1 Compute the products

a.

$$\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}\begin{pmatrix}-2&\frac{4}{3}&\frac{-1}{3}\\2&\frac{-5}{3}&\frac{2}{3}\\-1&2&-1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

b.

$$\begin{pmatrix}1&2&3&4&5\end{pmatrix}\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}=55$$

c.

$$\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}\begin{pmatrix}1&2&3&4&5\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\2&4&6&8&10\\3&6&9&12&15\\4&8&12&16&20\\5&10&15&20&25\end{pmatrix}$$

d.

$$\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}\begin{pmatrix}-2&\frac{4}{3}&\frac{-1}{3}\\2&\frac{-5}{3}&\frac{2}{3}\\-1&2&-1\end{pmatrix}\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}=\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}$$

e.

$$\begin{pmatrix}x&y&z\end{pmatrix}\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=2(x^2+y^2+z^2+xy+yz+xz)$$

f.

$$\begin{pmatrix}2&1&2&1&2\\0&2&1&2&1\\0&1&2&1&2\\0&2&1&2&1\\0&0&0&0&2\end{pmatrix}\begin{pmatrix}1&2&1&2&1\\0&1&2&1&2\\0&2&1&2&1\\0&1&2&1&2\\0&0&0&0&1\end{pmatrix}=\begin{pmatrix}2&10&8&10&10\\0&6&9&6&10\\0&6&6&6&8\\0&6&9&6&10\\0&0&0&0&2\end{pmatrix}$$

g.

$$\begin{pmatrix}-2&\frac{4}{3}&\frac{-1}{3}\\2&\frac{-5}{3}&\frac{2}{3}\\-1&2&-1\end{pmatrix}\begin{pmatrix}4&\frac{2}{3}&\frac{-2}{3}\\6&\frac{5}{3}&\frac{-2}{3}\\12&\frac{-16}{3}&\frac{10}{3}\end{pmatrix}\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}=\begin{pmatrix}-4&\frac{8}{3}&\frac{-2}{3}\\6&-5&2\\4&\frac{28}{3}&\frac{-16}{3}\end{pmatrix}\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}=\begin{pmatrix}2&0&0\\0&3&0\\4&12&12\end{pmatrix}$$

3 Symmetry

a. Let,

$$A=\begin{pmatrix}1&2&0\\3&-1&4\end{pmatrix}$$

We now compute, AA^T and A^TA.

$$AA^T=\begin{pmatrix}1&2&0\\3&-1&4\end{pmatrix}\begin{pmatrix}1&3\\2&-1\\0&4\end{pmatrix}=\begin{pmatrix}5&1\\1&26\end{pmatrix}$$

$$A^TA=\begin{pmatrix}1&3\\2&-1\\0&4\end{pmatrix}=\begin{pmatrix}1&2&0\\3&-1&4\end{pmatrix}\begin{pmatrix}10&-1&12\\-1&5&-4\\12&-4&16\end{pmatrix}$$

b. Let M be any $m\times n$ matrix. Show that M^TM and MM^T are symmetric.

Proof. Consider, by definition of symmetry, a matrix A is symmetric if and only if $A = A^T$. Then,

$$\begin{aligned} (M^TM)^T &= M^T(M^T)^T && \text{By definition of transpose} \\ \Leftrightarrow &= M^TM && \text{Transpose of a transpose is the original matrix} \end{aligned}$$

And,

$$\begin{aligned} (MM^T)^T &= (M^T)^TM^T && \text{By definition of transpose} \\ \Leftrightarrow &= MM^T && \text{Transpose of a transpose is the original matrix} \end{aligned}$$

So,

$$(M^TM)^T = M^TM, \text{ and, } (MM^T)^T = MM^T$$

Thus, these products are symmetric. □

Next, we consider the size of the resultant matrices, $M^TM = (n\times m)\times(m\times n) = n\times n$ and $MM^T = (m\times n)\times(n\times m) = m\times m$.

Finally, we consider the traces of both M^TM and MM^T. We note that in the text that we proved that,

$$tr(AB) = tr(BA) \Rightarrow tr(M^TM) = tr(MM^T)$$

thus, their traces are equal.

4 Dot product

Consider, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, column vectors. Prove that $x \cdot y = x^TIy$

Proof. Consider first, $x \cdot y$.

$$x \cdot y = \sum_{i=1}^{n} x_iy_i$$

Now, we consider, x^TI. By definition of I, this is x^T. Then,

$$x^TIy = x^Ty = \sum_{i=1}^{n} x_iy_i$$

Thus, we see that $x \cdot y = x^TIy$ □

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