4.6

Let $\mathbf{X}$ be distributed as $N_3(\mu, \Sigma)$ where $\mu' = [1, -1, 2]$. And,

$$\Sigma = \begin{bmatrix} 4 & 0 & -1 \\ 0 & 5 & 0 \\ -1 & 0 & 2 \end{bmatrix}$$

Given that Σ is the Variance-Covariance matrix of $\mathbf{X}$, we consider the following cases,

a. X_1 and X_2
Consulting the Σ matrix, we find,

$$Cov(X_1,\ X_2) = \sigma_{12} = 0$$

Thus, we see that these variables are independent of each other.

b. X_1 and X_3
Again, we compute,

$$Cov(X_1,\ X_3) = \sigma_{13} = -1$$

Hence, we see that these variables are not independent of eachother.

c. X_2 and X_3
Again,

$$Cov(X_2,\ X_3) = \sigma_{23} = 0$$

Thus, we see that these variables are independent.

d. $(X_1,\ X_3)$ and X_2
We shall now convert our Σ matrix such that we can compare the $(X_1,\ X_3)$ and X_2. Here,

$$\Sigma \to \left[\begin{array}{cc|c} 4 & 1 & 0 \\ \text{-1} & 2 & 0 \\ \hline 0 & 0 & 5 \end{array}\right]$$

A "2x2" matrix for comparison. Then, we have,

$$Cov((X_1,\ X_3),\ X_2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which from the text, we see is independent.

e. X_1 and $X_1 + 3X_2 - 2X_3$
We see that the covariance can be computed as,

$$Cov(X_1,\ X_1 + 3X_2 - 2X_3) = \sigma_{11} + 3\sigma_{12} - 2\sigma_{13} = 6$$

Thus, they are not independent.

4.7

a. Compute the conditional distribution of X_1, given $X_3 = x_3$.
We consider the joint distribution of $(X_1,\ X_3)$ is,

$$(X_1,\ X_3) \sim N_2\left(\begin{bmatrix} \mu_1 \\ \mu_3 \end{bmatrix}, \begin{bmatrix} \Sigma_{11} & \Sigma_{13} \\ \Sigma_{31} & \Sigma_{33} \end{bmatrix}\right)$$

Thus, the conditional distribution of $f(X_1|X_3 = x_3)$ becomes,

$$\begin{aligned} f(X_1|X_3 = x_3) &\sim N\left(\mu_1 + \frac{\sigma_{13}}{\sigma_{33}}(x_3 - \mu_3),\ \sigma_{11} - \frac{\sigma_{13}^2}{\sigma_{33}}\right) \\ &\sim N\left(1 + \frac{-1}{2}(x_3 - 2),\ 4 - \frac{(-1)^2}{2}\right) \\ &\sim N(1 - .5(x_3 - 2),\ 3.5) \end{aligned}$$

b. Similarly,

$$\begin{aligned} f(X_1|X_2=x_2,X_3=x_3) &\sim N\left(\mu_1+\frac{\sigma_{13}}{\sigma_{33}}(x_3-\mu_3)+\frac{\sigma_{12}}{\sigma_{22}}(x_2-\mu_2),\ \sigma_{11}-\frac{\sigma_{13}^2}{\sigma_{33}}-\frac{\sigma_{12}^2}{\sigma_{22}}\right) \\ &\sim N\left(1+\frac{-1}{2}(x_3-2)+\frac{0}{5}(x_2+1),\ 4-\frac{(-1)^2}{2}-\frac{0^2}{5}\right) \\ &\sim N(1-.5(x_3-2),\ 3.5) \end{aligned}$$

The same as in part a. We see that this is due to the fact that the covariance is 0 for X_1 and X_2.

4.17

Let $\mathbf{X}_1,\mathbf{X}_2,\mathbf{X}_3,\mathbf{X}_4,\mathbf{X}_5$ be independent and identically distributed random vectors with mean vector μ and covariance matrix Σ. Consider now the mean and covariances of,

a. $V_1=\sum_i \frac{1}{5}\mathbf{X}_i$
The mean and covariance of this vector is given by,

$$V_1\sim N\left(\sum_{i=1}^5 a_i\mu_i,\left(\sum_{i=1}^5 a_i^2\right)\Sigma\right),\ a_i=\frac{1}{5}$$

So,

$$\begin{aligned} \mu(V_1) &= \sum_{i=1}^5 a_i\mu_i && \text{Given that } \mu_i \text{ are identical} \\ &= \mu\sum_{i=1}^5\frac{1}{5} \\ &= \mu \end{aligned}$$

and,

$$\begin{aligned} Cov(V_1) &= \left(\sum_{i=1}^5 a_i^2\right)\Sigma && \text{Similarly to before, we factor } \Sigma \\ &= \Sigma\sum_{i=1}^5\frac{1}{5^2} \\ &= \frac{\Sigma}{5} \end{aligned}$$

b. $V_2=\mathbf{X}_1-\mathbf{X}_2+\mathbf{X}_3-\mathbf{X}_4+\mathbf{X}_5$
As before,

$$\begin{aligned} \mu(V_2) &= \sum_{i=1}^5 c_i\mu_i && \text{Given that } \mu_i \text{ are identical} \\ &= \mu\sum_{i=1}^5 c_i \\ &= \mu(1-1+1-1+1) \\ &= \mu \end{aligned}$$

And,

$$\begin{aligned} Cov(V_2) &= \left(\sum_{i=1}^5 c_i^2\right)\Sigma && \text{Similarly to before, we factor } \Sigma \\ &= \Sigma(1+1+1+1+1) \\ &= 5\Sigma \end{aligned}$$

c. Finally, we consider the covariance between the two vectors as,

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \sim N\left(\begin{bmatrix} \mu \\ \mu \end{bmatrix}, \Sigma^*\right)$$

Where,

$$\Sigma^* = \begin{bmatrix} (\sum a_i^2)\Sigma & (a'c)\Sigma \\ \\ (a'c)\Sigma & (\sum c_i^2)\Sigma \end{bmatrix} = \begin{bmatrix} \frac{1}{5}\Sigma & \frac{1}{5}\Sigma \\ \\ \frac{1}{5}\Sigma & 5\Sigma \end{bmatrix}$$

4.18

Consider,

$$\mathbf{X} = \begin{bmatrix} 3 & 6 \\ 4 & 4 \\ 5 & 7 \\ 4 & 7 \end{bmatrix}$$

To find the maximum likelihood estimates of μ and Σ we use the equations,

$$\hat{\mu} = \bar{\mathbf{X}}$$
$$\hat{\Sigma} = \frac{1}{n}\sum_{j=1}^{n}(X_j - \bar{X})(X_j - \bar{X})'$$

Using R, we easily compute both values to be,

$$\hat{\mu} = [4,\ 6]$$
$$\hat{\Sigma} = \frac{1}{4}\begin{bmatrix} 2 & 1 \\ 1 & 6 \end{bmatrix}$$

4.19

Let $\mathbf{X}_1,\ \mathbf{X}_2,\ \dots,\ \mathbf{X}_{20}$ be a random sample of size $n = 20$ from a $N_6(\mu, \Sigma)$ population. We consider the following,

a. Distribution of $(\mathbf{X}_1 - \mu)'\Sigma^{-1}(\mathbf{X}_1 - \mu)$.
Consider from the text, Result 4-7. Given a measurement $\mathbf{X}$ distributed $N_p(\mu, \Sigma)$, then, $(\mathbf{X} - \mu)'\Sigma^{-1}(\mathbf{X} - \mu) \sim \chi_p^2$ Thus, we can consider the distribution to be,

$$(\mathbf{X}_1 - \mu)'\Sigma^{-1}(\mathbf{X}_1 - \mu) \sim \chi_6^2$$

b. Distribution of $\bar{\mathbf{X}}$ and $\sqrt{n}(\bar{\mathbf{X}} - \mu)$.
Consider first, $\bar{\mathbf{X}}$. From the text, we see that box 4-23 tells us that

$$\bar{\mathbf{X}} \sim N_p(\mu, (1/n)\Sigma) = N_6(\mu, \frac{1}{20}\Sigma)$$

Similarly, we may now consider, $\sqrt{n}(\bar{\mathbf{X}} - \mu)$. Box 4-28 tells us that,

$$\sqrt{n}(\bar{\mathbf{X}} - \mu) \sim N_p(0, \Sigma) = N_6(0, \Sigma)$$

This makes sense, given the structure of the distribution of $\bar{\mathbf{X}}$. Shifting by μ will reduce the mean, and scaling by $\sqrt{n}$ will scale the covariance appropriately.

c. Distribution of $(n-1)\mathbf{S}$. From box 4-23, we see that this is distributed as a Wishart random matrix with $(n-1)$ degrees of freedom. So,

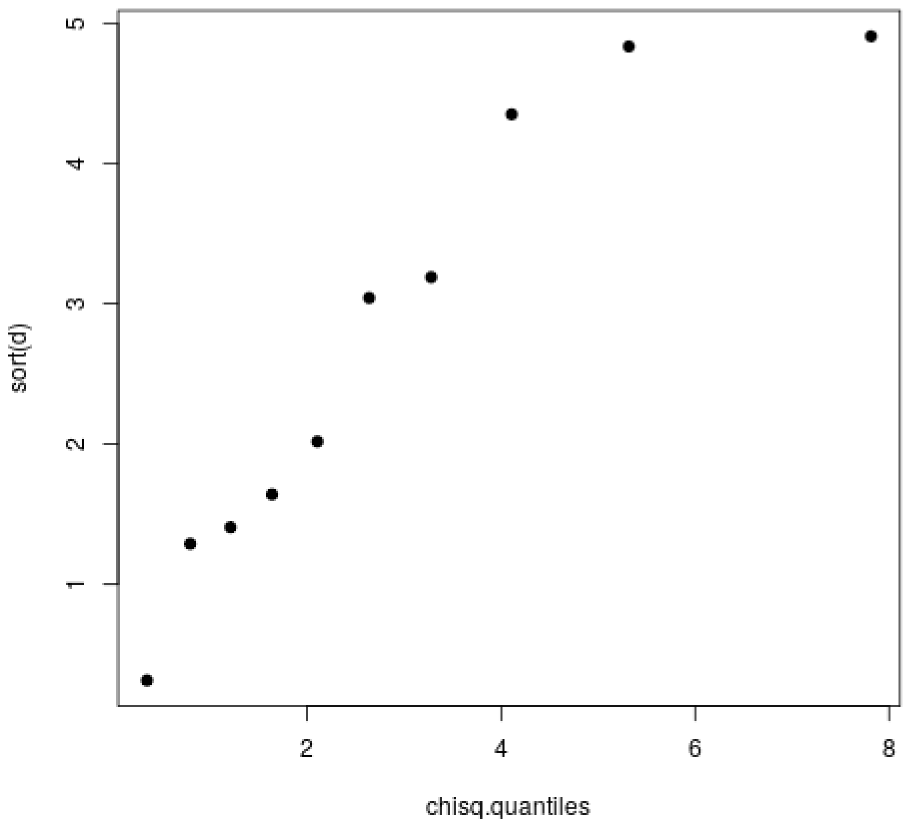
$$(n-1)S \sim W_{19}(\ \cdot\ |\Sigma)$$

4.25

We consider the data from problem 1.4. Using all three variables and the quantile cuttoffs,

$$< 0.3518, .7978, 1.2125, 1.6416, 2.1095, 2.6430, 3.2831, 4.1083, 5.317, 7.8147 >$$

We construct the chi-square plot of all three variables with the cuttofs above, shown below,



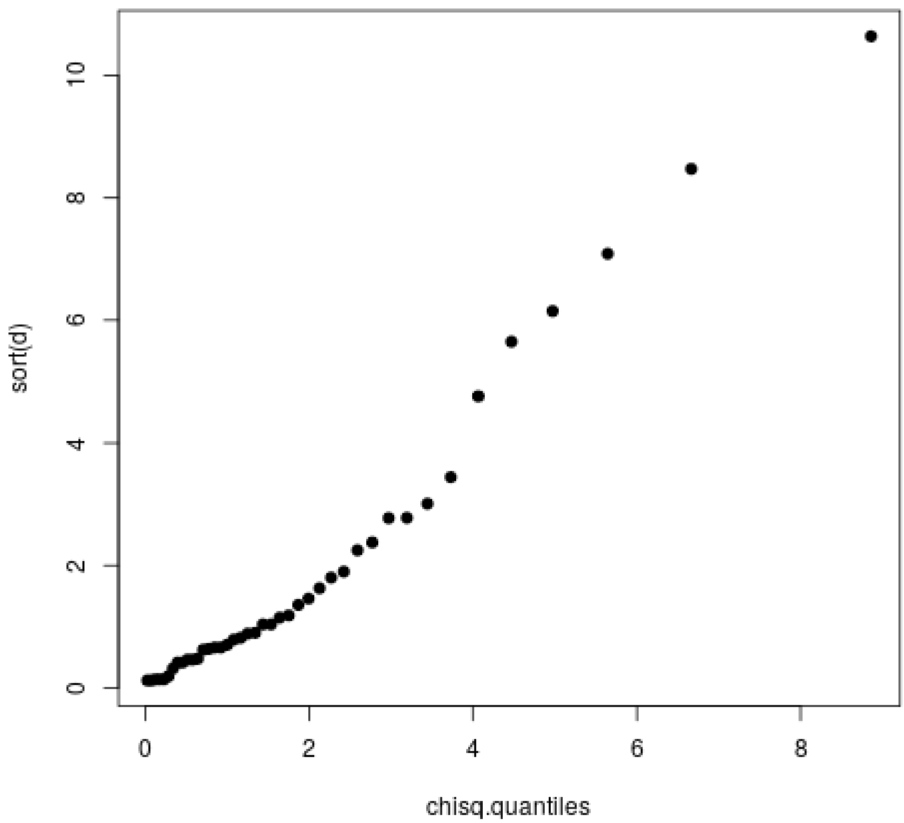
4.29

We consider the data from Table 1.5 specifically the pairs from $X_5 = NO_2$ and $X_6 = O_3$. We consider the follow,

a. Compute the statistical distances $(x_j - \bar{x})S^{-1}(x_j - \bar{x})$.
   The distances are,
   $< 0.4606524, 0.6592206, 2.3770610, 1.6282902, 0.4135364, 0.4760726, 1.1848895, 10.6391792, 0.1388339, 0.8162468, 1.3566301, 0.6228096, 5.6494392, 0.3159498, 0.4135364, 0.1224973, 0.8987982, 4.7646873, 3.0089122, 0.6592206, 2.7741416, 1.0360061, 0.7874152, 3.4437748, 6.1488606, 1.0360061, 0.1388339, 0.8856041, 0.1379719, 2.2488867, 0.1901188, 0.4606524, 1.1471939, 7.0857237, 1.4584229, 0.1224973, 1.8984708, 2.7782596, 8.4730649, 0.6370218, 0.7032485, 1.8013611$ This data is more concisely contained in the attached R code as well under the variable d in section 4.29.

b. The cutoff for a bivariate normal distribution with a .5 probability contour is 1.38629. For this data, we have that 0.6190476 of the data falls within the 50% contour.

c. The Chi-Squared plot of the data from a is shown below,


sort(d)
chisq.quantiles
0
2
4
6
8
10