Question 3. Let $L : \mathbb{R}^2 \to \mathbb{R}^2$ with

$$L\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}x\cos(\theta) + y\sin(\theta)\\ -x\sin(\theta) + y\cos(\theta)\end{pmatrix}$$

a. We write the matrix of L using the basis $\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}$ as,

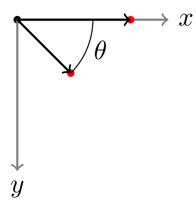
$$L = \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}$$

We note that this is the rotation matrix in two dimensions.

b. When $\theta \neq 0$, we consider how the matrix affects a given vector in the plane. We see that the matrix will actually rotate a vector in the plane by $-\theta$. This is shown below, when the operation,

$$L\begin{pmatrix}1\\0\end{pmatrix} = \begin{bmatrix}\cos(\theta) & \sin(\theta)\\ -\sin(\theta) & \cos(\theta)\end{bmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}\cos(\theta)\\ -\sin(\theta)\end{pmatrix}$$

Is applied with $\theta = \frac{\pi}{4}$.



As we can see, the vector is rotated 45 degrees in the negative direction.

c. Given the nature of the matrix, we expect that the matrix will have invariant directions when $\theta = 0$ and $\theta = \pi$. This corresponds to a rotation of 0 and 180 degrees, or coefficients $-1, 1$ acting on the vector itself.

d. Next, we attempt to find the eigenvalues of L by solving for the characteristic equation. So,

$$\begin{aligned}\begin{vmatrix}\cos(\theta) - \lambda & \sin(\theta)\\ -\sin(\theta) & \cos(\theta) - \lambda\end{vmatrix} &= (\cos(\theta) - \lambda)(\cos(\theta) - \lambda) - (-\sin^2(\theta))\\ &= \cos^2(\theta) - 2\cos(\theta)\lambda + \lambda^2 + \sin^2(\theta)\\ 0 &= \lambda^2 - 2\cos(\theta)\lambda + 1\end{aligned}$$

Solving for lambda,

$$\lambda = \frac{2\cos(\theta) \pm \sqrt{4\cos^2(\theta) - 4}}{2} = \cos(\theta) \pm \sqrt{-\sin^2(\theta)}$$

Which has no real solutions. However, by allowing $i = \sqrt{-1}$, we may find the solutions to be,

$$\lambda_{1,2} = \cos(\theta) \pm i\sin(\theta)$$

Question 4. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ where,

$$L\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}x+y\\x+z\\y+z\end{pmatrix}$$

a. Now, let e_i be a vector of all zeros, save a one in the i^{th} position. We now compute,

$$L\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}1\\1\\0\end{pmatrix}$$
$$L\begin{pmatrix}0\\1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\1\end{pmatrix}$$
$$L\begin{pmatrix}0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}$$

b. We now consider the matrix, $M = \begin{pmatrix} m_1^1 & m_2^1 & m_3^1 \\ m_1^2 & m_2^2 & m_3^2 \\ m_1^3 & m_2^3 & m_3^3 \end{pmatrix}$ From this, we note that Me_i becomes the vector $\begin{pmatrix} m_i^1 \\ m_i^2 \\ m_i^3 \end{pmatrix}$

c. We now compute the matrix representation of L as,

$$M = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

d. We now compute the eigenvalues and vectors of M. So,

$$\begin{aligned} \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & -\lambda & 1 \\ 0 & 1 & 1-\lambda \end{vmatrix} &= (1-\lambda)(-\lambda(1-\lambda)-1)-(1-\lambda) \\ &= -\lambda(1-\lambda)^2-(1-\lambda)-(1-\lambda) \\ &= -\lambda+2\lambda^2-\lambda^3-2+2\lambda \\ &= -\lambda^3+2\lambda^2+\lambda-2 \\ 0 &= -(\lambda-2)(\lambda-1)(\lambda+1) \end{aligned}$$

So, $\lambda = 2, 1, -1$. Then,

$$\begin{aligned} \begin{pmatrix} 1-2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & 1-2 \end{pmatrix} &= \begin{pmatrix} -1 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -1 \end{pmatrix} \\ &= \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So, we see that $x = z$ and $y = z$. Thus,

$$\lambda = 2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Next,

$$\begin{aligned} \begin{pmatrix} 1-1 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 1-1 \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So, we see that $x = -z$ and $y = 0$. Thus,

$$\lambda = 1 \Rightarrow v_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Finally,

$$\begin{aligned} \begin{pmatrix} 1+1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1+1 \end{pmatrix} &= \begin{pmatrix} 2 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So, we see that $x = z$ and $y = -2z$. Thus,

$$\lambda = 2 \Rightarrow v_2 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Question 8. Let,

$$M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

Then,

$$\begin{aligned} \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} &= (a-\lambda)(d-\lambda) - bc \\ &= ad - a\lambda - d\lambda + \lambda^2 - bc \\ P_M(\lambda) &= \lambda^2 - a\lambda - d\lambda + ad - bc \end{aligned}$$

Now, let us set $\lambda = M$ and compute,

$$P_M(M) = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} - \begin{pmatrix} a^2 & ab \\ ac & ad \end{pmatrix} - \begin{pmatrix} ad & bd \\ cd & d^2 \end{pmatrix} + \begin{pmatrix} ad-bc & 0 \\ 0 & ad-bc \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

So, we see that $P_M(M) = 0$ for 2×2 matrices. We now consider the space of 3×3 matrices. For simplicity, we consider matrices of the form,

$$\begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$$

Which for convenience we will write as vectors of the form,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix}$$

As before, we compute the equation,

$$P_M(\lambda) = abc - ac\lambda - bc\lambda + c\lambda^2 - ab\lambda + a\lambda^2 + b\lambda^2 - \lambda^3$$

So, we may compute,

$$\begin{aligned} P_M(M) &= \begin{pmatrix} abc \\ abc \\ abc \end{pmatrix} - \begin{pmatrix} a^2c \\ abc \\ ac^2 \end{pmatrix} - \begin{pmatrix} abc \\ b^2c \\ bc^2 \end{pmatrix} + \begin{pmatrix} a^2c \\ b^2c \\ c^3 \end{pmatrix} - \begin{pmatrix} a^2b \\ ab^2 \\ abc \end{pmatrix} + \begin{pmatrix} a^3 \\ ab^2 \\ ac^2 \end{pmatrix} + \begin{pmatrix} a^2b \\ b^3 \\ bc^2 \end{pmatrix} - \begin{pmatrix} a^3 \\ b^3 \\ c^3 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

So, we see that the equivalence $P_M(M) = 0$ seems to hold for 3×3 matrices.