PROOF OF THE QUADRATIC EQUATION

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ABSTRACT. In the paper that follows, an analytical proof of the so called "quadratic formula" shall be derived. Implementing the method of completing the square [1], and some clever algebraic manipulation, we will derive the classic formula used to compute roots of quadratic equations.

1. QUADRATIC THEOREM

A quadratic equation in mathematics is a polynomial equation wherein the highest order power of the unknown variable is 2. Thus, we note that in general, a quadratic equation takes the form, $ax^2 + bx + c = 0$, with $a \neq 0$. With this definition in mind, we consider the following theorem.

Theorem 1.1 (Quadratic Theorem).

Let an equation of the form,

$$Ax^2 + Bx + C = 0 \tag{1}$$

exist, with $A, B, C \in \mathbb{C}$, $A \neq 0$. Then, the solutions of this equation shall take the form,

$$x = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

2. Proof

Proof. Let A, B, C exist, such that $A, B, C \in \mathbb{C}$, and $A \neq 0$. Then, we consider equation (1), and perform some basic algebraic manipulation.

$$\begin{aligned}
Ax^2 + Bx + C &= 0 \\
x^2 + \frac{B}{A}x + \frac{C}{A} &= 0 && \text{Divide by } A \\
x^2 + \frac{B}{A}x &= -\frac{C}{A} && \text{Subtract the constant}
\end{aligned}$$

Next, we shall "Complete the Square" [1], by introducing a new constant to both sides, and manipulating the LHS of the equation into the square of a binomial,

$$\begin{aligned}
x^2 + \frac{B}{A}x + \left(\frac{B}{2A}\right)^2 &= -\frac{C}{A} + \left(\frac{B}{2A}\right)^2 && \text{Add the new constant} \\
\left(x + \frac{B}{2A}\right)^2 &= \left(\frac{B}{2A}\right)^2 - \frac{C}{A} && \text{Factor into the binomial form}
\end{aligned}$$

Finally, we may begin to solve for x,

$$\begin{aligned}
\left(x + \frac{B}{2A}\right)^2 &= \frac{B^2 - 4AC}{4A^2} && \text{Simplify RHS} \\
x + \frac{B}{2A} &= \pm\sqrt{\frac{B^2 - 4AC}{4A^2}} && \text{Take the square root} \\
x &= -\frac{B}{2A} \pm \frac{\sqrt{B^2 - 4AC}}{2A} && \text{Simplify} \\
x &= \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}
\end{aligned}$$

Thus, we have arrived at our desired result, and shown that the solutions of a quadratic equation adhere to the "Quadratic Formula". □

References

[1] Stapel, E. *Completing the Square: Solving Quadratic Equations — Purplemath*. [online] Purplemath. Available at: https://www.purplemath.com/modules/sqrquad.htm

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