1 Compute the products

a.
$$\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}\begin{pmatrix}-2&\frac{4}{3}&\frac{-1}{3}\\2&\frac{-5}{3}&\frac{2}{3}\\-1&2&-1\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

b.
$$\begin{pmatrix}1&2&3&4&5\end{pmatrix}\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}=55$$

c.
$$\begin{pmatrix}1\\2\\3\\4\\5\end{pmatrix}\begin{pmatrix}1&2&3&4&5\end{pmatrix}=\begin{pmatrix}1&2&3&4&5\\2&4&6&8&10\\3&6&9&12&15\\4&8&12&16&20\\5&10&15&20&25\end{pmatrix}$$

d.
$$\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}\begin{pmatrix}-2&\frac{4}{3}&\frac{-1}{3}\\2&\frac{-5}{3}&\frac{2}{3}\\-1&2&-1\end{pmatrix}\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}=\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}$$

e.
$$\begin{pmatrix}x&y&z\end{pmatrix}\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=2(x^2+y^2+z^2+xy+yz+xz)$$

f.
$$\begin{pmatrix}2&1&2&1&2\\0&2&1&2&1\\0&1&2&1&2\\0&2&1&2&1\\0&0&0&0&2\end{pmatrix}\begin{pmatrix}1&2&1&2&1\\0&1&2&1&2\\0&2&1&2&1\\0&1&2&1&2\\0&0&0&0&1\end{pmatrix}=\begin{pmatrix}2&10&8&10&10\\0&6&9&6&10\\0&6&6&6&8\\0&6&9&6&10\\0&0&0&0&2\end{pmatrix}$$

g.
$$\begin{pmatrix}-2&\frac{4}{3}&\frac{-1}{3}\\2&\frac{-5}{3}&\frac{2}{3}\\-1&2&-1\end{pmatrix}\begin{pmatrix}4&\frac{2}{3}&\frac{-2}{3}\\6&\frac{5}{3}&\frac{-2}{3}\\12&\frac{-16}{3}&\frac{10}{3}\end{pmatrix}\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}=\begin{pmatrix}-4&\frac{8}{3}&\frac{-2}{3}\\6&-5&2\\4&\frac{28}{3}&\frac{-16}{3}\end{pmatrix}\begin{pmatrix}1&2&1\\4&5&2\\7&8&2\end{pmatrix}=\begin{pmatrix}2&0&0\\0&3&0\\4&12&12\end{pmatrix}$$

3 Symmetry

a. Let,
$$A=\begin{pmatrix}1&2&0\\3&-1&4\end{pmatrix}$$

We now compute, AA^T and A^TA.

$$AA^T=\begin{pmatrix}1&2&0\\3&-1&4\end{pmatrix}\begin{pmatrix}1&3\\2&-1\\0&4\end{pmatrix}=\begin{pmatrix}5&1\\1&26\end{pmatrix}$$
$$A^TA=\begin{pmatrix}1&3\\2&-1\\0&4\end{pmatrix}=\begin{pmatrix}1&2&0\\3&-1&4\end{pmatrix}\begin{pmatrix}10&-1&12\\-1&5&-4\\12&-4&16\end{pmatrix}$$

b. Let M be any $m\times n$ matrix. Show that M^TM and MM^T are symmetric.

Proof. Consider, by definition of symmetry, a matrix A is symmetric if and only if $A = A^T$. Then,

$$\begin{aligned}(M^TM)^T &= M^T(M^T)^T && \text{By definition of transpose}\\ \Leftrightarrow &= M^TM && \text{Transpose of a transpose is the original matrix}\end{aligned}$$

And,

$$\begin{aligned}(MM^T)^T &= (M^T)^TM^T && \text{By definition of transpose}\\ \Leftrightarrow &= MM^T && \text{Transpose of a transpose is the original matrix}\end{aligned}$$

So,

$$(M^TM)^T = M^TM, \text{ and, } (MM^T)^T = MM^T$$

Thus, these products are symmetric. □

Next, we consider the size of the resultant matrices, $M^TM = (n\times m)\times(m\times n) = n\times n$ and $MM^T = (m\times n)\times(n\times m) = m \times m$.

Finally, we consider the traces of both M^TM and MM^T. We note that in the text that we proved that,

$$tr(AB) = tr(BA) \Rightarrow tr(M^TM) = tr(MM^T)$$

thus, their traces are equal.

4 Dot product

Consider, $x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ and $y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}$, column vectors. Prove that $x \cdot y = x^TIy$

Proof. Consider first, $x \cdot y$.

$$x \cdot y = \sum_{i=1}^{n} x_iy_i$$

Now, we consider, x^TI. By definition of I, this is x^T. Then,

$$x^TIy = x^Ty = \sum_{i=1}^{n} x_iy_i$$

Thus, we see that $x \cdot y = x^TIy$ □

6 Matrix of a linear transformation

Let V be a vector space where $B = (v_1, v_2)$ is an ordered basis. Define,

$$L : V \to V$$

with,

$$\begin{aligned}L(v_1) &= v_1 + v_2\\ L(v_2) &= 2v_1 + v_2\end{aligned}$$

We now consider the following,

$$(L(v_1), L(v_2) = \big((v_1, v_2)\begin{pmatrix}1\\1\end{pmatrix}, (v_1, v_2)\begin{pmatrix}2\\1\end{pmatrix}\big) = (v_1, v_2)\begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix}$$

Hence,

$$L = \begin{pmatrix}1 & 2\\ 1 & 1\end{pmatrix},\ tr(L) = 2$$

We now conside the new basis,

$$B' = (av_1 + bv_2, cv_1 + dv_2)$$

Now,

$$\begin{aligned} L(av_1 + bv_2) &= a(v_1 + v_2) + b(2v_1 + v_2) &&= (a+2b)v_1 + (a+b)v_2 \\ L(cv_1 + d_v2) &= c(v_1 + v_2) + d(2v_1 + v_2) &&= (c+2d)v_1 + (c+d)v_2 \end{aligned}$$

So, as before, we compute,

$$L = \begin{pmatrix} a+2b & c+2d \\ a+b & c+d \end{pmatrix},\ tr(L) = a + 2b + c + d$$

We see that the trace of this matrix is heavily influenced by the basis from which the transformation was constructed. Without making note of the bases used to compute the transformation, the trace is meaningless.