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Irreversible investment and Knightian uncertainty ☆

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Abstract

When firms make a decision about irreversible investment, they may not have complete confidence about their perceived probability measure describing future uncertainty. They may think other probability measures perturbed from the original one are also possible. Such uncertainty, characterized by not a single probability measure but a set of probability measures, is called "Knightian uncertainty." The effect of Knightian uncertainty on the value of irreversible investment opportunity is shown to be drastically different from that of traditional uncertainty in the form of risk. Specifically, an increase in Knightian uncertainty decreases the value of investment opportunity while an increase in risk increases it.

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1. Introduction and summary

The investment decision of any firm typically involves three features. First, future market conditions are uncertain. Second, the cost of investment is sunk and thus investment is irreversible. Third, investment opportunity does not vanish at once and when to invest becomes a critical decision. This irreversibility of investment under uncertainty and resulting optimal

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investment timing problem have attracted considerable attention in recent years, especially after McDonald and Siegel [15] successfully applied financial option pricing techniques to this problem and Dixit and Pindyck [4] related option-theoretic results to neoclassical investment theory.

Most irreversible investment studies, however, assume more than that future market conditions are uncertain. In these studies, future uncertainty is characterized by a certain probability measure over states of nature. This amounts to assuming that the firm is *perfectly certain* that future market conditions are governed by this particular probability measure. However, this assumption may be farfetched: the firm may not be so sure about future uncertainty. It may think other probability measures are also likely and have no idea of the relative "plausibility" of these measures. Uncertainty that is *not* reducible to a single probability measure and thus characterized by a *set* of probability measures is often called *Knightian uncertainty* (see [14,12,13]), or ambiguity in some cases. In contrast, uncertainty that *is* reducible to a single probability measure with known parameters is referred to as *risk* . That is, a firm may face Knightian uncertainty in contemplating investment, facing not a single probability measure but a set of probability measures.

The purpose of this paper is to show that the effect of uncertainty on the value of irreversible investment opportunity differs drastically between risk and Knightian uncertainty. Specifically, the standard result that increase in uncertainty increases the value of irreversible investment opportunities is reversed if uncertainty is not risk but Knightian uncertainty. That is, an increase in Knightian uncertainty (properly defined) reduces the value of an irreversible investment opportunity, while the opposite is true for an increase in risk in the form of an increase in variance. In contrast, both of them have the same effect on the value of waiting: they increase the value of waiting and make it more likely.

In this paper, we take a patent as an example of irreversible investment. To highlight the effect of Knightian uncertainty, the firm is assumed to be risk-neutral but *uncertainty-averse* in the sense that it computes the expected profit by using the "worst" element in the set of the probability measures characterizing Knightian uncertainty and chooses its strategy to maximize it (maximin criterion). [1]

Following the standard procedure of irreversible investment studies, we assume that (1) to utilize a patent, the firm has to build a factory and construction costs are sunk after its completion, and (2) the profit flow after the construction is characterized by a geometric Brownian motion with a drift. Then, the firm first calculates the value of the utilized patent, and then contemplates when to build a factory by taking into account the value of the utilized patent and the cost of investment. The firm's problem is thus formulated as an optimal stopping problem in continuous time. [2]

Unlike the standard case, however, we assume that the firm is not perfectly certain that the profit flow is generated by a particular geometric Brownian motion with say, variance σ^2 and drift μ, or equivalently, by a probability measure underlying this geometric Brownian motion,

[1] For axiomatization of such behavior, see Gilboa and Schmeidler [9]. Such behavior is also closely related to the one represented by Choquet-expected-utility maximization. See Schmeidler [21, first appeared in 1982 as a working paper] and Gilboa [8].

[2] The standard procedure is to apply financial option pricing techniques to this problem, exploiting the fact that an *un*-utilized patent can be considered a call option whose primal asset is a *utilized* patent that generates a stochastic flow of profits, and whose exercise price is a fixed cost of building a factory to produce patented products. (For example, see [4].) This approach and the optimal stopping approach are two ways of formulating the same problem and produce the same result.

say P. The firm may think that the profit flow is generated by other probability measures slightly different from P. The firm has no idea about which of these probability measures is "true." Thus, the firm faces Knightian uncertainty with respect to probability measures characterizing the profit flow.

We assume that the firm thinks these probability measures are not far from P. Firstly, we assume that these probability measures agree with P with respect to zero probability events. (That is, if a particular event's probability is zero with P, then it is also zero with these probability measures.) Then, these probability measures can be shown as a perturbation of P by a particular "density generator." Second, the deviation of these probabilities from P is not large in the sense that the corresponding density generator's move is confined in a range, $[-\kappa, \kappa]$, where κ can be described as a degree of this Knightian uncertainty. This specification of Knightian uncertainty in continuous time is called κ-ignorance by Chen and Epstein [3] in a different context.

These two assumptions, though they seem quite general, have strong implications. Under the first assumption, for each of the probability measures constituting the firm's Knightian uncertainty, the profit flow is characterized by a "geometric Brownian motion" of the same variance σ^2 with respect to this probability measure. Thus, "geometric Brownian motions" corresponding to these probability measures differ only in the drift term. (In fact, this is a direct consequence of well-known Girzanov's theorem in the literature of mathematical finance. See for example, [11].) Under the second assumption, the minimum drift term among them becomes $\mu - \kappa\sigma$. Note that the uncertainty-averse firm evaluates the present value of the patent according to the "worst" scenario. Loosely speaking, this amounts to calculating the patent's value using the probability measure corresponding to a geometric Brownian motion with variance σ^2 and minimum drift $\mu - \kappa\sigma$. Thus, an increase in κ, the degree of Knightian uncertainty, leads to a lower value of the utilized patent at the time of investment, since it is evaluated by a less favorable Brownian motion process governing the profit flow from the utilized patent. Consequently, the value of the unutilized patent is also reduced.

This is in sharp contrast with the positive effect of an increase in risk (that is, an increase in σ) on the value of the unutilized patent, when there is no Knightian uncertainty. An increase in σ under no Knightian uncertainty implies, when the firm waits, it can undertake investment only when market conditions are more favorable than before (since it does not have to undertake investment when market conditions are less favorable). Consequently, an increase in σ increases the value of the unutilized patent.

Despite such differences, both an increase in risk and in Knightian uncertainty similarly raise the value of waiting and thus make the firm more likely to postpone investment. However, the reason for waiting is critically different. An increase in risk (σ) under no Knightian uncertainty leaves the value of a utilized patent unchanged but increases the value of an unutilized patent, and thus makes waiting more profitable. An increase in Knightian uncertainty (κ) reduces both the value of the utilized patent and that of the unutilized patent, but it lowers the former more than the latter. This is because the value of the unutilized patent depends not only on the proceeds from undertaking investment (the utilized patent), but also on the proceeds from not undertaking investment, which is independent of the value of the utilized patent. Since the value of the utilized patent is reduced more than that of the unutilized patent, the firm finds waiting more profitable.

While in the current paper an increase in Knightian uncertainty raises the value of waiting, the opposite holds true in Nishimura and Ozaki's [19] search model, which is set up in a discrete-time infinite-horizon framework. They show that an increase in Knightian uncertainty lowers the reservation wage and hence shortens waiting. The value of waiting is thus reduced.

Although both the job search model in Nishimura and Ozaki [19] and the irreversible investment models in the current paper are formulated as optimal stopping problems, there is a fundamental difference between the two in the nature of uncertainty. In Nishimura and Ozaki [19], the decision-maker determines when she stops the search and resolves uncertainty. Thus, an increase in Knightian uncertainty makes the uncertainty-averse decision-maker more likely to stop the search and to resolve uncertainty. In contrast, in the current paper, the decision-maker contemplates when to begin investment and face uncertainty. Thus, an increase in Knightian uncertainty makes the uncertainty-averse decision-maker more likely to postpone investment to avoid facing uncertainty.

This paper is organized as follows. In Section 2, we present a simple two-period, two-state example and explain intuitions behind the result of this paper. In Section 3, we formulate the firm's irreversible investment problem in continuous time. In the same section, we formally define Knightian uncertainty in continuous time, derive an explicit formula for a utilized patent, and investigate the optimal investment timing problem. In Section 4, we conduct a sensitivity analysis and present the main result of this paper: differing effects of uncertainty between an increase in risk and Knightian uncertainty. Appendix A contains derivations of important formulae in Section 3. The concept of rectangularity of a set of density generators, [3] of which the κ-ignorance is a special case, plays an important role in our analysis. Appendix B provides some results on rectangularity for the sake of readers' convenience and to make exposition self-contained.

2. A two-period, two-state illustrative example

This section offers an illustrating example of the differing effects of risk and Knightian uncertainty. The example is a simple patent-pricing one and essentially the same as the widget-factory example of Dixit and Pindyck [4, Chapter 2]. We compare the effect of an increase in risk on the value of a patent with that of an increase in Knightian uncertainty. We show that an increase in Knightian uncertainty reduces the value of the patent, while an increase in risk increases its value. In contrast, both have the same effect on the value of waiting: they make waiting more likely.

Consider a risk-neutral firm contemplating whether or not to buy a patent (or a venture firm, vacant lot, etc). [4] After purchasing the patent, the firm has to spend a large amount of money to utilize it. The firm may have to build a new factory to produce patented products. The factory is product-specific and cannot be used for other purposes. Thus, the investment is irreversible and becomes sunk afterward.

Suppose that there are two periods, period 0 and period 1. There is no uncertainty in period 0 and the operating profit from producing and selling the products is π_0. There is uncertainty in period 1, where the state is either *boom* (b) or *slump* (s). Let π_1 be the operating profit in period 1, which equals π_b in boom and π_s in slump.

In order to utilize this patent, the firm has to build a factory to produce the product. Let I be the cost of building the factory. We assume that $\pi_s < I < \pi_b$. The firm has a choice between building the factory in period 0 and in period 1. Let p_b be the probability of boom in period 1 and

[3] These concepts are developed and discussed in Chen and Epstein [3].

[4] Nishimura and Ozaki [19] discuss the effect of an increase in both risk and Knightian uncertainty in a discrete-time search model where the decision-maker is both risk-averse and uncertainty-averse.

let r be the rate of interest. Then, if the firm builds a factory in period 0, the expected discounted cash flow from this patent is

$$(\pi_0 - I) + \frac{1}{1+r}\left(p_b \pi_b + (1-p_b)\,\pi_s\right), \tag{1}$$

while if it postpones investment until period 1, the expected discounted cash flow of this patent is

$$\frac{p_b}{1+r}\left(\pi_b - I\right), \tag{2}$$

since period 1's cash flow is $\pi_b - I$ in boom and 0 in slump (that is, the firm does not want to build the factory in slump since $\pi_s < I$).

If the firm is perfectly certain that the boom probability is p_b, the model is exactly the same as the Dixit–Pindyck example. The firm compares (1) and (2) and determines the optimal timing for investment. Then, the value of the unutilized patent is determined accordingly. Thus, if

$$\pi_0 - \left(1 - \frac{p_b}{1+r}\right) I + \frac{1}{1+r}\left(1-p_b\right)\pi_s < 0 \tag{3}$$

holds, then postponement is the optimal strategy. If otherwise, investment in period 0 is optimal. Consequently, if (3) holds true (that is, postponement is optimal), then (2) is the value of the patent at period 0. If not, (1) is the value of the patent at period 0.

In the real world, however, it is highly unlikely that the firm is absolutely certain about boom probability. The firm may think that p_b is likely to be boom probability, but at the same time may consider that another probability, say, p'_b, is also plausible. Moreover, the firm may not be at all certain whether a particular boom probability is "more plausible" than others. In sum, the firm may have a set of boom probabilities, instead of having one boom probability as in the Dixit–Pindyck example. Moreover, the firm may not be certain about the "relative plausibility" of these boom probabilities. Such a multiplicity of probability distributions is called *Knightian uncertainty*.

Let $\mathscr{P}$ be a compact set of boom probabilities that the firm thinks plausible. It is known (see [9]) that in multiple-probability-measure cases of this kind, if the firm acts in accordance with certain sensible axioms, then its behavior can be characterized as being *uncertainty-averse*: when the firm evaluates its position, it will use a probability corresponding to the "worst" scenario. This means (1) is replaced by

$$\begin{aligned}(\pi_0 - I) &+ \frac{1}{1+r}\min_{p_b\in\mathscr{P}}\left(p_b\pi_b + (1-p_b)\,\pi_s\right)\\ &= (\pi_0 - I) + \frac{1}{1+r}\left[\pi_s + \left(\min_{p_b\in\mathscr{P}} p_b\right)(\pi_b - \pi_s)\right]\end{aligned} \tag{4}$$

since $\pi_b > \pi_s$, and (2) is now

$$\frac{1}{1+r}\min_{p_b\in\mathscr{P}} p_b\left(\pi_b - I\right) = \frac{1}{1+r}\left[\left(\min_{p_b\in\mathscr{P}} p_b\right)(\pi_b - I)\right] \tag{5}$$

since $\pi_b > I$. Consequently, the postponement criterion is now

$$(\pi_0 - I) + \frac{\pi_s}{1+r} + \frac{1}{1+r}\left(\min_{p_b\in\mathscr{P}} p_b\right)(I - \pi_s) < 0. \tag{6}$$

Let us now consider in turn an increase in risk and uncertainty. To simplify exposition, let us further assume $p_b = \frac{1}{2}$ and $\mathscr{P} = \left[\left(\frac{1}{2}\right) - \varepsilon, \left(\frac{1}{2}\right) + \varepsilon\right]$. Here, $\varepsilon \in (0, \frac{1}{2})$ is a real number which can be described as the degree of "contamination" of confidence in $p_b = \frac{1}{2}$. We hereafter call this specification the *ε-contamination.* [5] An increase in ε can be considered as an increase in Knightian uncertainty. [6]

An increase in risk is characterized by a mean-preserving spread of the second-period operating profit π_1. Suppose that there is no Knightian uncertainty and risk is increased so that (π_s, π_b) is now spread to $(\pi_s - \gamma, \pi_b + \gamma)$. It is evident from (2) that the mean-preserving spread always increases the value of an *un*utilized patent, or the value of a patent when the firm postpones investment. Intuitively, this is because an increase in risk implies that, when the firm waits, it can undertake investment only when market conditions are more favorable than before. At the same time, it leaves the value of the utilized patent unchanged (see (1)). Consequently, the value of investment opportunity increases with an increase in risk.

Furthermore, (1) and (2) show that the firm is more likely to find it profitable to postpone investment than before when the mean-preserving spread takes place. This is also clear from (3). Its left-hand side is decreased by the mean-preserving spread and hence the criterion for postponement is more easily satisfied.

Next, suppose that there is Knightian uncertainty, and thus the value of the utilized patent is (4) and that of the unutilized patent is (5). Suppose further that Knightian uncertainty is increased in the sense that the degree of confidence contamination, ε, is increased. [7] It is plain to see from (4) and (5) that an increase in ε always decreases the value of both the utilized and unutilized patent. Therefore, the value of investment opportunity *de*creases rather than increases with increases in Knightian uncertainty. Intuitively, an increase in Knightian uncertainty leads to a lower value of the utilized patent at the time of investment since it is evaluated by a less favorable profit from the utilized patent, and hence, the value of the unutilized patent is also reduced.

It is also evident from (4) and (5) that reduction in value is larger in the utilized patent than in unutilized patent, implying that the firm is again more likely to postpone investment than before. This is also clear from (6). Its left-hand side is decreased by an increase in ε and hence the criterion for postponement is more easily satisfied.

We therefore observe that an increase in Knightian uncertainty and an increase in risk have opposite effects on the value of an unutilized patent (the investment opportunity) although they have the same effect on the timing of investment. In the following sections, we argue that basically the same observations hold true in general continuous-time models.

[5] The concept of ε-contamination can be applied to multiple-state cases. Let $\mathscr{M}$ be the set of all probability measures over states, and let $P_0 \in \mathscr{M}$. Then, the *ε-contamination* of P_0, $\{P_0\}^\varepsilon$, is defined by

$$\{P_0\}^\varepsilon \equiv \{\,(1-\varepsilon)P_0 + \varepsilon Q \mid Q \in \mathscr{M}\,\},$$

where $\varepsilon \in [0, 1)$. Some behavioral foundation for ε-contamination is provided by Nishimura and Ozaki [20]. Also, Nishimura and Ozaki [19] apply ε-contamination to a discrete-time search model and Nishimura and Ozaki [17] explore learning behavior under ε-contamination.

[6] More generally, an expansion of $\mathscr{P}$ (in terms of set-inclusion) can be interpreted as an increase in Knightian uncertainty. Ghirardato and Marinacci [7] develop the notion of comparative Knightian-uncertainty aversion, or in their terms, "comparative ambiguity aversion," and relate it to this set-expansion, which provides some behavioral foundation of our notion of an increase in Knightian uncertainty.

[7] The following observations still hold with some minor modification (such as a replacement of "increases" by "non-decreases") even if an increase in Knightian uncertainty is modeled more generally by an expansion of $\mathscr{P}$. See footnote 6.

3. A general continuous-time model of irreversible investment under Knightian uncertainty

In this section, the simple two-period, two-state example of building a factory for patented products is generalized to a continuous-time model. In Section 3.1, Knightian uncertainty is introduced into the continuous-time model, and three ways to characterize it are discussed: strong rectangularity, *i.i.d.* uncertainty, and κ-ignorance. κ-ignorance is a special case of *i.i.d.* uncertainty, and *i.i.d.* uncertainty is in turn a special case of strong rectangularity.

In Section 3.2, the value of a utilized patent is derived for the case of strong rectangularity and the optimal investment decision is formulated and characterized in Section 3.3. In Section 3.4, an analytic formula determining the value of an unutilized patent is derived for the case of κ-ignorance and infinite horizon. This formula will be utilized in the sensitivity analysis of Section 4, which examines the effect of risk and Knightian uncertainty.

3.1. Knightian uncertainty in continuous time: strong rectangularity, i.i.d. uncertainty, and $\boldsymbol{\kappa}$-ignorance

As in the simple example of Section 2, it is not likely that the firm has perfect confidence in the probability measure P. There may be other candidate probability measures that the firm considers probable. Thus, the firm faces Knightian uncertainty, in which the firm confronts a set of probability measures instead of a single probability measure P. However, these candidate probability measures are not likely to be wildly different from P, but it is rather a small deviation from P, like ε-contamination in the example of the previous section.

To model this type of Knightian uncertainty in the continuous-time framework, we follow Chen and Epstein [3] who characterized Knightian uncertainty in continuous time in a different context. Firstly, we assume that the firm considers only a set of probability measures that have perfect agreement with P with respect to zero probability events. This amounts to assuming that probability measures we consider are absolutely continuous with respect to P and one another. Two probability measures are called equivalent if they are absolutely continuous with each other. Thus, we are concerned with probability measures equivalent to P. [8]

It is known (Girsanov's theorem) in the mathematical finance literature that an equivalent measure is generated by a "density generator" from an original probability measure, and that a Brownian motion (with respect to the original probability measure) perturbed by this density generator is still a Brownian motion (with respect to the generated equivalent measure). Moreover, all equivalent measures to P can be generated in this way. This property in turn implies that "geometric Brownian motions" corresponding to probability measures equivalent to P, which describe the movement of profit π_t under these probability measures, are different only in their drift term, not in the volatility term.

Secondly, we assume that the firm considers only small perturbations from P: that is, the density generator is confined to a small range. In particular, we are mostly concerned with a continuous counterpart to the ε-contamination in Section 2, which Chen and Epstein call κ-ignorance. Specifically, κ-ignorance assumes that the density generator moves only in the range $[-\kappa, \kappa]$ and thus κ can be considered as the degree of Knightian uncertainty or ignorance. To clarify the meaning of κ-ignorance, we also consider weaker conditions of *i.i.d.* uncertainty and strong rectangularity.

[8] This is a weaker concept of Knightian uncertainty than that in the discrete-time model (see, for example, [19]).

3.1.1. Density generators and characterization of Knightian uncertainty

Let T be the expiration date of the patent, which is assumed to be finite for the time being. For simplicity, we assume the patent produces no profit after its expiration date T.[9] Later, we consider a case in which T is infinite.

Let $(\Omega, \mathcal{F}_T, P)$ be a probability space and $(B_t)_{0\leqslant t\leqslant T}$ be a standard Brownian motion with respect to P. As a filtration $(\mathcal{F}_t)_{0\leqslant t\leqslant T}$, we take the *standard filtration* for (B_t).[10]

Let us now define the "density generators" that are utilized in formulating Knightian uncertainty in this paper. Let $\mathcal{L}$ be the set of real-valued, measurable,[11] and $(\mathcal{F}_t)$-adapted stochastic processes on $(\Omega, \mathcal{F}_T, P)$ with an index set $[0, T]$, and let $\mathcal{L}^2$ be a subset of $\mathcal{L}$ which is defined by

$$\mathcal{L}^2 = \left\{ (\theta_t)_{0\leqslant t\leqslant T} \in \mathcal{L} \;\middle|\; \int_0^T \theta_t^2\, dt < +\infty \ P\text{-a.s.} \right\}.$$

Given $\theta = (\theta_t) \in \mathcal{L}^2$, we define a stochastic process $\left(z_t^\theta\right)_{0\leqslant t\leqslant T}$ by

$$(\forall t) \quad z_t^\theta = \exp\left(-\frac{1}{2}\int_0^t \theta_s^2\, ds - \int_0^t \theta_s\, dB_s\right), \tag{7}$$

where the stochastic integral, $\int_0^t \theta_s\, dB_s$, is well-defined for each t since $\theta \in \mathcal{L}^2$.[12] A stochastic process $(\theta_t) \in \mathcal{L}^2$ is called a *density generator* if $\left(z_t^\theta\right)$ thus defined is a $(\mathcal{F}_t)$-martingale.[13]

As the name suggests, a density generator generates another probability measure from a given probability measure, and the resulting measure is equivalent to the original measure. To see this, let θ be a density generator and define the probability measure Q^θ by

$$(\forall A \in \mathcal{F}_T) \quad Q^\theta(A) = \int_A z_T^\theta(\omega)\, dP(\omega). \tag{9}$$

Since $\left(z_t^\theta\right)$ is a martingale, $Q^\theta(\Omega) = E^P[z_T^\theta] = z_0^\theta = 1$, and hence, Q^θ is certainly a probability measure and it is absolutely continuous with respect to P. Furthermore, since z_T^θ is strictly positive, P is absolutely continuous with respect to Q^θ. Therefore, Q^θ is equivalent to P. Conversely, any

[9] To incorporate the possibility of after-expiration profit flow is straightforward but makes analysis cumbersome.

[10] A filtration $(\mathcal{F}_t)$ is the *standard filtration* for (B_t) if for each $t \geqslant 0$, $\mathcal{F}_t$ is the smallest σ-algebra that contains all P-null sets and with respect to that $(B_k)_{0\leqslant k\leqslant t}$ are all measurable.

[11] A real-valued stochastic process $(X_t)_{0\leqslant t\leqslant T}$ on $(\Omega, \mathcal{F}_T, P)$ is *measurable* if a function $(t, \omega) \mapsto X_t(\omega)$ is $(\mathcal{B}([0, T]) \otimes \mathcal{F}_T)$-measurable, where $\mathcal{B}([0, T])$ is the Borel σ-algebra on $[0, T]$.

[12] Equivalently, $\left(z_t^\theta\right)_{0\leqslant t\leqslant T}$ is defined as a unique solution to the stochastic differential equation: $dz_t^\theta = -z_t^\theta \theta_t\, dB_t$ with $z_0^\theta = 1$.

[13] A sufficient condition for $\left(z_t^\theta\right)$ to be $(\mathcal{F}_t)$-martingale and thus for (θ_t) to be a density generator is *Novikov's condition*:

$$E^P\left[\exp\left(\frac{1}{2}\int_0^T \theta_s^2\, ds\right)\right] < +\infty \tag{8}$$

(see [11, p. 199, Corollary 5.13]).

probability measure which is equivalent to P can be generated via (9) by some density generator (see [5, p. 289]).

We assume that the firm's set of probability measures describing its Knightian uncertainty consists of probability measures equivalent to P. The standard results described in the previous paragraph then imply that the firm's Knightian uncertainty is characterized as an "expansion" of the set of probability measures from a singleton set $\{P\}$ through density generators.

Let Θ be a set of density generators. We then define the set of probability measures generated by Θ, $\mathscr{P}^{\Theta}$, on $(\Omega, \mathcal{F}_T)$ by

$$\mathscr{P}^{\Theta} = \{ Q^{\theta} \mid \theta \in \Theta \}, \tag{10}$$

where Q^{θ} is derived from P according to (9). Thus, the firm's Knightian uncertainty is characterized by $\mathscr{P}^{\Theta}$ for some Θ.

3.1.2. A corresponding set of stochastic differential equations

Let μ and σ be real numbers. We assume that $\sigma \geqslant 0$ without loss of generality [14] and $\sigma \neq 0$ to exclude a deterministic case. Then, suppose that the operating profit from the utilized patent is a real-valued stochastic process $(\pi_t)_{0 \leqslant t \leqslant T}$ that is generated by a geometric Brownian motion such that

$$d\pi_t = \mu \pi_t \, dt + \sigma \pi_t \, dB_t,$$

where $\pi_0 > 0$ and (B_t) is a Brownian motion with respect to the probability measure P. Since $dB_t^{\theta} = dB_t + \theta_t \, dt$ by Girsanov's theorem, [15] we have for any $\theta \in \Theta$,

$$d\pi_t = (\mu - \sigma\theta_t)\pi_t \, dt + \sigma \pi_t \, dB_t^{\theta}. \tag{12}$$

Thus, (π_t) is also the solution of the stochastic differential equation (12) if Q^{θ} is the underlying probability measure because in this case, $\left(B_t^{\theta}\right)$ is a Brownian motion with respect to Q^{θ} by Girsanov's theorem. Under uncertainty characterized by Θ, the decision-maker considers all stochastic differential equations, (12), with $\theta \in \Theta$ varying. It should be noted that θ affects only the drift term, and not the volatility term, in (12).

Let $\theta \in \Theta$. Then, by (12) and an application of Ito's lemma to the logarithm of π_t by regarding Q^{θ} as the true probability measure, we obtain

$$(\forall t \geqslant 0) \quad \pi_t = \pi_0 \exp\left(\left(\mu - \frac{1}{2}\sigma^2\right) t - \sigma \int_0^t \theta_s \, ds + \sigma B_t^{\theta}\right). \tag{13}$$

3.1.3. Strong rectangularity, i.i.d. uncertainty and κ-ignorance

We assume that the firm considers "small deviation" from the original P. This means that the range that density generators can move is restricted to some "neighborhood" set of P. In particular, we consider a counterpart to the ε-contamination of Section 2, which is called κ-ignorance [3].

[14] If necessary, take $(-B_t)$ instead of (B_t) in the following.

[15] Girsanov's theorem states that if we define, for each $\theta \in \Theta$, a stochastic process $\left(B_t^{\theta}\right)_{0 \leqslant t \leqslant T}$ by

$$(\forall t) \quad B_t^{\theta} = B_t + \int_0^t \theta_s \, ds, \tag{11}$$

then $\left(B_t^{\theta}\right)$ turns out to be a standard Brownian motion with respect to $(\mathcal{F}_t)$ on $(\Omega, \mathcal{F}_T, Q^{\theta})$ [11, p. 191, Theorem 5.1].

It is now necessary to formally define κ-ignorance. However, since some of our results in this paper hold true under weaker conditions than κ-ignorance, it is worthwhile to also define these conditions, called rectangularity and *i.i.d.* uncertainty by Chen and Epstein.

A set of density generators, Θ, is called *strongly rectangular* if there exist a non-empty compact subset $\mathcal{K}$ of $\mathbb{R}$ and a compact-valued, measurable [16] correspondence $K : [0, T] \twoheadrightarrow \mathcal{K}$ such that

$$\Theta = \left\{ (\theta_t) \in \mathcal{L}^2 \;\middle|\; \theta_t(\omega) \in K_t \quad (m \otimes P)\text{-a.s.} \right\}, \tag{14}$$

where m denotes the Lebesgue measure restricted on $\mathcal{B}([0, T])$. [17] Any element of a set Θ defined by (14) satisfies Novikov's condition (8) since $(\forall t)\ K_t \subseteq \mathcal{K}$ and $\mathcal{K}$ is compact, and hence, it is certainly a density generator. We denote by $\Theta^{(K_t)}$ the set defined by (14).

To see an important implication of strong rectangularity, let $0 \leqslant s \leqslant t \leqslant T$, let x be an $\mathcal{F}_T$-measurable random variable and let Θ be a strongly rectangular set of density generators. Then, it holds that

$$\begin{aligned} \min_{\theta \in \Theta} E^{Q^\theta}[\, x \mid \mathcal{F}_s \,] &= \min_{\theta \in \Theta} E^{Q^\theta}\left[E^{Q^\theta}[\, x \mid \mathcal{F}_t \,] \;\middle|\; \mathcal{F}_s \right] \\ &= \min_{\theta \in \Theta} E^{Q^\theta}\left[\min_{\theta' \in \Theta} E^{Q^{\theta'}}[\, x \mid \mathcal{F}_t \,] \;\middle|\; \mathcal{F}_s \right] \end{aligned} \tag{15}$$

as long as the minima exist (Lemma B3 in Appendix B). Note that while the first equality always holds by the law of iterated integrals, the second equality may not hold if Θ is not strongly rectangular. [18] The "recursive" structure (15) under strong rectangularity is exploited when we solve an optimal stopping problem for the firm.

The *i.i.d.* uncertainty is a special case of strong rectangularity, in which K_t is independent of time t. To be precise, the uncertainty characterized by Θ is said to be an *i.i.d. uncertainty* if there exists a compact subset K of $\mathbb{R}$ such that $0 \in K$ and

$$\Theta = \left\{ (\theta_t) \in \mathcal{L}^2 \;\middle|\; \theta_t(\omega) \in K \quad (m \otimes P)\text{-a.s.} \right\}.$$

Finally, κ-*ignorance* is a special case of *i.i.d.* uncertainty, where the set K is further specified as

$$K = [-\kappa, \kappa]$$

for some $\kappa \geqslant 0$. It is evident that if $\kappa = 0$, Knightian uncertainty vanishes. If κ increases, it means that the firm is less certain than before that candidate probability measures are close to P. Thus, κ-ignorance can be considered as a continuous counterpart of ε-contamination in Section 2.

[16] A compact-valued correspondence $K : [0, T] \twoheadrightarrow \mathcal{K}$ is *measurable* if $\{ t \in [0, T] \mid K_t \cap U \neq \phi \} \in \mathcal{B}([0, T])$ holds for any open set U.

[17] Chen and Epstein [3] adopt a weaker definition of rectangularity than ours. That is, a set of density generators, Θ, is *rectangular* if there exists a set-valued stochastic process $(K_t)_{0 \leqslant t \leqslant T}$ such that

$$\Theta = \left\{ (\theta_t) \in \mathcal{L}^2 \;\middle|\; \theta_t(\omega) \in K_t(\omega) \quad (m \otimes P)\text{-a.s.} \right\}$$

and, for each t, $K_t : \Omega \twoheadrightarrow \mathbb{R}$ is compact-valued and satisfies some additional regularity conditions. Our definition of strong rectangularity further restricts K_t to be degenerate (that is, non-stochastic). This restriction makes the following analysis much easier.

[18] Note that the second equality in (15) also holds under the weaker requirement of rectangularity of Chen and Epstein [3] (see the previous footnote). Strong rectangularity is needed to show Proposition 1, where the rectangularity is not sufficient.

3.2. The value of a utilized patent under strong rectangularity

Let us now consider the value of a utilized patent under Knightian uncertainty described in the previous subsection. We first derive its exact formula. Here, it turns out that strong rectangularity, of which *i.i.d.* uncertainty (and thus κ-ignorance) is a special case, is sufficient to obtain a simple formula.

Suppose that $\mathscr{P}^{\Theta}$ is Knightian uncertainty that the firm faces, where $\mathscr{P}^{\Theta}$ is the set of measures defined by (10) with some strongly rectangular set of density generators, $\Theta = \Theta^{(K_t)}$. Then, the corresponding profit process (π_t) follows (12). As in Section 2, we assume that the firm is uncertainty-averse. Then, the value at time t of the utilized patent with expiration time T, which has the current profit flow π_t, is defined by

$$W(\pi_t, t) = \inf_{Q \in \mathscr{P}^{\Theta}} E^Q \left[\int_t^T e^{-\rho(s-t)} \pi_s \, ds \,\middle|\, \mathcal{F}_t \right], \tag{16}$$

where $\rho > 0$ is the firm's discount rate and $E^Q[\cdot|\mathcal{F}_t]$ denotes the expectation with respect to Q conditional on $\mathcal{F}_t$. We assume $\rho > \mu$. The infimum operator reflects the firm's uncertainty aversion.

Before presenting Proposition 1 that gives an exact formula, we need to define an "upper-rim density generator," which plays a pivotal role in our analysis. Given (K_t), we define an upper-rim density generator, $\left(\theta_t^*\right)$, by

$$(\forall t) \quad \theta_t^* \equiv \arg\max\{ \sigma x \mid x \in K_t \} = \max K_t, \tag{17}$$

where equality holds by $\sigma > 0$ and the compact-valuedness of K. Note that we write $\max K_t$ instead of $\{\max K_t\}$. Then, $\left(\theta_t^*\right)$ turns out to be a degenerate (that is, non-stochastic) measurable process,[19] and hence, $\left(\theta_t^*\right) \in \mathcal{L}$. Therefore, it follows that $\left(\theta_t^*\right) \in \Theta^{(K_t)}$ by (17). Obviously, if Knightian uncertainty is κ-ignorance so that $K_t = [-\kappa, \kappa]$, then $\theta_t^* = \kappa$.

We are now ready to present the exact formula. The proof is relegated to Appendix A (A.1).

Proposition 1. *Suppose that the firm faces Knightian uncertainty characterized by* $\Theta^{(K_t)}$, *where* $\Theta^{(K_t)}$ *is a strongly rectangular set of density generators defined by* (14) *for some* (K_t). *Then, the value of the utilized patent in* (16) *is given by*

$$W(\pi_t, t) = \int_t^T \pi_t \exp\left(-(\rho - \mu)(s - t) - \int_t^s \sigma \theta_r^* \, dr \right) ds, \tag{18}$$

where $\left(\theta_t^*\right)$ *is defined by* (17).

We notice that even though the firm is assumed to be risk neutral, the risk factor σ sneaks in and an increase in risk also influences the value of the unutilized patent under Knightian uncertainty, whereas the risk factor does not influence the value under no Knightian uncertainty, that is, when $\theta^* = 0$. Thus, Knightian uncertainty in the continuous-time framework makes the value of the utilized patent depend not only on Knightian uncertainty itself, but also on risk.

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[19] Let $a \in \mathbb{R}$. Then, $\{ t \mid \max K_t > a \} = \{ t \mid K_t \cap (a, +\infty) \neq \phi \} \in \mathcal{B}([0, T])$ by measurability of K (see footnote 16), which shows that $\max K_t$ is $\mathcal{B}([0, T])$-measurable.

3.3. The optimal investment decision under strong rectangularity

In this section, we formulate the investment problem of the firm as an optimal stopping problem under Knightian uncertainty, and relate the investment problem to the value of the utilized patent described in the previous section.

Consider the investment of building a factory to produce patented products, which costs I and in return generates a profit flow $(\pi_s)_{s\geqslant t}$ when made at time t. The firm faces the same Knightian uncertainty as in the previous section: $(\pi_s)_{s\geqslant t}$ follows (12) with strongly rectangular $\Theta = \Theta^{(K_t)}$. The firm possessing the patent has an investment opportunity and contemplates optimal timing of this investment.

Then, at time t, the firm faces the optimal stopping problem of maximizing

$$\min_{Q\in\mathcal{P}^\Theta} E^Q\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\,ds - e^{-\rho(t'-t)}I \,\middle|\, \mathcal{F}_t\right]$$

by choosing an $(\mathcal{F}_t)$-stopping time,[20] t' ($t' \in [t, T]$), when the investment is to be made.[21] The maximum of this problem is denoted by V_t:

$$V_t = \max_{t'\geqslant t}\ \min_{Q\in\mathcal{P}^\Theta} E^Q\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\,ds - e^{-\rho(t'-t)}I \,\middle|\, \mathcal{F}_t\right]. \tag{19}$$

Then, V_t is the value of the investment opportunity.

Now consider the two options available to the firm: invest now (at time t) or wait for a short time interval, dt, and reconsider whether to invest or not after that (at time $t + dt$). Then, as A.2 in Appendix A shows, V_t solves a version of the *Hamilton–Jacobi–Bellman equation*:

$$V_t = \max\left\{ W_t - I,\ \min_{Q\in\mathcal{P}^\Theta} E^Q[\,dV_t \mid \mathcal{F}_t\,] + V_t - \rho V_t\,dt \right\}. \tag{20}$$

Here, the first term in the right-hand side is the value of "stopping right now" and the second term is the value of "waiting," each of which corresponds to one of the two options mentioned above.

3.4. The value of an unutilized patent under κ-ignorance and infinite horizon

In general, it is difficult to derive an analytic solution of the functional equation (20) and get a simple formula for the value of the unutilized patent. However, analysis is greatly simplified if (a)

[20] That is, t' is such that $(\forall t\geqslant 0)\ \{t'\leqslant t\} \in \mathcal{F}_t$.

[21] An alternative approach to modeling the firm's behavior under uncertainty is to apply the *robust control theory*. Then, the firm's objective would be to maximize:

$$\min_{Q\in\mathcal{P}^\Theta}\left(E^Q\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\,ds - e^{-\rho(t'-t)}I \,\middle|\, \mathcal{F}_t\right] + \eta R_t(Q)\right),$$

where η is a constant and $R_t(\cdot)$ measures the degree of deviation of the "true" probability measure Q from the "approximating" probability measure P, or the cost of model-misspecification when Q is the true probability measure. The preference behind this control problem is *not* based on the maximin expected utility à la Gilboa and Schmeidler [9], which underlies the specification of Knightian uncertainty in the current paper. However, both preferences generate the same optimal path for some class of optimization problems. See Hansen et al. [10].

underlying Knightian uncertainty is further restricted to κ-ignorance,[22] (b) the planning horizon is infinite and (c) the patent never expires. (In the terminology of option theory, we are now considering irreversible investment as an American option.) In this section, we explicitly solve the optimal stopping problem in such a case and get a simple pricing formula of the unutilized patent.

3.4.1. κ-ignorance and infinite horizon

Under κ-ignorance, the upper-rim density generator θ_t^* defined by (17) is independent of t and given by

$$\theta^* = \arg\max\{\, \sigma x \mid x \in [-\kappa, \kappa]\} = \kappa. \tag{21}$$

Therefore, (18) is simplified to

$$W(\pi_t, t) = \int_t^T \pi_t e^{(-\rho+\mu-\kappa\sigma)(s-t)}\, ds = \frac{\pi_t}{\lambda}\left(1 - e^{-\lambda(T-t)}\right), \tag{22}$$

where $\lambda \in \mathbb{R}$ is defined by

$$\lambda \equiv \rho - (\mu - \kappa\sigma)\,. \tag{23}$$

In what follows, we let T go to infinity, and assume that relations between variables in the limit hold true in the infinite horizon case. We have assumed that $\rho > \mu$, and then, λ defined by (23) turns out to be positive since $\kappa \geqslant 0$. Consequently, under κ-ignorance, as T approaches infinity, (22) approaches

$$W(\pi_t) = \frac{\pi_t}{\lambda}\,, \tag{24}$$

which is independent of time. We hereafter assume that (24) holds in the infinite horizon case.[23]

Then, applying Ito's lemma by regarding P as the true probability measure, we have that (W_t) solves

$$dW_t = \mu W_t\, dt + \sigma W_t\, dB_t \tag{25}$$

with $W_0 = \pi_0/\lambda$. Hence, (11) implies that for any $\theta \in \Theta$, (W_t) solves

$$dW_t = (\mu - \sigma\theta_t) W_t\, dt + \sigma W_t\, dB_t^\theta. \tag{26}$$

This shows that when Q^θ is the true probability measure, (W_t) solves the stochastic differential equation defined by (26).

[22] This section can easily be extended to the case of *i.i.d.* uncertainty. See Nishimura and Ozaki [18].

[23] When the planning horizon and the patent expiration date are infinite in κ-ignorance, W becomes independent of t. However, conditions guaranteeing that (24) holds in the infinite horizon case are not yet known. We do not attempt to derive such conditions here, and instead we simply assume they are satisfied in our model. This is because we need more mathematical apparatus than we have in this paper. For example, Girsanov's theorem is usually stated in a finite-horizon framework and thus some sophistication is required to extend it to an infinite horizon (see [11, p. 192, Corollary 5.2]).

3.4.2. Time-homogeneous Hamilton–Jacobi–Bellman equation

Let us return to the optimal stopping problem of the previous subsection. If the planning horizon is infinite and (W_t) follows (25) (and (26)), then V_t defined by (19) depends only on W_t, and not on physical time t. Therefore, we are allowed to write it as $V_t = V(W_t)$ with some $V : \mathbb{R}_+ \to \mathbb{R}$. In this case, the Hamilton–Jacobi–Bellman equation turns out to be

$$V(W_t) = \max \left\{ W_t - I, \ \min_{Q \in \mathscr{P}^\Theta} E^Q [\, dV_t \mid \mathcal{F}_t \,] + V(W_t) - \rho V(W_t)\, dt \right\}. \tag{27}$$

Let us now solve the above Hamilton–Jacobi–Bellman equation. We conjecture that there exists W^* such that the optimal strategy of the firm takes the form of “stop right now if $W_t \geqslant W^*$ and wait if $W_t < W^*$.” This conjecture shall be verified to be true later.

In the continuation region, that is, when $W_t < W^*$, it holds from (27) that

$$\min_{Q \in \mathscr{P}^\Theta} E^Q [\, dV_t \mid \mathcal{F}_t \,] = \rho V(W_t)\, dt. \tag{28}$$

Here, the left-hand side is the minimum capital gain of holding the investment opportunity during $[t, t + dt]$, and the right-hand side is its opportunity cost measured in terms of the firm’s discount rate. Eq. (28) shows that both must be equal in the continuation region.

We now derive from Eq. (28) a non-stochastic ordinary differential equation for V. Conjecture that V is twicely differentiable in the continuation region. Then, by Ito’s lemma and (26), we have for each θ,

$$dV_t = V'(W_t) \left((\mu - \sigma\theta_t) W_t\, dt + \sigma W_t\, dB_t^\theta \right) + \tfrac{1}{2} \sigma^2 W_t^2 V''(W_t)\, dt. \tag{29}$$

At this stage, we conjecture that V' is positive. Then, the left-hand side of (28) is further rewritten (see A.3 in Appendix A) as

$$\min_{Q \in \mathscr{P}^\Theta} E^Q [\, dV_t \mid \mathcal{F}_t \,] = V'(W_t)\, (\mu - \kappa\sigma) W_t\, dt + \tfrac{1}{2} \sigma^2 W_t^2 V''(W_t)\, dt. \tag{30}$$

From (28) and (30), we obtain a (non-stochastic) second-order ordinary differential equation for V such that

$$\tfrac{1}{2} \sigma^2 W_t^2 V''(W_t) + (\mu - \kappa\sigma) W_t V'(W_t) - \rho V(W_t) = 0, \tag{31}$$

which must hold in the continuation region.

In order to solve (31) for V, we need two boundary conditions. One boundary condition is given by the condition that if the utilized patent has no value, the investment opportunity also has no value[24]:

$$V(0) = 0. \tag{32}$$

The other boundary condition comes from the Hamilton–Jacobi–Bellman equation (27) as

$$V(W^*) = W^* - I, \tag{33}$$

where W^* is the “reservation value” whose existence is now assumed.

[24] From (26), it follows that $W_T = 0$ for any $T \geqslant t$ if $W_t = 0$. Hence, it is optimal for the firm never to invest, leading to (32) by (19).

Since W^* must be optimally chosen by the firm, (33) serves as a free-boundary condition. In order to determine the value of W^*, we need an additional condition for V, which is obtained from the optimization with respect to W^*. To find this, consider the gain the firm would obtain if it made an investment upon observing W_t. It would be the value of the project minus the value of the investment opportunity (that is, the value of not making investment now) and it is given by $W_t - V(W_t)$. Since W^* should be chosen so as to maximize this, it must hold that

$$V'(W^*) = 1 \tag{34}$$

from the first-order condition for maximization. [25]

3.4.3. The optimal strategy

The ordinary differential equation (31) with boundary conditions (32), (33) and (34) can be explicitly solved to obtain (see A.4 in Appendix A)

$$V(W_t) = \left(\frac{I}{\alpha - 1}\right)^{1-\alpha} \alpha^{-\alpha} W_t^{\alpha} \equiv A W_t^{\alpha} \tag{35}$$

as far as $W_t < W^*$, where the reservation value W^* is given by

$$W^* = \frac{\alpha I}{\alpha - 1} \tag{36}$$

and α is a constant defined by

$$\alpha = \frac{-\left\{(\mu - \kappa\sigma) - \frac{1}{2}\sigma^2\right\} + \sqrt{\left\{(\mu - \kappa\sigma) - \frac{1}{2}\sigma^2\right\}^2 + 2\rho\sigma^2}}{\sigma^2}. \tag{37}$$

Under the maintained assumption that $\rho > \mu$ and $\kappa \geqslant 0$, it holds that $\alpha > 1$ (see A.5 in Appendix A). Hence, W^* and V are well-defined.

Therefore, the value of the investment opportunity or the patent, V, is given by

$$V(W_t) = \begin{cases} \left(\dfrac{I}{\alpha - 1}\right)^{1-\alpha} \alpha^{-\alpha} W_t^{\alpha} & \text{if} \quad W_t < W^*, \\ W_t - I & \text{if} \quad W_t \geqslant W^* \end{cases} \tag{38}$$

(see Fig. 1). Recall that we have made three conjectures: (a) There exists a unique reservation value W^*; (b) V is twicely differentiable in the continuation region; and (c) V' is positive. These conjectures are easily verified to hold true from (38) ((b) and (c) are immediate and see Fig. 1 for (a)).

To sum up, we have proved:

Proposition 2. *Assume that Knightian uncertainty the firm faces is κ-ignorance, and further assume that relations among variables in the finite-horizon case converge, as the horizon goes to infinity, to those in the infinite-horizon case. Then, in the case of infinite horizon, the value of the unutilized patent, that is, $V(W_t)$ in the continuation region ($W_t < W^*$), is given by* (35) *with W^* and α defined by* (36) *and* (37), *respectively.*

[25] We will find later that V is convex, and hence, the second-order condition turns out to be satisfied.

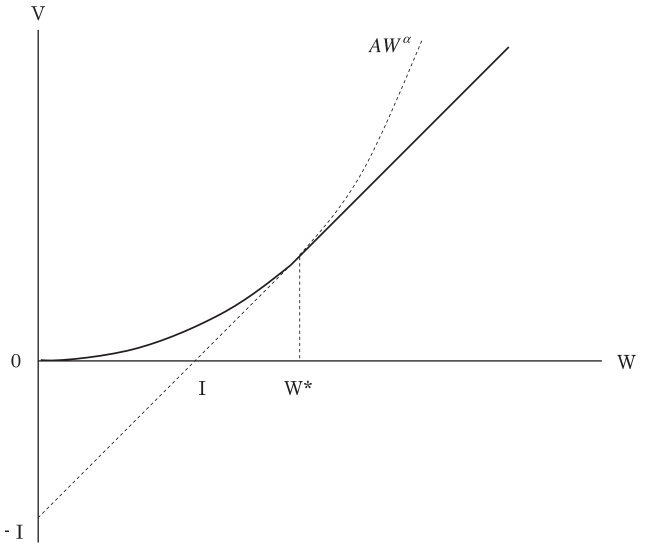

Fig. 1. Value function.

4. Sensitivity analysis

This section compares the effect of an increase in Knightian uncertainty and that of an increase in risk on the value of a patent and on optimal timing of investment. We show that the same result holds in the continuous-time case as in the simple two-period, two-state example of Section 2.

4.1. An increase in risk

Let us consider the effect of an increase in risk when there is no Knightian uncertainty, or when we assume that $\kappa = 0$. It is evident from (23) and (24) that σ^2 does not influence the value of the utilized patent. In contrast, σ^2 changes the value of the unutilized patent. With some calculation, we have

$$\frac{\partial \alpha}{\partial \sigma^2} < 0. \tag{39}$$

We also obtain [26] $\partial V(W_t)/\partial \alpha < 0$ for the unutilized patent (that is, when $W_t < W^*$). Combining these, we have $\partial V(W_t)/\partial \sigma^2 > 0$ for the value of the unutilized patent. Thus, an increase in σ^2 increases the value of the unutilized patent. Finally, since $\partial W^*/\partial \alpha < 0$ (see (36)), we obtain $\partial W^*/\partial \sigma^2 > 0$ by (39). Therefore, we get

[26] The claim holds since

$$\begin{aligned}\partial \ln V(W_t)/\partial \alpha &= -\ln I + \ln(\alpha - 1) - \ln \alpha + \ln W_t \\ &< -\ln I + \ln(\alpha - 1) - \ln \alpha + \ln W^* \\ &= -\ln I + \ln(\alpha - 1) - \ln \alpha + \ln\left(\frac{\alpha I}{\alpha - 1}\right) = 0.\end{aligned}$$

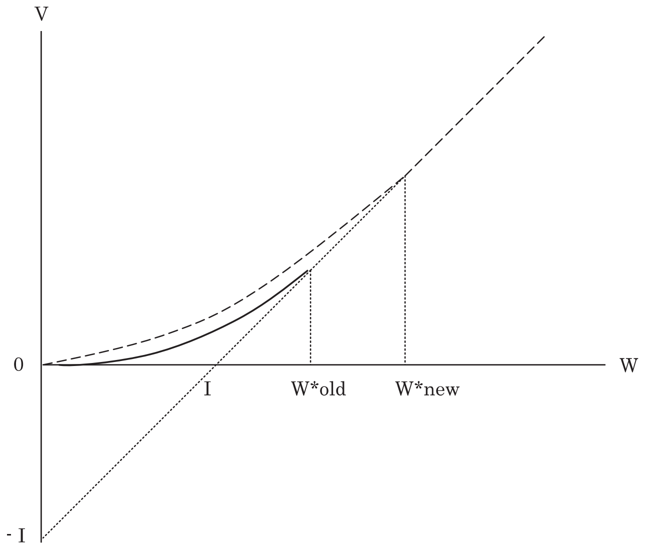

Fig. 2. An increase in risk.

Proposition 3. *In the case of no Knightian uncertainty, an increase in* σ^2, *that is, an increase in risk, induces* (a) *an increase in the value of the unutilized patent and no change in the value of the utilized patent, and* (b) *an increase in the reservation value* W^* (*see Fig.* 2).

4.2. *An increase in Knightian uncertainty (κ-ignorance)*

Let us now consider a case where Knightian uncertainty the firm faces is characterized as κ-ignorance, which is a continuous counterpart of the ε-contamination seen in the two-period, two-state example of Section 2.

Consider first the value of the utilized patent. We have from (23) and (24)

$$W_t = W(\pi_t) = \frac{\pi_t}{\rho - (\mu - \kappa\sigma)} .$$

Consequently, an increase in κ reduces the value of the utilized patent. It is contrastive to the case of no Knightian uncertainty where an increase in σ^2 has no effect on the value of the utilized patent.

Let us now turn to the value of the unutilized patent. It is evident from (35) that an increase in κ reduces the value of the unutilized patent by lowering W. Moreover, an increase in κ further reduces the value of the unutilized patent through α. It follows from (37) with some calculation that

$$\frac{\partial \alpha}{\partial \kappa} > 0 . \tag{40}$$

Thus, the effect of an increase in κ on α is just in the opposite direction to that of an increase in σ^2 under no Knightian uncertainty. Combining this effect and $\partial V(W_t)/\partial\alpha < 0$ when $W_t < W^*$ from (35), we have $\left(\partial V(W_t)/\partial\alpha\right)\left(\partial\alpha/\partial\kappa\right) < 0$ when $W_t < W^*$. Consequently, an increase in κ

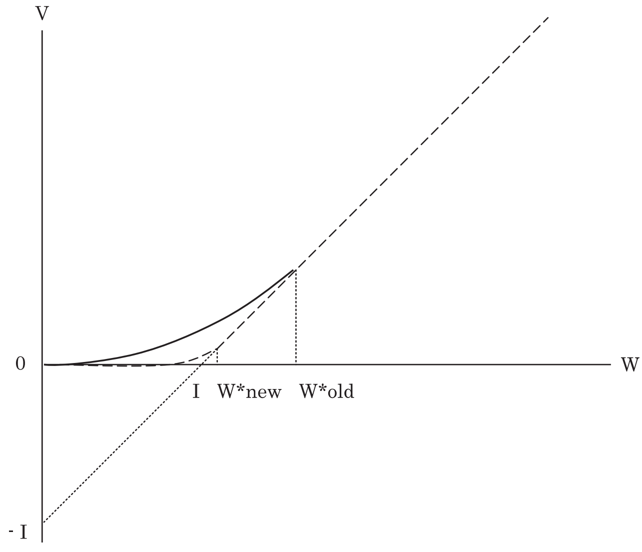

Fig. 3. An increase in Knightian uncertainty.

reduces the value of the unutilized patent (that is, $V(W_t)$ when $W_t < W^*$) by raising α. The direct effect of lowering W and the indirect effect of raising α both reduce the value of the unutilized patent.

Finally, since $\partial W^*/\partial\alpha < 0$, we obtain $\partial W^*/\partial\kappa < 0$ by (40). Thus, we conclude that the next proposition holds.

Proposition 4. *Assume the same assumptions as Proposition* 2. *Then an increase in* κ, *that is*, *an increase in Knightian uncertainty*, *induces* (a) *a decrease in the value of utilized and unutilized patents and* (b) *a decrease in the reservation value* W^* (*see Fig.* 3).

This proposition provides sharp contrast to an increase in the "risk" in the previous subsection. While an increase in risk increases the value of the unutilized patent (and leaves the value of the utilized patent unchanged), an increase in Knightian uncertainty (an increase in κ) decreases the value of the unutilized patent (as well as the value of the utilized patent).

4.3. *Value of waiting*

Let us now turn to the issue of the value of waiting. To analyze this, it is worthwhile to re-interpret the reservation value W^* in Propositions 3 and 4 in terms of the reservation profit flow π^* defined as

$$\pi^* = \begin{cases} (\rho-\mu)W^* & \text{under no Knightian uncertainty,} \\ \{\rho-(\mu-\kappa\sigma)\}\, W^* & \text{under Knightian uncertainty}(\kappa\text{ -ignorance}). \end{cases}$$

Note that $W_t = \pi_t/(\rho-\mu)$ under no Knightian uncertainty and $W_t = \pi_t/\{\rho-(\mu-\kappa\sigma)\}$ under κ-ignorance. When the current profit flow, π_t, is less than π^*, the value of the utilized patent, W_t, is less than W^* since $\rho-\mu > 0$ under no Knightian uncertainty and $\{\rho-(\mu-\kappa\sigma)\} > 0$

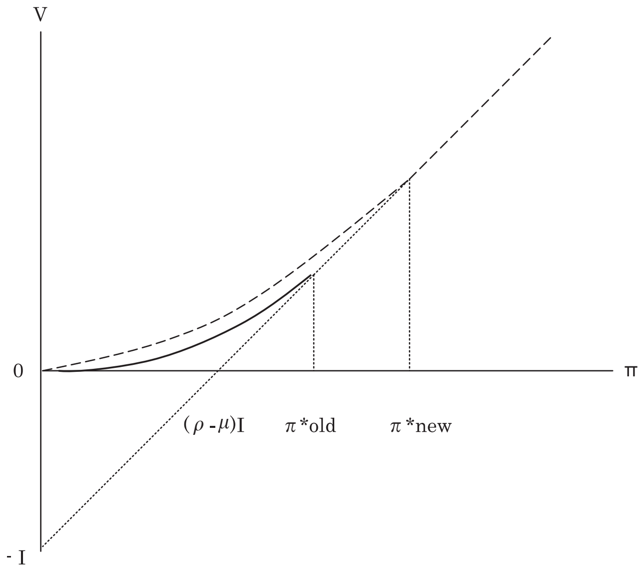

Fig. 4. An increase in risk (x-axis is measured in terms of profit flow).

under κ-ignorance. Hence waiting is the optimal strategy. On the other hand, when the current profit flow is greater than π^*, the value of the utilized patent is greater than W^* and hence "stopping right now" is the optimal strategy. Therefore, π^* thus defined serves as the reservation profit flow.

It follows that $\partial\pi^*/\partial\sigma^2 > 0$ under no Knightian uncertainty since $\partial W^*/\partial\sigma^2 > 0$ (see Section 4.1), and calculation shows that $\partial\pi^*/\partial\kappa > 0$ under κ-ignorance.[27] Thus, both an increase in risk under no Knightian uncertainty and an increase in κ-ignorance under Knightian uncertainty *in*crease the reservation profit flow, and thus the value of waiting is increased (see Figs. 4 and 5). This result conforms to that of Section 2, where an increase in both risk and Knightian uncertainty makes it more profitable for the firm to postpone irreversible investment.

This waiting-enhancing effect of an increase in Knightian uncertainty is in stark contrast to its waiting-reducing effect in the job search model investigated by Nishimura and Ozaki [19]. They show that in a discrete-time infinite-horizon job search model, an increase in Knightian uncertainty unambiguously reduces the reservation wage and thus shortens waiting.

Although both the job search and irreversible investment models are formulated as optimal stopping problems, there is a fundamental difference between the two as to the nature of uncertainty. In the job search model, the decision-maker determines when to stop the search and thus resolve uncertainty. Thus, an increase in Knightian uncertainty makes the uncertainty-averse decision-maker more likely to stop the search and resolve uncertainty. In contrast, in the irreversible investment model, the decision-maker contemplates when to begin investment and face uncertainty. Thus, an increase in Knightian uncertainty makes the uncertainty-averse decision-maker more likely to postpone investment to avoid facing uncertainty.

[27] In fact, it can be shown that $\partial\pi^*/\partial\kappa > 0$ if $\alpha > 1$.

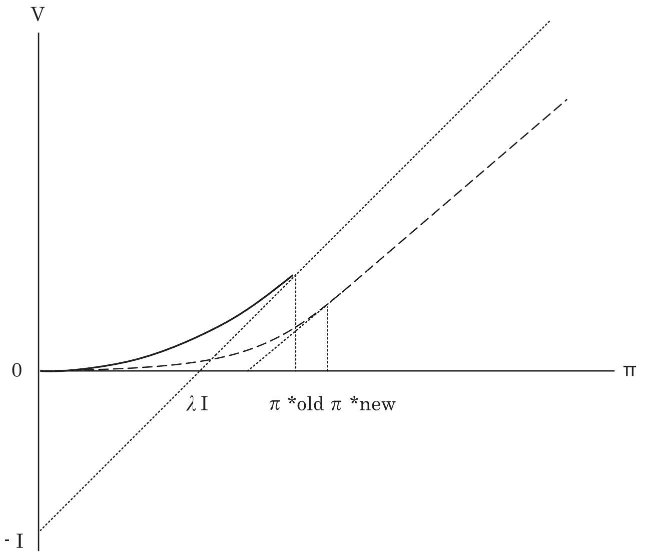

Fig. 5. An increase in Knightian uncertainty (x-axis is measured in terms of profit flow).

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Appendix A. Derivations

A.1. Proof of Proposition 1

We first show that for any $s \geqslant t$ and any $\theta \in \Theta^{(K_t)}$,

$$E^{Q^\theta}\left[\exp\left(-\int_t^s \sigma\theta_r \, dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right) \middle| \mathcal{F}_t \right]$$

$$\geqslant E^{Q^{\theta^*}}\left[\exp\left(-\int_t^s \sigma\theta_r^* \, dr + \sigma\left(B_s^{\theta^*} - B_t^{\theta^*}\right)\right) \middle| \mathcal{F}_t \right]. \tag{41}$$

To do this, let $s \geqslant t$ and $\theta \in \Theta^{(K_t)}$. Note that by the definition of $\left(\theta_t^*\right)$, it holds that

$$(\forall\omega) \quad \exp\left(-\int_t^s \sigma\theta_r \, dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right) \geqslant \exp\left(-\int_t^s \sigma\theta_r^* \, dr + \sigma\left(B_s^\theta - B_t^\theta\right)\right).$$

Thus, the monotonicity of conditional expectation [1, p. 468, Theorem 34.2(iii)] implies that

$$
\begin{aligned}
&E^{Q^{\theta}}\left[\exp\left(-\int_t^s \sigma\theta_r\,dr+\sigma\left(B_s^{\theta}-B_t^{\theta}\right)\right)\,\middle|\,\mathcal{F}_t\right]\\
&\geqslant E^{Q^{\theta}}\left[\exp\left(-\int_t^s \sigma\theta_r^*\,dr+\sigma\left(B_s^{\theta}-B_t^{\theta}\right)\right)\,\middle|\,\mathcal{F}_t\right]\\
&=\exp\left(-\int_t^s \sigma\theta_r^*\,dr\right)\exp\left(\frac{1}{2}\sigma^2(s-t)\right)\\
&=E^{Q^{\theta^*}}\left[\exp\left(-\int_t^s \sigma\theta_r^*\,dr+\sigma\left(B_s^{\theta^*}-B_t^{\theta^*}\right)\right)\,\middle|\,\mathcal{F}_t\right],
\end{aligned}
$$

where we invoked the “degeneracy” of $\left(\theta_t^*\right)$ to show the equalities. Consequently, (41) holds.

We can now rewrite W as follows:

$$
\begin{aligned}
&W(\pi_t,t)\\
&=\inf_{\theta\in\Theta} E^{Q^{\theta}}\left[\int_t^T e^{-\rho(s-t)}\pi_s\,ds\,\middle|\,\mathcal{F}_t\right]\\
&=\inf_{\theta\in\Theta}\int_t^T E^{Q^{\theta}}\left[e^{-\rho(s-t)}\pi_s\,\middle|\,\mathcal{F}_t\right]ds\\
&=\inf_{\theta\in\Theta}\int_t^T \pi_t\exp\left(\left(\mu-\rho-\frac{1}{2}\sigma^2\right)(s-t)\right)\\
&\quad\times E^{Q^{\theta}}\left[\exp\left(-\int_t^s \sigma\theta_r\,dr+\sigma\left(B_s^{\theta}-B_t^{\theta}\right)\right)\,\middle|\,\mathcal{F}_t\right]ds\\
&=\int_t^T \pi_t\exp\left(\left(\mu-\rho-\frac{1}{2}\sigma^2\right)(s-t)\right)\\
&\quad\times E^{Q^{\theta^*}}\left[\exp\left(-\int_t^s \sigma\theta_r^*\,dr+\sigma\left(B_s^{\theta^*}-B_t^{\theta^*}\right)\right)\,\middle|\,\mathcal{F}_t\right]ds\\
&=\int_t^T \pi_t\exp\left(\left(\mu-\rho-\frac{1}{2}\sigma^2\right)(s-t)-\int_t^s \sigma\theta_r^*\,dr\right)\\
&\quad\times E^{Q^{\theta^*}}\left[\exp\sigma\left(B_s^{\theta^*}-B_t^{\theta^*}\right)\,\middle|\,\mathcal{F}_t\right]ds\\
&=\int_t^T \pi_t\exp\left(\left(\mu-\rho-\frac{1}{2}\sigma^2\right)(s-t)-\int_t^s \sigma\theta_r^*\,dr\right)\exp\left(\frac{1}{2}\sigma^2(s-t)\right)ds\\
&=\int_t^T \pi_t\exp\left(-(\rho-\mu)(s-t)-\int_t^s \sigma\theta_r^*\,dr\right)ds, \qquad (42)
\end{aligned}
$$

where the first equality holds by (10); the second equality holds by Fubini's theorem for conditional expectation [6, p. 74, Proposition 4.6] [28] ; the third equality holds by (13); the fourth equality is derived from (41); the fifth equality holds by the fact that $\left(\theta_t^*\right)$ is a "degenerate" stochastic process; and the sixth equality holds by the fact that $\left(B_t^{\theta^*}\right)$ is the Brownian motion with respect to Q^{θ^*} and the formula for the expectation of a lognormal distribution (see, for example, [16]). Thus, Proposition 1 is proved.

A.2. Derivation of (20)

We obtain

$$
\begin{aligned}
V_t
&= \max_{t' \geqslant t} \min_{Q \in \mathscr{P}^\Theta} E^Q\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\, ds - e^{-\rho(t'-t)} I \,\middle|\, \mathcal{F}_t \right] \\
&= \max\left\{ \min_{Q \in \mathscr{P}^\Theta} E^Q\left[\int_t^T e^{-\rho(s-t)}\pi_s\, ds \,\middle|\, \mathcal{F}_t \right] - I,\ \max_{t' \geqslant t+dt} \min_{Q \in \mathscr{P}^\Theta} \right. \\
&\quad \left. \times\, E^Q\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\, ds - e^{-\rho(t'-t)} I \,\middle|\, \mathcal{F}_t \right]\right\} \\
&= \max\left\{ W_t - I,\ \max_{t' \geqslant t+dt} \min_{Q \in \mathscr{P}^\Theta} E^Q\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\, ds - e^{-\rho(t'-t)} I \,\middle|\, \mathcal{F}_t \right]\right\} \\
&= \max\left\{ W_t - I,\ \max_{t' \geqslant t+dt} \min_{\theta \in \Theta} E^{Q^\theta}\left[\int_{t'}^T e^{-\rho(s-t)}\pi_s\, ds - e^{-\rho(t'-t)} I \,\middle|\, \mathcal{F}_t \right]\right\} \\
&= \max\left\{ W_t - I,\ e^{-\rho\, dt} \max_{t' \geqslant t+dt} \min_{\theta \in \Theta} E^{Q^\theta} \right. \\
&\quad \left. \times \left[E^{Q^\theta}\left[\int_{t'}^T e^{-\rho(s-t-dt)}\pi_s\, ds - e^{-\rho(t'-t-dt)} I \,\middle|\, \mathcal{F}_{t+dt} \right] \,\middle|\, \mathcal{F}_t \right]\right\} \\
&= \max\left\{ W_t - I,\ e^{-\rho\, dt} \max_{t' \geqslant t+dt} \min_{\theta \in \Theta} E^{Q^\theta}\left[\min_{\theta' \in \Theta} E^{Q^{\theta'}} \right.\right. \\
&\quad \left.\left. \times \left[\int_{t'}^T e^{-\rho(s-t-dt)}\pi_s\, ds - e^{-\rho(t'-t-dt)} I \,\middle|\, \mathcal{F}_{t+dt} \right] \,\middle|\, \mathcal{F}_t \right]\right\}
\end{aligned}
$$

---

[28] Since the Brownian motion is a measurable process [1, p. 530, Theorem 37.2] and (θ_t) is a measurable process by assumption, (π_t) is also a measurable process by (13), and hence, a function $(s, \omega) \mapsto e^{-\rho(s-t)}\pi_s(\omega)$ is $(\mathcal{B}([t, T]) \otimes \mathcal{F}_T)$-measurable. Furthermore, it also holds from (13) that

$$\int_t^T E^{Q^\theta}[\,|e^{-\rho(s-t)}\pi_s|\,]\, ds < +\infty.$$

Therefore, we may invoke Ethier and Kurtz [6, p. 74, Proposition 4.6] to conclude that there exists a function $f : [t, T] \times \Omega \to \mathbb{R}$ such that f is $(\mathcal{B}([t, T]) \otimes \mathcal{F}_t)$-measurable, $(\forall s, \omega)\ f(s, \omega) = E^{Q^\theta}[e^{-\rho(s-t)}\pi_s | \mathcal{F}_t](\omega)$, $\int_t^T |f(s, \omega)|\, ds < +\infty$ P-a.s., and

$$(\forall \omega) \quad \int_t^T f(s, \omega)\, ds = E^{Q^\theta}\left[\int_t^T e^{-\rho(s-t)}\pi_s\, ds \,\middle|\, \mathcal{F}_t \right](\omega),$$

justifying the second equality in (42).

$$
\begin{aligned}
&= \max \left\{ W_t - I,\ e^{-\rho\, dt} \min_{\theta\in\Theta} E^{Q^\theta} \left[\max_{t' \geqslant t+dt} \min_{\theta'\in\Theta} E^{Q^{\theta'}} \right.\right.\\
&\quad \times \left.\left. \left[\int_{t'}^{T} e^{-\rho(s-t-dt)} \pi_s\, ds - e^{-\rho(t'-t-dt)} I \,\middle|\, \mathcal{F}_{t+dt} \right] \,\middle|\, \mathcal{F}_t \right] \right\}\\
&= \max \left\{ W_t - I,\ e^{-\rho\, dt} \min_{\theta\in\Theta} E^{Q^\theta} \left[V_{t+dt} \mid \mathcal{F}_t \right] \right\}\\
&= \max \left\{ W_t - I,\ (1-\rho\, dt) \left(\min_{\theta\in\Theta} E^{Q^\theta} [\, dV_t \mid \mathcal{F}_t \,] + V_t \right) \right\}\\
&= \max \left\{ W_t - I,\ \min_{\theta\in\Theta} E^{Q^\theta} [\, dV_t \mid \mathcal{F}_t \,] + V_t - \rho V_t dt \right\},
\end{aligned}
$$

where each equality holds by: the definition of V_t, (19) (first); splitting the decision between investing now (at time t) and waiting for a short time interval and reconsidering whether to invest or not after it (at time $t + dt$) (second); the definition of W_t, (16) (third); the definition of $\mathscr{P}^\Theta$, (10) (fourth); the law of iterated integrals (fifth); the rectangularity, (15) (sixth); the fact that t' is restricted to be greater than or equal to $t + dt$ (seventh); the definition of V_t, (19), with t replaced by $t + dt$ (eighth); writing V_{t+dt} as $V_t + dV_t$ and approximating $e^{-\rho dt}$ by $(1 - \rho dt)$ (such an approximation is justified since we let dt go to zero) (ninth); and eliminating the term which is of a higher order than dt (tenth).

A.3. Derivation of (30)

We get

$$
\begin{aligned}
&\min_{Q\in\mathscr{P}^\Theta} E^{Q} [\, dV_t \mid \mathcal{F}_t \,] = \min_{\theta\in\Theta} E^{Q^\theta} [\, dV_t \mid \mathcal{F}_t \,]\\
&\quad = \min_{\theta\in\Theta} E^{Q^\theta} \left[V'(W_t) \left((\mu - \sigma\theta_t) W_t\, dt + \sigma W_t\, dB_t^\theta \right) + \tfrac{1}{2}\sigma^2 W_t^2 V''(W_t)\, dt \,\middle|\, \mathcal{F}_t \right]\\
&\quad = \min_{\theta\in\Theta} V'(W_t)\, (\mu - \sigma\theta_t) W_t\, dt + \tfrac{1}{2}\sigma^2 W_t^2 V''(W_t)\, dt\\
&\quad = V'(W_t)\, (\mu - \kappa\sigma) W_t\, dt + \tfrac{1}{2}\sigma^2 W_t^2 V''(W_t)\, dt,
\end{aligned}
$$

where the first equality holds by the definition of $\mathscr{P}^\Theta$; the second equality holds by the fact that (29) holds for each θ; the third equality holds by the fact that $\left(B_t^\theta\right)$ is the Brownian motion with respect to Q^θ; and the last equality holds by (21) and the conjecture that V' is positive.

A.4. Derivation of (35), (36) and (37) as a solution of (31)

To solve (31) with (32), (33) and (34), consider the following quadratic equation called the *characteristic equation* for (31):

$$
\tfrac{1}{2}\sigma^2 x\,(x-1) + (\mu - \kappa\sigma)\, x - \rho = 0,
$$

the solutions to which are given by α defined by (37) and its conjugate, β. It turns out that $\alpha > 1$ and $\beta < 0$ (see A.5). Furthermore, it can be easily verified that both W_t^α and W_t^β solve (31) and that the Wronskian of W_t^α and W_t^β is non-zero for any $W_t > 0$. (Here, the *Wronskian* of the two functions f_1 and f_2 is defined by $W(f_1, f_2) = f_1 f_2' - f_1' f_2$, and $W(W_t^\alpha, W_t^\beta) = (\beta - \alpha) W_t^{\alpha+\beta-1}$.)

Hence, any solution to (31) can be expressed uniquely as a linear combination of W_t^{α} and W_t^{β}, that is,

$$V(W_t) = AW_t^{\alpha} + BW_t^{\beta}, \tag{43}$$

where A and B are some reals [2, p. 116, Theorem 3.4]. Conversely, it is obvious that any function of the form (43) is a solution to (31). We conclude that (43) exhausts all the solutions to (31).

We now turn to the boundary conditions. The negativity of β and (32) immediately imply that $B = 0$ and hence

$$V(W_t) = AW_t^{\alpha}, \tag{44}$$

where A still remains undetermined.

Next, (44), (33) and (34) imply the following two equations:

$$A\left(W^*\right)^{\alpha} = W^* - I \quad \text{and} \quad \alpha A\left(W^*\right)^{\alpha-1} = 1.$$

By solving these equations, we obtain the solution given in the text.

A.5. Proof of $\boldsymbol{\alpha} > 1$ and $\boldsymbol{\beta} < 0$

To see that $\alpha > 1$, note that

$$\begin{aligned}\alpha &> \frac{-\left(\mu - \kappa\sigma - \frac{1}{2}\sigma^2\right) + \sqrt{\left(\mu - \kappa\sigma - \frac{1}{2}\sigma^2\right)^2 + 2(\mu - \kappa\sigma)\sigma^2}}{\sigma^2} \\ &= \frac{-\left(\mu - \kappa\sigma - \frac{1}{2}\sigma^2\right) + \left|\mu - \kappa\sigma + \frac{1}{2}\sigma^2\right|}{\sigma^2},\end{aligned}$$

where the strict inequality holds since $\rho > \mu - \kappa\sigma$ by the assumption that $\rho > \mu$ and $\kappa \geqslant 0$. Our claim follows since the last term is unity if $\mu - \kappa\sigma \geqslant -\frac{1}{2}\sigma^2$ and more than unity if $\mu - \kappa\sigma < -\frac{1}{2}\sigma^2$.

Also, β, the conjugate of α, is negative since

$$\begin{aligned}\beta &= \frac{-\left(\mu - \kappa\sigma - \frac{1}{2}\sigma^2\right) - \sqrt{\left(\mu - \kappa\sigma - \frac{1}{2}\sigma^2\right)^2 + 2\rho\sigma^2}}{\sigma^2} \\ &< \frac{-\left(\mu - \kappa\sigma - \frac{1}{2}\sigma^2\right) - \left|\mu - \kappa\sigma - \frac{1}{2}\sigma^2\right|}{\sigma^2} \leqslant 0,\end{aligned}$$

where the strict inequality holds since $\rho > 0$.

Appendix B. Recursive structure under strong rectangularity

In this appendix, we prove that recursive structure (15) holds true under strong rectangularity. Let θ be a density generator, let $\left(z_t^{\theta}\right)$ be defined by (7) and define the measure Q_t^{θ} by

$$(\forall t \in [0, T])(\forall A \in \mathcal{F}_T) \quad Q_t^{\theta}(A) = \int_A z_t^{\theta}\, dP.$$

Then, the next lemma shows that Q_t^θ is a probability measure that coincides with Q^θ over $\mathcal{F}_t$. This is a direct implication of the martingale property of $\left(z_t^\theta\right)$.

Lemma B1. *The measure Q_t^θ is a probability measure satisfying*

$$(\forall A \in \mathcal{F}_t) \quad Q_t^\theta(A) = Q^\theta(A),$$

where Q^θ is defined by (9).

Proof. Since $\left(z_t^\theta\right)$ is a martingale, it follows that $Q_t^\theta(\Omega) = E^P\left[z_t^\theta\right] = z_0^\theta = 1$, and hence, Q_t^θ is certainly a probability measure. To show the claimed equality, let $t \in [0, T]$ and $A \in \mathcal{F}_t$. Then,

$$Q_t^\theta(A) = \int_A z_t^\theta \, dP = \int_A E^P\left[z_T^\theta \middle| \mathcal{F}_t\right] dP = \int_A z_T^\theta \, dP = Q^\theta(A),$$

where the second equality holds by the fact that $\left(z_t^\theta\right)$ is a martingale and the third equality holds by the definition of conditional expectation and the assumption that $A \in \mathcal{F}_t$. □

Lemma B1 implies the next lemma, which shows that, loosely speaking, the expectation with respect to Q^θ of an $\mathcal{F}_t$-measurable random variable conditional on $\mathcal{F}_s$ depends on the density generator θ only between s and t.

Lemma B2. *Let $0 \leqslant s \leqslant t \leqslant T$ and x be an $\mathcal{F}_t$-measurable random variable. Then, $E^{Q^\theta}[x \mid \mathcal{F}_s]$ depends only on $(\theta_u)_{s \leqslant u < t}$.*

Proof. First, note that

$$E^{Q^\theta}[x \mid \mathcal{F}_s] = \frac{1}{z_s^\theta} E^P\left[x z_t^\theta \middle| \mathcal{F}_s\right]. \tag{45}$$

To see this, let $A \in \mathcal{F}_s$. Then,

$$\begin{aligned}
\int_A \frac{1}{z_s^\theta} E^P\left[x z_t^\theta \middle| \mathcal{F}_s\right] dQ^\theta &= \int_A \frac{1}{z_s^\theta} E^P\left[x z_t^\theta \middle| \mathcal{F}_s\right] dQ_s^\theta \\
&= \int_A \frac{1}{z_s^\theta} E^P\left[x z_t^\theta \middle| \mathcal{F}_s\right] z_s^\theta \, dP \\
&= \int_A E^P\left[x z_t^\theta \middle| \mathcal{F}_s\right] dP \\
&= \int_A x z_t^\theta \, dP \\
&= \int_A x \, dQ_t^\theta \\
&= \int_A x \, dQ^\theta,
\end{aligned}$$

where each equality holds by: Lemma B1 and the fact that the integrand is $\mathcal{F}_s$-measurable (first); the definition of Q_s^θ (second); cancellation (third); the definition of conditional expectation and the fact that $A \in \mathcal{F}_s$ (fourth); the definition of Q_t^θ (fifth); and Lemma B1 and the fact that the integrand is $\mathcal{F}_t$-measurable (sixth). Since the whole equality holds for any $A \in \mathcal{F}_s$, (45) follows by the definition of conditional expectation.

Second, the right-hand side of (45) is rewritten as

$$\frac{1}{z_s^\theta} E^P \left[x z_t^\theta \,\middle|\, \mathcal{F}_s \right] = E^P \left[x \exp\left(-\frac{1}{2} \int_s^t \theta_u \, du - \int_s^t \theta_u \, dB_u \right) \,\middle|\, \mathcal{F}_s \right]$$

by the definition of $\left(z_t^\theta\right)$. This shows that $E^{Q^\theta}[x \mid \mathcal{F}_s]$ depends only on $(\theta_u)_{s \leqslant u \leqslant t}$. In fact, it depends only on $(\theta_u)_{s \leqslant u < t}$ since stochastic processes $\left(\int_0^t \theta_u \, du\right)_t$ and $\left(\int_0^t \theta_u \, dB_u\right)_t$ have a continuous sample path for each $\omega \in \Omega$ [11, p. 140, Remark 2.11]. This completes the proof. □

Using the above two lemmas we prove the main result of this appendix.

Lemma B3. *Let $0 \leqslant s \leqslant t \leqslant T$ and x be an $\mathcal{F}_T$-measurable random variable. Also assume that Θ is strongly rectangular. Then, under the assumption that the minima exist, it holds that*

$$\min_{\theta \in \Theta} E^{Q^\theta} \left[E^{Q^\theta}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right] = \min_{\theta \in \Theta} E^{Q^\theta} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right].$$

Proof. To show that "$\geqslant$" holds, let θ^* be such that

$$E^{Q^{\theta^*}} \left[E^{Q^{\theta^*}}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right] = \min_{\theta \in \Theta} E^{Q^\theta} \left[E^{Q^\theta}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right].$$

Then,

$$\begin{aligned} \min_{\theta \in \Theta} E^{Q^\theta} \left[E^{Q^\theta}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right] &= E^{Q^{\theta^*}} \left[E^{Q^{\theta^*}}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right] \\ &\geqslant E^{Q^{\theta^*}} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right] \\ &\geqslant \min_{\theta \in \Theta} E^{Q^\theta} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right]. \end{aligned}$$

To show that "$\leqslant$" holds, let θ^* and θ'^* be such that

$$E^{Q^{\theta^*}} \left[E^{Q^{\theta'^*}}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right] = \min_{\theta \in \Theta} E^{Q^\theta} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right].$$

Also, define θ^{**} by: $\left(\theta_u^{**}\right)_{0 \leqslant u < t} = \left(\theta_u^*\right)_{0 \leqslant u < t}$ and $\left(\theta_u^{**}\right)_{t \leqslant u \leqslant T} = \left(\theta_u'^*\right)_{t \leqslant u \leqslant T}$. Then, strong rectangularity implies that $\theta^{**} \in \Theta$. Furthermore,

$$\begin{aligned} \min_{\theta \in \Theta} E^{Q^\theta} \left[\min_{\theta' \in \Theta} E^{Q^{\theta'}}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right] &= E^{Q^{\theta^*}} \left[E^{Q^{\theta'^*}}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right] \\ &= E^{Q^{\theta^{**}}} \left[E^{Q^{\theta^{**}}}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right] \\ &\geqslant \min_{\theta \in \Theta} E^{Q^\theta} \left[E^{Q^\theta}[x \mid \mathcal{F}_t] \,\middle|\, \mathcal{F}_s \right], \end{aligned}$$

where the second equality holds by Lemma B2 and the inequality holds by the fact that $\theta^{**} \in \Theta$. □

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