1. Evaluate exactly $\lim_{x\to\infty} erf(x)$, where,

$$erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt$$

To begin, we consider the function $I = erf(R)$. Then, we begin by squaring I and analyzing the resultant integral.

$$I^2 = \frac{4}{\pi}\int_0^R e^{-x^2}dx\int_0^R e^{-y^2}dy$$

Fubini's Theorem allows us to combine these integrals into a double integral.

$$I^2 = \frac{4}{\pi}\int_0^R\int_0^R e^{-(x^2+y^2)}dxdy$$

Noting the sum of squares in the exponent, we shall consider a transformation of coordinates to polar coordinates using $x = r\cos(\theta),\ y = r\sin(\theta),\ dxdy = rdrd\theta$. Given that the original double integral would cover the region of quadrant I, we shall use the bounds on the integrals, $0 \leq r \leq R,\ 0 \leq \theta \leq \frac{\pi}{2}$, taking the limit as r tends towards infinity at the end to compute the final value. Thus, we compute the integral as follows.

$$\begin{aligned}
\frac{\pi I^2}{4} &= \int_0^R\int_0^R e^{-(x^2+y^2)}dxdy \\
&= \int_0^{\frac{\pi}{2}}\int_0^R re^{-(r^2\sin(\theta)^2+r^2\cos(\theta)^2)}drd\theta \\
&= \int_0^{\frac{\pi}{2}}\int_0^R re^{-r^2}drd\theta \\
&= \frac{\pi}{2}\int_0^R re^{-r^2}dr
\end{aligned}$$

Noting that this integrand has a clear antiderivative, we may solve directly,

$$\begin{aligned}
&= \frac{\pi}{2}\left(-\frac{1}{2e^{r^2}}\Big|_0^R\right) \\
&= \frac{\pi}{2}\left(-\frac{1}{2e^{R^2}}+\frac{1}{2}\right) \\
&= \frac{\pi}{4}\left(-\frac{1}{e^{R^2}}+1\right)
\end{aligned}$$

Thus, we see,

$$I^2 = -\frac{1}{e^{R^2}}+1$$

Taking the limit as $R\to\infty$, we see,

$$I^2 = 1$$

Thus, we may compute $I = 1$, as the erf function is strictly positive for positive x. So,

$$erf(\infty) = \frac{2}{\sqrt{\pi}}\int_0^\infty e^{-t^2}dt = 1$$

2. We shall compute the following Taylor expansions based on our known series,

$$e^x = \sum_{k=0}^\infty \frac{x^k}{k!},\ \sin(x) = \sum_{k=0}^\infty \frac{(-1)^k x^{2k+1}}{(2k+1)!},\ \cos(x) = \sum_{k=0}^\infty \frac{(-1)^k x^{2k}}{(2k)!}$$

(a) Degree 2 approximation of $f(x) = e^{-3x}$ at $x_0 = 0$
Using our known exponential expansion, we substitute this new exponent as,

$$e^{-3x} = \sum_{k=0}^\infty \frac{(-3x)^k}{k!} = \sum_{k=0}^\infty \frac{3^k(-1)^k(x)^k}{k!}$$

Then, our degree 2 expansion is,

$$f(x) \approx 1 - 3x + \frac{9x^2}{2}$$

So,

$$f(0.02) \approx 1 - 3(0.02) + \frac{9(0.02)^2}{2} \approx 0.9418$$

(b) Degree 4 approximation of $f(x) = \frac{sin(x)}{x}$ with $x_0 = 0$
Using our known $\sin(x)$ expansion, we substitute this new function as,

$$\frac{\sin(x)}{x} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{x(2k+1)!} = \sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k+1)!}$$

Then our degree 4 expansion is,

$$f(x) \approx 1 - \frac{x^2}{6} + \frac{x^4}{120}$$

So,

$$f(0.5) \approx 1 - \frac{0.5^2}{6} + \frac{0.5^4}{120} \approx 0.958854$$

(c) Degree 12 approximation of $f(x) = \cos(x^3)$ with $x_0 = 0$
Using our known $\cos(x)$ expansion, we substitute this new function as,

$$\cos(x^3) = \sum_{k=0}^{\infty} \frac{(-1)^k (x^3)^{2k}}{(2k)!} = \sum_{k=0}^{\infty} \frac{(-1)^k x^{6k}}{(2k)!}$$

Then our degree 12 approximation is,

$$f(x) \approx 1 - \frac{x^6}{2} + \frac{x^{12}}{24}$$

So,

$$f(0.1) \approx 1 - \frac{(0.1)^6}{2} + \frac{(0.1)^{12}}{24} \approx 0.9999995$$

3. Expand,

$$\frac{4f(x-h) - f(x-2h) - 3f(x)}{2h}$$

To the third order term.
We factor each term individually,

$$f(x-h) = f(x) - hf'(x) + \frac{h^2}{2}f''(x) - \frac{h^3}{6}f'''(x)$$
$$f(x-2h) = f(x) - 2hf'(x) + \frac{4h^2}{2}f''(x) - \frac{8h^3}{6}f'''(x)$$

Combining these terms with the above coefficients, we arrive at,

$$4f(x-h) - f(x-2h) = \Big(4f(x) - f(x)\Big) + \Big(-4hf'(x) + 2hf'(x)\Big) + \Big(2h^2 f''(x) - 2h^2 f''(x)\Big) + \Big(\frac{-4h^3}{6}f'''(x) + \frac{8h^3}{6}f'''(x)\Big)$$

Combining,

$$4f(x-h) - f(x-2h) = 3f(x) - 2hf'(x) + \frac{4h^3}{6}f'''(x)$$

Finishing off the numerator, we arrive at,

$$4f(x-h) - f(x-2h) - 3f(x) = -2hf'(x) + \frac{4h^3}{6}f'''(x)$$

Then,

$$\frac{4f(x-h) - f(x-2h) - 3f(x)}{2h} = -f'(x) + \frac{h^2}{3}f'''(x)$$

4. Estimate $erf(1)$

$$erf(1) = \frac{2}{\sqrt{\pi}} \int_0^1 e^{-t^2} dt$$

   (a) Using the RHR with 6 nodes: $erf(1) \approx 0.692442699200466$
   (b) Using the LHR with 6 nodes: $erf(1) \approx 0.7977961256718922$
   (c) Using the Midpoint Rule with 6 nodes: $erf(1) \approx 0.747677083350702$
   (d) Using the Trapezoidal Rule with 6 nodes: $erf(1) \approx 0.7451194124361791$
   (e) Using the Taylor expansion to the 8th degree polynomial: $erf(1) \approx 0.8382245241280951$
   (f) Comparing the values computed above to the value computed in the *SCIPY.SPECIAL* package, we obtain the following table of errors,

Approximation	Absolute Error
Right Hand	0.1502580937492488
Left Hand	0.04490466727782261
Midpoint	0.09502370959901274
Trapezoidal	0.09758138051353571
Taylor	0.004476268821619667

   So we see that the integration of the Taylor Polynomial is the most accurate of this set of approximations.

5. Prove that the function $f(x) = x^3 + 2x + k$ has exactly one zero regardless of the value of k.

Proof. Let $k > 0$ and consider the function $f(x) = x^3 + 2x + k$. Suppose, without loss of generality, that $k > 0$. Now, we consider,

$$f(k) = \underbrace{k^3}_{+} + \underbrace{2k}_{+} + \underbrace{k}_{+}$$

Thus, $f(k) > 0$. Next, we consider,

$$f(-k) = \underbrace{(-k)^3}_{-} + \underbrace{2(-k)}_{-} + \underbrace{(-k)}_{-}$$

Thus, $f(-k) < 0$.
So, by the intermediate value theorem, we may conclude that $f(x)$ has a zero over the interval $[-k, k]$. Now, we shall claim that there is exactly one zero.
Then, we suppose to the contrary that there are at least two roots, r_1, r_2 such that $f(r_1) = f(r_2) = 0$. Since f is differentiable, we may apply Rolle's Theorem, and conclude that $\exists\, \varepsilon \in (r_1, r_2)$ such that $f'(\varepsilon) = 0$. But,

$$f'(x) = 3x^2 + 2 > 0\ \forall x$$

a contradiction.
Thus, we see that there may only be one zero of this function. So, we have shown that there is only one zero of $f(x) = x^3 + 2x + k$ regardless of your choice in k. $\square$

6. What is the error in a Taylor polynomial of degree four where,

$$f(x) = \sqrt{x},\ x_0 = \frac{9}{16},\ x \in [\frac{1}{4}, 1]$$

We consider the function defining absolute error as follows,

$$E_n(x_0; \xi) = |R_n(x_0; \xi)| = \left| \frac{f^{n+1}(\xi)}{(n+1)!}(x - x_0)^{n+1} \right|$$

Substituting $n = 4$ and $f(x) = \sqrt{x}$ We arrive at the function,

$$E_4(\frac{9}{16}; \xi) = \left| \frac{105}{5!32\xi^{9/2}}(\xi - \frac{9}{16})^5 \right|$$

We note that the function within the absolute value signs increases from a negative value slowly until $x = 1$ The value of the function at 1 is very close to zero $E \approx 0.00044$, whereas the function at $1/4$ is relatively large comparatively, $E \approx 0.041$. Thus, we note that the largest possible error occurs at $x = \frac{1}{4} \to E \approx 0.041$

7. Calculate the Taylor expansion for $S(x)$,

$$S(x) = \int_0^x \sin(\frac{\pi}{2}t^2)dt$$

We consider the known expansion of $\sin(x)$,

$$\sin(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

Substituting into this formula, we arrive at

$$\sin(\frac{\pi}{2}x^2) = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{\pi}{2}x^2)^{2k+1}}{(2k+1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{\pi}{2})^{2k+1}x^{4k+2}}{(2k+1)!}$$

Integrating from zero to x, we arrive at,

$$S(x) = \sum_{k=0}^{\infty} \frac{(-1)^k(\frac{\pi}{2})^{2k+1}x^{4k+3}}{(4k+3)(2k+1)!}$$

Thus, we have arrived at a Taylor Expansion definition for $S(x)$.

8. Using a Jupyter Notebook, we compute the true value of $S(1)$ noting that S is the Fresnel sin function, and has a "true" representation in Python. Then, it is a simple matter to compute the error for any squence length.
The error drops below the accuracy limit after 6 terms of the Taylor expansion are used.

9. The error function may also be written as,

$$erf(x) = \frac{2}{\sqrt{\pi}}e^{-x^2}\sum_{k=0}^{\infty} \frac{2^k x^{2k+1}}{1\cdot 3\cdot 5\cdots(2k+1)}$$

We now compare this representation to the sum derived earlier in problem 4,

$$erf(x) = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)k!}$$

Based on the computations performed in a Jupyter Notebook, we see that the the original sum requires only 5 terms, whereas the new method requires 6 terms to be within this error tolerance. We use the absolute error for this comparison, as the values that we are computing are on a similar order of magnitude, and the alternating nature of the sum makes the relative error harder to interpret for a tolerance comparison. It is also worth noting that the tolerance is given as a number, as opposed to a ratio of difference, and as such we must use absolute error to obtain a true numeric difference.

10. Prove the following lemmas:

(a)

Lemma 1. *If $g(x)$ is an odd, integrable function over [-L,L], then*

$$\int_{-L}^{L} g(x)dx = 0$$

Proof. Given that $g(x)$ is an odd function, we note,

$$g(-x) = -g(x)$$

Then, we consider the following integrals,

$$\int_0^L g(x)dx = I$$

$$\int_{-L}^0 g(x)dx = \int_{-L}^0 -g(-x)dx$$

We then perform the following u-substitution,

$$u = -x$$
$$\frac{du}{dx} = -1$$
$$du = -dx$$

Then,

$$\int_{-L}^{0} -g(-x)dx = \int_{L}^{0} g(u)du$$

Next, we reverse the limits of the integral, negating it.

$$\int_{L}^{0} g(u)du = -\int_{0}^{L} g(u)du = -I$$

Then,

$$\int_{-L}^{L} g(x)dx = \int_{0}^{L} g(x)dx - \int_{0}^{L} g(u)du = I - I = 0$$

As desired. □

(b)

Lemma 2. *If g(x) is an even, integrable function over [-L,L], then*

$$\int_{-L}^{L} g(x)dx = 2\int_{0}^{L} g(x)dx$$

Proof. Given that $g(x)$ is an even function, we note that $g(-x) = g(x)$. Then, we consider the following,

$$\int_{-L}^{L} g(x)dx = \int_{0}^{L} g(x)dx + \int_{-L}^{0} g(x)dx = I + \int_{-L}^{0} g(x)dx$$

We then consider the following u-substitution,

$$\begin{aligned} u &= -x \\ \frac{du}{dx} &= -1 \\ du &= -dx \end{aligned}$$

Now,

$$\int_{-L}^{0} g(x)dx = -\int_{L}^{0} g(-u)du = -\int_{L}^{0} g(u)du$$

Reversing the limits on the integrand,

$$-\int_{L}^{0} g(u)du = \int_{0}^{L} g(u)du = I$$

Thus,

$$\int_{-L}^{L} g(x)dx = \int_{0}^{L} g(x)dx + \int_{-L}^{0} g(x)dx = I + I = 2\int_{0}^{L} g(x)dx$$

As desired. □