1. Consider the space B^n, the set of $n \times 1$ vectors with entries from the field $\mathbb{Z}_2$.

   (a) We consider the cardinality of B^n. Because each entry in a vector u from B^3 is from the set $(0, 1)$, we may calculate $|B^3| = 2^3 = 8$.

   (b) We now consider a possible span S over the space as,

   $$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

   (c) We now test whether this set actually spans B^3. Let the three vectors from above be denoted as $e_1,\ e_2,\ e_3$.

   $$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = e_1$$
   $$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = e_2$$
   $$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = e_3$$
   $$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = e_1 + e_2$$
   $$\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} = e_1 + e_3$$
   $$\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} = e_2 + e_3$$
   $$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} = 0e_1 + 0e_2 + 0e + 3$$
   $$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = e_1 + e_2 + e_3$$

   Hence, we see that the vectors span the space.

   (d) Finally, we consider the idea of spanning the space with two vectors. Let u and v be two vectors that supposedly span B^3. By the definition of the field $\mathbb{Z}_2$, we know that multiplying u or v by an integer greater than or equal to 2 may be mapped to the multiplication by 0 or 1. Thus, there are 2^2 possible linear combinations of u and v. i.e.,

   $$\begin{matrix} 0u + 0v & 1u + 0v \\ 1u + 0v & 1u + 1v \end{matrix}$$

   But, we know the cardinality of $B^3 = 8$. So, we see that u and v cannot span the space. So, we see that B^3 cannot be spanned by two vectors.

2. Consider now, $e_i \in \mathbb{R}^n$ being the vector of zeros with a 1 in the i^{th} position.

   (a) Let $E = \{e_1, e_2, \ldots, e_n\}$. We shall test for linear independence.
   Let Γ be an arbitrary element of E. If E is linearly dependent, then we may write,

   $$\Gamma = \sum_i k_i \hat{e}_i,\ \hat{e}_i \in E \setminus \Gamma$$

Without loss of generality, we shall assume that Γ has a 1 in the 1^{st} position. Then, $E \setminus \Gamma$, contains vectors with $1's$ in every other position. But, any sum of these elements will have the following form,

$$\begin{pmatrix} 0 \\ \lambda_2 \\ \lambda_3 \\ \vdots \\ \lambda_n \end{pmatrix}$$

But, for any set λ_i this can never equal Γ. Thus, the set is linearly independent.

(b) Consider now, a vector $v \in \mathbb{R}^n$. This vector takes the form,

$$v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$$

Consider now,

$$\sum_{i=1}^{n} (v \cdot e_i) e_i$$

By the definition of e_i, we know that,

$$v \cdot e_i = v_i$$

As the only non-zero entry of e_i is a 1 in the i^{th} position. Then, $(v \cdot e_i)e_i$ is the vector of zeros with the i^{th} position equal to v_i. So,

$$\sum_{i=1}^{n} (v \cdot e_i) e_i = \begin{pmatrix} v_1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ 0 \\ \vdots \\ 0 \end{pmatrix} + \ldots + \begin{pmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ v_n \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \\ \vdots \\ v_n \end{pmatrix} = v$$

(c) Finally, we note,

$$span(\{e_1, e_2, \ldots, e_n\}) = \mathbb{R}^n$$