6.1

Proof. Let P be an orthogonal projector, and consider the value of $I-2P$. First, consider,

$$\begin{aligned}(I-2P)^* &= -2P*+I* && \text{By Theorem 6.1, } P^*=P\\ &= I-2P\end{aligned}$$

So, we see that $(I-2P)^* = (I-2P)$ so it is a projector. Next, we consider,

$$\begin{aligned}(I-2P)(I-2P)^* &= I^2-4IP+4P^2 && \text{By definition of a projector, } P^2=P\\ &= I-4P+4P\\ &= I\end{aligned}$$

Thus, we see that the quantity $(I-2P)$ is unitary because its complement is itself and its inverse. □

We now consider the geometric meaning of $(I-2P)$. We know that the operator P projects a vector v onto the basis of P. We also know that $(I-P)v$ is orthogonal to Pv, an almost mirrored projection of v about v. If we consider the flashlight analogy, Pv is a light being shined towards the basis of P. Then, mirroring the light position about v, $(I-P)v$ projects v in the orthogonal direction. As such, we know that $(I-2P)$ will project it in the orthoginal direction, but twice as far.

6.3

Consider, $A\in\mathbb{C}^{m\times n}$ with $m\geq n$. First,
Consider A^*A to be singular. Then, by definition of singularity, we have a vector $x\neq 0$ such that $A^*Ax=0$. Thus, we see that $x^*A^*Ax=0$ as well. Because of this, we see that A^*A is not full rank as the determinant is zero.
Second,
Consdider A^*A as not full rank. Then, $\exists x\neq 0$ such that $Ax=0$. Then, we may also conclude $A^*Ax=0$. Because $x\neq 0$, we see that the matrix A^*A must be singular. So, we have shown that the contrapositive is true in both cases.

6.4

We consider the matrices,

$$A=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix},\ B=\begin{bmatrix}1&2\\0&1\\1&0\end{bmatrix}$$

From the text, we know that for a matrix M, the associated projection matrix is,

$$P=M(M^*M)^{-1}M^*$$

So, we compute as follows,

1. Matrix A,
   (a) We compute,
   $$(A^*A)^{-1}=\begin{bmatrix}\frac{1}{2}&0\\0&1\end{bmatrix}$$
   Then,
   $$P_A=A(A^*A)^{-1}A^*=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix}\begin{bmatrix}\frac{1}{2}&0\\0&1\end{bmatrix}\begin{bmatrix}1&0&1\\0&1&0\end{bmatrix}=\begin{bmatrix}\frac{1}{2}&0&\frac{1}{2}\\0&1&0\\\frac{1}{2}&0&\frac{1}{2}\end{bmatrix}$$
   (b) We then consider,
   $$P_A\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}\frac{1}{2}+\frac{3}{2}\\2\\\frac{1}{2}+\frac{3}{2}\end{bmatrix}=\begin{bmatrix}2\\2\\2\end{bmatrix}$$
2. Matrix B,
   (a) We compute,
   $$(B^*B)^{-1}=\frac{1}{6}\begin{bmatrix}5&-2\\-2&2\end{bmatrix}$$
   Then,
   $$P_B=B(B^*B)^{-1}B^*=\frac{1}{6}\begin{bmatrix}1&2\\0&1\\1&0\end{bmatrix}\begin{bmatrix}5&-2\\-2&2\end{bmatrix}\begin{bmatrix}1&0&1\\2&1&0\end{bmatrix}=\frac{1}{6}\begin{bmatrix}5&2&1\\2&2&-2\\1&-2&5\end{bmatrix}$$

(b) We then consider,

$$P_B\begin{bmatrix}1\\2\\3\end{bmatrix}=\begin{bmatrix}\frac{5+4+3}{6}\\\frac{2+4-6}{6}\\\frac{1-4+15}{6}\end{bmatrix}=\begin{bmatrix}2\\0\\2\end{bmatrix}$$

6.5

We consider P to be a nonzero projector. Let x be a nonzero vector such that $Px=\alpha\neq 0$. Then, by definition of a projector,

$$P^2x=Px=\alpha$$

But,

$$P^2x=P(Px)=P\alpha=\alpha$$

So, we have shown that $Px=x$. Then,

$$\frac{||Px||}{||x||}=1$$

Because we know that $||Px||\leq||P||\ ||x||$, we conclude that $||P||\geq 1$. Next, consider P as orthogonal. Then, we know $P^*=P$. Consider the SVD of P, $P=U\Sigma V^*$. Given the nature of this decomposition $UU^*=VV^*=I$. Then,

$$||P||_2=||P^2||_2=||PP^*||_2=||U\Sigma V^*V\Sigma^*U^*||_2=||U\Sigma\Sigma^*U^*||_2=||\Sigma\Sigma^*||_2=\sigma_1^2$$

The largest singular value squared. But, $||P||_2=||U\Sigma V^*||_2=||\Sigma||_2=\sigma_1$. So, we have,

$$\sigma_1^2=\sigma_1=1$$

Thus, with orthagonality, we have equality to one.

7.1

Using Graham Schmidt, we compute,

1. Matrix A,

$$A=\begin{bmatrix}1&0\\0&1\\1&0\end{bmatrix},\ a_1=\begin{bmatrix}1\\0\\1\end{bmatrix},\ a_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$$

Then,

$$\begin{aligned}u_1&=\begin{bmatrix}1\\0\\1\end{bmatrix}\\e_1&=\begin{bmatrix}1/\sqrt{2}\\0\\1/\sqrt{2}\end{bmatrix}\\u_2&=a_2-(a_2\cdot e_1)e_1\\&=\begin{bmatrix}0\\1\\0\end{bmatrix}\\e_2&=\begin{bmatrix}0\\1\\0\end{bmatrix}\end{aligned}$$

Then,

$$\hat{Q}=\begin{bmatrix}1/\sqrt{2}&0\\0&1\\1/\sqrt{2}&0\end{bmatrix},\ \hat{R}=\begin{bmatrix}\sqrt{2}&0\\0&1\end{bmatrix}$$

Expanding to the full QR factorization, $q_3=q_1\times q_2$,

$$Q=\begin{bmatrix}1/\sqrt{2}&0&-1/\sqrt{2}\\0&1&0\\1/\sqrt{2}&0&1/\sqrt{2}\end{bmatrix},\ R=\begin{bmatrix}\sqrt{2}&0\\0&1\\0&0\end{bmatrix}$$

2. Matrix B,

$$B = \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix},\ b_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},\ b_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

Then,

$$\begin{aligned} u_1 &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \\ e_1 &= \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix} \\ u_2 &= a_2 - (a_2 \cdot e_1)e_1 \\ &= \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} \\ e_2 &= \begin{bmatrix} 1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \end{bmatrix} \end{aligned}$$

Then,

$$\hat{Q} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} \\ 0 & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{3} \end{bmatrix},\ \hat{R} = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}$$

Expanding to the full QR factorization, $q_3 = q_1 \times q_2$,

$$Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & -1/\sqrt{6} \\ 0 & 1/\sqrt{3} & 2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{3} & 1/\sqrt{6} \end{bmatrix},\ R = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \\ 0 & 0 \end{bmatrix}$$

7.3

We consider a matrix $A,\ m \times m$ with QR factorization $A = QR$. Then,

$$det(A) = det(Q)det(R) = det(R) = \Pi_{j=1}^{m}$$

Then,

$$a_j = \sum_{i=1}^{j} r_{ij} q_j$$

So, we may expand,

$$||a_j||_2^2 = \sum_{i=1}^{j} ||r_{ij}q_j||_2^2 = \sum_{i=1}^{j} ||r_{ij}||_2^2 \geq ||r_{jj}||_2^2$$

We note that the 2-norm of a larger set of numbers must be greater than or equal to only one number. Taking the product of both sides then,

$$\Pi_{j=1}^{m} ||a_j||_2^2 \geq \Pi_{j=1}^{m} ||r_{jj}||_2^2$$

From before,

$$\Pi_{j=1}^{m} ||a_j||_2^2 \geq \Pi_{j=1}^{m} ||r_{jj}||_2^2 = |det(A)|$$

As desired. Geometrically, we see that this states that the volume of a rectangular parallelepiped is greater than or equal to the volume of a general parallelepiped with the same side lengths.

7

We consider the Householder matrix $H = I - 2\frac{vv^*}{v^*v}$.

a. $Hv = -v$

Proof. Let $H = I - 2\frac{vv^*}{v^*v}$, and consider,

$$\begin{aligned} Hv &= (I - 2\frac{vv^*}{v^*v})v \\ &= v - 2\frac{vv^*v}{v^*v} \\ &= v - 2v \\ &= -v \end{aligned}$$

□

b. $u \perp v \Rightarrow Hu = u$

Proof. Let $H = I - 2\frac{vv^*}{v^*v}$, and consider $u \perp v$. Then,

$$\begin{aligned} Hu &= (I - 2\frac{vv^*}{v^*v})u \\ &= u - 2\frac{vv^*u}{v^*v} && \text{Note that } v^*u = 0 \\ &= u \end{aligned}$$

□

c. $H(\gamma x) = Hx$ for $\gamma \in \mathbb{C}$.

Proof. Let $H = I - 2\frac{vv^*}{v^*v}$, and consider $\gamma \in \mathbb{C}$. Then,

$$\begin{aligned} H(\gamma v) &= (I - 2\frac{vv^*}{v^*v})(\gamma v) \\ &= \gamma v - 2\gamma\frac{vv^*v}{v^*v} \\ &= \gamma v - 2\gamma v \\ &= -\gamma v \end{aligned}$$

Let $y = \gamma v$. We know that $Hy = -y = -\gamma v$ as we desire. □

d. $H^*H = H^2 = I$.

Proof. Let $H = I - 2\frac{vv^*}{v^*v}$ and consider,

$$H^* = I^* - 2\frac{(v^*)^*v^*}{v^*(v^*)^*} = I - 2\frac{vv^*}{v^*v} = H$$

Then,

$$H^*H = HH = H^2 = (I - 2\frac{vv^*}{v^*v})(I - 2\frac{vv^*}{v^*v})$$

Solving,

$$\begin{aligned} (I - 2\frac{vv^*}{v^*v})(I - 2\frac{vv^*}{v^*v}) &= I - 4\frac{vv^*}{v^*v} + 4\frac{vv^*vv^*}{v^*vv^*v} \\ &= I - 4\frac{vv^*}{v^*v} + 4\frac{v(v^*v)v^*}{(v^*v)^2} \\ &= I - 4\frac{vv^*}{v^*v} + 4\frac{vv^*}{v^*v} \\ &= I \end{aligned}$$

As desired. □

e. $H^{-1} = H$.

Proof. From (d), we have shown $H^*H = HH = I$, which is the same as saying H is its own inverse. □

f. We now compute the eigenvalues, determinant and singular values of H. First, we note that there are three types of vectors in relation to the Householder matrix.

   (a) Perpindicular to v

   (b) Parallel to v

   (c) Sum of the two.

   We recall from above, that for a parallel vector, we have $Hv = -v \Rightarrow \lambda = -1$, and for a perpindicular vector, $Hu = u \Rightarrow \lambda = 1$. Finally, for a vector that is a function of both, we can write $y = \alpha(v) + \beta(u)$, which has one eigenvalue of -1 and the rest as positive ones.

   Next, we see that the determinant is the product of the eigenvalues, which is, -1.

   Finally, we compute the singular values of the matrix as the positive roots of the eigenvalues of H^*H. Because $H * H = I$, we see that the only values are $\sigma = 1$.

10.2

See Jupyter Notebook

10.3

We consider,

$$Z = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 7 \\ 4 & 2 & 3 \\ 4 & 2 & 2 \end{bmatrix}$$

Using our constructed QR code, we arrive at,

$$Q = \begin{bmatrix} -0.10101525 & -0.31617307 & 0.5419969 & -0.68420846 & -0.35767115 \\ -0.40406102 & -0.3533699 & 0.51618752 & 0.32800841 & 0.58122744 \\ -0.70710678 & -0.39056673 & -0.52479065 & 0.00939722 & -0.26826124 \\ -0.40406102 & 0.55795248 & 0.38714064 & 0.36559727 & -0.49181753 \\ -0.40406102 & 0.55795248 & -0.12044376 & -0.53899869 & 0.46946506 \end{bmatrix}$$

$$R = \begin{bmatrix} -9.89949494 & -9.49543392 & -9.69746443 \\ 0 & -3.29191961 & -3.01294337 \\ 0 & 0 & 1.97011572 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Using Python's built in QR function, we see that we obtain the same results. The major difference between the two is that the integers in Z must be cast as doubles for the calculations in our constructed function, which is slightly more inconvenient than simply typing the integers.

11.3

We consider the function $b = \cos(4t)$ over the interval $[0, 1]$. First, we construct the X matrix using the flip and vandermonde commands. We construct the matrix such that,

$$X = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 1 & 0.0204 & 4.16e-4 & \dots & 2.55766e-19 \\ 1 & 0.0408 & 1.66e-3 & \dots & 5.23e-16 \\ \dots 1 & 1 & 1 & \dots & 1 \end{bmatrix}$$

We then consider the matrix equation,

$$\begin{aligned} Xw &= b \\ Aw &= X^T b \\ QRw &= X^t b \\ Rw &= (XQ)^T b \\ w &= R^{-1}(XQ)^T b \end{aligned}$$

Using this analysis, we solve for the weights in the Jupyter Notebook.

***Note: For the different methods described in this problem, we obtain slightly different values for each of the weights.

We see that this is because the high powers of t in our X matrix are around machine epsilon. This results in error that propogates throughout the computation due to its repeated use in the computation of not only X, but also the A, Q, R matrices as well.