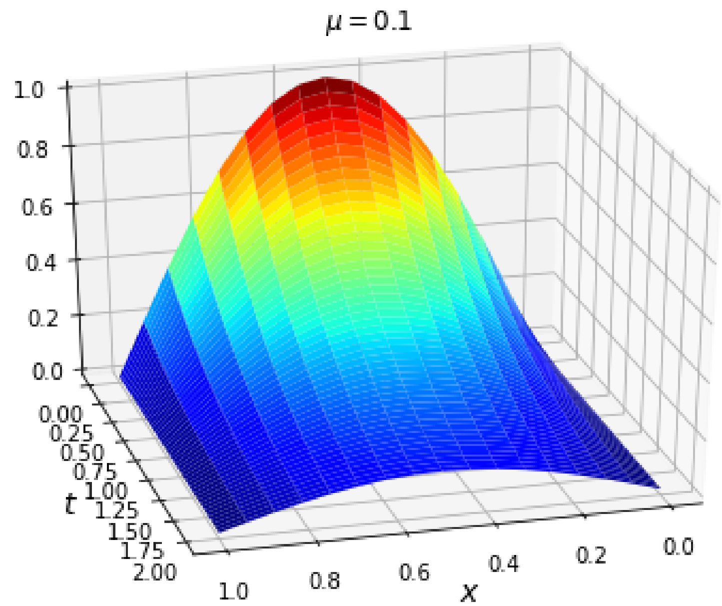
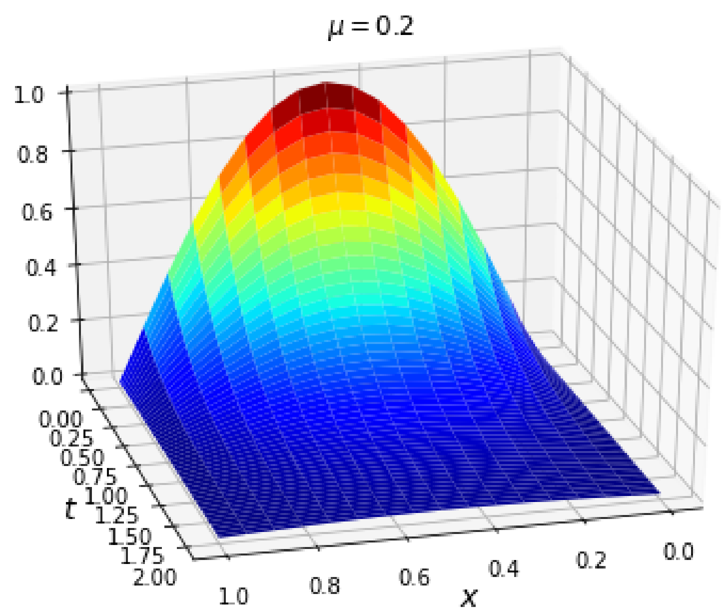
1. Numerically solve the concentration (heat or diffusion) equation with homogeneous boundary conditions. Run the simulation up to a final time of $t_f = 2$.

$$\begin{aligned} u_t &= \mu \frac{\partial^2 u}{\partial x^2}, \quad \mu \in (0,1] && (x,t) \in (0,1) \times (0,\infty) \\ u(0,t) &= 0 && t \in (0,\infty) \\ u(1,t) &= 0 && t \in (0,\infty) \\ u(x,0) &= \sin(\pi x) && x \in [0,1] \end{aligned}$$

We construct the matrix that defines the second spatial derivative using the CD2 stencil $L = \frac{1}{h^2}[1,-2,1]$. Then, we compute $u_{xx} = \mu * L \cdot u$ with a for loop over the interior of the u matrix. We then apply the forward euler method to increment u through time. We see that the equation behaves as we would expect, starting with a large peak of energy and diffusing down at the pace dictated by μ. Below, we see the difference of two different μ as they are applied to the equation.





We see that as we increase μ, the system diffuses faster, resulting in the steady state earlier.

2. See jupyter notebook.

3. We construct the second order accurate differentiation matrix with constant boundary conditions as seen below.

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, the first and last rows of the matrix represent the constant boundary conditions, and the interior rows represent the incremental derivatives $u'_1,\ u'_2, \dots, u'_{N-1}$

4. Numerically solve Poisson's equation with homogeneous boundary conditions.

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= & x \in (-1,1) \\ u(-1) &= 0 \\ u(1) &= 0 \end{aligned}$$

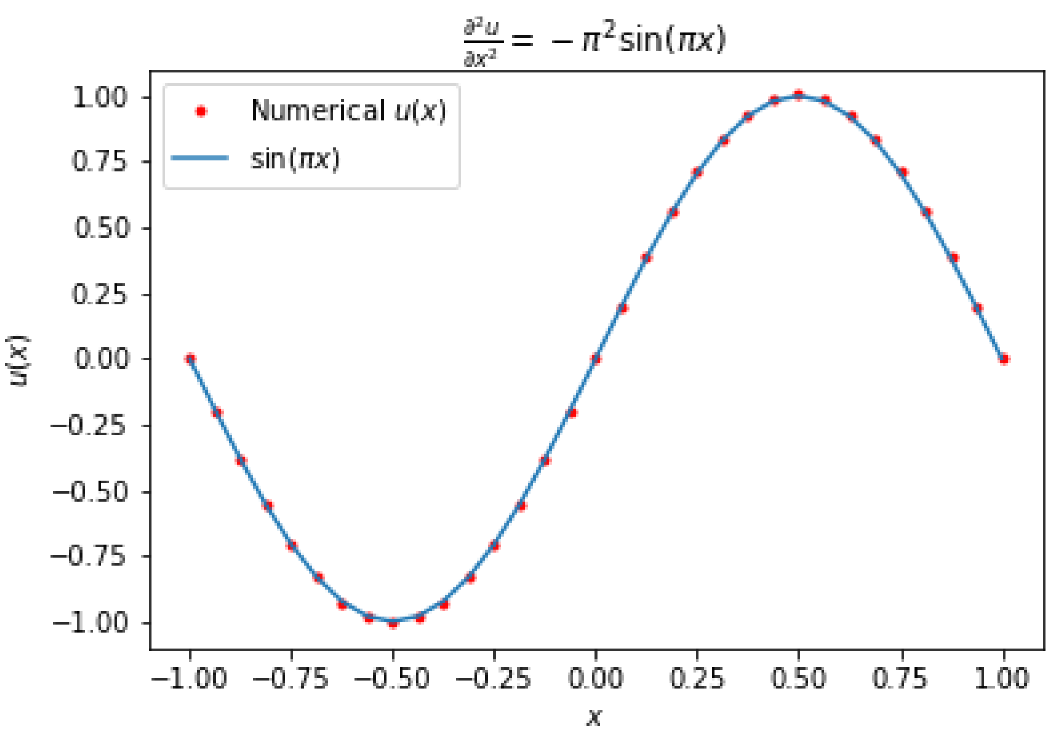
We may analytically solve this eqation,

$$u(x) = \sin(\pi x)$$

Thus, we may compare our solution numerically to the true solution to test accuracy. To numerically solve this equation, we write the equation in matrix form as,

$$Du = f, \quad f = -\pi^2 \sin(\pi x)$$

We then use *numpy.linalg.solve* to compute the u matrix. Finally, we plot the solution over the true solution for comparison.



So, we see that the computed solution is accurate.

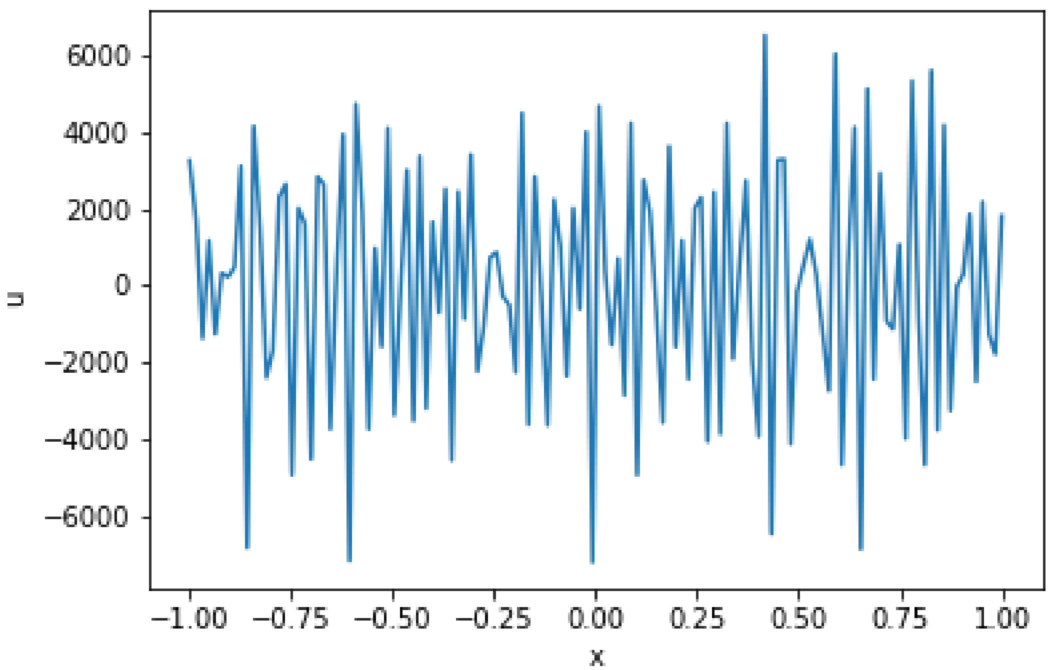
5. We now consider numerically solving the Poisson eqation defined as,

$$\frac{\partial^2 u}{\partial x^2} + \lambda u = f(x) \qquad x \in (-1,1)$$
$$u(-1) = 0.808$$
$$u(1) = 0.449$$

We will attempt to solve the equation as before, considering now the Matrix equation,

$$Du + \lambda u = f(x) \Leftrightarrow (D + \lambda I)u = f(x)$$

Then, we use the *linalg.solve* function to compute u. The result is as follows.



6. We solve Poisson's equation between two circular pipes. We assume the flow is steady and axisymmetric, i.e. $\frac{\partial w}{\partial \theta} = 0$. Poisson's equation in polar coordinates is,

$$\nabla^2 w = \frac{G}{\mu} \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{G}{\mu}$$

Based on our assumption, we can reduce this to,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) = \frac{G}{\mu}$$

Let us now call $\frac{G}{\mu} = \eta$ and solve as,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) = \eta$$
$$\Leftrightarrow$$
$$\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) = \eta r$$

Integrating with respect to r, we arrive at,

$$\frac{\partial w}{\partial r} = \frac{\eta r^2}{2} + c_1$$

Integrating again with respect to r, we arrive at,

$$w(r) = \frac{\eta r^3}{6} + c_1 r + c_2$$

We now impose the boundary conditions,

$$\begin{aligned} \frac{\eta a^3}{6} + c_1 a + c_2 &= 0 & w(a) = 0 \\ \frac{\eta b^3}{6} + c_1 b + c_2 &= W & w(b) = W \end{aligned}$$

We now solve this system of equations in matrix form as,

$$\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{\eta a^3}{6} \\ W - \frac{\eta b^3}{6} \end{bmatrix}$$

Subtracting $R_1 = R_1 - R_2$ and $R_2 = R_2 - \frac{b}{a} R_1$

$$\left[\begin{array}{cc|c} a-b & 0 & -\frac{\eta a^3}{6} + \frac{\eta a^3}{6} - W \\ 0 & 1 - \frac{b}{a} & (W - \frac{\eta b^3}{6}) + \frac{b}{a}\frac{\eta a^3}{6} \end{array}\right]$$

Simplifying and solving,

$$\begin{aligned} c_1 &= \frac{\eta}{6}\frac{b^3 - a^3}{a-b} - \frac{W}{a-b} \\ c_2 &= \frac{W}{1-\frac{b}{a}} + \frac{\eta}{6}\frac{ba^2 - b^3}{1 - \frac{b}{a}} \end{aligned}$$

Thus, our final solution is,

$$w(r) = \frac{\eta r^3}{6} + \left(\frac{\eta}{6}\frac{b^3 - a^3}{a-b} - \frac{W}{a-b}\right) r + \left(\frac{W}{1-\frac{b}{a}} + \frac{\eta}{6}\frac{ba^2 - b^3}{1-\frac{b}{a}}\right)$$

7. We derive the Schwarzchild radius by the Laplacian method. First, we take the force equation and compute the squared velocity as follows.

$$\begin{aligned} m\frac{d^2 r}{dt^2} &= -\frac{GMm}{r^2} \\ \frac{d}{dt}\frac{dr}{dt} &= -\frac{GM}{r^2} \\ \frac{d}{dt} v &= -GMr^{-2} \\ v &= -GM\int r^{-2} dt \\ v^2 = v\frac{dr}{dt} &= -GM\int r^{-2}\frac{dr}{dt} dt = -GM\int r^{-2} dr \\ v^2 &= \frac{GM}{r} + C \end{aligned}$$

We now implement the initial condition $v(R) = v_e$ and solve for C.

$$\begin{aligned} v_e^2 &= \frac{GM}{R} + C \\ C &= v_e^2 - \frac{GM}{R} \end{aligned}$$

So,

$$v^2 = \frac{GM}{r} - \frac{GM}{R} + v_e^2$$

Next, we consider the limit as $r \to \infty$. Here, $v(r) \to 0$. Thus,

$$\lim_{r\to\infty} v(r)^2 = \lim_{r\to\infty}\left[\frac{GM}{r} - \frac{GM}{R} + v_e^2\right] \Rightarrow 0 = v_e^2 - \frac{GM}{R} \Rightarrow v_e = \sqrt{\frac{GM}{R}}$$

We now set $v_e = c$, and solve for R

$$R_s = \frac{GM}{c^2}$$

We now compute the radii for various objects.

$Object$	R_s	$Swarzchild\ radius$
$Sun(M = 1.989 \times 10^{30} kg)$	$1476.923m$	$2953.845m$
$Earth(M = 5.972 \times 10^{24} kg)$	$0.0044m$	$0.0089m$
$Betelgeuse(M = 2.387 \times 10^{31} kg)$	17724.556	35449.112