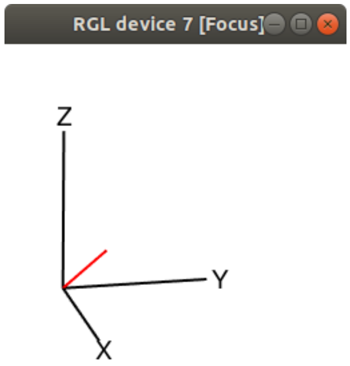
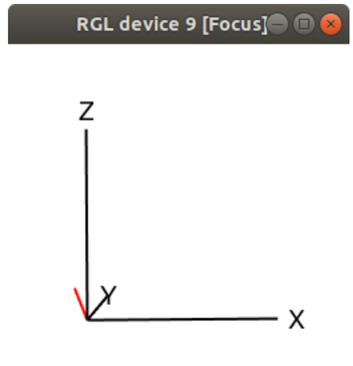
Problem 2.1

Let,

$$x = \begin{bmatrix} 5 & 1 & 3 \end{bmatrix}, \; y = \begin{bmatrix} -1 & 3 & 1 \end{bmatrix}$$

Plotting, we find,





Using R, we compute that,

$$\begin{aligned} |x| &= \sqrt{5^2 + 1^2 + 3^2} \\ &= \sqrt{35} \approx 5.91608 \\ |y| &= \sqrt{-1^2 + 3^2 + 1^2} \\ &= \sqrt{11} \approx 3.3166 \end{aligned}$$

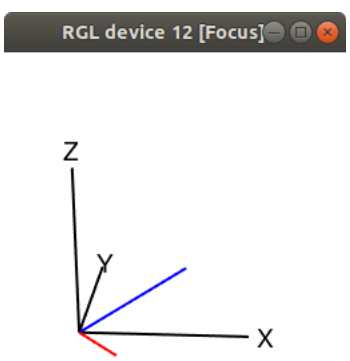
The angle is then,

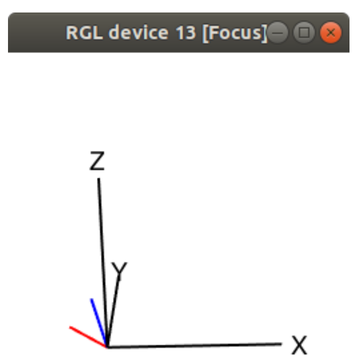
$$\cos(\theta) = \frac{x'y}{|x||y|} = \frac{1}{\sqrt{35}\sqrt{11}} \approx 0.050964 \Rightarrow \theta = 1.51981rad = 87.07867\,\text{deg}$$

Finally, the projection of x onto y is computed as,

$$proj = \frac{x'y}{L_y^2} y = \begin{bmatrix} \frac{-1}{11} \\ \frac{2}{11} \\ \frac{1}{11} \end{bmatrix}$$

Plotting the adjusted vectors in red with the original blue,





We see that these are inverses of eachother.

Problem 2.5

We consider,

$$\mathbf{Q} = \begin{bmatrix} \frac{5}{13} & \frac{12}{13} \\ \frac{-12}{13} & \frac{5}{13} \end{bmatrix}$$

If this matrix is Orthogonal, then $\mathbf{Q'Q} = I$. Using, R, we compute,

$$\mathbf{Q'Q} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

As desired.

Problem 2.6

We shall let,

$$A = \begin{bmatrix} 9 & -2 \\ -2 & 6 \end{bmatrix}$$

We see that A is symmetric, as $A = A'$. Using R, we compute the eigenvalues to be,

$$\Lambda = \begin{bmatrix} 10 \\ 5 \end{bmatrix}$$

Because these eigenvalues are greater than zero, we see that the matrix is indeed positive definite.

Problem 2.8

Consider the matrix

$$A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$$

We compute the eigenvalues as follows:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{bmatrix} 0 \qquad = (1-\lambda)(-2-\lambda) - 4$$

Solving, we find that $\Lambda = [2, -3]$ Plugging this in, we compute the eigenvectors as,

$$A - 2I = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \qquad = \begin{bmatrix} 1 & -2 \\ 0 & 0 \end{bmatrix}$$

Thus, we see that the eigenvector for this value is,

$$e_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Similarly, the vector for the other eigenvalue is computed to be,

$$e_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

Normalizing, we find that these vectors become,

$$e_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix},\ e_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{-2}{\sqrt{5}} \end{bmatrix}$$

Finally, the spectral decomposition of A becomes,

$$A = \lambda_1 e_1' e_1 + \lambda_2 e_2' e_2$$

Which is easily verified in R.

Problem 2.9

Using A as above, we compute A inverse using the equation presented in the text,

$$A^{-1} = \frac{1}{|A|} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}$$

A^{-1} is then,

$$A^{-1} = \begin{bmatrix} 1/3 & 1/3 \\ 1/3 & -1/6 \end{bmatrix}$$

The eigenvalues and associated eigenvectors are then,

$$\Lambda = \begin{bmatrix} 1/2 \\ -1/3 \end{bmatrix}$$
$$e_1 = \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix}$$
$$e_2 = \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}$$

The spectral decomposition is then,

$$A = \lambda_1 e_1' e_1 + \lambda_2 e_2' e_2$$

We see that the eigenvectors are the same as before, and the eigenvalues are merely the inverses of those from before.

Problem 2.15

Consider the quadratic form,

$$3x_1^2 + 3x_2^2 - 2x_1x_2$$

We first convert this to an equation of the form,

$$x'Ax$$

Then,

$$A = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}$$

We shall now compute the eigenvalues with R,

$$\Lambda = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Since these are both positive values, we see that the quadratic is indeed positive definite.

Problem 2.16

We consider the arbitrary matrix A with dimension $n \times p$. Then, $A'A$ is a symmetric matrix with dimension $p \times p$. Now, let $y = Ax$ as described. We now compute,

$$\begin{aligned} y'y &= (Ax)'Ax \\ &= (x'A')Ax \end{aligned}$$

The equation $y'y$ looks similar to the form of the quadratic $x'Ax$ from the text. If the quadratic is non-negative then we may conclude A is non-negative. Here, "A" is $A'A$. Now, we note that,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{bmatrix}$$

A vector of dimension p. So,

$$y'y = y_1^2 + y_2^2 + \dots y_p^2$$

This function is a sum of squares, and is thus greater than or equal to zero. Now,

$$y'y = x'A'Ax \geq 0$$

We see that the matrix $A'A$ is non-negative definite as desired.

Problem 2.18

We consider the points $(x_1,\ x_2)$ defined by the distance to the origin by,

$$c^2 = 4x_1^2 + 3x_2^2 - 2\sqrt{2}x_1x_2$$

We note that the RHS of this equation can be rewritten as,

$$x'Ax = [x_1,\ x_2] \begin{bmatrix} 4 & -\sqrt{2} \\ -\sqrt{2} & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

The matrix A then has the eigenvalues $\Lambda = [2,\ 5]$ And normalized eigenvectors,

$$e_1 = \begin{bmatrix} 1/\sqrt{3} \\ \sqrt{2}/\sqrt{3} \end{bmatrix},\ e_2 = \begin{bmatrix} \sqrt{2}/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix}$$

For $c^2 = 1$, the major axis has length,

$$\frac{c}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{2}} = 0.7071$$

And the minor,

$$\frac{c}{\sqrt{\lambda_2}} = \frac{1}{\sqrt{5}} = 0.4472$$

These axes follow the ordained eigenvectors from above.
For $c^2 = 4$, the major axis has length,

$$\frac{c}{\sqrt{\lambda_1}} = \frac{2}{\sqrt{2}} = 1.4142$$

And the minor,

$$\frac{c}{\sqrt{\lambda_2}} = \frac{2}{\sqrt{5}} = 0.89443$$

Thus, we see that as c^2 increases, the lengths of the sides increase as well.

Problem 2.21

We consider the matrix,

$$A = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}$$

Then,

$$A'A = \begin{bmatrix} 9 & 1 \\ 1 & 9 \end{bmatrix}$$

With associated eigen values and vectors,

$$\Lambda = \begin{bmatrix} 10 \\ 8 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix},\ e_2 = \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

We now compute the same things, for the value AA'.

$$AA' = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 8 & 0 \\ 4 & 0 & 8 \end{bmatrix}$$

$$\Lambda = \begin{bmatrix} 10 \\ 8 \\ 0 \end{bmatrix}$$

$$e_1 = \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix},\ e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},\ e_3 = \begin{bmatrix} 2/\sqrt{5} \\ 0 \\ -1/\sqrt{5} \end{bmatrix}$$

The decomposition is then,

$$A = \sqrt{10} \begin{bmatrix} 1/\sqrt{5} \\ 0 \\ 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + \sqrt{8} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 2 & -2 \\ 2 & 2 \end{bmatrix}$$

As desired.

Problem 2.23

We consider the covariance matrix defined as,

$$\Sigma = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pp} \end{bmatrix}$$

And the standard deviation matrix

$$V^{1/2} = \begin{bmatrix} \sqrt{\sigma_{11}} & 0 & \dots & 0 \\ 0 & \sqrt{\sigma_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \sqrt{\sigma_{pp}} \end{bmatrix}$$

We now consider,

$$(V^{1/2})^{-1}\Sigma(V^{1/2})^{-1} = \begin{bmatrix} 1/\sqrt{\sigma_{11}} & 0 & \dots & 0 \\ 0 & 1/\sqrt{\sigma_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/\sqrt{\sigma_{pp}} \end{bmatrix} \begin{bmatrix} \sigma_{11} & \sigma_{12} & \dots & \sigma_{1p} \\ \sigma_{21} & \sigma_{22} & \dots & \sigma_{2p} \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{p1} & \sigma_{p2} & \dots & \sigma_{pp} \end{bmatrix} \begin{bmatrix} 1/\sqrt{\sigma_{11}} & 0 & \dots & 0 \\ 0 & 1/\sqrt{\sigma_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1/\sqrt{\sigma_{pp}} \end{bmatrix}$$

Because of the zeros on the nondiagonal entries in the V matrices, this simplifies to,

$$\begin{bmatrix} \frac{\sigma_{11}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{11}}} & \frac{\sigma_{12}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{22}}} & \dots & \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} \\ \frac{\sigma_{21}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{11}}} & \frac{\sigma_{22}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{22}}} & \dots & \frac{\sigma_{2p}}{\sqrt{\sigma_{22}}\sqrt{\sigma_{pp}}} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\sigma_{1p}}{\sqrt{\sigma_{11}}\sqrt{\sigma_{pp}}} & \dots & \dots & \frac{\sigma_{pp}}{\sqrt{\sigma_{pp}}\sqrt{\sigma_{pp}}} \end{bmatrix} = \rho$$

We note that the entries on the diagonal are equal to one, and the remaining entries are the various population correlation values we have previously computed. Now,

$$\begin{aligned} (V^{1/2})^{-1}\Sigma(V^{1/2})^{-1} &= \rho && \Leftrightarrow \\ V^{1/2}(V^{1/2})^{-1}\Sigma(V^{1/2})^{-1}V^{1/2} &= V^{1/2}\rho V^{1/2} && \Leftrightarrow \\ \Sigma &= V^{1/2}\rho V^{1/2} \end{aligned}$$

As desired.

Problem 2.25

Let X have the covariance matrix,

$$\Sigma = \begin{bmatrix} 25 & -2 & 4 \\ -3 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix}$$

Using the formulas defined in the previous problem, we compute the matrices,

$$V^{1/2} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

$$\rho = \begin{bmatrix} 1 & -1/5 & 4/15 \\ -1/5 & 1 & 1/6 \\ 4/15 & 1/6 & 1 \end{bmatrix}$$

Using R, we compute,

$$V^{1/2}\rho V^{1/2} = \begin{bmatrix} 25 & -2 & 4 \\ -3 & 4 & 1 \\ 4 & 1 & 9 \end{bmatrix} = \Sigma$$

As desired.

Problem 2.26

From before,

$$\sigma_{13} = \frac{4}{15}$$

We now consider the correlation between X_1 and $1/2X_2 + 1/2X_3$

$$Cor(X_1,\ 1/2X_2 + 1/2X_3) = \frac{cov(X_1,\ 1/2X_2 + 1/2X_3)}{\sqrt{Var(X_1)}\sqrt{Var(1/2X_2 + 1/2X_3)}}$$

From before, we know that $Var(X_1) = 25 = \sigma_{11}$.

Next,

$$Var(1/2X_2 + 1/2X_3) = \frac{1}{2^2}\sigma_{22} + \frac{1}{2}\sigma_{33} + 2(\frac{1}{2^2}\sigma_{23}) = 1 + \frac{9}{4} + \frac{1}{2} = \frac{15}{4}$$

Finally,

$$cov(X_1,\ 1/2X_2 + 1/2X_3) = \frac{\sigma_{12}}{2} + \frac{\sigma_{13}}{2} = -1 + 2 = 1$$

So,

$$Cor(X_1,\ 1/2X_2 + 1/2X_3) = \frac{1}{\sqrt{25}\sqrt{15/4}} = 0.10328$$