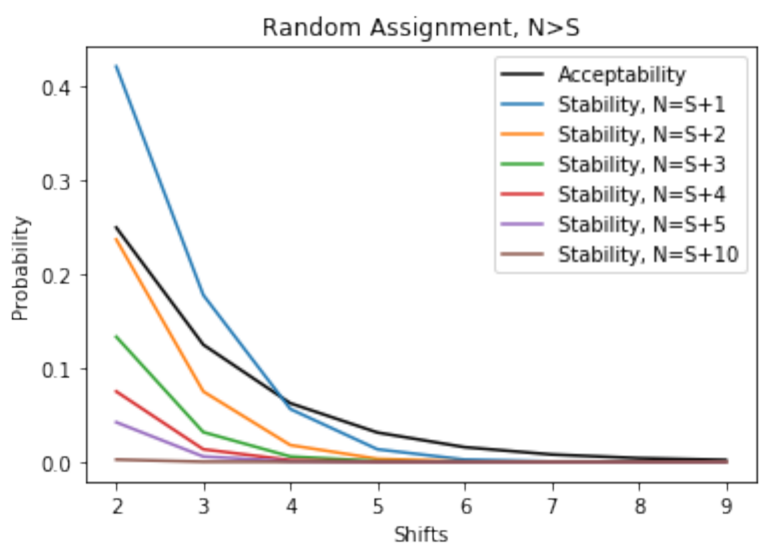


Figure 2:



Recall that the definition of acceptability implies that each employee earns a wage higher than her cost for the shift to which she is assigned. Acceptability is thus a heuristic that is adjustable with the wage. If an employer wants to make outcomes more acceptable, then she can adjust the wage. While stability is independent of the wage, there are indeed mechanisms that could induce more stable outcomes. One of these mechanisms is trading cycles, where employees trade allocations in order to improve their welfare.

3.3 Random Assignment with Trades

Trades resolve instability. Take that (e_i, s_j) and (e_k, s_ℓ) exist. Define a trade between e_i and e_j as a change in allocation from (e_i, s_j) and (e_k, s_ℓ) to (e_i, s_ℓ) and (e_k, s_j).

Trades spawn from the existence of cycles. Recall the definition of "pointing" to an employee: e_i points to e_k if $c_i^j > c_i^\ell$. A cycle is present if there is a sequence of at least three employees $e_i, e_{i+1}, ..., e_z$ such that e_i points to e_{i+1} for all $i \in \{1, ..., z-1\}$ and $e_i = e_z$. A trade means that e_i gets the assignment of e_{i+1} for $i \in \{1, ..., z-1\}$. $z - i - 1$ trades occur if the new assignment is such that e_i gets the assignment of e_{i+1} for all $i \in \{1, ..., z-1\}$. Of course, swaps between two employees can be trivially understood in terms of two-sized cycles.

An employee trades only if she can acquire a shift for which she has a lower cost. Thus trades should not only guarantee stability, but should also increase the acceptability of outcomes.

Proposition 3.3.1. *The probability that random assignment with trades leads to an acceptable outcome is bounded below by*

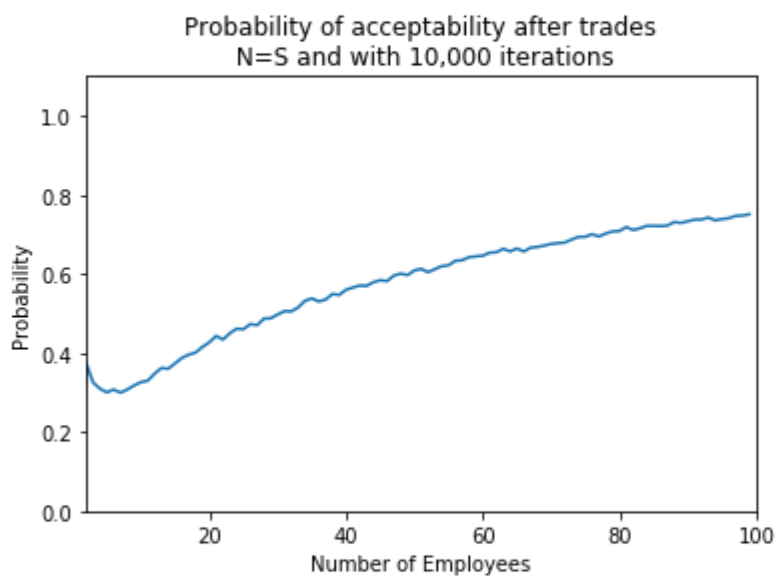
$$\left(\frac{1}{2}\right)^S + \left(\frac{1}{2}\right)^S * S! * \prod_{i=1}^{S} \left(\frac{1}{2}\right)^i + \left(\frac{1}{2}\right)^S * \frac{S}{S-1} * \frac{1}{8}. \tag{2}$$

Proof: See Appendix B.

Note that there are multiple scenarios where the aggregate allocation is unacceptable: as few as one employee may have an unacceptable allocation, or as many as S allocations may be unacceptable. I determine the lower bound by calculating a few cases where acceptability arises, and take into account the possible permutations of trades that may generate acceptability across all employees. A discrete calculation is difficult for two reasons: the probabilities of trades are conditioned on the trades which already occurred (an employee who has made a trade is happier with her new shift; she is harder to please); and there are many permutations of possible trades, as there is no limit to the number of trades that employees can make. In Appendix B I provide sample calculations for $N = S = 2$ and $N = S = 3$ to further justify the lower bound.

Nonetheless, each term in equation (2) is strictly positive, so trades strictly increase the probability of acceptability. This is expected: the cost that one has for her assigned shift weakly decreases after trades. I simulate the amount that trades improve acceptability, available in Figure 3.

Figure 3:



The improvement upon simple random assignment is obvious; trades drastically improve the probability that an outcome results in acceptability. Note that there are two counteracting factors that influence acceptability when employees are increased: the options of trade increases, while the number of employees who need to find acceptable outcomes also increases. The initial dip that acceptability suffers when trades are included is because the addition of employees outweighs the additional options. Over time, though, the increased trading options for employees affect acceptability more significantly.

The expectation is that acceptability improves with the number of shifts over time, excluding the initial dip at $N = S = 3$. I calculate my lower bound by the examination of trades wherein someone with an unacceptable outcome trades shifts with someone who has an acceptable outcome. At higher S values especially, many potential trades are discounted.

It thus seems like a trivial decision on part of the employer to allow trades among employees. However, there may be some cost to trading shifts that discourages its implementation. For instance, shift trading tools may not fit into the company's software architecture, or employers may not want employees to trade shifts behind the scenes so that they know who is working at any point in time. Nonetheless, some number of trades may fit within the budget. I now prepare a lower bound expectation for the number of trades that an employer can expect her employees to make.

Proposition 3.3.2. *With trade conditions relaxed, the expected total number of cycles which induce more acceptable outcomes has a lower bound of*

$$\frac{\left(\frac{S}{4}\right)^{S+1} - \left(\frac{S}{4}\right)^{\frac{S}{2}}}{\frac{S}{4} - 1}. \tag{3}$$

Proof: See Appendix B.

I relax trade conditions such that employees merely want to guarantee an acceptable outcome for as many people as possible. I assume here that employees are comfortable sacrificing utility up until their respective outcomes are unacceptable.

The expectation of the total number of trades strictly increases in S. By the same logic which constitutes the lower bound as applicable in **Proposition 3.3.2**, the number of cycles strictly increases in N, as well.[29] Thus this lower bound applies for all cases where $N > S$, as well.

However, these cycles are not indicative of the actual number of trades which would ever have to occur, especially given normal behavior. Shapley and Scarf (1974) found that the maximum number of trades which have to be made in order to achieve stability is N.[30]

In Figure 4 I provide the expected number of trades for the total number of employees. Note that the number of trades begins to surpass the total number of employees/shifts once the numbers become large. Employees begin to trade multiple times once different allocations become available. The number of trades increases in the number of shifts and in the number of employees.

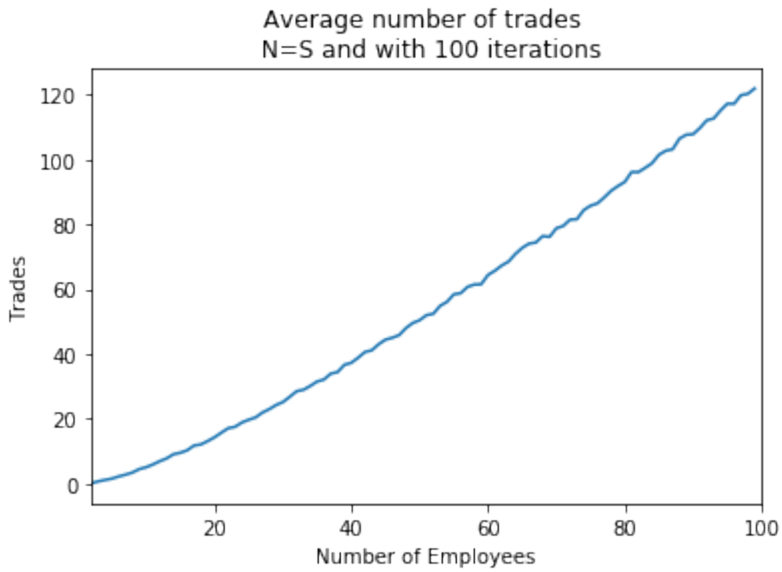
Shapley and Scarf (1974) found that a Top Trading Cycle can find an allocation that results in stability in as many rounds as there are agents. Hence the maximum number of trades required to reach stability is N. Even after N cycles are cleared, though, the resulting allocation may not be acceptable. And oftentimes employees make unnecessary trades along the way, in that employees trade for other shifts prematurely before trading again.

Note also that trades strictly decrease the chance of absenteeism. A trade between (e_i, s_j) and (e_k, s_ℓ) results in new allocation (e_i, s_ℓ) and (e_k, s_j). The former allocation has absenteeism rate $A_i^j = 0.06c_i^j$ and $A_k^\ell = 0.06c_k^\ell$. For the

[29] More employees implies that there are more people with whom employees can trade.

[30] Note that **Proposition 3.3.3** entails trades beyond stable outcomes.

Figure 4:



new allocation $A_k^j = 0.06c_k^j$ and $A_i^\ell = 0.06c_i^\ell$. By definition, $c_i^j > c_i^\ell$ and $c_k^l > c_k^j$. Thus $A_i^j > A_k^j$ and $A_k^\ell > A_i^\ell$. It is trivially true that the same logic applies for trades among cycles. I demonstrate the change in average cost of assigned shifts and in average absenteeism after trades below.

Figure 5 shows that the average cost of assigned shifts significantly decreases in the number of trades. Employees only trade when their costs decrease, and with many employees in the market, there are many opportunities to conduct a suitable trade that benefits both parties. Random assignment without trades renders the average cost of an assigned shift as 1, whereas random assignment with trades leads to an average cost below 0.5 with just 20 shifts and employees.

Figure 6 demonstrates that absenteeism decreases significantly from the expected 0.06 rate in random assignment.[31] Trades are expected to cut absen-

[31] An average cost of 1 gives an average absenteeism rate of $A_i^j = 0.06$

Figure 5:

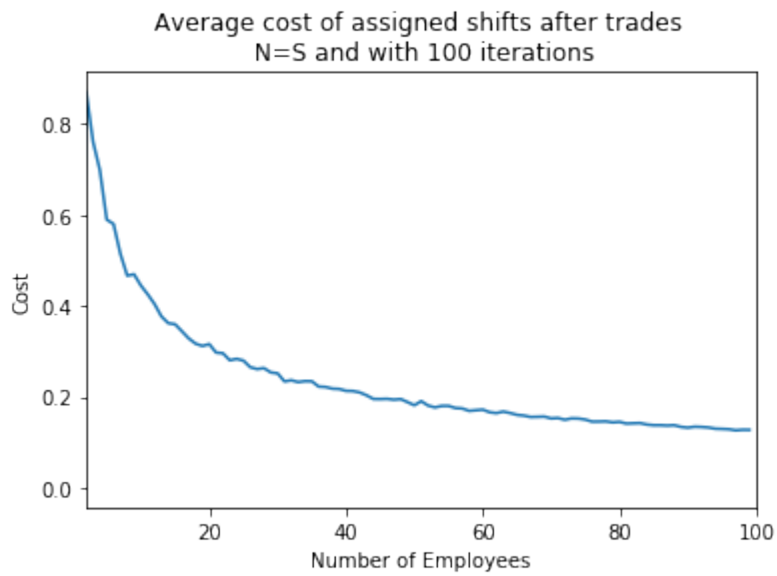
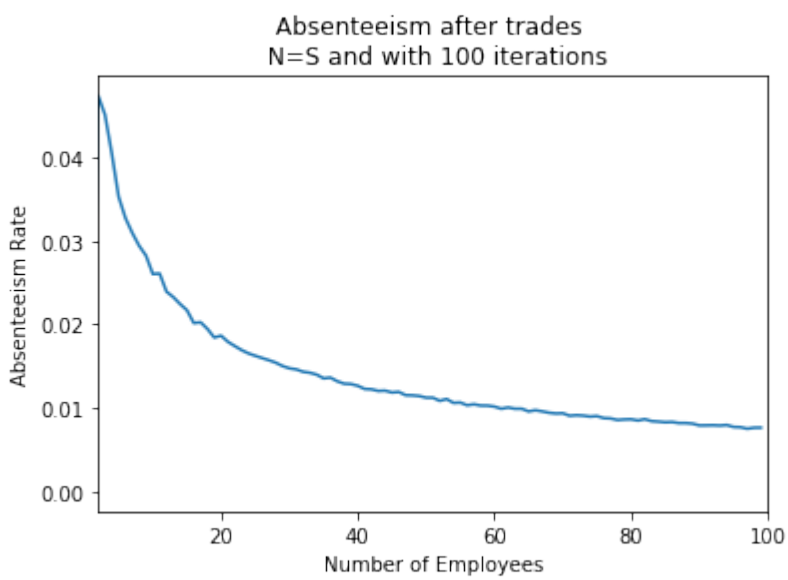


Figure 6:



teeism in half with just 20 employees and shifts.

So, trades lead to stability, increase the probability at which an allocation arrives at acceptability, and decrease absenteeism. I now consider if a Dutch auction mechanism improves upon these outcomes.

3.4 Dutch Auction

I define a Dutch auction mechanism which may improve employer and employee outcomes.

The notation, with some brief revisions on the prior mechanisms, follows. An employer posts S shifts, each denoted s_j with $j = (1, ..., S)$. There are N employees, each denoted e_i with $i = (1, ..., N)$, each of whom can obtain no more than one shift. It must be true that $N \geq S$, and each shift needs to be assigned.

Each employee e_i has private cost $c_i^j \sim Uni\,(0,\,2)$ for working s_j, which is independently and identically distributed from the costs of other employees; likewise, her cost for s_j is independently and identically distributed from her costs for shifts $s_{\ell \neq j}$. Shifts have no priority for one employee over another.

Each shift pays w_t which increases linearly along $(0, 2)$ as time t increases along $(0, 2)$. Starting at $t = 0$, each employee e_i has the option to bid for s_j, at which point she is assigned s_j, earns w_t, and the auction for s_j closes, at which point no employee can bid for s_j. Define the assignment of e_i to s_j as (e_i, s_j). The utility that e_i receives from claiming s_j at time t is defined as

$$u\left(c_i^j, t\right) = w_t - c_i^j = t - c_i^j.$$

The utility that e_i receives from obtaining no shift, defined as the matching $(e_i, \emptyset)$, is similarly defined as

$$u_i^{\emptyset} = 0.$$

Each employee e_i can participate in the auctions for all shifts s_j with $j =$

$(1, ..., S)$ simultaneously. If the auction results in just one employee and one shift remaining, the auction concludes and the final shift is assigned to the remaining employee and pays out the w_t which the most recently acquired shift paid out.

I first examine optimal bidding behavior with respect to one's cost for a shift.

Proposition 3.4.1. *With one shift and N employees, it is a weakly dominated strategy to bid for an item at $t = c_i^j$.*

Proof: See Appendix C.

The utility that e_1 receives from bidding on s_1 at $t = c_1^1$ is equal to $u(t) = t - c_1^1 = 0$. The utility that e_1 receives from acquiring no shift is also 0. But the utility that e_1 receives from bidding at $t' > c_1^1$ when s_1 is still available is equal to $u(t') = t' - c_1^1$., and there is some nonzero chance that s_1 is available at that point in time.[32]

Because $t' - c_1^1 > 0$, and because there is some positive probability (call it $a(t')$ that the shift is around slightly after $t = c_1^1$, then $\left(t' - c_1^1\right) a(t) > 0$. Thus waiting to bid until t' weakly dominates bidding at t. The employee has incentive to wait; the concern is then to investigate how long to wait.

Proposition 3.4.2. *With one shift and N employees, it is an equilibrium bidding strategy to bid*

$$b(c) = 2 - \frac{n-1}{n}(2-c). \tag{4}$$

Proof: See Appendix C.

The decision in response to "How long to wait to bid on a shift?" must

[32] Note that costs are independently and identically distributed. Thus if an employee has the lowest cost for a certain shift, it is likely that the shift will be available briefly after the point where $t = c_1^1$.

maximize the product of her expected utility and the probability that s_j is still available at t'. This bid is strictly increasing in her cost for a shift, meaning she is going to bid at a later time when her cost for the shift is higher.

It follows that one's bidding strategy should only consider the shift for which she has the lowest cost. She has no incentive to work a shift for which she has a higher cost so long as costs are independently and identically distributed across shifts.[33]

Lemma 1. *At any point in time, the employee should only consider bidding for the lowest cost shift which has not yet been allocated to anyone else.*

Proof: See Appendix C.

With independently and identically distributed costs, employees have no incentive to bid on a shift for which she has a higher cost.

It is so far clear that an employee should only focus on the auction of her lowest cost shift. Up until the conclusion of that auction, an employee will not participate in the auction of any other shift. But when there are multiple shifts remaining, an employee's bid depends not only on her cost for her lowest cost shift, but on her cost for all other shifts, and also on her expected profit from winning any one of the future auctions. If more auctions follow, the employee is expected to pursue greater profit (and bid later) in the earlier stages.

Proposition 3.4.3. *As the number of shifts remaining decreases, employee bids decrease to the single-object case prescribed in equation (4).*

Proof: See Appendix C.

---

[33] If costs were not independently and identically distributed, then there may be incentive to bid for a shift for which one has a higher cost. Consider a case where an employee wants two shifts. If there is less demand—or higher average cost—for the shift for which she has a lower cost, then she may want to wait to bid on the shift even longer. If this is the case, she may bid on the shift for which she has a higher cost first.

An employee's strategy for s_j where $(j = 1, ..., S-1)$ auction depends on the profit she expects from the $S - j$ shifts that are not yet allocated. Because there are only S shifts, the auction concludes after shift s_S is allocated. And so an employee's strategy during the last auction is the same as the single-object Dutch, defined in equation (4).

The bidding behavior for the first $S - 1$ auctions differs from that for the single object case, and depends on her probability of winning any one of the remaining auctions that are yet to be conducted. If she is still able to win another shift after losing out on her preferred shift, she is likely to wait a little bit longer for her preferred shift.

Pairing **Propositions 3.4.2** and **3.4.3**, I am able to determine the minimum amount that the employer should expect to pay out in the Dutch auction mechanism. I calculate a lower bound using equation (4) as the least wage someone will earn from a given shift.[34] Using $S = 10$ and $N = [10, 100]$, I graph the expectations for the average paid wages in Figure 7.

With many employees, the lower bound for the average wage is significantly below the wage in random assignment and with trades. The average wage decreases as the ratio of employees to shifts increases. This makes sense: employees need to bid earlier to secure a shift when there are many competing bidders.

Using similar logic, I conclude that the average cost of each assignment decreases as the number of employees increases. Figure 8 demonstrates the average cost of each assignment using the same specifications of S and N.

As the number of employees increases, the average cost of each assignment approaches 0. The employee with the minimum cost always wins the shift in

[34] Equation (5) holds the most conservative bid; when more shifts remain, employees are expected to pursue more profit in their allocation.

Figure 7:

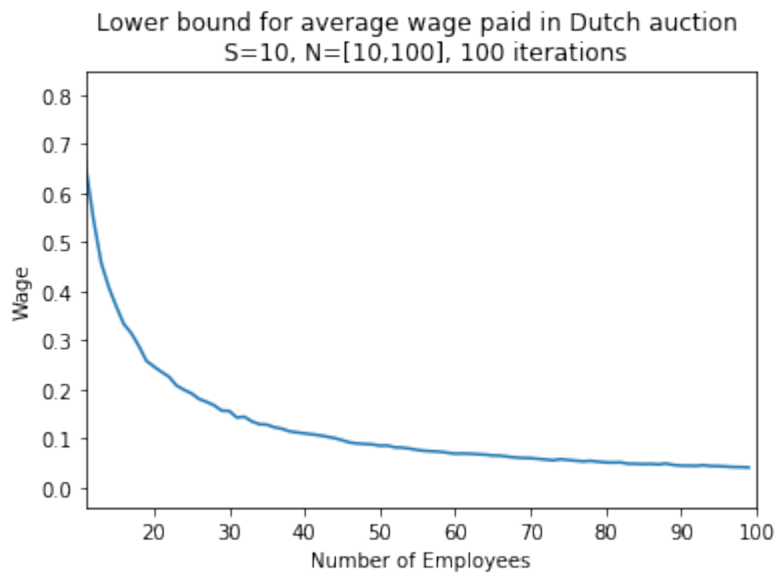
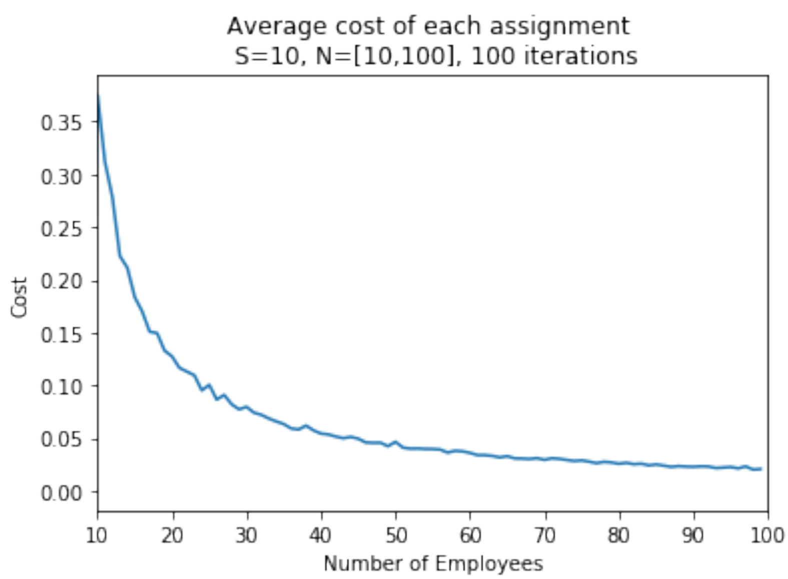


Figure 8:



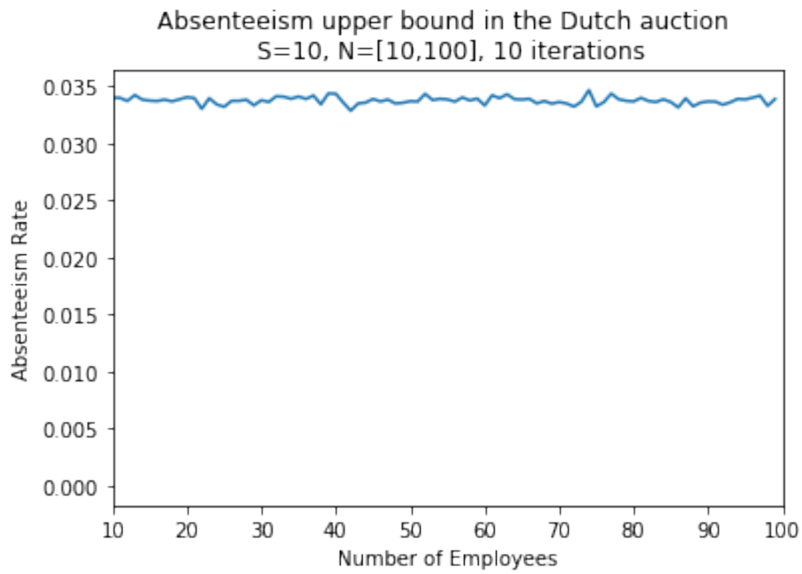
equilibrium bidding strategy.[35] Thus as the number of employees increases, a

[35] This is a trivial extension of **Lemma 1**.

lesser cost is more likely to win a given shift.

Constructing a graph of expected absenteeism rates in the Dutch auction is simple once one has expectations of the average cost and average wage of assignments. The expected absenteeism level when $S = 10$ and $N = [10, 100]$ is available in Figure 9. The absenteeism rate is bounded above right around 0.03.

Figure 9:



Recall that A_i^j decreases in wage paid and increases in the cost of assignment. In the Dutch auction mechanism, employees may earn less (Figure 7) but their costs decrease at about double the rate (Figure 8). And so the absenteeism rate decreases by at most around a factor of two. Note that this is an upper bound because the wage paid was a lower bound, since it was calculated only using equation (4). Note from **Lemma 1** that employees are only pursuing the shifts which maximize their welfare at any given point in time. Thus with no prior allocation, employees will never trade shifts. It is clear, then, that the allocation is always stable.

Proposition 3.4.4. *In any symmetric equilibrium of the Dutch auction, the resulting allocation is stable.*

Lemma 1 states that employees only bid on remaining shifts for which they have the lowest cost. And because equilibrium strategies are symmetric with respect to their costs, employee e_i always acquires her best available shift; any shift s_j is claimed prior by some employee e_m who has cost c_m^j such that $c_m^j < c_i^j$. Thus at least one employee of any employee pair will be unwilling to trade, as each employee acquires the least-cost shift which is available to her at the time.

By similar logic, the allocations are also always acceptable. No employee will pursue a shift which guarantees her negative utility.

Proposition 3.4.5. *In any symmetric equilibrium of the Dutch auction, the resulting allocation is acceptable.*

An allocation is acceptable if the wage w that e_i receives for working s_j exceeds her cost c_i^j for that shift, or $w - c_i^j \geq 0$. It was proven already that it is a weakly dominated strategy to bid on shifts at the time which guarantees $w = c_i^j$. Indeed, equilibrium strategy entails employees bidding on shifts at a time $t' > c_i^j$. Thus every allocation is acceptable.

Of course, the trade-off is that wages may vary. In the **Results** section, I compare the three assignment mechanisms, with an eye to the stability, acceptability, and incurred costs of each.

4 Results

The employer, not the employees, typically decides which mechanism the business should use in order to assign shifts to employees. It makes sense, then, to compare the utility that the employer derives from each assignment mechanism. I can generally claim that her utility is weakly increasing in the probabilities of stability and acceptability of assignment, and weakly decreasing in wages paid, absenteeism, and the number of trades that employees are expected to make after assignment. The effect which these have on employers may differ by sector or even by company; my goal is to delineate which mechanism is optimal for the employer given her preferences for these outcomes.

Define employer utility as

$$V = \alpha C + \beta T - \gamma A - W - \eta R,$$

where $\alpha, \beta, \gamma, \eta \geq 0$, and with the following variable assignments:

C: probability of acceptability;

T: probability of stability;

A: sum of the absenteeism rates of assigned employees;

W: sum of the wages paid to employees;

R : number of trades that are expected to occur.

I first analyze each variable generally.

Acceptability occurs with probability $\frac{1}{2}^S$ in random assignment (**Proposition 3.2.1**), strictly improves when trades are allowed (**Proposition 3.3.1**), and is guaranteed in the Dutch mechanism (**Proposition 3.4.5**). An employer with a high α value is likely to prefer the Dutch mechanism.

Similarly, stability is very unlikely in random assignment (**Proposition 3.2.3**), but is guaranteed after trades and via the Dutch mechanism (**Proposition**

3.4.4). A high β value is likely to lead the employer to avoid simple random assignment.

It should be noted that employers are unlikely to directly care about stability or acceptability. These measures are instead meaningful in the kinds of behaviors they imply: stability implies a certain optimality, in that no one wants to trade; acceptability implies that each employee derives positive utility from her assignment. Acceptability is useful in that it speaks to general employee morale; employees are at least weakly satisfied with their shifts. Similarly, acceptability is useful in that it is negatively correlated with the absenteeism rate. Though there may be alternative outcomes that are unacceptable but guarantee lower absenteeism, acceptability at least guarantees that the average absenteeism rate is at most 0.06. If the allocation is unacceptable, then there is at least one allocation for which the absenteeism rate exceeds 0.06. This correlation is readily apparent in Figures 3 and 6: trades improved acceptability while also decreasing absenteeism. Nonetheless, for an employer, an analysis of absenteeism is likely to be more useful than the proxy heuristic of acceptability.

Recall that absenteeism is defined for e_i who is assigned to s_j as $A_i^j(c_i^j, w) = \frac{0.06c_i^j}{w}$. In cases of random assignment and random assignment with trades, the wage paid to employees is always $w = 1$. In these mechanisms, $A_i^j(c_i^j) = 0.06c_i^j$. The average absenteeism rate in random assignment is 0.06 because the average cost of each assignment is 1. After trades, absenteeism decreases according to Figure 6. And when shifts are assigned via the Dutch mechanism, absenteeism decreases according to Figure 9. The calculation there is nontrivial: not only do costs change in the Dutch auction, but so do the wages. Still, a high γ means that an employer should avoid random assignment, and may prefer random assignment with trades over a Dutch auction mechanism, dependent on the number of employees in her business.

Wages in random assignment and random assignment with trades are stable. However, in the Dutch auction mechanism, not only are the wages dynamic, but they are largely dependent on the number of employees in the auction, as evident in **Propositions 3.4.2** and **3.4.3**. Figure 7 reveals the lower bound for the average wage paid in the Dutch auction, which is steadily below the wage defined in random assignment and random assignment with trades. However, there is some uncertainty here; employers cannot guarantee paying less in wages via the Dutch mechanism. Nonetheless, a high δ may convince an employer to deploy a Dutch auction mechanism for shift assignment.

Lastly, an employer should consider the cost of trades. Figure 4 depicts the number of trades that an employer should expect when $N = S$. If there is a high cost to trades, meaning a high η, an employer should avoid relying on trades to allocate shifts.

In general, random assignment is useful when the employer has a fairly strong signal that employees will not miss shifts, regardless of their assignment. Under random assignment, the employer does not have to spend more on labor in order to ensure attendance. Random assignment with trades is a strict improvement over random assignment when the act of trading is of sufficiently low cost to employers and employees. The Dutch auction mechanism is an improvement over the other two mechanisms if the employer highly values the stability and acceptability of schedules, is comfortable with the risks of dynamic wages, and highly values attendance.

I now consider each of the five factors—stability, acceptability, wages, absenteeism, and the number of trades—more specifically.

4.1 Stability

From **Proposition 3.2.2** and Figure 2, random assignment is only likely to result in a stable allocation when $N = S = 2$. For any other values of N and S, the allocation is likely to be unstable. Random assignment with trades and the Dutch mechanism lead to stable outcomes when employees behave rationally, i.e. conduct trades when it is optimal for them, and always bid on their most preferred shift. If an employer strongly values stability in her allocations, she should opt for either of the two latter mechanisms.

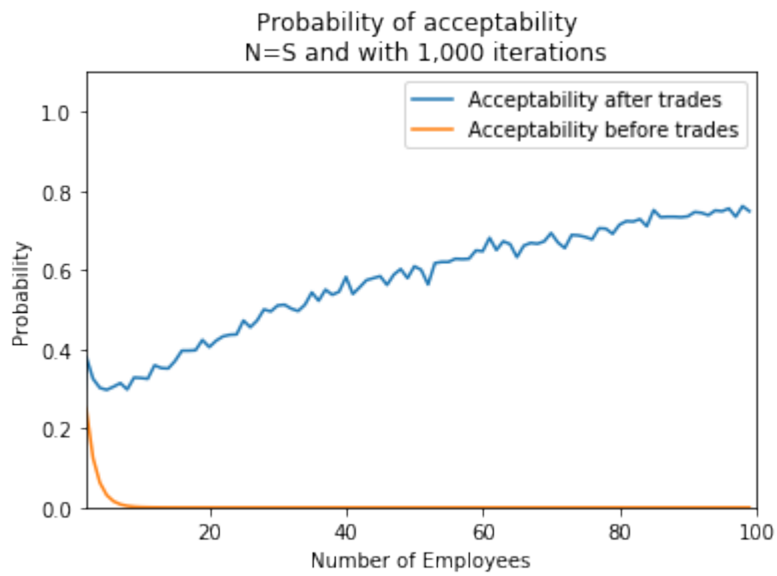
It should be noted that bidders are generally unaffected by the outcomes of their competitors. Perhaps it is sufficient to know that the outcome of their allocation was indeed stable, but it is unlikely for employees to significantly lose utility when they become aware that they were better off swapping shifts. It is useful when employers reflect on turnover, however it is not easy to discern stability in practice.[36] Thus it is rare that stability is the main concern for either employees or employers.

4.2 Acceptability

Recall that an an individual allocation is acceptable if her cost for the shift she is assigned is less than or equal to the wage she earns from that shift. Generally, the achievement of acceptability is making everyone just happy enough. In Figure 10 I compare the probability of acceptability graphically between random assignment and random assignment with trades. The improvement which trades have on acceptability are obvious. From **Proposition 3.4.5**, the Dutch auction always results in acceptability. If acceptability is particularly valuable to an employer, the Dutch auction and random assignment with trades are

[36] Costs inform strategic behavior, but costs are hardly explicitly reported.

Figure 10:



preferable when N is large. If N is small, the Dutch auction may be optimal; acceptability is still unlikely after trades when $N \leq 10$.

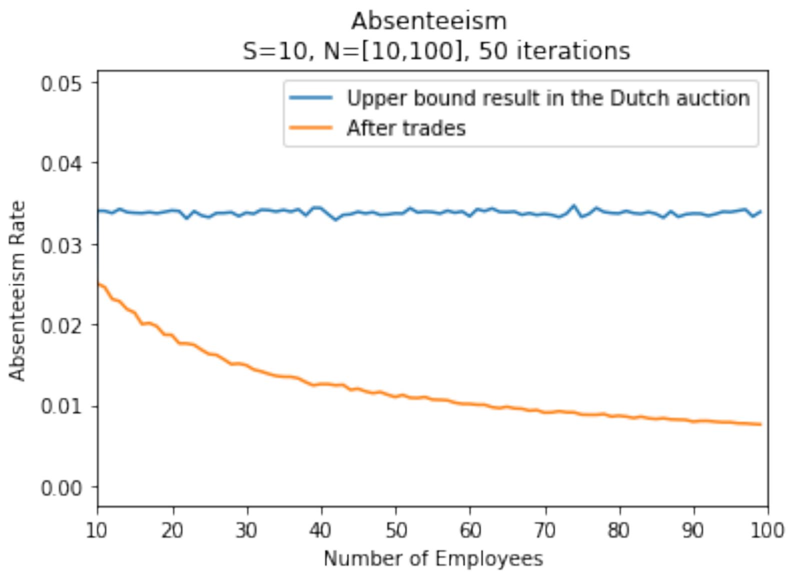
4.3 Wages

In random assignment and random assignment with trades, the wage is constant, set in my model at $w = 1$. From Figure 7, the Dutch auction has a lower bound for the average wage that is beneath 1. However, this lower bound does not rely on the variant bid behavior referenced in **Proposition 3.4.3**, that bidders seek out more profit when more auctions remain and when there are less competitors in the market. In general, if an employer fears the potential for distributed wages which may exceed the standard w, random assignment and random assignment with trades are strict improvements over the Dutch auction. However, Figure 7 reflects that a large amount of competition may keep the wages down.

4.4 Absenteeism

The average absenteeism rate in random assignment is trivially 0.06. It is worth analyzing Figures 6 and 9 together in order to discern which of the other two mechanisms is strongest with respect to absenteeism. The depiction of absenteeism after trades and via the Dutch auction is available in Figure 11. At all employee values $N \geq 10$, the absenteeism rate after trades is below the

Figure 11:



upper bound of the absenteeism rate in the Dutch auction mechanism. Each is rather useful for minimizing absenteeism, and provides a strict improvement on the absenteeism rate in simple random assignment.

4.5 Trades

It has been well observed so far that trades strictly improves upon the outcomes of random assignment, and sometimes even on those of the Dutch auction. However, the benefits of trades may be negated if there is some cost to trades.

If trades detract from employer utility, then the employer may prefer the Dutch auction or random assignment mechanisms.

5 Profit Maximization

It is also possible to analyze the mechanism selection decision from the lens of profit maximization. I provide sample analyses and simulations, though there is potential for further exploration in this space.

The employer will choose the mechanism that maximizes her firm's profit. The total production Q of the firm can be modeled via the Cobb-Douglas production function

$$Q = AK^{\alpha}L^{\beta},$$

where A is the firm's productivity, K is the capital input, L is the labor input, and α and β (where $\alpha, \beta > 0$) are the respective output elasticities for capital and labor. I have already discussed that there is a direct relationship between employee satisfaction and production; research shows that high morale directly affects productivity.[37] I thus pose that there is another factor M, the quality of labor, which directly affects a firm's production.[38] Thus I redefine the production function as

$$Q = AK^{\alpha}L^{\beta}M^{\gamma},$$

where γ is the output elasticity of the quality of labor.

The firm's profit π can be modeled as the expectation of her production less the amount that she pays for labor, or

$$\pi = \mathbb{E}\left[AK^{\alpha}L^{\beta}M^{\gamma} - Lw\right],$$

where w is the wage paid for every unit of labor. In this paper I am concerned with the labor market decision and am not concerned with the capital decision.

[37] See Weakliem (2006) and Yang (1997) for more.

[38] Hellerstein and Neumark (2004) derive a similar equation.

Net of K and α,

$$\pi = \mathbb{E}\left[AL^{\beta}M^{\gamma} - Lw\right].$$

I can draw general conclusions about the behavior of the relevant variables. L is inversely related to the absenteeism rate. I pose that M is correlated with the acceptability and stability of the allocation, or $M = (\lambda C + \xi T)$, where C is the probability of acceptability and T is the probability of stability, and where λ, $\xi > 0$. The wage w can be adjusted according to the mechanism used. Thus I can rewrite the firm's profit equation as

$$\pi = \mathbb{E}\left[AL^{\beta}(\lambda \mathrm{C} + \xi T)^{\gamma} - Lw\right].$$

Note that in random assignment or after trading cycles, it holds that $\mathbb{E}[Lw] = 0.94\mathbb{E}\left[c_i^j \mid (e_i, s_j)\right]$. Different wage-setting mechanisms have different L, C, T, and w. The scheduling decision is made unique for each firm by the elasticity and parameter values.

I am able to draw general observations related to the parameters from the profit equation. With high β, minimizing absenteeism is paramount. As shown in Figure 11, trading cycles and the Dutch auction are most useful to minimizing absenteeism. When β is 0, the employer should focus on increasing the quality of labor, as well as reducing $\mathbb{E}[Lw]$. When there is a large amount of competition, the Dutch auction may be most useful because stability and acceptability are guaranteed, and wages may decrease. Trading cycles are strictly preferred to random assignment in this case because absenteeism decreases while wages stay the same. With high γ, the Dutch auction is likely to be preferred. Trivially, with λ, $\xi = 0$, the quality of labor $M = 0$, and so the firm should minimize the wages paid to employees. Any mechanism that minimizes w is preferred; this is most likely in the Dutch auction with high competition.

In perfect equilibrium, the random assignment of schedules generates $\pi = 0$. A absorbs all absolute differences between the parameters in the profit equation; my concern, then, is for the relative differences between the terms. I can simplify the equation by setting $\gamma = 1 - \beta$. The expectation of profit becomes

$$\pi = \mathbb{E}\left[AL^{\beta}\left(\lambda \mathrm{C} + \xi T\right)^{1-\beta} - Lw\right].$$

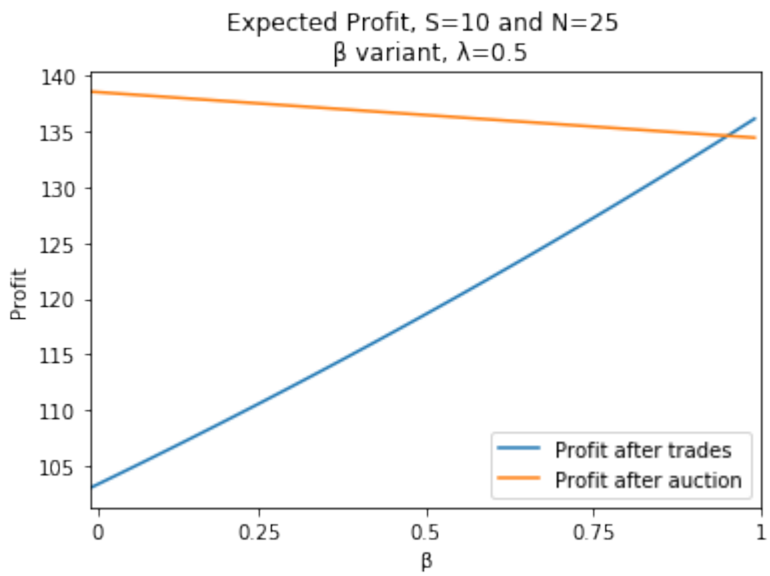
Similarly, the relative weights of λ and ξ are important, not their absolute values. And so the expectation of profit becomes

$$\pi = \mathbb{E}\left[AL^{\beta}\left(\lambda \mathrm{C} + (1-\lambda)\, T\right)^{1-\beta} - Lw\right].$$

The adjustment of the scheduling mechanism is expected to change the profit from $\pi = 0$. Under the zero profit condition and negligent of the quality of labor, it holds that $AL^{\beta} = Lw$. The marginal effect of additional labor on the left-hand side is $\beta AL^{\beta-1}$; on the right-hand side, it is w. Since $AL^{\beta} = Lw$ when $\pi = 0$, the relative marginal effects are βw and w, respectively. When $\beta = [0, 1]$, the marginal effect of the right-hand side is always greater, so adding labor should be less valuable to the firm than reducing wages. But, accounting for the quality of labor, there is a slight change in effects, since relative productivity changes. Using $S = 10$ and $N = 25$, I simulate the profits that the firm should expect given parameter values for β and λ, with L, C, T, and w determined by the scheduling mechanism in place.

Figure 12 demonstrates the difference in profit when $\lambda = 0.5$ but β spans $[0, 1]$. When β is small, the auction mechanism is generally preferred to trading cycles. The auction mechanism always guarantees acceptable and stable outcomes; with low β, these heuristics are paramount. But as β increases, these heuristics are less important. In the case of $S = 10$ and $N = 25$, trades are

Figure 12:



expected to be preferred to the Dutch auction right around $\beta = 0.9$. This makes sense: absenteeism is expected to be lower from trades than from the auction mechanism, specifically because wages decrease in the auction window and not from trades.

Figure 13:

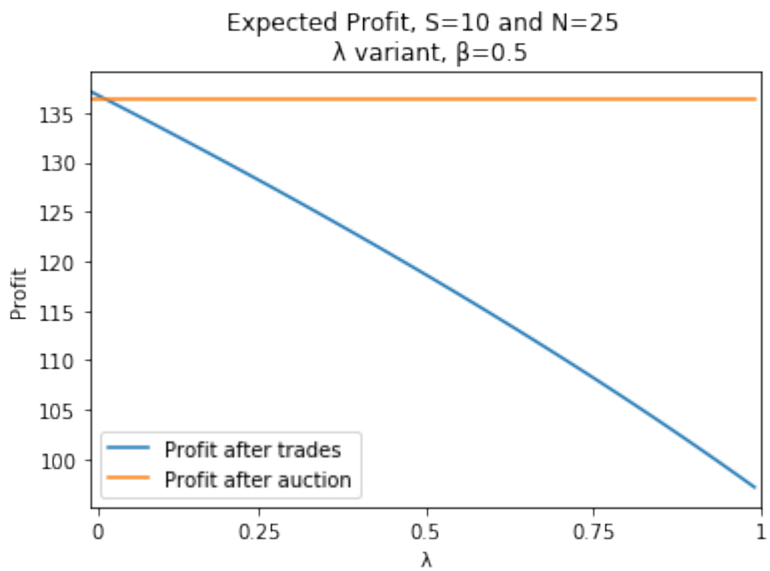


Figure 13 demonstrates the difference in profit when $\beta = 0.5$ but λ spans $[0, 1]$. When λ is small, stability is most important; each mechanism guarantees stable outcomes, and so they generate roughly the same profit. But when λ is large, acceptability is most important, and only the Dutch auction guarantees acceptability. And so the Dutch auction is expected to be an improvement over trading cycles as λ increases.

Recall that only a lower bound for the wages in the Dutch auction was calculated. In practice, $\mathbb{E}[Lw]$ will increase more than shown here, but so will L; as a result, the profit is expected to change. More robust results from the profit maximization model are possible with more discrete wage values.

6 Conclusion

The three mechanisms for shift allocation described in this paper—random assignment, random assignment with trades, and a Dutch auction—may suit different employers and industries. Random assignment is useful when the employer has a fairly strong signal that employees will not miss shifts, regardless of their assignment or the wage that they are paid. Trading cycles guarantee stability, while improving acceptability and decreasing absenteeism, and do not change the wages paid. Trades may only decrease employer utility if trading is costly. The Dutch auction is a useful mechanism for shift assignment: it guarantees stability and acceptability while decreasing absenteeism, and perhaps decreasing the wages paid to employees.

The Dutch auction mechanism described in this paper eliminates many of the scheduling problems that shift workers face today, including a lack of input into their schedules, as well as the Friday Night Problem. The mechanism imposes dynamic wages in order to guarantee that each employee earns a wage which exceeds her cost for working that shift; simultaneously no employee group needs to communicate in order to conduct trades. While manual scheduling procedures fail to consistently meet employee preferences, the auction mechanism decentralizes the process, and employees only choose to work then they are satisfied with their assignments. At the same time, each employee has an equal opportunity to bid, providing a fairer alternative to self-scheduling mechanisms that many companies currently use. Trades are similarly useful for employees: they can swap shifts among themselves in ways that strictly improve their welfare.

There have been many pieces of economic literature, especially in auctions, that explore if theoretical conditions hold in practice. In the future, I would like to explore the practical implementation of each mechanism described in this paper.

The cases where the Dutch auction mechanism breaks down may be worth exploring. For instance, with an unbounded cost function and a finite wage, the guarantee of an acceptable aggregate allocation no longer holds. It is also worth exploring the collusive measures that employees could undertake in order to maximize their wages, and the measures that the employer ought take in response. I also want to explore the ways in which employees can declare their costs for shifts with strategyproof behavior such that trading cycles can be fixed on a randomized outcome, therefore always resulting in stable outcomes without extensive effort on part of the employees.

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Appendix

A Random Assignment

Proof of ***Proposition 3.2.1***: The average c_i^j that e_i has for s_j is w. Thus $Pr\left(c_i^j < w\right) = \frac{1}{2}$. By definition, an individual allocation is acceptable if $w \geq c_i^j$ when (e_i, s_j), so an individual allocation is acceptable with probability $\frac{1}{2}$. For the aggregate allocation to be acceptable, each individual allocation must be acceptable. Because the number of assignments is equal to the number of shifts, efficient aggregate assignment occurs at probability $\left(\frac{1}{2}\right)^S$.

Proof of ***Proposition 3.2.2***: By symmetry, each one of the $S!$ allocations has the same probability of being stable. Take for example that $N = S = 2$, and that (e_1, s_1) and (e_2, s_2) exist. The allocation is unstable if $c_1^1 > c_2^1$ and $c_1^2 < c_2^2$. Because each cost is independently and identically distributed from the other, $Pr\left(c_1^1 > c_2^1\right) = \dfrac{1}{2}$ and $Pr\left(c_1^2 < c_2^2\right) = \dfrac{1}{2}$. The probability of each occurring is $\frac{1}{2} * \frac{1}{2} = \frac{1}{4}$. Thus the matching is stable with probability $1 - \frac{1}{4} = \frac{3}{4}$.

Take for another example that $N = S = 3$, and that (e_1, s_1), (e_2, s_2) and (e_3, s_3) exist. The matching is unstable if any of the following are true: $c_1^1 > c_2^1$ and $c_1^2 < c_2^2$, or $c_1^1 > c_3^1$ and $c_1^3 < c_3^3$, or $c_2^2 > c_3^2$ and $c_2^3 < c_3^3$, or $c_1^1 > c_2^1$ and $c_2^2 > c_3^2$ and $c_3^3 > c_1^3$, or $c_1^1 > c_3^1$ and $c_2^2 > c_1^2$ and $c_3^3 > c_2^3$.

The probability of stability is *ostensibly* equal to the probability that not any of the above instances occur, or $\left(1 - \frac{1}{2^2}\right)^3 \left(1 - \frac{1}{2^3}\right)^2$. However, the probability that three employees want to trade is conditioned on the nonexistence of a two-cycle between any of the employees in the three-cycle. If this correlation is not taken into account at the three-sized group level, then the probability is overestimated.

For the general case $N = S$, the probability that two employees want to trade is $\frac{1}{2^2}$, and there are $\binom{S}{2}$ combinations of employee-shift pairs. So the probability of instability at the two-sized group level is $\frac{1}{2^2}^{\binom{S}{2}}$. A lower bound for aggregate stability is thus

$$\frac{1}{2^2}^{\binom{S}{2}}.$$

I nonetheless provide probabilities that the allocation is stable at the r-sized group level, meaning that there are not r-sized groups of employees for whom trades are possible, regardless of the stability among groups of size less than r.

The probability that three employees want to trade is $\frac{1}{2^3}$, and there are $\binom{S}{3} * 2!$ possible combinations of employee-shift triplets. (The concern here is not for total permutations, but for the number of *cyclic permutations* possible from the set of employees. Recall that, though order matters in our creation of groups, a trading cycle (e_1, e_2, e_3) is equivalent to (e_3, e_1, e_2).)

So the probability of instability at the three-sized group level is $\frac{1}{2^3}^{\binom{S}{3}*2}$. For the analysis of r-sized groups, the probability that r employees want to trade is $\frac{1}{2^r}$,and there are $\binom{S}{r} * (r-1)\,!$ combinations of employee-shift r-sized groups. So the probability of instability at the r-sized group level is $\frac{1}{2^r}^{\binom{S}{r}*(r-1)!}$. By definition of stability, the probability that the outcome is stable for just the r-sized group is $1 - \left(\frac{1}{2^r}\right)^{\binom{S}{r}*(r-1)!}$.

Proof of ***Proposition 3.2.3***:

Again, by symmetry, each allocation has the same probability of being stable. The probability of stability among the assignments of $e_1, ..., e_S$ to $s_1, ..., s_S$ is equal to $\frac{1}{2^2}^{\binom{S}{2}}$. But now there are employees $e_{N-S}, ..., e_N$ who are not assigned shifts. Consider an assignment (e_i, s_j) where $1 \leq i \leq S$. There may be an employee e_q where $N - S \leq q \leq S$ for whom $c_q^j < c_i^j$.

The allocation is unstable if $c_q^j < c_i^j$ and $c_i^j > w$. Because costs are independent and identically distributed, and because the allocation is independent of costs, $Pr\left(c_q^j < c_i^j\right) = \frac{1}{2}$, and $Pr\left(c_i^j > w\right) = \frac{1}{2}$. Thus there is stability between e_q and e_i with probability $\frac{1}{2^2} = \frac{3}{4}$. There are $N - S$ possible values for q, and there are S employees for whom employee e_q can cause instability. And so the probability that the allocation is stable is bounded above by

$$\frac{3}{4}^{\binom{S}{2}} \frac{3}{4}^{(N-S)S}.$$

Much like the case in **Proposition 3.2.2**, the inclusion of the instability caused at the three-sized group level and above requires conditioning the probabilities of instability at these levels on the probability of stability at the two-sized group level. The two-sized group level provides a sufficient upper bound for stability.

The equation simplifies:

$$\begin{aligned} \frac{3}{4}^{\binom{S}{2}} \frac{3}{4}^{(N-S)S} &= \frac{3}{4}^{\frac{S!}{2(S-2)!}+(N-S)S} \\ &= \frac{3}{4}^{\frac{S(S-1)}{2}+(N-S)S} \\ &= \frac{3}{4}^{S\left((N-S)+\frac{S-1}{2}\right)} \\ &= \frac{3}{4}^{S\left(N+\frac{-S-1}{2}\right)} \\ &= \frac{3}{4}^{S\left(N-\frac{S+1}{2}\right)} \end{aligned} \tag{1}$$

B Random Assignment with Trades

Proof of ***Proposition 3.3.1***: First consider the case where $N = S = 2$, with assignments (e_1, s_1) and (e_2, s_2).

The allocation is acceptable outright if $c_1^1 < 1$ and $c_2^2 < 1$. Because costs are uniformly distributed about the wage, $Pr\left(c_1^1 < 1\right) = \frac{1}{2}$ and $Pr\left(c_2^2 < 1\right) = \frac{1}{2}$.

The probability of this outcome is $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

There is thus a $\frac{3}{4}$ probability that the initial allocation is unacceptable. There are three cases: only the individual allocation for e_1 is unacceptable, only the allocation for e_2 is unacceptable, and both allocations are unacceptable.

First consider the case where $c_1^1 > 1$ and $c_2^2 > 1$, which, by the same logic as above, occurs with probability $\frac{1}{4}$. Trades make this allocation acceptable if $c_1^2 < 1$ and $c_2^1 < 1$, each of which occurs with probability $\frac{1}{2}$. The outcome can thus be made acceptable with probability $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$.

Now consider the case where just e_1 has an unacceptable allocation. (By symmetry, all results hold for the case where just e_2 has an unacceptable allocation.) It must be true that $c_1^1 > 1$ and $c_2^2 < 1$. Each occurs with probability $\frac{1}{2}$, so this outcome occurs with probability $\frac{1}{4}$. Note that e_2 will only trade s_2 for s_1 if $c_2^1 < c_2^2$. Trades thus make this allocation acceptable if $c_1^2 < 1$ and $c_2^1 < c_2^2$. $Pr\left(c_1^2 < 1\right) = \frac{1}{2}$ because costs are uniformly distributed about the wage. But it is known already that $c_2^2 < 1$. Using Bayes' Theorem,

$$Pr\left(c_2^1 < c_2^2 \mid c_2^2 < 1\right) = \frac{Pr\left(c_2^2 < 1 \mid c_2^1 < c_2^2\right) * Pr\left(c_2^1 < c_2^2\right)}{Pr\left(c_2^2 < 1\right)} = \frac{\frac{1}{2} * \frac{1}{2}}{\frac{1}{2}} = \frac{1}{4}.$$

So the case where just e_1 has an unacceptable allocation can be made acceptable via trades with probability $\frac{1}{2} * \frac{1}{4} = \frac{1}{8}$. By symmetry, the case where just e_2 has an unacceptable allocation can also be made acceptable via trades with probability $\frac{1}{2} * \frac{1}{4} = \frac{1}{8}$.

Before trades, the allocation is acceptable with probability $\left(\frac{1}{2}\right)^2 = \frac{1}{4}$. After trades, the allocation is acceptable with probability $\left(\frac{1}{2}\right)^2 + \frac{1}{4} * \frac{1}{4} + 2 * \frac{1}{4} * \frac{1}{8} = \frac{3}{8}$.

Now consider the case where $N = S = 3$, with assignments $(e_1, s_1), (e_2, s_2)$, and (e_3, s_3).

The allocation is acceptable outright if $c_1^1 < 1$, $c_2^2 < 1$, and $c_3^3 < 1$. Because costs are uniformly distributed about the wage, the probability of each occurring is $\frac{1}{2}$. The probability of this outcome is thus $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$.

There is thus a $\frac{7}{8}$ probability that the initial allocation is unacceptable. There are many cases that generate unacceptability: all allocations are unacceptable, only two are unacceptable, and just one is unacceptable.

Each allocation is individually unacceptable with probability $\frac{1}{2}$. The case where $c_1^1 > 1$, $c_2^2 > 1$, and $c_3^3 > 1$ thus occurs with probability $\frac{1}{8}$. The allocation can be made acceptable in a variety of ways. First, e_1 and e_2 can trade such that each receives an acceptable allocation, and then one of them can trade with e_3. Or e_2 and e_3 can trade, such that each receives an acceptable allocation, and then one of them can trade with e_1. Or e_1 and e_3 can trade, such that each receives an acceptable allocation, and then one of them can trade with e_3. Consider the case where e_1 and e_2 initially trade. They trade such that each receives an acceptable allocation with probability $\frac{1}{4}$. The probability that e_1 is then willing to trade with e_3 is $\frac{1}{4}$, from the same use of Bayes' Rule as shown above. e_3 is trivially willing to trade with e_1 with probability $\frac{1}{2}$, so the probability that e_1 and e_3 trade is $\frac{1}{8}$. Thus the probability that e_1 and e_2 trade—and then e_1 and e_3 trade—such that the resulting outcome is acceptable is $\frac{1}{4} * \frac{1}{8} = \frac{1}{32}$. There are 3! possible orderings of trades that may occur when each allocation is acceptable. And so the probability that the allocation can be made acceptable when each is individually unacceptable is $3! * \frac{1}{32} = \frac{3}{16}$.

Now consider the case where exactly one individual allocation is unacceptable. There are $\binom{3}{1} = 3$ combinations of this, so this outcome occurs with probability $\frac{3}{8}$. Say e_1 is the only allocation that is unacceptable. (Note that the results hold by symmetry for e_2 and e_3.) The allocation can be made acceptable if $c_1^2 < 1$ and $c_2^1 < c_2^2$, or if $c_1^3 < c_1^1$ and $c_3^1 < c_3^3$. Each occurs with probability $\frac{1}{2} * \frac{1}{4} = \frac{1}{8}$, so with probability $\frac{1}{4}$ the case where only e_1 has an unacceptable allocation can be made acceptable.

Now consider the case where exactly two individual allocations are unacceptable. There are $\binom{3}{2} = 3$ combinations of this, so this outcome occurs with probability $\frac{3}{8}$. Say that e_1 and e_2 have unacceptable allocations. The probability that the allocation can be made acceptable is complex. e_1 and e_2 can trade and resolve the unacceptability by themselves, which is possible with probability $\frac{1}{4}$. e_1 and e_3 can trade with probability $\frac{1}{8}$ (see the application of Bayes' Rule above) and then either e_1 and e_2 can trade (probability $\frac{1}{4}$) or e_2 and e_3 can trade (probability $\frac{1}{16}$). Note that this final probability is $Pr\left(c_3^2 < c_3^1 \mid c_3^3 < 1, c_3^3 > c_3^1\right)$; by trivial extension of Bayes' Rule above, the product is compounded by a factor of $\frac{1}{2}$. The possible permutations of trades complicates even the $N = S = 3$ case. I pose a lower bound as an alternative to a discrete calculation in order to demonstrate the effectiveness of trades with respect to acceptability.

For $N = S = 3$, the allocation is acceptable outright with probability $\frac{1}{8}$. There is a $\frac{1}{8}$ probability that the allocation is unacceptable for each individual, which can be resolved with probability $\frac{3}{16}$. Exactly one allocation is unacceptable with probability $\frac{3}{8}$, which can be made acceptable with probability $\frac{1}{4}$. So the probability of acceptability after trades for the $N = S = 3$ case is bounded below by $\frac{1}{8} + \frac{1}{8} * \frac{3}{16} + \frac{3}{8} * \frac{1}{4} = \frac{31}{128}$.

For the general $N = S$ case, the probability that the allocation is outright acceptable is $\left(\frac{1}{2}\right)^S$. The probability that the allocation is unacceptable for each individual is $\left(\frac{1}{2}\right)^S$, and there are $S!$ possible permutations of ways to solve it.The probability that the allocation can be made acceptable from these trades can be written generally as $\prod_{i=1}^{S}\left(\frac{1}{2}\right)^i$, merely a generalization of the result from Bayes' Rule. The probability that there is exactly one unacceptable outcome is $S\left(\frac{1}{2}\right)^S$, and there is a $\frac{1}{8}$ probability that she can conduct a trade with any of the other $S-1$ employees. And so the probability of acceptability after trades is bounded below by

$$\left(\frac{1}{2}\right)^{S} + \left(\frac{1}{2}\right)^{S} * S! * \prod_{i=1}^{S} \left(\frac{1}{2}\right)^{i} + \left(\frac{1}{2}\right)^{S} * \frac{S}{S-1} * \frac{1}{8}.$$

Note that each term is strictly positive. The lower bound affirms that trades strictly improve upon the probability of acceptability; meanwhile the simulation results in Figure 3 demonstrate the effects of trades when S is large.

Recall that the inclusion of more employees such that $N > S$ strictly decreased the probability that the allocation rendered a stable outcome. By similar logic it can be inferred that the addition of employees increases the number of possible trades which can be made, thus increasing the chance of acceptability for the aggregate allocation. For example, take $N = 3$ and $S = 2$, and that (e_1, s_1), (e_2, s_2) and $(e_3, \emptyset)$ exist. With an extra employee, a cycle may form which enables e_3 to obtain a shift for which neither e_1 or e_2 had an acceptable outcome, or it may hold that $c_3^2 < w$ or $c_3^1 < w$ when either $c_2^2 > w$ or $c_1^1 > w$. Thus the probability of arriving at acceptability through trades strictly increases in N, so the lower bound still holds.

*Proof of **Proposition 3.3.2***:

Take that (e_i, s_j) and (e_k, s_ℓ) exist. Trade conditions are relaxed such that e_i and e_j will trade shifts if $c_k^\ell > w$, $w > c_i^\ell$ and $w > c_k^j$. This means that employees are willing to swap shifts if it makes the other employee's matching acceptable while not making her own unacceptable.

Take (e_i, s_j) as an acceptable allocation, or that $c_i^j < w$. e_i is willing to trade for s_ℓ if $c_i^\ell < w$. And, $Pr\left(c_i^\ell < w\right) < \frac{1}{2}$. Take (e_k, s_ℓ) as an unacceptable allocation, or that $c_k^\ell > w$. Still, e_k is willing to trade for s_j if $c_k^j < w$, and it nonetheless holds that $Pr\left(c_k^j < w\right) < \frac{1}{2}$.

Thus we can say that e_i "points" to s_ℓ with probability $\frac{1}{2}$, and that e_i "points" to s_ℓ with probability $\frac{1}{2}$. Let $D \sim D\left(S, p\right)$ be a directed Erdős–Rényi random graph with S vertices and the probability p that there is a directed edge between any two ordered pairs of vertices. Each vertex is an individual allocation (e_i, s_j), which is acceptable at probability $\frac{1}{2}$. It has been determined already that each e_i points to s_ℓ with probability $\frac{1}{2}$; this is the probability that there is a directed edge between e_i and s_ℓ.

Let X be the total number of cycles. The expected total number of cycles is equal to the sum of the number of cycles that traverse over k vertices, and each cycle traverses at least two vertices. Letting X_k be the number of cycles of length k in D, the expected total number of cycles is then

$$E[X] = \sum_{k=2}^{S} E[X_k].$$

By symmetry, one can reduce the question of the expected number of cycles in $D(S, p)$ to the expected number of cycles in $D(k, p)$. Let Y_k be the number of cycles of length k in a random graph of size k. The expected number of cycles of length k is then

$$E[X_k] = \binom{S}{k} * E[Y_k].$$

Much like the essence of former proofs, a cycle of length k is a permutation of numbers from 1 to k, but duplicity must be discounted. The probability of each edge being present is $\frac{1}{2}$, so the probability of k edges being present is $\frac{1}{2}^k$.

$$E[Y_k] = (k-1)! \, \frac{1}{2}^k.$$

Substituting $E[Y_k]$ into the first equation, the result is

$$E[X] = \sum_{k=2}^{S} \binom{S}{k} E[Y_k].$$

Substitutions continue:

$$\begin{aligned}
E[X] &= \sum_{k=2}^{S} \binom{S}{k} (k-1)! \, \frac{1}{2}^k \\
&= \sum_{k=2}^{S} \frac{S!}{k! \, (S-k)\,!} \, (k-1)\,! \, \frac{1}{2}^k \\
&= \sum_{k=2}^{S} \frac{S!}{k \, (S-k)\,!} \frac{1}{2}^k \\
&= \sum_{k=2}^{S} \frac{S * (S-1) * ... * (S-k+1)}{k} \frac{1}{2}^k .
\end{aligned}$$

A conservative lower bound can be calculated:

$$\begin{aligned}
E[X] &\geq \sum_{k=S/2}^{S} \frac{S * (S-1) * ... * (S-k+1)}{k} \frac{1}{2}^k \\
&\geq \sum_{k=S/2}^{S} \frac{\left(\frac{S}{2}\right)^k}{S} \frac{1}{2}^k \\
&\geq \frac{1}{S} \sum_{k=S/2}^{S} \left(\frac{S}{4}\right)^k \\
&\geq \frac{\left(\frac{S}{4}\right)^{S+1} - \left(\frac{S}{4}\right)^{\frac{S}{2}}}{\frac{S}{4} - 1} .
\end{aligned} \tag{3}$$

By the same logic which constitutes the lower bound as applicable in equation (2) for $N > S$, the number of cycles strictly increases in N. Thus this lower bound applies for all cases where $N > S$, as well.

C Dutch Auction

Proof of ***Proposition 3.4.1***:

Assume s_1 is the only shift available. Consider the following cases: e_1 bids at $t = c_1^1$ for s_1, e_1 bids at $t > c_1^1$, or e_1 waits too long to bid and s_1 is taken by another employee.

The utility that e_1 receives from bidding at $t = c_1^1$ is equal to

$$u(t) = t - c_1^1 = 0.$$

The utility that e_1 receives from bidding at $t' > c_1^1$ when s_1 is still available is equal to

$$u(t') = t' - c_1^1.$$

There, of course, is the chance that e_1 waits too long to bid on s_1 such that someone bids sooner, and the result is $(e_i, \emptyset)$. The utility in this case is equal to

$$u((e_i, \emptyset)) = 0.$$

Thus the consideration is whether e_1 should bid for s_1 at $t = c_1^1$, or wait to bid and risk not acquiring the shift. The utility of waiting until some $t' > t$ is defined as

$$\begin{aligned} u(\text{wait until } t' \text{ to bid on } s_1) = u(t') * Pr(\text{ no one else bid on } s_1 \text{ by } t') + \\ u((e_i, \emptyset)) * Pr(\text{someone one else bid on } s_1 \text{ by } t') \end{aligned}$$

$$= \left(t' - c_1^1\right) * Pr(\text{no one else bid on } s_1 \text{ by } t').$$

This value is positive for all $t' > t$, thus waiting to bid until t' weakly dominates bidding at t. The employee has incentive to wait; the concern is then to investigate how long to wait.

Proof of ***Proposition 3.4.2***: From Reiß and Schöndube (2008), the equilibrium bid in a Dutch auction with two bidders and with independently and identically distributed costs according to $H(c)$, c $\in [\underline{c}, \bar{c}]$ is

$$b(c) = c + \frac{\int_c^{\bar{c}} [1 - H(x)]}{1 - H(c)},$$

where $1 - H(c) = 1 - \Pr(\tilde{c} \leq c) = (1 - c/2)^{N-1}$. Converting to the $N > 2$ case and with $\bar{c} = 2$ gives

$$\begin{aligned} b(c) &= \frac{\int_c^2 (1 - H(x))^{n-1} dx}{(1 - H(c))^{n-1}} \\ &= \frac{\int_c^2 \left(1 - \frac{x}{2}\right)^{n-1} dx}{\left(1 - \frac{c}{2}\right)^{n-1}}. \end{aligned}$$

Define $u = 1 - \frac{x}{2}$ such that $dx = -2du$. So

$$\int \left(1 - \frac{x}{2}\right)^{n-1} dx = -2 \int u^{n-1} du = -2 \frac{u^n}{n}.$$

Rewriting in terms of x gives

$$\begin{aligned} \int_c^2 \left(1 - \frac{x}{2}\right)^{n-1} dx &= -\frac{2\left(1 - \frac{x}{2}\right)^n}{n} \Big|_{x=c}^2 \\ &= \frac{(2 - c)^n 2^{1-n}}{n}. \end{aligned}$$

So

$$\begin{aligned} b(c) &= \frac{(2 - c)^n 2^{1-n}}{n \left(1 - \frac{c}{2}\right)^{n-1}} \\ &= \frac{2^{n-1} (2 - c) 2^{1-n}}{n} \\ &= \frac{2 - c}{n}. \end{aligned}$$

Because the auction is procurement (a higher cost means a higher bid), simply add c to the right-hand side.

$$b(c) = \frac{2-c}{n} + c.$$

Grouping the c terms gives

$$\begin{aligned} b(c) &= \frac{2-c+cn}{n} \\ &= \frac{2+c(n-1)}{n} \\ &= \frac{2}{n} + c\frac{n-1}{n} \\ &= 2 - \frac{n–1}{n}(2–c). \end{aligned} \tag{4}$$

With t $(0,2)$, the time at which the employee bids in equilibrium is equal to $b(c)$.

Proof of ***Proposition 3.4.3***:

As an example, let $N = 3$ and $S = 2$, and e_1 exist such that $c_1^1 < c_1^2$. Employees have an identical distribution of costs across all shifts, and because costs are independent and identically distributed, each shift which is not obtained in the first stage has the same distribution of costs as every other shift. There is no special information to gain from bidding on one shift over another.

The probability that e_1 is assigned a shift is independent of her cost for that shift, but merely a function of her decision to bid for that shift. This is the previously defined $a(t)$. Thus it is true that

$$Pr(\text{win } s_1 \mid \text{bid } s_1) = Pr(\text{win } s_2 \mid \text{bid } s_2).$$

While probabilities of shift acquisition are independent of one's cost, it is indeed true that utility is dependent on cost. The utility of bidding on s_1 is thus

$$
\begin{gathered}
u\,(\text{bid } s_1) = \\
Pr\,(\text{win}s_1 \mid \text{ bid } s_1) * \left(w_t - c_1^1\right) + \\
Pr\,(\text{do not win } s_1, s_2 \text{ unassigned } \mid \text{bid } s_1) * \left(\text{ profit } \mid s_2 \text{ unassigned, } c_1^2\right) + \\
Pr\,(\text{do not win } s_1, s_2 \text{ assigned to someone else} \mid \text{bid } s_1) * \\
(\text{ profit } \mid s_1 \text{ and } s_2 \text{ assigned to someone else})\,.
\end{gathered}
$$

Similarly, the utility of bidding on s_2 is

$$
\begin{gathered}
u\,(\text{bid } s_2) = \\
Pr\,(\text{win } s_2 \mid \text{bid } s_2) * \left(w_t - c_1^2\right) + \\
Pr\,(\text{do not win } s_2, s_1 \text{ unassigned } \mid \text{bid } s_2) * \left(\text{profit } \mid s_1 \text{ unassigned, } c_1^1\right) + \\
Pr\,(\text{do not win } s_2, s_1 \text{ assigned to someone else } \mid \text{bid } s_2) * \\
(\text{profit } \mid s_1 \text{ and } s_2 \text{ assigned to someone else})\,.
\end{gathered}
$$

It has already been asserted that $u\,((e_i, \emptyset)) = 0$. Thus the profit that e_1 receives when both s_1 and s_2 are unassigned to her is 0, or

$$(\text{profit } \mid s_1 \text{ and } s_2 \text{ assigned to someone else}) = 0.$$

So with the according transformations,

$$
\begin{gathered}
u\,(\text{bid } s_1) = \\
Pr\,(\text{win } s_1 \mid \text{ bid } s_1) * \left(w_t - c_1^1\right) + \\
Pr\,(\text{do not win } s_1, s_2 \text{ unassigned } \mid \text{ bid } s_1) * \left(\text{profit } \mid s_2 \text{ unassigned, } c_1^2\right)
\end{gathered}
$$

and

$$u\,(\text{bid } s_2) =$$

$$Pr\,(\text{win } s_2 \mid \text{bid } s_2) * \left(w_t - c_1^2\right) +$$

$$Pr\,(\text{do not win } s_2, s_1 \text{unassigned } \mid \text{bid } s_2) * \left(\text{profit } \mid s_1 \text{unassigned, } c_1^1\right).$$

Recall that costs are independently and identically distributed across shifts and employees, thus

$$Pr\,(\text{win } s_1 \mid \text{ bid } s_1) = Pr\,(\text{win } s_2 \mid \text{ bid } s_2)\,,$$

and

$$Pr\,(\text{do not win } s_1, s_2 \text{ unassigned } \mid \text{ bid } s_1) =$$

$$Pr\,(\text{do not win } s_2, s_1 \text{ unassigned } \mid \text{ bid } s_2)\,.$$

It is also clear that $\left(\text{profit } \mid s_2 \text{ unassigned, } c_1^2\right) = w_{t_2'} - c_1^2$, and $\left(\text{profit } \mid s_1 \text{ unassigned, } c_1^1\right) = w_{t_1'} - c_1^1$, where $t < t_1', t_2', \leq 2$. Hence $u\,(\text{bid } s_1) > u\,(\text{bid } s_2)$ if and only if

$$Pr\,(\text{win } s_1 \mid \text{ bid } s_1) * \left(w_{t_1} - c_1^1\right) +$$

$$Pr\,(\text{do not win } s_1, s_2 \text{ unassigned } \mid \text{ bid } s_1) * \left(w_{t_2'} - c_1^2\right) >$$

$$Pr\,(\text{win } s_1 \mid \text{ bid } s_1) * \left(w_{t_2} - c_1^2\right) +$$

$$Pr\,(\text{do not win } s_1, s_2 \text{ unassigned } \mid \text{ bid } s_1) * \left(w_{t_1'} - c_1^1\right),$$

or

$$Pr\left(\text{win } s_1 \mid \text{ bid } s_1\right) * \left(\left(w_t - c_1^1\right) - \left(w_t - c_1^2\right)\right) >$$
$$Pr\left(\text{do not win } s_1, s_2 \text{ unassigned } \mid \text{ bid } s_1\right) * \left(\left(w_{t'_2} - c_1^2\right) - \left(w_{t'_1} - c_1^1\right)\right).$$

This simplifies to

$$Pr\left(\text{win } s_1 \mid \text{ bid } s_1\right) * \left(c_1^2 - c_1^1\right) >$$
$$Pr\left(\text{do not win } s_1, s_2 \text{ unassigned } \mid \text{ bid } s_1\right) * \left(c_1^1 - c_2^1 + w_{t'_2} - w_{t'_1}\right).$$

Assume each employee pursues symmetric strategy and bids exclusively on her least-cost shift. Then at t and with her least-cost shift s_j available, e_i wins s_j if and only if $c_i^j < c_k^j$ for $1 \leq k \neq i \leq N$. Thus $Pr\left(\text{win } s_1 \mid \text{ bid } s_1\right) = \frac{1}{N}$. The probability that e_1 does not win s_1 is $1 - \frac{1}{N}$, or $\frac{N-1}{N}$. s_2 is still available after e_1 misses out on s_1 when $c_1^2 < c_m^2$ for $1 \leq m \leq N-1$. This occurs with *ex ante* probability of $\frac{1}{N-1}$. $Pr\left(\text{do not win } s_1, s_2 \text{ unassigned } \mid \text{ bid } s_1\right) = \frac{N-1}{N} * \frac{1}{N-1} = \frac{1}{N}$. Thus the equation simplifies to

$$\frac{1}{N} * \left(c_1^2 - c_1^1\right) > \frac{1}{N} * \left(c_1^1 - c_2^1 + w_{t'_2} - w_{t'_1}\right),$$

or

$$c_1^2 - c_1^1 > c_1^1 - c_2^1 + w_{t'_2} - w_{t'_1}.$$

Bidding on s_1 is thus strictly better than s_2 if

$$\frac{c_1^2 - c_1^1}{2} > \left(w_{t'_2} - w_{t'_1}\right).$$

It has already been determined that the equilibrium bidding strategy is to bid on a remaining shift at time

$$b\left(t'\right) = \frac{N-1}{Nc^{N-1}} - c\frac{N-1}{N}.$$

And so the wage which e_1 expects for s_1 evaluates to $\frac{1}{2c_1^1} - \frac{c_1^1}{2}$,and for s_2 she expects $\frac{1}{2c_1^2} - \frac{c_1^2}{2}$. So

$$\frac{c_1^2 - c_1^1}{2} > \left(w_{t_2'} - w_{t_1'}\right)$$

$$\frac{c_1^2 - c_1^1}{2} > \frac{1}{2c_1^2} - \frac{c_1^2}{2} - \frac{1}{2c_1^1} + \frac{c_1^1}{2}$$

$$c_1^2 - c_1^1 > \frac{1}{c_1^2} - c_1^2 - \frac{1}{c_1^1} + c_1^1$$

Simplifying, this equals

$$2c_1^2 - 2c_1^1 > \frac{1}{c_1^2} - \frac{1}{c_1^1},$$

which holds for all $c_1^2 > c_1^1$.

Proof of ***Proposition 3.4.3***:

Because there are auctions of multiple shifts, an employee's bids depend not only on her cost for that shift, but on her cost for all other shifts, and her expected profit from winning any one of the future auctions.

Thus an employee's strategy for s_j where $(j = 1, ..., S-1)$ auction depends on the profit she expects from the $S-j$ shifts that are not yet allocated. Because there are only S shifts, the auction concludes after shift s_S is allocated. And so an employee's strategy during the last auction is the same as the single-object Dutch, defined in equation (5).

The bidding behavior for the first $S-1$ auctions differs from that for the single object case, and depends on her probability of winning any one of the remaining auctions that are yet to be conducted. This probability is likewise dependent on the number of employees who have not yet been assigned shifts. Define n as the number of employees who have not been assigned shifts. Let $\zeta(y, j, S, n)$ be the employee's *ex ante* probability of winning s_y, where $j \leq y \leq S$, in the series of auctions from s_j to s_S before s_j is claimed. For $\zeta(1, 1, S, n)$, it is known that no shifts have been claimed, thus $n = N$. It has already been shown that the employee with the lowest cost for a shift wins the respective shift. The probability that e_i has the lowest cost for s_j is $\frac{1}{N}$, thus $\zeta(1, 1, S, n) = \frac{1}{N}$.

Further, $\zeta(2, 1, S, n)$ depicts the probability that an employee wins s_2 in the series of auctions from s_1 to s_S. The probability that she does not win s_1 is $1-\frac{1}{N}$; the probability that she does win s_2 thereafter is $\frac{1}{N-1}$. Thus $\zeta(2, 1, S, n) = \frac{1}{N}$. [39] More generally, for $1 \leq y \leq S$,

$$\zeta(y, 1, S, n) = \prod_{k=1}^{y-1} \left(1 - \frac{1}{n-y+k+1}\right) * \frac{1}{n-y+1},$$

and for $j \leq y \leq S$,

$$\zeta(y, j, S, n) = \prod_{k=j}^{y-1} \left(1 - \frac{1}{n-y+k+1}\right) * \frac{1}{n-y+1}.$$

This probability is relevant to the calculation of one's expected profit for the auction of s_y. Define $\theta_i^j(i, j, n, S)$ as the profit that e_i expects from obtaining s_j when there are n employees who have not yet acquired shifts and S shifts remaining. Similarly, let $\phi_i^j(i, j, n, S)$ denote the *ex ante* expected profit for e_i for winning *any* auction from the series of auctions from the s_j (for $1 \leq j \leq S$) to s_S. Thus

$$\phi_i^j(i, j, n, S) = \sum_{y=j}^{S} \zeta(y, 1, S, n) * \theta_i^j(i, y, n, S)$$

[39] Note that this probability is the same as that used to derive **Lemma 1**.

Consider the sequential auction of $s_1, ..., s_S$. At the auction of s_S, only s_S remains. $N - S + 1$ employees remain, and so the equilibrium bidding strategy is

$$b(t') = \frac{(N-S)}{(N-S+1)\,c^{N-S}} - c\frac{(N-S)}{N-S+1}.$$

For s_{S-1}, an employee e_i bids at time t if

$$b(t) \leq w_t - c_i^{S-1} - \phi(i, S, S, n)$$

Bids decrease by a factor of $\phi(i, j+1, S, n)$. Similarly, for s_{S-2}, the equilibrium bid is decreased by $\phi(i, j+1, S, n)$. Thus bids strictly decrease as the number of items remaining decreases.