SCHOOL CHOICE WITH CONSENT*

ONUR KESTEN

Abstract

An increasingly popular practice for student assignment to public schools in the U.S. is the use of school choice systems. The celebrated Gale-Shapley student-optimal stable mechanism (SOSM) has recently replaced two deficient student assignment mechanisms that were in use in New York City and Boston. We provide theoretical evidence that the SOSM outcome may produce large welfare losses. Then we propose an efficiency adjusted deferred acceptance mechanism (EADAM) that allows a student to consent to waive a certain priority that has no effect on his assignment. Under EADAM a consenting student causes himself no harm, but may help many others benefit as a consequence. We show that EADAM can recover any welfare losses due to SOSM while also preserving immunity against strategic behavior in a particular way. It is also possible to use EADAM to eliminate welfare losses due to randomly breaking ties in student priorities.

I. INTRODUCTION

In the last two decades many U.S. states have adopted intra- and inter-district public *school choice systems*[1] that allow parents to enroll their children in a public school other than the school they

[1]The term *school choice* is more broadly used to describe a wide array of programs that give parents the opportunity to choose the school their children will attend. Other forms of school choice (than the one we focus in

would have been assigned to based on their residence area.[2] In a public school choice system, each student submits a list of *preferences* of schools to a central placement authority, such as the school district, which then determines which student will be placed to which school. Since school capacities are limited, making it impossible to place each student to his first choice school, in addition to the preferences of students, student *priorities* for schools also need to be taken into consideration. There may be several criteria in determining a priority order for a school. For example, in Boston, the first priority for a school is given to the students who are in the same walk zone and who have a sibling attending that school; the second priority is given to those who only have a sibling attending that school; the third priority is given to those who are only in the same walk zone; and the fourth priority is given to the remaining students. For those students who are in the same priority group, the order is determined by a random lottery. Today, many cities such as New York City, Boston, Seattle, Cambridge, Charlotte, Denver, Minneapolis, and Columbus are using centralized (public) school choice systems. Given the growing popularity of these systems, the significance of the student assignment method that a school district will employ is apparent. This paper addresses this issue from a mechanism design perspective.

A *school choice problem* is a pair consisting of a *preference profile* of students and a collection of *priority orders* for schools. At a *matching* each student is placed to only one school, and the number of students placed to a particular school does not exceed the capacity of that school. A *school choice mechanism* or simply, a mechanism, is a systematic way of selecting a matching for a given school choice problem.

A closely related problem is the well-known *college admissions problem* due to Gale and Shapley (1962).[3] The main difference, however, is that in a college admissions problem, student 'priorities' for schools are replaced by school 'preferences' over students, which call for different strategic and welfare considerations. In the present context, student priorities are enforced by local/state laws,

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this paper) include vouchers, tuition tax credits, charter schools, and home schooling.

[2]The popularity of these systems is also on the rise. According to a survey by the NCES, from 1993 to 2007, the percentage of students attending a "chosen" public school (a public school other than their assigned public school) increased from 11% to 16%, while the percentage of children attending an assigned public school decreased from 80% to 73%. (http://nces.ed.gov/programs/coe/2009/pdf/32_2009.pdf)

[3]See Roth and Sotomayor (1990) for an excellent survey on two-sided matching.

hence no school has a say on the way its priority order is determined. Therefore schools are viewed as 'objects' to be consumed.

The pioneering work of Abdulkadiroğlu and Sönmez (2003) on school choice problems examines some of the real-life student placement mechanisms, and offers two alternative competing mechanisms as attractive replacements.[4] One of these mechanisms, the student-optimal stable mechanism (SOSM) has taken an early lead over its competitors by recently replacing two controversial mechanisms that were in use in New York City (NYC) which has the largest public school system in the country with over a million students (Abdulkadiroğlu, Pathak, and Roth 2005), and in Boston which has over 60,000 students enrolled in the public school system (Abdulkadiroğlu et al. 2005, 2006). A major reason for these replacement decisions is the fact that the SOSM outcome never gives rise to situations of "priority violation." More precisely, given a school choice problem *the priority of student i is violated (or, disrespected)* at a matching if there is a student j who is assigned to a school s at this matching such that (a) student i prefers school s to his current assignment, and (b) student i has higher priority than j for school s. From a fairness standpoint, situations of priority violation can be argued to produce an obvious conflict with the very role of priorities, and thus may induce parents to seek legal action when faced with such situations.

In addition to the success of SOSM at respecting student priorities, matching theory gives two strong reasons to further support the transition decisions of the NYC and Boston school districts. First, SOSM is the most favorable mechanism to students among those that eliminate situations of priority violation (Gale and Shapley 1962), and second, it is strategy-proof (Roth 1982), i.e., it is a dominant strategy for each student to state preferences truthfully. This second feature has made SOSM quite attractive in particular for the Boston school district where student assignment used to be viewed as a "high-stakes gamble"[5] (e.g., Ergin and Sönmez [2006]; Pathak and Sönmez [2008]) prior to the adoption of SOSM. The SOSM outcome is determined through the following

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[4]A partial list of other related papers is: Erdil and Ergin (2008), Ehlers (2006), Chen and Sönmez (2006), Ergin and Sönmez (2006), Ergin (2002), Kesten (2006, 2007), Haeringer and Klijn (2007), Pathak (2008), Kojima and Pathak (2009), Pathak and Sönmez (2008a,b), and Fetherstone and Niederle (2008).

[5]See "Understanding the Options for a new BPS Assignment Method" by the Boston Public Schools Strategic Planning Manager Valerie Edwards, presented at the October 13, 2004 dated Boston Public Schools school committee meeting.

student-proposing **deferred acceptance (DA)** algorithm:

At the first step, each student applies to his first choice school. For each school s with a capacity q_s, those q_s applicants who have the highest priority for s are tentatively placed to s, and the rest are rejected.

In general,

At the k-th step, each student who was rejected at step $k - 1$ applies to his next choice school. For each school s, the highest priority q_s applicants among the new applicants and those who were tentatively placed at a previous step are tentatively placed to s, and the rest are rejected.

The algorithm terminates when no student is rejected any more. The following simple example shows that SOSM is not problem free however.

EXAMPLE 1. Consider a school choice problem with four students $(i_1, \ldots, i_4)$ and four schools $(s_1, \ldots, s_4)$ each with one seat. The priority orders (e.g., $\succ_s$ denotes the priority order for school s) and student preferences (e.g., P_i denotes the preferences of student i) are as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$		P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
i_4	i_2	i_3	$\vdots$		s_1	s_1	s_2	s_3
i_1	i_3	i_4	$\vdots$		$\underline{s_4}$	$\underline{s_2}$	$\underline{s_3}$	$\underline{s_1}$
i_2	$\vdots$	$\vdots$	$\vdots$		$\vdots$	$\vdots$	$\vdots$	$\vdots$

The following table illustrates the steps of the DA algorithm applied to this problem. [The columns of the table represent the schools, and the rows represent the steps of the algorithm. Any student tentatively placed to a school at a particular step is shown in a box at the corresponding entry of the table.]

	School s_1	School s_2	School s_3	School s_4
Step 1	$\underline{i_1}$, i_2	$\underline{i_3}$	$\underline{i_4}$	
Step 2	⋮	$\underline{i_2}$, i_3	⋮	
Step 3		⋮	$\underline{i_3}$, i_4	
Step 4	$\underline{i_4}$, i_1		⋮	
Step 5	$\underline{i_4}$	$\underline{i_2}$	$\underline{i_3}$	$\underline{i_1}$

In words, at Step 1 students i_1 and i_2 apply to school s_1, and as a result the lower priority student i_2 is rejected between the two. At the next step student i_2 applies to school s_2 and displaces student i_3 (who was tentatively placed to s_2 at the previous step). Then student i_3 applies to school s_3 and displaces student i_4 (who was tentatively placed to s_3 at Step 1). Finally, student i_4 applies to school s_1 and displaces student i_1 (who was tentatively placed to s_1 at Step 1). The algorithm terminates at the next step when student i_1 applies to school s_4.

The matching that SOSM recommends is underlined above on the given preference profile (i.e., i_1 is assigned to s_4, i_2 to s_2, i_3 to s_3, and i_4 to s_1.). Note however that this matching is not Pareto efficient. For example, assigning students i_2, i_3 and i_4 to their first choices without changing the assignment of student i_1 would clearly make three students better off. ◇

In this paper we show that the extent of the welfare loss due to SOSM can be troublingly large. In fact, for any given set of schools one can find situations in which every student is assigned to either his last choice or his second last choice under SOSM (Proposition 2). A recent empirical exercise by Abdulkadiroğlu, Pathak, and Roth (2009) also shows that inefficiency of SOSM is not merely a theoretical concern. Based on student preference data from the NYC school district, the authors observe that potential welfare gains over the SOSM matching for the NYC school district indeed exist and are significant.[6]

One possible solution to the inefficiency problem is considering a move to a *Pareto efficient* mechanism such as the second proposal of Abdulkadiroğlu and Sönmez (2003). This class of

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[6]They report that in 2006-07 for example, an alternative matching would assign over 4,000 grade 8 students to a better school than the one they were already assigned by SOSM without hurting any of the remaining students.

mechanisms is based on Gale's *top trading cycles* idea (Shapley and Scarf 1974) that allows students to 'trade priorities' for different schools among themselves.[7] From a theoretical perspective, due to a well-known result in two-sided matching (e.g., Gale and Shapley [1962]; Roth [1982]; Balinski and Sönmez [1999]) *Pareto efficient mechanisms cannot guarantee complete elimination of priority violations*, and thus do not rule out the possibility of potential legal action by upset parents. From a practical perspective, trading-based mechanisms might also raise ethical concerns. In his May 25, 2005 dated memorandum to the School Committee, regarding his take on a trading-based mechanism, the (then) Superintendent of Boston Thomas Payzant writes:

"There may be advantages to this approach... It may be argued, however, that certain priorities - e.g., sibling priority - apply only to students for particular schools and should not be traded away."

The Boston Public Schools (BPS) Strategic Planning Team's May 11, 2005 dated recommendation report further states that:[8]

"The trading mechanism can have the effect of 'diluting' priorities' impacts, if priorities are to be 'owned' by the district as opposed to being 'owned' by parents; shifts the emphasis onto the priorities and away from the goals BPS is trying to achieve by granting these priorities in the first place; and could lead to families believing they can strategize by listing a school they don't want in hopes of a trade."

The nature of the school choice problem causes education officials as well as mechanism designers to face a difficult dilemma due to the incompatibility of efficiency and the aversion to priority violations. In this paper we propose a practical intermediate solution to this dilemma. The intuition for this solution can be seen through a close examination of Example 1. Once student i_1 is tentatively placed to school s_1 at step 1, he initiates a rejection chain that eventually causes

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[7]These mechanisms are also strategy-proof, and therefore induce straightforward action by students. Gale's top trading cycles idea has also proved quite useful in other resource allocation applications such as *on-campus housing* (e.g., Abdulkadiroğlu and Sönmez [1999]) and *kidney exchange* (e.g., Roth, Sönmez, and Ünver [2004]).

[8]The following three remarks do not appear in the same order in the original report as summarized below. For the complete report see "Recommendation to implement a new BPS assignment algorithm" at http://www.archive.org/details/recommendationto00bost.

himself to be rejected from school s_1. In other words, the application of student i_1 to school s_1 does not bring any benefit to himself, but only hurts three other students, namely students i_2, i_3 and i_4. In fact, if student i_1 were to sacrifice his priority for school s_1 and never apply to this school in the first place, then the SOSM outcome would be both free of any priority violations and Pareto superior to the initial matching. We call any student like student i_1 of Example 1, who interrupts a desirable settlement among other students at no gain to himself as an *interrupter*. The mechanism we propose is based on the idea of identifying interrupters in any given school choice problem and neutralizing their adverse effect on the outcome. Such a task becomes rather cumbersome and challenging as the problem size gets large since the DA algorithm may then contain multiple chains nested in one another. Interestingly for example, a student who serves as an interrupter at a particular instance of the DA algorithm, may himself actually suffer a welfare loss due to the presence of another interrupter at a different instance.

A crucial point for the above idea to work is that an interrupter student (e.g., student i_1 of our example) should actually *consent* to waive his priority for a critical school (e.g., school s_1 of our example). On the other hand, since a student can be identified as an interrupter only after every student submits his preferences, there is no way of telling who is an interrupter and who is not prior to the central procedure. Therefore any consenting decisions for priority waiving should be handled a priori. In practice this would mean that school districts should ask students to sign a consent form according to which the student gives permission for his priority for a school to be waived in case he turns out to be an interrupter for that school. To make an interesting analogy, the consenting decision of an individual can be viewed similarly to the decision of donating one's organs after death for transplantation or research. While such a decision cannot possibly hurt the donating individual, it may help many others as a consequence. Nonetheless, this is a right every individual is given by law and cannot be forced to waive.

This paper proposes an *efficiency adjusted deferred acceptance mechanism (EADAM)* that closely mimics SOSM and makes adjustments to recover artificial welfare losses caused only by those interrupters who give consent for priority waiving. We show that EADAM Pareto dominates

SOSM, i.e., no student is ever worse off under EADAM when compared with his assignment under SOSM. The size of the Pareto improvement over SOSM increases as the rate of consenting students increases. When all students consent, the EADAM outcome is Pareto efficient (Theorem 1). Under EADAM no student's priority is ever violated. However, a consenting student's priority for a particular school may be violated with his permission (Theorem 1). Nevertheless, a consenting student is never hurt by such a sacrifice: No consenting student ever gains by choosing not to consent (Proposition 3).

No Pareto efficient mechanism that can Pareto improve upon SOSM is fully immune to strategic action (Proposition 4). Theoretical and empirical studies have shown that failure to satisfy dominant strategy incentive compatibility does not necessarily entail easy manipulability of a mechanism in practice.[9] We next re-model the school choice problem in a 'limited information' setting that allows for correlation among students' preferences, and show for any student that a truth-telling strategy stochastically dominates any other strategy when other students are also truthful. Hence, truth-telling is an ordinal Bayesian Nash equilibrium of the preference revelation game under EADAM (Theorem 2). This kind of a strategic immunity is not a feature of the widely-used student assignment method known as the Boston mechanism.

Incorporating individual students' consenting decisions into SOSM as a way to achieve welfare improvements without introducing priority violations is the key innovation of EADAM. In general, there is no easy way of utilizing individual consents since an individual's consent for priority waiving cannot guarantee that other non-consenting students' priorities for the same school will be respected. By simply tracing the steps of SOSM backwards EADAM explores possible improvement paths in which a consenting student makes it possible for a critical school to be assigned to the next most deserving student in terms of priority. In that regard, two aspects of EADAM distinguish it from exhaustive matching search algorithms in which priorities and consents are respected: its computationally practical polynomial-time algorithm and the unharmed student incentives for giving consent.

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[9]See, for example, Roth and Peranson (1999) and Roth and Rothblum (1999).

In a related paper, Erdil and Ergin (2008) underline the efficiency cost suffered by SOSM when ties in student priorities are broken via some random draw. They propose an intuitive and practical mechanism that restores such artificial welfare losses. We also show that EADAM can alternatively be used as a way to recover welfare losses originating only from random tie-breaking (Proposition 5). Consequently, EADAM can be conveniently customized to eliminate welfare losses that stem from ties in priorities and/or those from the intrinsic dynamics of SOSM itself.

The rest of the paper is organized as follows: Section II introduces the school choice problem. Section III presents a trade-off result among the properties of strategy-proofness, Pareto efficiency, and fairness. Section IV studies the size of the welfare loss under SOSM. Section V introduces the new mechanism, presents its main properties, and shows how it can be modified to recover welfare losses due to tie-breaking. Section VI concludes. All the proofs are relegated to Appendix.

II. SCHOOL CHOICE PROBLEM

Let $I \equiv \{i_1, i_2, \ldots, i_n\}$ denote the finite set of students. A generic element in I is denoted by i. Let $S \equiv \{s_1, s_2, \ldots, s_m\}$ with $|S| \geq 2$ denote the finite set of schools. A generic element in S is denoted by s. Each school has a finite number of available seats. Let q_s be the number of available seats at school s, or the **capacity** of s. We assume throughout that the total number of seats is no less than the number of students (i.e., $n \leq \Sigma_{s \in S} q_s$). For each school there is a strict priority order (complete, transitive and antisymmetric relations) of all students, and each student has strict preferences (complete, transitive and antisymmetric relations) over all schools. Let us denote the preferences of student i by P_i. Let R_i denote the at-least-as-good-as relation associated with P_i. The priority orders are determined according to state/local laws and certain criteria of school districts. Let us denote the priority order for school s by $\succ_s$.

A **school choice problem** or, simply a problem, is a pair $((\succ_s)_{s \in S}, (P_i)_{i \in I})$ consisting of a collection of priority orders and a preference profile.[10] For a given problem, at a **matching** each

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[10] The school choice problem is also closely related to *house allocation problems* and *housing markets* in which there is a set of objects that are collectively or privately owned. In these models however, the capacity of each house is one. See for example, Shapley and Scarf (1974), Pápai (2000), Abdulkadiroğlu and Sönmez (1998, 1999),

student is placed to only one school and the number of students placed to a particular school does not exceed the number of available seats at that school. Formally, a matching $\mu : I \longrightarrow S$ is a function such that for each $i \in I$, $\mu(i) \in S$ and for each $s \in S$, $|\mu^{-1}(s)| \leq q_s$. A matching is **Pareto efficient** if there is no other matching at which all students are at least as well off, and at least one student better off.

A very closely related problem is the well-known *college admissions problem* due to Gale and Shapley (1962). The crucial difference between the two problems is that in a college admissions problem, schools are active and have preferences over students whereas here, schools are passive and viewed merely as objects (each of which has multiple copies) to be consumed. The central concept in college admissions is "stability." A matching is **stable** if there is no student-school pair (i, s) such that student i prefers school s to the school he is placed to, and school s prefers student i to at least one student who is placed to it. The natural counterpart of stability in our context is "fairness" (Balinski and Sönmez 1999). Given a matching μ, **the priority of student i for school s is violated** (or, **disrespected**) if i would rather be placed to s (i.e., $s \, P_i \, \mu(i)$), and yet there is some student j placed to s who has lower priority for s than student i (i.e., $\mu(j) = s$ and $i \succ_s j$). A matching is **fair** if no student's priority for any school is violated.

A school choice mechanism or, simply a **mechanism** φ, is a systematic way of selecting a matching for each problem. A mechanism is Pareto efficient if it always selects Pareto efficient matchings. A mechanism is fair if it always selects fair matchings. A mechanism φ is **strategy-proof** if no student can ever gain by misstating his preferences, i.e., there do not exist a problem $(\succ = (\succ_s)_{s \in S}, \, P = (P_i)_{i \in I})$, a student i, and preferences P_i' such that $\varphi_i(\succ, P_i', P_{-i}) \, P_i \, \varphi_i(\succ, P)$.

III. TRADE-OFFS AMONG PROPERTIES

A common mechanism among school districts in the U.S. is the so-called **Boston mechanism**, which was in use in the Boston school district before the adoption of the Gale-Shapley student-optimal stable mechanism (SOSM) in 2006. The Boston mechanism and its slight variants are still

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and Ergin (2000).

in use in many places such as Seattle, Minneapolis, Lee County, and Florida. A long overlooked drawback of this mechanism has been underscored in recent research: It gives strong incentives to students for misstating preferences (e.g., Ergin and Sönmez [2006]; Abdulkadiroğlu et al. [2006]). Loosely, under the Boston mechanism a student who initially has high priority for a school faces a high risk of losing his priority advantage if he decides not to list that school as his first choice. As a consequence many students, in fear of losing their priorities, tend to submit preferences that are not representative of their true choices.[11]

Following intensive policy discussions at a number of meetings, in July 2005 the Boston School Committee voted to replace the Boston mechanism with SOSM which, by its strategy-proofness, strongly discourages strategic attempts through preference manipulation (Roth 1982). A major advantage of SOSM is that it is not only fair, but also every student prefers his assignment under SOSM to that at any other fair matching (Gale and Shapley 1962).

A well-known negative result for the school choice context (inherited from two-sided matching) is the incompatibility between Pareto efficiency and fairness: A Pareto efficient and fair matching may not always exist, and if it exists, it is unique (Roth 1982). Hence, SOSM is not Pareto efficient as shown in Example 1. On the other hand, Pareto efficient and strategy-proof mechanisms proposed so far (e.g., Abdulkadiroğlu and Sönmez [2003]; Kesten [2007]) are based on Gale's *top trading cycles idea* that enables a student to obtain a higher priority (than his own) at a desirable school by trading his priority for a less desirable school with another student. By the earlier incompatibility, these mechanisms unfortunately cannot avoid situations in which a student's priority is violated. A second concern about this kind of a mechanism has been raised by officials in Boston who showed reluctance to the idea of 'trading priorities' for reasons outlined previously.

We next present another incompatibility result that shows a three-way tension among our desiderata. It is not unreasonable to expect a *good* mechanism to select the Pareto efficient and fair matching whenever it exists. SOSM clearly meets this requirement. However, as the next result shows not only those that are trading-based, but *all* strategy-proof and Pareto efficient

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[11]For more on the Boston mechanism, see Chen and Sönmez (2003), Abdulkadiroğlu et al. (2005, 2006), Ergin and Sönmez (2006), and Pathak and Sönmez (2008a).

mechanisms fail this requirement, and introduce unnecessary and otherwise avoidable situations of priority violation. All proofs are given in Appendix.

PROPOSITION 1. *There is no Pareto efficient and strategy-proof mechanism that selects the Pareto efficient and fair matching whenever it exists.*

IV. HOW INEFFICIENT IS SOSM?

It is clear that "fairness" is an ideal equity criterion in this context. Yet, it is in conflict with Pareto efficiency. If one values fairness over Pareto efficiency, then SOSM is unquestionably the most natural choice. Nonetheless, the fact that we care about equity should not mean that the welfare aspects of the problem can be totally neglected. This raises a question about the price one needs to pay for achieving fairness. The following example illustrates a striking situation in which every student is unsatisfied at the most favorable fair matching (for students) that one can possibly find.

EXAMPLE 2 (SOSM may result in severe efficiency loss). Let $I \equiv \{i_1, i_2, \ldots, i_{12}\}$ and $S \equiv \{s_1, s_2, \ldots, s_5\}$ where each school except s_5 has two seats and school s_5 has four seats. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
i_5	i_2	i_{11}	i_4	i_{12}
i_7	i_{10}	i_1	i_8	i_9
i_{12}	i_9	i_6	i_3	i_6
i_4	i_7	i_{10}	i_1	i_3
i_8	i_{11}	i_5	i_2	$\vdots$
i_9	i_3	i_{12}	i_6	
i_{10}	i_1	i_4	i_7	
i_{11}	i_8	i_2	i_5	
i_6	i_{12}	i_3	i_9	
i_1	i_4	i_7	i_{10}	
i_2	i_5	i_8	i_{11}	
i_3	i_6	i_9	i_{12}	

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	P_{i_7}	P_{i_8}	P_{i_9}	$P_{i_{10}}$	$P_{i_{11}}$	$P_{i_{12}}$
$\boxed{s_1}$	$\boxed{s_1}$	s_1	$\boxed{s_2}$	$\boxed{s_2}$	s_2	$\boxed{s_3}$	$\boxed{s_3}$	s_3	$\boxed{s_4}$	$\boxed{s_4}$	s_4
s_2	s_3	s_3	s_3	s_4	s_1	s_4	s_2	s_4	s_1	s_1	s_2
s_4	s_4	s_2	s_1	s_3	s_4	s_2	s_1	s_1	s_3	s_2	s_3
$\underline{s_3}$	$\underline{s_2}$	s_4	$\underline{s_4}$	$\underline{s_1}$	s_3	$\underline{s_1}$	$\underline{s_4}$	s_2	$\underline{s_2}$	$\underline{s_3}$	s_1
s_5	s_5	$\boxed{s_5}$	s_5	s_5	$\boxed{s_5}$	s_5	s_5	$\boxed{s_5}$	s_5	s_5	$\boxed{s_5}$

The outcome of SOSM for this problem is the above underlined matching. (See Appendix for details.) It is indeed disappointing to see that SOSM places each student to either his last choice or to his second last choice. This matching is clearly Pareto inefficient. For example, the matching, marked above with boxes, which places eight of the students to their first choices and that does not change the placement of the remaining four, Pareto dominates this matching. ◇

In general, it is possible to construct arbitrarily large school choice problems for which SOSM

results in high welfare losses. The next result shows that for any given set of schools one can always find a problem for which the SOSM outcome is extremely unfavorable to every student.

PROPOSITION 2. *Given any set of schools S and capacity vector $q = (q_s)_{s \in S}$ there always exists a set of students I and a problem $((\succ_s)_{s \in S}, (P_i)_{i \in I})$ for which the student-optimal stable mechanism places each student to either his worst choice or to his second worst choice.*

It is also worth noting that the statement of Proposition 2 is tight. Indeed, SOSM never places each student to his worst choice. This observation simply comes from a well-known result that the outcome of SOSM is weakly Pareto optimal (Roth 1982).

V. A DEFERRED ACCEPTANCE MECHANISM WITH CONSENT

Leaving the main drawback of SOSM aside, we have seen that it stands promising in terms of fairness and strategic immunity which have been two main reasons that influenced the decisions of the school officials in Boston and New York City. We next study the reasons behind the efficiency loss of this mechanism.

Let us once again recall Example 1 where the outcome of SOSM is not Pareto efficient. When the DA algorithm is applied to this problem, student i_1 causes student i_2 to be rejected from school s_1, and starts a chain of rejections which ends back at school s_1 forming a full cycle and causing student i_1 himself to be rejected. There such a cycle has resulted in loss of efficiency. As the following illustrations will show, it is indeed this kind of cyclical rejection chains that lie behind the inefficiency of SOSM.[12]

To sum up, what is going on in Example 1 is that, by applying to school s_1, student i_1 "interrupts" a desirable settlement among students i_2, i_3, and i_4 without affecting his own placement,

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[12]Ergin (2002) offers a restriction on the priority structure that is sufficient as well as necessary to guarantee that such rejection cycles never form.

and artificially introduces inefficiency into the outcome. The key idea behind the mechanism we are going to introduce is based on preventing students like student i_1 of this example from interrupting settlements among other students. Coming back to Example 1, suppose school s_1 is to be removed from student i_1's preferences without affecting the relative ranking of the other schools in his preferences. Note that, when we re-run the DA algorithm replacing the preferences of student i_1 by his new preferences, there is no change in the placement of student i_1. But, because the previously mentioned cycle now disappears, students i_2, i_3, and i_4 each move one position up in their preferences. Moreover, the new matching is now Pareto efficient.

Given a problem to which the DA algorithm is applied, let i be a student who is tentatively placed to a school s at some Step t and rejected from it at some later Step t'. If there is at least one other student who is rejected from school s after Step $t - 1$ and before Step t', i.e., rejected at a step $l \in \{t, t+1, \ldots, t'-1\}$, then we call student i an **interrupter** for school s, and the pair (i, s) an **interrupting pair** of Step t'.

As we will argue shortly, if the outcome of SOSM is inefficient for a problem, then this means there needs to be at least one interrupting pair in the corresponding DA algorithm, even though the converse is not necessarily true (i.e., an interrupting pair does not always result in efficiency loss.).[13] The mechanism we shall propose relies on the idea of identifying interrupting pairs in the DA algorithm, and canceling the applications of the interrupters in these pairs to the corresponding critical schools (to be made precise shortly). Even though it may seem straightforward at a first glance, such a job becomes quite challenging as the size of the problem increases. For example, for a given problem there may be more than one interrupter for the same school, and the rejection chains in the DA algorithm could have a nested and complicated structure making it difficult to identify which interrupter (or interrupters) is (are) the actual reason(s) for the inefficiency. To illustrate, we give an example.

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[13]For example, consider an interrupting pair (i, s) : it is possible that student i's rejection from school s (at Step t' according to the above definition) could be caused by some student j whose application to school s has not been directly or indirectly triggered by the student that student i displaced from school s when he is tentatively admitted. In such cases as these the SOSM outcome does not suffer efficiency loss due to the presence of an interrupter.

EXAMPLE 3 (A problem where the associated DA algorithm contains nested rejection chains). Let $I \equiv \{i_1, i_2, \ldots, i_6\}$ and $S \equiv \{s_1, s_2, \ldots, s_5\}$ where each school except s_5 has only one seat and school s_5 has two seats. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
i_2	i_3	i_1	i_4	$\vdots$
i_1	i_6	i_6	i_3	
i_5	i_4	i_2	i_6	
i_6	i_1	i_3	$\vdots$	
i_4	$\vdots$	$\vdots$		
i_3				

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}
$\boxed{s_2}$	$\boxed{s_3}$	s_3	$\boxed{s_1}$	s_1	s_4
s_1	$\underline{s_1}$	$\boxed{s_4}$	s_2	$\boxed{s_5}$	s_1
s_3	s_5	$\underline{s_2}$	$\underline{s_4}$	$\vdots$	s_3
$\vdots$	$\vdots$	$\vdots$			s_2
					$\boxed{s_5}$

The DA algorithm applied to this problem is summarized in the following table. [If a student remains tentatively placed to a school for a certain number of steps, we use vertical dots to denote this.] The SOSM outcome is the underlined matching above.

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}, i_4$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	
2	$\vdots$	$i_1, \boxed{i_4}$	$\vdots$	$i_6, \boxed{i_3}$	
3	$i_5, i_6, \boxed{i_1}$	$\vdots$		$\vdots$	
4	$\vdots$		$i_2, \boxed{i_6}$		$\boxed{i_5}$
5	$\boxed{i_2}, i_1$		$\vdots$		$\vdots$
6	$\vdots$		$\boxed{i_1}, i_6$		
7		$\boxed{i_6}, i_4$	$\vdots$		
8		$\vdots$		$\boxed{i_4}, i_3$	
9		$i_6, \boxed{i_3}$		$\vdots$	
10	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_5}, \boxed{i_6}$

Here, for example, there are two interrupters for school s_1 : student i_5 (because student i_4 was rejected while he was tentatively placed to school s_1) and student i_1 (students i_5 and i_6 were rejected while he was tentatively placed to school s_1). Similarly, students i_4 and i_6 are interrupters for school s_2, and students i_2 and i_6 are interrupters for school s_3 etc. $\diamond$

How can we prevent the interrupter students from causing inefficiency? Recall that for an interrupter student, his application to the school for which he is an interrupter does not affect his placement but may affect those of others resulting in inefficiency. Then, how about removing from each interrupter student's preferences the school(s) which he is an interrupter for (without changing the relative ranking of other schools), and applying SOSM to the revised problem? The following simple variant of Example 1 employs this natural approach as well as another alternative to solve the inefficiency problem. Neither approach works.

EXAMPLE 4 (How to neutralize the interrupters? A challenge!). Let $I \equiv \{i_1, i_2, i_3\}$ and $S \equiv \{s_1, s_2, s_3\}$ where each school has only one seat. The priorities for the schools and the preferences of the students are given as follows:

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$		P_{i_1}	P_{i_2}	P_{i_3}
i_3	i_1	$\vdots$		$\boxed{s_1}$	s_1	$\boxed{s_2}$
i_1	i_2			s_2	s_2	s_1
i_2	i_3			s_3	$\boxed{s_3}$	s_3

The corresponding DA table is as follows:

Step	s_1	s_2	s_3
1	$\boxed{i_1}, i_2$	$\boxed{i_3}$	
2	$\boxed{i_1}$	$i_3, \boxed{i_2}$	
3	$i_1, \boxed{i_3}$	$\boxed{i_2}$	
4	$\boxed{i_3}$	$i_2, \boxed{i_1}$	
5	$\boxed{i_3}$	$\boxed{i_1}$	$\boxed{i_2}$

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The outcome of SOSM for this problem is the underlined matching above. It is easy to see that this matching is not Pareto efficient. (For instance, compare it to the matching marked with boxes). There are two interrupting pairs within the algorithm: (i_1, s_1) [because student i_2 was rejected while student i_1 was tentatively placed to school s_1] and (i_2, s_2) [because student i_3 was rejected while student i_2 was tentatively placed to school s_1]. Now consider the revised problem when we remove school s_1 from student i_1's preferences and school s_2 from those of student i_2. The DA table corresponding to the revised problem is as follows:

Step	s_1	s_2	s_3
1	i_2	i_1, i_3	
2	i_2, i_3	i_1	
3	i_3	i_1	i_2

The outcome does not change (i.e., still inefficient) even though there are no interrupters left in the new algorithm. Hence, we are stuck. Now let us consider another approach. Instead of handling all the interrupters simultaneously, this time let us start with the *earliest* interrupter in the algorithm. Note that student i_1 was identified as an interrupter at step 3 before student i_2 who was identified at step 4. Thus, let us then consider the revised problem when we only remove school s_1 from student i_1's preferences. The DA table corresponding to the revised problem is as follows:

Step	s_1	s_2	s_3
1	i_2	i_1, i_3	
2	i_2, i_3	i_1	
3	i_3	i_1, i_2	
4	i_3	i_1	i_2

Once again, there is no change in the outcome. Hence, this approach does not work either. Finally, let us do the updating exercise, this time, starting with the *latest* interrupter in the algorithm. Hence, we now consider the revised problem when we only remove school s_2 from the

preferences of student i_2. The DA table corresponding to the revised problem is as follows:

Step	s_1	s_2	s_3
1	$\boxed{i_1}$, i_2	$\boxed{i_3}$	
2	$\boxed{i_1}$	$\boxed{i_3}$	$\boxed{i_2}$

The outcome is the matching marked with boxes above. The inefficiency in the SOSM outcome now disappears. ◇

Notice that when a student's preference list is subjected to the above updating exercise, this inevitably introduces the possibility of a priority violation at the final outcome (for example, at the above matching marked with boxes the priority of student i_2 for school s_2 is violated while everyone else's is respected.) Therefore in real-life applications it is imperative that each student is asked for permission to waive his priority for a critical school in cases similar to above. We incorporate this aspect of the problem into the procedure by dividing the set of students into two groups: those students who *consent* for priority waiving and those who do not. The following algorithm summarizes the informal discussion so far, and describes the above procedure more precisely for a given problem and a given set of consenting students:

Round 0: Run the DA algorithm.

Round 1: Find the last step (of the DA algorithm run in Round 0) at which a consenting interrupter is rejected from the school for which he is an interrupter. Identify all interrupting pairs of that step each of which contains a consenting interrupter. If there are no interrupting pairs, then stop. For each identified interrupting pair (i, s), remove school s from the preferences of student i without changing the relative order of the remaining schools. [Do not make any changes in the preferences of the remaining students.] Re-run the DA algorithm with the new preference profile.

In general,

Round k, k ≥ 2: Find the last step (of the DA algorithm run in Round k-1) at which a consenting interrupter is rejected from the school for which he is an interrupter. Identify all interrupting pairs of that step each of which contains a consenting interrupter. If there are no interrupting pairs, then stop. For each identified interrupting pair (i, s), remove school s from the preferences of student i without changing the relative order of the remaining school. [Do not make any changes in the preferences of the remaining students.] Re-run the DA algorithm with the new preference profile.

Since the number of schools and students is finite, the algorithm eventually terminates in a finite number of steps. At termination, the outcome obtained at the final round is the outcome of the algorithm. We call the mechanism that associates to each problem the outcome of the above algorithm as the **efficiency adjusted deferred acceptance mechanism (EADAM)**.

A remark in terms of the actual computation of the EADAM outcome is in order. Since the DA algorithm run in two consecutive rounds of the EADAM algorithm are identical until the first step a consenting interrupter applies to the school for which he is an interrupter, in practice the EADAM outcome can be conveniently computed by only re-running the relevant last steps of the DA algorithm. Note also that each round of the EADAM algorithm consists of a run of the DA algorithm which is a polynomial-time procedure (e.g., see Gusfield and Irving [1989]). Then since a student can be identified as an interrupter at most $|S|$ times, these iterations need to be done at most $|I||S|$ times giving us a computationally simple polynomial-time algorithm.[14] We next give a detailed example to illustrate how EADAM works.

EXAMPLE 5. Let us find the outcome of EADAM for the problem given in Example 3 assuming for simplicity that all students consent.

Round 0: We run the DA algorithm. The table given in Example 3 shows the steps of the DA algorithm for this problem.

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[14]Furthermore, the memory space required is also polynomial since the identification of blocking pairs at each step requires at most $|I||S|$ recordings.

Round 1: The last step at which an interrupter is rejected from the school he is an interrupter for is Step 9 where the interrupting pair is (i_6, s_2). [Student i_6 is an interrupter for school s_2 because there is a student (namely, student i_4) who was rejected from school s_2 at the step student i_6 was tentatively placed to school s_3.] We remove school s_2 from the preferences of student i_6 (and keep the preferences of the remaining students the same). We then re-run the DA algorithm with the new preference profile:

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}, i_4$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	
2	$\vdots$	$i_1, \boxed{i_4}$	$\vdots$	$i_6, \boxed{i_3}$	
3	$i_5, i_6, \boxed{i_1}$	$\vdots$			
4	$\vdots$		$i_2, \boxed{i_6}$		$\boxed{i_5}$
5	$\boxed{i_2}, i_1$		$\vdots$		$\vdots$
6	$\vdots$		$\boxed{i_1}, i_6$		
7	$\boxed{i_2}$	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_3}$	$\boxed{i_5}, \boxed{i_6}$

Round 2: The last step at which an interrupter is rejected (from the school he is an interrupter for) is Step 6 where the interrupting pair is (i_6, s_3). We remove school s_3 from the (updated) preferences of student i_6. We then re-run the DA algorithm with the new preference profile:

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_5}, i_4$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	
2	$\vdots$	$i_1, \boxed{i_4}$	$\vdots$	$i_6, \boxed{i_3}$	
3	$i_5, i_6, \boxed{i_1}$	$\vdots$		$\vdots$	
4	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_5}, \boxed{i_6}$

Round 3: The last step at which an interrupter is rejected (from the school he is an interrupter for) is Step 3 where the interrupting pair is (i_5, s_1). We remove school s_1 from the preferences of

student i_5, and keep the preferences of the remaining students the same. We then re-run the DA algorithm with the new preference profile:

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	$\boxed{i_5}$
2	$\vdots$	$\vdots$	$\vdots$	$i_6, \boxed{i_3}$	$\vdots$
3	$\boxed{i_6}, i_4$			$\vdots$	
4	$\vdots$	$i_1, \boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_3}$	
5	$i_6, \boxed{i_1}$				
6	$\boxed{i_1}$	$\boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_5}, i_6$

Round 4: The last step at which an interrupter is rejected (from the school he is an interrupter for) is Step 5 where the interrupting pair is (i_6, s_1). We remove school s_1 from the (updated) preferences of student i_6. We then re-run the DA algorithm with the new preference profile:

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\boxed{i_6}$	$\boxed{i_5}$
2	$\vdots$	$\vdots$	$\vdots$	$i_6, \boxed{i_3}$	$\vdots$
3	$\boxed{i_4}$	$\boxed{i_1}$	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_5}, i_6$

Round 5: There are no interrupting pairs, hence we stop. The outcome of EADAM is the matching obtained at the end of Round 4. This matching is marked with boxes on the preference profile given in Example 3. ◇

EADAM very much mimics SOSM. This allows it to inherit a great deal of the fairness property of SOSM. From a welfare perspective, at each round of the algorithm no student is made worse off as compared to his placement in a previous round, but some students may be made better off. This continues until all relevant consents have been carried out, or there is no room left for a possible Pareto improvement in the outcome.

THEOREM 1. *The efficiency adjusted deferred acceptance mechanism (EADAM) Pareto dominates*[15] *the student-optimal stable mechanism (SOSM) as well as any fair mechanism. If no student consents, the two mechanisms are equivalent. If all students consent, then the EADAM outcome is Pareto efficient. At the EADAM outcome all non-consenting students' priorities are respected, however there may be consenting students whose priorities for some schools are violated with their permission.*

As the proportion of consenting students increases, the improvement rate of EADAM over SOSM increases, and attains its maximum when all students consent. EADAM also passes the earlier test to which we have subjected Pareto efficient mechanisms.

COROLLARY 1. *If all students consent, then EADAM selects the fair and Pareto efficient matching whenever it exists.*

Theorem 1 suggests that the EADAM matching also offers a sense of "reasonable fairness." This is because a (consenting) student whose priority is violated under EADAM is in fact never better off at any fair matching that completely eliminates such situations. Moreover, it is even possible that a (consenting) student who suffers from a priority violation under EADAM may be placed to an even worse school for him under SOSM which offers him the best placement he can possibly get under a fair mechanism. This point is illustrated in the next example.

EXAMPLE 6 (A student who suffers a priority violation under EADAM is placed to an even worse school under SOSM). Let $I \equiv \{i_1, i_2, i_3, i_4, i_5\}$ and $S \equiv \{s_1, s_2, s_3, s_4, s_5\}$ where each school has only one seat. The priorities for schools and the preferences of students are given as follows:

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[15]Formally, mechanism Φ *Pareto dominates* mechanism Ψ iff there is no problem $(\succ, P)$ and no student i such that $\Psi_i(\succ, P) \ P_i \ \Phi_i(\succ, P)$.

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$
i_3	i_2	i_5	i_1	$\vdots$
i_1	i_3	i_4	i_2	
i_2	$\vdots$	i_1	$\vdots$	
$\vdots$		$\vdots$		

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}
s_1	$\boxed{s_1}$	$\boxed{s_2}$	s_3	$\boxed{s_4}$
$\boxed{s_3}$	$\underline{s_2}$	$\underline{s_1}$	$\boxed{s_5}$	$\underline{s_5}$
$\underline{s_4}$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\vdots$				

The DA algorithm applied to this problem is given in the following table.

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_1}, i_2$	$\boxed{i_3}$	$\boxed{i_4}$	$\boxed{i_5}$	
2	$\vdots$	$i_3, \boxed{i_2}$	$\vdots$	$\vdots$	
3	$i_1, \boxed{i_3}$	$\vdots$			
4	$\vdots$		$i_1, \boxed{i_4}$		
5			$\vdots$	$i_5, \boxed{i_1}$	
6			$\boxed{i_5}, i_4$		
7	$\boxed{i_3}$	$\boxed{i_2}$	$\boxed{i_5}$	$\boxed{i_1}$	$\boxed{i_4}$

The outcomes of SOSM (the underlined matching) and EADAM when all students consent (the matching marked with boxes) are shown above. Note that at the EADAM matching, student i_1 has higher priority than student i_2 who is placed to school s_1. However, student i_1 is placed to school s_4 (a school worse for him than his placement under EADAM) under SOSM.

Put differently, under EADAM by consenting student i_1 helps some other students (namely, students i_2 and i_3) without hurting himself, and similarly at another instance, another student (namely, student i_4) helps student i_1 (as well as student i_5) by consenting without hurting himself.⋄

V.A. To Consent or Not

The next proposition shows that EADAM does not give a student any disincentive to give consent. Indeed, a student has nothing to lose by waiving his priority right at certain occasions but may

in fact allow many others to improve their assignments as a consequence.

PROPOSITION 3. *Under EADAM no consenting student ever gains by instead not consenting.*
More precisely, the own placement of a student does not change whether he consents or not.

V.B. Strategic Issues

EXAMPLE 7 (EADAM is not strategy-proof). Let $I \equiv \{i_1, i_2, i_3\}$ and $S \equiv \{s_1, s_2, s_3\}$ where each school has only one seat. The priorities for the schools and the preferences of the students are given as follows.

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_3	i_2	i_2
i_1	i_3	i_1
i_2		

P_{i_1}	P_{i_2}	P'_{i_2}	P_{i_3}
$\underline{s_1}$	s_1	$\boxed{s_1}$	$\boxed{s_3}$
$\boxed{s_2}$	$\underline{s_2}$	s_3	s_1
	s_3	s_2	

The corresponding DA table is as follows:

Step	s_1	s_2	s_3
1	$\boxed{i_1}$, i_2		$\boxed{i_3}$
2	$\boxed{i_1}$	$\boxed{i_2}$	$\boxed{i_3}$

Since the outcome of the DA algorithm is already Pareto efficient, we stop at the end of Round 0. The outcome of EADAM as well as that of SOSM when all students truthfully report their preferences is given by the underlined matching. Now, suppose that student i_2 reports preferences P'_{i_2} in which he exchanges the places of s_2 and s_3 in his true preferences. Let us re-calculate the outcome of EADAM. The DA table corresponding to this new problem is as follows:

Step	s_1	s_2	s_3
1	$\boxed{i_1}$, i_2		$\boxed{i_3}$
2	$\boxed{i_1}$		i_3, $\boxed{i_2}$
3	i_1, $\boxed{i_3}$		$\vdots$
4	$\boxed{i_3}$	$\boxed{i_1}$	$\boxed{i_2}$

Round 1: Since student i_1 is rejected from school s_1 at Step 3 and since student i_2 has been rejected from school s_1 while student i_1 was tentatively placed to school s_1, we identify (i_1, s_1) as the last and the only interrupting pair. Suppose student i_1 consents. Then we remove school s_1 from student i_1's preferences, and re-run the DA algorithm with the new preference profile. The corresponding DA table is as follows:

Step	s_1	s_2	s_3
1	$\boxed{i_2}$	$\boxed{i_1}$	$\boxed{i_3}$

Round 2: There are no interrupting pairs and we stop. The outcome of EADAM when student i_2 misstates his preferences is the matching marked with boxes above. ⋄

In the above example student i_2 was able to manipulate EADAM by misstating his preferences. The reason this happened is that by switching the positions of two schools in his preferences, student i_2 initiated a rejection chain which eventually caused student i_1 to be rejected from school s_1. This, in turn, caused us to identify (i_1, s_1) as an interrupting pair, and prevent student i_1 from applying to school s_1, which thus has benefited student i_2.

The following proposition shows that the vulnerability to dominant strategy incentive compatibility cannot be avoided by any Pareto efficient mechanism that aims to improve upon SOSM. This result is an immediate consequence of Proposition 1.

PROPOSITION 4. *There is no Pareto efficient and strategy-proof mechanism that Pareto dominates SOSM.*

Recently, Abdulkadiroğlu, Pathak, and Roth (2009) strengthen Proposition 4 by extending

it to problems with weak priority orders and dropping Pareto efficiency.[16] A critical point to underline is that violation of strategy-proofness does not necessarily imply easy manipulability. It is well-known that in two-sided matching markets stability, a vital requirement for the survival of a mechanism, is incompatible with strategy-proofness. On the other hand, mechanisms based on the DA algorithm and its variants are successfully in use in several applications such as a number of U.S. medical markets (e.g., Roth [1984]; Roth [1991]; Roth and Peranson [1999]; Niederle and Roth [2005]).[17] It turns out that these mechanisms do not offer much chance for manipulation,[18] and any possible manipulation opportunities rapidly vanish as the market size increases (e.g., Immorlica and Mahdian [2005]; Kojima and Pathak [2009]).

Roth and Rothblum (1999) search for the reason behind the success of DA based mechanisms (despite violating strategy-proofness), and argue that this may be due to the fact that agents lack complete information about the preferences of other agents in the market, which considerably reduces the scope of potentially profitable strategic behavior. They show that regardless of the attitude of an agent toward risk, in a low information environment it is never profitable for an agent to simply switch the positions of two alternatives in his preference ranking.[19] Ehlers (2008) extends this approach to general mechanisms, and proposes sufficiency conditions under which a mechanism is immune to strategic behavior in a limited information environment.

In the real world a student is often unable to distinguish between two schools in the sense that he is unsure as to how other students rank the two schools in their stated preferences or, how the two schools' priority orders differ. Roth and Rothblum (1999) model such situations in an incomplete information setting using a notion of "symmetric information." A student's information (or, belief) is said to be *symmetric* for two schools s and s' if given that his own preferences are

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[16] Also see Erdil and Ergin (2008) for a related result.

[17] Two such mechanisms are the *hospital-* and *intern-optimal stable mechanisms* which have been in use (with certain changes over the years) in the assignment of medical interns to hospital positions. The way these mechanisms can be manipulated by interns/hospitals is based on a very similar idea to the way EADAM can be manipulated by a student. An intern/hospital by misstating his/its true preferences initiates a rejection chain which eventually causes a more preferable hospital/intern to propose to himself/itself.

[18] In an empirical study on the hospital- and intern-optimal stable mechanisms, Roth and Peranson (1999) find that only about 0.01% of the non-proposing side had a chance of successful manipulation.

[19] Also see Ehlers (2004) for an extension of the analysis of Roth and Rothblum (1999) to more general information structures.

fixed, his information assigns the same probability to any problem and to its symmetric problem in which the positions of s and s' are exchanged, i.e., he is unable to deduce any difference between the two schools from his information. (See Appendix for a formal treatment.)

Nonetheless it may not be realistic to assume that any two schools are always viewed to be symmetric. In real life, for example, students share certain common perceptions about the quality of different schools. Specifically, some schools are deemed to be better than some others by many students and their parents. Hence it may not be reasonable to assume that a student's information is symmetric for two schools that have apparent difference in terms of popularity. In fact, Ehlers (2008) shows that if a student's information is completely symmetric (i.e., symmetric for any two schools), then one cannot distinguish between two mechanisms such as SOSM and the controversial Boston mechanism on the basis of their strategic immunity even though there is strong experimental and empirical evidence (e.g., Chen and Sönmez [2006]; Roth and Peranson [1999]; Abdulkadiroğlu et al. [2006]) that suggests otherwise. Consequently, he shows that the assumption of completely symmetric information may not necessarily serve as a useful benchmark for analyzing strategic behavior in a low information environment.

We propose the following information setting which can be seen as an intermediate case between a 'complete information' setting and a 'completely symmetric incomplete information' setting. The set of schools is partitioned into quality classes. It is common knowledge among students that any student prefers any school in some quality class to any other school that belongs to a lower quality class. However, all students' information about any two schools within the same quality class is symmetric. We next investigate the strategic opportunities for manipulating EADAM in this new setting.

We now interpret the stated preferences of a student as his strategy. Given a problem $((\succ_s)_{s \in S}, (P_i)_{i \in I})$, a student i, and two preferences P_i' and P_i'', we say strategy P_i' *stochastically dominates* strategy P_i'' if the probability distribution induced on the placements of student i when he states P_i', stochastically dominates the probability distribution induced on his placements when he states P_i'' where the comparison is based on preferences P_i of student i (see Appendix for a precise definition).

As the next result and the following remark suggest, our information setting allows us to draw a clear line between EADAM and many other mechanisms including the Boston mechanism in terms of immunity to strategic action.

THEOREM 2. *Suppose that the following is common knowledge among students. The set of schools is partitioned into quality classes as follows: Let $\{S_1, S_2, \ldots, S_m\}$ be a partition of S. Given any $k, l \in \{1, \ldots, m\}$ such that $k < l$, each student prefers any school in S_k to any school in S_l. Moreover, each student's information is symmetric for any two schools s and s' such that $s, s' \in S_r$ for some $r \in \{1, \ldots, m\}$. Then for any student the strategy of truth-telling stochastically dominates any other strategy when other students behave truthfully. Thus, truth-telling is an ordinal Bayesian Nash equilibrium of the preference revelation game under EADAM.*[20]

REMARK 1. It is easy to see that a similar statement cannot be made for the (old) Boston mechanism. For example, if all students have identical preferences (i.e., there are $|S|$ quality classes), and if each school has the same priority order, the Boston mechanism can be easily manipulated by a low priority student when other students behave truthfully. Note that in such a case no student can gain by misstating his preferences under EADAM when others behave truthfully. Thus truth-telling is a Nash equilibrium of the preference revelation game under EADAM in this case.

REMARK 2. A commonly studied manipulation strategy in two-sided matching markets is a "truncation strategy." First suppose that students are also allowed to announce certain schools as unacceptable in their preferences.[21] Then a student using a *truncation strategy* simply ranks the schools in the same order as his true preferences but announces all schools ranked below a certain threshold school as unacceptable.[22] It is also easy to see that no student would ever benefit from a truncation strategy under EADAM at any problem. This is simply because if a student truncates his true preferences, then either the rounds of EADAM are completely unaffected (when

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[20] A strategy profile is an ordinal Bayesian Nash equilibrium if it is a Bayesian Nash equilibrium for every possible von Neumann-Morgenstern utility representation of students' true preferences. See, for example, Ehlers and Massó (2007).

[21] Note that the model and the analysis of this paper can be straightforwardly extended to this case.

[22] More precisely, if the true preferences P_i of student i ranks k schools as acceptable, then a *truncation strategy* P_i' of him ranks the best $l \leq k$ of these k schools as acceptable and in the same order as in P_i.

his truncation threshold is weakly below his Round 0 placement) or, he remains unassigned at the end of Round 0 (when his truncation threshold is strictly above his Round 0 placement) which in turn implies that he remains unassigned under EADAM. In either case, the student's placement does not improve.

V.C. Alternative Ways to Improve upon SOSM and the Idea of Consent

We next discuss the possibility of using alternative methods to recover welfare losses caused by interrupter students under SOSM. Ignoring the strategic aspects of the problem for now, it is easy to see that any Pareto improvement over the SOSM assignment can be obtained by executing improvement cycles in which a group of students trade their SOSM assignments among themselves in exchange for better ones. One popular such method is based on Gale's celebrated *top trading cycles*[23] idea which is also adopted to school choice by Abdulkadiroğlu and Sönmez (2003) as a way to achieve a Pareto efficient mechanism. The top trading cycles is an iterative procedure that works as follows: Each student points to the student who is assigned his favorite school. This leads to at least one cycle to form. Trades within each cycle are carried out by assigning each student in a cycle to the school he points to. Next, participants in a cycle are removed, and the same procedure is applied to the reduced problem and so on.

Once the SOSM assignment is computed, it is indeed plausible to implement an improvement cycle selection procedure such as the top trading cycles so long as the resulting priority violation pertains to students who have chosen to consent. In what follows we point out two major complications that arise with improvement cycle selection procedures. The first one is that selecting improvement cycles in some arbitrary way can significantly limit the scope of implementation. We first give an example before discussing the second issue.

EXAMPLE 8 (Improvement cycles should not be chosen arbitrarily).

Let $I \equiv \{i_1, i_2, i_3, i_4, i_5, i_6, i_7, i_8\}$ and $S \equiv \{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$ where schools s_1 through s_6 each

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[23]Due to its compelling efficiency and incentive features, this method has also been a commonly used tool in the related problem of *house allocation*. See, for example, Pápai (2000) and Roth, Sönmez, and Ünver (2004) for two applications of this method in deterministic settings, and see Kesten (2009) for an application in a random setting.

has one seat, and school s_7 has two seats. The priorities for schools and the preferences of students are given below. The outcome of SOSM for this problem is the underlined inefficient matching.

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$	$\succ_{s_5}$	$\succ_{s_6}$	$\succ_{s_7}$
i_4	i_2	i_3	i_8	i_6	i_7	$\vdots$
i_1	i_3	i_4	i_7	i_7	i_8	
i_2	$\vdots$	$\vdots$	i_6	$\vdots$	$\vdots$	
i_3			i_5			
i_7			i_2			
i_5			i_1			
i_8			i_4			
i_6			i_3			

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}	P_{i_5}	P_{i_6}	P_{i_7}	P_{i_8}
s_1	s_1	s_1	$\cdot s_4$	s_4	s_4	s_1	$\cdot s_1$
s_4	s_4	s_2	s_3	$\cdot \underline{s_7}$	$\cdot \underline{s_5}$	s_5	s_6
$\cdot \underline{s_7}$	$\cdot \underline{s_2}$	$\cdot \underline{s_3}$	$\underline{s_1}$	$\vdots$	$\vdots$	$\cdot \underline{s_6}$	$\underline{s_4}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$			$\vdots$	$\vdots$

For this problem applying the top trading cycles procedure to the SOSM outcome results in the pairwise trading cycle between (i_4, s_1) and (i_8, s_4) leading to the Pareto efficient final matching indicated above by dots. It is easy to check that such an exchange however completely destroys the fairness of the former SOSM matching since now this exchange results in violation of every other student's priority for some school involved in this exchange. In particular, priorities of students i_1, i_2, i_3, and i_7 for school s_1 are violated, and similarly priorities of students i_1, i_2, i_5, and i_6 for school s_4 are violated as a result of this improvement. In other words, implementing such a trade would be possible only if *all* six students suffering from priority violation at either school gave consent for priority waiving at the same time. Assuming that some students may choose not to consent, this exchange may never be possible. ⋄

To make things even worse, consider the case when larger exchanges (than a pairwise cycle) are to be implemented. For each school involved in such a trade, it would be imperative to obtain consent of every relevant student whose priority is to be violated when all the trades end. In such cases the likelihood of getting stuck without gaining much Pareto improvement clearly increases

with the cycle size.

The above example shows the uneasiness of allowing students to simply trade their SOSM assignments without violating priorities of those who choose not to consent. One way to solve this problem is via the use of meticulous and exhaustive algorithms that search for feasible matchings (or improvement cycles for that matter) at which priorities and consent decisions are simultaneously respected. Such approaches however have two disadvantages. The first one is their impracticality due to the obvious computational complexity given the size of the set of possible matchings and the number of students involved.

A second and more crucial issue for an improvement cycle selection algorithm is the resulting incentives for consenting behavior. Unless a student is guaranteed that his consent decision will in no way affect his final assignment, he may abstain from this practice for obvious reasons. Therefore mechanisms based on search algorithms may naturally cause students to exhibit aversion to consenting since such a decision might result in being left out of an improvement cycle whereas not consenting cannot hurt. Consequently, once incentives for giving consent are disrupted, the whole purpose behind the idea of consent is lost.

On the other hand, the EADAM algorithm can also be thought as an iterative way to select Pareto improvement cycles over the SOSM assignment. A key difference from the above approaches is that EADAM constructs its own improvement cycles based on priorities using the recent history of the DA algorithm rather than its final outcome. For example, if EADAM obtains a Pareto improvement over SOSM at some round, this means that once the relevant interrupter(s) consent(s), a group of at least two students can form a cycle of trades such that each school involved in the trade goes to the next most deserving (according to priority) student among all students but the consenting interrupter(s). In other words, this guarantees that regardless of the size of the improvement cycle, when a Pareto improvement is considered at some round of the EADAM algorithm, the resulting priority violation is always relevant for exactly one school, namely for the school for which the consent of the corresponding interrupting pair(s) is (are) required. To see this point, consider the DA table corresponding to the above problem.

Step	s_1	s_2	s_3	s_4	s_5	s_6	s_7
1	$\boxed{i_1}, i_2, i_3, i_7, i_8$			$\boxed{i_5}, i_4, i_6, i_7$			
2	⋮	$\boxed{i_3}$	$\boxed{i_4}$	$\boxed{i_5}, i_2$	$\boxed{i_6}, i_7$	$\boxed{i_8}$	
3		$i_3, \boxed{i_2}$	⋮	$\boxed{i_5}$	⋮	$\boxed{i_7}, i_8$	
4		⋮	$i_4, \boxed{i_3}$	$\boxed{i_8}, i_5$		⋮	
5	$\boxed{i_4}, i_1, i_5$		⋮	$\boxed{i_8}$			
6	⋮			$\boxed{i_8}, i_1$			$\boxed{i_5}$
7	$\boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_3}$	$\boxed{i_8}$	$\boxed{i_6}$	$\boxed{i_7}$	$\boxed{i_1}, i_5$

The EADAM algorithm identifies two interrupters: i_1 for s_1 and i_5 for s_4. If student i_1 consents, this means we can carry out the trades in the cycle $\{(i_4, s_1), (i_3, s_3), (i_2, s_2)\}$. Indeed, once student i_1 is out of the picture, i_4 has the next highest priority for s_3 after i_3; i_3 has the next highest priority for s_2 after i_2; and i_2 has the next highest priority for s_1 after i_4. The resulting priority violation is relevant only for school s_1. Similarly, if student i_5 consents, this means we can carry out the trades in the cycle $\{(i_8, s_4), (i_7, s_6), (i_6, s_5)\}$. As before, the resulting priority violation is relevant for exactly one school, namely for school s_4 in this case.

We have discussed the important role interrupter students play for implementing the idea of consenting under EADAM. But why is it important to always solicit consent from the most recent interrupter(s) in the DA algorithm? The answer is simple. Suppose (i, s) is the last interrupting pair identified at the first round, and the consent of student i would indeed lead to a Pareto improvement. Now consider any possible Pareto improving trade over the SOSM outcome among a group of students that involves school s (e.g., the one selected by the top trading cycles procedure). Such a trade cannot be implemented unless student i consents or the trade includes him. In other words, student i is the critical student for all the trades involving school s. Should student i consent, this role goes to the next most recently identified interrupting pair involving school s and so on.

Going back to the alternative improvement cycle considered in Example 8, note that the

pairwise top trading cycle $\{(i_4, s_1), (i_8, s_4)\}$ in fact mingles the two cycles EADAM selects by choosing one element from each cycle. Since this trade depends only on the outcome obtained at the last step of the DA algorithm but not on any of the earlier steps, it adds new students to the list of those to ask for consent. In general, selection of improvement cycles may inevitably cause one to tie his hopes for improvement to a simultaneous consent acquisition with regards to a multiplicity of schools, whereas EADAM brings this burden down to a single school for each possible improvement cycle at each round. This low consenting friction faced by EADAM at each round allows EADAM to explore possible paths to improvement proceeding from the most critical student(s) to less by tracing the steps of SOSM backwards. Most notably, by Proposition 3 such an exploration does not harm student incentives for consenting.

V.D. Extension to Weak Priority Orders

Erdil and Ergin (2008) and Abdulkadiroğlu, Pathak, and Roth (2009) point out an important source of efficiency loss SOSM may suffer in practical applications. In the U.S. policies mandated by school districts often give rise to priority orders with broad indifference classes that contain students who have identical characteristics. Since the implementation of the DA algorithm requires the priority orders to be strict, school districts typically use a random draw to break the ties within indifference classes. This however may result in artificial welfare losses the cause of which, as we will show shortly, can be explained by the very same idea we have used to explain as to why the SOSM outcome may be Pareto inefficient when priority orders are strict. Erdil and Ergin (2008) argue that arbitrary tie-breaking may cause the welfare of a significant number of students to be adversely affected under SOSM. The following simple example demonstrates this point.

EXAMPLE 9 (Arbitrary tie-breaking may introduce artificial interrupters). Consider the following problem with $I \equiv \{i_1, i_2, i_3\}$ and $S \equiv \{s_1, s_2, s_3\}$ where each school has one seat. Students i_1 and i_2 share equal priority for school s_1.

	$\succsim_{s_1}$	$\succsim_{s_2}$	$\succsim_{s_3}$		P_{i_1}	P_{i_2}	P_{i_3}
	i_3	i_2	$\vdots$		$\underline{s_1}$	$\boxed{s_1}$	$\boxed{s_2}$
	i_1, i_2	i_3			$\boxed{s_3}$	$\underline{s_2}$	$\underline{s_1}$
		i_1			$\vdots$	$\vdots$	$\vdots$

Let us suppose that the random draw for tie-breaking results in favor of i_1. The steps of the DA algorithm applied to this problem are given in the following table, and the resulting inefficient matching is underlined in the above preference profile:

Step	s_1	s_2	s_3
1	$\boxed{i_1}, i_2$	$\boxed{i_3}$	
2	$\boxed{i_1}$	$\boxed{i_2}, i_3$	
3	$\boxed{i_3}, i_1$	$\vdots$	
4	$\boxed{i_3}$	$\boxed{i_2}$	$\boxed{i_1}$

Observe that the rejection chain that leads to the inefficiency is caused by student i_1 who serves as an interrupter for school s_1. But this is clearly an artificial efficiency loss. For example, if the tie-breaker instead favored student i_2, the new SOSM outcome (shown in boxes above) would in fact Pareto dominate the earlier one without violating any student's priority. $\diamond$

The above example suggests that the innovation in the EADAM idea can also be used to recover the artificial efficiency losses introduced by random tie-breaking when priority orders are weak. The only difference now is in the way interrupter students are handled. If it is possible to replace an interrupter student with an equal priority student who was rejected at some step while the interrupter is tentatively placed to the critical school, then we simply update this interrupter's preferences in the usual way without any need for seeking his consent. The following simple variant of the EADAM algorithm makes this idea more precise for a given problem with a general priority structure (possibly with indifferences):

Round 0: Randomly choose a tie-breaker, and run the induced DA algorithm.

Round k, $k \geq 1$: Find the last step of the DA algorithm run in Round k-1 at which an interrupting pair (i, s) is identified such that at least one student with the same priority with student

i for school s (according to the priority structure before tie-breaking) has been rejected while student i was tentatively placed to s. Remove school s from the preferences of student i without changing the relative order of the remaining schools, and re-run the DA algorithm with the new preference profile. If there are no such interrupting pairs, then stop.

Indeed the above procedure eliminates situations of unnecessary welfare loss caused by random tie-breaking . When the procedure terminates (which again happens in finite number of steps), there are no further improvements possible without violating a student's priority. This is stated in the next result. We omit the proof as it is similar to that of Theorem 1.

PROPOSITION 5. *Assume that priority orders are weak. Then the outcome of the above variant of EADAM is fair. Furthermore, there is no other fair matching that Pareto dominates this outcome.*

Finally, it is also straightforward to combine the earlier EADAM algorithm proposed (for the case when priority orders are strict) with the above one as a way to neutralize both types of interrupters as a way to achieve full (rather than constrained) Pareto efficiency. This is easily done as follows: Starting from the last step of the DA algorithm, whenever an interrupter is identified, we check whether he is an interrupter due to tie-breaking, or an interrupter in the usual sense. In the former case, we always update this student's preferences without seeking his consent. In the latter case, we update the preferences only if the student consents. Our last example illustrates both sources of efficiency loss that the SOSM may suffer, and exemplifies various degrees of improvement that the EADAM approach can provide.

EXAMPLE 10 (How EADAM handles either source of efficiency loss). Consider the following problem with $I \equiv \{i_1, i_2, i_3, i_4\}$ and $S \equiv \{s_1, s_2, s_3, s_4\}$ where each school has one seat. Suppose that the random tie-breaking favors i_1 over i_4.

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$	$\succ_{s_4}$
i_3	i_4	i_2	$\vdots$
i_1	i_2	i_1, i_4	
i_2	i_3	i_3	
i_4	i_1		

P_{i_1}	P_{i_2}	P_{i_3}	P_{i_4}
s_1	$\boxed{s_1}$	$\boxed{s_2}$	$\cdot\boxed{s_3}$
s_3	$\cdot s_2$	$\cdot\underline{s_1}$	$\underline{s_2}$
$\cdot\boxed{s_4}$	$\underline{s_3}$	$\vdots$	$\vdots$
s_2	s_4		

The DA algorithm applied to this problem is summarized in the following table. The SOSM outcome is the underlined matching above.

Step	s_1	s_2	s_3	s_4
1	$\boxed{i_1}, i_2$		i_4	
2	$\boxed{i_1}$	$\boxed{i_2}, i_3$	$\vdots$	
3	$\boxed{i_3}, i_1$	$\vdots$		
4	$\vdots$		$\boxed{i_1}, i_4$	
5		$\boxed{i_4}, i_2$	$\boxed{i_1}$	
6			$\boxed{i_2}, i_1$	
7	$\boxed{i_3}$	$\boxed{i_4}$	$\boxed{i_2}$	$\boxed{i_1}$

The latest interrupting pair is (i_1, s_3). Since i_4 is rejected at a step at which equal priority student i_1 is tentatively placed to s_3, we conclude that i_1 is an artificial interrupter, and remove school s_3 from his preferences. This leads to the matching marked with dots above. It is easy to check that this matching is also fair and Pareto dominates the outcome of SOSM with random tie-breaking. The new DA table now contains only one interrupting pair, namely (i_1, s_1). This time we seek consent from student i_1 since the interrupting behavior does not stem from tie-breaking. If he consents, the final outcome further improves, and is now fully efficient (shown in boxes above).

VI. CONCLUSION

The celebrated Gale-Shapley student-optimal stable mechanism (SOSM) is becoming a central student assignment mechanism in major school districts in the U.S. as well as in the school choice literature. A main reason for this success is the fact that this mechanism can fully respect student priorities. We have provided theoretical evidence that the SOSM outcome may suffer large welfare losses.[24] Matching theory further suggests that this efficiency loss cannot be avoided by any other fair mechanism that completely eliminates violation of student priorities.

The proposed efficiency adjusted deferred acceptance mechanism (EADAM) incorporates into SOSM an idea of *consenting* to priority waiving by offering each student the opportunity to 'reconsider' those priorities that cannot possibly help him secure a better assignment otherwise. The more students consent, the higher is the welfare improvement over SOSM. When the consent of some student is honored by EADAM, this leads to a Pareto improvement for a group of students who exchange their SOSM assignments with those who deserve it next most in terms of priority. By the construction of the EADAM algorithm such a modification neither reduces the fairness achieved by SOSM nor disrupts the incentives for giving consent. The price for consenting however is the loss of dominant strategy incentive compatibility (i.e., strategy-proofness) in complete information environments. In this sense EADAM internalizes the three-way tension between fairness, efficiency, and incentives. In low information environments such as practical applications EADAM also preserves the strategic immunity of SOSM. The idea behind EADAM can also be adopted to avoid artificial welfare losses due to random tie-breaking.

A practical advantage of our proposal is that it could easily be adopted by a school district that is currently using SOSM (e.g., Boston and NYC) without any significant transition costs. In such a district, for example, students would not need to change the way in which they report their preferences.[25] The school choice system in NYC also involves active schools (that state preferences over students) in addition to passive schools (that we have assumed throughout the paper). Prior

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[24]The situation for the widely-used Boston mechanism may be even worse. Ergin and Sönmez (2006) argue that its deficiency due to its lack of efficiency is even more serious than that of it due to strong vulnerability to strategic behavior. They argue that this large efficiency loss can be recovered via a transition to SOSM.

[25]I thank Dennis Epple for making this point.

to the adoption of SOSM, the unfairness (instability in that context) of the matching in NYC gave schools incentive to circumvent the assignment process by concealing capacity (Abdulkadiroglu, Pathak and Roth 2005). On the other hand, despite its stability, schools can still manipulate SOSM (as well as any other stable mechanism) by concealing capacity (Sönmez 1997). Interestingly however, one can check that unlike SOSM, EADAM (when all students consent) is immune to capacity manipulations in a two-sided version of the present model (when schools are assumed to have responsive preferences (Roth [1985])). A thorough investigation of all such incentive issues (for the school side) that may arise in a two-sided matching context is left for future work.

APPENDIX

Proof of Proposition 1. Suppose there exists one such mechanism φ. Consider the following problem. When each student truthfully submits his preferences, then the outcome of the DA algorithm (as well as that of φ) is the underlined matching below which is Pareto efficient.

$\succ_{s_1}$	$\succ_{s_2}$	$\succ_{s_3}$
i_3	i_2	i_2
i_1	i_3	i_1
i_2	i_1	i_3

P_{i_1}	P_{i_2}	P'_{i_2}	P''_{i_2}	P_{i_3}	P'_{i_3}
$\underline{s_1}$	s_1	s_1	s_3	$\underline{s_3}$	s_1
$\boxed{s_2}$	$\underline{s_2}$	$\boxed{s_3}$	$\vdots$	$\boxed{s_1}$	$\vdots$
s_1	s_3	s_2		s_2	

Hence, φ has to select this matching. Suppose student i_2 submits fake preferences P'_{i_2}. Then the outcome of the DA algorithm for the new problem is the matching marked with boxes, which is not Pareto efficient. For the same problem, consider the outcome of φ. If φ places student i_2 to school s_1, then student i_2 gains; if φ places student i_2 to school s_3, then since φ is Pareto efficient, student i_1 has to be placed to school s_1 and student i_3 to school s_2. In this case, consider the preference profile $(P_{i_1}, P'_{i_2}, P'_{i_3})$. The SOSM matching for the corresponding problem is Pareto efficient and places student i_3 to school s_1. Hence, student i_3 gains by reporting P'_{i_3} when his true preferences are P_{i_3}. If φ places student i_2 to school s_2, then consider the preference profile $(P_{i_1}, P''_{i_2}, P_{i_3})$. The SOSM matching for the corresponding problem is Pareto efficient and places

student i_2 to school s_3. Then student i_2 gains by submitting P''_{i_2} instead of P'_{i_2}.

<div align="center">**Q.E.D.**</div>

EXAMPLE 1 (The detailed DA table).

Step	s_1	s_2	s_3	s_4	s_5
1	$\boxed{i_1, i_2}, i_3$	$\boxed{i_4, i_5}, i_6$	$\boxed{i_7, i_8}, i_9$	$\boxed{i_{10}, i_{11}}, i_{12}$	
2	$\boxed{i_1, i_6}, i_2$	$\boxed{i_4, i_{12}}, i_5$	$\boxed{i_7, i_3}, i_8$	$\boxed{i_{10}, i_9}, i_{11}$	
3	$\boxed{i_{11}, i_6}, i_1$	$\boxed{i_8, i_{12}}, i_4$	$\boxed{i_2, i_3}, i_7$	$\boxed{i_5, i_9}, i_{10}$	
4	$\boxed{i_{11}, i_{10}}, i_6$	$\boxed{i_1, i_8}, i_{12}$	$\boxed{i_2, i_4}, i_3$	$\boxed{i_5, i_7}, i_9$	
5	$\boxed{i_9, i_{10}}, i_{11}$	$\boxed{i_1, i_3}, i_8$	$\boxed{i_4, i_{12}}, i_2$	$\boxed{i_6, i_7}, i_5$	
6	$\boxed{i_8, i_9}, i_{10}$	$\boxed{i_3, i_{11}}, i_1$	$\boxed{i_{12}, i_5}, i_4$	$\boxed{i_2, i_6}, i_7$	
7	$\boxed{i_4, i_8}, i_9$	$\boxed{i_{11}, i_7}, i_3$	$\boxed{i_{10}, i_5}, i_{12}$	$\boxed{i_1, i_2}, i_6$	
8	$\boxed{i_4, i_{12}}, i_8$	$\boxed{i_7, i_9}, i_{11}$	$\boxed{i_{10}, i_6}, i_5$	$\boxed{i_1, i_3}, i_2$	
9	$\boxed{i_{12}, i_5}, i_4$	$\boxed{i_2, i_9}, i_7$	$\boxed{i_6, i_{11}}, i_{10}$	$\boxed{i_8, i_3}, i_1$	
10	$\boxed{i_5, i_7}, i_{12}$	$\boxed{i_2, i_{10}}, i_9$	$\boxed{i_{11}, i_1}, i_6$	$\boxed{i_8, i_4}, i_3$	
11	$\boxed{i_5, i_7}$	$\boxed{i_2, i_{10}}$	$\boxed{i_{11}, i_1}$	$\boxed{i_8, i_4}$	$\boxed{i_3, i_6, i_9, i_{12}}$

Proof of Proposition 2. Let a school set $S = \{s_1, s_2, \ldots, s_{K+1}\}$ where $K \geq 1$, and a capacity vector $q = (q_s)_{s \in S}$ be given. Assume without loss of generality that $q_{s_1} \leq q_{s_2} \leq \cdots \leq q_{s_{K+1}}$. Let $I \equiv \{i_1, i_2, \ldots, i_n\}$ be a set of students such that $n > \Sigma_{s \neq s_{K+1}} q_s$ and $n \leq \Sigma_s q_s$. Consider a partition of $I = \{I_1, I_2, \ldots, I_{K+1}\}$ where $I_1 = \cup_{t=1}^{q_{s_1}} \{i_t\}$, $I_2 = \cup_{t=q_{s_1}+1}^{q_{s_1}+q_{s_2}} \{i_t\}, \ldots, I_K = \cup_{t=\Sigma_{m=1}^{K-1} q_{s_m}+1}^{\Sigma_{m=1}^{K} q_{s_m}} \{i_t\}$, and $I_{K+1} = \cup_{t=\Sigma_{s \neq s_{K+1}} q_s+1}^{n} \{i_t\}$. Note that $|I_{K+1}| \geq 1$. For simplicity, we first construct a problem for the case when $q_s = q_{s'}$ for all $s, s' \in S' \equiv S \backslash \{s_{K+1}\}$, and later describe the necessary modifications for the general case. For the first case with uniform school capacities we choose $|I_{K+1}| = 1$. Consider the following problem: Suppose that within each set in the partition of I, students have the same preferences as indicated in the preference table given below (left). Among the sets in the partition of I, students in I_1 and I_{K+1} have the same preferences. Note also that s_{K+1} is the last choice of each student. Suppose that school priorities are as indicated in the table given below (right) where for any school in S', and any distinct pair $I_f, I_g \in I$, students in I_f either all have

higher priority than all students in I_g or, all have lower priority than all students in I_g. Suppose also that within each set in the partition of I, a student has higher priority for every school in S' than any other student (in the same set) who has a lower index. Suppose finally that the priority order for school s_{K+1} is arbitrary.

$i \in I_1$	$i \in I_2$	$\cdots$	$i \in I_K$	$i \in I_{K+1}$
s_1	s_2		s_K	s_1
s_2	s_3		s_1	s_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
	s_K			
s_K	s_1		s_{K-1}	s_K
s_{K+1}	s_{K+1}	$\cdots$	s_{K+1}	s_{K+1}

$\succ_{s_1}$	$\succ_{s_2}$	$\cdots$	$\succ_{s_K}$	$\succ_{s_{K+1}}$
I_2	I_3		I_{K+1}	$\vdots$
I_3	I_4		I_1	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
	I_{K+1}			
I_{K+1}	I_1			
I_1	I_2	$\cdots$	I_K	

At the first step of the DA algorithm, students in I_2 apply to school s_2, and all are tentatively placed to s_2; students in I_3 apply to school s_3, and all are tentatively placed to s_3; ...; and students in I_K apply to school s_K, and all are tentatively placed to s_K. Students in $I_1 \cup I_{K+1}$ apply to school s_1 and the lowest indexed student in I_1 (i.e., student i_1) is rejected from s_1. At the second step, this student (who is rejected from s_1) applies to s_2 and (by our choice of student priorities) causes the lowest indexed student in I_2 to be rejected from s_2. Continuing in this way, at the K-th step the lowest indexed student in I_{K-1} applies to school s_K and (by our choice of student priorities) causes the lowest indexed student in I_K to be rejected from s_K. Any student in I_K rejected from s_K, in turn, applies to s_1 and causes a student from $I_1 \cup I_{K+1}$ to be rejected from school s_1. Note that by our choice of the problem any student rejected from a school at any step of the algorithm is tentatively placed to the school he next applies to causing some student to be rejected. Hence, this process continues until student $i_1 \in I_1$ is rejected from s_K (and next applies to school s_{K+1} where he is permanently admitted). Note that the student that caused this rejection (by applying to school s_K) is student $i_n \in I_{K+1}$. After this point, no student is rejected from any school, and the algorithm terminates. Observe that student i_1 ends up at his worst choice, and students i_2

through i_n end up at their second worst choices. In general, the larger the size of the set I_{K+1}, the more students end up at their worst choices. For the more general case with non-uniform school capacities, the only necessary modification is to choose $|I_{K+1}| = q_{s_K} - q_{s_1}$. It is easy to show using the same problem that by the choice of the problem (and since $n > \Sigma_{s \neq s_{K+1}} q_s$), each student in I applies to each school in S' at some step of the DA algorithm. The algorithm terminates when $|I_{K+1}|$ students have applied to their last choices. At this point, the remaining students are at their second worst choices: The highest indexed $q_{s_{K-1}}$ students in I_K end up at school s_{K-1}; the highest indexed $q_{s_{K-2}}$ students in I_{K-1} end up at school s_{K-2}; $\ldots$; the highest indexed q_{s_1} students in I_2 end up at school s_1; and all q_{s_K} students in $I_1 \cup I_{K+1}$ end up at school s_K. We omit the details.

<div align="center">

Q.E.D.

</div>

Proof of Theorem 1. The next two lemmas establish the proof of Theorem 1. But before, a word of clarification is in order. When we say *student i is an interrupter of Round t*, this means that student i is identified as an interrupter during Round $t+1$ in the DA algorithm that was run at the end of Round t.

LEMMA 1. *Given a problem, the matching obtained at the end of Round r, $r \geq 1$, of the EADAM algorithm places each student to a school which is at least as good for him as the school he was placed to at the end of Round $r - 1$.*

Proof. Suppose by contradiction that there is a problem, a Round r, $r \geq 1$, of the EADAM algorithm, and a student i_1 such that the school student i_1 is placed to at Round r is worse for him than the school s_1^{r-1} he was placed to at Round $r - 1$. This means when we run the DA algorithm at Round r, student i_1 is rejected from school s_1^{r-1}. Then, there is a student $i_2 \in I \backslash \{i_1\}$ who is placed to school s_1^{r-1} at Round r and who was placed to a school s_2^{r-1} (at Round $r - 1$) which is better for him than school s_1^{r-1}. Then, this means there is a student $i_3 \in I \backslash \{i_1, i_2\}$ who is placed to school s_2^{r-1} at Round r and who was placed to a school s_3 which is better for him than school s_2^{r-1} at Round $r - 1$, and so on. Then, there must be a student $i_k \in I \backslash \{i_1, \ldots, i_{k-1}\}$ who is the *first* student to apply to a school s_{k-1}^{r-1} which is worse for him than the school s_k^{r-1} he was placed

to at Round $r - 1$.

We consider two cases:

Case 1. Student i_k is not an interrupter of Round $r - 1$: The preferences of student i_k is the same in Rounds r and $r - 1$. Then, there is a student who is placed to school s_k^{r-1} at Round r and who did not apply to it at Round $r - 1$. But then, this contradicts the assumption that student i_k is the first student to apply to a school which is worse for him than the school he was placed to Round $r - 1$.

Case 2. Student i_k is an interrupter of Round $r - 1$: At Round r, student i_k, instead of applying to the school he is an interrupter for, applied to his next choice, say school s^*. Student i_k also applied to school s^* at Round $r - 1$. Then, there is a student who is placed to school s_k^{r-1} at Round r and who did not apply to it at Round $r - 1$. But then, this again contradicts the assumption that student i_k is the first student to apply to a school which is worse for him than the school he was placed to at Round $r - 1$.

Q.E.D.

Hence, by Lemma 1, at each round of the algorithm, no student is placed to a school which is worse for him than the school he is placed to under SOSM. Then EADAM Pareto dominates SOSM.

Q.E.D.

LEMMA 2: *Suppose all students consent. Then the EADAM outcome is Pareto efficient.*

Proof. Suppose by contradiction that there is a problem for which the matching selected by EADAM is not Pareto efficient. Let α denote this matching. Hence, there is another matching β that Pareto dominates matching α. Suppose the algorithm terminates in $R \geq 1$ rounds. Given $r \in \{1, 2, \ldots, R\}$, let α_r denote the matching obtained at the end of Round r of the algorithm. By Lemma 1, matching β also Pareto dominates each matching α_r with $r \in \{1, 2, \ldots, R\}$.

We first show that at matching β, no interrupter of Round r, $r \in \{1, 2, \ldots, R\}$, is placed to the school for which he is an interrupter at Round r. We argue by induction. Suppose, at matching

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β, there is an interrupter i_1 of Round 1 who is placed to a school s_1 for which he is an interrupter at this round. Note that at matching α_1, all the seats of school s_1 are full. (Otherwise, student i_1 would not be rejected from it at Round 0.) Since matching β Pareto dominates matching α_1, there is a student i_2 who is placed to school s_1 at matching α_1, and who is placed to a school s_2 which is better for him, at matching β. Note again that at matching α_1, all the seats of school s_2 are also full. Then, there is a student i_3 who is placed to school s_2 at matching α_1, and who is placed to a school s_3 which is better for him, at matching β. Continuing in a similar way, we conclude that because matching β Pareto dominates matching α_1, there is a student i_k who is placed to school s_{k-1} at matching α_1, and who is placed to school s_1 which is better for him, at matching β. That is, there is a cycle of students $(i_1, i_2, \ldots, i_k)$, $k \geq 2$, such that each student prefers the school the next student in the cycle (for student i_k it is i_1) is placed to at matching α_1, to the school he is placed to at the same matching. Let us consider the DA algorithm run in Round 0 of the algorithm. Let $i \in \{i_1, i_2, \ldots, i_k\}$ be the student in this cycle who is the *last* (or, one of the last, if there are more than one such students) to apply to the school, say school s, he is placed to at the end of this round. Then the student in the above cycle who prefers school s to the school he is placed to at matching α_1, was rejected from there at an earlier step. Then, when student i applies to school s, all the seats are already full and since student i is placed to this school at the end of the round, some student i' is rejected. Then, student i' is an interrupter for school s. Furthermore, by the assumption about student i, student i' is rejected from school s, at a step later than the step the interrupter i_1 is rejected from school s_1 for which he is an interrupter. But then, student i_1 cannot be an interrupter of Round 1.

Suppose that at matching β, no interrupter of some Round k, $0 \leq k \leq r-1$, is placed to the school for which he is an interrupter at Round k. We want to show that at matching β, no interrupter of Round r is placed to the school for which he is an interrupter at Round r. Consider matching α_r. Since β Pareto dominates each matching α_r, using the same argument as in the previous paragraph, there is a cycle of students $(i'_1, i'_2, \ldots, i'_k)$ such that each prefers the school the next student in the cycle (for student i'_k it is i'_1) is placed to at matching α_r, to the the school he is placed to at the same matching. Furthermore, due to our supposition about the interrupters

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of earlier rounds, none of the students in this cycle is an interrupter for the school he prefers. Then, for each of the students in the cycle there is a corresponding step of the DA algorithm run in Round $r - 1$ such that he is rejected from the school he prefers, at that step. But then, we can again apply the same argument we used in the previous paragraph to conclude that student i_1' cannot be an interrupter of Round r.

At the end of Round R of the algorithm, there are no interrupters left and we obtain the matching α. Since matching β Pareto dominates matching α, there is again a cycle $(i_1'', i_2'', \ldots, i_k'')$, $k \geq 2$, of students each of whom prefers the school the next student in the cycle (for student i_k'' it is i_1'') is placed to at matching α to the the school he is placed to at the same matching. Note that by what we just proved in the previous paragraph, no student in $\{i_1'', i_2'', \ldots, i_k''\}$ can be an interrupter at any round for the school the next student in the cycle is placed to at matching α. Therefore, each student in $\{i_1'', i_2'', \ldots, i_k''\}$ applies to the the school the next student in the cycle is placed to at a step of the DA algorithm run in Round R. Let $i'' \in \{i_1'', i_2'', \ldots, i_k''\}$ be the student in this cycle who is the *last* to be rejected from the school the next student in the cycle is placed to. When student i'' applies to the school he is placed to at the end of Round R, because the student in $\{i_1'', i_2'', \ldots, i_k''\}$ who prefers this school to the school he is placed to was rejected from here at an earlier step, there is a student i''' who is rejected from this school. Then student i''' is an interrupter for this school, contradicting this round being the last round. **Q.E.D.**

Proof of Proposition 3. Consider a problem, and take some student i. We contrast the rounds of the EADAM algorithm applied to this problem when student i consents with those when he does not. If student i is never identified as an interrupter when he consents, then clearly his placement is the same in the case when he does not consent. In the case he consents, suppose student i is identified as an interrupter for some school s at some Round t, $t \geq 1$ for the *first* time. Note that the rounds of the EADAM algorithm when student i consents (case 1) are identical to the rounds of the EADAM algorithm when he does not consent (case 2) until Round t. Note also that for both cases when the DA algorithm is run at the end of Round $t - 1$, the steps of the DA algorithm are also identical until the step at which student i applies to school s. Let us call it step k.

Comparing the steps of the DA algorithm after step k for the two cases, in the first case at step $k + 1$ student i applies to his next choice school right after school s. In the second case at step $k + 1$ student i applies to school s and gets rejected from it at a later step, and ends up at the same school as in case 1. In order for student i to have a different placement under EADAM in the second case, the consenting interrupter(s) identified in Round $t + 1$ must be different than the consenting interrupter(s) identified in Round $t + 1$ of the first case. Since the two cases are identical until step k of the DA algorithm, any possible new interrupter (as compared to case 1) that could be identified in Round $t + 1$ of the second case needs to be some student j_1 who is rejected from some school s_1 due to the application of student i to school s at step k which causes some student j_0 to be rejected from school s and apply to school s_1 (causing j_1 to be rejected, and giving rise to the interrupting pair (j_1, s_1)) or, some student j_2 who is rejected from some school s_2 because of the application of student j_1 to it after being rejected from school s_1 (giving rise to the interrupting pair (j_2, s_2)) or, some student j_3 who is rejected from some school s_3 because of the application of student j_2 after being rejected from school s_2 to it (giving rise to the interrupting pair (j_3, s_3)) and so on. In other words, the new interrupter must be a student that is displaced from a school because of the rejection chain initiated by the application of student i to school s at step k. The new interrupting pair, say (j, s') is identified at Round $t + 1$, and school s' is removed from the preferences of student j. In order for this to affect the placement of student i at the end of Round $t + 1$, it must be that when student j applied to school s' in the DA algorithm run at the end of Round t, he must indeed be initiating a rejection chain which displaced student i and caused him to apply to school s. But then, the aforementioned rejection chain initiated by student i (through his application to school s) is in fact part of a larger rejection chain which was initiated by student j (through his application to school s'). In other words, in the DA algorithm run at the end of Round t, student j applies to school s' and initiates a rejection chain that causes student i to initiate a rejection chain which eventually causes student j to be rejected from school s. Then after school s' is removed from the preferences of student j at Round $t + 1$ and the DA algorithm is run, by applying to his next choice school after school s' student j again initiates a chain that causes the tentatively placed student at school s to be rejected, which in turn causes

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the tentatively placed student at school s' to be rejected, which in turn causes student i to be rejected until he applies to school s from which he is also rejected. This brings us back to the situation in case 1. (This is very much similar to the situation analyzed in Example 4 when one considers the earliest interrupter to solve the inefficiency problem.)

Thus not consenting does not change the placement of student i at Round $t + 1$. Repeatedly applying the same reasoning to the remaining rounds for student i, we conclude that not consenting does not change his placement under EADAM.

Q.E.D.

Proof of Theorem 2. Before we prove Theorem 2, following Roth and Rothblum (1999) and Ehlers (2008), we formalize the discussion in the main text.[26] In this strategic setup, each student is now a player: Given $s \in S$, let $\mathcal{B}_s$ be the class of all strict priority orders for school s. Given $i \in I$, let $\mathcal{P}_i$ be the class of all strict preferences for student i. Let $\mathcal{X}_{-i} \equiv (\mathcal{B}_s)_{s \in S} \times (\mathcal{P}_{i'})_{i' \in I \setminus \{i\}}$. A random (school choice) problem is a probability distribution $\widetilde{P}_{-i}$ over $\mathcal{X}_{-i}$. Here, $\widetilde{P}_{-i}$ is interpreted as student i's information (or, his belief) about the stated preferences of the other students and the priority orders for all schools. Let $\mathcal{A}$ be the set of all matchings. A random matching $\widetilde{a}$ is a probability distribution over $\mathcal{A}$. Let $\widetilde{a}(i)$ be the distribution which $\widetilde{a}$ induces on the set of student i's placements S. Let φ be a mechanism. Given a problem (P_i, P_{-i}) where $P_{-i} \in \mathcal{X}_{-i}$, let $\varphi(P_i, P_{-i})$ be the matching selected by φ for this problem. Also, let $\varphi(P_i, P_{-i})(i)$ denote student i's placement at this matching. Given a mechanism φ and a student i with preferences P_i, each random preference profile $\widetilde{P}_{-i}$ induces a random matching $\varphi(P_i, \widetilde{P}_{-i})$ in the following way: For all $a \in \mathcal{A}$, $\Pr\{\varphi(P_i, \widetilde{P}_{-i}) = a\} = \Pr\{\widetilde{P}_{-i} = P_{-i} \text{ and } \varphi(P_i, P_{-i}) = a\}$. Let $\varphi(P_i, \widetilde{P}_{-i})(i)$ be the distribution which $\varphi(P_i, \widetilde{P}_{-i})$ induces over student i's set of placements. Given $i \in I$, P_i, P_i', $P_i'' \in \mathcal{P}_i$, and a random preference profile $\widetilde{P}_{-i}$, we say that strategy P_i' **stochastically** P_i-**dominates** strategy P_i'' if for all $s \in S$, $\Pr\{\varphi(P_i', \widetilde{P}_{-i})(i) \ R_i \ s\} \geq \Pr\{\varphi(P_i'', \widetilde{P}_{-i})(i) \ R_i \ s\}$.

We consider a model where a student cannot distinguish between two schools (i.e., is not sure about how other students rank the two schools). In such a case, we say that his information about

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[26]We note two differences between our model and that of Ehlers (2008). First, although his analysis pertains to the the case when each school (hospital in his framework) has unit quota, his results equally apply to our case as well. Second, our model assumes that all schools are acceptable for every student.

the two schools is *symmetric*. Then , such a student believes that any problem is equally likely as its symmetric problem in which the roles of the two schools are exchanged. Formally (Roth and Rothblum 1999), given $i \in I$, $P_i \in \mathcal{P}_i$, and $s, s' \in S$, let $P_i^{s \leftrightarrow s'}$ denote the preferences in which the positions of s and s' are exchanged and the other positions in P_i are unchanged. Let $P_{-i}^{s \leftrightarrow s'}$ denote the profile such that each student $i' \in N \backslash \{i\}$ exchanges the positions of s and s' in his preferences, schools s and s' exchange their priority orders and capacities (i.e., $\succ_s$ becomes the priority order and q_s the capacity of school s' in $P_{-i}^{s \leftrightarrow s'}$, and $\succ_{s'}$ becomes the priority order and $q_{s'}$ the capacity of school s in $P_{-i}^{s \leftrightarrow s'}$), and the priority orders and the capacities of the other schools remain unchanged. Given $i \in I$ and $s, s' \in S$, student i's information for schools s and s' is **symmetric** if P_{-i} and $P_{-i}^{s \leftrightarrow s'}$ are equally probable, i.e., $\Pr\{\widetilde{P}_{-i} = P_{-i}\} = \Pr\{\widetilde{P}_{-i} = P_{-i}^{s \leftrightarrow s'}\}$.[27]

Ehlers (2008) gives two conditions that are sufficient for a mechanism to be immune to strategic behavior due to a switch of two alternatives in the preferences of a student when he has limited information about other students. That is, if a student's information for two schools is symmetric, then under any mechanism satisfying the two conditions, it is never beneficial for him to switch the true ranking of those two schools in the preferences he states.

The two conditions Ehlers proposes are "anonymity" and "positive association." Anonymity requires that the mechanism should treat all schools equally. That is, the names of schools should not matter. Next, we formalize these two conditions (given in Theorem 3.1 of Ehlers). Given $a \in \mathcal{A}$ and $s, s' \in S$, let $a^{s \leftrightarrow s'}$ denote the matching such that for all $i \in I$, (i) if $a(i) \notin \{s, s'\}$, then $a^{s \leftrightarrow s'}(i) = a(i)$, (ii) if $a(i) = s$, then $a^{s \leftrightarrow s'}(i) = s'$, and (iii) if $a(i) = s'$, then $a^{s \leftrightarrow s'}(i) = s$.

ANONYMITY. For all $i \in I$, all $P_i \in \mathcal{P}_i$, all $P_{-i} \in \mathcal{X}_{-i}$, and all $s, s' \in S$, if $\varphi(P_i, P_{-i}) = a$, then $\varphi(P_i^{s \leftrightarrow s'}, P_{-i}^{s \leftrightarrow s'}) = a^{s \leftrightarrow s'}$.

Next we define the second condition of Ehlers. It says that given a student i with preferences P_i, if the position of a school s that student i is placed to is exchanged with that of another school

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[27]It should be noted that it may not be realistic to assume that student i's information for schools s and s' is symmetric when the two schools have apparent differences in the number of available seats. Thus our analysis is more sensible when one also assumes that the two schools (that are viewed to be symmetric) have comparable capacities.

s' which he prefers to s, then the student's placement should not change.

POSITIVE ASSOCIATION. For all $i \in I$, all $P_i \in \mathcal{P}_i$, all $P_{-i} \in \mathcal{X}_{-i}$, and all $s, s' \in S$, if $\varphi(P_i, P_{-i})(i) = s$ and $s' \; P_i \; s$, then $\varphi(P_i^{s \leftrightarrow s'}, P_{-i})(i) = s$.

Suppose that $\{S_1, S_2, \ldots, S_m\}$ is a partition of S such that each student's preferences and information satisfy the conditions given in Theorem 2. Then the following proposition is a straightforward corollary of Theorem 3.1 of Ehlers (2008) for our framework.

PROPOSITION A.1 (Ehlers 2008). *Consider a student i with true preferences P_i and information $\widetilde{P}_{-i}$ satisfying the conditions in Theorem 2. Under any mechanism satisfying anonymity and positive association, the strategy P_i stochastically P_i-dominates any other strategy P_i' that ranks every school in S_r above every school in S_k for all $r < k$.*

PROPOSITION A.2. *Given a student i with preferences P_i, let P_{-i} be a realization of $\widetilde{P}_{-i}$. Suppose that student i is placed to some school $x \in S_r$ at the problem $P = (P_i, P_{-i})$ under EADAM. Suppose student i considers submitting the preference list $P_i^{s \leftrightarrow s'}$ in which the positions of two schools s and s' (that may or may not belong to the same quality class) are switched. Suppose that student i is placed to some school y at the problem $(P_i^{s \leftrightarrow s'}, P_{-i})$ under EADAM. If school x is not in the same quality class with s or s' (i.e., $S_r \cap \{s, s'\} = \emptyset$), then we have $x \; R_i \; y$.*

Before proving Proposition A.2, we first make a useful observation.

LEMMA 3. *At problem P the school to which student i is placed under SOSM and the school he is placed to under EADAM belong to the same quality class.*

Proof. Note first that any rejection chain that causes a student to be identified as an interrupter for some school under EADAM contains only those schools that belong to the same quality class. This is because if some student j gets rejected, say from some school $a \in S_k$ with $1 \le k < m$, and if the next school he applies to, say some $b \in S_l$, belongs to a lower quality class $l > k$, then by the restriction on the preferences of the remaining students, no student who could be rejected from the lower quality school b would ever apply to a school in S_k. By similar reasoning, no subsequently rejected student would ever apply to a school in S_k.

Suppose by contradiction that EADAM places student i to some school $x \in S_r$ that belongs to a different quality class than his SOSM placement, say some school $y \in S_p$ with $p > r$. This means that there is some Round t, $t \geq 1$ of the EADAM algorithm applied to this problem such that student i is placed to school x at the end of this round. By the argument in the previous paragraph, this means that at the end of Round $t - 1$, student i must be placed to a school also from S_r. Again applying the same reasoning, at the end of any previous round student i must indeed be placed to a school from S_r. This contradicts the fact that student i is placed to school y at the end of Round 0.

<p align="center">**Q.E.D.**</p>

Proof of Proposition A.2. Suppose without loss of generality that $s \; P_i \; s'$. We consider three cases. *Case 1.* If $x \; P_i \; s$, then by the working of the SOSM algorithm, Round 0 of the EADAM algorithm is the same for both P and (P_i', P_{-i}). Clearly, the remaining rounds are also the same. *Case 2.* If $s' \; P_i \; x$, then in Round 0 of both problems student i applies to the same schools (where the application ordering is reversed for s and s'), and ends up at school x at the end of this round. By Lemma 3 this means that student i cannot possibly initiate a new rejection chain at (P_i', P_{-i}) that might benefit him. *Case 3.* If $s \; P_i \; x \; P_i \; s'$, then Round 0 of both problems are identical until student i applies to school s for P and to school s' for (P_i', P_{-i}). By applying to school s' at (P_i', P_{-i}), by the argument in the first paragraph of the proof of Lemma 3, he cannot initiate a new rejection chain that can possibly affect student placements at schools in S_r. If student i ends up at school s' at the end of Round 0, by the reasoning in the first paragraph of the proof of Lemma 3, his EADAM placement is also s'. If student i gets subsequently rejected from school s', then the next schools he applies to in Round 0 are identical to those he applies to at (P_i', P_{-i}) after school s. Thus his placement does not change.

<p align="center">**Q.E.D.**</p>

LEMMA 4. *EADAM satisfies anonymity and positive association.*

Proof. It is easy to see that EADAM satisfies anonymity. We show that EADAM satisfies positive association. First, note that SOSM already satisfies this requirement. This directly

follows from its *strategy-proofness*. An indirect way to see this is the following: When the ranking of the school he is placed to (under SOSM) improves in the preferences of a student i, because other students' preferences (as well as the priority orders) remain unchanged, at the new problem (i.e., at the problem where the two positions of the two schools in student i's preferences are exchanged) no student applies to a school he did not apply to at the initial problem (i.e., at the problem where the two positions of the two schools in student i's preferences are not exchanged) and moreover, a student may even apply to less schools now. Note further that the DA algorithms for the two problems are identical until the step of the DA algorithm (applied to the new problem) at which student i applies to the school he is placed to at the initial problem. Then student i cannot be rejected from that school and he is placed to the same school at the new problem. An important observation here is that the school a student is placed to is not affected by which schools he was rejected from before applying to that school.[28]

Suppose a student i is placed to a school s at the EADAM outcome. By Theorem 1, at Round 0 he must have applied to school s at some step of the DA algorithm, from which he could have been rejected at a later step in that round (in the case his EADAM placement is better than that of SOSM). Loosely, if student i was rejected from school s at some step of the DA algorithm run in Round 0, then this means he is placed back at school s at a later round, because of the schools he applied to after school s that allow him to be part of a rejection chain. That is, after being rejected from school s student i must have been part of a rejection chain that eventually causes an interrupter to be identified at some round of EADAM, which in turn, must have enabled each participant of that chain to end up at a better school for him. Thus student i's choices after school s matter for determining his placement at the EADAM outcome.

When we exchange the positions of schools s and s' where $s'\, P_i\, s$ in student i's preferences, the ranking of the schools which he used to rank worse than school s are unaffected. We consider two cases. We refer to the problem where the positions of schools s and s' are not exchanged as the *initial problem* and to the other one as the new problem:

Case 1. At the end of Round 0 of the initial problem, student i is placed to school s : That is for

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[28]This may affect the placement of other students, though.

the initial problem EADAM and SOSM place student i to the same school, namely school s. Since SOSM satisfies positive association, at the end of Round 0 of the EADAM algorithm applied to the new problem, student i is also placed to school s. (Thus, at no round of the EADAM algorithm applied to the new problem student i ever applies to a school worse than school s for him.) Then by Lemma 1, his placement under EADAM for the new problem is at least as good as school s for him. Note that the steps of the DA algorithm applied to the initial and the new problem are identical until student i applies to school s. At the initial problem, student i failed to be part of a rejection chain that would (under EADAM) in turn place him to a school better than s. Since at the new problem the schools he applies to before applying to school s are the same, and since the preferences of all the remaining students are unchanged, he still fails to participate in such a chain and thus his placement does not change.

Case 2. At the end of Round 0 of the initial problem, student i is placed to a worse school for him than school s: By the strategy-proofness of SOSM, the school student i is placed to at the end of Round 0 is the same for both problems. Furthermore, the schools student i applies to in this round are also the same for both problems (though, his applications are not in the same order). This means all other students' placements are also the same at the end of Round 0 for both problems.[29]

Hence at Round 0 of the EADAM algorithm applied to the new problem, student i is again rejected from school s, and after being rejected from school s, because other students' preferences are unaffected, student i is also rejected from the schools he used to rank between schools s and s' (and, he cannot initiate any rejection chain that would cause any student to be identified as an interrupter for these schools since if this was possible, it would also be the case at the initial problem). Further, at the initial problem, after being rejected from school s, student i must have been part of a rejection chain that eventually causes an interrupter to be rejected from the school for which he is an interrupter, which in turn, must have caused each participant of that chain

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[29]For if a student had ended up at a better (worse) school for him at the new problem, this would mean that there is another student who, at the new problem, did not apply to (applied to) a school he applied to (did not apply to) at the initial problem. Iteratively applying this argument, since the schools student i applies to in this round are also the same for both problems, we would reach a contradiction.

to end up at a better school for himself. At the new problem the only way for student i not to be identified as a participant of the same chain at some round of the EADAM algorithm is when a new rejection chain appears that causes some participant of this chain to be identified as an interrupter. But note that at Round 0 of the EADAM algorithm of both problems, each student applies to the same schools. Then there cannot be any student who would start a new rejection chain at the new problem. Then at some round of the EADAM algorithm applied to the new problem student i is again part of a rejection chain that eventually causes an interrupter to be rejected from the school for which he is an interrupter, which in turn, causes student i to be placed back to school s.

<div align="center">

Q.E.D.

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Now we are ready to complete the proof of Theorem 2. Let i be a student with true preferences P_i and P_{-i} a realization of $\widetilde{P}_{-i}$. Suppose that student i is placed to some school $x \in S_r$ at the problem $P = (P_i, P_{-i})$ under EADAM. Given two schools s and s' with $s\ P_i\ s'$, consider the alternative strategy $P_i^{s \leftrightarrow s'}$ for student i. Suppose that student i is placed to some school y at the problem $(P_i^{s \leftrightarrow s'}, P_{-i})$. If $S_r \cap \{s, s'\} = \{s\}$, then by similar reasoning to Case 3 in the proof of Proposition A.2, we have $x\ R_i\ y$. If $\{s'\} \subset S_r \cap \{s, s'\}$, then by the working of EADAM strategy $P_i^{s \leftrightarrow s'}$ is equivalent for student i to some strategy that P_i' that ranks every school in S_r above every school in S_k for all $r < k$. Then Propositions A.1, Lemma 4, and Proposition A.2 together imply that for any $s, s' \in S$, strategy P_i stochastically P_i−dominates strategy $P_i^{s \leftrightarrow s'}$. Using a simple induction argument (similar to the proof of part (b) of Theorem 3.1 of Ehlers), we conclude that strategy P_i stochastically P_i−dominates any other strategy $P_i' \in \mathcal{P}_i$.

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Q.E.D.

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