Question 2. Let $L : V \to W$ be a linear transformation. Show that $ker(L) = \{0_V\}$ if and only if L is one-to-one.

Proof.
$\Rightarrow$
Let L be the linear transformation defined above, and consider first the case wherein $ker(L) = \{0_V\}$. Then, let $x, y \in V$ exist, such that $f(x) = f(y)$. Then,

$$\begin{aligned} f(x) &= f(y) \\ f(x) - f(y) &= 0 \\ f(x-y) &= 0 && \text{By Linearity} \\ x - y &\in ker(L) && \text{By definition of the kernel} \end{aligned}$$

Then, because $ker(L) = \{0_V\}$,

$$x - y = 0_V x \qquad = y$$

Thus, we have shown that the trivial kernel induces an injective transformation.
$\Leftarrow$
Now, let us consider the case where L is injective. Let $x \in ker(L)$. Then, by definition,

$$L(x) = 0$$

But, because L is linear, we know that

$$L(0_V) = 0$$

So,

$$L(x) = 0 = L(0_V)$$

Because L is injective, we note,

$$x = 0_V$$

So,

$$ker(L) = 0_V$$

Thus, L being injective has induced the trivial kernel, as desired. □

Question 6. Let $P_n(x)$ be the space of polynomials in x of degree less than or equal to n, and let the derivative operator,

$$\frac{d}{dx} : P_n(x) \to P_n(x)$$

a. First, we consider the kernel of this operator. By definition, this is the set of elements in $P_n(x)$ which map to zero under the derivative. For polynomials, this is trivially the set of constants, or any function of the form,

$$P_0(x) = cx^0, \ c \in \mathbb{R}$$

Thus, we conclude that the kernel of the derivative has degree 0.

b. Next, we consider the image of this operator. For polynomials, the degree of the derivative is simply one less than the original polynomial. Thus, for a polynomial of degree less than or equal to n, the derivative will have degree less than or equal to $n-1$. So, the dimension of the image of the derivative is $n-1$. If the target space is changed from $P_n(x)$ to either $P_{n-1}(x)$ or $P_{n+1}(x)$, the dimension of the image will remain the same, as it is entirely dependent on the domain of the transformation.

We now consider $P_2(x, y)$ which is the space of polynomials of degree ≤ 2, and the operator

$$L := \frac{\partial}{\partial x} + \frac{\partial}{\partial y} : P_2(x, y) \to P_2(x, y)$$

We consider the kernel of this operator. An element $\eta \in ker(L)$ must be defined, such that

$$\frac{\partial \eta}{\partial x} + \frac{\partial \eta}{\partial y}$$

From this definition, we may conclude that the kernel of L is the set,

$$ker(L) := \{0,\ x+y,\ x-y\}$$

Writing this in basis form, we need,

$$\{x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ -1 \end{bmatrix}\}$$

From our set definition, we see that the kernel of the transformation has degree 1. We also note that $L(P_2(x,y))$ will map to $P_1(x,y)$ due to the polynomial structure of P. Then, we consider the dimension formula,

$$dim(V) = dim(ker(V)) + dim(L(V))$$

$$dim(P_2) = dim(ker(P_2)) + dim(P_1)$$

$$2 = 1 + 1$$

As desired.