1. Determine if $x - x^3 \in span\{x^2, 2x + x^2, x + x^3\}$
   Let $\alpha = x^2$, $\beta = 2x + x^2$, and $\gamma = x + x^3$, and consider,

$$-\alpha + \beta - \gamma = -(x^2) + (2x + x^2) - (x + x^3) = (2x - x) + (x^2 - x^2) + (-x^3) = x - x^3$$

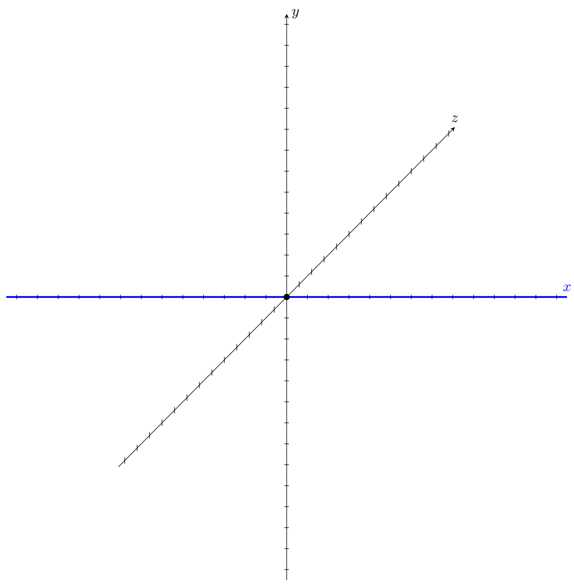
   Thus, we see that $x - x^3$ is the linear combination of elements of the set, and thus is in the span.

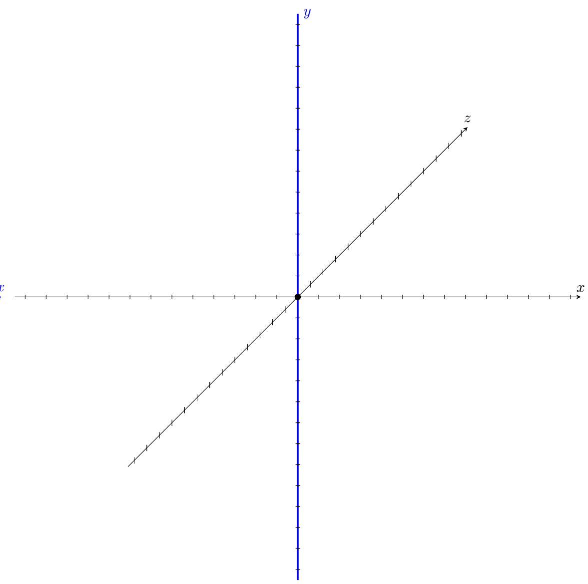
2. Let U and W be subspaces of V. Are:

   (a) $U \cup W$

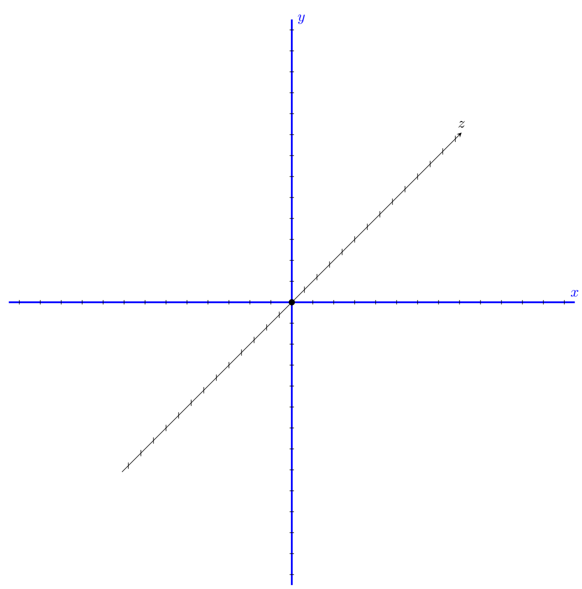
   (b) $U \cap W$

   (a) No. Consider the vector space, $\mathbb{R}^3$, with subspaces consisting of the x-axis and y-axis respectibely. i.e. $U = \{\{x, 0, 0\} | x \in \mathbb{R}\}$, $W = \{\{0, y, 0\} | y \in \mathbb{R}\}$ (Shown below)

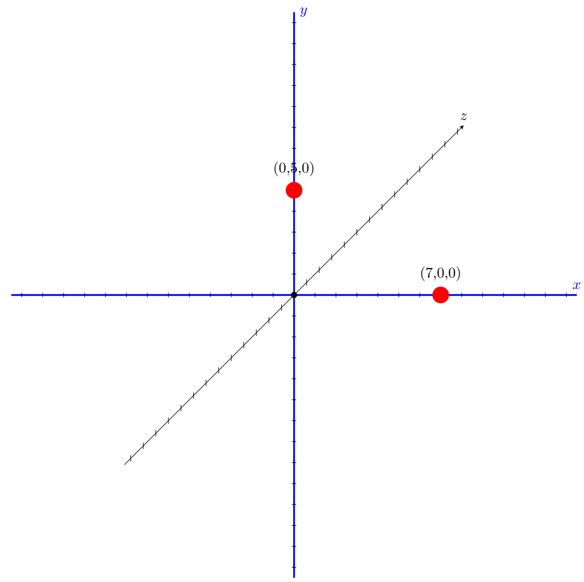




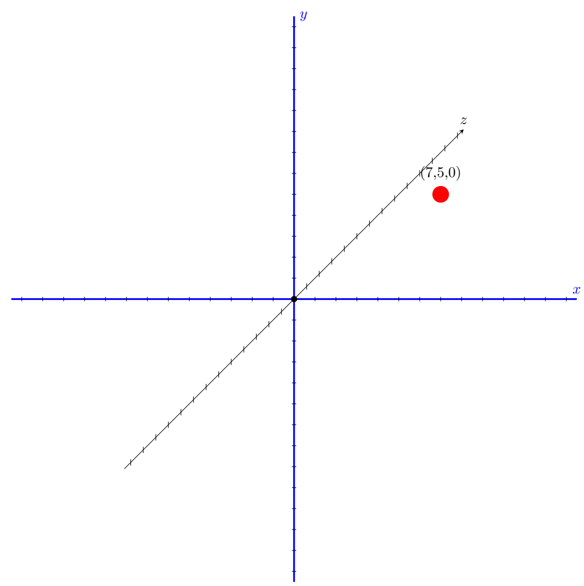
   Then, we consider their union (shown below)



   Consider now the elements $\{7, 0, 0\} \in U$ and $\{0, 5, 0\} \in W$



Which are both contained in the union. But their sum, $\{7, 5, 0\}$ is not contained in the union as seen below.



This violates the closure under addition property of a vector space. Thus, we conclude the the union of two vector spaces is not a subspace in general.

(b) Let $\eta = U \cup W$, where $U, W \subseteq V$. Let us now consider whether η is itself a subspace of V. First, we note, $\vec{0} \in U, W$. Then, by construction, $\vec{0} \in \eta$. Then, we know the zero element is in η. Next, consider an arbitrary $e \in \eta$. Then, we know that $e \in U \wedge e \in W$. Because U and W are both subspaces, $\forall e \in U, W\ \lambda e \in U, W$ for a scalar λ. Because $\lambda e \in U, W\ \lambda e \in \eta$. Thus, η is closed under scalar multiplication. Finally, consider $x, y \in \eta$. Then, $x, y \in U, W \Rightarrow x + y \in U, W$ because they are both subspaces. Thus, $x + y \in \eta$. Thus it is closed under addition. So, we see that η is a subset as well.

3. Let $L : \mathbb{R}^3 \to \mathbb{R}^3$ where,

$$L(x, y, z) = (x + 2y + z,\ 2x + y + z,\ 0)$$

Find $ker(L)$, $im(L)$ and the eigenspaces of $\mathbb{R}^3_{-1}, \mathbb{R}^3_3$.

(a) To begin, we construct the matrix defined by L as,

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We then set this matrix equal to the zero vector and reduce,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & -3 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 0 & \frac{1}{3} & 0 \\ 0 & 1 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From this reduction, we glean the following equations,

$$v_1 = -\frac{1}{3}v_3,\ v_2 = -\frac{1}{3}v_3$$

Thus, the kernel of the transformation is,

$$\lambda \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix},\ \lambda \in \mathbb{R}$$

(b) We now consider the image of L as,

$$im(L) = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 2y + z \\ 2x + y + z \\ 0 \end{bmatrix}$$

(c) Finally, we consider the two eigenspaces.

i. $\mathbb{R}^3_{-1}$: Consider the difference, $(L + I)v = 0$

$$L + I = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 2 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

Then,

$$\left[\begin{array}{ccc|c} 2 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then we arrive at the equations,

$$v_1 = -v_2,\ v_3 = 0$$

So our eigenvector is,

$$\eta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \eta \in \mathbb{R}$$

So the eigenspace is $span\{\eta \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} | \eta \in \mathbb{R}\}$.

ii. R^3_3 : Consider the difference, $(L - 3I)v = 0$

$$L - 3I = \begin{bmatrix} -2 & 2 & 1 \\ 2 & -2 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

Then,

$$\left[\begin{array}{ccc|c} -2 & 2 & 1 & 0 \\ 2 & -2 & 1 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Then we arrive at the equations,

$$v_1 = v_2,\ v_3 = 0$$

So our eigenvector is,

$$\eta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \eta \in \mathbb{R}$$

So the eigenspace is $span\{\eta \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} | \eta \in \mathbb{R}\}$.