9.1

Consider the covariance matrix,

$$\rho = \begin{bmatrix} 1.0 & 0.63 & 0.45 \\ 0.63 & 1.0 & 0.35 \\ 0.45 & 0.35 & 1.0 \end{bmatrix}$$

Consider the random variables,

$$\begin{aligned} Z_1 &= .9F_1 + \varepsilon_1 \\ Z_2 &= .7F_1 + \varepsilon_2 \\ Z_3 &= .5F_1 + \varepsilon_3 \end{aligned}$$

and,

$$\Psi = Cov(\varepsilon) = \begin{bmatrix} 0.19 & 0 & 0 \\ 0 & 0.51 & 0 \\ 0 & 0 & .75 \end{bmatrix}$$

We construct the L matrix as,

$$L = \begin{bmatrix} 0.9 \\ 0.7 \\ 0.5 \end{bmatrix}$$

Then,

$$\begin{aligned} \rho &= LL^T + \Psi \\ &= \begin{bmatrix} 0.9 \\ 0.7 \\ 0.5 \end{bmatrix} \begin{bmatrix} 0.9 & 0.7 & 0.5 \end{bmatrix} + \begin{bmatrix} 0.19 & 0 & 0 \\ 0 & 0.51 & 0 \\ 0 & 0 & .75 \end{bmatrix} \\ &= \begin{bmatrix} 0.81 & 0.63 & 0.45 \\ 0.63 & 0.49 & 0.35 \\ 0.45 & 0.35 & 0.25 \end{bmatrix} + \begin{bmatrix} 0.19 & 0 & 0 \\ 0 & 0.51 & 0 \\ 0 & 0 & .75 \end{bmatrix} \\ &= \begin{bmatrix} 1.0 & 0.63 & 0.45 \\ 0.63 & 1.0 & 0.35 \\ 0.45 & 0.35 & 1.0 \end{bmatrix} \end{aligned}$$

As desired.

9.2

Now, using the formula,

$$h_i^2 = \sum_{j=1}^{m} l_{ij}^2$$

Then,

$$\begin{aligned} h_1^2 &= 0.9^2 \\ &= 0.81 \\ h_2^2 &= 0.7^2 \\ &= 0.49 \\ h_3^2 &= 0.5^2 \\ &= 0.25 \end{aligned}$$

This is to say that the first factor accounts for 81% of the variance in the observed variable, the second accounts for 49% and the third 25%.
Next, we consider,

$$\begin{aligned} Corr(Z_i, F_1) &= Cov(Z_i, F_1) \\ &= Cov((l_{i1}F_1 + \varepsilon_i), F_1) \\ &= l_{i1}Var(F_1) + Cov(\varepsilon_i, F_1) && \text{From before, we know these values} \\ &= l_{i1} \end{aligned}$$

Thus,

$$Corr(Z_1, F_1) = 0.9$$
$$Corr(Z_2, F_1) = 0.7$$
$$Corr(Z_3, F_1) = 0.5$$

Based on this, we will expect that the first variable Z_1 has the most weight in "naming" the factor as it has the largest correlation to the factor.

9.3

Given the eigenvalues and associated vectors to the ρ matrix, we consider an $m = 1$ factor model and construct the loading matrix as,

$$\begin{aligned} L &= \sqrt{\lambda_1} e_1 \\ &= \sqrt{1.96} \begin{bmatrix} 0.625 \\ 0.593 \\ 0.507 \end{bmatrix} \\ &= \begin{bmatrix} 0.8757 \\ 0.8311 \\ 0.7111 \end{bmatrix} \end{aligned}$$

Then,

$$\Psi = \rho - LL^T = \begin{bmatrix} 0.2330860 & -0.0978302 & -0.1727161 \\ -0.0978302 & 0.3092618 & -0.2409810 \\ -0.1727161 & -0.2409810 & 0.4943692 \end{bmatrix}$$

Taking the diagonal entries of this matrix,

$$\Psi = \begin{bmatrix} 0.2330860 & 0 & 0 \\ 0 & 0.3092618 & 0 \\ 0 & 0 & 0.4943692 \end{bmatrix}$$

We see that this matrix is different than the original from 9.1. We next compute the proportion of total variance explained by the first factor as,

$$\frac{\lambda_1}{\sum \lambda_i} = \frac{1.963283}{3} = 0.6544277$$

9.4

We compute,

$$\tilde{\rho} = \begin{bmatrix} 0.81 & 0.63 & 0.45 \\ 0.63 & 0.49 & 0.35 \\ 0.45 & 0.35 & 0.25 \end{bmatrix}$$

We then compute the eigenvalues and eigenvectors,

$$\lambda_i = 1.55, \ 0, \ 0$$

We want the eigenvector for the largest eigenvalue,

$$e_1 = \begin{bmatrix} 0.7228974 \\ 0.5622535 \\ 0.4016097 \end{bmatrix}$$

Then, $L = \sqrt{\lambda_1} e_1 = \begin{bmatrix} 0.9 \\ 0.7 \\ 0.5 \end{bmatrix}$ This aligns with the original values in 9.1, as we would expect for $m = 1$.

9.12

We begin by computing S_n as,

$$S_n = \frac{23}{24} S = \begin{bmatrix} 0.010610667 & 0.007684875 & 0.007820000 \\ 0.007684875 & 0.006149625 & 0.005754792 \\ 0.007820000 & 0.005754792 & 0.006490792 \end{bmatrix}$$

And,

$$\Psi = S_n - LL^T = \begin{bmatrix} 0.0001658267 & -5.650000e-07 & 1.700000e-06 \\ -0.0000005650 & 4.945850e-04 & 1.991667e-06 \\ 0.0000017000 & 1.991667e-06 & 6.385417e-04 \end{bmatrix}$$

Taking the diagonals, we construct the following table,

Variable	Specific Variance
ln(Length)	0.0001658267
ln(Width)	0.0004945850
ln(Height)	0.0006385417

Next, we compute the communalities for the variables as before as,

Variable	Communality
ln(Length)	$0.1022^2 = 0.01044484$
ln(Width)	$0.0752^2 = 0.00565504$
ln(Height)	$0.0765^2 = 0.00585225$

Next, we compute the proportion of variance explained by the factor as,

$$\frac{\sum h_i^2}{\sum S_{n_{ii}}} = \frac{0.02195213}{0.02325108} = 0.9441336$$

Finally, we compute the residual matrix as,

$$Resid = S_n - LL^T - Psi = \begin{bmatrix} 0 & -5.650000e-07 & 1.700000e-06 \\ -5.65e-07 & 0 & 1.991667e-06 \\ 1.70e-06 & 1.991667e-06 & 0 \end{bmatrix}$$

9.13

We consider the following hypotheses,

$$H_0 = \Sigma = LL^T + \Psi$$
$$H_A = \Sigma \text{ unrestricted}$$

We compute the test statistic,

$$(n - 1(2p + 4m + 5)/6) \ln \left[\frac{|LL^T + \Psi|}{|S_n|} \right] = 0.0530115$$

We also test our condition,

$$\begin{aligned} m &< \frac{1}{2}\left(2p + 1 - \sqrt{8p+1}\right) \\ &= \frac{1}{2}(2(3) + 1 - \sqrt{8(3)+1}) \\ &= \frac{1}{2}(7-5) \\ &= 1 \end{aligned}$$

But, since we have defined $m = 1$ in our null hypothesis, this condition does not hold.

9.20

We consider the data from Table 1.5 for variables 1, 2, 5, and 6. To begin, we construct the S matrix in R,

$$\begin{bmatrix} 2.5000000 & -2.780488 & -0.5853659 & -2.231707 \\ -2.7804878 & 300.515679 & 6.7630662 & 30.790941 \\ -0.5853659 & 6.763066 & 11.3635308 & 3.126597 \\ -2.2317073 & 30.790941 & 3.1265970 & 30.978513 \end{bmatrix}$$

We then compute the PCA solution for a $m = 1$ model by computing the following,

Variable	Factor 1	Communality
X_1	-0.1749782	0.03061737
X_2	17.3246829	300.14463897
X_5	0.4213923	0.17757147
X_6	1.9587473	3.83669086

Here, we see that the variance of this model is 304.1895 which accounts for 88.07955% of the total variance.
Next, we consider the $m = 2$ model,

Variable	Factor 1	Factor 2	Communality
X_1	-0.1749782	-0.4048141	0.1944918
X_2	17.3246829	-0.6085601	300.5149843
X_5	0.4213923	0.7421918	0.7284201
X_6	1.9587473	5.1867451	30.7390159

This model has a combined variance of 332.18, which accounts for 96.18343% of the total variance.
We now consider the factor analysis of this model with $m = 1$,

Variable	Factor 1	Communality
X_1	-0.3241708	0.10508670
X_2	0.4096201	0.16778859
X_5	0.2316952	0.05368266
X_6	0.7710411	0.59450442

This model has a variance of 0.921 which accounts for 23% of the overall variance.
We now consider the $m = 2$ model,

Variable	Factor 1	Factor 2	Communality
X_1	-0.398	0.347	0.27841057
X_2	0.479	0.256	0.29469219
X_5	0.255	-0.024	0.06556664
X_6	0.656	0.021	0.43043576

We may then tabulate Ψ for each of these methods as,

$$\begin{aligned}\Psi_1 &= \begin{bmatrix} 1-.10508670 & & & \\ & 1-0.16778859 & & \\ & & 1-0.05368266 & \\ & & & 1-0.59450442 \end{bmatrix} \\ &= \begin{bmatrix} 0.8949133 & & & \\ & 0.8322114 & & \\ & & 0.9463173 & \\ & & & 0.4054956 \end{bmatrix} \\ \Psi_2 &= \begin{bmatrix} 1-0.27841057 & & & \\ & 1-0.29469219 & & \\ & & 1-0.06556664 & \\ & & & 1-0.43043576 \end{bmatrix} \\ &= \begin{bmatrix} 0.7215894 & & & \\ & 0.7053078 & & \\ & & 0.9344334 & \\ & & & 0.5695642 \end{bmatrix}\end{aligned}$$

We see that for the PCA model with $m = 2$, X_2 and X_6 are the largest in factor 1, and for factor two it is X_5 and X_6. For the MLE analysis, we see that the X_2 and X_6 dominate Factor 1, and X_1 and X_2 dominate factor 2.

9.21

Next, we consider the $m = 2$ case, and perform the varimax rotation. Using built in R functions, we compute the rotation matrix to be,

$$T = \begin{bmatrix} 0.9724650 & 0.2330489 \\ -0.2330489 & 0.9724650 \end{bmatrix}$$

Our PCA two factor analysis is then transformed to,

Variable	Factor 1	Factor 2	Communality
X_1	-0.07581869	-0.4344461	0.1944918
X_2	16.98947231	3.4456951	300.515
X_5	0.23682229	0.8199605	0.7284201
X_6	0.69604788	5.5004121	30.73902

Here, factor 1 has variance 289.1885 which accounts for 83.73593% of the variance. Factor 2 then has variance 42.98843 which accounts for 12.44751%. Using R, we perform this varimax analysis on the max likelyhood analysis, arriving at,

Variable	Factor 1	Factor 2	Communality
X_1	-0.112	-0.516	0.2784106
X_2	0.537	0.080	0.2946922
X_5	0.190	0.172	0.06556664
X_6	0.538	0.375	0.4304358

Here, factor 1 has variance 0.626 which accounts for 15.7% of the variance. Factor 2 then has variance 0.443 which accounts for 11.1%. We also note that these results are fairly similar to those from before.

9.23

We now construct the correlation matrix R as,

$$R = \begin{bmatrix} 1.0000000 & -0.1014419 & -0.1098249 & -0.2535928 \\ -0.1014419 & 1.0000000 & 0.1157320 & 0.3191237 \\ -0.1098249 & 0.1157320 & 1.0000000 & 0.1666422 \\ -0.2535928 & 0.3191237 & 0.1666422 & 1.0000000 \end{bmatrix}$$

Our 1 and 2 factor PC solutions are then,

Variable	Factor 1	Factor 2
X_1	-0.5638413	0.2427710
X_2	0.6450049	0.5212837
X_5	0.4769735	-0.7351875
X_6	0.7708354	0.1963048
Variance	1.555639	0.9097107
% Explained	38.89098	0.2274277

The maximal likelyhood estimates will be the same as before, as the R matrix is computaionally singular for the fractanal funtion in R. We see that compared to 9.20, the relevant variables for each factor are different, which suggests that different results are obtained for different matrices.