Question 1.

a. State and prove the Schwarz inequality.
For any non-zero vectors u and v with innerproduct $\langle , \rangle$

$$|\langle x, y\rangle| \leq ||u|| \; ||v||$$

Proof. Let u and v be non-zero vectors with inner product, $\langle , \rangle$, and let x be a scalar. Consider now,

$$\langle u + xv, u + xv\rangle = \langle u, u\rangle + x\langle u, v\rangle + x\langle u, v\rangle + x^2\langle v, v\rangle = \langle u, u\rangle + 2x\langle u, v\rangle + x^2\langle v, v\rangle$$

By definition of the inner product, we know that it is greater than zero, so we have,

$$\langle u, u\rangle + 2x\langle u, v\rangle + x^2\langle v, v\rangle \geq 0$$

We shall expand this into the familiar quadratic form,

$$\underbrace{||v||}_{\text{a}} x^2 + \underbrace{2\langle u, v\rangle}_{\text{b}} x + \underbrace{||u||^2}_{\text{c}} \geq 0$$

For a quadratic function to be greater than or equal to zero, we need the $D \leq 0$, where D is the discriminant of the quadratic formula. Because this condition is satisfied, we see that,

$$4\langle u, v\rangle^2 - 4||u||^2||v||^2 \leq 0$$
$$\Leftrightarrow \langle u, v\rangle^2 \leq ||u||^2||v||^2$$

Taking the principle root of both sides, we arrive at the equation,

$$|\langle u, v\rangle| \leq ||u||||v||$$

Thus, we have proven the Schwarz inequality □

b. Prove for any p and q in $\mathbb{R}^n$,

$$||p - q|| \geq ||p|| - ||q||$$

where $||||$ is the magnitude of a vector in $\mathbb{R}^n$ defined by $||x|| = \left(\sum_i x_i\right)^{1/2}$ and the dot product of two vectors x, y is defined as $x \cdot y = \sum_i x_i y_i$, which is the inner product in $\mathbb{R}^n$.

Proof. We consider,

$$\big(||p|| - ||q||\big)^2$$

Expanding, we arrive at,

$$||p||^2 - 2||p|| \; ||q|| + ||q||^2$$

Note, from the Schwarz inequality,

$$|\langle x, y\rangle| \leq ||u|| \; ||v|| \Leftrightarrow -2|\langle x, y\rangle| \geq -2||u|| \; ||v||$$

Thus,

$$||p||^2 - 2||p|| \; ||q|| + ||q||^2 \leq ||p||^2 - 2\langle p, q\rangle + ||q||^2$$

But,

$$||p - q||^2 = ||p||^2 - 2p \cdot q + ||q||^2 = ||p||^2 - 2\langle p, q\rangle + ||q||^2$$

So, we have

$$||p - q||^2 \geq \big(||p|| - ||q||\big)^2$$

Taking the principle root, we arrive at,

$$||p - q|| \geq \big|||p|| - ||q||\big|$$

Finally, we note that $x \leq |x| \; \forall x \in \mathbb{R}$. So,

$$||p - q|| \geq \big|||p|| - ||q||\big| \geq ||p|| - ||q||$$

As desired. □

c. Now, let θ be the angle between vector p and q. We define,

$$\cos(\theta) = \frac{p \cdot q}{||p||\ ||q||}$$

Consider now, the Schwarz inequality as applied to $\mathbb{R}^n$. Our inner product becomes the dot product, and we have,

$$\begin{aligned}|p \cdot q| &\leq ||p||\ ||q|| \\ -||p||\ ||q|| \leq p \cdot q &\leq ||p||\ ||q|| \\ -1 \leq \frac{p \cdot q}{||p||\ ||q||} &\leq 1\end{aligned}$$

So, we see that the range of $\cos(\theta)$ is $[-1, 1]$

Question 2.

a. A subset U of a vector space V is a subspace of V if U is a vector space with the same operations as V.

b. Prove that the intersection between two subspaces of a vector space is also a subspace.

Proof. Let A and B be distinct subspaces of the vector space V, and let $W = A \cap B$.
First, we note that by the definition of a subspace, $\vec{0} \in A, B$. Thus, $\vec{0} \in W$. So, we have the zero vector.
Second, consider arbitrary elements $x, y \in W$. By construction of W, we know that $x, y \in A$ and $x, y \in B$. Because A and B are subspaces of V, we know that $x + y \in A$ and $x + y \in B$. Thus, we see that $x + y \in W$. So, W is closed under addition.
Finally, we consider c, a scalar value. Then, $\forall x \in W$, we know that both x and cx are in A and B because they are subspaces. Thus, $cx \in W\ \forall x \in W$, and we see that W is closed under scalar multiplication.
So, we may conclude that W is also a subspace of V. □

c. Prove that the union between two subspaces of a vector space is also a subspace.

Proof. We consider an example. Let $A = \{(x, 0)|x \in \mathbb{R}\}$, and $B = \{(0, y)|y \in \mathbb{R}\}$, i.e. the x and y axes. We know that these sets are subspaces of the vector space $\mathbb{R}^2$. Let us now construct,

$$W = \{(x, 0), (0, y)|x, y \in \mathbb{R}\} = A \cup B$$

and consider,

$$x^* = (1, 0),\ y^*(0, 1)$$

By definition, x^* and y^* are in W, but,

$$x^* + y^* = (1, 1) \notin W$$

So, W is not closed under addition, a violation of the definition of a vector subspace. Thus, we have reached a contradiction. Hence, in general, a set formed from the union of two vector subspaces will not also be a subspace. □

Question 3. Let,

$$A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix}$$

a. Compute $A^T A^{-1}$. First, we compute A^{-1} by finding $det(A)$.

$$\begin{vmatrix} 2 & 1 \\ 3 & -1 \end{vmatrix} = (2)(-1) - (1)(3) = -5$$

Then,

$$A^{-1} = \frac{1}{-5} \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix}$$

So, we may compute,

$$A^T A^{-1} = \frac{1}{-5} \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ -3 & 2 \end{bmatrix} = \frac{1}{-5} \begin{bmatrix} -2-9 & -2+6 \\ -1+3 & -1-2 \end{bmatrix} = \begin{bmatrix} \frac{11}{5} & \frac{-4}{5} \\ \frac{-2}{5} & \frac{3}{5} \end{bmatrix}$$

b. A matrix A is called symmetric if
$$A = A^T$$
Here,
$$A = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \neq \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix} = A^T$$
So A is not symmetric.

c. Let $f(x) = 2x^2 - 2$. What is the trace of $f(A)$?
We begin by computing
$$f(A) = 2A^2 - 2I$$
$$A^2 = \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 4+3 & 2-1 \\ 6-3 & 3+1 \end{bmatrix} = \begin{bmatrix} 7 & 1 \\ 3 & 4 \end{bmatrix}$$
Then,
$$f(A) = 2A^2 - 2I = 2\begin{bmatrix} 7 & 1 \\ 3 & 4 \end{bmatrix} - 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 2 \\ 6 & 8 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 12 & 2 \\ 6 & 6 \end{bmatrix}$$
Then,
$$tr(f(A)) = \sum_i A_{ii} = 12 + 6 = 18$$

Question 4. Let A be any 2x2 matrix. Then,
$$\frac{1}{2}(tr(A))^2 - \frac{1}{2}(tr(A^2)) - det(A) = 0$$

Proof. Let A be an arbirtrary matrix of dimension 2×2, with elements a, b, c, d constructed as,
$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$
Consider first,
$$tr(A) = a + d$$
Then
$$\frac{1}{2}(tr(A))^2 = \frac{1}{2}(a+d)^2 = \frac{a^2}{2} + ad + \frac{d^2}{2}$$
Next,
$$A^2 = AA = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{bmatrix}$$
So,
$$tr(A^2) = a^2 + 2bc + d^2 \Leftrightarrow \frac{1}{2}(tr(A^2)) = \frac{a^2}{2} + bc + \frac{d^2}{2}$$
Finally,
$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$
So,
$$\begin{aligned} \frac{1}{2}(tr(A))^2 - \frac{1}{2}(tr(A^2)) - det(A) &= (\frac{a^2}{2} + ad + \frac{d^2}{2}) - (\frac{a^2}{2} + bc + \frac{d^2}{2}) - (ad - bc) \\ &= (ad - bc) - (ad - bc) \\ &= 0 \end{aligned}$$
As desired. □

Question 5. Let L be the linear transformation from $\mathbb{R}^3 \to \mathbb{R}^3$ given by,

$$L(x,y,z) = (-2x - y + z, -x - 2y + z, x + y - 2z)$$

We consider the construction of L in matrix form, i.e.

$$L = \begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix}$$

Such that,

$$\begin{bmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2x - y + z \\ -x - 2y + z \\ x + y - 2z \end{pmatrix}$$

From this definition, it is clear that the rows of L are constructed as the coefficients for each row of the output vector, thus,

$$L = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

Consider now, the computations of the eigenvalues of this matrix as $det(L - \lambda I) = 0$ So,

$$\begin{aligned} \det(L - \lambda I) &= \begin{vmatrix} -2-\lambda & -1 & 1 \\ -1 & -2-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{vmatrix} \\ &= (-2-\lambda)((-2-\lambda)^2 - 1) - (-1)(2 + \lambda - 1) + (1)(-1 + 2 + \lambda) \\ &= -\lambda^3 - 6\lambda^2 - 9\lambda - 4 \end{aligned}$$

Plotting this characteristic polynomial, we see that $\lambda = -4$ is a root, as well as $\lambda = -1$ being a multiple root. Thus, we have our three roots of this polynomial, $\lambda_1 = -4,\ \lambda_2 = \lambda_3 = -1$. We begin construction of the eigenvectors as follows,

$$(L - \lambda_1 I)\vec{e}_1 = \vec{0},\ (L - \lambda_2 I)\vec{e}_2 = \vec{0},\ (L - \lambda_3 I)\vec{e}_3 = \vec{0}$$

$$\begin{aligned} (L - \lambda_1 I)\vec{e}_1 &= \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} \\ &= \begin{bmatrix} 2 & -1 & 1 \\ 0 & \frac{3}{2} & \frac{3}{2} \\ 0 & \frac{3}{2} & \frac{3}{2} \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} && R_2{+}= \frac{1}{2}R_1,\ R_3{-}= \frac{1}{2}R_1 \\ &= \begin{bmatrix} 2 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} && R_2 = \frac{2}{3}R_2,\ R_3{-}= R_2 \\ &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} && R_1{+}= R_2,\ R_1 = \frac{1}{2}R_1 \end{aligned}$$

Setting equal to $\vec{0}$ we find,

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\varepsilon_1 = -\varepsilon_3, \quad \varepsilon_2 = -\varepsilon_3$$

So, taking $\varepsilon_3 = 1$ for simplicity,

$$\lambda_1 \Rightarrow \vec{e}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Similarly,

$$\begin{aligned}(L-\lambda_{2,3}I)\vec{e}_{2,3} &= \begin{bmatrix}-1 & -1 & 1\\ -1 & -1 & 1\\ 1 & 1 & -1\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix}\\ &= \begin{bmatrix}1 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix} && R_2- = R_1,\ R_3+ = R_1,\ R_1 = -R_1\end{aligned}$$

Setting equal to $\vec{0}$ we find,

$$\begin{bmatrix}1 & 1 & -1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\end{bmatrix} = \begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$$

$$\varepsilon_1+\varepsilon_2=\varepsilon_3$$

Here, we must fix two variables to arrive at a vector, thus we have multiple eigenvectors:

$$\lambda_2,\ \varepsilon_1=0,\ \varepsilon_2=1 \Rightarrow \vec{e}_2 = \begin{bmatrix}0\\ 1\\ 1\end{bmatrix}$$

$$\lambda_3,\ \varepsilon_1=1,\ \varepsilon_2=0 \Rightarrow \vec{e}_3 = \begin{bmatrix}1\\ 0\\ 1\end{bmatrix}$$

Finally, as a check, we apply the linear transformation L onto each of these vectors to see if they are true eigenvectors.

$$L\vec{e}_1 = \begin{bmatrix}-2 & -1 & 1\\ -1 & -2 & 1\\ 1 & 1 & -2\end{bmatrix}\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix} = \begin{bmatrix}2+1+1\\ 1+2+1\\ -1-1-2\end{bmatrix} = \begin{bmatrix}4\\ 4\\ -4\end{bmatrix} = -4\begin{bmatrix}-1\\ -1\\ 1\end{bmatrix} = \lambda_1\vec{e}_1$$

$$L\vec{e}_2 = \begin{bmatrix}-2 & -1 & 1\\ -1 & -2 & 1\\ 1 & 1 & -2\end{bmatrix}\begin{bmatrix}0\\ 1\\ 1\end{bmatrix} = \begin{bmatrix}0-1+1\\ 0-2+1\\ 0+1-2\end{bmatrix} = \begin{bmatrix}0\\ -1\\ -1\end{bmatrix} = -1\begin{bmatrix}0\\ 1\\ 1\end{bmatrix} = \lambda_2\vec{e}_2$$

$$L\vec{e}_3 = \begin{bmatrix}-2 & -1 & 1\\ -1 & -2 & 1\\ 1 & 1 & -2\end{bmatrix}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} = \begin{bmatrix}-2+0+1\\ -1+0+1\\ 1+0-2\end{bmatrix} = \begin{bmatrix}-1\\ 0\\ -1\end{bmatrix} = -1\begin{bmatrix}1\\ 0\\ 1\end{bmatrix} = \lambda_3\vec{e}_3$$

Hence, we hace found the eigenvalues and the eigenvectors associated with them. To test whether these vectors are linearly independent, we need,

$$det([\vec{e}_1,\vec{e}_2,\vec{e}_3]) \neq 0$$

Checking,

$$\begin{vmatrix}-1 & 0 & 1\\ -1 & 1 & 0\\ 1 & 1 & 1\end{vmatrix} = -1(1-0)+0+1(-1-1) = -1-2 = -3 \neq 0$$

Thus, our eigenvectors are independent.