6.3

a. If $p\begin{pmatrix}1\\2\end{pmatrix} = 1$ and $p\begin{pmatrix}2\\4\end{pmatrix} = 3$ is p linear?
If p is a linear function, then

$$p(cx) = cp(x)$$

Assuming to the contrary that p is linear, we note,

$$p\begin{pmatrix}2\\4\end{pmatrix} = 2p\begin{pmatrix}1\\2\end{pmatrix}$$

But,

$$2p\begin{pmatrix}1\\2\end{pmatrix} = 2(1) = 2 \neq 3$$

Thus, linearity has been violated. So p is not linear.

b. If $Q(x^2) = x^3$ and $Q(2x^2) = x^4$ is Q linear?
If Q is a linear function, then

$$Q(cx^2) = cQ(x^2)$$

Assuming to the contrary that Q is linear, we note,

$$Q(2x^2) = 2Q(x^2)$$

But,

$$2Q(x^2) = 2x^3 \neq x^4, \text{ in general}$$

Since this does not hold for $x \neq 2$, Q is not linear in general.

6.5

Let P_n be the space of polynomials of degree n or less on t. Let $L : P_2 \to P_3$ such that $L(1) = 4,\ L(t) = t^3,\ L(t^2) = t - 1$.

a.

$$\begin{aligned} L(1 + t + 2t^2) &= L(1) + L(t) + L(2t^2) \\ &= L(1) + L(t) + 2L(t^2) \\ &= 4 + t^3 + 2(t-1) \\ &= t^3 + 2t + 2 \end{aligned}$$

b.

$$\begin{aligned} L(a + bt + ct^2) &= L(a) + L(bt) + L(ct^2) \\ &= aL(1) + bL(t) + cL(t^2) \\ &= a(4) + bt^3 + c(t-1) \\ &= bt^3 + ct + 4a - c \end{aligned}$$

c. Find a, b, c such that, $L(a + bt + ct^2) = 1 + 3t + 2t^3$

From above, we have $L(a + bt + ct^2) = bt^3 + ct + 4a - c$. Then,

$$bt^3 + ct + 4a - c = 1 + 3t + 2t^3$$

So, we have

$$\begin{aligned} b &= 2 \\ c &= 3 \\ 4a - c &= 1 \\ \Leftrightarrow 4a - 3 &= 1 \\ \Leftrightarrow a &= 1 \end{aligned}$$

6.6

Let $\mathcal{I} : f \to \mathcal{I}f(x)$ where, $\mathcal{I}f(x) := \int_0^x f(t)dt$, where f is a continuous function. Then we shall consider,

$$\begin{aligned}\mathcal{I}(af+bg) &= \int_0^x af(t)+bg(t)dt \\ &= \int_0^x af(t)dt + \int_0^x bg(t)dt \\ &= a\int_0^x f(t)dt + b\int_0^x g(t)dt \\ &= a\mathcal{I}f + b\mathcal{I}g\end{aligned}$$

As required by the definition of a linear operator. Thus, $\mathcal{I}$ is indeed a linear operator on the space of continuous functions.

6.7

Let $z \in \mathbb{C}$, and let $\bar{z} = x - iy$ and $c : \mathbb{R}^2 \to \mathbb{R}^2$ such that $c(x, y) = (x, -y)$.

a. Consider the following, $\alpha = (ax, ay)$ and $\beta = (bx, by)$. Then,

$$\begin{aligned}c(\alpha+\beta) &= c(ax+bx, ay+by) \\ &= (ax+bx, -(ay+by)) \\ &= (ax, -ay) + (bx, -by) \\ &= ac(x, y) + bc(x, y)\end{aligned}$$

As required by the definition of a linear map. Thus, c is a linear map from $\mathbb{R}^2 \to \mathbb{R}^2$.

b. Consider the conjugate operator $\bar{z}$. Let us assume that $\bar{z}$ is a linear operator on $\mathbb{C}$. Then,

$$\bar{z}(\alpha+\beta) = \bar{z}(\alpha) + \bar{z}(\beta)$$

$$\bar{z}(c\alpha) = c\bar{z}(\alpha),\ c \in \mathbb{C}$$

Consider now, $\alpha = x + iy,\ c = i,\ i^2 = -1$. Then,

$$\bar{z}(i\alpha) = \bar{z}(ix - y) = -y - ix$$

But,

$$i\bar{z}(\alpha) = i\big(x - iy\big) = y + ix$$

A clear contradiction to the definition of a linear operator. Thus, $\bar{z}$ cannot be a linear operator on $\mathbb{C}$.