1. For which values of a does $U = \text{span}\{< 2,0,2 >, < 2,4,-6 >, < a,2,0 >\} = \mathbb{R}^3$? For any special values at which $U \neq \mathbb{R}^3$, express U as the span of the least number of vectors possible. Give the dimension of U for these cases.

   (a) To begin, we consider placing the vectors in U in matrix form and row reducing. Thus, we compute,

$$\begin{aligned}
\begin{bmatrix} 2 & 0 & 2 \\ 2 & 4 & -6 \\ a & 2 & 0 \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & -3 \\ a & 2 & 0 \end{bmatrix} & R_1 = \frac{R_1}{2},\ R_2 = \frac{R_2}{2} \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 2 & -a \end{bmatrix} & R_2 = R_2 - R_1,\ R_3 = R_3 - aR_1
\end{aligned}$$

   At this point, we clearly see that in order for U to span $\mathbb{R}^3$, $a \neq 4$, as this would reduce the dimension of this matrix. So, assuming $a \neq 4$,

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 2 & -a \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & -a+4 \end{bmatrix} & R_3 = R_3 - R_2 \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 0 & 1 \end{bmatrix} & R_3 = \frac{R_3}{-a+4} \\
&= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & R_1 = R_1 - R_3,\ R_2 = R_2 - R_3,\ R_2 = \frac{R_2}{2}
\end{aligned}$$

   So, we see that for all values of $a \neq 4$, U spans $\mathbb{R}^3$, as it reduces to the trivial basis.

   (b) We now consider the case of $a = 4$. In this case, we return to the calculations from before, and evaluate,

$$\begin{aligned}
\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 2 & -a \end{bmatrix} &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & -4 \\ 0 & 2 & -4 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} & R_3 = R_3 - R_2,\ R_2 = \frac{R_2}{2}
\end{aligned}$$

   From here, we see that U is the span of only two vectors, $U = \text{span}\{< 1,0,1 >, < 0,1,-2 >\}$

   (c) From here, we see that the dimension of U is two.

2. Let L be the linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$,

$$L(x,y,z) = (2x + y - z, -2x - 4y + 2z, -x - y + 2z)$$

Find the eigenvalues and eigenvectors.

To begin, we first write L in matrix form as,

$$L = \begin{bmatrix} 2 & 1 & -1 \\ -2 & -4 & 2 \\ -1 & -1 & 2 \end{bmatrix}$$

We then compute the eigenvalues as the solution to $(L - \lambda I)v = 0$. This is done by solving the determinant of the $L - \lambda I$ matrix as follows:

$$\begin{aligned}\begin{vmatrix} 2-\lambda & 1 & -1 \\ -2 & -4-\lambda & 2 \\ -1 & -1 & 2-\lambda \end{vmatrix} &= (2-\lambda)\big((-4-\lambda)(2-\lambda) - 2(-1)\big) - \big((2-\lambda)(-2) - 2(-1)\big) - \big((-2)(-1) - (-1)(-4-\lambda)\big) \\ &= (2-\lambda)\big(\lambda^2 + 2\lambda - 6\big) - (2\lambda - 2) - (-\lambda - 2) \\ &= -\lambda^3 + 9\lambda - 8 \\ 0 &= -(\lambda - 1)(\lambda^2 + \lambda - 8)\end{aligned}$$

Solving this, we see that trivially $\lambda_1 = 1$ is a solution. We then use the quadratic formula to solve for the remaining two solutions as,

$$\lambda_{2,3} = \frac{-1 \pm \sqrt{1^2 - 4(1)(-8)}}{2} = \frac{-1 \pm \sqrt{33}}{2}$$

With these eigenvalues in hand, we may compute the corresponding eigenvectors as follows.
$\lambda_1 = 1$

$$\begin{aligned} L - \lambda_1 I &= \begin{bmatrix} 1 & 1 & -1 \\ -2 & -5 & 2 \\ -1 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix} & R_3 = R_3 + R_1,\ R_2 = R_2 + 2R_1 \end{aligned}$$

This yields the solution system,

$$v_1 + v_2 - v_3 = 0$$

$$-3v_2 = 0 \Rightarrow v_2 = 0$$

Back substituting, we find then

$$v_1 - v_3 = 0 \Rightarrow v_1 = v_3$$

The corresponding eigenvector is then,

$$\lambda_1 = 1 \Rightarrow e_1 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

Next, $\lambda_2 = \frac{-1+\sqrt{33}}{2}$

$$\begin{aligned}
L-\lambda_2 I &= \begin{bmatrix} 2-\frac{-1+\sqrt{33}}{2} & 1 & -1 \\ -2 & -4-\frac{-1+\sqrt{33}}{2} & 2 \\ -1 & -1 & 2-\frac{-1+\sqrt{33}}{2} \end{bmatrix} \\
&= \begin{bmatrix} \frac{5-\sqrt{33}}{2} & 1 & -1 \\ -2 & \frac{-7-\sqrt{33}}{2} & 2 \\ -1 & -1 & \frac{5-\sqrt{33}}{2} \end{bmatrix} \\
&= \begin{bmatrix} -2 & \frac{-7-\sqrt{33}}{2} & 2 \\ \frac{5-\sqrt{33}}{2} & 1 & -1 \\ -1 & -1 & \frac{5-\sqrt{33}}{2} \end{bmatrix} && R_1 \leftrightarrow R_2 \\
&= \begin{bmatrix} -2 & \frac{-7-\sqrt{33}}{2} & 2 \\ 0 & \frac{3+\sqrt{33}}{4} & \frac{3-\sqrt{33}}{2} \\ -1 & -1 & \frac{5-\sqrt{33}}{2} \end{bmatrix} && R_2 = R_2 + \frac{5-\sqrt{33}}{2}R_1 \\
&= \begin{bmatrix} -2 & \frac{-7-\sqrt{33}}{2} & 2 \\ 0 & \frac{3+\sqrt{33}}{4} & \frac{3-\sqrt{33}}{2} \\ 0 & \frac{3+\sqrt{33}}{4} & \frac{3-\sqrt{33}}{2} \end{bmatrix} && R-3 = R_3 - \frac{R_1}{2} \\
&= \begin{bmatrix} -2 & \frac{-7-\sqrt{33}}{2} & 2 \\ 0 & 1 & \frac{-7+\sqrt{33}}{2} \\ 0 & 0 & 0 \end{bmatrix} && R_3 = R_3 - R_2,\ R_2 = \frac{R_2}{\frac{3+\sqrt{33}}{4}} \\
&= \begin{bmatrix} -2 & 0 & -2 \\ 0 & 1 & \frac{-7+\sqrt{33}}{2} \\ 0 & 0 & 0 \end{bmatrix} && R_1 = R_1 - \frac{-7-\sqrt{33}}{2}R_2 \\
&= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{-7+\sqrt{33}}{2} \\ 0 & 0 & 0 \end{bmatrix} && R_1 = \frac{-R_1}{2}
\end{aligned}$$

This yields the solution equations,

$$v_1 + v_3 = 0 \Rightarrow v_1 = -v_3$$

$$v_2 + \frac{-7+\sqrt{33}}{2}v_3 = 0 \Rightarrow v_2 = \frac{7-\sqrt{33}}{2}v_3$$

Which yield the eigenvector,

$$\lambda_2 = \frac{-1+\sqrt{33}}{2} \Rightarrow e_2 = \begin{pmatrix} -1 \\ \frac{7-\sqrt{33}}{2} \\ 1 \end{pmatrix}$$

Finally, $\lambda_3 = \frac{-1-\sqrt{33}}{2}$

$$L-\lambda_3 I = \begin{bmatrix} 2-\frac{-1-\sqrt{33}}{2} & 1 & -1 \\ -2 & -4-\frac{-1-\sqrt{33}}{2} & 2 \\ -1 & -1 & 2-\frac{-1-\sqrt{33}}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 1 & -1 \\ -2 & \frac{-7+\sqrt{33}}{2} & 2 \\ -1 & -1 & \frac{5+\sqrt{33}}{2} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 1 & -1 \\ 0 & \sqrt{33}-6 & \frac{9-\sqrt{33}}{2} \\ -1 & -1 & \frac{5+\sqrt{33}}{2} \end{bmatrix} \qquad R_2 = R_2 - \frac{5-\sqrt{33}}{2}R_1$$
$$= \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 1 & -1 \\ 0 & \sqrt{33}-6 & \frac{9-\sqrt{33}}{2} \\ 0 & \frac{-9+\sqrt{33}}{4} & \frac{15+\sqrt{33}}{4} \end{bmatrix} \qquad R_3 = R_3 - \frac{5-\sqrt{33}}{4}$$
$$= \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 1 & -1 \\ 0 & 1 & \frac{-7-\sqrt{33}}{2} \\ 0 & \sqrt{33}-6 & \frac{9-\sqrt{33}}{2} \end{bmatrix} \qquad R_2 \leftrightarrow R_3,\ R_2 = \frac{R_2}{\frac{-9+\sqrt{33}}{4}}$$
$$= \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 1 & -1 \\ 0 & 1 & \frac{-7-\sqrt{33}}{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad R_3 = R_3 - (\sqrt{33}-6)R_2$$
$$= \begin{bmatrix} \frac{5+\sqrt{33}}{2} & 0 & \frac{5+\sqrt{33}}{2} \\ 0 & 1 & \frac{-7-\sqrt{33}}{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 - R_2$$
$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & \frac{-7-\sqrt{33}}{2} \\ 0 & 0 & 0 \end{bmatrix} \qquad R_1 = \frac{R_1}{\frac{5+\sqrt{33}}{2}}$$

This yields the solution equations,

$$v_1 + v_3 = 0 \Rightarrow v_1 = -v_3$$
$$v_2 + \frac{-7-\sqrt{33}}{2}v_3 = 0 \Rightarrow v_2 = \frac{7+\sqrt{33}}{2}v_3$$

The corresponding eigenvector is then,

$$\lambda_3 = \frac{-1-\sqrt{33}}{2} \Rightarrow e_3 = \begin{pmatrix} -1 \\ \frac{7+\sqrt{33}}{2} \\ 1 \end{pmatrix}$$

We now test the independence of the vectors with the determinant of the matrix they produce,

$$\begin{vmatrix} 1 & -1 & -1 \\ 0 & \frac{7-\sqrt{33}}{2} & \frac{7+\sqrt{33}}{2} \\ 1 & 1 & 1 \end{vmatrix} = \frac{7-\sqrt{33}}{2} - \frac{7+\sqrt{33}}{2} - \frac{7+\sqrt{33}}{2} + \frac{7-\sqrt{33}}{2} = -2\sqrt{33} \neq 0$$

So, we see that they are independent.

4. Let L be a linear transformation from vector space V to vector space W, with V a finite dimensional vector space. Define $\ker(L)$ and $L(V)$ and show that they are vector spaces. Then prove that $dim(V) = dim(ker(L)) + dim(L(V))$

   (a) To begin, we consider the kernel and image of V under L.

      i. The kernel of L, denoted as $\ker(L)$ is the set of elements in V which map to the zero vector of W.

$$\ker(L) = \{v \in V \big| L(v) = 0_W\}$$

      We now test for the conditions of a vector space, namely closure under addition and scalar multiplication, and that the zero vector of V is in the kernel.
      First, let $x, y \in \ker(L)$. Then,

$$\begin{aligned} L(x+y) &= L(x) + L(y) && \text{By linearity} \\ &= 0_W + 0_W && \text{By definition of the kernel} \\ &= 0_W \end{aligned}$$

      Thus, we see that the kernel is closed under addition.
      Next, let $x \in \ker(L)$ and let λ be a scalar value. Then,

$$\begin{aligned} L(\lambda x) &= \lambda L(x) && \text{By linearity} \\ &= \lambda 0_W && \text{By construction of } x \\ &= 0_W \end{aligned}$$

      Thus, we see that the kernel is closed under scalar multiplication.
      Finally, consider an arbitrary $v \in V$. Then,

$$L(0_V) = L(v + (-v)) \quad v - v = 0_V \quad = L(v) + L(-v) \quad \text{By linearity} = L(v) - L(v) \quad \text{By linearity} \quad = 0_W$$

      Thus, we see that the zero vector of V is in the kernel of the transformation. Thus, we conclude that the kernel is a vector space as well.

      ii. Next, we consider the image of V under L, $L(V)$. We shall test the same conditions as above to see that it is a vector space. First, we consider $x, y \in L(V)$. By construction, we know that there exist $\eta, \gamma \in V$ such that,

$$L(\eta) = x,\ L(\gamma) = y$$

      Then,

$$L(\eta) + L(\gamma) = L(\eta + \gamma) \in L(V)$$

      Thus, we see that $L(V)$ is closed under addition. Next, we consider $x \in L(V)$ and a scalar value λ. Then, as before, we know that $\exists \eta \in V$ such that $L(\eta) = x$. Then,

$$L(\lambda\eta) = \lambda L(\eta) = \lambda x$$

      Thus, we see that $L(V)$ is closed under scalar multiplication. Finally, we consider the zero vectors inclusion. We note that the kernel of L is clearly contained in $L(V)$. Thus, from before, we know that the zero vector in the kernel, and thus in $L(V)$. So we may conclude that $L(V)$ is a vector space as well.

   (b) We now consider the "Dimension Formula"

$$\dim V = \dim[\ker L] + \dim L(V)$$

   Proof. Let $L : V \to W$, with V being a finite dimensional vector space. First, consider the set

$$S := \{b_1, \ldots, b_p, e_1, \ldots, e_q\}$$

   Where the elements b_i are a basis for the kernel of L, and the elements e_i extend the set to be a full basis for the space V. It is worth noting now that,

$$\dim V = p + q$$

   and

$$\dim[\ker L] = p$$

Now, we consider an arbitrary vector $w \in L(V)$. Then, we may write w as follows:

$$\begin{aligned}
w &= L(c_1b_1 + \cdots + c_pb_p + d_1e_1 + \cdots + d_qe_q) & \\
&= L(c_1b_1) + \cdots + L(c_pb_p) + L(d_1e_1) + \cdots + L(d_qe_q) & \text{By linearity} \\
&= c_1L(b_1) + \cdots + c_pL(b_p) + d_1L(e_1) + \cdots + d_qL(e_q) & \text{Also by linearity} \\
&= d_1L(e_1) + \cdots + d_qL(e_q) & \text{By definition of the kernel}
\end{aligned}$$

So,

$$L(V) = \text{span}\{L(e_1), \ldots, L(e_q)\}$$

To show independence, we assume to the contrary that it is not. Then, by definition there are constants d_i which are not all zero such that

$$\begin{aligned}
0 &= d_1L(e_1) + \ldots + d_qL(e_q) \\
&= L(d_1e_1 + \ldots + d_qe_q)
\end{aligned}$$

But, from before, we defined e_i to be linearly independent. So, $d_1e_1 + \ldots + d_qe_q \neq 0$. Because $d_1e_1 + \ldots + d_qe_q$ is not the zero vector, but still maps to zero, we know that $d_1e_1 + \ldots + d_qe_q$ is in the kernel of L. Thus, from before, $d_1e_1 + \ldots + d_qe_q$ is within the span of $\{b_1, \ldots, b_p\}$ as b_i is a basis for the kernel. This means that S is not a basis for V, a contradition. Thus, we conclude that $L(V)$ is linearally independent. So, $\dim L(V) = q$. So, we have,

$$\begin{aligned}
\dim V &= \dim[\ker L] + \dim L(V) \\
p + q &= (p) + (q)
\end{aligned}$$

As desired. $\square$

5. Let $\{a_n\}_{n\geq 1}$ be a subadditive sequence of positive real numbers. Prove that $\lim_{n\to\infty} \frac{a_n}{n}$ exists.
A subadditive sequence is a sequence where

$$a_{n+m} \leq a_n + a_m,\ n, m \geq 1$$

i.e. the future values of the sequence are limited by the past values. We shall now attempt the proof.

Proof. Let $\{a_n\}_{n\geq 1}$ be a subadditive sequence of positive real numbers, and let M be defined as the smallest value $\frac{a_n}{n}$ for $n \geq 1$. Then, consider

$$M + \delta > \frac{a_n}{n},\ \delta > 0$$

For some n that depends on δ. If we move δ away from our mimimum value of a_n/n, we have the inequality above. Then, we arrive at $a_n < n(M + \delta)$ for our chosen n. We now consider, $\mu = max(a_i)$ for $i \in 1, 2, \ldots, n$, and $p \geq n$. By definition of the integers, we may now write, $p = kn + r$, with $k, r \in \mathbb{Z}$, and $0 \leq r \leq n$. Then,

$$a_p = a_{n+n+\ldots+n+r} \leq ka_n + a_r$$

Noting that r is bounded by n, we conclude that $a_r \leq \mu$. Thus,

$$a_p \leq ka_n + a_r \leq ka_n + \mu$$

We now divide this equality to obtain a more familiar form,

$$\frac{a_p}{p} \leq \frac{ka_n}{p} + \frac{\mu}{p}$$

From before, we know that a_n is strictly bounded by $(M + \delta)n$. So, we may write,

$$\frac{a_p}{p} < \frac{kn(M + \delta)}{p} + \frac{\mu}{p}$$

Finally, we take the limit as $p \to \infty$. On the right, we may clearly see that μ/p will tend to zero as p increases. This leaves

$$\lim_{p\to\infty} \frac{kn}{p}(M + \delta)$$

From our definition of p, we see that this fraction will simplify to 1 in the limit, and then we are left with $M + \delta$. So,

$$\lim_{p\to\infty} \frac{a_p}{p} < M + \delta$$

Because δ is arbitrary, we may allow it to be very small, resulting in our limit being bounded above by the smallest value of the original sequence a_n/n. Because M exists and is finite, we know that the limit is finite as well. □

7. Let X be the shift space over the finite alphabet $\{0,1\}$ with the only restriction being the block $\{00\}$. Find the entropy of the space.

   (a) We consider the shift space X from above. We have our finite alphabet as the set $\mathcal{A} = \{0,1\}$ and our restrictive set $\mathcal{F} = \{00\}$. Let $B_n(x)$ be a block of length n in X, with elements x from the alphabet $\mathcal{A}$. Let us consider the first few values of B in order that we may get a feel for how it looks.

$$B_1(x) = \{0\},\ \{1\}$$

$$B_2(x) = \{01\},\ \{10\},\ \{11\}$$

We now consider an arbitrary B_{n+2} and consider the size of said set. To progress from B_n to B_{n+1} for any n, we must add either a 1 or a 0 to the exisiting block, noting the restriction. In order for a block to be in B_{n+2}, it falls into two categories. The first of which is the block,

$$\underbrace{\dots}_{n+1} \cup 1$$

Here, we have taken an arbitrary element of B_{n+1} and appended a 1 to it. We know that this block will be in B_{n+2}, because the addition of a 1 at the end of any string in B_{n+1} cannot violate our restriction on the space. The only other option for this space are blocks of the form,

$$\underbrace{\underbrace{\dots}_{n} 1}_{n+1} \cup 0$$

From before, we know that we can always add a 1 to a block and remain in X. But, in order to add a zero to the end of a block, we must first check to make sure that the set ends in a 1, else, we will be in violation of our restriction. So, when adding a 1 to a block, we have $|B_{n+1}|$ possibilities to add to, whereas in order to add a 0 to the set, we have only $|B_n|$ possible sets to append to. Thus, we arrive at the function for the size of B_{n+2},

$$|B_{n+2}| = |B_{n+1}| + |B_n|$$

The entropy of this system is,

$$E = \frac{|B_{n+1}|}{|B_n|}$$

We note now that our formula for the size of B_{n+2} is actually a formula for the Fibonacci sequence. From above, we know that

$$|B_1| = 2,\ |B_2| = 3$$

So, we see that

$$|B_3| = 5,\ |B_4| = 8, \dots$$

Because this is functionally similar to the Fibonacci sequence, we may consider the form,

$$F_n = \alpha c_1^n + \beta c_2^n,\ n \geq 1$$

Then, F_{n+2} becomes

$$\alpha c_1^{n+2} + \beta c_2^{n+2} = \alpha c_1^{n+1} + \beta c_2^{n+1} + \alpha c_1^n + \beta c_2^n$$

Simplifying, we find,

$$\alpha c_1^n (c_1^2 - c_1 - 1) + \beta c_2^n (c_2^2 - c_2 - 1) = 0$$

This is only the case when the quadratic expressions are zero, and $c_1 \neq c_2$. Solving, we find the solutions to be,

$$c_1, c_2 = \frac{1 \pm \sqrt{5}}{2}$$

Let c_1 be the larger of the two values. Then,

$$F_n = \alpha \frac{1+\sqrt{5}}{2}^n + \beta \frac{1-\sqrt{5}}{2}^n$$

Rewriting this symbolically, and then factoring, we arrive at,

$$F_n = \alpha c_1^n (1 + \frac{\beta}{\alpha}\left(\frac{c_2}{c_1}\right)^n)$$

Because we have defined $c_1 > c_2$, we note that in the limit of n,

$$F_n = \alpha c_1^n$$

Finally then,

$$\frac{F_{n+1}}{F_n} = \frac{\alpha c_1^{n+1}}{\alpha c_1^n} = c_1 = \frac{1+\sqrt{5}}{2}$$

Our entropy is then defined as,

$$E = \frac{|B_{n+1}|}{|B_n|} = \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$