1. We consider the derivation of,

$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} + Ch^2$$

To begin this derivation, we consider the fourth order Taylor expansions of both $u(x+h)$ and $u(x-h)$ as follows,

$$u(x+h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{3}u'''(x) + \frac{h^4}{24}u^{(4)}(\xi)$$
$$u(x-h) = u(x) + hu'(x) + \frac{h^2}{2}u''(x) + \frac{h^3}{3}u'''(x) + \frac{h^4}{24}u^{(4)}(\xi)$$

Where ξ is the value of x on the domain that maximizes the fourth derivative. Then, we add the functions together, noting that the first and third derivatives cancel out,

$$u(x+h) + u(x-h) = 2u(x) + h^2u''(x) + \frac{h^4}{12}u^{(4)}(\xi)$$
$$h^2u''(x) = u(x+h) - 2u(x) + u(x-h) - \frac{h^4}{12}u^{(4)}(\xi)$$
$$u''(x) = \frac{u(x+h) - 2u(x) + u(x-h)}{h^2} - \frac{h^2}{12}u^{(4)}(\xi)$$

Hence, we see that,

$$C = -\frac{u^{(4)}(\xi)}{12}$$

2. We now consider the application of this to the Laplacian,

$$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$$

From problem 1, we may write,

$$\frac{\partial^2 u}{\partial x^2} = \frac{u(x+h,y) - 2u(x,y) + u(x-h,y)}{h^2}$$
$$\frac{\partial^2 u}{\partial y^2} = \frac{u(x,y+h) - 2u(x,y) + u(x,y-h)}{h^2}$$

Thus, we find the centered difference to be,

$$\nabla^2 u = \frac{u(x+h,y) + u(x,y+h) - 4u(x,y) + u(x-h,y) + u(x,y-h)}{h^2}$$

3. Consider,

$$u(x,y,t) = e^{-2a^2\pi^2 t}\sin(\pi x)\sin(\pi y)$$

Then,

$$\frac{\partial u}{\partial t} = (-2a^2\pi^2)e^{-2a^2\pi^2 t}\sin(\pi x)\sin(\pi y) = (-2a^2\pi^2)u$$

We now compute $\nabla^2 u$ as follows,

$$\frac{\partial u}{\partial x} = \pi e^{-2a^2\pi^2 t}\cos(\pi x)\sin(\pi y)$$
$$\frac{\partial^2 u}{\partial x^2} = -\pi^2 u$$
$$\frac{\partial u}{\partial y} = \pi e^{-2a^2\pi^2 t}\sin(\pi x)\cos(\pi y)$$
$$\frac{\partial^2 u}{\partial y^2} = -\pi^2 u$$

Thus,

$$\nabla^2 u = -2\pi^2 u \Leftrightarrow a^2\nabla^2 u = -2a^2\pi^2 u = \frac{\partial u}{\partial t}$$

As desired.

8. We consider the differential equation,

$$\frac{\partial \omega}{\partial t} = \nu \nabla^2 \omega + [\psi, \omega]$$

Where,

$$[\psi, \omega] = \frac{\partial \psi}{\partial y}\frac{\partial \omega}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial \omega}{\partial y}$$

Let $\psi(x,y) = y - x$. Then,

$$[\psi, \omega] = \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y}$$

So, we see that our equation from above becomes,

$$\frac{\partial \omega}{\partial t} = \nu\left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2}\right) + \frac{\partial \omega}{\partial x} + \frac{\partial \omega}{\partial y}$$

We now implement a centered difference finite difference scheme on the RHS of this equation as,

$$\begin{aligned}\frac{\partial \omega}{\partial t} =& \nu\left(\frac{\omega(x+h,y) + \omega(x,y+h) - 4\omega(x,y) + \omega(x-h,y) + \omega(x,y-h)}{h^2}\right) \\ &+ \frac{\omega(x+h,y) - \omega(x-h,y)}{2h} + \frac{\omega(x,y+h) - \omega(x,y-h)}{2h}\end{aligned}$$