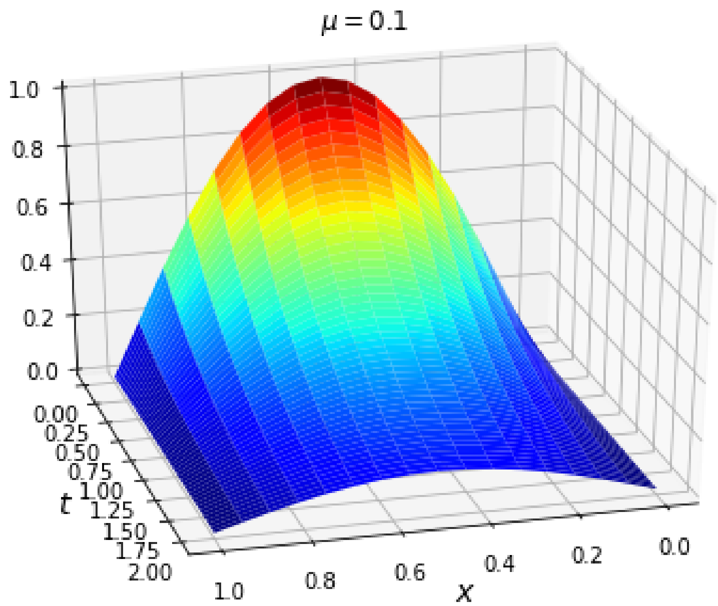
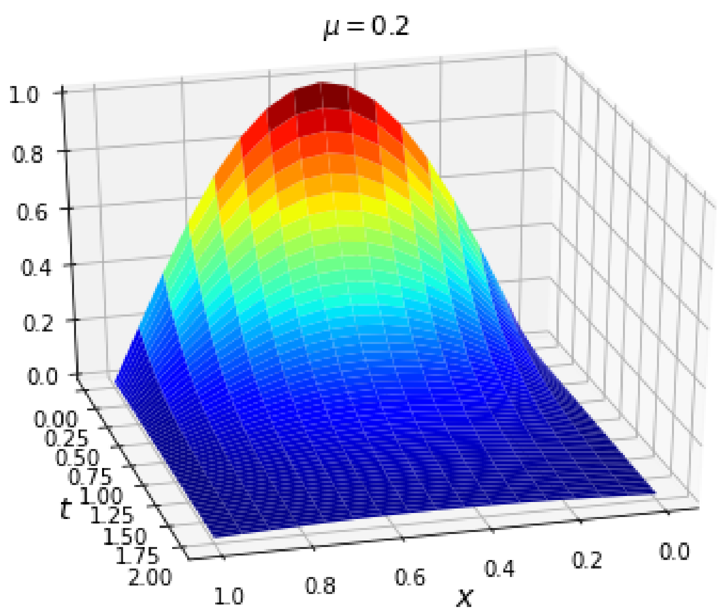
1. Numerically solve the concentration (heat or diffusion) equation with homogeneous boundary conditions. Run the simulation up to a final time of $t_f = 2$.

$$\begin{aligned}
u_t &= \mu \frac{\partial^2 u}{\partial x^2}, \quad \mu \in (0,1] && (x,t) \in (0,1) \times (0,\infty) \\
u(0,t) &= 0 && t \in (0,\infty) \\
u(1,t) &= 0 && t \in (0,\infty) \\
u(x,0) &= \sin(\pi x) && x \in [0,1]
\end{aligned}$$

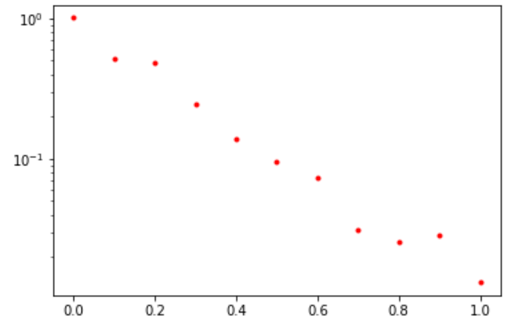
We construct the matrix that defines the second spatial derivative using the CD2 stencil $L = \frac{1}{h^2}[1,-2,1]$. Then, we compute $u_{xx} = \mu * L \cdot u$ with a for loop over the interior of the u matrix. We then apply the forward euler method to increment u through time. We see that the equation behaves as we would expect, starting with a large peak of energy and diffusing down at the pace dictated by μ. Below, we see the difference of two different μ as they are applied to the equation.



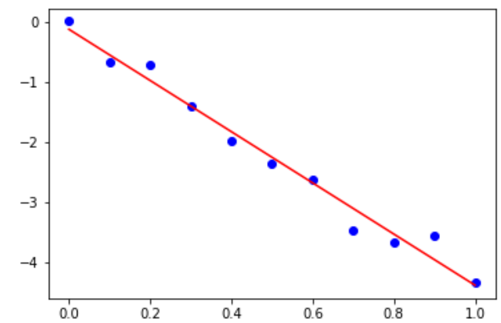


We see that as we increase μ, the system diffuses faster, resulting in the steady state earlier.

2. We consider first, a semilog plot of the data, as seen below.



We see that there is a clear linear trend in this logged data. So, we perform a simple linear regression to solve the slope of the data. So, we use gradient descent to reduce the squared error. This result follows.

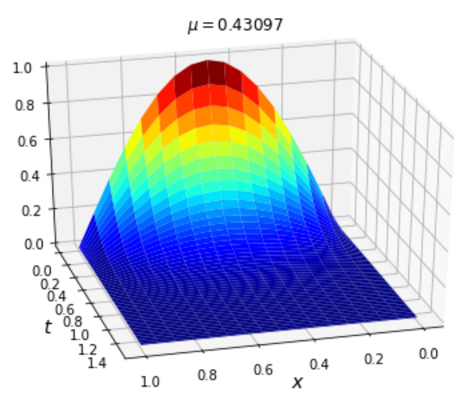


$$y \approx -0.12413451 - 4.2535229t$$

We know from the construction of the concentration profile that the logged peak data has slope equal to $-\pi^2\mu$. Thus, we compute μ to be,

$$\mu = \frac{4.2535229}{\pi^2} = 0.43097222048680517$$

Finally, we use the same method as problem 1 to compute the propogation of the data. The results are as follows:



From this computation we see that the peak data at $t = 1.5$ is,

$$p(1.5) = 0.0016992207655039224$$

3. We construct the second order accurate differentiation matrix with constant boundary conditions as seen below.

$$D = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & \dots & 0 & 0 & 0 & 0 \\ \vdots & & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$$

Here, the first and last rows of the matrix represent the constant boundary conditions, and the interior rows represent the incremental derivatives $u_1',\ u_2', \dots, u_{N-1}'$

4. Numerically solve Poisson's equation with homogeneous boundary conditions.

$$\frac{\partial^2 u}{\partial x^2} = \qquad x \in (-1,1)$$
$$u(-1) = 0$$
$$u(1) = 0$$

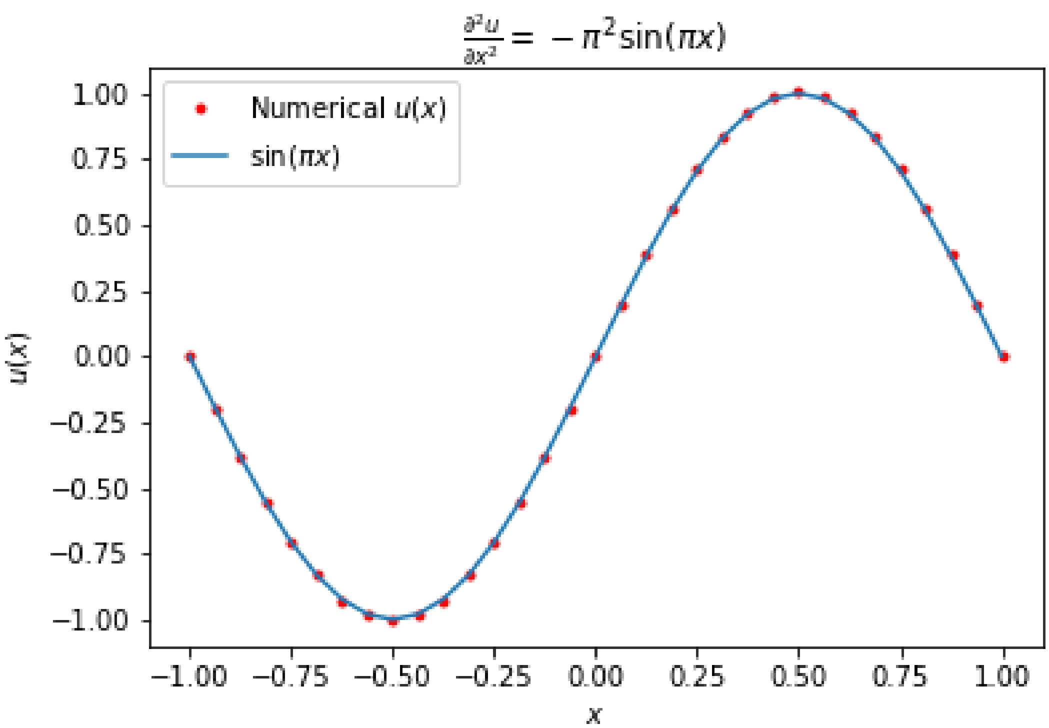
We may analytically solve this eqation,

$$u(x) = \sin(\pi x)$$

Thus, we may compare our solution numerically to the true solution to test accuracy. To numerically solve this equation, we write the equation in matrix form as,

$$Du = f, \quad f = -\pi^2 \sin(\pi x)$$

We then use *numpy.linalg.solve* to compute the u matrix. Finally, we plot the solution over the true solution for comparison.



So, we see that the computed solution is accurate.

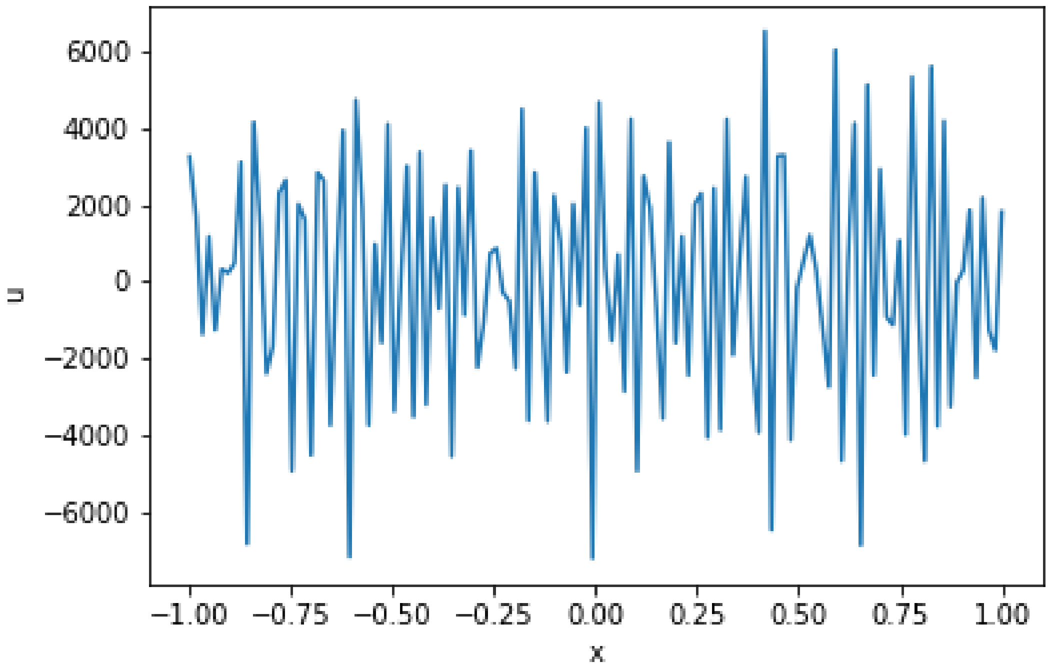
5. We now consider numerically solving the Poisson eqation defined as,

$$\frac{\partial^2 u}{\partial x^2} + \lambda u = f(x) \qquad x \in (-1,1)$$
$$u(-1) = 0.808$$
$$u(1) = 0.449$$

We will attempt to solve the equation as before, considering now the Matrix equation,

$$Du + \lambda u = f(x) \Leftrightarrow (D + \lambda I)u = f(x)$$

Then, we use the *linalg.solve* function to compute u. The result is as follows.



6. We solve Poisson's equation between two circular pipes. We assume the flow is steady and axisymmetric, i.e. $\frac{\partial w}{\partial \theta} = 0$. Poisson's equation in polar coordinates is,

$$\nabla^2 w = \frac{G}{\mu} \Rightarrow \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} = \frac{G}{\mu}$$

Based on our assumption, we can reduce this to,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) = \frac{G}{\mu}$$

Let us now call $\frac{G}{\mu} = \eta$ and solve as,

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) = \eta$$

$$\Leftrightarrow$$

$$\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial r}\right) = \eta r$$

Integrating with respect to r, we arrive at,

$$\frac{\partial w}{\partial r} = \frac{\eta r^2}{2} + c_1$$

Integrating again with respect to r, we arrive at,

$$w(r) = \frac{\eta r^3}{6} + c_1 r + c_2$$

We now impose the boundary conditions,

$$\frac{\eta a^3}{6} + c_1 a + c_2 = 0 \qquad\qquad w(a) = 0$$
$$\frac{\eta b^3}{6} + c_1 b + c_2 = W \qquad\qquad w(b) = W$$

We now solve this system of equations in matrix form as,

$$\begin{bmatrix} a & 1 \\ b & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -\frac{\eta a^3}{6} \\ W - \frac{\eta b^3}{6} \end{bmatrix}$$

Subtracting $R_1 = R_1 - R_2$ and $R_2 = R_2 - \frac{b}{a} R_1$

$$\left[\begin{array}{cc|c} a-b & 0 & -\frac{\eta a^3}{6} + \frac{\eta a^3}{6} - W \\ 0 & 1 - \frac{b}{a} & (W - \frac{\eta b^3}{6}) + \frac{b}{a}\frac{\eta a^3}{6} \end{array}\right]$$

Simplifying and solving,

$$c_1 = \frac{\eta}{6}\frac{b^3 - a^3}{a-b} - \frac{W}{a-b}$$
$$c_2 = \frac{W}{1-\frac{b}{a}} + \frac{\eta}{6}\frac{ba^2 - b^3}{1 - \frac{b}{a}}$$

Thus, our final solution is,

$$w(r) = \frac{\eta r^3}{6} + \left(\frac{\eta}{6}\frac{b^3 - a^3}{a-b} - \frac{W}{a-b}\right) r + \left(\frac{W}{1-\frac{b}{a}} + \frac{\eta}{6}\frac{ba^2 - b^3}{1-\frac{b}{a}}\right)$$

7. We derive the Schwarzchild radius by the Laplacian method. First, we take the force equation and compute the squared velocity as follows.

$$m\frac{d^2 r}{dt^2} = -\frac{GMm}{r^2}$$
$$\frac{d^2 r}{dt^2} = -\frac{GM}{r^2}$$

Note,

$$\frac{d^2 r}{dt^2} = \frac{dv}{dt}$$

and,

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt} = v\frac{dv}{dr}$$

So,

$$v\frac{dv}{dr} = -\frac{GM}{r^2}$$
$$\Leftrightarrow$$
$$\int v\,dv = -GM \int r^{-2} dr$$
$$\frac{v^2}{2} = GMr^{-1} + C$$
$$v^2 = 2GMr^{-1} + C$$

We now implement the initial condition $v(R) = v_e$ and solve for C.

$$v_e^2 = \frac{2GM}{R} + C$$
$$C = v_e^2 - \frac{2GM}{R}$$

So,

$$v^2 = \frac{2GM}{r} - \frac{2GM}{R} + v_e^2$$

Next, we consider the limit as $r \to \infty$. Here, $v(r) \to 0$. Thus,

$$\lim_{r\to\infty} v(r)^2 = \lim_{r\to\infty}\left[\frac{2GM}{r} - \frac{2GM}{R} + v_e^2\right] \Rightarrow 0 = v_e^2 - \frac{2GM}{R} \Rightarrow v_e = \sqrt{\frac{2GM}{R}}$$

We now set $v_e = c$, and solve for R

$$R_s = \frac{2GM}{c^2}$$

We now compute the radii for various objects.

$Object$	$Swarzchild\ radius$
$Sun(M = 1.989 \times 10^{30} kg)$	$2953.845m$
$Earth(M = 5.972 \times 10^{24} kg)$	$0.0089m$
$Betelgeuse(M = 2.387 \times 10^{31} kg)$	35449.112
$Ant - Man(M = 78kg)$	1.158×10^-25

Ant-mans radius is approximately equal to the cross sectional radius of 200 1MeV neutrinos.