1. Which term grows faster as $n \in \mathbb{N}$ goes to infinity, $n!$ or a^n for $a > 1$?
Consider,

$$\lim_{n\to\infty} \frac{a^n}{n!}$$

Next, we note that,

$$\Gamma(n+1) = n! = \int_0^\infty x^n e^{-x} dx$$

We now consider the behavior of the integrand over the domain of integration by finding the critical points.

$$\begin{aligned}
\frac{d}{dx} x^n e^{-x} &= nx^{n-1}e^{-x} - x^n e^{-x} && \text{Now set equal to zero and solve} \\
0 &= nx^{n-1}e^{-x} - x^n e^{-x} \\
\Leftrightarrow x^n e^{-x} &= nx^{n-1}e^{-x} \\
\Leftrightarrow x &= n
\end{aligned}$$

Thus, our critical values are $x = 0, n$. Evaluating the integrand at these critical points,

$$\begin{aligned}
f(0) &= 0^n e^{-0} = 0 \\
f(n) &= n^n e^{-n} > 0
\end{aligned}$$

So, we know that the integrand grows from 0 to $n^n e^{-n}$ as x goes from 0 to n, and then it decreases while remaining positive as $x \to \infty$. Thus, we may draw the following conclusion,

$$\int_0^\infty x^n e^{-x} dx \geq \int_0^n x^n e^{-x} dx$$

Next, we consider the behavior of e^{-x} over the interval $x \in [0, n]$. Because we know that $\frac{d}{dx} e^{-x} = -e^{-x}$, we know that this function is monotonically decreasing, so, on the interval $x \in [0, n]$, we know that e^{-x} is minimized at e^{-n}. So,

$$\int_0^\infty x^n e^{-x} dx \geq \int_0^n x^n e^{-x} dx \geq \int_0^n x^n e^{-n} dx$$

Next, we manipulate the function as follows,

$$\begin{aligned}
\int_0^n x^n e^{-n} dx &= e^{-n} \int_0^n x^n dx \\
&= e^{-n} \frac{n^{n+1}}{n+1} \\
\Rightarrow n! &\geq e^{-n} \frac{n^{n+1}}{n+1} \\
\Rightarrow \frac{1}{n!} &\leq e^n \frac{n+1}{n^{n+1}}
\end{aligned}$$

So,

$$0 \leq \frac{1}{n!} \leq e^n \frac{n+1}{n^{n+1}}$$

Multiplying by a^n and taking the limit,

$$\lim_{n\to\infty} 0 \leq \lim_{n\to\infty} \frac{a^n}{n!} \leq \lim_{n\to\infty} a^n e^n \frac{n+1}{n^{n+1}}$$

$$\lim_{n\to\infty} a^n e^n \frac{n+1}{n^{n+1}} = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)\left(\frac{ae}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{1}{n}\right) \lim_{n\to\infty} \left(\frac{ae}{n}\right)^n = \lim_{n\to\infty} \left(\frac{ae}{n}\right)^n$$

Given that we have fixed $a \in \mathbb{R}$, ae is a constant. Thus, in the limit, n will evenutally surpass ae, and the limit will tend to zero. So, by the squeeze theorem, we may conclude $n!$ grows faster.

2. Which term grows faster as $n \in \mathbb{N}$ goes to infinity, $n!$ or n^n?
Consider,

$$\lim_{n\to\infty} \frac{n!}{n^n}$$

Next, we note that,

$$\Gamma(n+1) = n! = \int_0^\infty x^n e^{-x} dx$$

So, we may now compute the upper bound of this with respect to n. Note,

$$\int_0^\infty x^n e^{-x} dx = \int_0^\infty x^n e^{-\frac{x}{2}} e^{-\frac{x}{2}} dx$$

Now, we maximize the function $f(x) = x^n e^{\frac{-x}{2}}$ within the limits of integration, $x \in [0, \infty)$

$$\begin{aligned}
\frac{d}{dx} x^n e^{\frac{-x}{2}} &= nx^{n-1} e^{\frac{-x}{2}} - \frac{1}{2} x^n e^{\frac{-x}{2}} && \text{Now set equal to zero and solve} \\
0 &= nx^{n-1} e^{\frac{-x}{2}} - \frac{1}{2} x^n e^{\frac{-x}{2}} \\
\Leftrightarrow \frac{1}{2} x^n e^{\frac{-x}{2}} &= nx^{n-1} e^{\frac{-x}{2}} \\
\Leftrightarrow x^n e^{\frac{-x}{2}} &= 2nx^{n-1} e^{\frac{-x}{2}} \\
\Leftrightarrow x = 2n
\end{aligned}$$

Thus, our critical points for this function are $x = 0, 2n$. We now compute $f(0),\ f(2n)$.

$$f(0) = 0^n e^{\frac{-0}{2}} = 0$$
$$f(2n) = (2n)^n e^{-n} > 0$$

Thus, $x = 2n$ is the maximal value. Then,

$$\begin{aligned}
\int_0^\infty x^n e^{-\frac{x}{2}} e^{-\frac{x}{2}} dx &\le (2n)^n e^{-n} \int_0^\infty e^{-x/2} dx \\
&= 2(2n)^n e^{-n} \\
&= 2n^n \frac{2^n}{e^n}
\end{aligned}$$

Then, $0 \le n! \le 2n^n \left(\frac{2}{e}\right)^n$. Dividing by n^n and taking the limit, we find,

$$\lim_{n\to\infty} 0 \le \lim_{n\to\infty} \frac{n!}{n^n} \le \lim_{n\to\infty} 2\left(\frac{2}{e}\right)^n$$

We note, $2 < e \Rightarrow \lim_{n\to\infty} \left(\frac{2}{e}\right)^n \to 0$. So, we have,

$$0 \le \lim_{n\to\infty} \frac{n!}{n^n} \le 0$$

Then, by the squeeze theorem, we may conclude,

$$lim_{n\to\infty} \frac{n!}{n^n} = 0$$

Thus, we see that n^n grows faster than $n!$.

3. Using Steffensen's method with a tolerance of 10^{-8} and a starting root guess of 1.75, we end up with $x^* = 2.0663938632$. As desired.

4. Using Steffensen's method with a tolerance of 10^{-8} and a starting root guess of .060538, we arrive at $x^* = .060538000$

5. We find that the interpolating value of $x^* = 1.3964831$, with $f(x^*) = g(x^*) \approx .14222413$

6. Given n data points $(x_1, y_1), \ldots, (x_n, y_n)$ the linear least-squares line-of-best fit is the linear function

$$f(x) = w_1 x + w_0$$

that minimizes the sum of squares

$$E(w_0, w_1) = \sum_{j=1}^{n} \left(y_j - f(x_j)\right)^2 .$$

Show that the minimum value of E occurs for w_0 and w_1 satisfying the two equations

$$\begin{aligned} w_0 n + w_1 n\bar{x} &= n\bar{y} \\ w_0 n\bar{x} + w_1 \sum_{j=1}^{n} x_j^2 &= \sum_{j=1}^{n} x_j y_j \end{aligned}$$

We consider the error function defined above, expanding $f(x_j)$ in terms of w_j. Then,

$$E(w_0, w_1) = \sum_{i=1}^{n} (y_i - w_1 x_i - w_0)^2 = \sum_{i=1}^{n} y_i^2 - 2w_1 \sum_{i=1}^{n} x_i y_i - 2w_0 \sum_{i=1}^{n} y_i + w_1^2 \sum_{i=1}^{n} x_i^2 + 2w_0 w_1 \sum_{i=1}^{n} x_i + w_0^2 \sum_{i=1}^{n} 1$$

In order to compute w_0, w_1 to minimize this function, we must derive E with respect to both variables. These computations follow,

$$\begin{aligned} \frac{\partial E}{\partial w_0} &= -2\sum_{i=1}^{n} y_i + 2w_1 \sum_{i=1}^{n} x_i + 2w_0 n && \text{Set equal to zero and solve} \\ 0 &= -2\sum_{i=1}^{n} y_i + 2w_1 \sum_{i=1}^{n} x_i + 2w_0 n \\ 2\sum_{i=1}^{n} y_i &= 2w_1 \sum_{i=1}^{n} x_i + 2w_0 n \\ \frac{n\sum_{i=1}^{n} y_i}{n} &= \frac{n\sum_{i=1}^{n} x_i w_1}{n} + 2nw_0 \\ n\bar{y} &= nw_0 + n\bar{x}w_1 \end{aligned}$$

$$\begin{aligned} \frac{\partial E}{\partial w_1} &= -2\sum_{i=1}^{n} x_i y_i + 2w_1 \sum_{i=1}^{n} x_i^2 + 2w_0 \sum_{i=1}^{n} x_i && \text{Set equal to zero and solve} \\ 0 &= -2\sum_{i=1}^{n} x_i y_i + 2w_1 \sum_{i=1}^{n} x_i^2 + 2w_0 \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i y_i &= w_1 \sum_{i=1}^{n} x_i^2 + w_0 \sum_{i=1}^{n} x_i \\ \sum_{i=1}^{n} x_i y_i &= nw_0\bar{x} + w_1 \sum_{i=1}^{n} x_i^2 \end{aligned}$$

Thus, we have arrived at the equations desired from above. Next, we shall solve for the values with Cramer's rule. First, we compute into matrix form,

$$\begin{bmatrix} n & n\bar{x} \\ n\bar{x} & \sum_{i=1}^{n} x_i^2 \end{bmatrix} \begin{bmatrix} w_0 \\ w_1 \end{bmatrix} = \begin{bmatrix} n\bar{y} \\ \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

Then,

$$D = n\sum_{i=1}^{n} x_i^2 - n^2\bar{x}^2$$

Substituting by Cramer's rule,

$$\begin{bmatrix} n\bar{y} & n\bar{x} \\ \sum_{i=1}^{n} x_i y_i & \sum_{i=1}^{n} x_i^2 \end{bmatrix}$$

Then,

$$D_{w_0} = n\bar{y}\sum_{i=1}^{n} x_i^2 - n\bar{x}\sum_{i=1}^{n} x_i y_i$$

And,

$$\begin{bmatrix} n & n\bar{y} \\ n\bar{x} & \sum_{i=1}^{n} x_i y_i \end{bmatrix}$$

Then,

$$D_{w_1} = n\sum_{i=1}^{n} x_i y_i - n^2\bar{x}\bar{y}$$

Finally, we may compute,

$$w_0 = \frac{D_{w_0}}{D} = \frac{n\bar{y}\sum_{i=1}^{n} x_i^2 - n\bar{x}\sum_{i=1}^{n} x_i y_i}{n\sum_{i=1}^{n} x_i^2 - n^2\bar{x}^2}$$
$$w_1 = \frac{D_{w_1}}{D} = \frac{n\sum_{i=1}^{n} x_i y_i - n^2\bar{x}\bar{y}}{n\sum_{i=1}^{n} x_i^2 - n^2\bar{x}^2}$$

7. Using our 2D gradient descent algorithm, we compute the equation of best fit to be,

$$f(x_j) \approx 2.80677502x_j - 3.0665807$$

Using then the direct equations for the coefficients to compute directly and arrive at the equation,

$$\hat{f}(x_j) \approx 2.80684869x_j - 3.066561824$$

As such, we see that our gradient descent algorithm is accurate.